# PARTITION FUNCTION. (semi-canonical).

$$\Xi = \sum_{n_s=0}^{\infty} \sum_{n_\pm=0}^{\infty} \frac{e^{\mu_s n_s}\, e^{\mu_\pm n_\pm}}{v^{n_s} n_s!\; v^{n_\pm} n_\pm!} \cdot \frac{1}{v^{n_p(N+1)} n_p!} \cdot$$

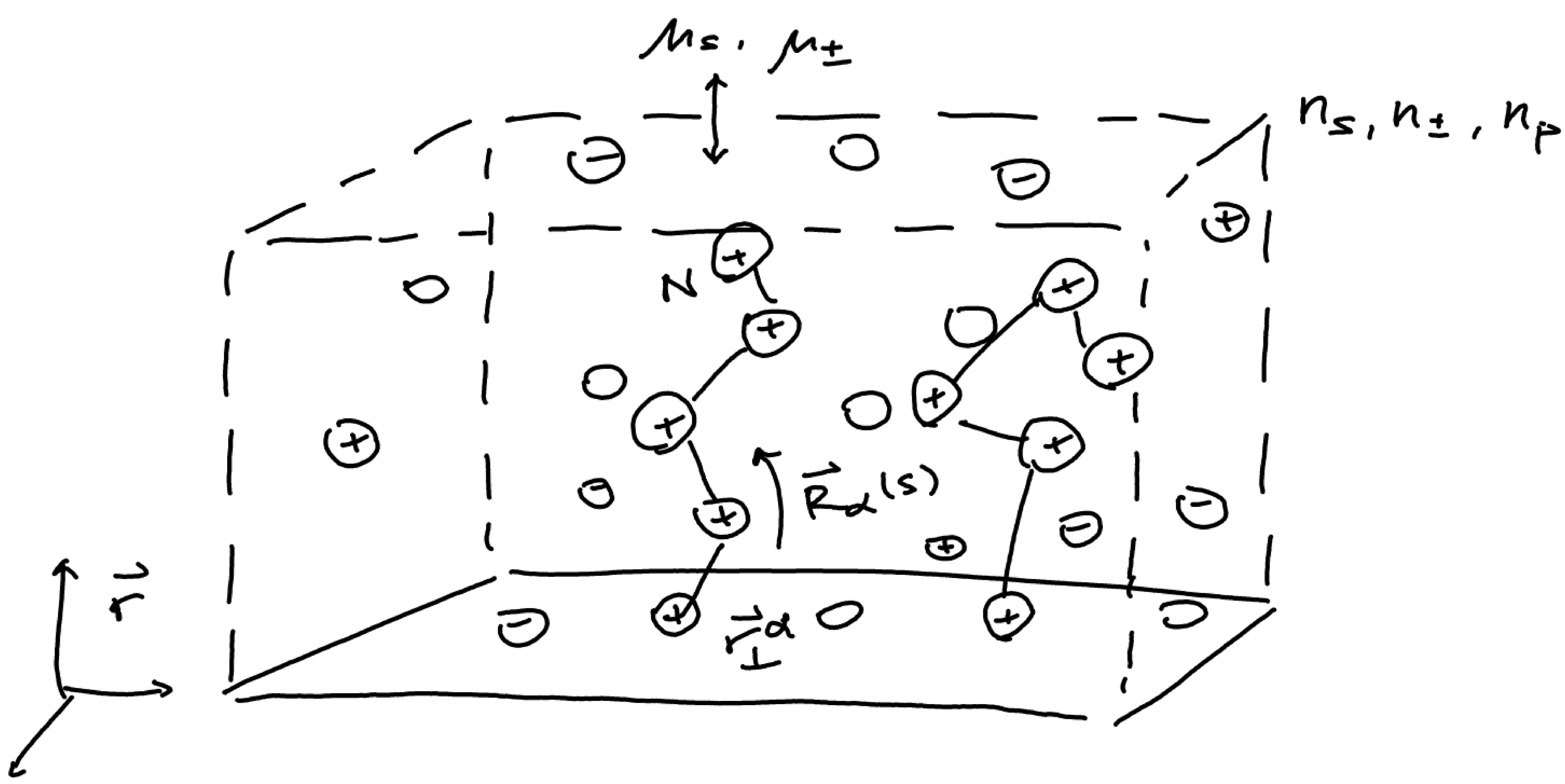
$$\prod_{\alpha=1}^{n_p} \prod_{s=0}^{N} \int d\vec{R}(s) \prod_{\beta=1}^{n_s} \int d\vec{r}_\beta \prod_{\gamma_\kappa=1}^{n_\pm} \int d\vec{r}_{\gamma_\kappa} \cdot$$

$$\exp\{-\beta \mathcal{H}\}\; \delta[\hat{\varphi}_p + \hat{\varphi}_s - 1],$$

where
$$\beta\mathcal{H} := \sum_{s=0}^{N-1} \frac{3}{2b^2}\left[\vec{R}(s+1) - \vec{R}(s)\right]^2$$
$$+ \frac{\chi}{v} \int d\vec{r}\; \hat{\varphi}_p \hat{\varphi}_s$$
$$+ \frac{1}{2} \int d\vec{r}\, d\vec{r}\,'\; \hat{\rho}_c(\vec{r})\, C(\vec{r},\vec{r}\,')\, \hat{\rho}_c(\vec{r}\,')$$

and
$$\hat{\varphi}_p(\vec{r}) := \sum_{\alpha=1}^{n_p} v_p \Big( \sum_{s=1}^{N} \delta(\vec{R}_\alpha(s) - \vec{r}) + \delta(\vec{R}_\alpha(0) - \vec{r}_\perp^{\,\alpha}) \Big)$$
$$\hat{\varphi}_s(\vec{r}) := \sum_{\beta=1}^{n_s} v_s\, \delta(\vec{r}_\beta - \vec{r})$$
$$\hat{\rho}_c(\vec{r}) := \sum_{\gamma_\kappa=1}^{n_\pm} \pm z_\pm\, \delta(\vec{r}_{\gamma_\kappa} - \vec{r}) + \frac{\alpha}{v_p} \hat{\varphi}_p$$

$$-\nabla \cdot [\varepsilon(\vec{r}) \nabla C(\vec{r},\vec{r}\,')] \equiv \delta(\vec{r} - \vec{r}\,').$$



PARTICLE-TO-FIELD TRANSFORMATION.

