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# Training Neural Networks as Learning Data-Adaptive Kernels: Provable Representation and Approximation Benefits

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## ABSTRACT

Consider the problem: given the data pair  $(\mathbf{x}, \mathbf{y})$  drawn from a population with  $f_*(x) = \mathbf{E}[\mathbf{y}|\mathbf{x} = x]$ , specify a neural network model and run gradient flow on the weights over time until reaching any stationarity. How does  $f_t$ , the function computed by the neural network at time  $t$ , relate to  $f_*$ , in terms of approximation and representation? What are the provable benefits of the adaptive representation by neural networks compared to the prespecified fixed basis representation in the classical nonparametric literature? We answer the above questions via a dynamic reproducing kernel Hilbert space (RKHS) approach indexed by the training process of neural networks. First, we show that when reaching any local stationarity, gradient flow learns an adaptive RKHS representation and performs the global least-squares projection onto the adaptive RKHS, simultaneously. Second, we prove that as the RKHS is data-adaptive and task-specific, the residual for  $f_*$  lies in a subspace that is potentially much smaller than the orthogonal complement of the RKHS. The result formalizes the representation and approximation benefits of neural networks. Finally, we show that the neural network function computed by gradient flow converges to the kernel ridgeless regression with an adaptive kernel, in the limit of vanishing regularization. The adaptive kernel viewpoint provides new angles of studying the approximation, representation, generalization, and optimization advantages of neural networks. Supplementary materials for this article are available online.

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## 1. Introduction

Consider iid data pairs drawn from a joint distribution  $(\mathbf{x}, \mathbf{y}) \sim P = P_x \times P_{y|x}$  on the space  $\mathcal{X} \times \mathcal{Y}$ . At the intersection of statistical learning theory (Vapnik 1998) and approximation theory (Cybenko 1989), the following *approximation* problem requires to be first understood, before any further statistical results to be established. For a model class  $\mathcal{F}$ , one is interested in whether there exists  $f \in \mathcal{F} : \mathcal{X} \rightarrow \mathcal{Y}$  such that the population squared loss is small,

$$L(f) = \mathbf{E}_{(\mathbf{x}, \mathbf{y}) \sim P} \frac{1}{2} (\mathbf{y} - f(\mathbf{x}))^2 = \mathbf{E}_{\mathbf{x} \sim P_x} \frac{1}{2} (f_*(\mathbf{x}) - f(\mathbf{x}))^2 + \mathbf{E}_{(\mathbf{x}, \mathbf{y}) \sim P} \frac{1}{2} (\mathbf{y} - f_*(\mathbf{x}))^2, \quad (1)$$

with the conditional expectation (or Bayes estimator) defined as  $f_*(x) := \mathbf{E}[\mathbf{y}|\mathbf{x} = x]$ . Equation (1) generally reads as approximating  $f_*$  in the mean squared error sense.

Statistically, researchers approach the above question mainly in two ways. The first is by assuming that the conditional expectation  $f_*$  lies in the correct model class  $\mathcal{F}$ . For example, say  $\mathcal{F}$  consists of linear models or splines with a particular order of smoothness, or more broadly functions lying in a reproducing kernel Hilbert space (RKHS). Conceptually, this “well-specification” assumption requires substantial knowledge about what model class  $\mathcal{F}$  might be suitable for the regression task

at hand, which is often unavailable in practice. Within each framework, minimax optimal rates and extensive study have been established in Stone (1980) and Wahba (1990). The second way, which extends the first approach further, considers all  $f_*$  under some mild conditions. Building upon certain *universal approximation theorem*, one studies a sequence of model classes  $\mathcal{F}_\epsilon$  called sieves with  $\epsilon$  changing (Geman and Hwang 1982), such that the class  $\mathcal{F}_\epsilon$  contains an  $\epsilon$ -approximation to any  $f_*$  under some metric. A final result usually requires a careful balancing of the approximation and stochastic error by tuning  $\epsilon$ . Particular cases for the latter approach include polynomials (Stone–Weierstrass, Bernstein), radial-basis (Park and Sandberg 1991; Niyogi and Girosi 1996), and two-layer and multilayer neural networks (Cybenko 1989; Hornik, Stinchcombe, and White 1989; Rahimi and Recht 2008; Anthony and Bartlett 2009; Daniely, Frostig, and Singer 2016; Bach 2017; Poggio et al. 2017; Farrell, Liang, and Misra 2018; Koehler and Risteski 2018).

However, the following significant drawbacks of the above current theory make it inadequate to present an *adaptive* and realistic explanation of the practical success of *neural networks*. First, the function computed in practice could be very different from that claimed in the approximation theory, either by the existence or by constructions. To see this, consider the multilayer neural networks. It is hard to conceive that the function, computed in practice via now-standard stochastic gradient descent (SGD) training procedure, is close to the one asserted

by the universal approximation results. Second, in practice, researchers usually explore different model classes  $\mathcal{F}$  to learn which representation best suits the data. For example, using different kernels machines, random forests, or specify certain architectures then run SGD on neural networks. In this case, strictly speaking, the choice of the model class depends on the data in an *adaptive* way, without prior knowledge about the basis. There have been substantial advances made to address the above two concerns—for instance, Jones (1992) on the first and Huang, Cheang, and Barron (2008) and Barron et al. (2008) on the second—for  $\mathcal{F}$  being a linear span of a library of candidate functions (union of various set of basis that can be correlated), with greedy selection rules. Nevertheless, the current theory still falls short of describing the approximation and adaptivity for the non-convex and possibly nonsmooth gradient descent training on all-layer weights of the neural networks, as done in practice.

We take a step to bridge the above mismatch in the current theory and practice for neural networks and to establish a theoretical framework where the model classes adapt to the data. In particular, we answer the following *algorithmic approximation* question:

Given data pair  $(\mathbf{x}, \mathbf{y}) \sim P$ , denote  $f_*(\mathbf{x}) = \mathbf{E}[\mathbf{y}|\mathbf{x} = \mathbf{x}]$ . Specify a neural networks model, and run gradient flow until any stationarity ( $t \rightarrow \infty$ ). Denote the computed function to be  $f_t(\mathbf{x})$ . How does  $f_t(\mathbf{x})$  relate to  $f_*(\mathbf{x})$ , in terms of approximation and representation?

Also, we aim to formalize and shed light on the *representation benefits* of neural networks:

What are the provable benefits of the adaptive representation learned by training neural networks compared to the classical nonparametric prespecified fixed basis representation?

The intimate connection between two-layer neural networks and RKHSs has been studied in the literature (see, e.g., Rahimi and Recht 2008; Cho and Saul 2009; Daniely, Frostig, and Singer 2016; Bach 2017; Jacot, Gabriel, and Hongler 2018). However, to the best of our knowledge, known results are mostly based on a *fixed* RKHS (in our notation  $K_0$  in Section 5.1). In that sense, random features for kernel learning (Rahimi and Recht 2008, 2009; Rudi and Rosasco 2017) can be viewed as neural networks with fixed random sampled first layer weights, and tunable second layer weights. From the neural networks side, Rotskoff and Vanden-Eijnden (2018), Mei, Montanari, and Nguyen (2018), and Sirignano and Spiliopoulos (2019) studied the mean-field theory for two-layer neural networks, and Jacot, Gabriel, and Hongler (2018); Du et al. (2018); Chizat and Bach (2018); Ghorbani et al. (2019) studied the linearization of neural networks around the initialization and draw connections to RKHS  $K_0$  in various over-parameterized settings. In contrast, we will establish a general theory with the *dynamic* and *data-adaptive* RKHS  $K_t$  obtained via training neural networks, with standard gradient flow on weights of *both* layers. Connections and distinctions to the literature that motivates our study are further discussed with details in Section 5. As a distinctive feature of the adaptive theory, we emphasize that all  $f_* \in L^2(P_{\mathbf{x}})$  is considered, without prespecified structural assumptions.

## 1.1. Problem Formulation

In this article, we consider the time-varying function  $f_t$  to approximate  $f_*$ , parameterized by a two-layer rectified linear unit (ReLU) neural network (NN).

$$f_t(\mathbf{x}) := \sum_{j=1}^m w_j(t) \sigma(\mathbf{x}^T \mathbf{u}_j(t)). \quad (2)$$

The time index  $t$  corresponds to the evolution of parameters driven by the gradient flow/descent (GD) training dynamics. Here, each individual pair  $(w_j \in \mathbb{R}, \mathbf{u}_j \in \mathbb{R}^d)$  in the summation is associated with a *neuron*. Consider the gradient flow as the training dynamics for the weights of the neurons: for the loss function  $\ell(y, f) = (y - f)^2/2$  and the random variable  $\mathbf{z} := (\mathbf{x}, \mathbf{y})$ , the parameters  $(w_j, \mathbf{u}_j)$  evolve with time as follows

$$\begin{aligned} \frac{dw_j(t)}{dt} &= -\mathbf{E}_{\mathbf{z}} \left[ \frac{\partial \ell(\mathbf{y}, f_t)}{\partial f} \sigma(\mathbf{x}^T \mathbf{u}_j(t)) \right], \\ \frac{d\mathbf{u}_j(t)}{dt} &= -\mathbf{E}_{\mathbf{z}} \left[ \frac{\partial \ell(\mathbf{y}, f_t)}{\partial f} w_j(t) \mathbb{1}_{\mathbf{x}^T \mathbf{u}_j(t) \geq 0} \mathbf{x} \right]. \end{aligned} \quad (3)$$

Equivalently, we can rewrite the function computed by NN at time  $t$  as

$$f_t(\mathbf{x}) := \int \sigma(\mathbf{x}^T \mathbf{u}) \tau_t(d\mathbf{u}), \quad (4)$$

where  $\tau_t = \sum_{j=1}^m w_j(t) \delta_{\mathbf{u}_j(t)}$  is a signed combination of delta measures. We will define a careful rescaling of  $\tau_t$  denoted as  $\rho_t$  (Equation (19)), then derive the corresponding distribution dynamic for  $\rho_t$  driven by the gradient flow later in Section 5.2. The rescaled formulation naturally extends to the infinite neurons case with  $m \rightarrow \infty$ .

In this article, by considering various distributions of  $\mathbf{z}$ , we study two following problems: approximation and empirical risk minimization (ERM).

### 1.1.1. Function Approximation

The data pair  $\mathbf{z} \sim P$  is sampled from the population joint distribution. We are going to answer how  $f_t$  approximates  $f_*(\mathbf{x}) = \mathbf{E}[\mathbf{y}|\mathbf{x} = \mathbf{x}]$  in function spaces, induced by the gradient flow on neuron weights

$$\mathbf{E}_{\mathbf{z} \sim P} (\mathbf{y} - f_t(\mathbf{x}))^2 = \|f_t - f_*\|_{L_\mu^2}^2 + \mathbf{E}_{\mathbf{z} \sim P} (\mathbf{y} - f_*(\mathbf{x}))^2. \quad (5)$$

Here, we denote  $\mu := P_{\mathbf{x}}$ , and remark that all  $f_* \in L_\mu^2$  are considered without additional assumptions.

