# Exact Inference

## 3007/7059 Artificial Intelligence

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# Outline

- Recap of Bayesian Networks
- Inference by enumeration
- Inference by variable elimination

# Recap:Global semantic of a Bayesian network

A Bayesian Network encodes our knowledge on how the variables interact. This is specified in terms of conditional independence assertions.

The global semantics of a network define a joint distribution of all variables as the product of local conditional distributions.

The joint distribution defined by a Bayesian Network with variables $X_1, \ldots, X_n$ is:

$$\begin{aligned} P(X_1, \ldots, X_n) &= P(X_1|Parents(X_1)) \times P(X_2|Parents(X_2)) \\ &\quad \times \cdots \times P(X_n|Parents(X_n)) \\ &= \prod_{i=1}^{n} P(X_i|Parents(X_i)) \end{aligned}$$

where $Parents(X_i)$ are parents of $X_i$ as specified by the particular Bayesian Network.

# Recap: Inference

Recall the general rule of statistical inference:

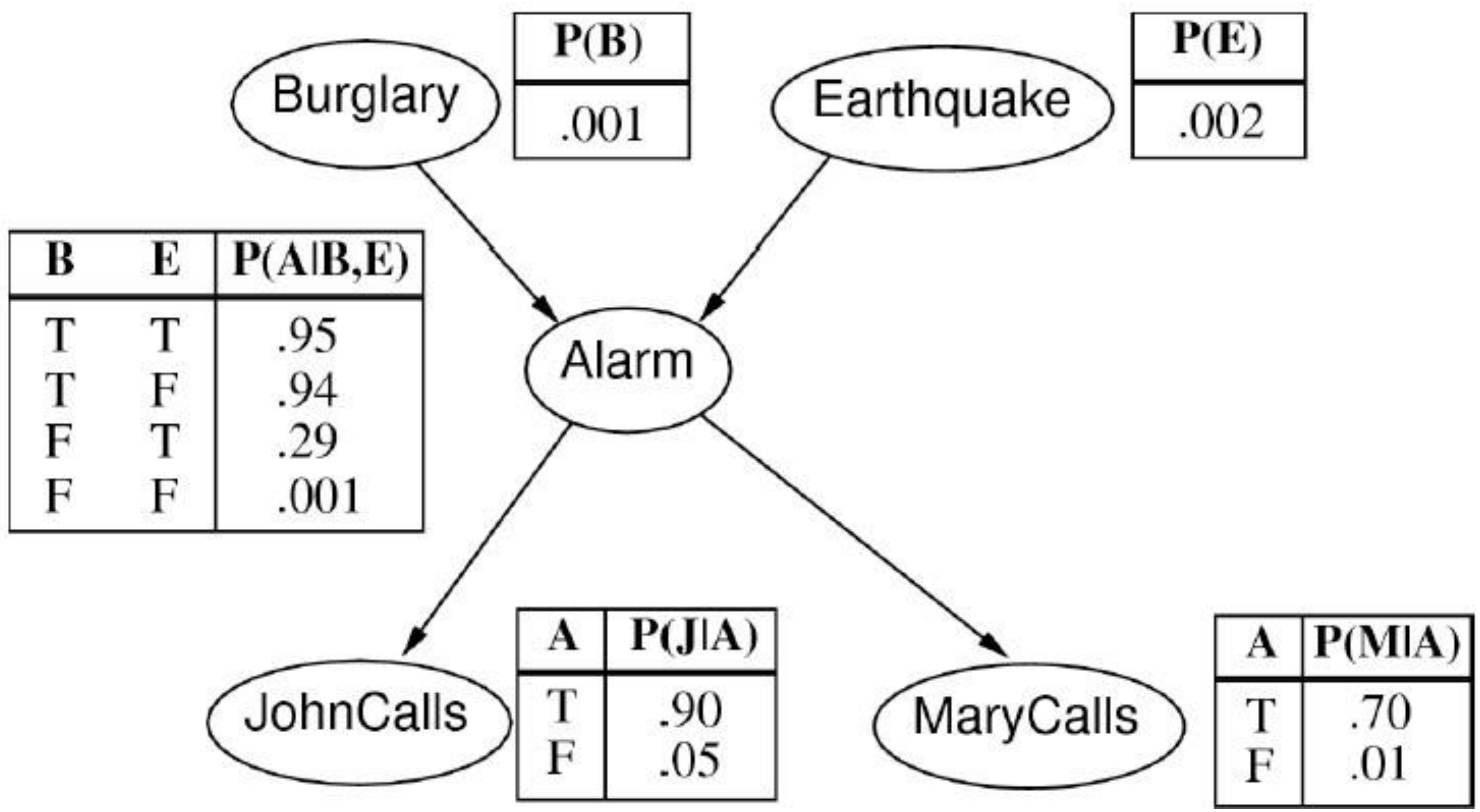
$$P(X|e) = \alpha \sum_{\forall Y} P(X, e, Y)$$

where $X$ is the query variable, $e$ the observed values for the evidence variables, and $Y$ the unobserved variables. As usual $\alpha$ is a normalisation constant that we solve for at the end.