$$\cdot \quad 1 \equiv \int D\Phi \; \delta\left[\Phi(\vec{r}) - \hat{\Phi}(\vec{r})\right] \quad \Xi$$

$$\delta\left[\Phi(\vec{r}) - \hat{\Phi}(\vec{r})\right] \equiv \int D\Psi \exp\left[\frac{1}{v}\int d\vec{r}\; i\Psi\left(\Phi(\vec{r}) - \hat{\Phi}(\vec{r})\right)\right]$$

$$\cdot \quad \Xi' = \Xi \cdot \int D\varphi_p \; \delta\left[\varphi_p(\vec{r}) - \hat{\varphi}_p(\vec{r})\right]$$

$$\cdot \int D\varphi_s \; \delta\left[\varphi_s(\vec{r}) - \hat{\varphi}_s(\vec{r})\right].$$

$$= \int D\varphi_p D\varphi_s \, \Xi \cdot \exp\left[\frac{1}{v}\int d\vec{r}\; i w_p\left(\hat{\varphi}_p(\vec{r}) - \varphi_p(\vec{r})\right)\right]$$

$$\cdot \exp\left[\frac{1}{v}\int d\vec{r}\; i w_s\left(\hat{\varphi}_s(\vec{r}) - \varphi_s(\vec{r})\right)\right]$$

$$\downarrow \quad \hat{\Phi}(r)\; \delta\left[\hat{\Phi}(\vec{r}) - \Phi(\vec{r})\right] = \Phi(\vec{r})\; \delta\left[\hat{\Phi}(\vec{r}) - \Phi(\vec{r})\right]$$

$$= \int D\varphi_p Dw_p D\varphi_s Dw_s \sum_{n_s, n_\pm = 0}^{\infty} \frac{e^{\mu_s n_s} e^{\mu_\pm n_\pm}}{v_s^{n_s} n_s! \, v_\pm^{n_\pm} n_\pm!} \frac{1}{v_p^{n_p(N+1)} n_p!}$$

$$\prod_{\beta=1}^{n_s} \int d\vec{r}_\beta \prod_{\gamma, k=1}^{n_\pm} \int d\vec{r}_\pm \prod_{\alpha=1}^{n_p} \prod_{s=0}^{N} \int d\vec{R}_\alpha \exp\left\{-\left[\sum_{s=0}^{N-1} \frac{3}{2b^2}\left(R_\alpha(s+1) - R_\alpha(s)\right)^2\right.\right.$$

$$+ \frac{\chi}{v}\int dr\, \varphi_p \varphi_s + \frac{1}{2}\int dr\, dr'\, \hat{\rho}_c(\vec{r})\, C(\vec{r}, \vec{r}')\, \hat{\rho}_c(\vec{r}')$$

$$\left.\left. - \frac{1}{v}\int d\vec{r}\;\; i w_p\left(\hat{\varphi}_p(\vec{r}) - \varphi_p(\vec{r})\right) - \frac{1}{v}\int d\vec{r}\;\; i w_s\left(\hat{\varphi}_s(\vec{r}) - \varphi_s(\vec{r})\right)\right]\right\}$$

$$\cdot \; \delta\left[\varphi_p(\vec{r}) + \varphi_s(\vec{r}) - 1\right]$$

$$= \int D\varphi_p Dw_p D\varphi_s Dw_s \; \sum (\cdots) \prod_{\Gamma} \int D\Gamma \; (\cdots).$$

$$\cdot \exp\left\{-\left[\sum_{s=0}^{N-1} \frac{3}{2b^2}\left(\vec{R}_\alpha(s+1) - \vec{R}_\alpha(s)\right)^2 + \frac{\chi}{v}\int d\vec{r}\; \varphi_p \varphi_s\right.\right.$$

$$+ \frac{1}{2}\int d\vec{r}\, d\vec{r}'\, \hat{\rho}_c(\vec{r})\, C(\vec{r}, \vec{r}')\, \hat{\rho}_c(\vec{r}')$$

$$\left.\left. - \frac{1}{v}\int d\vec{r}\; \left(i w_p\left(\hat{\varphi}_p(\vec{r}) - \varphi_p(\vec{r})\right) + i w_s\left(\hat{\varphi}_s(\vec{r}) - \varphi_s(\vec{r})\right)\right)\right]\right\}$$

$$\cdot \int D\eta \exp\left[\frac{1}{v}\int d\vec{r}\;\; i\eta\left(\varphi_p(\vec{r}) + \varphi_s(\vec{r}) - 1\right)\right].$$

$$\Xi' = \int D\varphi_p DW_p D\varphi_s DW_s D\eta$$

$$\cdot \sum_{n_\pm=0}^{\infty} \frac{e^{\mu_\pm n_\pm}}{v^{n_\pm} n_\pm!} \prod_{\gamma_k=1}^{n_\pm} \int d\vec{r}_{\gamma_k} \exp\left\{-\frac{1}{2}\int d\vec{r} d\vec{r}' \hat{\rho}_c(\vec{r}) C(\vec{r},\vec{r}') \hat{\rho}_c(\vec{r}')\right\}$$

$$\cdot \sum_{n_s=0}^{\infty} \frac{e^{\mu_s n_s}}{v_s^{n_s} n_s!} \prod_{\beta=1}^{n_s} \int d\vec{r}_\beta \exp\left\{-\frac{1}{v}\int d\vec{r}\; iW_s \hat{\varphi}_s(\vec{r})\right\}$$

$$\cdot \frac{1}{v_p^{n_p(N+1)} n_p!} \prod_{\alpha=1}^{n_p} \prod_{s=0}^{N} \int d\vec{R}_\alpha \exp\left\{-\frac{1}{v}\int d\vec{r}\; iW_p \hat{\varphi}_p(\vec{r}) - \sum_{s=0}^{N-1} \frac{3}{2b^2}\left(\vec{R}_\alpha(s+1) - \vec{R}_\alpha(s)\right)^2\right\}$$

$$\cdot \exp\left\{-\left[\frac{1}{v}\int d\vec{r}\left(\chi \varphi_p(\vec{r})\varphi_s(\vec{r}) - iW_p\varphi_p(\vec{r}) - iW_s\varphi_s(\vec{r}) - i\eta\left(\varphi_p(\vec{r}) + \varphi_s(\vec{r}) - 1\right)\right)\right]\right\}$$

## HUBBARD – STRATONOVICH TRANSFORMATION.

$$\exp\left(-\frac{1}{2}\int d\vec{r} d\vec{r}'\; \underline{X}(\vec{r}) A(\vec{r},\vec{r}') \underline{X}(\vec{r}')\right)$$

$$= C^{-1}\int D\underline{Y} \exp\left[-\frac{1}{2}\int d\vec{r} d\vec{r}'\; \underline{Y}(\vec{r}) A^{-1}(\vec{r},\vec{r}') \underline{Y}(\vec{r}') - i\int d\vec{r}\; \underline{X}(\vec{r}) \underline{Y}(\vec{r})\right]$$

