Ty Madsen
ECEN 370
HW #4

1) B 2.16) X is R.V.

$p_X(x) = \begin{cases} x^2/a, & \text{if } x = -3,-2,-1,0,1,2,3 \\ 0, & \text{otherwise} \end{cases}$

a) $E[X] = \sum_x x p_X(x) = \frac{x^3}{a}$ for $x = -3,-2 \cdots 2,3$

$\boxed{E[X] = 0}$ $= \cancel{\frac{-27}{a}} + \cancel{\frac{-8}{a}} + \cancel{\frac{-1}{a}} + 0 + \cancel{\frac{1}{a}} + \cancel{\frac{8}{a}} + \cancel{\frac{27}{a}} = 0$

$a = \sum_x p_X(x) = \sum_x x^2 = (-3)^2 + (-2)^2 + (-1)^2 + 0^2 + 1^2 + 2^2 + 3^2$

$\boxed{a = 28}$

b) $Z = (X - E[X])^2 = X^2 \Rightarrow Z = \{0,1,4,9\}$

$p_Z(z) = p_X(\sqrt{z}) + p_X(-\sqrt{z}) = \frac{z}{28} + \frac{z}{28}$ $\boxed{p_Z(z) = \begin{cases} \frac{z}{14}, & \text{for } z = \{1,4,9\} \\ 0, & \text{OTHERWISE} \end{cases}}$

c) $\mathrm{var}(X) = E[Z] = \sum_z z p_Z(z)$

$= \frac{1}{14}\sum z^2 = \frac{1}{14}(1 + 16 + 81) = \frac{98}{14} = \boxed{7}$

d) $\mathrm{var}(X) = \sum_x (x - E[X])^2 p_X(x)$

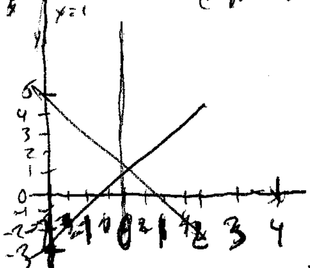
$= x^2 p_X(x) = \frac{1}{28}\sum_{x \in \{-3,-2,\ldots,3\}} x^4 = 2(81 + 16 + 1) = \frac{196}{28} = \boxed{7}$

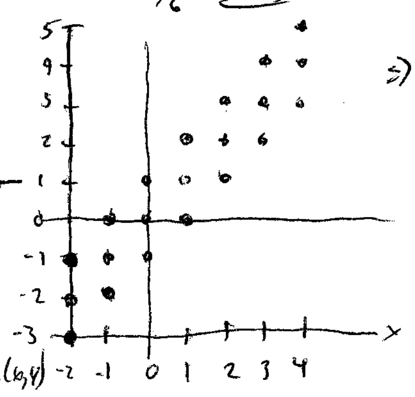
2) S-2.47) Roll fair die $p = \frac{1}{6}$

$u_X = E(X) = \sum_k x_k p_X(x_k)$ $\quad p_X(x) = (1-p)^{x-1} p$

$= \sum_{x=1} x q^{x-1} p = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p} \Rightarrow \frac{1}{1/6} = \boxed{6}$

3) B-2.24)





$\Rightarrow$ 21 POSSIBILITIES

PMF $X = \sum_y p_{XY}(x,y)$ $\quad$ PMF $Y = \sum_x p_{XY}(x,y)$

$p_{X,Y}(x,y) = \begin{cases} 1/21 & \text{if } (x,y) \text{ is in} \\ 0 & \text{otherwise} \end{cases}$

a) $\boxed{p_X(x) = \begin{cases} 3/21, & x = -2,-1 \cdots 4 \\ 0 & \text{OTHERWISE} \end{cases}}$ $E[X]$ = MIDDLE OF $[-2,4]$ $\boxed{E[X] = 1}$

$p_Y(y) = \begin{cases} 1/21, & y = -3,5 \\ 2/21 & y = -2,4 \\ 3/21 & y = -1,0 \cdots 3 \\ 0 & \text{OTHERWISE} \end{cases}$ $\quad E[Y] = \frac{1}{21}[-3+5] + \frac{2}{21}(-2+4) + \frac{3}{21}[-1+1+2+3]$

$= \frac{4}{21} + \frac{15}{21} = \frac{21}{21} = 1$

$\boxed{E[Y] = 1}$

b) $P = 100X + 200Y$

$= 100E[X] + 200E[Y] = \boxed{300}$

4) B-2.26) $p_X(k) = \begin{cases} P(X>k-1) - P(X>k) & \text{for } k=101,\ldots,110 \\ 0 & \text{otherwise} \end{cases}$

$P(X>100) = 1$

$P(X>k) = P(X_1>k, X_2>k, X_3>k)$
$= P(X_1>k)P(X_2>k)P(X_3>k)$
$= \frac{(110-k)^3}{10^3}$