# Recap:Inference problem

I'm at work, neighbour John called to say my alarm is ringing, but neighbour Mary didn't call. Sometimes it's set off by minor earthquakes. Is there a burglar?

$$P(b|j, \neg m)$$




| P(B) |
|---|
| .001 |

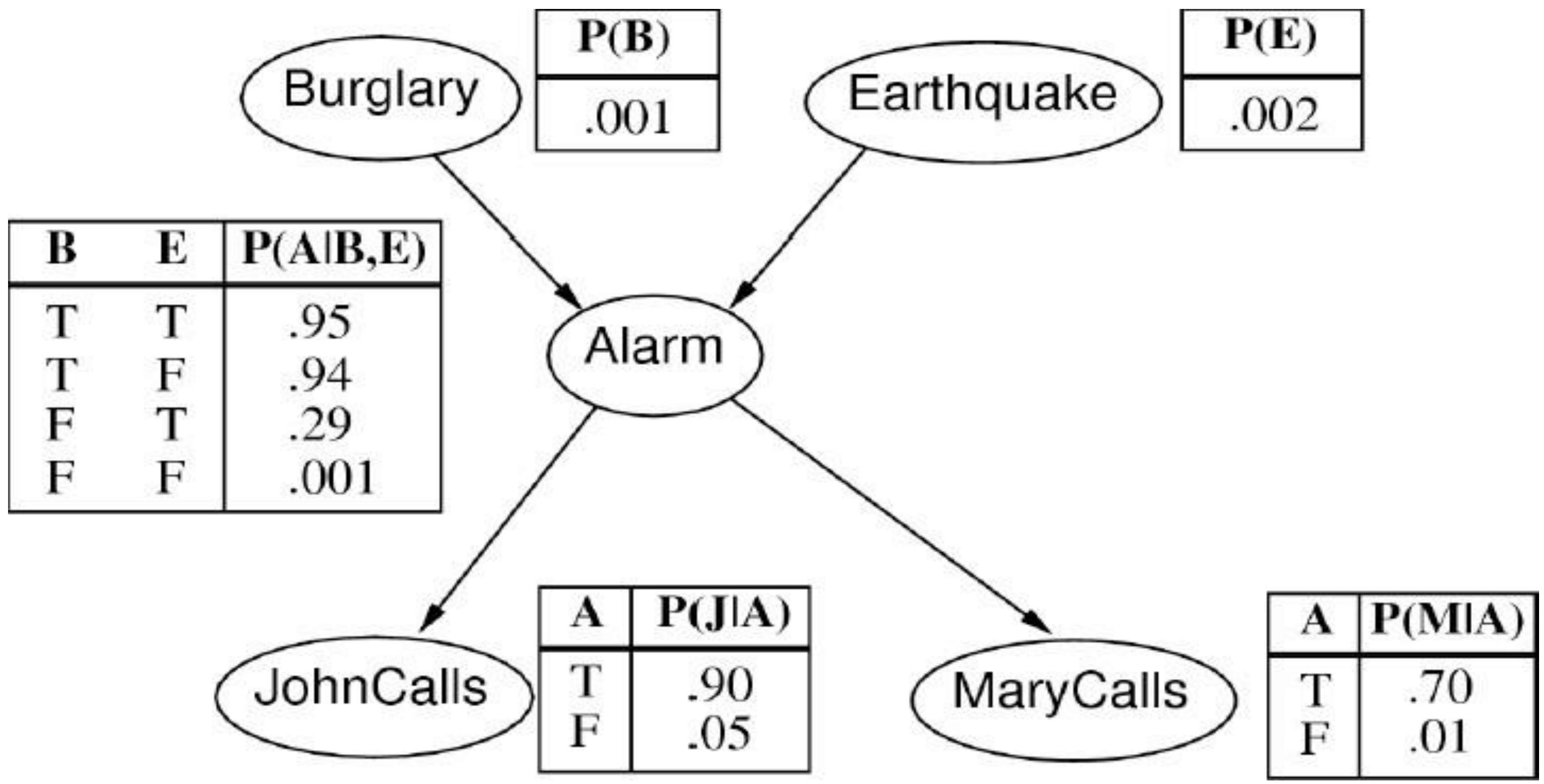
| P(E) |
|---|
| .002 |

| B | E | P(A\|B,E) |
|---|---|---|
| T | T | .95 |
| T | F | .94 |
| F | T | .29 |
| F | F | .001 |

| A | P(J\|A) |
|---|---|
| T | .90 |
| F | .05 |

| A | P(M\|A) |
|---|---|
| T | .70 |
| F | .01 |

# Performing inference




| P(B) |
|---|
| .001 |

| P(E) |
|---|
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| B | E | P(A\|B,E) |
|---|---|---|
| T | T | .95 |
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| A | P(J\|A) |
|---|---|
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| A | P(M\|A) |
|---|---|
| T | .70 |
| F | .01 |

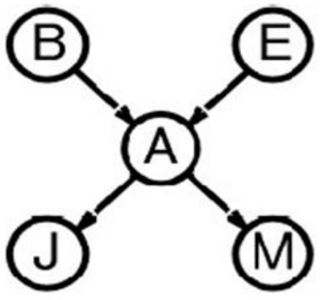
$P(B, E, A, J, M)?$ $\quad P(b|j, \neg m)?$

$$P(J, M, A, B, E) = P(J|A)P(M|A)P(A|B, E)P(B)P(E)$$

# Performing inference on Bayesian networks

A direct application of the general rule yields

$$P(b|j, \neg m) = \alpha \sum_e \sum_a P(b, j, \neg m, e, a)$$




Observe that the summands are joint probabilities of all the variables. Hence, we introduce the global semantics of the network:

$$P(b|j, \neg m) = \alpha \sum_e \sum_a P(b)P(e)P(a|b, e)P(j|a)P(\neg m|a)$$

# Inference by enumeration

Expanding by enumerating the summands we obtain

$$P(b|j,\neg m) = \alpha \sum_e \sum_a P(b)P(e)P(a|b,e)P(j|a)P(\neg m|a)$$

$$= \alpha\,[P(b)P(e)P(a|b,e)P(j|a)P(\neg m|a)$$ e, a

$$+P(b)P(e)P(\neg a|b,e)P(j|\neg a)P(\neg m|\neg a)$$ e, -a

$$+P(b)P(\neg e)P(a|b,\neg e)P(j|a)P(\neg m|a)$$ -e, a

$$+P(b)P(\neg e)P(\neg a|b,\neg e)P(j|\neg a)P(\neg m|\neg a)]$$ -e, -a

For each of the components on the RHS, we read its value from the appropriate CPT, yielding $P(b|j,\neg m) = \alpha 0.00025677$.