, where $C := (2\pi)^{N/2} \det(A)^{1/2} = \text{const.}$
and $N$ is dim. of functional integral.

$$\exp\left(-\frac{1}{2}\int d\vec{r} d\vec{r}'\; \hat{\rho}_c(\vec{r}) C(\vec{r},\vec{r}') \hat{\rho}_c(\vec{r}')\right)$$

$$= C_\chi^{-1}\int D\psi \exp\left[-\frac{1}{2}\int d\vec{r} d\vec{r}'\; \psi(\vec{r})\, C^{-1}(\vec{r},\vec{r}')\, \psi(\vec{r}') - i\int d\vec{r}\; \psi(\vec{r}) \hat{\rho}_c(\vec{r})\right]$$

, where $C_\chi = (2\pi)^{N/2} \det(C)^{1/2}$
and $C^{-1}(\vec{r},\vec{r}') = -\nabla\cdot[\varepsilon(\vec{r})\nabla]\,\delta(\vec{r}-\vec{r}')$.

$$= C_\chi^{-1}\int D\psi \exp\left[-\frac{1}{2}\int d\vec{r} d\vec{r}'\; \psi(\vec{r})\left(-\nabla\cdot(\varepsilon(\vec{r})\nabla)\,\delta(\vec{r}-\vec{r}')\right)\psi(\vec{r}') - i\int d\vec{r}\; \psi(\vec{r})\left(\pm z_\pm \sum_{\gamma_k=1}^{n_\pm} \delta(\vec{r}_{\gamma_k} - \vec{r}) + \frac{\alpha}{v_p}\varphi_p(\vec{r})\right)\right].$$

$$= C_\chi^{-1}\int D\psi \exp\left[\int d\vec{r}\; \frac{1}{2}\psi\nabla\cdot(\varepsilon(\vec{r})\nabla\psi) - \frac{\alpha}{v_p}\varphi_p(\vec{r})\, i\psi \mp \int d\vec{r}\; z_\pm \sum_{\gamma_k=1}^{n_\pm} \delta(\vec{r}_{\gamma_k} - \vec{r})\, i\psi.\right]$$

$$\Xi' = \int D\varphi_p \, Dw_p \, D\varphi_s \, Dw_s \, D\eta \, D\psi$$

$$\cdot \sum_{n_\pm=0}^{\infty} \frac{e^{\mu_\pm n_\pm}}{v^{n_\pm} n_\pm!} \prod_{\kappa=1}^{n_\pm} \int d\vec{r}_\kappa^\gamma \cdot C_\psi^{-1} \exp\Big\{ \int d\vec{r} \, \frac{1}{2} \psi \nabla \cdot \varepsilon(\vec{r}) \psi$$

$$- \frac{\alpha}{v_p} \varphi_p(\vec{r}) \, i\psi \mp \int d\vec{r} \, z_\pm \sum_{\kappa=1}^{n_\pm} \delta(\vec{r}_\kappa^\gamma - \vec{r}) \, i\psi \Big\}$$

$$\cdot \sum_{n_s=0}^{\infty} \frac{e^{\mu_s n_s}}{v^{n_s} n_s!} \prod_{\beta=1}^{n_s} \int d\vec{r}_\beta \exp\Big\{ -\frac{1}{v} \int d\vec{r} \, i w_s v_s \sum_{\beta=1}^{n_s} \delta(\vec{r}_\beta - \vec{r}) \Big\}$$

$$\cdot \frac{1}{v_p^{n_p(N+1)} n_p!} \prod_{\alpha=1}^{n_p} \prod_{s=0}^{N} \int d\vec{R}_\alpha(s) \exp\Big\{ -\frac{1}{v} \int d\vec{r} \, i w_p \hat{\varphi}_p(\vec{r})$$

$$- \sum_{s=0}^{N-1} \frac{3}{2b^2} \left( \vec{R}_\alpha(s+1) - \vec{R}_\alpha(s) \right)^2 \Big\}$$

$$\cdot \exp\Big\{ - \Big[ \frac{1}{v} \int d\vec{r} \, \chi \varphi_p(\vec{r}) \varphi_s(\vec{r}) - i w_p \varphi_p(\vec{r}) - i w_s \varphi_s(\vec{r})$$

$$- i\eta \left( \varphi_p(\vec{r}) + \varphi_s(\vec{r}) - 1 \right) \Big] \Big\} .$$

$$\underline{\sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x}$$

Redundant

$$\cdot \sum_{n_\pm=0}^{\infty} \frac{1}{n_\pm!} \prod_{\kappa=1}^{n_\pm} \Big\{ \frac{e^{\mu_\pm}}{v_\pm} \int d\vec{r}_\kappa^\gamma \cancel{\prod_{\kappa=1}^{n_\pm}} \exp\left( \mp \int d\vec{r} \, z_\pm \delta(\vec{r}_\kappa^\gamma - \vec{r}) i\psi \right) \Big\}$$

$$= \sum_{n_\pm=0}^{\infty} \frac{1}{n_\pm!} \prod_{\kappa=1}^{n_\pm} \Big\{ \frac{e^{\mu_\pm}}{v_\pm} \int d\vec{r}_\kappa^\gamma \exp\left( \mp z_\pm \, i\psi(\vec{r}_\kappa^\gamma) \right) \Big\}$$

$$= \exp\Big\{ \frac{e^{\mu_\pm}}{v_\pm} \int d\vec{r}_\kappa^\gamma \exp\left( \mp z_\pm \, i\psi(\vec{r}_\kappa^\gamma) \right) \Big\}$$

Redundant

$$\cdot \sum_{n_s=0}^{\infty} \frac{1}{n_s!} \prod_{\beta=1}^{n_s} \Big\{ \frac{e^{\mu_s}}{v_s} \int d\vec{r}_\beta \cancel{\prod_{\beta=1}^{n_s}} \exp\Big\{ -\frac{1}{v} \int d\vec{r} \, v_s \delta(\vec{r}_\beta - \vec{r}) i w_s \Big\}$$

$$\downarrow \; v = v_s = v_p \quad (\text{Assume})$$

$$= \exp\Big\{ \frac{e^{\mu_s}}{v} \int d\vec{r}_\beta \exp\left( -i w_s(\vec{r}_\beta) \right) \Big\}$$

$$\Xi' = C_\chi^{-1} \int D\varphi_p\, Dw_p\, D\varphi_s\, Dw_s\, D\eta\, D\psi$$

$$\cdot \exp\Big\{ -\Big[ -\frac{e^{\mu_s}}{v} \overbrace{\underline{\int d\vec{r}\, \exp(-iw_s)}}^{Q_s} + \frac{1}{v}\int d\vec{r}\, \Big( \chi \varphi_p(\vec{r})\varphi_s(\vec{r})$$