### 1.1.2. ERM and Interpolation

The data pair  $\mathbf{z} \sim \frac{1}{n} \sum_{i=1}^n \delta_{\mathbf{x}=x_i, \mathbf{y}=y_i}$  follows the empirical distribution. We will study gradient flow for the ERM

$$\frac{1}{2n} \sum_{i=1}^n (y_i - f_t(x_i))^2. \quad (6)$$

In this case, the target reduces to  $\widehat{\mathbf{E}}[\mathbf{y}|\mathbf{x} = x_i] = y_i$  with  $\widehat{\mathbf{E}}$  as the empirical expectation. When the minimizer of Equation (6) achieves the zero loss, we call it the *interpolation* problem (Zhang et al. 2016; Ma, Bassily, and Belkin 2017; Belkin et al. 2018; Belkin, Ma, and Mandal 2018; Liang and Rakhlin 2018;

Rakhlin and Zhai 2018). Here, we are interested in when and how  $f_t(x_i)$  interpolates  $y_i$ , for  $1 \leq i \leq n$ .

Finally, we remark that in practice, extending the gradient flow results to the (1) positive step size GD, and (2) mini-batch stochastic GD, are standalone interesting research topics. The reasons are that the optimization is nonsmooth for the ReLU activation and that the interplay between the batch size and step size is less transparent in nonconvex problems.

## 2. Preliminaries and Summary

### 2.1. Notations

We use the boldface lower case  $\mathbf{x}$  to denote a random variable or vector. The normal letter  $x$  can either be a scalar or a vector when there is no confusion. The transpose of a matrix  $\mathbf{A}$ , resp. vector  $u$  is denoted by  $\mathbf{A}^T$ , resp.  $u^T$ .  $\mathbf{A}^+$  denotes the Moore Penrose inverse. For  $n \in \mathbb{N}$ , let  $[n] := \{1, \dots, n\}$ . We use  $\mathbf{A}[i, j]$  to denote the  $i, j$ th entry of a matrix. We denote  $\mathbb{1}_{\mathcal{D}}$  as the indicator function of set  $\mathcal{D}$ . We call symmetric positive semidefinite functions  $K(\cdot, \cdot), H(\cdot, \cdot) : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$  kernels, and use calligraphy letter  $\mathcal{K}, \mathcal{H}$  to denote Hilbert spaces. We use  $\langle f, g \rangle_{\mu} = \int f(x)g(x)\mu(dx)$  to denote the inner product in  $L^2_{\mu}$  (or  $L^2(P_x)$ ).  $\hat{\mu}$  denotes the empirical distribution for  $\mu$ . Notation  $\mathbf{E}_{\mathbf{x}}$  is the expectation w.r.t. random variable  $\mathbf{x}$ , and  $\mathbf{E}_{\mathbf{x}, \tilde{\mathbf{x}}} h(\mathbf{x}, \tilde{\mathbf{x}}) = \int \int h(x, \tilde{x})\mu(dx)\mu(d\tilde{x})$ . For a signed measure  $\rho = \rho_+ - \rho_-$  with the positive and negative parts, define  $|\rho| = \rho_+ + \rho_-$ .

### 2.2. Preliminaries

We use the signed measure  $\rho_t$ , defined by the neuron weights at training time  $t$  collectively, to construct a *dynamic RKHS*. The mathematical definition of  $\rho_t$  is deferred to Sections 5.1 and 5.2 (specifically, Equation (19)). The stationary signed measure at  $t \rightarrow \infty$  is denoted as  $\rho_{\infty}$ . For completeness, we walk through the construction of the dynamic kernel and RKHS with  $\rho_t$ . Define the linear operator  $\mathcal{T} : L^2_{\mu}(x) \rightarrow L^2_{|\rho_t|}(\Theta)$ , such that for any  $f(x) \in L^2_{\mu}(x)$

$$(\mathcal{T}f)(\Theta) := \int f(x)\|\Theta\|\sigma(x^T\Theta)\mu(dx), \quad \forall \Theta \in \text{supp}(\rho_t).$$

One can define the adjoint operator  $\mathcal{T}^* : L^2_{|\rho_t|}(\Theta) \rightarrow L^2_{\mu}(x)$ , such that for  $p(\Theta) \in L^2_{|\rho_t|}(\Theta)$ ,

$$(\mathcal{T}^*p)(x) := \int p(\Theta)\|\Theta\|\sigma(x^T\Theta)|\rho_t|(d\Theta).$$

Note that both  $\mathcal{T}$  and  $\mathcal{T}^*$  are compact operators under the finite total variation and compact support assumptions. For the finite neurons case (2), the operator is of finite rank. We define the compact integral operator  $\mathcal{T}^*\mathcal{T}$  with the corresponding kernel

$$H_t(x, \tilde{x}) = \int \|\Theta\|^2 \sigma(x^T\Theta)\sigma(\tilde{x}^T\Theta)|\rho_t|(d\Theta), \text{ and} \\ (\mathcal{T}^*\mathcal{T}f)(x) := \int H_t(x, \tilde{x})f(\tilde{x})\mu(d\tilde{x}). \quad (7)$$

The dynamic RKHS  $\mathcal{H}_t$  can be readily constructed via  $H_t$ . Let the eigen decomposition of  $\mathcal{T}^*\mathcal{T}$  be the countable sum  $\mathcal{T}^*\mathcal{T} = \sum_{i=1}^E \lambda_i e_i e_i^*$ . Here,  $E$  can be a nonnegative integer or  $\infty$ , and

**Table 1.** Nature of the results studied in this article.

	Finite neurons $m$	Infinite neurons $m \rightarrow \infty$
Finite samples $n$	Interpolation (finite rank kernel, Theorems 3.1, 3.2, and Proposition 4.1)	Interpolation (finite rank kernel, Theorems 3.1, 3.2, and Proposition 4.1)
Infinite samples $n \rightarrow \infty$	Approximation (finite rank kernel, Theorems 3.1 and 3.2)	Approximation (possibly universal kernel, <sup>a</sup> Theorems 3.1 and 3.2)

<sup>a</sup>Whether the kernel is universal in the  $m, n \rightarrow \infty$  case still depends on  $f_*$  and the data distribution  $P$ . See the simulations of Maennel, Bousquet, and Gelly (2018).

$\lambda_i > 0$ .  $e_i$  without confusion can represent either an eigen function or a linear functional. Similarly, we have the singular value decomposition for  $\mathcal{T} = \sum_{i=1}^E \sqrt{\lambda_i} t_i e_i^*$  and  $\mathcal{T}^*$  as well. For a detailed discussion (see, e.g., Casselman 2014). Again,  $t_i$  is a function in  $L^2_{|\rho_t|}(\Theta)$  or a linear functional. The RKHS can be specified as follows.

$$\mathcal{H}_t = \left\{ h \mid h(x) = \sum_i h_i e_i(x), \sum_i \frac{h_i^2}{\lambda_i} < \infty \right\}.$$

We refer to  $H_{\infty}$  as the stationary RKHS kernel, and  $\mathcal{H}_{\infty}$  as the stationary RKHS. One can view that the gradient flow training dynamics—on the parameters of NN—induces a sequence of functions  $\{f_t : t \geq 0\}$  and dynamic RKHS  $\{\mathcal{H}_t : t \geq 0\}$ , indexed by the time  $t$ .

### 2.3. Organization and Summary

We will prove three results, which are summarized informally in this section (see also Table 1). We remark that Theorems 3.1 and 3.2 are stated for the approximation problem. However, as done in Corollary 3.1, by substituting  $\mathcal{P}, \mu$  by the empirical counterparts, one can easily state the analog for the ERM problem. Recall  $f_*(x) = \mathbf{E}[y|x = x]$ .

#### 2.3.1. Gradient Flow on NN Converges to Projection Onto Data-Adaptive RKHS

Theorem 3.1 shows that has done in practice training NN with simple gradient flow, in the limit of any *local* stationarity, learns the adaptive representation, and performs the *global* least squares projection simultaneously. Define  $f_{\infty} = \lim_{t \rightarrow \infty} f_t$  as the function computed by ReLU networks (defined in (2), or more generally in (20)) until any stationarity of the gradient flow dynamics (defined in (3), with the squared loss) for the population distribution  $(\mathbf{x}, \mathbf{y}) \sim P$ . Define the corresponding stationary RKHS  $\mathcal{H}_{\infty} = \lim_{t \rightarrow \infty} \mathcal{H}_t$  (defined in (7)).

[Informal version of Theorem 3.1] Consider  $f_* \in L^2_{\mu}$ , for any local stationarity of the gradient flow dynamics (3) on the weights of neural networks (2), the function computed by NN at stationarity  $f_{\infty}$  satisfies

$$f_{\infty} \in \arg \min_{g \in \mathcal{H}_{\infty}} \|f_* - g\|_{L^2_{\mu}}^2.$$



### 2.3.2. Representation Benefits of Data-Adaptive RKHS

**Theorem 3.2** illustrates the provable benefits of the learned data-adaptive representation/basis  $\mathcal{H}_\infty$ . We emphasize that  $\mathcal{H}_\infty$ , as obtained by training neural networks on the data  $(\mathbf{x}, \mathbf{y}) \sim P$ , depends on the data in an implicit way such that there are advantages of representing and approximating  $f_*$ .

[Informal version of **Theorem 3.2**] Consider  $f_* \in L_\mu^2$  and the same setup as **Theorem 3.1**. Decompose  $f_*$  into the function  $f_\infty$  computed by the neural network and the residual  $\Delta_\infty$

$$f_* = f_\infty + \Delta_\infty.$$

Then there is another RKHS (defined in (11))  $\mathcal{K}_\infty \supset \mathcal{H}_\infty$ , such that

$$f_\infty \in \mathcal{H}_\infty, \quad \Delta_\infty \in \text{Ker}(\mathcal{K}_\infty) \subset \text{Ker}(\mathcal{H}_\infty),$$

with a gap in the spaces  $\mathcal{H}_\infty \oplus \text{Ker}(\mathcal{K}_\infty) \neq L_\mu^2$ .

### 2.3.3. Convergence to Ridgeless Regression With Adaptive Kernels

**Proposition 4.1** establishes that in the vanishing regularization  $\lambda \rightarrow 0$  limit, the neural network function computed by gradient flow converges to the kernel ridgeless regression with an adaptive kernel (denoted as  $\hat{f}_\infty^{\text{rkhs}}(x)$ ). Consider using the gradient flow on the weights of the neural network function  $f_t(x) = \sum_{j=1}^m w_j(t) \sigma(x^T u_j(t))$ , to solve the  $\ell_2$ -regularized ERM

$$\frac{1}{2n} \sum_{i=1}^n (y_i - f_t(x_i))^2 + \frac{\lambda}{2m} \sum_{j=1}^m [w_j(t)^2 + \|u_j(t)\|^2].$$

Denote the function computed by NN at any local stationarity of ERM as  $\hat{f}_\infty^{\text{nn},\lambda}(x)$ , we answer the extrapolation question at a new point  $x$ , with the generalization error discussed in **Proposition 4.2**. The result is extendable to the infinite neurons case.

[Informal version of **Proposition 4.1**] Consider only the bounded assumption on initialization that  $|w_j^2(0) - \|u_j\|^2(0)| < \infty$  for all  $1 \leq j \leq m$ . At stationarity, denote the corresponding adaptive kernel as  $\hat{H}_\infty^\lambda$ . The neural network function  $\hat{f}_\infty^{\text{nn},\lambda}(x)$  has the following expression,

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \hat{f}_\infty^{\text{nn},\lambda}(x) &= \hat{H}_\infty(x, X) \hat{H}_\infty(X, X)^+ Y \\ &=: \hat{f}_\infty^{\text{rkhs}}(x) \text{ (ridgeless regression with kernel } \hat{H}_\infty). \end{aligned}$$

## 3. Main Results: Benefits of Adaptive Representation

We formally state two main results of the article, **Theorems 3.1** and **3.2**.

