Gröbner Fan, related ideas and an algorithm

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Abstract

In this technical report we discuss the Gröbner fan construction. We examine the connections between the Gröbner basis algorithm, monomial orderings, Euclidean geometry and linear programming used for the Gröbner fan construction. We also provide some examples and a detailed explanation of the algorithm by Mora and Robbiano [1].

Keywords— commutative algebra, computational algebraic geometry, linear programming

1 Introduction

The Gröbner fan of an ideal was introduced in [1]. This idea was motivated to study all possible reduced Gröbner basis for all monomial orders since the latter heavily depends on a particular monomial order [2] since the main component in many Gröbner basis algorithms relies on a division algorithm. A simple (and quite expected) observation notices that different monomial orders lead different Gröbner basis. Moreover, the performance of certain algorithms is faster using certain monomial orders. In addition, current complexity results [3] indicate instances where th runtime for ideals with polynomials of degree at most d is $\mathcal{O}(2^{2^d})$. Hence, it is interesting to obtain a characterization of how to transform one Gröbner basis to another with different monomial order due to the expensive nature of the algorithm. For ther latter, some techniques which are based on the Gröbner fan, like Gröbner walk, provide a solution. More efficient methods might rely on the use of Universal Gröbner basis. However, the Gröbner fan has proven to be useful in other areas of mathematics like tropical algebra, auction design and optimization problems [4].

Among of the main outcomes of [1] are:

1. There is a one-to-one correspondance between reduced marked Gröbner bases with initial ideals.
2. The set of initial ideals is finite (hence the set of all reduced marked Gröbner bases is finite too).
3. For every initial ideals in $k[x_1, \ldots, x_n]$ there is a corresponding positive vector in $\mathbb{R}^n$.

Similarly, in [5], the authors motivated the above points through three questions. An important fact mentioned in both [5, 6], which received little attention, was about the encoding of monomials orders. I consider this crucial since the main focus of our study is to characterize Gröbner bases independently of the monomial order, hence it is necessary to give a formal representation of the latter. For instance in [5] this was achieved by representing monomial orders using matrices, whereas [6] considered a recursive definition using *vectors* and the standard *dot product* in $\mathbb{Z}^n$ assuming the existance of an arbitrary term order, which tipically is the lexicographical order. In the end, both approaches are useful since the main property of these encodings is the relevance of the first row of the matrix (initial vector respectively) to define an implicit Gröbner basis, which is the so called *marked Gröbner basis*.

Given all these properties, the algorithm by Mora and Robianno [1] for computing Gröbner fans exploits the fact that the Gröbner basis are finite and so a naive algorithm of three steps will always terminate. Their approach can be studied as follows:

1. Given a set of polynomials S, compute all possible marked Gröbner bases by enumerating all possible initial monomials and filter the ones that are feasible using linear programming techniques. The enumeration is a combinatorial process of choosing a initial monomial m for each polynomial f in S and making the correspondant inequalities with each non-initial monomial in f.

2. Extend the previous marked Gröbner bases using a Gröbner basis algorithm to find new marked Gröbner basis. This step is motivated by a result in [5] which states that the Gröbner fan cover the positive orthant of $\mathbb{R}^n$. Hence, if this step cannot find more extensions it means the set of marked Gröbner basis has already covered this section of $\mathbb{R}^n$.

3. Filter the previous marked Gröbner basis by computing the reduced Gröbner basis. This step give us additionally a complete geometrically characterization of the reduced marked Gröbner bases and it is important since many of the marked Gröbner bases computed in the previous step might be the equivallent with different monomial orders.

We will discuss several examples to illustrate the above algorithm in Section 4. In Section 2 we will discuss the relevance of the monomial orders, initial monomials, and its relevance with the geometric structure that entails the finiteness of the construction [1]. In Section 3 we will discuss an implementation of the Mora and Robbiano algorithm using Python will additional support of the computer algebra system *Sage* [8] and the linear programming library scipy [9].

2 The Gröbner Fan of an ideal: A mix of commutative algebra, combinatorics, and linear programming

For this discussion, we will choose to represent monomial orders as matrices with real entries. There are different conditions for particular entries in such matrices to define a monomial order [5].

In order to compare two terms it is just necessary to use their exponents. Let M be a matrix and α, β the exponents of two polynomials. A matrix order works as follows: $\alpha \prec_M \beta$ if there exists a row vector ω in M, say the ith row, such that $\alpha \cdot \omega < \beta \cdot \omega$ and all previous rows $\omega^{'}$ in M we have that $\alpha \cdot \omega^{'} = \beta \cdot \omega^{'}$.