$$- i w_p \varphi_p - i w_s \varphi_s - i\eta\big(\varphi_p(\vec{r}) + \varphi_s(\vec{r}) - 1\big) \Big)$$

$$+ \int d\vec{r} \Big( -\frac{1}{2}\psi \nabla\cdot\varepsilon\nabla\psi + \frac{\alpha}{v}\varphi_p(\vec{r})\, i\psi - \underbrace{\frac{e^{\mu_\pm}}{v_\pm}\exp(\mp z_\pm i\psi)}_{\rho_\pm(\vec{r})} \Big) \Big] \Big\}$$

$$\cdot \frac{1}{n_p!} \prod_{\alpha=1}^{n_p} \prod_{s=0}^{N} \underbrace{\frac{1}{v}\int dR_\alpha\, \exp\Big[ -\frac{3}{2b^2}\sum_{s=0}^{N-1}\big(\vec{R}_\alpha(s+1) - \vec{R}_\alpha(s)\big)^2 - \frac{1}{v}\int d\vec{r}\, i w_p \hat{\varphi}_p(\vec{r}) \Big]}_{Q_p^\alpha}$$

$$= \frac{C_\psi^{-1}}{n_p!} \int D\varphi_p\, Dw_p\, D\varphi_s\, Dw_s\, D\eta\, D\psi$$

$$\cdot \exp\Big\{ -\Big[ -\sum_{\alpha=1}^{n_p} \ln Q_p^\alpha - e^{\mu_s} Q_s + \frac{1}{v}\int d\vec{r}\, \Big( \chi \varphi_p(\vec{r})\varphi_s(\vec{r})$$

$$- i w_p \varphi_p(\vec{r}) - i w_s \varphi_s(\vec{r}) - i\eta\big(\varphi_p(\vec{r}) + \varphi_s(\vec{r}) - 1\big) \Big)$$

$$+ \int d\vec{r} \Big( -\frac{1}{2}\psi\nabla\cdot\varepsilon\nabla\psi + \frac{\alpha}{v}\varphi_p(\vec{r})\, i\psi - \rho_\pm(\vec{r}) \Big) \Big] \Big\}$$

<u>SADDLE POINT APPROXIMATION</u>

- $\Xi^* = \Xi(\Psi^*)$ & $\Psi^* = \cancel{Re(x)}^{\,0} + i\,Im(-y)$ (Purely imaginary)

  $\hookrightarrow\ i\Psi^* \rightarrow \Psi^*$

$$\Xi^* = \frac{C_\psi^{-1}}{n_p!} \exp\Big\{ -\Big[ -\sum_{\alpha=1}^{n_p} \ln Q_p^\alpha - e^{\mu_s} Q_s + \frac{1}{v}\int d\vec{r}\, \Big( \chi\varphi_p\varphi_s$$

$$- w_p\varphi_p - w_s\varphi_s - \eta(\varphi_p + \varphi_s - 1) \Big) + \int d\vec{r}\, \Big( -\frac{1}{2}\psi\nabla\cdot\varepsilon\nabla\psi$$

$$+ \frac{\alpha}{v}\varphi_p\psi - \rho_+ - \rho_- \Big) \Big] \Big\}$$

$F = -\ln \Xi^*$

$$F = -\sum_{\alpha=1}^{n_p} \ln Q_p^\alpha - e^{\mu_s} Q_s + \frac{1}{v}\int d\vec{r}\,\big(\chi \varphi_p \varphi_s - w_p \varphi_p - w_s \varphi_s$$
$$- \eta(\varphi_p + \varphi_s - 1)\big) + \int d\vec{r}\,\Big(-\frac{\psi}{2}\nabla\cdot\varepsilon\nabla\psi + \frac{\alpha}{v}\varphi_p \psi - \rho_+ - \rho_-\Big)$$

- Integration by-parts: $-\frac{1}{2}\int d\vec{r}\, \psi\nabla\cdot\varepsilon\nabla\psi = -\frac{1}{2}(\psi\cdot\varepsilon\nabla\psi)_{dA} + \frac{1}{2}\int d\vec{r}\, \varepsilon\nabla\psi\cdot\nabla\psi$

$\downarrow \nabla\psi|_{dA} = \emptyset$

$= \frac{1}{2}\int d\vec{r}\, \varepsilon|\nabla\psi|^2$

$\frac{DF}{D\Psi} = \emptyset.$

$\varphi_p$ : $\emptyset = \frac{1}{v}\big(\chi\varphi_s(\vec{r}) - w_p(\vec{r}) - \eta(\vec{r})\big) - \frac{1}{2}\frac{d\varepsilon}{d\varphi_p}|\nabla\psi|^2 + \frac{\alpha}{v}\psi(\vec{r})$

$\hookrightarrow \boxed{w_p(\vec{r}) = \chi\varphi_s(\vec{r}) - \eta(\vec{r}) - \frac{v}{2}\frac{d\varepsilon}{d\varphi_p}|\nabla\psi|^2 + \alpha\psi(\vec{r}).}$

$\varphi_s$ : $\emptyset = \frac{1}{v}\big(\chi\varphi_p(\vec{r}) - w_s(\vec{r}) - \eta(\vec{r})\big)$

$\hookrightarrow \boxed{w_s(\vec{r}) = \chi\varphi_p(\vec{r}) - \eta(\vec{r}).}$

$\eta$ : $\boxed{\emptyset = \varphi_p(\vec{r}) + \varphi_s(\vec{r}) - 1.}$

$w_s$ : $\emptyset = -e^{\mu_s}\frac{\delta Q_s}{\delta w_s} - \frac{1}{v}\varphi_s = -e^{\mu_s}\big(-\frac{1}{v}\exp(-w_s)\big) - \frac{\varphi_s}{v}$

$\hookrightarrow \boxed{\varphi_s(\vec{r}) = e^{\mu_s}\exp(-w_s(\vec{r}))}$

$\psi$ : $\emptyset = \nabla\cdot\varepsilon(\vec{r})\nabla\psi(\vec{r}) + \frac{\alpha}{v}\varphi_p(\vec{r}) + z_+\frac{e^{\mu_+}}{v_+}e^{-z_+\psi(\vec{r})} - z_-\frac{e^{\mu_-}}{v_-}e^{z_-\psi(\vec{r})}$

$\hookrightarrow \boxed{-\nabla\cdot\varepsilon(\vec{r})\nabla\psi(\vec{r}) = \frac{\alpha}{v}\varphi_p(\vec{r}) + z_+\lambda_+ e^{-z_+\psi(\vec{r})} - z_-\lambda_- e^{z_-\psi(\vec{r})}}$