Note that the result does not yet amount to a probability value as we haven't solved for $\alpha$.

# Inference by enumeration

Calculate the scaling factor

To compute $\alpha = \frac{1}{P(j, \neg m)}$ we obtain the marginal probability

$$P(j, \neg m) = \sum_b \sum_e \sum_a P(b, e, a, j, \neg m)$$

# Inference by enumeration

An alternative is to realise that $\langle P(b|j,\neg m), P(\neg b|j,\neg m)\rangle$ is a probability distribution and that $\alpha$ is a normalisation constant that ensures that the probability distribution sums to 1.

Hence, compute $P(\neg b|j,\neg m) = \alpha 0.0498$, using the global semantics and enumerating the summands as before, yielding

$$\langle P(b|j,\neg m), P(\neg b|j,\neg m)\rangle = \alpha\langle 0.00025677, 0.0498\rangle = \langle 0.0051, 0.9949\rangle$$

where $\alpha$ is solved as $\frac{1}{0.00025677+0.0498}$.

# Complexity of inference by enumeration

In the burglar-alarm example, we evaluate the expression

$$
\begin{aligned}
P(b|j,\neg m) &= \alpha \sum_e \sum_a P(b)P(e)P(a|b,e)P(j|a)P(\neg m|a) \\
&= \alpha\,[P(b)P(e)P(a|b,e)P(j|a)P(\neg m|a) \\
&\quad +P(b)P(e)P(\neg a|b,e)P(j|\neg a)P(\neg m|\neg a) \\
&\quad +P(b)P(\neg e)P(a|b,\neg e)P(j|a)P(\neg m|a) \\
&\quad +P(b)P(\neg e)P(\neg a|b,\neg e)P(j|\neg a)P(\neg m|\neg a)]
\end{aligned}
$$

by adding up 4 terms, each obtained by multiplying 5 numbers— In total we need 16 multiplications and 3 additions (excludes the contribution due to term $\alpha$).

# Complexity of inference by enumeration

In the burglar-alarm example, we evaluate the expression

$$P(b|j, \neg m) = \alpha \sum_e \sum_a P(b)P(e)P(a|b,e)P(j|a)P(\neg m|a)$$

In the worst case, where we have to sum out almost all of the $n$ variables (where we assume they are all Boolean), the complexity of inference by enumeration is $\mathcal{O}(n2^n)$.

This means we will not be able to perform inference by enumeration except for the smallest networks!

# Depth-first Evaluation

An improvement can be achieved by observing that $P(b)$ is a constant that can be moved outside the summations over $E$ and $A$, while $P(e)$ can be moved outside the summation over $A$:

$$P(b|j,\neg m) = \alpha \sum_e \sum_a P(b)P(e)P(a|b,e)P(j|a)P(\neg m|a)$$
$$= \alpha P(b) \sum_e P(e) \sum_a P(a|b,e)P(j|a)P(\neg m|a)$$

Thus we evaluate by looping through the variables in order, multiplying CPT entries as we go.

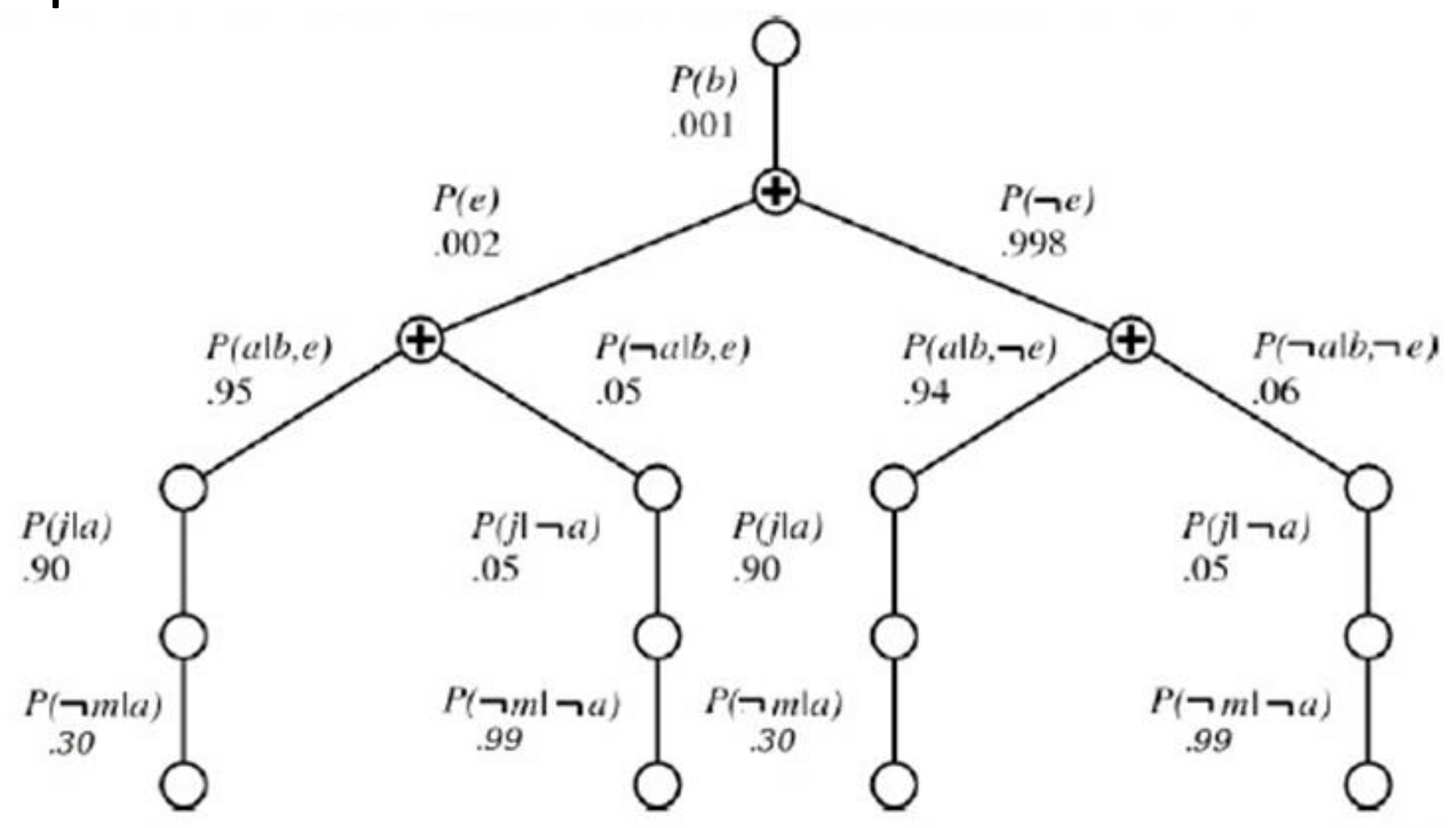
Now evaluating the expression involves 11 multiplications and 3 additions (again excluding the contribution due to term $\alpha$).