Example 2.0.1. *Let us consider the following matrix* $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. *This matrix encodes the* graded lexicographical order $x > y > z$ *for the polynomial ring* $k[x, y, z]$. *We can see that the first row essentially compares the total grade in the exponent of terms and the rest of the rows break tie using the lexicographic order.*

Example 2.0.2. *On the other hand, not all matrices define a monomial order. For instance, let us consider the matrix* $M^{'} = \begin{pmatrix} 1 & 2 & -1 & -2 \\ 2 & -1 & -2 & 1 \\ -1 & -2 & 1 & 2 \\ -2 & 1 & 2 & -1 \end{pmatrix}$. *We notice that multiplying* $M^{'}$ *with any vector with the same elements in each entry will give us a zero vector. Hence* $M^{'}$ *cannot distinguish these set of vectors, which is problematic according to the definition of a monomial order. In general it should be desirable that the kernel of these matrices intersecting the correspondant* $\mathbb{N}^n$ *is empty. The latter will provide injectivity to the matrix in order to distinguish different vectors. The latter property is common and shared among different monimonial order in matrices with different domains (rational, real, etc).*

We will like to study Gröbner bases without *explicitly* specifying the monomial ordering. For the latter the idea of a *marked Gröbner basis* was introduced.

Definition 2.1. *[5] A* (reduced) marked Gröbner basis *is a set of pairs of polynomials and monomials* $\{(f_i, g_i) | i = 1, \ldots, r\}$ *such that* $\{f_i | i = 1, \ldots, r\}$ *is a (reduced) Gröbner basis and* $g_i = LT_{\prec}(f_i)$ *for some monomial order* $\prec$.

The latter definition helps providing an implicit monomial ordering, i.e. the monomial order used to compute the Gröbner basis is not available. Is this information enough to uniquely determine the

[1] It is worth mentioning the set of monomial ideals is finite for the commutative case. However, for the non commutative case this construction is not finite as noted by Weispfenning in [7]

original monomial order used to compute the Gröbner basis? As shown in [5], having the information of the leading monomial we can produce a set of inequalities that, if satisfiable, the solution of such system of inequalities corresponds to the first row of matrix order.

Example 2.1.1. *We will highlight the leading monomial of a polynomial using parenthesis. Let us consider the marked Gröbner basis* $\{(xy)-z^2, (wyz)-x^3, (wy^2)-x^2z, (wz^3)-x^4\}$. *The set of inequalities entailed by the marked Gröbner basis is:*

- $(0,1,1,0)\cdot(a,b,c,d) \geq (0,0,0,2)\cdot(a,b,c,d)$
- $(1,1,1,0)\cdot(a,b,c,d) \geq (0,3,0,0)\cdot(a,b,c,d)$
- $(1,0,2,0)\cdot(a,b,c,d) \geq (0,2,0,1)\cdot(a,b,c,d)$
- $(1,0,0,3)\cdot(a,b,c,d) \geq (0,4,0,0)\cdot(a,b,c,d)$

Using a linear solver we compute a solution for the above system of inequalities:

```
>>> linprog(np.array([1, 1, 1, 1]), A_ub = np.array([[0, -1, -1, 2], [-1, 2, -1, 0], [-1, 2, -2, 1], [-1, 4, 0, -3]]), b_ub = np.array([0, 0, 0, 0]), bounds = (0.01, None))
     fun: 0.040000000000000008
 message: 'Optimization terminated successfully.'
     nit: 8
   slack: array([  0.00000000e+00,   0.00000000e+00,   2.31296463e-18,
          2.31296463e-18,   0.00000000e+00,   2.89120579e-19,
          5.78241159e-19,   0.00000000e+00])
  status: 0
 success: True
       x: array([ 0.01,  0.01,  0.01,  0.01])
```

Figure 1: The solution found by scipy.optimize.linprog [9] is $x = [0.01, 0.01, 0.01, 0.01]$

We notice then that the first row for the previous matrix order are all the same elements. Thus we can suspect an equi-graded order (i.e. an order which doesn't prefer an indeterminate while computing the total grade) is used. This phenomenon happens quite frequently, specifically with equi-graded orders. For example, consider the graded lexicographical order $x > y > z$ and the graded lexicographical order $y > z > x$. The first row of their matrix order will be the same, but the rest of their matrices will be different.

From the previous observation, if we take all the vectors in the $\mathbb{R}^n$ such that they satisfy the set of inequalities demanded by a matrix order we will obtain a *cone* in $\mathbb{R}^n$. In [5] it is shown that the geometry properties of these cones form a fan, which is a structure such that the intersection of cones (known as faces) belong to the structure as well as each of the faces of the cones belong to the structures. In hindsight, the faces of this structure correspond to elements of the cones such that the the dot product cannot distinguish between exponent vectors, hence it is important to rely on the interior points of these cones to fully distinguish/classify exponent vectors.