$w_p$ : $\emptyset = -\sum_{\alpha=1}^{n_p}\frac{1}{Q_p^\alpha}\frac{\delta Q_p^\alpha}{\delta w_p} - \frac{1}{v}\varphi_p(\vec{r})$

$\hookrightarrow \varphi_p(\vec{r}) = -\sum_{\alpha=1}^{n_p}\frac{v}{Q_p^\alpha}\frac{\delta Q_p^\alpha}{\delta w_p}$

## $Q_p^\alpha$

$$Q_p^\alpha = \frac{1}{v^{N+1}} \prod_{s=0}^{N} \int d\vec{R}_\alpha(s) \exp\Big[-\frac{3}{2b^2}\sum_{s=0}^{N-1}(\vec{R}_\alpha(s+1)-\vec{R}_\alpha(s))^2 - \frac{1}{v}\int d\vec{r}\, w_p(\vec{r})\,\hat{\varphi}_p(\vec{r})\Big].$$

$$= \frac{1}{v^{N+1}} \prod_{s=0}^{N} \int d\vec{R}_\alpha(s) \exp\Big[-\frac{3}{2b^2}\sum_{s=0}^{N-1}(\vec{R}_\alpha(s+1)-\vec{R}_\alpha(s))^2 - \frac{1}{v}\int d\vec{r}\, w_p(\vec{r})\, v\Big(\sum_{s=1}^{N}\delta(\vec{R}_\alpha(s)-\vec{r}) + \delta(\vec{R}_\alpha(0)-\vec{r}_\perp^\alpha)\Big)\Big]$$

$$= \frac{1}{v^{N+1}} \int D\vec{R}_\alpha(1\cdots N) \exp\Big[-\frac{3}{2b^2}\sum_{s=0}^{N-1}(\vec{R}_\alpha(s+1)-\vec{R}_\alpha(s))^2\Big]$$

$$\cdot \prod_{s=1}^{N}\int d\vec{R}_\alpha(s) \exp[-w_p(\vec{R}_\alpha(s))]$$

$$\cdot \int d\vec{R}_\alpha(0) \exp[-w_p(\vec{r}_\perp^\alpha)] \exp\Big[-\frac{3}{2b^2}(\vec{R}_\alpha(1)-\vec{r}_\perp^\alpha)^2\Big]$$

$\downarrow$ Define an $s'$ at $\vec{r}'$

$$= \frac{1}{v^{s'}} \int D\vec{R}_\alpha(1\cdots s')\Big\{ e^{-w_p(\vec{R}_\alpha(s'))} \prod_{s=1}^{s'-1} \exp\Big[-\frac{3}{2b^2}(\vec{R}_\alpha(s+1)-\vec{R}_\alpha(s))^2 - w_p(\vec{R}_\alpha(s))\Big].$$

$$\cdot \frac{1}{v}\int d\vec{r}_\perp^\alpha \exp\Big[-\frac{3}{2b^2}(\vec{R}_\alpha(1)-\vec{r}_\perp^\alpha)^2 - w_p(\vec{r}_\perp^\alpha)\Big]\Big\}$$

$$\cdot \frac{1}{v^{N-s'}}\int D\vec{R}_\alpha(s'+1\cdots N)\Big\{e^{-w_p(\vec{R}_\alpha(N))}\prod_{s=s'}^{N-1}\exp\Big[-\frac{3}{2b^2}(\vec{R}_\alpha(s+1)-\vec{R}_\alpha(s))^2 - w_p(\vec{R}_\alpha(s))\Big]\Big\}$$

$$\cdot \int d\vec{r}'\,\delta(\vec{r}'-\vec{R}_\alpha(s')) \cdot \underbrace{e^{w_p(\vec{R}_\alpha(s'))}}_{\text{Balance extra}}$$

$\downarrow$ Shorthand : $E(s+1,s) := \Phi(s+1,s)\cdot\exp[-w_p(\vec{R}_\alpha(s))]$

, where $\Phi(s+1,s) := \exp\Big[-\frac{3}{2b^2}(\vec{R}_\alpha(s+1)-\vec{R}_\alpha(s))^2\Big]$

$$= \frac{1}{v^{s'+1}}\int d\vec{r}_\perp^\alpha \int D\vec{R}_\alpha(1\cdots s')\Big\{e^{-w_p(\vec{R}_\alpha(s'))} E(1,0)\cdot\prod_{s=1}^{s'-1}E(s+1,s) \cdot \int d\vec{r}'\,\delta(\vec{r}'-\vec{R}_\alpha(s'))\Big\}$$

$$\cdot \frac{1}{v^{N-s'}}\int D\vec{R}_\alpha(\underbrace{s'+1\cdots N}))\Big\{e^{-w_p(\vec{R}_\alpha(N))}\prod_{s=s'}^{N-1}E(s+1,s)\Big\} e^{w_p(\vec{R}_\alpha(s'))}$$

$$= \frac{1}{v^{s'+1}} \int d\vec{r}_\perp^\alpha \int D\vec{R}(1\cdots s') \left\{ e^{-w_p(\vec{R}_\alpha(s'))} E(1,0) \prod_{s=1}^{s'-1} E(s+1,s) \right.$$

$$\left. \cdot \int d\vec{r}' \, \delta(\vec{R}(s') - \vec{r}') \right\}$$

$$\cdot \frac{1}{v^{N-s'}} \cdot \frac{1}{v} \int D\vec{R}_\alpha(\underset{m}{s'} \cdots N) \left\{ e^{-w_p(\vec{R}_\alpha(N))} \prod_{s=s'}^{N-1} E(s+1,s) \, v\,\delta(\vec{R}(s') - \vec{r}') \right\}$$

$$\cdot e^{w_p(\vec{R}_\alpha(s'))}$$

$\downarrow$ Define
$$q_\perp^\alpha(\vec{r}'; s') = \frac{1}{v^{s'+1}} \int d\vec{r}_\perp^\alpha \int D\vec{R}_\alpha(1\cdots s') \left\{ e^{-w_p(\vec{R}_\alpha(s'))} \right.$$

$$\left. E(1,0) \prod_{s=1}^{s'-1} E(s+1,s) \cdot \underset{\text{m}}{v} \, \delta(\vec{R}_\alpha(s') - \vec{r}') \right\}$$

↖ Extra

$$\cdot \int dR_\alpha(0) \, \delta(\vec{R}_\alpha(0) - \vec{r}_\perp^\alpha).$$

(Propagator starting at grafted end.)