### 3.1. Gradient Flow, Projection and Adaptive RKHS

We study how the function  $f_t$  computed from gradient flow on NN represents  $f_*$  when reaching any stationarity, under the squared loss. Consider the gradient flow dynamics (23) reaching *any stationarity*. Assume that the corresponding signed measure in (19) satisfies  $\text{TV}(\rho_\infty) < \infty$  with a compact support. The mathematical details about  $\rho_\infty$  are postponed to **Section 5.2**.

We employ the notation  $\rho_\infty$  since reaching stationarity can be viewed as  $t \rightarrow \infty$ .

We would like to emphasize that this stationary signed measure  $\rho_\infty$  is *task adaptive*: it implicitly depends on the regression task  $f_*$  and the data distribution  $P$ , rather than being prespecified by the researcher as in Bach (2017), Daniely, Frostig, and Singer (2016), and Cho and Saul (2009). With the RKHS established in **Section 2.2**, we are ready to state the following theorem.

**Theorem 3.1 (Approximation).** For any conditional mean  $f_*(x) = \mathbf{E}[y|\mathbf{x} = x] \in L_\mu^2$ , consider solving the approximation problem (5), with the ReLU NN function  $f_t$  defined in (2) where  $w_j(t)$  and  $\theta_j(t)$  are the weights for  $t \geq 0, 1 \leq j \leq m$ . For any signed measure  $\rho_0$  with  $\text{TV}(\rho_0) < \infty$ , consider the infinitesimal initialization weights  $u_j(0) = \Theta_j/\sqrt{m}$ , and  $w_j(0) = \text{sgn}(\rho_0(\Theta_j))\|\Theta_j\|/\sqrt{m}$ , with  $\Theta_j \sim \rho_0$  sampled independently. When the training dynamics (3) reaches any stationarity, it defines a stationary signed measure  $\rho_\infty^{(m)}$  (on the collective weights) with  $\text{TV}(\rho_\infty^{(m)}) < \infty$ , and a corresponding stationary RKHS  $\mathcal{H}_\infty$  with the kernel defined in Equation (7), such that:

1. the function computed by neural networks at stationarity has the form

$$f_\infty(x) = \int \|\Theta\| \sigma(x^T \Theta) \rho_\infty^{(m)}(d\Theta); \quad (8)$$

2.  $f_\infty$  is a global minimizer of approximating  $f_*$  within the RKHS  $\mathcal{H}_\infty$

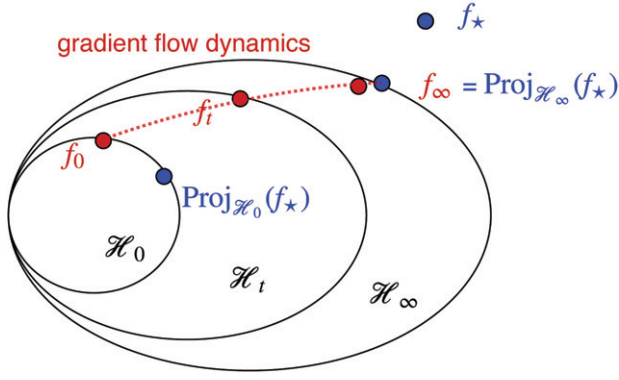
$$f_\infty \in \arg \min_{g \in \mathcal{H}_\infty} \|f_* - g\|_{L_\mu^2}^2. \quad (9)$$

In addition, the same results extend to the infinite neurons case with  $m \rightarrow \infty$  where the limit for  $\rho_\infty^{(m)}$  can be defined in the weak sense.

**Remark 3.1.** The above theorem shows that  $\lim_{t \rightarrow \infty} f_t$  obtained by training on two-layer weights over time until any stationarity, is the same as projecting  $f_*$  onto the stationary RKHS  $\mathcal{H}_\infty$ . The projection is the solution to the classic nonparametric least squares, had one known the adaptive representation  $\mathcal{H}_\infty$  beforehand. Conceptually, this is *distinct* from the theoretical framework in the current statistics and learning theory literature: we do not require the structural knowledge about  $f_*$  (say, smoothness, sparsity, reflected in  $\mathcal{F}$ ). Instead, we run gradient descent on neural networks to learn an adaptive representation for  $f_*$ , and show how the computed function represents  $f_*$  in this adaptive RKHS  $\mathcal{H}_\infty$ .

In other words, as done in practice training NN with simple gradient flow, in the limit of any *local* stationarity, learns the adaptive representation, and performs the *global* least-squares projection simultaneously. Training NN is learning a dynamic representation (quantified by  $\mathcal{H}_t$ ), at the same time updating the predicted function  $f_t$ , as shown in **Figure 1**.

A final note on the infinite neuron case: for any fixed time  $t$ , with the proper random initialization, setting  $m \rightarrow \infty$  defines a proper distribution dynamics on the weak limit  $\rho_t$  shown in **Lemma 5.3**. Then set  $t \rightarrow \infty$  to obtain the stationarity RKHS  $\mathcal{H}_\infty$ .



**Figure 1.** Illustration of Theorem 3.1. Red dotted line denotes the function  $f_t$  computed along the gradient flow dynamics on the weights of NN. Along training, one learns a sequence of dynamic RKHS representation  $\mathcal{H}_t$ 's. Over time,  $f_t$  converges to the projection of  $f_*$  onto  $\mathcal{H}_\infty$ . We emphasize that the initial function  $f_0$  computed by NN is very different from the projection of  $f_*$  onto the initial RKHS  $\mathcal{H}_0$ .

From the above, we have the following natural decomposition,

$$\Delta_\infty(x) = f_*(x) - f_\infty(x) \in \text{Ker}(\mathcal{H}_\infty). \quad (10)$$

Surprisingly, as we show in the next section,  $\Delta_\infty$  actually lies in a smaller subspace of  $\text{Ker}(\mathcal{H}_\infty)$ , characterized by  $\text{Ker}(\mathcal{K}_\infty)$ . We call this the *representation and approximation benefits* of the data-adaptive RKHS learned by training neural networks.

Before moving next, we briefly discuss the above theorem when applied to the empirical measure, to solve the ERM problem. First, as a direct corollary, the following holds.

**Corollary 3.1 (ERM).** Consider the ERM problem (6), with the other settings the same as in Theorem 3.1. One can define the finite dimensional RKHS  $\widehat{\mathcal{H}}_\infty$  (at most rank  $n$ ) as in (7) with  $\widehat{\mu} = \frac{1}{n} \sum_{i=1}^n \delta_{x_i}$  substituting  $\mu$ . When reaches any stationarity, the solution satisfies

$$\widehat{f}_\infty \in \arg \min_{g \in \widehat{\mathcal{H}}_\infty} \frac{1}{n} \sum_{i=1}^n (y_i - g(x_i))^2.$$

More importantly, we will show in Proposition 4.1 that the function computed by training neural networks with gradient descent on the empirical risk objective  $\widehat{f}_\infty(x)$  until any stationarity (with vanishing  $\ell_2$  regularization), can be shown to be the kernel ridgeless regression with the data-adaptive RKHS  $\widehat{\mathcal{H}}_\infty$ . Hence, studying the out of sample performance for GD on NN reduces to the generalization of kernel ridgeless regression with adaptive kernels.

### 3.2. Representation Benefits of Adaptive RKHS

We now define another adaptive RKHS  $\mathcal{K}_\infty$  named as the GD kernel, which turns out to be different from  $\mathcal{H}_\infty$  in (7). Interestingly, the difference in these two kernels sheds light on the representation benefits of the adaptive RKHS. The new RKHS  $\mathcal{K}_\infty$  is motivated by the gradient training dynamics. Recall the associated signed measure  $\rho_\infty$  at the stationarity, the GD kernel

is defined as

$$K_\infty(x, \tilde{x}) = \int (\|\Theta\|^2 \mathbb{1}_{x^T \Theta \geq 0} \mathbb{1}_{\tilde{x}^T \Theta \geq 0} x^T \tilde{x} + \sigma(x^T \Theta) \sigma(\tilde{x}^T \Theta)) \times |\rho_\infty|(d\Theta) \neq H_\infty(x, \tilde{x}), \quad (11)$$

which is different than the stationary RKHS kernel  $H_\infty$  in (7). We use  $\mathcal{K}_t : L_\mu^2(x) \rightarrow L_\mu^2(x)$  to denote the integral operator associated with  $K_t$ ,

$$(\mathcal{K}_t f)(x) := \int K_t(x, \tilde{x}) f(\tilde{x}) \mu(d\tilde{x}).$$

With a slight abuse of notation, we denote the corresponding RKHS to be  $\mathcal{K}_t$  as well. Now we are ready to state the main theorem on the representation benefits.

**Theorem 3.2 (Representation benefits).** Consider  $f_* \in L_\mu^2$  and the same setting as in Theorem 3.1. Consider the approximation problem (5) with either finite or infinite neurons, and the gradient flow dynamics (23) (equivalently (3)) with data pair  $(\mathbf{x}, \mathbf{y}) \sim P$  drawn from the population distribution. When reaching any stationary signed measure  $\rho_\infty$ ,  $f_*$  is decomposed into the function  $f_\infty$  computed by the neural network and the residual  $\Delta_\infty$

$$f_* = f_\infty + \Delta_\infty.$$

Recall the RKHS  $\mathcal{H}_\infty$  in (7) and the GD RKHS  $\mathcal{K}_\infty$  in (11), all learned from the data  $(\mathbf{x}, \mathbf{y}) \sim P$  and  $f_*$  adaptively. The following holds,

$$f_\infty \in \mathcal{H}_\infty, \quad \Delta_\infty \in \text{Ker}(\mathcal{K}_\infty) \subset \text{Ker}(\mathcal{H}_\infty),$$

with  $\mathcal{H}_\infty \oplus \text{Ker}(\mathcal{K}_\infty) \neq L_\mu^2$ . In other words, GD on NN decomposes  $f_*$  into two parts, and each lies in a space that is NOT the orthogonal complement to the other.

**Remark 3.2.** As we can see  $\text{Ker}(\mathcal{K}_\infty)$  and  $\text{Ker}(\mathcal{H}_\infty)$  are not the same. Therefore, the decomposition  $f_\infty + \Delta_\infty$  is not a trivial orthogonal decomposition to the RKHS  $\mathcal{H}_\infty$  and its complement.

Recall Theorem 3.1, projecting  $f_*$  to the RKHS  $\mathcal{H}_\infty$  with the data-adaptive kernel

$$H_\infty(x, \tilde{x}) = \int \sigma(x^T \Theta) \sigma(\tilde{x}^T \Theta) |\rho_\infty|(d\Theta)$$

associated with  $|\rho_\infty|$  is the same as the function constructed by neural networks (GD limit as  $t \rightarrow \infty$ ). However, the residual lies in a possibly much smaller space due to Theorem 3.2, which is the null space of the RKHS  $\mathcal{K}_\infty$

$$K_\infty(x, \tilde{x}) = \int (\|\Theta\|^2 \mathbb{1}_{x^T \Theta \geq 0} \mathbb{1}_{\tilde{x}^T \Theta \geq 0} x^T \tilde{x} + \sigma(x^T \Theta) \sigma(\tilde{x}^T \Theta)) \times |\rho_\infty|(d\Theta).$$

In other words, as the learned adaptive basis  $\mathcal{H}_\infty$  (from GD) depends on the data distribution and the task  $f_*$  implicitly, it has the advantage of representing  $f_*$  by squeezing the residual into a smaller subspace in the null space of  $\mathcal{H}_\infty$ . A pictorial illustration can be found in Figure 2. This representation and approximation benefit helps with explaining the better interpolation results obtained by neural networks (Zhang et al. 2016;

Liang and Rakhlin 2018; Belkin et al. 2018; Belkin, Ma, and Mandal 2018): (1) the adaptive basis is tailored for the task  $f_*$ , thus the residual/interpolation error lies in a smaller space; (2) in view of the ODE in Corollary 5.2, the second layer of NN adds *implicit regularization* to the smallest eigenvalues of  $K_t$ , thus improving the converging speed of  $\Delta_t$  to zero.

