1 Difference of Powers formula

Difference of Power is the generalization:

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}) \tag{1}$$

$$= (a-b)\sum_{k=0}^{n-1} a^{n-1-k}b^k \tag{2}$$

where $\{n : n \in \mathbb{N}\, n \text{ and } \geq 2\}$

In fact, we can use this result to show an upper bound and lower bound for $b^n - a^n$. If $0 < a < b$ then

$$(b-a)na^{n-1} < b^n - a^n < (b-a)nb^{n-1} \tag{3}$$

Proof. We assume that (2) is true and we try to obtain the left hand side of (1). then

$$\begin{aligned}
(a-b)\sum_{k=0}^{n-1} a^{n-1-k}b^k &= a\sum_{k=0}^{n-1} a^{n-1-k}b^k - b\sum_{k=0}^{n-1} a^{n-1-k}b^k \\
&= \sum_{k=0}^{n-1} a^{n-k}b^k - \sum_{k=0}^{n-1} a^{n-1-k}b^{k+1} \\
&= a^n + \left(\sum_{k=1}^{n-1} a^{n-k}b^k - \sum_{k=0}^{n-2} a^{n-1-k}b^{k+1}\right) - b^n \\
&= a^n - b^n
\end{aligned}$$

So we are able to show the relation. Now we want to bound the for (3). We can do this by supposing $0 < a < b \in \mathbb{R}$ and that (2) holds.

To determine an upper bound, we replace a with b (since $b > a$). Multiply both sides of (2) by -1.

$$\begin{aligned}
(b-a)\sum_{k=0}^{n-1} b^{n-1-k}a^k &< (b-a)\sum_{k=0}^{n-1} b^{n-1-k}b^k \\
&= (b-a)\sum_{k=0}^{n-1} b^{n-1} \\
&= (b-a)nb^{n-1}
\end{aligned}$$

So we see that $(b-a)nb^{n-1}$ is an upper bound for $b^n - a^n$.

Without loss of generality we assume the reverse. For a lower bound we do the same thing but replace b with a. Then we see that $b^n - a^n$ can be bounded from below.

□

2 Bernoulli Inequality

3 Every Real number has n^{th} root for all natural numbers