↳ for $s' = 0$,
$$q_\perp^\alpha(\vec{r}'; 0) = \frac{1}{v} \int d\vec{r}_\perp^\alpha \int d\vec{R}_\alpha(0) \, e^{-w_p(\vec{r}_\perp^\alpha)}$$

$$\cdot v\, \delta(\vec{R}_\alpha(0) - \vec{r}') \, \delta(\vec{R}_\alpha(0) - \vec{r}_\perp^\alpha)$$

$$= \int d\vec{R}_\alpha(0) \cancel{\int d\vec{R}_\alpha(0)} \, e^{-w_p(\vec{r}')} \, \cancel{\delta(\vec{R}_\alpha(0) - \vec{r}')}$$

$$\cdot \delta(\vec{r}' - \vec{r}_\perp^\alpha)$$

I.C.

$$\boxed{q_\perp^\alpha(\vec{r}; 0) = e^{-w_p(\vec{r}')} \, v \, \delta(\vec{r}' - \vec{r}_\perp^\alpha).}$$

$\downarrow$ Define
$$q^*(\vec{r}'; s') = \frac{1}{v^{N-s'+1}} \int D\vec{R}_\alpha(s' \cdots N) \left\{ e^{-w_p(\vec{R}_\alpha(N))} \right.$$

$$\left. \cdot \prod_{s=s'}^{N-1} E(s+1,s) \cdot v\, \delta(\vec{R}_\alpha(s') - \vec{r}') \right\}$$

(Propagator starting at free end.)

↳ for $s' = N$,
$$q^*(\vec{r}'; N) = \int d\vec{R}_\alpha(N) \, e^{-w_p(\vec{R}_\alpha(N))}$$

$$\cdot \delta(\vec{R}(N) - \vec{r}')$$

I.C.

$$\boxed{q^*(\vec{r}'; N) = e^{-w_p(\vec{r}')}}$$

$$Q_p^\alpha = \underset{\text{m}}{\frac{1}{v}} \int d\vec{r}' \, e^{w_p(\vec{r}')} \, q_\perp^\alpha(\vec{r}'; s') \, q^*(\vec{r}'; s') \qquad \forall s'$$

$\downarrow$ ↖ from extra

$\downarrow$ Choose $s' = \emptyset$.

$$= \cancel{\frac{1}{\nu}} \int d\vec{r}\,' \cancel{e^{W_p(\vec{r}\,')}} \left( \cancel{e^{-W_p(\vec{r}\,')}} \cancel{\nu}\, \delta(\vec{r}\,' - \vec{r}_\perp^{\,\alpha}) \right) q^*(\vec{r}\,'; \emptyset)$$

$$\boxed{Q_p^\alpha = q^*(\vec{r}_\perp^{\,\alpha}; \emptyset)}$$

$\underline{\dfrac{\delta Q_p^\alpha}{\delta W_p}}$

$$\frac{\delta}{\delta W_p} \left\{ \frac{1}{\nu^N} \prod_{s=\emptyset}^{N-1} \int dR_\alpha(s)\, E(s+1, s)\, \frac{1}{\nu} \int dR_\alpha(N)\, e^{-W_p(\vec{R}_\alpha(N))} \right\}$$

$\downarrow$ Shorthand $F[\vec{R}_\alpha(s)] := \begin{cases} \frac{1}{\nu} \int d\vec{R}_\alpha(s)\, e^{-W_p(\vec{R}_\alpha(s))} & \text{if } s = N \\ \frac{1}{\nu} \int d\vec{R}_\alpha(s)\, E(s+1, s) & \text{otherwise} \end{cases}$

$$= \frac{\delta}{\delta W_p} \left\{ \prod_{s=\emptyset}^{N} F[\vec{R}_\alpha(s)] \right\}$$

$\downarrow$ Product rule $\dfrac{\delta (G_1[g]\, G_2[g])}{\delta\, g(\vec{r})} = G_1 \dfrac{\delta G_2}{\delta g} + G_2 \dfrac{\delta G_1}{\delta g}$

$$= \sum_{s=\emptyset}^{N} \left( \prod_{s' \neq s} F[\vec{R}_\alpha(s')] \right) \frac{\delta F[\vec{R}_\alpha(s)]}{\delta W_p(\vec{r})}$$

$\downarrow$ Chain rule $\dfrac{\delta F[\vec{R}_\alpha(s)]}{\delta W_p(\vec{r})} = \displaystyle\int d\vec{R}_\alpha(s) \frac{\delta F[\vec{R}_\alpha(s)]}{\delta W_p(\vec{R}_\alpha(s))} \overset{\delta(\vec{R}_\alpha(s) - \vec{r})}{\cancel{\frac{\delta W_p(\vec{R}_\alpha(s))}{\delta W_p(\vec{r})}}}$

$\downarrow$ $\dfrac{\delta F[\vec{R}_\alpha(s)]}{\delta W_p[R_\alpha(s)]} = - F[\vec{R}_\alpha(s)]$

$$= \int d\vec{R}_\alpha(s) \left( - F[\vec{R}_\alpha(s)] \right) \delta(\vec{R}_\alpha(s) - \vec{r})$$

$$= - \sum_{s'=\emptyset}^{N} \frac{1}{\nu^{N+1}} \int D\vec{R}_\alpha(\emptyset \cdots N) \prod_{s=\emptyset}^{N-1} E(s+1, s) \cdot e^{-W_p(\vec{R}_\alpha(N))} \cdot \int d\vec{R}_\alpha(s')\, \delta(\vec{R}_\alpha(s') - \vec{r}).$$

$\downarrow$ Split by the $s'$ at $\vec{r}$

$$= -\sum_{s'=0}^{N} \Big\{ \frac{1}{v^{s'+1}} \int D\vec{R}_\alpha(0 \cdots s') \prod_{s=0}^{s'-1} E(s+1,s)\, e^{-W_p(\vec{R}_\alpha(s'))}$$

$$\cdot \frac{1}{v^{N-s'}} \int D\vec{R}_\alpha(s'+1, N) \prod_{s=s'}^{N-1} E(s+1,s)\, e^{-W_p(\vec{R}_\alpha(N))} \Big\}$$

$$\cdot \int d\vec{R}_\alpha(s')\, \delta(\vec{R}_\alpha(s') - \vec{r}) \,.\, \cdot\, e^{W_p(\vec{R}_\alpha(s'))}$$

$$\left( \cdot \int d\vec{r}\,'\ \delta(\vec{R}_\alpha(s') - \vec{r}\,') \right)$$

$$\downarrow\ q_\perp^\alpha(\vec{r}\ ; s') = \frac{1}{v^{s'+1}} \int d\vec{r}_\perp^{\,\alpha} \int D\vec{R}_\alpha(1 \cdots s') \Big\{ e^{-W_p(\vec{R}_\alpha(s'))}$$

