Contents

1 Power Inequality

Theorem 1.1 (Power Inequality)**.** *For every $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$ if $0 \leq x < y$, then $0 \leq x^n < y^n$*

To prove this let us start with a lemma.

Lemma 1.1.1. *Suppose $a, b, c \in \mathbb{R}$. If $a < b$ and $c > 0$, then $ac < bc$.*

Proof. Suppose towards a contradiction that $ac = bc$. Then by the equality we require $bc \leq ac$. Since we have $c > 0$, then $c^{-1} > 0$. Then we have

$$b = bcc^{-1} \leq acc^{-1} = a$$

which contradicts the assumption that $a < b$. □

Proof. Let $P(n)$: for all $x, y \in \mathbb{R}$, if $0 \leq x < y$, then $x^n < y^n$.

Base Case
Take $P(1) : 0 \leq x < y$ which implies $0 \leq x^1 < y^1$. Which is true based on the given relationship of x and y.

Inductive Step
Suppose $P(n)$ holds, that is $\forall x, y, \in \mathbb{R}$, if $0 \leq x < y$, then $0 \leq x^n < y^n$. Let $x, y \in R$, $0 \leq x < y$. Then

$$\begin{aligned} x^{n+1} &= x^n \cdot x \leq x^n \cdot y && \text{(using } x < y \text{, and } 0 \leq x^n\text{)} \\ &< y^n = y^{n+1} \cdot y && \text{(by the claim using } x^n < y^n \text{ and } 0 < y\text{)} \end{aligned}$$

This then implies $x^{n+1} < y^{n+1}$ (1). Finally, since $0 \leq x^n$, and $0 \leq x$, then $0 \leq x^{n+1}$ (2). Taking these two (1 and 2) we have the desired result.

□

2 The Real Numbers and the Completeness Axiom

Let $S \subseteq \mathbb{R}$, where $S \neq \emptyset$.

1. The largest element of S (if there is one), is called the *maximum of* S, $\max S$
2. The least element of S (if there is one), is called the *minimum of* S, $\min S$

Example 2.1. Every finite non-empty $S \subseteq \mathbb{R}$ has a maximum and minimum.

Example 2.2. Let $a < b \in \mathbb{R}$. Then consider the closed interval

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$

has minimum $\min[a, b] = a$ and maximum $\max[a, b] = b$.

Example 2.3. The open interval

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

has neither a max nor a min.

Example 2.4. $\mathbb{Z}, \mathbb{Q}, \subseteq \mathbb{R}$ do not have a min or max. Consider $a \in \mathbb{Z}$. Then there exists and $a + 1 \in \mathbb{Z}$ hence there is no max. Conversely the same arg for min.

Definition 2.1 (Bounds)**.** Let $S \subseteq \mathbb{R}$ where $S \neq \emptyset$.

1. We call $M \in \mathbb{R}$ **an upper bound for** S if $M \geq S$ for all $s \in S$.
2. We call $m \in \mathbb{R}$ **a lower bound for** S if $m \leq S$ for all $s \in S$.
3. We say that S is bounded if S is bounded from above and from below.

Example 2.5. The max of a set (if it exists) is an upper bound. The min of a set (if it exists) is a lower bound.

Example 2.6. Let $a < b \in \mathbb{R}$ then a is a lower bound for $[a, b]$ and (a, b). b is an upper bound for both sets as well.

Example 2.7. Neith of the set $\mathbb{Z}$ or $\mathbb{Q}$ are bounded from below or above.

Definition 2.2. Let $S \subseteq \mathbb{R}$ where $S \neq \emptyset$.

1. If S is bounded from above and has a least upper bound, s_0, then s_0 is the **supremum** of S, $s_0 = \sup S$.
2. If S is bounded from below and has a greatest lower bound, s_1, then s_1 is the **infinum** of S, $s_1 = \inf S$.

Remark 1 *1. Every $S \subseteq \mathbb{R}, S \neq \emptyset$, can have a most one supremum and one infinum.*

2. If $S \subseteq \mathbb{R}$ has a max, then $\max S = \sup S$.

3. The following are equivalent:

- $s_0 = \sup S$
- $s_0 \geq s \forall s \in S$, *and if* $s_1 \geq s \forall s \in S$, *then* $s_1 \geq s_0$
- $s_0 \geq s \forall s \in S$, *and if* $s_1 < s_0$, *then* $s_1 < s$ *for some* $s \in S$.

Example 2.8. For $a < b \in \mathbb{R}$,

1. $\sup[a, b] = \sup(a, b) = b$
2. $\inf[a, b] = \inf(a, b) = a$

Example 2.9. Let $A = \{\frac{1}{n^2} : n \in \mathbb{N}, n \geq 3\}$ A is bounded from above and from below. $\max A = 1/3$, however is there is no minimum.