$$= \alpha P(b) \sum_e P(e)[P(a|b,e)P(j|a)P(\neg m|a)+P(\neg a|b,e)P(j|\neg a)P(\neg m|\neg a)]$$

# Depth-first Evaluation

$$P(b|j,\neg m) = \alpha P(b) \sum_e P(e) \sum_a P(a|b,e) P(j|a) P(\neg m|a)$$

The process can be illustrated as an evaluation tree.




The evaluation proceeds top-down, multiplying values along each path and summing at the "+" nodes

# Complexity of Depth-first Evaluation

$$P(b|j,\neg m) = \alpha \sum_e \sum_a P(b)P(e)P(a|b,e)P(j|a)P(\neg m|a)$$

$$\begin{aligned} = \ & \alpha\,[P(b)P(e)P(a|b,e)P(j|a)P(\neg m|a) \\ & +P(b)P(e)P(\neg a|b,e)P(j|\neg a)P(\neg m|\neg a) \\ & +P(b)P(\neg e)P(a|b,\neg e)P(j|a)P(\neg m|a) \\ & +P(b)P(\neg e)P(\neg a|b,\neg e)P(j|\neg a)P(\neg m|\neg a)] \end{aligned}$$

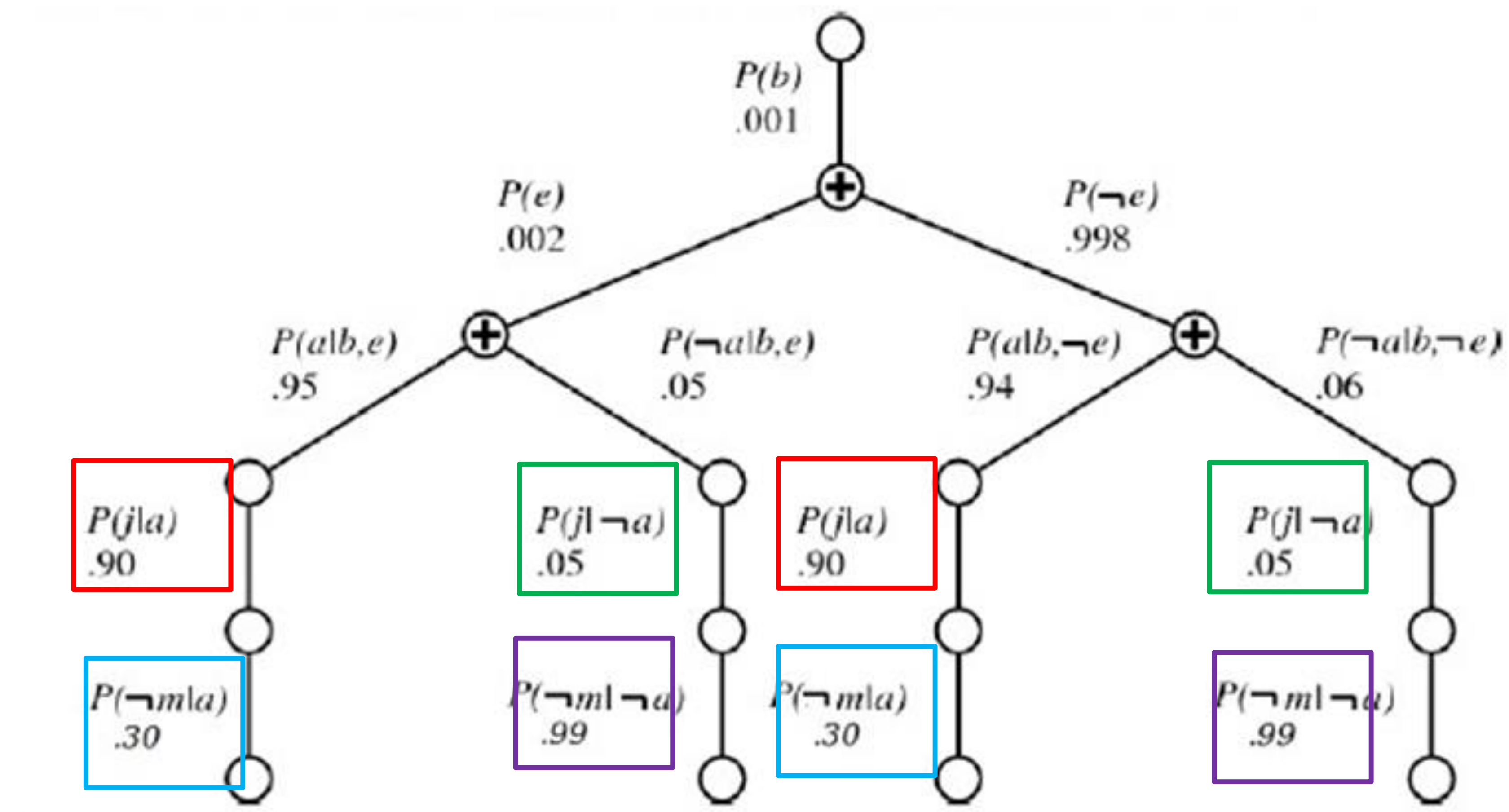
$$O(n\,2^n)$$

$$P(b|j,\neg m) = \alpha P(b) \sum_e P(e) \sum_a P(a|b,e)P(j|a)P(\neg m|a)$$

$$O(2^n)$$

# Problem of DF

$$P(b|j,\neg m) = \alpha P(b) \sum_e P(e) \sum_a P(a|b,e) P(j|a) P(\neg m|a)$$



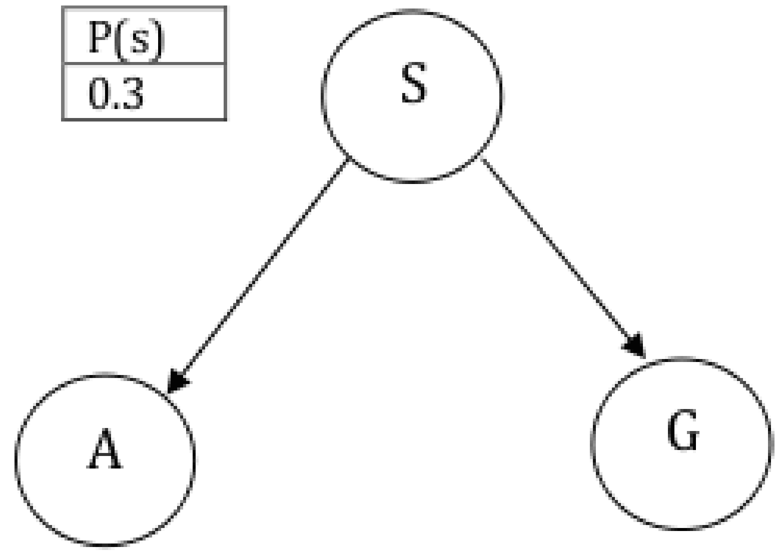

Repeat computation!

# Variable elimination

- Recursively eliminate variables and merge terms into factors
- Store factors to avoid repeating computations
- Complexity for single connected n Boolean variables: is linear in the size (number of CPT entries) of the network

