Ty Madsen | HW#9 | ECEN 370

1) S-4.32

$Z = X^2 + Y^2$

$\mu_x = \mu_y = 0 \quad \sigma_x^2 = \sigma_y^2 = \sigma_z$

$f_z(z) = f_{xy}(x,y) = f_x(x) f_y(y) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(x^2+y^2)}$

$f_{xy}(\sqrt{z-y^2}, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(z-y^2+y^2)} = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2} z}$

$f_{xy}(-\sqrt{z-y^2}, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(+z-y^2+y^2)} = \nearrow$

$f_z(z) = \int_{-\sqrt{z}}^{\sqrt{z}} \frac{1}{2\sqrt{z-y^2}} \left(2 \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2} z} \right) dy = \frac{1}{\pi\sigma^2} e^{-\frac{1}{2\sigma^2}} \int_0^{\sqrt{z}} \frac{1}{\sqrt{z-y^2}} dy$

if $y = \sqrt{z} \sin\theta \Rightarrow \sqrt{z-y^2} = \sqrt{z - z\sin^2\theta} = \sqrt{z(1-\sin^2\theta)} = \sqrt{z}\cos\theta \quad dy = \sqrt{z}\cos\theta \, d\theta$

$\int_0^{\sqrt{z}} \frac{1}{\sqrt{z-y^2}} dy = \int_0^{\pi/2} 1 \, d\theta = \pi/2$

$$\boxed{f_z = \frac{1}{2\sigma^2} e^{-\frac{1}{2\sigma^2} z} \quad z > 0}$$

2) B-4.17

$\sigma_x^2 = \sigma_y^2 = \sigma^2$

$V = X - Y \quad W = X + Y$

$E[V] = E[X-Y] = E[X] - E[Y]$

$\mathrm{Cov}[X-Y, X+Y] = E[(X-Y)(X+Y)] - E[X-Y]E[X+Y]$

$= E[X^2] - 2E[XY] - E[Y^2] - (E[X] - E[Y])(E[X] + E[Y])$

$-2E[XY] + \underbrace{E[X^2] - (E[X])^2}_{\cancel{VAR(X)}} - \underbrace{(E[Y^2] - E[Y])^2 - 2E[XY])}_{\cancel{VAR(Y)}}$ $\quad (E[X])^2 - 2E[X]E[Y] - (E[Y])^2$

$-2E[XY] + 2E[XY] = \boxed{0 \Rightarrow \text{NON CORRELATED}}$

3) B-4.18)

$E[W] = E[X] = E[Y] = E[Z] = 0 \qquad R = W + X \quad S = X + Y \quad T = Y + Z$

VAR(") = 1

$\rho(R,S) = \frac{\mathrm{Cov}(R,S)}{\sqrt{\mathrm{var}(R)\,\mathrm{var}(S)}} =$ $\quad E[(W+X)(X+Y)] - E[W+X]E[X+Y]$ $\quad (E[W]+E[X])(E[X]+E[Y])$ $\quad 0 \quad 0 \quad 0$

$E[WX] + E[X^2] + E[WY] + E[XY] - (E[\cancel{WX}] + E[WY]) + E[\cancel{X}]^2 + E[XY])$

$0 \quad 1 \quad 0 \quad 0$

$= \frac{1}{\sqrt{(1+1)(1+1)}} = \boxed{\frac{1}{2}}$

$\rho(R,T) = \frac{0}{\sqrt{4}} = \boxed{0}$ $\quad E[(W+X)(Y+Z)]$

$E[WY] + \cancel{E[WZ]} + \cancel{E[XY]} + E[XZ] - E[W+X]E[Y+Z]$

$0 \qquad (E[W]+E[X])(E[Y]+E[Z])$

$0 \quad 0 \quad 0 \quad 0$

4) B-4.19 $E[X]=0$ $E[X^2]=1$ $E[X^3]=0$ $E[X^4]=3$

$Y=a+bX+cX^2$

$Y^2=(a+bX+cX^2)(a+bX+cX^2)$

$a^2+abX+acX^2+abX+b^2X^2+bcX^3+acX^2+bcX^3+c^2X^4$

$a^2+2abX+(2ac+b^2)X^2+2bcX^3+c^2X^4$

$$\rho(X,Y)=\frac{cov(X,Y)}{\sqrt{VAR(X)\,VAR(Y)}}=\frac{E[X(a+bX+cX^2)]-\cancel{E[X]}E[Y]}{}$$

$=E[aX+bX^2+cX^3]=aE[X]+bE[X^2]+cE[X^3]$
$\quad 0 \quad b(1) \quad 0$

$E[Y]=[a+bX+cX^2]=c$

$VAR(X)=E[X^2]-(E[X])^2 = 1$

$(a+c)^2=a^2+c^2+2ac$

$VAR(Y)=E[a^2+2abX+(2ac+b^2)X^2+2bcX^3+c^2X^4]-c$
$\quad a^2 \quad 2ab \quad (2ac+b^2) \quad 0 \quad 3c^2$