1. $\inf A = 0$. A is the set of values where each value is equivalent to $1/n^2$ for values of $n \geq 3$. So we can choose n to be sufficiently large. Because $n \to \infty$, $\frac{1}{n^2} \to 0$. So we know that 0 is the greatest lowerbound.
2. $\sup A = \max A = 1/3$

Definition 2.3 (Completeness Axiom)**.** Every non-empty subset of $\mathbb{R}$ which is bounded from above has a least upper bound. This is equivalent to: Given $S \subseteq \mathbb{R}$, $S \neq \emptyset$, if S has at least one upper bound, then $\sup S$, exists.

Remark 2
The Completeness Axiom failes for $\mathbb{Q}$.

$$\begin{aligned} A &= \{r \in \mathbb{Q} : 0 \leq r \text{ and } r^2 \leq 2\} \\ &= \{r \in \mathbb{Q} : 0 \leq r \leq \sqrt{2}\} \end{aligned}$$

A is bounded from above (e.g. $\frac{3}{2}$ is an upper bound), but the $\sup A = \sqrt{2} \notin \mathbb{Q}$.

Corollary 2.0.1. *Every* $\emptyset \neq S \subseteq \mathbb{R}$ *that is bounded from below has a greatest lower bound* $\inf S$

Proof. Given $S \subseteq \mathbb{R}$, let $-S = \{-s : s \in S\}$. Given that S is bounded below, $\exists m \in \mathbb{R}$ such that $m \leq s$ for $s \in S$. This implies that $-m \geq -s$ for all $s \in S$.

So $-m \geq u$, $\forall u \in -S$. Thus $-S$ is bounded from above by $-m$. By the Completeness Axiom for $-S$, the $\sup -S$ exists.

Let $s_0 = \sup -S$. What we need to show is that

1. $-s_0$ is a lower bound of $S(-s_0 \leq s, \forall s \in S)$
2. $-s_0$ is the greatest lower bound (if $t \leq s$, $\forall s \in S$ then $t \leq -s_0$)

.

We take $s_0 \geq -s$ for all $s \in S$. And this implies the condition (1) by multiply both sides by -1.

For the second part assume $t \leq s$, $\forall s \in S$. This is equivalent to

$$\begin{aligned} & -t \geq -s, \forall s \in S \\ \implies & -t \geq u, \forall u \in -S \\ \implies & -t \geq s_0 \\ \implies & t \leq s_0 \end{aligned}$$

□

2.1 What is $\mathbb{R}$

$\mathbb{R}$ is a number system containing $\mathbb{Q}$ and satisfying the Completeness Axiom.

Definition 2.4 (Archimedian Property)**.** The following properties of $\mathbb{R}$ hold.

1. For every positive real number $a > 0$, there is a natural number n such that $n > a$.
2. For every $a, b > 0$ in $\mathbb{R}$, there is an $n \in \mathbb{N}$ such that $n \cdot a > b$
3. For every $\epsilon > 0 \in \mathbb{R}$ there is an $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$

Proof.

1. Assume towards contradiction that $\exists a > 0 \in \mathbb{R}$ such that $a \geq n$ for every $n \in \mathbb{N}$. Thus a is an upper bound for $\mathbb{N} \subseteq \mathbb{R}$. By the completeness axiom, there exist some $b \in \mathbb{R}$ such that $b = \sup \mathbb{N}$. As b is the least upper bound of the set $\mathbb{N}$ the number $b - 1/2$ is not an upper bound for $\mathbb{N}$. In particular, $\exists n \in \mathbb{N}$ such that $n > b - \frac{1}{2}$. This implies that $n+1 > b - \frac{1}{2} + 1 > b$. However, this says that $b \neq \sup N$, which is a contradiction.

2. Suppose $a, b > 0$, in particular $\frac{b}{a} > 0$. By the first property, $\exists n \in \mathbb{N}$ such that $n > \frac{b}{a}$. This rearanging, $n \cdot a > b$.

3. Suppose $\epsilon > 0$, $\epsilon \in \mathbb{R}$. Then $\frac{1}{\epsilon} > 0$. By (1) $\exists n \in \mathbb{N}$ such that math $n > \frac{1}{\epsilon}$. Rearanging, we can get $\epsilon > \frac{1}{n}$.

□

Corollary 2.0.2. *Suppose $a < b \in \mathbb{R}$ and $b - a > 1$. Then there is an integer m such that $a < m < b$.*

Proof. By the Archimedian Property, $\exists k > \max(|a|, |b|)$. Then we know that $-k < a < b < k$. Let the sets $K = \{j \in \mathbb{Z} : -k \leq j \leq k\}$ and $K' = \{j \in \mathbb{Z} : a \leq j\}$ such that both K, K' are finite and non empty as $k \in K' \subseteq K$. □

Theorem 2.1 (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** *For every real numbers $a, b \in \mathbb{R}$ with $a < b$, $\exists r \in Q)$ such that $a < r < b$*

Proof. We need to find a quotient of integers, $m, n \in \mathbb{Z}$ such that $n > 0$ and

$$a < \frac{m}{n} < b$$

We need to choose n such that the demoniator is large enough so that consecutive increments of $1/n$ are too close to step over in the interval (a, b).

Using the Archimedean Property we may pick $\frac{1}{n} < \epsilon$ where $\epsilon = b - a$. □