$P(X+1>k) = P(X_1+1>k, X_2+1>k, X_3+1>k)$
$= P(X_1+1>k)P(X_2+1>k)P(X_3+1>k)$
$= \frac{(111-k)^3}{10^3}$

a) $\boxed{p_X(k) = \begin{cases} \frac{(111-k)^3-(110-k)^3}{10^3} & \text{for } k=101,102,\ldots,110 \\ 0 & \text{otherwise} \end{cases}}$

b) $E[X_i] = \frac{(101+110)}{2} = 105.5$

$E[X] = \sum_{k=-\infty}^{\infty} k \cdot p_X(k) = \sum_{k=101}^{110} k \cdot p_X(k) = \sum_{k=101}^{110} k\left(\frac{(111-k)^3-(110-k)^3}{10^3}\right) = 103.025$

$E[X_i] - E[X] = \boxed{2.475}$

5) S-2.53

X = Poisson RV w/ λ. Find conditional PMF of $X \mid B = (X \text{ is even})$

$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!} \quad k=0,1,\ldots$

$P(B) = P(X=0,2,4\ldots) = \sum_{k=\text{even}}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!}$

$P(\text{ODD}) = P(X=1,3,5\ldots) = \sum_{k=\text{odd}}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!}$

$P(B)+P(A) = \sum_{k=\text{ev}}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} + \sum_{k=\text{odd}}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = e^{-\lambda}e^{\lambda} = 1$

$P(B)-P(A) = \sum_{k=\text{ev}}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} - \sum_{k=\text{odd}}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!} = e^{-\lambda}e^{-\lambda} = e^{-2\lambda}$

$\Rightarrow 2P(B) = 1+e^{-2\lambda} \Rightarrow$

$P(B) = \frac{1}{2}(1+e^{-2\lambda})$

$p_X(k|B) = \frac{P\{(X=k)\cap B\}}{P(B)} = \boxed{\begin{cases} \frac{P(X=x)}{P(B)} = \frac{2e^{-\lambda}\lambda^k}{(1+e^{-2\lambda})k!} & k \text{ even} \\ \frac{P(\emptyset)}{P(B)} = 0 & k = \text{odd} \end{cases}}$

6) B-2.51) 4 Rolls of 6-sided die. X = # of 1s Y = # of 2s $P_{XY}(x,y) = ?$

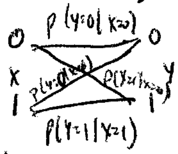
$p_Y(y) = \binom{4}{y}\left(\frac{1}{6}\right)^y\left(\frac{5}{6}\right)^{4-y} \quad y=0,1,2,3,4$

$(Y=y)$ X = # 1s given $4-y$ rolls w/ vals 1,3,4,5,6

$p_{X|Y}(x|y) = \binom{4-y}{x}\left(\frac{1}{5}\right)^x\left(\frac{4}{5}\right)^{4-y-x} \quad 0\le x+y\le 4$

$p_{X,Y}(x,y) = p_Y(y)\,p_{X|Y}(x|y) = \boxed{\begin{cases} \binom{4}{y}\left(\frac{1}{6}\right)^y\left(\frac{5}{6}\right)^{4-y}\binom{4-y}{x}\left(\frac{1}{5}\right)^x\left(\frac{4}{5}\right)^{4-y-x} & \text{for } 0\le x+y\le 4 \\ 0 & \text{otherwise} \end{cases}}$

7) S - 3.12 a,b)



$P(x=0) = .5$
$P(y=1|x=0) = .1$
$P(y=0|x=1) .2$

a) $p_{x,y}(x,y) = ?$

$P(x=1) = 1 - P(x=0) = .5$
$P(y=0|x=0) = .9$
$P(y=1|x=1) = .8$

$\Rightarrow$

$P(x=0, y=0) = P(y=0|x=0)P(x=0) = .9(.5) = .45 = p_{xy}(0,0)$
$P(x=0, y=1) = P(y=1|x=0)P(x=0) = .1(.5) = .05 = p_{xy}(0,1)$
$P(x=1, y=0) = P(y=0|x=1)P(x=1) = .2(.5) = .1 = p_{xy}(1,0)$
$P(x=1, y=1) = P(y=1|x=1)P(x=1) = .8(.5) = .4 = p_{xy}(1,1)$

b) $p_x(x) = \sum_y p_{xy}(x,y) = \begin{cases} .5 & x=0 \\ .5 & x=1 \end{cases}$

$p_y(y) = \sum_x p_{xy}(x,y) = \begin{cases} .55 & y=0 \\ .45 & y=1 \end{cases}$

8) S-3.13 abc) Draw 3: 2R 3W 4B $X = \#$ RED $Y = \#$ WHITE

a) $R_{xy} = \{(0,0), (0,1), (0,2), (0,3), (1,0), (1,1), (1,2), (2,0), (2,1)\}$

b) $p_{xy}(i,j) = P(x=i, y=j)$ $i = 0,1,2$ $j = 0,1,2,3$

$(0,0) = \binom{4}{3} / \binom{9}{3} = \frac{4!\,3!\,6!}{3!\,9!} = \frac{4!}{7 \cdot 8 \cdot 9} = \frac{4}{84}$