$VAR[Y]=\cancel{a^2}\cancel{2ac}+b^2+3c^2-(\cancel{a^2}+c^2+2ac)$

b^2+2c^2

$$=\frac{b}{\sqrt{1\,(b^2+2c^2)}}$$

$$\boxed{=\frac{b}{\sqrt{b^2+2c^2}}}$$

5) B-4.20

$(E[XY])^2\le E[X^2]E[Y^2] \to 0\le E\left[\left(X-\frac{E[XY]}{E[Y^2]}Y\right)^2\right]=E\left[X^2-2\frac{E[XY]}{E[Y^2]}XY+\frac{E[XY]^2}{(E[Y^2])^2}Y^2\right]$

$=E[X^2]-2\frac{E[XY]}{E[Y^2]}E[XY]+\frac{(E[XY])^2}{(E[Y^2])^2}E[Y^2]$

$=E[X^2]-\frac{(E[XY])^2}{E[Y^2]} \Rightarrow (E[XY])^2\le E[X^2]E[Y^2]$

FOR ~~IND~~ RVS:

$(E[XY])^2=(E[X]E[Y])^2=E[X]^2E[Y]^2$

NORMAL RVS.

$(E[XY])^2=(E[X]E[Y]+2cov(X,Y))^2 \Rightarrow (E[X]E[Y]+2E[XY]-2E[X]E[Y])^2$

$\Rightarrow (2E[XY]-E[X]E[Y])^2$

$cov(X,Y)=E[XY]-E[X]E[Y]$

$\Rightarrow \cancel{4E[XY]^2-4E[XY]E[X]E[Y]+E[X]^2E[Y]^2}$

$\cancel{(E[XY])^2=4E[XY]^2-4E[XY]E[X]E[Y]+(E[X])^2(E[Y])^2=(2(E[XY]))^2-4E[XY]E}$

$(E[XY])^2=(2E[XY]-E[X]E[Y])^2\Rightarrow$

6) B-5.1

a) $\sigma M_n=1/\sqrt{n}$ $\quad \sigma M_n\le .01 \Rightarrow \boxed{n\ge 10{,}000}$

b) $P(|M_n-h|\le .05)\ge .99$ $\quad h=E[M_n],\ \sigma^2 M_n=1/n$

$P(|M_n-h|\le .05)=P(|M_n-E[M_n]|\le .05)=1-P(|M_n-E[M_n]|\ge .05)$

$\ge 1-\frac{1/n}{(.05)^2} \Rightarrow 1-\frac{1/n}{(.05)^2}\ge .99 \Rightarrow \boxed{n\ge 40{,}000}$

c) a) $\sigma M_n=.3/\sqrt{n}\to \sigma M_n\le .01$ $\boxed{n\ge 900}$

b) $\sigma M_n=.3/\sqrt{n}$ $\quad \sigma M_n\le .01$ $\quad n\ge 900$

$1-\frac{.09/n}{(.05)^2}\ge .99 \Rightarrow \boxed{n\ge 3{,}600}$

7) B-4.81

$$\lim_{n\to\infty} P(|\bar{X}_n - \mu| > \varepsilon) = 0 \qquad \bar{X}_n = \frac{1}{n}(x_1 + \cdots + x_n) \qquad E\{X_i\} = \mu, \ Var(X_i) = \sigma^2$$

$$E(\bar{x}_n) = \mu \qquad Var(\bar{x}_n) = \frac{\sigma^2}{n} \Rightarrow P(|\bar{x}_n - \mu| > \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2}$$

$$\lim_{n\to\infty} \sigma^2/(n\varepsilon^2) = 0 \ \therefore \ \lim_{n\to\infty} P(|\bar{x}_n - \mu| > \varepsilon) = 0$$

8) S-4.82 $\quad f_X(x) = \qquad x_1 \cdots x_n$

$$\bar{X}_n = \frac{1}{n}(x_1 + \cdots + x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad \mu, \sigma^2 \qquad \varepsilon = \sigma/10$$

$$P\left(|x_n - \mu| > \frac{\sigma}{10}\right) = 1 - P\left(|x_n - \mu| \leq \frac{\sigma}{10}\right) \leq \frac{\sigma^2}{n\sigma^2/100} = \frac{100}{n}$$

$$P\left(|\bar{x}_n - \mu| \leq \frac{\sigma}{10}\right) \geq 1 - \frac{100}{n} \qquad \frac{100}{n} \leq .05 \qquad n \geq 100/.05 = 2000$$

$$\boxed{n \geq 2000}$$

9) B-5.4

$f \qquad M_n = S_n/n \qquad P(|M_n - \mu| \geq \varepsilon) \leq \delta \qquad \delta = \frac{\sigma^2}{\varepsilon^2} \qquad \sigma = 1/2\sqrt{n}$

a) $\varepsilon = 1/2\,\varepsilon_0 \qquad P(|M_n - f| \geq \varepsilon) \leq \frac{1}{4n\varepsilon^2} \rightarrow \frac{1}{4n\frac{1}{4}\varepsilon_0^2} \Rightarrow n = 4n_0$

n MUST INCREASE BY FACTOR OF 4

b) δ IS REDUCED TO $1/2\,\delta$

$$P(|M_n - f| \geq \varepsilon) \leq \frac{1}{4n\varepsilon^2} \cdot \frac{1}{2} = \frac{1}{8n\varepsilon^2} \Rightarrow n = \frac{1}{2} n_0$$

n MUST BE ~~REDUCED~~ DOUBLED ~~TO 1/2~~