$$E(1,0) \prod_{s=1}^{s'-1} E(s+1,s) \cdot \underbrace{v}_{\text{Extra}}\, \delta(\vec{R}_\alpha(s') - \vec{r}\ ) \Big\}$$

$$\cdot \int d\vec{R}_\alpha(0)\, \delta(\vec{R}_\alpha(0) - \vec{r}_\perp^{\,\alpha}).$$

$$\downarrow\ q^*(\vec{r}\ ; s') = \frac{1}{v^{N-s'+1}} \int D\vec{R}_\alpha(s' \cdots N) \Big\{ e^{-W_p(\vec{R}_\alpha(N))}$$

$$\cdot \prod_{s=s'}^{N-1} E(s+1,s) \cdot v\, \delta(\vec{R}_\alpha(s') - \vec{r}\ ) \Big\}$$

$$= -\sum_{s'=0}^{N} \Big\{ q_\perp^\alpha(\vec{r}\ ; s')\, q^*(\vec{r}\ ; s'\ )\, e^{W_p(\vec{r})} \cdot \frac{1}{\underbrace{v}_{\text{from extra}}} \Big\}$$

$$\frac{\delta Q_p^\alpha}{\delta W_p} = -\frac{e^{W_p(\vec{r})}}{v} \sum_{s=0}^{N} q_\perp^\alpha(\vec{r}\,; s)\, q^*(\vec{r}\,; s)$$

$$\hookrightarrow\ \therefore\ \varphi_p(\vec{r}) = \sum_{\alpha=1}^{n_p} \frac{1}{Q_p^\alpha}\, e^{W_p(\vec{r})} \sum_{s=0}^{N} q_\perp^\alpha(\vec{r}; s)\, q^*(\vec{r}; s).$$

$$= \sum_{\alpha=1}^{n_p} \left( q^*(\vec{r}_\perp^{\,\alpha}; 0) \right)^{-1} e^{W_p(\vec{r})} \sum_{s=0}^{N} q_\perp^\alpha(\vec{r}; s)\, q^*(\vec{r}; s)$$

## Quenched average

As $n_p, L_\perp \to \infty$, $\sum_{\alpha=1}^{n_p} f \to \sigma \int d\vec{r}_\perp\, f(\vec{r}_\perp)$, where $\sigma := \frac{n_p}{L_\perp^2}$

- $-\sum_{\alpha=1}^{n_p} \ln Q_p(\vec{R}_\alpha) \to -\sigma \int d\vec{r}_\perp \ln g^*(\vec{r}_\perp; 0)$

- $$\varphi_p(\vec{r}) = \sum_{\alpha=1}^{n_p} \frac{e^{W_p(\vec{r})}}{g^*(\vec{r}_\perp^\alpha; 0)} \sum_{s=0}^{N} g_\perp^\alpha(\vec{r}; s)\, g^*(\vec{r}; s)$$

$$\to \sigma \int d\vec{r}_\perp\, e^{W_p(\vec{r})} \sum_{s=0}^{N} \frac{g_\perp^\alpha(\vec{r}; s)}{g^*(\vec{r}_\perp^\alpha; 0)}\, g^*(\vec{r}; s)$$

$$= e^{W_p(\vec{r})} \sum_{s=0}^{N} \left( \sigma \int d\vec{r}_\perp \frac{g_\perp^\alpha(\vec{r}; s)}{g^*(\vec{r}_\perp^\alpha; 0)} \right) g^*(\vec{r}; s)$$

Define $g(\vec{r}; s) := \sigma \int d\vec{r}_\perp\, g_\perp^\alpha(\vec{r}; s) / g^*(\vec{r}_\perp^\alpha; 0)$

$\hookrightarrow g(\vec{r}; 0) = \sigma \int d\vec{r}_\perp\, v\, e^{-W_p(\vec{r}_\perp^\alpha)} \delta(\vec{r}_\perp^\alpha - \vec{r}) / g^*(\vec{r}_\perp^\alpha; 0)$.

$$\boxed{g(\vec{r}; 0) = \frac{\sigma v e^{-W_p(\vec{r})}}{g^*(\vec{r}; 0)}\, \delta(z - z^*)}$$

$$\boxed{\varphi_p(\vec{r}) = e^{W_p(\vec{r})} \sum_{s=0}^{N} g(\vec{r}; s)\, g^*(\vec{r}; s)}$$

<u>RECURSION</u>

$$g^{*}(\vec{r};s) = \frac{1}{v^{N-s+1}} \int D\vec{R}_\alpha(s\cdots N) \Big\{ e^{-W_p(\vec{R}_\alpha(N))} e^{-W_p(\vec{R}_\alpha(s))} \cdot \prod_{s'=s}^{N-1} E(s'+1,s') \cdot v \; \delta(\vec{R}_\alpha(s)-\vec{r}) \Big\}$$

· Consider an $s+1$ at $\vec{r}\,'$:

$$g^{*}(\vec{r}\,';s+1) = \frac{1}{v^{N-s}} \int D\vec{R}_\alpha(s+1\cdots N) \Big\{ e^{-W_p(\vec{R}_\alpha(N))} e^{-W_p(\vec{R}_\alpha(s+1))} \prod_{s'=s+1}^{N-1} E(s'+1,s') \cdot v \; \delta(\vec{R}_\alpha(s+1)-\vec{r}\,') \Big\}$$

$$g^{*}(\vec{r};s) = \frac{1}{v}\cdot\frac{1}{v^{N-s}} \int d\vec{R}_\alpha(s) \int D\vec{R}_\alpha(s+1\cdots N) \Big\{ e^{-W_p(\vec{R}_\alpha(N))} e^{-W_p(\vec{R}_\alpha(s))} E(s+1,s) \prod_{s'=s+1}^{N-1} E(s'+1,s')\, v \; \delta(\vec{R}_\alpha(s)-\vec{r}) \Big\}$$

$$= \frac{1}{v} \int d\vec{R}_\alpha(s) \Big\{ \frac{1}{v^{N-s}} \int D\vec{R}_\alpha(s+1\cdots N) e^{-W_p(\vec{R}_\alpha(N))} e^{-W_p(\vec{R}_\alpha(s+1))} \prod_{s'=s+2}^{N-1} E(s'+1,s')\cdot v \Big\} E(s+1,s) \; \delta(\vec{R}_\alpha(s)-\vec{r})$$

$$\cdot\, e^{W_p(\vec{R}_\alpha(s+1))} e^{-W_p(\vec{R}_\alpha(s))} \int d\vec{r}\,' \; \delta(\vec{R}_\alpha(s+1)-\vec{r}\,')$$

