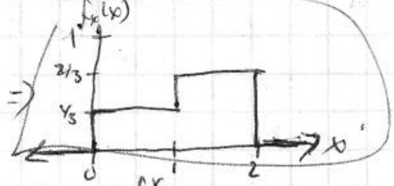
Ty Madsen | ECEN 370 | HW #6 | 2/23/15

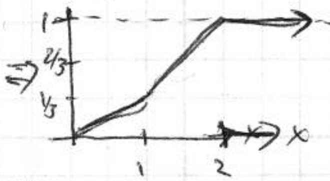
1.) S-2.21)

$$f_X(x) = \begin{cases} 1/3 & 0<x<1 \\ 2/3 & 1<x<2 \\ 0 & \text{OTHERWISE} \end{cases} \Rightarrow$$

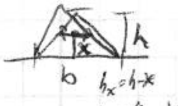


$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(x)\,dx = \int_0^x f_X(x)\,dx$

$$F_X(x) \Rightarrow \begin{cases} 0 & 0>x \\ 1/3\, x & 0<x<1 \\ 2/3\, x & 1<x<2 \\ 1 & 2>x \end{cases} \Rightarrow$$



2) B-3.5)



$A = bh/2$

$h_x = h - x$ $A_x = b(h-x)^2/2h$

$b_x = (h-x)b/h$

$F_X(x) = 1 - P(X > x) = 1 - \frac{A_x}{A} = 1 - \frac{b(h-x)^2/(2h)}{bh/2} = 1 - \left(\frac{h-x}{h}\right)^2$

CDF: $F_X(x) = \begin{cases} 0 & \text{for } x<0 \\ F \\ 1 & x>h \end{cases}$ CDF

PDF = $f_X(x) = \frac{dF_X(x)}{dx} = \begin{cases} \frac{2(h-x)}{h^2} & 0 \le x \le h \\ 0 & \text{OTHERWISE} \end{cases}$

3) B-3.6) $p = .5$ X: Time Y: # of People

$F(X \le x) = \begin{cases} 0 & x<0 \\ 1/2(2 - e^{-\lambda x}) & x \ge 0 \end{cases}$

$F_X(x) = P(X \le x) = 1/2\, \underbrace{P(X \le x \mid Y=0)}_{1} + 1/2\, \underbrace{P(X \le x \mid Y=1)}_{1 - e^{-\lambda x}}$

$1/2\,(2 - e^{-\lambda x})$

4) B-3.7)

X: DIST TO CENTER

$F_X(x) = P(X \le x) = \frac{\pi x^2}{\pi r^2} = \frac{x^2}{r^2}$ $0 \le x \le r$

$F_X(x) = \begin{cases} \left(\frac{x}{r}\right)^2 & x \in [0, r] \\ 0 & x<0 \\ 1 & x>r \end{cases}$

a)

$\frac{dF(x)}{dx} = \frac{2x}{r^2}$

$f_X(x) = \begin{cases} \frac{2x}{r^2} & 0 \le x \le r \\ 0 & \text{OTHERWISE} \end{cases}$

$E[X] = \int_0^r \frac{2x^2}{r^2}\,dx = \frac{2x^3}{3r^2}\Big|_0^r = \boxed{\frac{2r}{3}}$

$E(x^2) = \int_0^r \frac{2x^3}{r^2}\,dx = \frac{2x^4}{4r^2}\Big|_0^r \boxed{\frac{r^2}{2}}$

$VAR(X) = \frac{r^2}{2} - \frac{4r^2}{9} = \frac{r^2}{18} = \boxed{\frac{r^2}{18}}$ VAR(X)

b) $S = 1/x$ ~~$X \le t$~~

$S \ge 0$ OTHERWISE

$1/t < s$ $F_S(s) = P(S \le s) = P(X \le t)P(S \le s \mid X \le t) + P(X > t)P(S \le s \mid X > t)$

$P(X \le t) = \frac{t^2}{r^2}$ $P(X > t) = 1 - \frac{t^2}{r^2}$

$S \le 0 \because X > t$

$P(S \le s \mid X > t) = 1$

$P(S \le s \mid X \le t) = P(1/x \le s \mid X \le t) = \frac{P(1/s \le x \le t)}{P(x \le t)}$

$= \frac{\frac{\pi t^2 - \pi(1/s)^2}{\pi r^2}}{\frac{\pi t^2}{\pi r^2}} = 1 - 1/t^2 s^2$