# Variable elimination

| P(s) |
|---|
| 0.3 |




| S | P(a\|S) |
|---|---|
| t | 0.6 |
| f | 0.3 |

| S | P(g\|S) |
|---|---|
| t | 0.7 |
| f | 0.2 |

$$P(A) = \sum \sum P(A, s, g)$$

$$= \sum_s \sum_g P(s)P(A|s)P(g|s)$$

$$= \sum_s \left(P(s)P(A|s)P(g|s) + P(s)P(A|s)P(\neg g|s)\right)$$

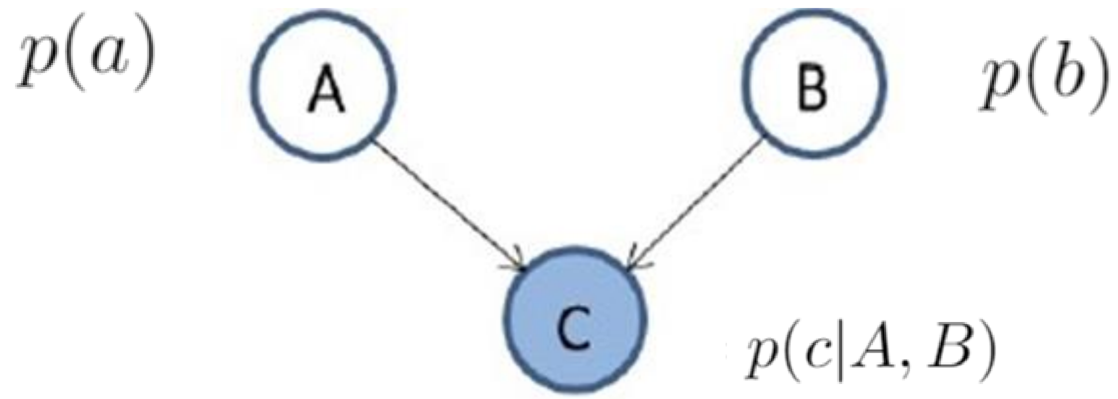
Eliminate G

$$= \sum_s \left( P(s)P(A|s) \sum_g P(g|s) \right)$$

Eliminate S

$$= \sum_s P(s)P(A|s) \quad = f(A)$$

# Variable elimination




$$P(C) = \sum_a \sum_b P(a, b, C)$$

$$= \sum_{a,b} P(a)P(b)P(C|a, b)$$

$$= \sum_b P(b) \sum_a P(a)P(C|a, b)$$

Eliminate A

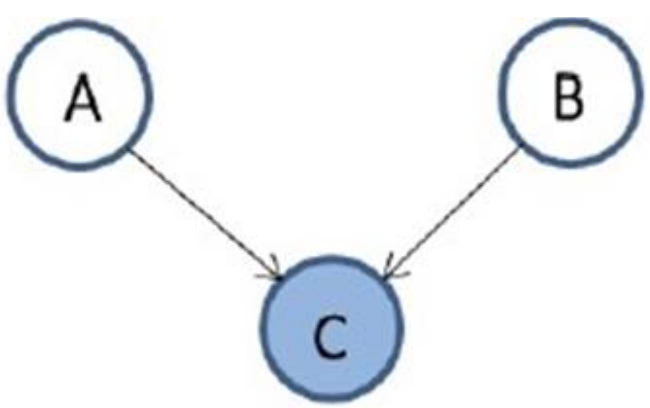
$$= \sum_b P(b) f_1(B, C)$$

Eliminate B

$$= f_2(C)$$

# Variable elimination -factor

- Factor associate a real value for each setting of its arguments.
- Factor in BN is corresponding to conditional probability distributions.
- The joint distribution is a product of factors.




$$P(A,B,C) = \underbrace{P(A)}_{f_1}\underbrace{P(B)}_{f_2}\underbrace{P(C|A,B)}_{f_3}$$

# Variable elimination -factor operation

- Let X, Y and Z are three random variables, and $\phi_1(X,Y)$ and $\phi_2(Y,Z)$ are two factors, their product is a new factor:

$$\psi(X,Y,Z) = \phi_1(X,Y)\phi_2(Y,Z)$$

- An Example:

$\phi_1$ has 3*2 = 6 entries

$\phi_2$ has 2*2 = 4 entries

yields:

$\psi$ has 3*2*2 = 12 entries

| | | |
|---|---|---|
| $a^1$ | $b^1$ | 0.5 |
| $a^1$ | $b^2$ | 0.8 |
| $a^2$ | $b^1$ | 0.1 |
| $a^2$ | $b^2$ | 0 |
| $a^3$ | $b^1$ | 0.3 |
| $a^3$ | $b^2$ | 0.9 |

| | | |
|---|---|---|
| $b^1$ | $c^1$ | 0.5 |
| $b^1$ | $c^2$ | 0.7 |
| $b^2$ | $c^1$ | 0.1 |
| $b^2$ | $c^2$ | 0.2 |

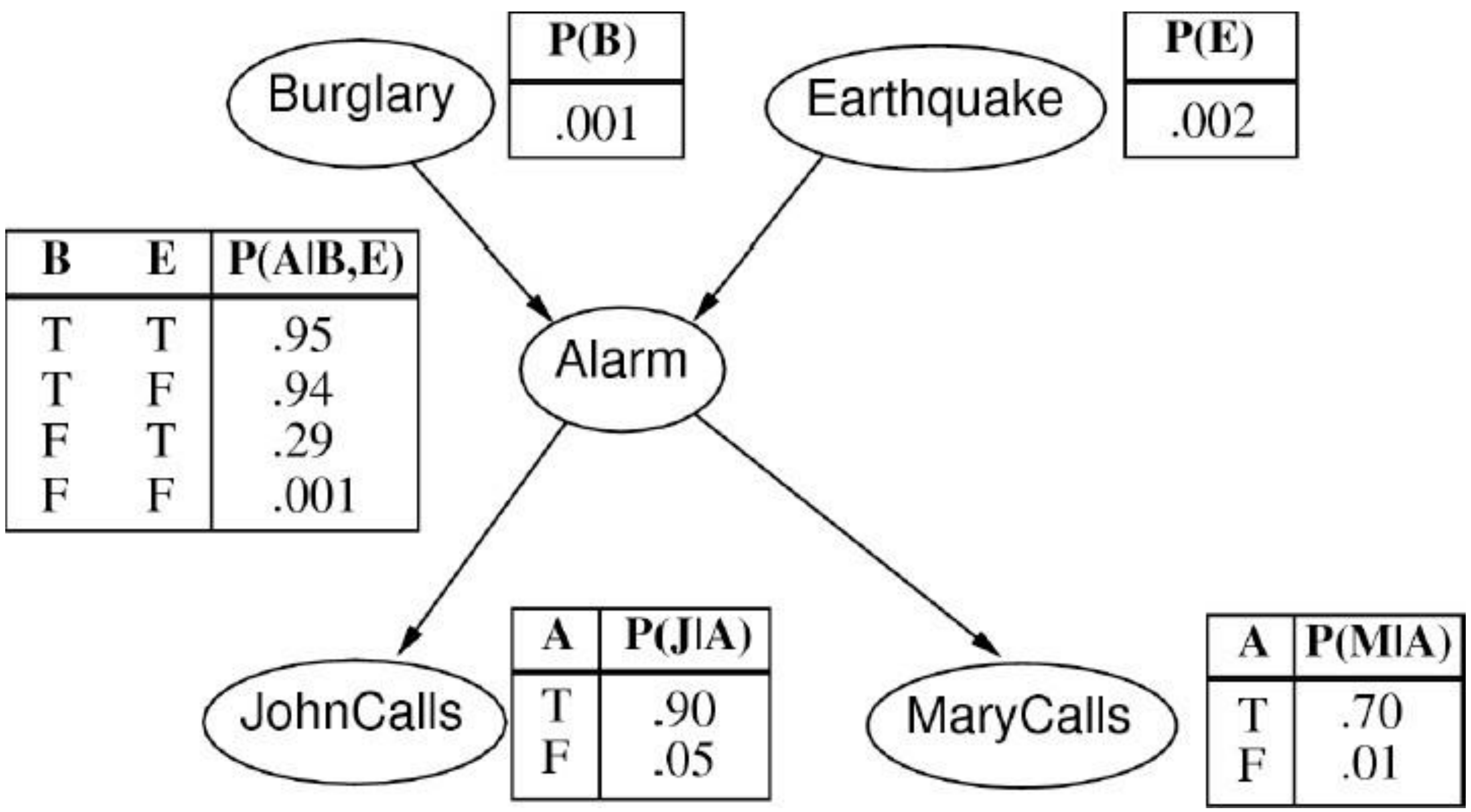
| | | | |
|---|---|---|---|
| $a^1$ | $b^1$ | $c^1$ | $0.5 \cdot 0.5 = 0.25$ |
| $a^1$ | $b^1$ | $c^2$ | $0.5 \cdot 0.7 = 0.35$ |
| $a^1$ | $b^2$ | $c^1$ | $0.8 \cdot 0.1 = 0.08$ |
| $a^1$ | $b^2$ | $c^2$ | $0.8 \cdot 0.2 = 0.16$ |
| $a^2$ | $b^1$ | $c^1$ | $0.1 \cdot 0.5 = 0.05$ |
| $a^2$ | $b^1$ | $c^2$ | $0.1 \cdot 0.7 = 0.07$ |
| $a^2$ | $b^2$ | $c^1$ | $0 \cdot 0.1 = 0$ |
| $a^2$ | $b^2$ | $c^2$ | $0 \cdot 0.2 = 0$ |
| $a^3$ | $b^1$ | $c^1$ | $0.3 \cdot 0.5 = 0.15$ |
| $a^3$ | $b^1$ | $c^2$ | $0.3 \cdot 0.7 = 0.21$ |
| $a^3$ | $b^2$ | $c^1$ | $0.9 \cdot 0.1 = 0.09$ |
| $a^3$ | $b^2$ | $c^2$ | $0.9 \cdot 0.2 = 0.18$ |

