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Definition 1.1 (The exponential Function)**.** The function $\exp : \mathbb{R} \to \mathbb{R}$ is is defined by $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Let $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$. We wish t that $\phi = \exp$

Proposition 1.1. *Let us fix $x \in \mathbb{R}$. Then:*

1. *For every natural number $n > |x|$, then $\left(1 + \frac{x}{n}\right)^n < (1 - \frac{x}{n})^{-n}$*

   Proof. Suppose $n > |x|$. Then $0 \leq \frac{x^2}{n^2} < 1$. Then using this fact, consider the product

   $$\begin{aligned} \left(1 + \frac{x}{n}\right)^n \cdot \left(1 - \frac{x}{n}\right)^n &= \left(\left(1 + \frac{x}{n}\right)\left(1 - \frac{x}{n}\right)\right)^n \\ &= \left(1 - \frac{x^2}{n^2}\right) \\ &< 1 \end{aligned}$$

   This follows from multiplying both parts of the previous inequality by the same power. If we divide both sides by $\left(1 - \frac{x}{n}\right)^n$, we conclude that $\left(1 + \frac{x}{n}\right)^n < \left(1 - \frac{x}{n}\right)^{-n}$.

   $\square$

2. *The sequence $\left(1 + \frac{x}{n}\right)^n$ for $n > |x|$ is strictly increasing.*

   Proof. Suppose $n > |x|$, we want to show that $(n + 1)$st element of the sequence is greather than the nth element of the sequence.

   That is $\left(x + \frac{x}{n}\right)^n \leq \left(x + \frac{x}{n+1}\right)^{n+1}$. The A + M inequality: Assume that $n \geq 2$ and given $a_1, \ldots a_n \in \mathbb{R}$, $(a_1 \ldots a_n)^{1/n} \leq \frac{a_1 + \ldots a_n}{n}$. Consider $a_1 = 1, a_2 = \left(1 + \frac{x}{n}\right), \ldots a_{n+1} = \left(1 + \frac{x}{n}\right)$. $\square$

3. *The sequence $\left(1 - \frac{x}{n}\right)^{-n}$ for $n > |x|$ is strictly decreasing.*

4. $\left(1 + \frac{x}{n}\right)^n - \left(1 - \frac{x}{n}\right)^{-n} \to 0$ *as $n \to \infty$. These two sequences approach each other from the left and right.*

By (2), (3), and (4) of the proposition for $n > |x|$, the sequence of the closed intervals,

$$\left[\left(1 + \frac{x}{n}\right)^n, \left(1 - \frac{x}{n}\right)^{-n}\right]$$

is nested. By the **Nested Interval Lemma**, there is a unique real number contained in the intersection of all of these intervals. The common limit of the sequences of left endpoints and the sequence of right endpoints.

This enables us to make the following defintion

Definition 1.2. Define the function $\phi : \mathbb{R} \to \mathbb{R}$ for each $x \in \mathbb{R}$ where

$$\phi(x) = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = \left(1 - \frac{x}{n}\right)^{-n} \tag{1}$$

Note: $\phi(0) = 1$. We also define $e = \phi(1)$ as the **Euler's constant**.

Remark 1 *1. For every natural number $n \in \mathbb{N}$ such that $n > |x|$*

$$\left(1 + \frac{x}{n}\right)^n \leq \phi(x) \leq \left(1 - \frac{x}{n}\right)^{-n} \tag{2}$$

2.