Before concluding this section, we remark that a similar result holds for the ERM problem (6). As we shall discuss in the next section, the gap between  $\mathcal{H}_\infty$  and  $\mathcal{K}_\infty$  can be large, even for the ERM problem.

#### 4. Implications of the Adaptive Theory

In this section, we will discuss some direct implications of the adaptive kernel theory for neural networks established in this article.

##### 4.1. Example: Gap in Spaces $\mathcal{H}_\infty$ and $\mathcal{K}_\infty$

In Theorem 3.2, it is established that  $\text{Ker}(\mathcal{K}_\infty) \subset \text{Ker}(\mathcal{H}_\infty)$ . We now construct a concrete case to illustrate the potentially significant gap in these two spaces as follows. Consider only one neuron with  $m = 1$ , solving ERM problem (6) with  $n$  samples, and  $\mathbf{x}$  with dimension  $d$ . In this case,  $\rho_\infty$  is supported on only one point, noted as  $\Theta_\infty \in \mathbb{R}^d$ . Denote  $X \in \mathbb{R}^{n \times d}$  as the data matrix, one can show that

$$H_\infty(X, X) = \underbrace{\sigma(X\Theta_\infty^T)}_{n \times 1} \underbrace{\sigma(X\Theta_\infty^T)^T}_{1 \times n}$$

has rank 1. In contrast,

$$K_\infty(X, X) \succeq \underbrace{\text{diag}(\mathbb{1}_{X\Theta_\infty^T \geq 0})}_{n \times d} X X^T \underbrace{\text{diag}(\mathbb{1}_{X\Theta_\infty^T \geq 0})}_{d \times n}$$

can be of rank  $d \wedge |\{i : x_i^T \Theta_\infty \geq 0\}|$ . Hence, the null space of  $K_\infty$  is much smaller than that of  $H_\infty$ . The gap can be large for many other settings of  $(n, m, d)$ .

##### 4.2. Connections to Min-norm Interpolation

The following result establishes the connections between the solution of gradient descent on neural networks (at local stationarity), and the kernel ridgeless regression (Belkin, Ma, and Mandal 2018; Liang and Rakhlin 2018; Hastie et al. 2019) with an adaptive kernel  $\hat{H}_\infty^\lambda$ . Empirical evidence on the similarity between the interpolation with kernels and neural networks was discovered in Belkin, Ma, and Mandal (2018). The following proposition provides a novel way of studying the generalization property of neural networks via adaptive kernels.

**Proposition 4.1 (Interpolation: Connection to kernel ridgeless regression).** Consider the gradient flow dynamics on all the weights of the neural network function  $f_t(x) = \sum_{j=1}^m w_j(t) \sigma(x^T u_j(t))$ , to solve the  $\ell_2$ -regularized ERM

$$\frac{1}{2n} \sum_{i=1}^n (y_i - f_t(x_i))^2 + \frac{\lambda}{2m} \sum_{j=1}^m [w_j(t)^2 + \|u_j(t)\|^2].$$

Consider only the bounded assumption on initialization that  $|w_j^2(0) - \|u_j\|^2(0)| < \infty$  for all  $1 \leq j \leq m$ . At stationarity, denote the signed measure as  $\hat{\rho}_\infty^\lambda$  and the corresponding adaptive kernel as  $\hat{H}_\infty^\lambda$ . Then the neural network function at stationarity  $\hat{f}_\infty^{\text{nn}, \lambda}(x)$  satisfies,

$$\hat{f}_\infty^{\text{nn}, \lambda}(x) = \hat{H}_\infty^\lambda(x, X) \left[ \frac{n}{m} \lambda \cdot I_n + \hat{H}_\infty^\lambda(X, X) \right]^{-1} Y.$$

In the vanishing regularization  $\lambda \rightarrow 0$  limit, the neural network function converges to the kernel ridgeless regression with the adaptive kernel, when  $\hat{H}_\infty(X, X) := \lim_{\lambda \rightarrow 0} \hat{H}_\infty^\lambda$  exists,

$$\lim_{\lambda \rightarrow 0} \hat{f}_\infty^{\text{nn}, \lambda}(x) = \hat{H}_\infty(x, X) \hat{H}_\infty(X, X)^+ Y = \hat{f}_\infty^{\text{rkhs}}(x).$$

Note that the generalization theory for the kernel ridgeless regression has been established in Liang and Rakhlin (2018) and Hastie et al. (2019). Here, the kernel  $\hat{H}_\infty(X, X)$  is data-adaptive (that adapts to  $f_*$ ) learned along training, instead of being fixed and prespecified.

##### 4.3. Connections to Random Kitchen Sinks

Let us introduce two function spaces, with the base measure  $\rho_0$  (fixed representation)

$$\Gamma_2(\rho_0) := \left\{ f(x) \mid f(x) = \int \sigma(x^T \Theta) w(\Theta) \rho_0(d\Theta), w \in L_{\rho_0}^2 \right\},$$

$$\Gamma_1(\rho_0) := \left\{ f(x) \mid f(x) = \int \sigma(x^T \Theta) w(\Theta) \rho_0(d\Theta), w \in L_{\rho_0}^1 \right\}.$$

In random kitchen sinks studied in Rahimi and Recht (2008, 2009), by assuming  $f_* \in \Gamma_2(\rho_0)$  that lies in the RKHS, the approximation error can be controlled by the existence of the following function with  $\theta_j, j \in [m]$  iid sampled from  $\rho_0$

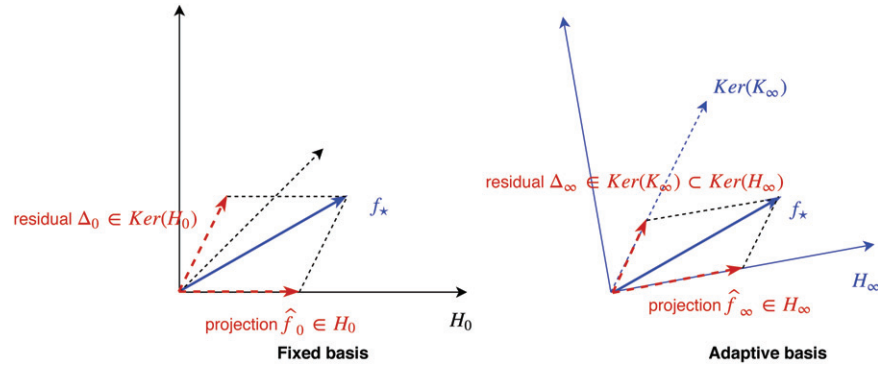
$$\hat{f}(x) = \frac{1}{m} \sum_{j=1}^m \sigma(x^T \Theta_j) w(\Theta_j) \in \Gamma_1(\rho_0), \text{ but } \hat{f}(x) \notin \Gamma_2(\rho_0).$$

Note that  $\hat{f}$  lies in a possibly much larger space  $\Gamma_1(\rho_0)$  though the target only lies in  $f_* \in \Gamma_2(\rho_0)$ . Similarly for two-layer neural networks function  $f_t(x)$  considered in Bach (2017, sec. 2.3), the RKHS space  $\Gamma_2(\rho_0)$  can be more restrictive compared to  $f_t \in \Gamma_1(\rho_0)$ .

In contrast, with the adaptive RKHS representation  $\mathcal{H}_\infty$ , we have shown that

$$f_\infty(x) \in \Gamma_1(|\rho_\infty|), \text{ and } f_\infty(x) \in \Gamma_2(|\rho_\infty|).$$

The extreme case of fully adaptive function space  $\Gamma_2(|\rho_*|)$  is defined with  $\rho_*$  tailored for  $f_*$ ,  $f_* = \int \sigma(x^T \Theta) \rho_*(d\Theta)$ . The adaptive representation learned by neural networks can be viewed as in between the fixed and the fully adaptive representation.



**Figure 2.** Illustration of Theorem 3.2: fixed basis versus adaptive learned basis. In classic statistics, one specifies the fixed function space/basis  $H_0$  then decompose  $f_*$  into the projection  $\hat{f}_0$  and residual  $\Delta_0 \in \text{Ker}(H_0)$ . However, for GD on NN, one learns the adaptive basis  $H_\infty$  that depends on  $f_*$ . Therefore, the residual  $\Delta_\infty$  lies in a subspace of  $\text{Ker}(H_\infty)$ .

#### 4.4. Adaptive Generalization Theory

Now we attempt to provide a new decomposition to study the generalization of NN via adaptive kernels. Recall we have shown that  $\hat{f}_\infty^{\text{rkhs}}(x) = \lim_{\lambda \rightarrow 0} \hat{f}_\infty^{\text{nn},\lambda}(x) = \hat{H}_\infty(x, X) \hat{H}_\infty(X, X)^+ Y$ , where  $\hat{H}_\infty(x, \tilde{x}) := \int \sigma(x^T \Theta) \sigma(\tilde{x}^T \Theta) \hat{\rho}_\infty^{(n,m)}(d\Theta)$ . Define the population limit  $\rho_\infty^{(m)}(d\Theta) := \lim_{n \rightarrow \infty} \hat{\rho}_\infty^{(n,m)}$  and  $H_\infty(x, \tilde{x}) := \int \sigma(x^T \Theta) \sigma(\tilde{x}^T \Theta) \rho_\infty^{(m)}(d\Theta)$ . Denote the ridgeless regression with the population adaptive kernel  $H_\infty$ ,

$$f_\infty^{\text{rkhs}}(x) = H_\infty(x, X) H_\infty(X, X)^+ Y.$$

Assume  $(y - f_*(\mathbf{x}))^2 \leq \sigma^2$  a.s. (can be relaxed). One can derive the following decomposition for generalization.

**Proposition 4.2 (Adaptive generalization).**

$$\begin{aligned} & \left\| \lim_{\lambda \rightarrow 0} \hat{f}_\infty^{\text{nn},\lambda} - f_* \right\|_\mu^2 \\ & \lesssim \underbrace{\left\| \hat{f}_\infty^{\text{rkhs}} - f_\infty^{\text{rkhs}} \right\|_\mu^2}_{\text{adaptive representation error}} + \underbrace{\left\| f_\infty - f_* \right\|_\mu^2}_{\text{adaptive approximation error}} \\ & \quad + (n \|f_\infty - f_*\|_\mu^2 + \sigma^2) \underbrace{\mathbf{E}_{\mathbf{x} \sim \mu} \|H_\infty(X, X)^{-1} H_\infty(X, \mathbf{x})\|_\mu^2}_{\text{adaptive variance}} \\ & \quad + \underbrace{\|H_\infty(x, X) H_\infty(X, X)^{-1} f_\infty(X) - f_\infty(x)\|_\mu^2}_{\text{adaptive bias}}. \end{aligned}$$

Note this result holds without requiring global optimization guarantees. The first term is the representation error, which corresponds to the closeness of the adaptive RKHS  $\hat{H}_\infty$  (using empirical distribution) and  $H_\infty$  (using population distribution). The second term is the adaptive approximation error studied in the current article. The third and fourth terms are the variance and bias expressions studied in Liang and Rakhlin (2018), Hastie et al. (2019), Rakhlin and Zhai (2018), Bartlett et al. (2019), and Liang, Rakhlin, and Zhai (2020), as if assuming the actual function lies in  $\mathcal{H}_\infty$ . This decomposition suggests the possibility of studying generalization without explicit global understanding of the optimization, and providing rates that adapts to  $f_*$  without structural assumptions.