$(0,1) = \binom{4}{2}\binom{3}{1} / \binom{9}{3} = \frac{18}{84}$

$(0,2) = \binom{3}{2}\binom{4}{1} / \binom{9}{3} = \frac{12}{84}$

$(0,3) = \binom{3}{3} / \binom{9}{3} = \frac{1}{84}$

$(1,0) = \binom{2}{1}\binom{4}{2} / \binom{9}{3} = \frac{12}{84}$

$(1,1) = \binom{2}{1}\binom{3}{1}\binom{4}{1} / \binom{9}{3} = \frac{24}{84}$

$(1,2) = \binom{2}{1}\binom{3}{2} / \binom{9}{3} = \frac{6}{84}$

$(2,0) = \binom{2}{2}\binom{4}{1} / \binom{9}{3} = \frac{4}{84}$

$(2,1) = \binom{2}{2}\binom{3}{1} / \binom{9}{3} = \frac{3}{84}$

i \ j	0	1	2	3
0	4/84	18/84	12/84	1/84
1	12/84	24/84	6/84	0
2	4/84	3/84	0	0

c) $p_x(i) = \begin{cases} 35/84 & i=0 \\ 42/84 & i=1 \\ 7/84 & i=2 \end{cases}$

$p_y(j) = \begin{cases} 20/84 & j=0 \\ 45/84 & j=1 \\ 18/84 & j=2 \\ 1/84 & j=3 \end{cases}$

9) S-2.35) $E[g(X,Y)] = \sum_x \sum_y g(x,y) p_{xy}(x,y)$ & $E[aX+bY] = aE[X] + bE[Y]$

$= \sum_y p_y(y) E[g(x,y) | Y=y] = \sum_y p_y(y) E[g(x,y) | Y=y] = \sum_y p_y(y) \sum_x g(x,y) p_{x|y}(x|y) = \sum_x \sum_y g(x,y) p_{xy}(x,y)$

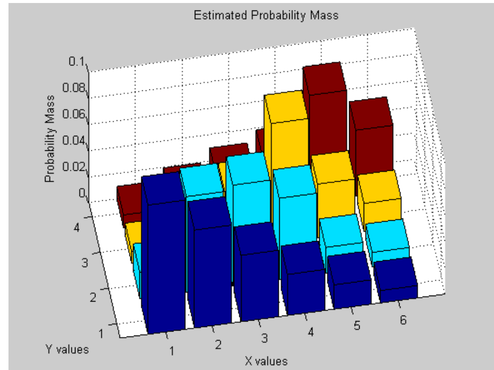
$E[aX+bY] = \sum_x \sum_y (ax+by) p_{x,y}(x,y) = a \sum_x x \sum_y p_{xy}(x,y) + b \sum_y y \sum_x p_{xy}(x,y)$

$= a \sum_x x p_x(x) + b \sum_y y p_y(y) = aE[X] + bE[Y]$

10.

a) Estimated joint PMF of burger/fry data

	1	2	3	4
1	0.0984	0.0201	0.0087	0.0095
2	0.0746	0.0729	0.0185	0.0189
3	0.0506	0.0769	0.0449	0.0293
4	0.0312	0.0621	0.0919	0.0383
5	0.0185	0.0199	0.041	0.0815
6	0.0094	0.0113	0.0215	0.0501



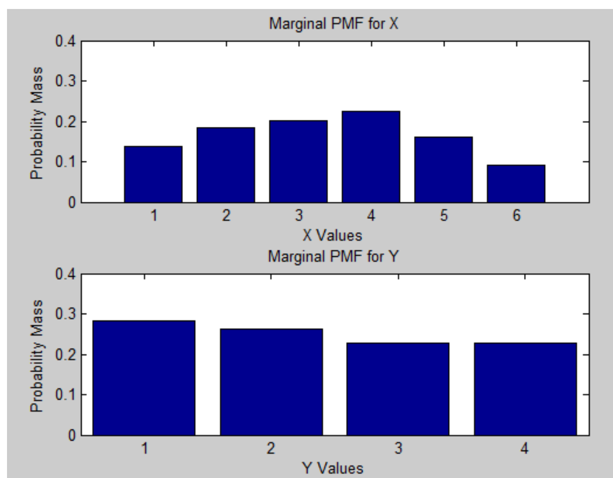
b) Probability that a normal customer will buy three burgers and two servings of fries: **0.0769 = 7.69%**

c) Marginal PMF for the number of burgers a normal customer will buy:

1	2	3	4	5	6
0.1367	0.1849	0.2017	0.2235	0.1609	0.0923

d) Marginal PMF for the number of servings of fries a customer will buy:

1	2	3	4
0.2827	0.2632	0.2265	0.2276



e) Expected number of burgers that a normal customer will buy: **3.3639 Burgers**

f) Expected number of servings of fries that a normal customer will buy: **2.3990 Fries**

g) If burgers cost $2.00 and fries cost $1.00, what is the expected amount of money that you will obtain from each normal customer:

2*E[X] + 1*E[Y] = 2*3.3639+2.3990 = **$9.13**

h) PMF of the number of burgers given 2 fries purchased:

Burgers	2 Fries
1	0.0201
2	0.0729
3	0.0769
4	0.0621
5	0.0199
6	0.0113