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1 Introduction

The Physical key definition underlying all modern physics.

> Spacetime is a four - dimensional topological manifold with a smooth atlas carrying a torsion-free connection compatible with a Lorentzian metric and a time orientation satisfying the Einstien Equations

2 Topological Spaces

Want to talk about space time and at the coarsest level, space time is a set. Each point is an element of the set. However, this set is not enough to talk about continuity of maps.

Why as Physicist why do we want to talk about maps. In classical physics, curves do not jump. Particles move along curves with no jumps. Structures on the set allows you to talk about the set.

Intrested in the Weakest structure we can establish on set which allows a good definition of continuitiy of maps. As mathematicians we can establish the weakest structure on a set called a **topology**.

Definition 2.1. Topology Let M be a set. A topology $\mathcal{O} \subseteq \mathcal{P}(M)$ where $\mathcal{P}(M)$ is the power set of M. So you make a choice of sets to be be in $\mathcal{O}$. These should satisfy:

1. $\emptyset \in \mathcal{O}$ is required and $M \in \mathcal{O}$
2. Let $U \in \mathcal{O}$ and $V \in \mathcal{O}$. Then the intersection of them is in $\mathcal{O}$, $U \bigcap V \in \mathcal{O}$.
3. Assume $U_\alpha \in \mathcal{O}$ $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{O}$

Example 2.1. 1. $M = \{1,2,3\}$, $\mathcal{O}_1 = \{\emptyset, \{1,2,3\}\}$ is a topology.

- The empty set is in $\mathcal{O}$ and $M \in \mathcal{O}$ so (1) qualifies.
- The intersection of both sets is the empty set so (2) checks out.
- The union of all sets is just (2) so (3) checks out.

2. $M = \{1,2,3\}$, $\mathcal{O}_2 = \{\emptyset, \{1\}, \{2\}\{1,2,3\}\}$ is not a topology.

Example 2.2. Let M be any set. Then

1. The chaotic topology is defined as: $\boxed{\mathcal{O}_{chaotic} = \{\emptyset, M\}}$.
2. The discrete topology is defined as: $\boxed{\mathcal{O}_{discrete} = \{\emptyset, \mathcal{P}(M)\}}$

Why do we care about these? They are utterly useless :). They are extreme cases. The chaotic has the fewest possible and the discrete topology has the most elements. They serve as test cases for topologies.

There is a very important example which will help reconcile continuity, the standard topology.

Definition 2.2. The Standard Topology Let $M = \mathbb{R}^d = \mathbb{R} \times \mathbb{R} ... \times \mathbb{R} = \{(p_1, ..., p_d) : p_i \in \mathbb{R}\}$. The standard topology is $\mathcal{O}_{standard} \subseteq \mathcal{P}(\mathbb{R}^d)$.

The standard topology contains non-countable many elements. So we define

1. The **softball**, $B_r(p)$, where $r \in \mathbb{R}^+$ and $p \in \mathbb{R}^d$ where $B_r(p) = \{(q_1, ..., q_d) : \sum_{i=1}^{d} (q_i - p_i)^2 < r^2\}$. You can think of r as the radius of the ball and p as the center of the ball. There is a softball for any r and any p.
2. $U \in \mathcal{O}_{standard}$ implies that $\forall p \in U \exists r \in \mathbb{R}^+$ where $B_r(p) \subseteq U$.

Example 2.3.

A simple exercies why the standard topology satisfies these three axioms.

Eventually we are going to think of M as points in spacetime. We defined topologies in order to define the continuitiy of maps. Topologies are chosen to provide structure.

3 Terminology

Table 1: caption

Symbol	Name	Definition
M	Set	(ZFC)
$\mathcal{O}$	Topology	A set of open sets.
$(M, \mathcal{O})$	Topological Space	
$U \in \mathcal{O}$, $U \subseteq M$	Open Set	
$M/A \in \mathcal{O}$, $A \subseteq M$	Closed Set	

4 Continous Maps

A map $f : M \to N$