#### 5. Time-Varying Kernels and Evolution

In this section, we lay out the mathematical details on the time-varying kernels and the evolution of the signed measure  $\rho_t$  supporting the main results. In the meantime, we will discuss in depth the relevant literature motivating our proof ideas.

First, we describe the motivation behind the dynamic RKHS  $\mathcal{K}_t$ , and the GD kernel induced by the gradient descent dynamics. Extensions to the multilayer perceptrons will be in Appendix 1.2.

**Lemma 5.1 (Dynamic kernel of finite neurons GD).** Consider the approximation problem (1) with a neural network function (2), and the training process (3) with population distribution. Let  $\Delta_t(x) = f_*(x) - f_t(x)$  be the residual. Define the time-varying kernel  $K_t(\cdot, \cdot) : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ ,

$$\begin{aligned} K_t(x, \tilde{x}) = & \sum_{j=1}^m \left[ \sigma(x^T u_j(t)) \sigma(\tilde{x}^T u_j(t)) \right. \\ & \left. + w_j(t)^2 \mathbb{1}_{x^T u_j(t) \geq 0} \mathbb{1}_{\tilde{x}^T u_j(t) \geq 0} x^T \tilde{x} \right]. \end{aligned} \quad (12)$$

Then the residual  $\Delta_t$  driven by the GD dynamics satisfies,

$$\frac{d\mathbf{E}_{\mathbf{x}} \left[ \frac{1}{2} \Delta_t(\mathbf{x})^2 \right]}{dt} = -\mathbf{E}_{\mathbf{x}, \tilde{\mathbf{x}}} [\Delta_t(\mathbf{x}) K_t(\mathbf{x}, \tilde{\mathbf{x}}) \Delta_t(\tilde{\mathbf{x}})]. \quad (13)$$

When running GD to solve the empirical risk minimization (ERM), the dynamics of the finite-dimensional sample residual  $\|\Delta_t\|_\mu^2$  has been established in Jacot, Gabriel, and Hongler (2018) and Du et al. (2018). Here, we generalize the result to optimize the weights of both layers, and to solve the infinite-dimensional population approximation problem rather than the empirical risk minimization problem. For a general loss function  $\ell(y, f)$  with curvature (say, logistic loss), similar results hold under slightly stronger conditions.

**Corollary 5.1.** Consider a general loss function  $\ell(y, f)$  that is  $\alpha$ -strongly convex in the second argument  $f$ , with  $K_t$  defined in (12). Assume in addition  $\frac{1}{n} K_t(X, X) \in \mathbb{R}^{n \times n}$  has smallest eigenvalue  $\lambda_t > 0$ . Define  $\Delta_t(x_i) := \frac{\partial \ell(y_i, f_t(x_i))}{\partial f}$ , then we have



for all  $f_* : \mathbb{R}^d \rightarrow \mathbb{R}$ ,

$$\begin{aligned} \frac{d\widehat{\mathbf{E}}[\ell(\mathbf{y}, f_t(\mathbf{x}))]}{dt} &= -\widehat{\mathbf{E}}_{\mathbf{x}, \tilde{\mathbf{x}}} [\Delta_t(\mathbf{x}) K_t(\mathbf{x}, \tilde{\mathbf{x}}) \Delta_t(\tilde{\mathbf{x}})] \\ &\leq -2\alpha\lambda_t \cdot \widehat{\mathbf{E}}[\ell(\mathbf{y}, f_t(\mathbf{x})) - \ell(\mathbf{y}, f_*(\mathbf{x}))]. \end{aligned}$$

### 5.1. Initialization, Rescaling and $K_0$

Now we describe the initialization and rescaling schemes used in the main theorems. Rewrite (1) according to the signs of the second layer weights

$$f_t(x) := \sum_{j=1}^{m_+} w_{+j}(t) \sigma(x^T u_{+j}(t)) + \sum_{j=1}^{m_-} w_{-j}(t) \sigma(x^T u_{-j}(t)).$$

#### 5.1.1. Initialization

We consider the “infinitesimal” initialization drawn iid from two probability measures  $\rho_{+,0}$  and  $\rho_{-,0}$  that do not depend on  $m$ :

$$\begin{aligned} u_{+j}(0) &= \frac{1}{\sqrt{m}} \Theta_{+j} \text{ where } \Theta_{+j} \sim \rho_{+,0}, \\ u_{-j}(0) &= \frac{1}{\sqrt{m}} \Theta_{-j} \text{ where } \Theta_{-j} \sim \rho_{-,0}. \end{aligned} \quad (14)$$

Here,  $m = m_+ + m_-$  with  $m_+ \asymp m_-$ . The  $1/\sqrt{m}$  rescaling factor turns out to be crucial when defining the infinite neurons limit for the evolution of signed measures. Remark that such initialization is w.l.o.g., and accounts for the infinitesimal nature used in practice when the number of neurons grows. For the second layer weights, we impose the “balanced condition” motivated by Maennel, Bousquet, and Gelly (2018),

$$w_{+j}(0) = \|u_{+j}(0)\| \geq 0, \quad w_{-j}(0) = -\|u_{-j}(0)\| \leq 0. \quad (15)$$

It turns out that with such initialization, the balanced condition holds throughout the training process induced by gradient flow, which is useful for the main theorems. Interestingly, in the proof of Proposition 4.1, we show that such balanced condition always holds at stationarity when training neural networks with  $\ell_2$  regularization, even for unbalanced initialization.

**Proposition 5.1 (Balanced condition).** For  $u_{+j}(t), u_{-j}(t), w_{+j}(t)$  and  $w_{-j}(t)$ , and the initialization specified above, at any time  $t$ , we have

$$w_{+j}(t) = \|u_{+j}(t)\|, \quad w_{-j}(t) = -\|u_{-j}(t)\|.$$

#### 5.1.2. Rescaling

To prepare for the distribution dynamic theory in the next section, we introduce a parameter rescaling with the  $\sqrt{m}$  factor. Let  $\theta_{+j}(t) = \sqrt{m}w_{+j}(t)$  and  $\theta_{-j}(t) = \sqrt{m}w_{-j}(t)$ , also define  $\Theta_{+j}(t) = \sqrt{m}u_{+j}(t)$  and  $\Theta_{-j}(t) = \sqrt{m}u_{-j}(t)$  sampled from  $\rho_{+,0}$  and  $\rho_{-,0}$  at  $t = 0$ . Under this representation,

$$\begin{aligned} f_t(x) &= \frac{1}{m} \sum_{j=1}^{m_+} \theta_{+j}(t) \sigma(x^T \Theta_{+j}(t)) \\ &\quad + \frac{1}{m} \sum_{j=1}^{m_-} \theta_{-j}(t) \sigma(x^T \Theta_{-j}(t)). \end{aligned} \quad (16)$$

By the positive homogeneity of ReLU, we have the corresponding dynamics on the rescaled parameters,

$$\begin{aligned} \frac{d\theta_{+j}}{dt} &= \sqrt{m} \frac{dw_{+j}}{dt} = -\sqrt{m} \mathbf{E}_{\mathbf{z}} \left[ \frac{\partial \ell(\mathbf{y}, f(\mathbf{x}))}{\partial f} \sigma(\mathbf{x}^T u_{+j}) \right] \\ &= -\mathbf{E}_{\mathbf{z}} \left[ \frac{\partial \ell(\mathbf{y}, f(\mathbf{x}))}{\partial f} \sigma(\mathbf{x}^T \Theta_{+j}) \right], \end{aligned} \quad (17)$$

$$\begin{aligned} \frac{d\theta_{-j}}{dt} &= \sqrt{m} \frac{dw_{-j}}{dt} = -\sqrt{m} \mathbf{E}_{\mathbf{z}} \left[ \frac{\partial \ell(\mathbf{y}, f(\mathbf{x}))}{\partial f} w_{-j} \mathbb{1}_{\mathbf{x}^T u_{-j} \geq 0} \right] \\ &= -\mathbf{E}_{\mathbf{z}} \left[ \frac{\partial \ell(\mathbf{y}, f(\mathbf{x}))}{\partial f} \theta_{-j} \mathbb{1}_{\mathbf{x}^T \Theta_{-j} \geq 0} \right]. \end{aligned} \quad (18)$$

Define at time  $t$

$$\rho_{+,t} := \frac{1}{m} \sum_{j=1}^{m_+} \delta_{\Theta_{+j}(t)}, \quad \rho_{-,t} := \frac{1}{m} \sum_{j=1}^{m_-} \delta_{\Theta_{-j}(t)} \quad (19)$$

as the empirical distribution over neurons on the parameter space  $\Theta$ . The  $\rho_{+,t}$  and  $\rho_{-,t}$  converge weakly to proper distributions in the infinite neurons limit  $m \rightarrow \infty$  (see, e.g., Bach 2017; Mei, Montanari, and Nguyen 2018). Through the balanced condition in Proposition 5.1 and Proposition 1.1 (see Appendix), we know by substituting  $\theta_j$  with  $\|\Theta_j\|$

$$\begin{aligned} f_t(x) &= \int \|\Theta\| \sigma(x^T \Theta) \rho_t(d\Theta), \\ \text{where the signed measure } \rho_t &:= \rho_{+,t} - \rho_{-,t}. \end{aligned} \quad (20)$$

The above motivates the study of the RKHS  $\mathcal{H}_t$  as in Theorem 3.1, with the kernel

$$H_t(x, \tilde{x}) = \int \|\Theta\|^2 \sigma(x^T \Theta) \sigma(\tilde{x}^T \Theta) |\rho_t|(d\Theta). \quad (21)$$

To conclude this section, we provide the explicit formula for the initial kernel matrix  $K_0$  under such infinitesimal random initialization. Specifically, consider the initialization with  $w_j$  being  $\pm 1/\sqrt{m}$  with equal chance and  $u_i \sim N(\mathbf{0}, 1/m \cdot \mathbf{I}_d)$  iid sampled. The initial kernel  $K_0$  has the following expression, in the infinite neurons limit.