$$= \frac{1}{v} \int d\vec{R}_\alpha(s) \int d\vec{r}\,' \; g^{*}(\vec{r}\,';s+1) \cdot \Phi(\vec{r}\,',s)\, \cancel{e^{-W_p(\vec{r}\,')}} \cdot \delta(\vec{R}_\alpha(s)-\vec{r}) \cdot \cancel{e^{W_p(\vec{r}\,')}} \cdot e^{-W_p(\vec{R}_\alpha(s))}$$

$$= \frac{1}{v} e^{-W_p(\vec{r})} \int d\vec{r}\,' \; \Phi(\vec{r},\vec{r}\,')\, g^{*}(\vec{r}\,';s+1)$$

$$\hookrightarrow \boxed{g^{*}(\vec{r};s-1) = v^{-1} e^{-W_p(\vec{r})} \int d\vec{r}\,' \; \Phi(\vec{r},\vec{r}\,')\, g^{*}(\vec{r};s)}$$

$$g_\perp^\alpha(\vec{r};s) = \frac{1}{v^{s+1}} \int d\vec{r}_\perp^\alpha \int D\vec{R}_\alpha(1\cdots s) \Big\{ e^{-W_p(\vec{R}_\alpha(s))}$$
$$\cdot E(1,\emptyset) \prod_{s'=1}^{s-1} E(s'+1,s') \cdot v\,\delta(\vec{R}_\alpha(s)-\vec{r}) \Big\}$$

- Consider an $s+1$ at $\vec{r}'$:

$$g_\perp^\alpha(\vec{r}';s+1) = \frac{1}{v^{s+2}} \int d\vec{r}_\perp^\alpha \int D\vec{R}_\alpha(1\cdots s+1) \Big\{ e^{-W_p(\vec{R}_\alpha(s+1))}$$
$$\cdot E(1,\emptyset) \prod_{s'=1}^{s} E(s'+1,s') \cdot v\,\delta(\vec{R}_\alpha(s+1)-\vec{r}') \Big\}$$

$$g_\perp^\alpha(\vec{r}';s+1) = \frac{1}{v} \int D\vec{R}_\alpha(s+1) \Big\{ \frac{1}{v^{s+1}} \int d\vec{r}_\perp^\alpha \int D\vec{R}_\alpha(1\cdots s)\, e^{-W_p(\vec{R}_\alpha(s))}$$
$$\cdot E(1,\emptyset) \prod_{s'=1}^{s-1} E(s'+1,s')\, v \Big\} e^{-W_p(\vec{R}_\alpha(s+1))}\, e^{W_p(\vec{R}_\alpha(s))}$$
$$\cdot E(s+1,s)\, \delta(\vec{R}_\alpha(s+1)-\vec{r}') \int d\vec{r}\, \delta(\vec{R}_\alpha(s)-\vec{r}).$$

$$= \frac{1}{v} \int d\vec{r} \int D\vec{R}_\alpha(s+1)\, g_\perp^\alpha(\vec{r};s)\, e^{-W_p(\vec{r}')} \cancel{e^{W_p(\vec{r})}}$$
$$\Phi(\vec{r},s+1)\, \cancel{e^{-W_p(\vec{r})}} \quad \delta(\vec{R}_\alpha(s+1)-\vec{r}')$$

$$= \frac{1}{v} e^{-W_p(\vec{r}')} \int d\vec{r}\, \Phi(\vec{r},\vec{r}')\, g_\perp^\alpha(\vec{r};s)$$

$$\Big\downarrow \quad g(\vec{r};s) := \sigma \int d\vec{r}_\perp\, g_\perp^\alpha(\vec{r};s) / g^e(\vec{r}_\perp^\alpha;\emptyset)$$

$$\boxed{g(\vec{r};s+1) = v^{-1} e^{-W_p(\vec{r}')} \int d\vec{r}'\, \Phi(\vec{r},\vec{r}')\, g(\vec{r}';s)}$$

# FREE ENERGY

$$F = -\sum_{\alpha=1}^{n_p} \ln Q_p^\alpha - e^{\mu_s} Q_s + \frac{1}{v}\int d\vec{r}\,\Big(\chi \varphi_p \varphi_s - w_p \varphi_p - w_s \varphi_s$$

$$- \eta(\varphi_p + \varphi_s - 1)\Big) + \int d\vec{r}\,\Big(\varepsilon \frac{|\nabla\psi|^2}{2} + \frac{\alpha}{v}\varphi_p \psi - \rho_+ - \rho_-\Big)$$

, where $Q_p^\alpha = q^*(\vec{r}_I^{\,\alpha}; 0)$

$$Q_s = \frac{1}{v}\int d\vec{r}\,\exp(-w_s(\vec{r}))$$

# SELF-CONSISTENT FIELD EQUATIONS

$$\begin{cases} w_p(\vec{r}) = \chi\varphi_s(\vec{r}) - \eta(\vec{r}) - \frac{v}{2}\frac{d\varepsilon}{d\varphi_p}|\nabla\psi|^2 + \alpha\psi(\vec{r}) \\ w_s(\vec{r}) = \chi\varphi_p(\vec{r}) - \eta(\vec{r}) \\ 0 = \varphi_p(\vec{r}) + \varphi_s(\vec{r}) - 1 \\ \varphi_p(\vec{r}) = e^{w_p(\vec{r})} \sum_{s=0}^{N} q(\vec{r};s)\, q^*(\vec{r};s) \\ \varphi_s(\vec{r}) = e^{\mu_s}\exp(-w_s(\vec{r})) \\ -\nabla\cdot\varepsilon(\vec{r})\nabla\psi(\vec{r}) = z_+\lambda_+ e^{-z_+\psi(\vec{r})} - z_-\lambda_- e^{-z_-\psi(\vec{r})} + \frac{\alpha}{v}\varphi_p(\vec{r}) \end{cases}$$

$$q(\vec{r}; s+1) = v^{-1} e^{-w_p(\vec{r})} \int d\vec{r}\,'\,\Phi(\vec{r},\vec{r}\,')\, q(\vec{r}\,'; s)$$

with $q(\vec{r}; 0) = \dfrac{\sigma v e^{-w_p(\vec{r})}}{q^*(\vec{r}; 0)}\,\delta(z - z^*)$

$$q^*(\vec{r}; s-1) = v^{-1} e^{-w_p(\vec{r})} \int d\vec{r}\,\Phi(\vec{r},\vec{r}\,')\, q^*(\vec{r}\,'; s)$$

with $q^*(\vec{r}; N) = e^{-w_p(\vec{r})}$

, where $\Phi(\vec{r},\vec{r}\,') = \exp\left[-\frac{3}{2b^2}(\vec{r} - \vec{r}\,')^2\right]$