Social Choice Theory Overview

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1 Introduction

1.1 Profiles

Let X (the alternatives) and V (the voters) be nonempty sets. Let $O(X)$ be the set of strict weak orders on X. A **profile** is a function $\mathbf{P} : ZV \to O(X)$.

- Write $x\mathbf{P}_i y$ for $\langle x, y\rangle \in \mathbf{P}(i)$.
- Let $\mathbf{P}(x, y) = \{i \in V | x\mathbf{P}_i y\}$.
- Let $\mathbf{P}_{|\{x,y\}}$ be the function mapping each $i \in V$ to $\mathbf{P}(i) \cap \{x, y\}^2$.

1.2 SWFs

A **social welfare function** (SWF) is a function $f : O(X)^V \to O(X)$.

- Think of $f(\mathbf{P})$ as society's preference ranking, given the profile $\mathbf{P}$ of individual preference rankings.
- Write $xf(\mathbf{P})y$ for $\langle x, y\rangle \in f(\mathbf{P})$.

2 Axioms

- Universal Domain (UD)
  - Idea: our SWF should work on any collection of ballots.
  - Formally: dom(f) is the set of all profiles.
- Independence of Irrelevant Alternatives (IIA)
  - Idea: the social preference for x vs. y shoudl depend only on how each voter ranked x vs. y.
  - Formally: for any profiles $\mathbf{P}$ and $\mathbf{P}'$ such that $\mathbf{P}_{|\{x,y\}} = \mathbf{P}'_{|\{x,y\}}$:

$$xf(\mathbf{P})y \iff xf(\mathbf{P}')y.$$

- Ranking (R)
  - Idea: our voting system should yield a *ranking* of the candidates.
  - Formally: for any profile $\mathbf{P}$, $f(\mathbf{P})$ is a strict weak order.
- Pareto (P)
  - Idea: if every voter prefers x to y, then society should prefer x to y.
  - Formally: for any profile $\mathbf{P}$ with $V \subseteq \mathbf{P}(x, y)$, we have $xf(\mathbf{P})y$.

- Nondictatorship (ND)
    - Idea: no one voter should have their way no matter how others vote.
    - Formally: there is no $i \in V$ such that for all profiles $\mathbf{P}$, if $x\mathbf{P}_i y$ then $xf(\mathbf{P})y$.

The main results are the following.

Theorem 1 (Arrow 1951)**.** If $|X| \geq 3$ and V is finite, then there is no SWF satisfying Universal Domain, Independence of Irrelevant Alternatives, Ranking, Pareto, and Nondictatorship.

Theorem 2 (Fishburn 1970)**.** If V is infinite, then there is a SWF satisfying Universal Domain, Independence of Irrelevant Alternatives, Ranking, Pareto, and Nondictatorship.