**Lemma 5.2 (Fixed kernel).** With initialization specified above, consider w.l.o.g.  $\|x\| = \|\tilde{x}\| = 1$ , and denote  $\Theta \sim \pi$  as the isotropic Gaussian  $N(\mathbf{0}, \mathbf{I}_d)$ . By the strong law of large number, we have almost surely,

$$\begin{aligned} \lim_{m \rightarrow \infty} K_0(x, \tilde{x}) &= \mathbf{E}_{\Theta \sim \pi} [\sigma(x^T \Theta) \sigma(\tilde{x}^T \Theta) + \mathbb{1}_{x^T \Theta > 0} \mathbb{1}_{\tilde{x}^T \Theta > 0} x^T \tilde{x}] \\ &= \left[ \frac{\pi - \arccos(t)}{\pi} t + \frac{\sqrt{1-t^2}}{2\pi} \right], \quad \text{where } t = x^T \tilde{x}. \end{aligned}$$

Much known results (Bengio et al. 2006; Rahimi and Recht 2008; Cho and Saul 2009; Daniely, Frostig, and Singer 2016; Bach 2017) on the connection between RKHS and two-layer NN focus on some fixed kernel, such as  $K_0$ . To instantiate useful statistical rates, one requires  $f_*$  to lie in the corresponding prespecified RKHS  $\mathcal{K}_0$ , which is nonverifiable in practice. In contrast, the dynamic kernel is less studied. We will establish a dynamic and adaptive kernel theory defined by GD, without making any structural assumptions on  $f_*$  other than  $f_* \in L^2_{\mu}$ .

## 5.2. Evolution of $\rho_t$

In this section, we derive the evolution of the signed measure  $\rho_t$  defined by the neurons at the training  $t$ , which in turn determines the dynamic kernel  $K_t$  defined in (12). To generalize the result to the case of infinite neurons, we follow and borrow tools from the mean-field characterization (Jordan, Kinderlehrer, and Otto 1998; Mei, Montanari, and Nguyen 2018; Rotskoff and Vanden-Eijnden 2018). The rescaling described in the previous section proves handy when defining such infinite neurons limit. We define the velocity field driven by the regression task and the interaction among neurons,

$$V(\Theta) = \mathbf{E}[\mathbf{y}\sigma(\mathbf{x}^T\Theta)], \quad U(\Theta, \tilde{\Theta}) = -\mathbf{E}[\sigma(\mathbf{x}^T\Theta)\sigma(\mathbf{x}^T\tilde{\Theta})]. \quad (22)$$

The following theorem casts the training process as distribution dynamics on  $\rho_{+,t}, \rho_{-,t}$ .

**Lemma 5.3 (Dynamic kernel and evolution).** Consider the approximation problem (1), and the gradient flow as the training dynamic (3). For  $\rho_{+,t}, \rho_{-,t}$  and  $\rho_t$  defined in (19) with possibly infinite neurons, we have the following PDE characterization on distribution dynamics of  $\rho_{+,t}, \rho_{-,t}$

$$\begin{aligned} \partial_t \rho_{+,t}(\Theta) &= -\nabla_{\Theta} \cdot \left[ \rho_{+,t}(\Theta) \cdot \|\Theta\| \left( \nabla_{\Theta} V(\Theta) \right. \right. \\ &\quad \left. \left. + \nabla_{\Theta} \int U(\Theta, \tilde{\Theta}) \|\tilde{\Theta}\| \rho_t(d\tilde{\Theta}) \right) \right], \\ \partial_t \rho_{-,t}(\Theta) &= \nabla_{\Theta} \cdot \left[ \rho_{-,t}(\Theta) \cdot \|\Theta\| \left( \nabla_{\Theta} V(\Theta) \right. \right. \\ &\quad \left. \left. + \nabla_{\Theta} \int U(\Theta, \tilde{\Theta}) \|\tilde{\Theta}\| \rho_t(d\tilde{\Theta}) \right) \right]. \end{aligned} \quad (23)$$

Moreover, the GD kernel  $K_t$  is defined as

$$K_t(x, \tilde{x}) = \int (\|\Theta\|^2 \mathbb{1}_{x^T\Theta \geq 0} \mathbb{1}_{\tilde{x}^T\Theta \geq 0} x^T \tilde{x} + \sigma(x^T\Theta)\sigma(\tilde{x}^T\Theta)) \times |\rho_t|(d\Theta). \quad (24)$$

**Remark 5.1.** As in Mei, Montanari, and Nguyen (2018) and Rotskoff and Vanden-Eijnden (2018), let's first show that in the infinite neurons limit  $m \rightarrow \infty$ ,  $\rho_{+,t}, \rho_{-,t}$  are properly defined, with Equation (23) characterizing the distribution dynamics. For simplicity, we assume the initialization  $\rho_{+,0}, \rho_{-,0}$  is with bounded support. Add the superscript  $m$ ,  $\rho_{+,t}^{(m)}, \rho_{-,t}^{(m)}, \rho_t^{(m)}$  to (19) to indicate their dependence on  $m$ . Consider that  $\nabla_{\Theta} V(\Theta)$ ,  $\nabla_{\Theta} U(\Theta, \tilde{\Theta})$  in (22) are bounded and uniform Lipchitz continuous as in Mei, Montanari, and Nguyen (2018, A3). With the same proof as in Mei, Montanari, and Nguyen (2018, Theorem 3), one can show that with  $m \rightarrow \infty$ , the initial distribution  $\rho_0^{(m)} \xrightarrow{d} \rho_0 = \rho_{+,0} - \rho_{-,0}$  by law of large number. And by the solution's continuity w.r.t. the initial value, we have  $\rho_t^{(m)} \xrightarrow{d} \rho_t$  as  $m \rightarrow \infty$  well defined, for any fixed  $t$ .

Note that our problem setting is slightly different from that in Mei, Montanari, and Nguyen (2018), where the authors consider the NN with fixed second layer weights to be  $1/m$ . We reiterate that the reparameterization via  $\theta$  and  $\Theta$  is crucial: (1) weights on both layers are optimized following the gradient flow; (2) infinitesimal random initialization is employed in practice. In

the setting of Mei, Montanari, and Nguyen (2018, eq. (3)), the training process is slightly different from the vanilla GD on weights, with an additional  $m$  factor in the velocity term. This subtlety is also mentioned in Rotskoff and Vanden-Eijnden (2018). In short, the rescaling looks at the dynamics where  $\Theta$ 's are on the invariant scale as  $m \rightarrow \infty$  for any fixed effective time  $t$  (that does not depend on  $m$ ). Here, we analyze the exact gradient flow on the two-layer weights, with infinitesimal random initialization as in practice, resulting in a different velocity field (22) compared to that in Mei, Montanari, and Nguyen (2018).

The proof of Theorem 3.1 makes use of (20)–(21) and the stationary condition implied by Lemma 5.3. The balanced condition is crucial in both Theorem 3.1 and Proposition 4.1. The details of the proof are deferred to Section 7.

## 5.3. Two RKHS: $\mathcal{K}_{\infty}$ and $\mathcal{H}_{\infty}$

In this section, we compare the two adaptive RKHS appeared  $\mathcal{K}_{\infty}$  in (24), and  $\mathcal{H}_{\infty}$  in (21). The comparison will lead to the proof of Theorem 3.2. We start with generalizing Lemma 5.1 with the possibly infinite neurons case via the distribution dynamics in (23).

**Corollary 5.2.** Consider the same setting as in Lemma 5.1 with possibly infinite neurons NN (20), and the training process (23). Define the time-varying kernel matrix  $K_t(\cdot, \cdot) : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ , with the signed measure  $\rho_t$  follows (23)

$$K_t(x, \tilde{x}) = \int (\|\Theta\|^2 \mathbb{1}_{x^T\Theta \geq 0} \mathbb{1}_{\tilde{x}^T\Theta \geq 0} x^T \tilde{x} + \sigma(x^T\Theta)\sigma(\tilde{x}^T\Theta)) \times |\rho_t|(d\Theta) \quad (25)$$

$$=: K_t^{(0)}(x, \tilde{x}) + K_t^{(1)}(x, \tilde{x}). \quad (26)$$

Then we still have  $d\mathbf{E}_{\mathbf{x}}[\frac{1}{2}\Delta_t(\mathbf{x})^2]/dt = -\mathbf{E}_{\mathbf{x},\tilde{\mathbf{x}}}[\Delta_t(\mathbf{x})K_t(\mathbf{x}, \tilde{\mathbf{x}})\Delta_t(\tilde{\mathbf{x}})]$ .

It turns out that the kernels  $K_{\infty}$  and  $H_{\infty}$ , defined in (11) and (7), respectively, satisfy the following inclusion property.

**Proposition 5.2.** Consider the training process reaches any stationarity  $\rho_{\infty} = \rho_{+, \infty} - \rho_{-, \infty}$  with compact support within radius  $D$  and finite total variation. We have

$$K_{\infty} \succeq K_{\infty}^{(0)} \succeq K_{\infty}^{(1)} \succeq \frac{1}{D^2} H_{\infty}, \quad (27)$$

with  $K_{\infty}^{(0)}, K_{\infty}^{(1)}$  defined in (26). Combining with the fact that  $H_{\infty} \neq K_{\infty}$  implies

$$\text{Ker}(\mathcal{K}_{\infty}) \subset \text{Ker}(\mathcal{H}_{\infty}).$$

The proof of Theorem 3.2 uses the following fact: when reaching stationarity, due to the ODE defined by GD in Lemma 5.1, the residual must satisfy

$$\Delta_{\infty}(x) = f_*(x) - f_{\infty}(x) \in \text{Ker}(\mathcal{K}_{\infty}). \quad (28)$$

The proof of Proposition 5.2 and Theorem 3.2 are deferred to Section 7.

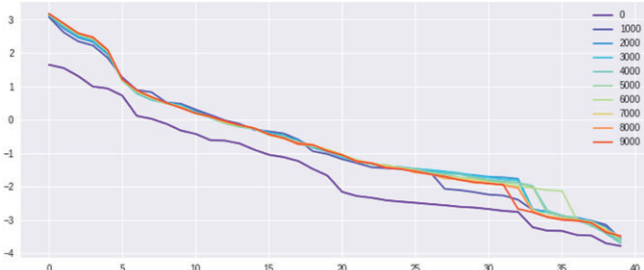
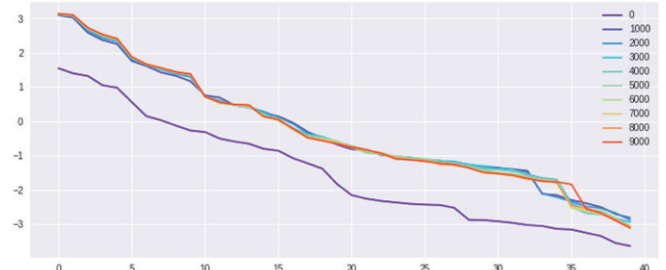
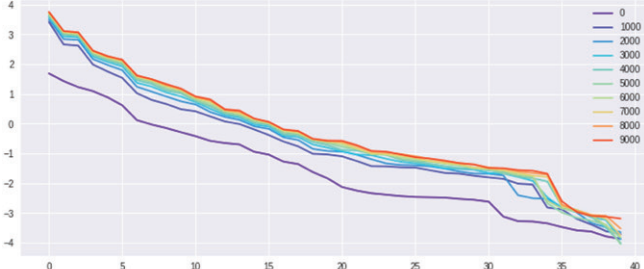
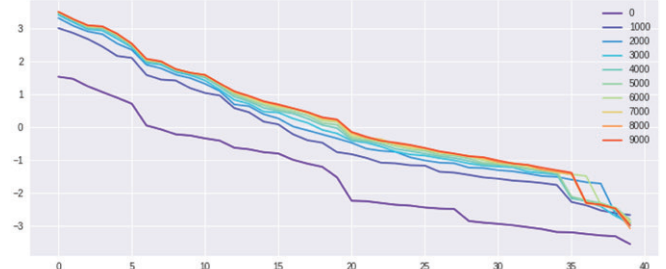
(a)  $J = 2$ (b)  $J = 4$ (c)  $J = 8$ (d)  $J = 16$ 

Figure 3. Log of the sorted top 80% eigenvalues of kernel matrix along training with different  $f_k$ .