$\Rightarrow 1 - \frac{1}{s^2 r^2}$

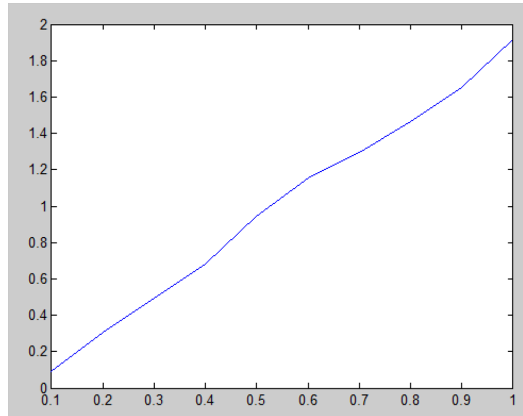
$F_S(s) = \begin{cases} 1 - P(X \le t) = 1 - \frac{t^2}{r^2} & 0 < s < 1/t \\ 1 - \frac{1}{s^2 r^2} & 1/t \le s \\ 0 & s<0 \end{cases}$

Problem 4)

```
clear;
N=10000;
hits = 0;
r = 1;
X = zeros(N,1);
pdf = zeros(10,1);
x_range = linspace(.1,1,10);

while hits < N
   x = rand()*2 - 1;
   y = rand()*2 - 1;
   dist = sqrt(x^2 + y^2);
   if(dist > r)  %ignore
   else  %add
      hits = hits+1;
      X(hits,1) = dist;
      idx = ceil(dist*10);
      pdf(idx) = pdf(idx)+1;
   end;
end;
PDF = pdf./1000;% = 2x/r^2 = (2x)
plot(x_range,PDF);
mean = sum(X)/N;% = 2r/3 = (2/3)
moment_2 = sum(X.^2/N);%E[X^2] = r^2/2 = (1/2)
var = moment_2 - mean^2;% = r^2/18 = (1/18)
```

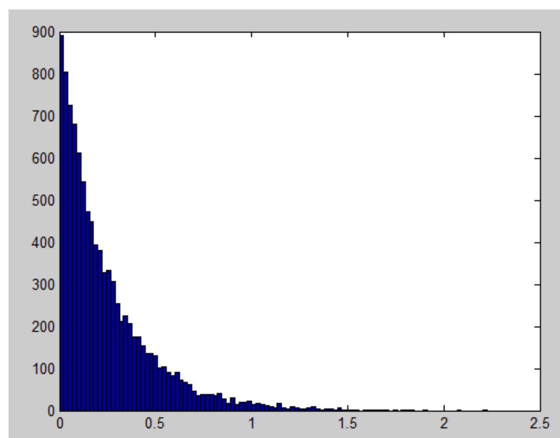
PDF:



PDF	~2x
Mean	.6682
Var	.0555

Problem 6)

```
clear;
N = 10000;
lambda = 4;%arbitrary
X = zeros(1,N);
for k=1:N
    u = rand();
    X(k) = -log(1-u)/lambda;
end;
hist(X, 100);
```



5) B-3.9) X = WAITING TIME A = FIND TAXI OR BUS AFTER 5 MIN

Y = FIND Z = TAXI AFTER < 5 MIN

$P(\text{TAXI TIME} > 5\text{ MIN}) = 1/2$

$P(A) = \frac{2}{3} + \frac{1}{3}\cdot\frac{1}{2} = \frac{5}{6}$

$p_Y(y) = \begin{cases} \frac{2}{3P(A)} & \text{if } y=0 \\ \frac{1}{6P(A)} & \text{if } y=5 \end{cases} = \begin{cases} \frac{12}{15} & y=0 \\ \frac{3}{15} & y=5 \end{cases}$

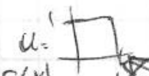
$P_Y(0) = \frac{P(Y=y \cap A)}{P(A)} = \frac{2}{3P(A)} = \frac{12}{15}$

$P_Y(5) = 3/15$

$f_Z(z) = \begin{cases} 1/5 & 0 \le z \le 5 \\ 0 & \text{ELSE} \end{cases}$ $F_X(x) = P(A)F_Y(x) + (1-P(A))F_Z(x)$

$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{5}{6}\cdot\frac{12}{15} + \frac{1}{6}\cdot\frac{x}{5} & 0 \le x \le 5 \\ 1 & 5 \le x \end{cases}$

$E[X] = P(A)E[Y] + (1-P(A))E[Z] = \frac{5}{6}\cdot\frac{3}{15}\cdot 5 + \frac{1}{6}\cdot\frac{5}{2} = \frac{15}{12}$