# Variable elimination

$$P(E|j,m) = \alpha P(E,j,m)$$

$$= \alpha \underbrace{P(E)}_{E} \sum_b \underbrace{P(b)}_{B} \sum_a \underbrace{P(a|b,E)}_{A} \underbrace{P(j|a)}_{J} \underbrace{P(m|a)}_{M}$$

$$f_E(E) \quad f_B(B) \quad f_A(A,B,E) \quad f_J(A) \quad f_M(A)$$



# Variable elimination

$$P(E|j,m) = \alpha P(E,j,m)$$

$$\alpha \underbrace{P(E)}_{E} \sum_b \underbrace{P(b)}_{B} \sum_a \underbrace{P(a|b,E)}_{A} \underbrace{P(j|a)}_{J} \underbrace{P(m|a)}_{M}$$

$$f_E(E) \quad f_B(B) \quad f_A(A,B,E) \quad f_J(A) \quad f_M(A)$$

| E | $f_E(E)$ |
|---|---|
| T | .002 |
| F | .998 |

| B | $f_B(B)$ |
|---|---|
| T | .001 |
| F | .999 |

| A | B | E | $f_A(A,B,E)$ |
|---|---|---|---|
| T | T | T | .95 |
| T | T | F | .94 |
| T | F | T | .29 |
| T | F | F | .001 |
| F | T | T | .05 |
| F | T | F | .06 |
| F | F | T | .71 |
| F | F | F | .999 |

| A | $f_J(A)$ |
|---|---|
| T | .9 |
| F | .05 |

| A | $f_M(A)$ |
|---|---|
| T | .7 |
| F | .01 |

# Variable elimination

$$P(E|j,m) = \alpha P(E,j,m)$$

$$= \alpha P(E) \sum_b P(b) \sum_a P(a|b,E) P(j|a) P(m|a)$$

$$= \alpha f_E(E) \sum_b f_B(B) \sum_a f_A(A,B,E) f_J(A) f_M(A)$$ factorize

# Variable elimination

$$P(E|j,m) = \alpha P(E,j,m)$$

$$= \alpha P(E) \sum_b P(b) \sum_a P(a|b,E) P(j|a) P(m|a)$$

$$= \quad \alpha f_E(E) \textstyle\sum_b f_B(B) \sum_a f_A(A,B,E) \boxed{f_J(A) f_M(A)}$$

factorize

$$= \quad \alpha f_E(E) \textstyle\sum_b f_B(B) \sum_a f_A(A,B,E) \boxed{f_{JM}(A)}$$

factor product

| A | $f_{JM}(A)$ |
|---|---|
| T | .9 * .7 |
| F | .05 * .01 |

=

| A | $f_J(A)$ |
|---|---|
| T | .9 |
| F | .05 |

| A | $f_M(A)$ |
|---|---|
| T | .7 |
| F | .01 |

$$f_{JM}(A) = f_J(A) f_M(A)$$

# Variable elimination

$$P(E|j,m) = \alpha P(E,j,m)$$

$$= \alpha P(E) \sum_b P(b) \sum_a P(a|b,E) P(j|a) P(m|a)$$

$$= \alpha f_E(E) \textstyle\sum_b f_B(B) \sum_a f_A(A,B,E) f_J(A) f_M(A)$$ factorize

$$= \alpha f_E(E) \textstyle\sum_b f_B(B) \sum_a \boxed{f_A(A,B,E) f_{JM}(A)}$$ factor product

$$= \alpha f_E(E) \textstyle\sum_b f_B(B) \sum_a \boxed{f_{AJM}(A,B,E)}$$ factor product

| A | B | E | $f_{AJM}(A,B,E)$ |
|---|---|---|---|
| T | T | T | .95 * .63 |
| T | T | F | .94 * .63 |
| T | F | T | .29 * .63 |
| T | F | F | .001 * .63 |
| F | T | T | .05 * .0005 |
| F | T | F | .06 * .0005 |
| F | F | T | .71 * .0005 |
| F | F | F | .999 * .0005 |