## 6. Experiments

We run experiments to illustrate the spectral decay of the dynamic kernels defined in  $K_t$  over time  $t$ . The exercise is to quantitatively showcase that during neural network training, one does learn the data-adaptive representation, which is task-specific depending on the true complexity of  $f_*$ . The training process is the same as the one we theoretically analyze: vanilla gradient descent on a two-layer NN of  $m$  neurons, with infinitesimal random initialization scales as  $1/\sqrt{m}$ .

The first experiment is a synthetic exercise with well-specified models. We generate  $\{x_i\}_{i=1}^{50}$  from isotropic Gaussian in  $\mathbb{R}^5$ , and  $y_i = f_*(x_i) = \sum_{j=1}^J w_j^* \sigma(x_i^T u_j^*)$  with different  $J$ . In other words, we choose different target  $f_*$  (task complexity) by varying  $J$ . We select  $m = 500$  in our experiment. The top 80% of the sorted eigenvalues of the kernel matrix  $K_t$  along the GD training process are shown in Figure 3. The  $x$ -axis is the index of eigenvalues in descending order, and the  $y$ -axis is the logarithmic values of the corresponding eigenvalues. Different color indicates the spectral decay of the  $K_t$  at different training time  $t$ . The eigenvalue-decays stabilize over time  $t$  means that the training process approaches stationarity. As we can see with  $f_*$  belongs to the NN family, the eigenvalues of the kernel matrix, in general, become larger during the training process. For a more complicated target function, it takes longer to reach stationarity.

The second experiment is another synthetic test on fitting random labels. We generate  $\{x_i\}_{i=1}^{50}$  from isotropic Gaussian in  $\mathbb{R}^5$ , as  $y_i$  takes  $\pm 1$  with equal chance. We select  $m = 200, 500$ , and  $n = 50, 200$  to investigate those parameters' influence on the kernel  $K_t$ . We want to point out two observations. First, fixed

$n$ , we investigate over-parameterized models ( $m = 200, 500$  large). Shown from Figure 4 along the row, the kernels for different  $m$ 's behave much alike. In other words, in the infinite neurons limit, the kernel will stabilize. Second, fixed  $m$ , we vary the number of samples  $n$ , to simulate different interpolation hardness. As seen from Figure 4 along the column, the kernels and the convergence over time are distinct, reflecting the different difficulty of the interpolation.

The third experiment (Figure 5) is regression using the MNIST dataset with different sample size  $n = 50, 200$ . We hope to investigate the influence of sample size on the kernel matrix along the training process. For a larger sample size  $N$ , it takes longer to reach stationarity.

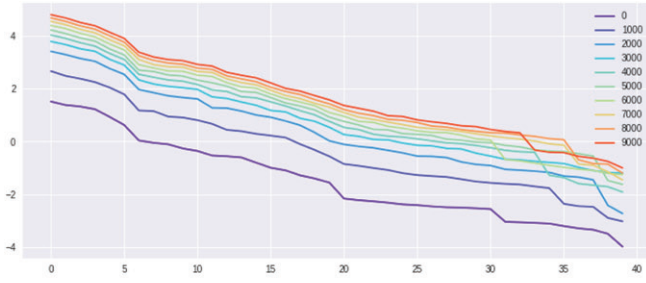
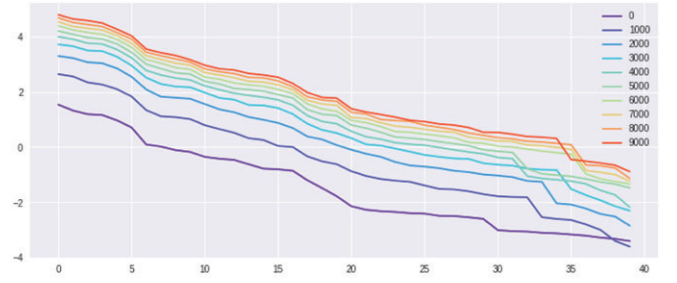
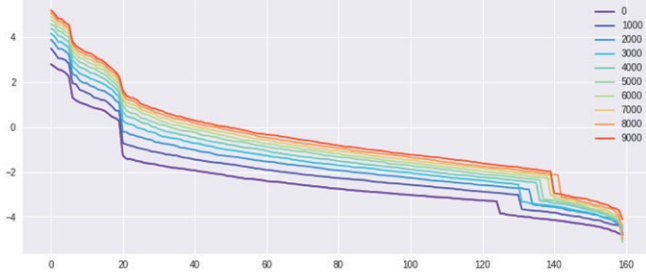
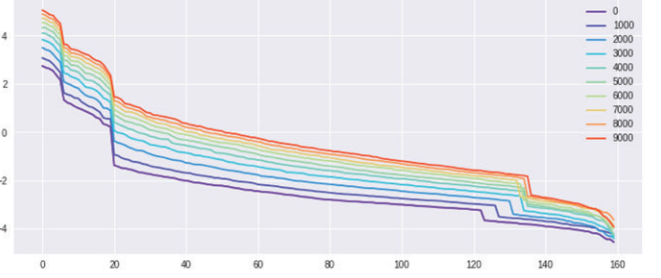
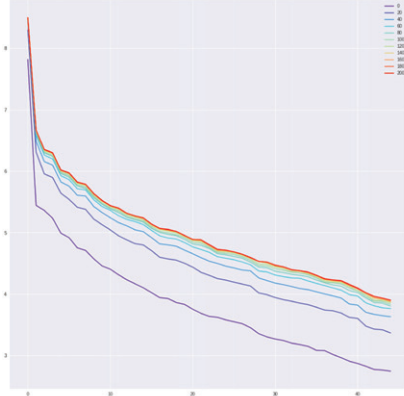
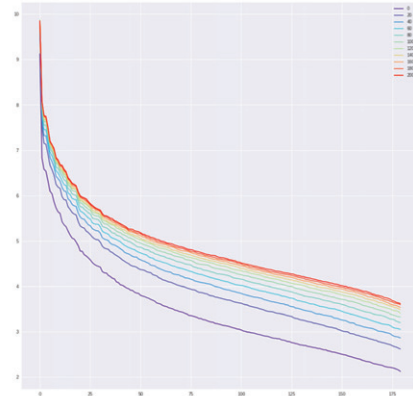
## 7. Main Proofs

*Proof of Theorem 3.1.* From the definition, we have  $\mathcal{T}^*p \in \mathcal{H}_\infty$  for any  $p \in L^2_{|\rho_\infty|}$ , and  $\mathcal{T}^*$  is a surjective mapping. Suppose that  $\hat{g} \in \mathcal{H}_\infty$  is a minimizer of (9), then we claim that for any  $p \in L^2_{|\rho_\infty|}$ , one must have

$$\langle f_* - \hat{g}, \mathcal{T}^*p \rangle_\mu = 0, \quad \forall p \in L^2_{|\rho_\infty|}. \quad (29)$$

This claim can be seen from the following argument. Suppose not, then for  $p$  that violates the above, construct

$$\hat{g}_\epsilon = \hat{g} + \epsilon \mathcal{T}^*p \in \mathcal{H}_\infty,$$

(a)  $N = 50, m = 200$ (b)  $N = 50, m = 500$ (c)  $N = 200, m = 200$ (d)  $N = 200, m = 500$ **Figure 4.** Log of the sorted top 80% eigenvalues of kernel matrix along training with random labels.(a)  $N = 50$ (b)  $N = 200$ **Figure 5.** Log of sorted top 90% eigenvalues of kernel matrix along training process for mnist.

we know

$$\|f_* - \hat{g}_\epsilon\|_\mu^2 = \|f_* - \hat{g}\|_\mu^2 - 2\epsilon \langle f_* - \hat{g}, \mathcal{T}^* p \rangle_\mu + \epsilon^2 \|\mathcal{T}^* p\|_\mu^2. \quad (30)$$

For  $\epsilon$  with the same sign as  $\langle f_* - \hat{g}, \mathcal{T}^* p \rangle_\mu \neq 0$  and small enough, one can see that  $\|f_* - \hat{g}_\epsilon\|_\mu^2 < \|f_* - \hat{g}\|_\mu^2$  which validates that  $\hat{g}$  is a minimizer. From the same argument, one can see that  $\hat{g}$  is a minimizer if and only if (29) holds, in other words,

$$\langle \mathcal{T}(f_* - \hat{g}), p \rangle_{\rho_\infty} = \langle f_* - \hat{g}, \mathcal{T}^* p \rangle_\mu = 0. \quad (31)$$

From PDE characterization (23) with ReLU activation, one knows that

$$\begin{aligned} V(\Theta) &= \mathbf{E}[\mathbf{y}\sigma(\mathbf{x}^T \Theta)] = \mathbf{E}[f_*(\mathbf{x})\sigma(\mathbf{x}^T \Theta)] \\ U(\Theta, \tilde{\Theta}) &= -\mathbf{E}[\sigma(\mathbf{x}^T \Theta)\sigma(\mathbf{x}^T \tilde{\Theta})] \end{aligned}$$

and the expression for the velocity field

$$\begin{aligned} \|\Theta\| \left( \nabla_\Theta V(\Theta) + \nabla_\Theta \int U(\Theta, \tilde{\Theta}) \|\tilde{\Theta}\| \rho_t(d\tilde{\Theta}) \right) \\ = \|\Theta\| \left( \int f_*(x) x \mathbb{1}_{x^T \Theta > 0} \mu(dx) \right. \\ \left. - \int \int x \mathbb{1}_{x^T \Theta > 0} \sigma(x^T \tilde{\Theta}) \|\tilde{\Theta}\| \rho_\infty(d\tilde{\Theta}) \mu(dx) \right). \end{aligned}$$



We know that any stationary point  $(\rho_{+, \infty}, \rho_{-, \infty})$  has the following property (Mei, Montanari, and Nguyen 2018):

$$\begin{aligned} \text{supp}(\rho_\infty) &\subseteq \left\{ \Theta : \int f_*(x) x \mathbb{1}_{x^T \Theta > 0} \mu(dx) \right. \\ &\quad \left. = \int \int x \mathbb{1}_{x^T \Theta > 0} \sigma(x^T \tilde{\Theta}) \|\tilde{\Theta}\| \rho_\infty(d\tilde{\Theta}) \mu(dx) \right\}. \end{aligned} \quad (32)$$

Multiplying both sides by  $\|\Theta\| \Theta^T$  and recall the property of ReLU, the above condition implies that for all  $\Theta \in \text{supp}(\rho_\infty)$ , we have

$$\begin{aligned} &\int f_*(x) \|\Theta\| \sigma(x^T \Theta) \mu(dx) \\ &= \int \int \|\Theta\| \sigma(x^T \Theta) \sigma(x^T \tilde{\Theta}) \|\tilde{\Theta}\| \rho_\infty(d\tilde{\Theta}) \mu(dx). \end{aligned} \quad (33)$$

One can see the stationary condition on  $\rho_\infty$  (fixed points of the dynamics) (33) translates to

$$\mathcal{T}f_*(\Theta) = \left( \mathcal{T} \mathcal{T}^* \frac{d\rho_\infty}{d|\rho_\infty|} \right) (\Theta), \quad \forall \Theta \in \text{supp}(\rho_\infty). \quad (34)$$