6) B-3.10) $U \in [0,1]$ u:  $F_X(x) = u$ $S = \{x \mid 0 < F_X(x) < 1\}$

a) $X \le x$ iff $F(X) \le F(x)$

$P(X \le x) = P(F(X) \le F(x)) = P(U \le F(x)) = F(x)$ SINCE U IS UNIFORM

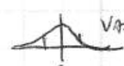
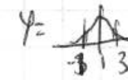
b) $F(x) = 1 - e^{-\lambda x}$ $x \ge 0$ $U \in (0,1)$ $1 - e^{-\lambda x} = u \Rightarrow x = \frac{\ln(1-u)}{-\lambda} = \boxed{-\ln(1-u)/\lambda}$

c) FOR EACH $u \in (0,1)$ THERE IS x_u s.t. $F(x_u - 1) < u < F(x_u)$

FOR ALL INTEGERS k:

$P(X=k) = P(F(k-1) < U < F(k)) = F(k) - F(k-1)$

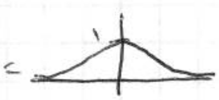
MATLAB: SEE MATLAB PROBLEM #6 ON ~~NEXT~~ PREV PAGE

7) B-3.11) X =  VAR=1 Y =  VAR=4 $\sigma = 2$

a) FIND $P(X \le 1.5)$ & $P(X \le -1)$ $\Phi(0) - (1 - \Phi(1)) = .5 - .1587$

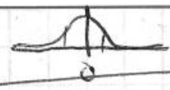
$P(X \le 1.5) = \sigma = 1 \Rightarrow \Phi(1.5) = .9332$

$P(X \le -1) = 1 - \Phi(1) = 1 - .8413 = .1587$

b) PDF of $(Y-1)/2$ = STANDARD NORMAL = 

c) $P(-1 \le Y \le 1) = \Phi(0) - [1 - \Phi(1)] = .5 - .1587$

$= .3413$

8) B-3.12)  $\sigma = \sigma$ $\quad k = 1,2,3$

$$\boxed{\{X \geq k\sigma\} = \begin{cases} .1587 & k=1 \\ .0228 & k=2 \\ .0013 & k=3 \end{cases} \qquad \{|X| \leq k\sigma\} = \begin{cases} .6826 & k=1 \\ .9544 & k=2 \\ .9974 & k=3 \end{cases}}$$

$k=1: (X \geq \sigma) = 1 - \Phi(1) = 1 - .8413 = .1587$

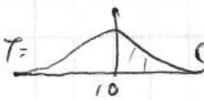
$\Rightarrow 1 - \Phi(k) = 1 - .9772 = .0228$

$1 - .9987 = .0013$

$(\Phi(k) - .5)2 = (.8413 - .5) \times 2 = .6826$

$= \Phi(k)2 - 1 \qquad (.9772)2 - 1 = 1.9544 - 1 = .9544$

$(.9987)2 - 1 = 1.9974 - 1 = .9974$

9) B-3.13) $T =$  $\sigma = 10$

$P(T \leq 59F) = \frac{(59-32)5}{9} = \frac{270}{9} \quad \frac{135}{9} = 15$

$P(T \leq 15C) = \Phi(.5) = \boxed{.6195}$

10) B-3.15)

a) $a = \pi r^2/2 \quad f_{(X,Y)}(x,y) = 2/\pi r^2$ IN SC $\quad f_{(X,Y)}(x,y) = 0$ else

$$\boxed{f_{(X,Y)}(x,y) = \begin{cases} 2/\pi r^2 & x,y \text{ IN SEMI CIRCLE} \\ 0 & \text{OTHERWISE} \end{cases}}$$

b) $[-\sqrt{r^2-y^2}, \sqrt{r^2-y^2}]$

$f_Y(y) = \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{2}{\pi r^2} dx = \begin{cases} \frac{4\sqrt{r^2-y^2}}{\pi r^2} & \text{if } 0 \leq y \leq r \\ 0 & \text{ELSE} \end{cases} \qquad \frac{2\sqrt{r^2-y^2}}{\pi r^2} + \frac{2\sqrt{r^2-y^2}}{\pi r^2} = \frac{4\sqrt{r^2-y^2}}{\pi r^2}$

$E[Y] = \frac{4}{\pi r^2} \int_0^r y\sqrt{r^2-y^2}\, dy = \boxed{\frac{4r}{3\pi}} \qquad z = r^2 - y^2$

c) D = SEMICIRCLE

$E[Y] = \iint_{(x,y \in D)} y f_{X,Y}(x,y)\, dx\, dy = \int_0^\pi \int_0^r \frac{2}{\pi r^2} s(\sin\theta) s\, ds\, d\theta = \int_0^\pi \frac{2r^3 \sin\theta}{3\pi r^2} d\theta = \frac{2r}{3\pi}(-\cos\theta)\Big|_0^\pi$

$\frac{2r}{3\pi}(1 - -1) = 2$

$= \boxed{\frac{4r}{3\pi}}$