=

| A | B | E | $f_A(A,B,E)$ |
|---|---|---|---|
| T | T | T | .95 |
| T | T | F | .94 |
| T | F | T | .29 |
| T | F | F | .001 |
| F | T | T | .05 |
| F | T | F | .06 |
| F | F | T | .71 |
| F | F | F | .999 |

| A | $f_{JM}(A)$ |
|---|---|
| T | .63 |
| F | .0005 |

$$f_{AJM}(A,B,E) = f_A(A,B,E) f_{JM}(A)$$

# Variable elimination

$$P(E|j,m) = \alpha P(E,j,m)$$

$$= \alpha P(E) \sum_b P(b) \sum_a P(a|b,E) P(j|a) P(m|a)$$

$$= \quad \alpha f_E(E) \textstyle\sum_b f_B(B) \sum_a f_A(A,B,E) f_J(A) f_M(A)$$ factorize

$$= \quad \alpha f_E(E) \textstyle\sum_b f_B(B) \sum_a f_A(A,B,E) f_{JM}(A)$$ factor product

$$= \quad \alpha f_E(E) \textstyle\sum_b f_B(B) \boxed{\sum_a f_{AJM}(A,B,E)}$$ factor product

$$= \quad \alpha f_E(E) \textstyle\sum_b f_B(B) \boxed{f_{\bar{A}JM}(B,E)}$$ factor marginalization, and elimitate A

| A | B | E | $f_{AJM}(A,B,E)$ |
|---|---|---|---|
| T | T | T | $.95 * .63$ |
| T | T | F | $.94 * .63$ |
| T | F | T | $.29 * .63$ |
| T | F | F | $.001 * .63$ |
| F | T | T | $.05 * .0005$ |
| F | T | F | $.06 * .0005$ |
| F | F | T | $.71 * .0005$ |
| F | F | F | $.999 * .0005$ |

| B | E | $f_{\bar{A}JM}(B,E)$ |
|---|---|---|
| T | T | $.95 * .63 + .05 * .0005 = .5985$ |
| T | F | $.94 * .63 + .06 * .0005 = .5922$ |
| F | T | $.29 * .63 + .71 * .0005 = .1830$ |
| F | F | $.001 * .63 + .999 * .0005 = .001129$ |

$$f_{\bar{A}JM}(B,E) = \sum_a f_{AJM}(A,B,E)$$

# Variable elimination

$$
\begin{aligned}
P(E|j,m) &= \alpha P(E,j,m) \\
&= \alpha P(E) \sum_b P(b) \sum_a P(a|b,E) P(j|a) P(m|a) \\
&= \alpha f_E(E) \textstyle\sum_b f_B(B) \sum_a f_A(A,B,E) f_J(A) f_M(A) \quad \text{factorize} \\
&= \alpha f_E(E) \textstyle\sum_b f_B(B) \sum_a f_A(A,B,E) f_{JM}(A) \\
&= \alpha f_E(E) \textstyle\sum_b f_B(B) \sum_a f_{AJM}(A,B,E) \\
&= \alpha f_E(E) \textstyle\sum_b f_B(B) f_{\bar{A}JM}(B,E) \quad \text{Eliminate A} \\
&= \alpha f_E(E) \textstyle\sum_b f_{B\bar{A}JM}(B,E) \\
&= \alpha f_E(E) f_{\bar{B}\bar{A}JM}(E) \quad \text{Eliminate B} \\
&= \alpha f_{E\bar{B}\bar{A}JM}(E)
\end{aligned}
$$

The process of evaluation is a process of summing out variables (right to left) from pointwise products of factors to produce new factors, eventually yielding a factor that is the solution, i.e., the posterior distribution over the query variable.

It is bottom-up in the evaluation tree.

# Variable elimination

- Recursively eliminate variables and merge terms  into factors
- Store factors to avoid repeating computations
- Complexity for single connected n Boolean variables: is linear in the size (number of CPT entries) of the network

# Variable elimination

Also useful for doing inference multiple times

e.g.

$$P(B|J,M) = \alpha P(B,J,M)$$
$$P(B|J) = \alpha P(B,J)$$
$$P(B|M) = \alpha P(B,M)$$

# Variable elimination

$$\begin{aligned}P(B|J,M) &= \alpha P(B,J,M)\\ &= \sum_a \sum_e P(B)P(e)P(a|B,e)P(J|A)P(M|a)\\ &= \alpha P(B)\sum_e P(e)\sum_a P(a|B,e)P(J|a)P(M|a)\\ &= \alpha P(B) f(J,M)\end{aligned}$$

$$\begin{aligned}P(B|J) &= \alpha P(B,J)\\ &= \sum_m \sum_a \sum_e P(B)P(e)P(a|B,e)P(J|A)P(m|a)\\ &= \alpha P(B)\sum_m \sum_e P(e)\sum_a P(a|B,e)P(J|a)P(m|a)\\ &= \alpha P(B)\sum_m f(J,m)\end{aligned}$$

$$\begin{aligned}P(B|M) &= \alpha P(B,M)\\ &= \sum_j \sum_a \sum_e P(B)P(e)P(a|B,e)P(j|A)P(M|a)\\ &= \alpha P(B)\sum_j \sum_e P(e)\sum_a P(a|B,e)P(j|a)P(M|a)\\ &= \alpha P(B)\sum_j f(j,M)\end{aligned}$$