Here, the function  $\frac{d\rho_\infty}{d|\rho_\infty|}$  is the Radon–Nikodym derivative. In addition, one can easily verify that, as  $\rho_\infty$  has bounded total variation

$$\frac{d\rho_\infty}{d|\rho_\infty|} \in L^2_{|\rho_\infty|}.$$

Therefore, combining all the above, one knows that

$$f_\infty(x) = \int \|\Theta\| \sigma(x^T \Theta) \rho_\infty(d\Theta) = \mathcal{T}^* \frac{d\rho_\infty}{d|\rho_\infty|} \in \mathcal{H}_\infty$$

and that for any  $p \in L^2_{|\rho_\infty|}$

$$\langle f_* - f_\infty, \mathcal{T}^* p \rangle_\mu = \langle \mathcal{T}(f_* - f_\infty), p \rangle_{|\rho_\infty|}, \quad (35)$$

$$= \left\langle \mathcal{T}f_* - \mathcal{T} \mathcal{T}^* \frac{d\rho_\infty}{d|\rho_\infty|}, p \right\rangle_{|\rho_\infty|}, \quad (36)$$

$$= \int \left( \mathcal{T}f_* - \mathcal{T} \mathcal{T}^* \frac{d\rho_\infty}{d|\rho_\infty|} \right) (\Theta) |\rho_\infty|(d\Theta) = 0$$

due to (34). (37)

We have proved that  $f_\infty = \mathcal{T}^* \frac{d\rho_\infty}{d|\rho_\infty|}$  satisfies normal condition for being a minimizer to (9).  $\square$

**Proof of Proposition 5.2.** The first inequality in (27) is trivial. For the second inequality, it suffices to show for any  $c = (c_1, \dots, c_p)^T, x_1, \dots, x_p, \Theta$ , we have

$$\sum_{ij} c_i c_j \|\Theta\|^2 x_i^T x_j \mathbb{1}_{x_i^T \Theta > 0} \mathbb{1}_{x_j^T \Theta > 0} \geq \sum_{ij} c_i c_j \sigma(x_i^T \Theta) \sigma(x_j^T \Theta). \quad (38)$$

The RHS equals

$$\sum_{ij} c_i c_j x_i^T \Theta x_j^T \Theta \mathbb{1}_{x_i^T \Theta > 0} \mathbb{1}_{x_j^T \Theta > 0} = \left( \sum_i c_i x_i^T \Theta \mathbb{1}_{x_i^T \Theta > 0} \right)^2, \quad (39)$$

$$= \langle \Theta, \sum_i c_i x_i \mathbb{1}_{x_i^T \Theta > 0} \rangle^2 \leq \|\Theta\|^2 \left\| \sum_i c_i x_i \mathbb{1}_{x_i^T \Theta > 0} \right\|^2 = \text{LHS}. \quad (40)$$

For the last inequality, with compactness condition on  $\rho_\infty$ , we have

$$\begin{aligned} &\sum_{ij} c_i c_j \int \|\Theta\|^2 \sigma(x_i^T \Theta) \sigma(x_j^T \Theta) |\rho_\infty|(\Theta) \\ &\leq D^2 \sum_{ij} c_i c_j \int \sigma(x_i^T \Theta) \sigma(x_j^T \Theta) |\rho_\infty|(\Theta). \end{aligned} \quad (41)$$

Therefore,  $D^2 K_\infty^{(1)} \geq H_\infty$ .  $\square$

**Proof of Theorem 3.2.** Let us rewrite Corollary 5.2 into

$$\frac{d}{dt} \|\Delta_t\|_\mu^2 = -2 \langle \Delta_t, \mathcal{K}_t \Delta_t \rangle_\mu = -2 \|\mathcal{K}_t^{1/2} \Delta_t\|_\mu^2, \quad (42)$$

here  $\mathcal{K}_t : L_\mu^2(x) \rightarrow L_\mu^2(x)$  denotes the integral operator associated with  $K_t$ ,

$$(\mathcal{K}_t f)(x) := \int K_t(x, \tilde{x}) f(\tilde{x}) \mu(d\tilde{x}). \quad (43)$$

From (42)

$$\frac{d}{dt} \|\Delta_\infty\|_\mu^2 = -2 \|\mathcal{K}_\infty^{1/2} \Delta_\infty\|_\mu^2, \quad (44)$$

we know that the RHS equals zero implies

$$\|\mathcal{K}_\infty^{1/2} \Delta_\infty\|_\mu^2 = 0$$

$$\langle \mathcal{K}_\infty^{1/2} g, \Delta_\infty \rangle_\mu = \langle g, \mathcal{K}_\infty^{1/2} \Delta_\infty \rangle_\mu = 0, \quad \forall g \in L_\mu^2.$$

This further implies  $\Delta_\infty$  lies in the kernel of RKHS  $\mathcal{K}_\infty$  as  $\mathcal{K}_\infty = \{\mathcal{K}_\infty^{1/2} g : g \in L_\mu^2\}$ .  $\square$

**Proof of Proposition 4.1.** The gradients on the original parameters are,

$$\begin{aligned} \frac{dw_j(t)}{dt} &= -\widehat{\mathbf{E}} \left[ \frac{\partial \ell(\mathbf{y}, f_t)}{\partial f} \sigma(\mathbf{x}^T u_j(t)) \right] - \frac{1}{m} \lambda w_j(t), \\ \frac{du_j(t)}{dt} &= -\widehat{\mathbf{E}} \left[ \frac{\partial \ell(\mathbf{y}, f_t)}{\partial f} w_j(t) \mathbb{1}_{\mathbf{x}^T u_j(t) \geq 0} \mathbf{x} \right] - \frac{1}{m} \lambda u_j(t). \end{aligned}$$

Clearly, on the rescaled parameter, the following holds

$$\begin{aligned} \frac{d\theta_j}{dt} &= \sqrt{m} \frac{dw_j}{dt} = -\widehat{\mathbf{E}} \left[ (f_t(\mathbf{x}) - \mathbf{y}) \sigma(\mathbf{x}^T \Theta_j(t)) \right] - \frac{1}{m} \lambda \theta_j, \\ \frac{d\Theta_j}{dt} &= \sqrt{m} \frac{du_j}{dt} = -\widehat{\mathbf{E}} \left[ (f_t(\mathbf{x}) - \mathbf{y}) \theta_j \mathbb{1}_{\mathbf{x}^T \Theta_j \geq 0} \mathbf{x} \right] - \frac{1}{m} \lambda \Theta_j. \end{aligned}$$

Multiply the first equation by  $\theta_j$ , and the second equation by  $\theta_j^T$ , take the difference, we can verify that

$$\frac{d(\theta_j^2 - \|\Theta_j\|^2)}{dt} = -\lambda/m(\theta_j^2 - \|\Theta_j\|^2), \quad (45)$$

$$\theta_j(t)^2 - \|\Theta_j(t)\|^2 = (\theta_j(0)^2 - \|\Theta_j(0)\|^2) \exp(-\lambda t/m). \quad (46)$$

Therefore, the balanced condition still holds at stationarity for arbitrary bounded initialization,

$$\theta_j(\infty)^2 - \|\Theta_j(\infty)\|^2 = 0, \quad \forall j.$$

Now the optimality condition for the velocity field reads the following, for any  $\Theta_j(\infty) \in \text{supp}(\widehat{\rho}_\infty^\lambda)$  (we abbreviate the  $\infty$

in the following display, note  $\tilde{\theta}(\infty)$  corresponds to the second layer weights w.r.t. to  $\tilde{\Theta}(\infty)$

$$\begin{aligned} & \theta_j \widehat{\mathbf{E}}[\mathbf{y} \mathbb{1}_{\mathbf{x}^T \Theta_j \geq 0} | \mathbf{x}] \\ &= \theta_j \int \tilde{\theta} \widehat{\mathbf{E}}[\mathbb{1}_{\mathbf{x}^T \Theta_j \geq 0} \mathbf{x} \sigma(\mathbf{x}^T \tilde{\Theta})] |\widehat{\rho}_\infty^\lambda|(d\tilde{\Theta}) + \frac{1}{m} \lambda \Theta_j, \\ & \text{Multiply by } \Theta_j^T, \\ &= \int \theta_j \tilde{\theta} \widehat{\mathbf{E}}[\sigma(\mathbf{x}^T \Theta_j) \sigma(\mathbf{x}^T \tilde{\Theta})] |\widehat{\rho}_\infty^\lambda|(d\tilde{\Theta}) + \frac{\lambda}{m} \|\Theta_j\|^2, \\ &= \int \theta_j \tilde{\theta} \widehat{\mathbf{E}}[\sigma(\mathbf{x}^T \Theta_j) \sigma(\mathbf{x}^T \tilde{\Theta})] |\widehat{\rho}_\infty^\lambda|(d\tilde{\Theta}) \\ & \quad + \lambda \int \theta_j \tilde{\theta} \mathbb{1}_{\tilde{\Theta}=\Theta_j} |\widehat{\rho}_\infty^\lambda|(d\tilde{\Theta}), \end{aligned}$$

where the last step uses the condition  $\theta_j^2(\infty) = \|\Theta_j(\infty)\|^2$ , and the fact that  $|\widehat{\rho}_\infty^\lambda| = \frac{1}{m} \sum_{j=1}^m \delta_{\Theta_j}$  and

$$\int \theta_j \tilde{\theta} \mathbb{1}_{\tilde{\Theta}=\Theta_j} |\widehat{\rho}_\infty^\lambda|(d\tilde{\Theta}) = \frac{1}{m} \theta_j^2 = \frac{1}{m} \|\Theta_j\|^2.$$

In the matrix form, where  $\widehat{\rho}_\infty^\lambda = \frac{1}{m} \sum_{l \in [m]} \text{sgn}(\theta_l) \delta_{\Theta_l}$

$$\sum_{l \in [m]} \left[ n \widehat{U}(\Theta_j, \Theta_l) + n \lambda \mathbb{1}_{\Theta_l=\Theta_j} \right] \theta_l / m = \sigma(\Theta_j^T X) Y.$$

Therefore, define  $\sigma(x^T \Xi) := [\sigma(x^T \Theta_1), \dots, \sigma(x^T \Theta_m)] \in \mathbb{R}^{1 \times m}$ , and  $\sigma(X \Xi) := [\sigma(x_1^T \Xi)^T, \dots, \sigma(x_n^T \Xi)^T] \in \mathbb{R}^{m \times n}$ , we have

$$\begin{aligned} \widehat{f}_\infty^{\text{nn}, \lambda}(x) &= \sum_{l \in [m]} \theta_l \sigma(x^T \Theta_l) / m \\ &= \sigma(x^T \Xi) [\sigma(X \Xi) \sigma(X \Xi)^T + n \lambda I_m]^{-1} \sigma(X \Xi) Y \\ &= \sigma(x^T \Xi) \sigma(X \Xi) [\sigma(X \Xi)^T \sigma(X \Xi) + n \lambda I_n]^{-1} Y \\ &= \widehat{H}_\infty^\lambda(x, X) [\widehat{H}_\infty^\lambda(X, X) + n/m \cdot \lambda I_n]^{-1} Y. \end{aligned}$$

The last line follows as  $\widehat{H}^\lambda(x, \tilde{x}) := \int \sigma(x^T \Theta) \sigma(\tilde{x}^T \Theta) |\widehat{\rho}_\infty^\lambda|(d\Theta) = 1/m \cdot \sigma(x^T \Xi) \sigma(\tilde{x}^T \Xi)^T$ .  $\square$

## Supplementary Materials

The remaining proofs are designated to the appendix, which is included in the supplementary materials.

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