3 Mora and Robianno Algorithm: Discussion and Implementation in Sage

Here is an implementation [2] of algorithm by Mora and Robbiano for computing Gröbner fan of an ideal:

```
1  # 'inputBasis' is an array of polynomials
2  def groebnerFan(inputBasis):
3
4      # Initialization
5      L = ([], [], [], {}, [])
6      Lnew = [L]
7      for polynomial in inputBasis:
8          Lold = Lnew
9          Lnew = []
10         for (G, M, E, Psi, B) in Lold:
11             for leadingMonomial in polynomial.monomials():
```

[2] Comments in Python begin with #

```
12                  Gnew = G[:]
13                  Gnew.append(polynomial)
14                  Mnew = M[:]
15                  Mnew.append(leadingMonomial)
16                  Enew = E[:]
17                  for nonLeadingMonomial in polynomial.monomials():
18                      if(nonLeadingMonomial != leadingMonomial):
19                          # We substract the Leading Monomial to the
20                          # Non Leading Monomials because the LP solver
21                          # has <= as default inequalities
22                          Enew.append(subtractExponents(nonLeadingMonomial,
23                                                        leadingMonomial))
24                  Psinew = copy.deepcopy(Psi)
25                  Psinew[polynomial] = leadingMonomial
26                  Bnew = B[:]
27                  for g in G:
28                      Bnew.append((g, polynomial))
29                  if isNotEmptyTO(Enew):
30                      L = (Gnew, Mnew, Enew, Psinew, Bnew)
31                      Lnew.append(L)
32
33      # Computation of the Groebner Bases
34      Lwork = Lnew
35      Lpartial = []
36      while (Lwork != []):
37          G, M, E, Psi, B = Lwork.pop()
38          f, g = B.pop()
39          T = lcm(Psi[f], Psi[g])
40          gCoeffPsiG = g.monomial_coefficient(Psi[g])
41          fCoeffPsiF = f.monomial_coefficient(Psi[f])
42          h = gCoeffPsiG*T*f//Psi[f] - fCoeffPsiF*T*g//Psi[g]
43          check, subtract = minimalPolynomialCheck(G, Psi, h)
44          while check:
45              h = h - subtract
46              check, subtract = minimalPolynomialCheck(G, Psi, h)
47          if h == 0 :
48              if (B == []):
49                  Lpartial.append((G, M, E, Psi))
50              else:
51                  Lwork.append((G, M , E, Psi, B))
52          else:
53              for leadingMonomial in h.monomials():
54                  Gnew = G[:]
55                  Gnew.append(h)
56                  Mnew = M[:]
57                  Mnew.append(leadingMonomial)
58                  Enew = E[:]
59                  for nonLeadingMonomial in h.monomials():
60                      if(nonLeadingMonomial != leadingMonomial):
61                          # We substract the Leading Monomial to the
62                          # Non Leading Monomials because the LP solver
63                          # has <= as default inequalities
64                          Enew.append(subtractExponents(nonLeadingMonomial,
65                                                        leadingMonomial))
66                  Psinew = copy.deepcopy(Psi)
67                  Psinew[h] = leadingMonomial
68                  Bnew = B[:]
69                  for g in G:
70                      Bnew.append((g, h))
71                  if isNotEmptyTO(Enew):
```

```
72                          Lwork.append((Gnew, Mnew, Enew, Psinew, Bnew))
73
74      # Computation of the Reduced Groebner Bases
75      # and of the Groebner Region
76      Loutput = []
77      Mon = []
78      while (Lpartial != []):
79          G, M, E, Psi = Lpartial.pop()
80          if (not membershipIdealArrayTest(M, Mon)):
81              polynomial, check = reducibilityCheck(G, M, Psi)
82              while check:
83                  G.remove(polynomial)
84                  M.remove(Psi[polynomial])
85                  del Psi[polynomial]
86                  polynomial, check = reducibilityCheck(G, M, Psi)
87              for g in G:
88                  G.remove(g)
89                  M.remove(Psi[g])
90                  gnew = red(G, M, g)
91                  coeffGNew = gnew.monomial_coefficient(Psi[g])
92                  gnew = 1/coeffGNew*gnew
93                  G.append(gnew)
94                  M.append(Psi[g])
95                  tempPsiG = Psi[g]
96                  del Psi[g]
97                  Psi[gnew] = tempPsiG
98              E = []
99              for g in G:
100                 for monomial in g.monomials():
101                     if(monomial != Psi[g]):
102                         E.append(subtractExponents(Psi[g],
103                                                    monomial))
104             Loutput.append((G, M, E, Psi))
105             Mon.append(M)
106     return Loutput
```

As mentioned before, the algorithm has three main components. The implementation is straightforward once is understood the high level idea. Many additional methods where needed to implement separately in order to provide a clean design.

It might be worth mentioning the main data structure used in the algorithm. In many parts of the algorithm we will see the common decomposition of an array of elements into the tuple (G, M, E, Psi, B). These components stand for the following:

- G is the set of polynomials for a Gröbner basis.
- M is the set of monomials keeping track of the leading monomials for the respective G.
- E is the set of inequalities produced by enumerating the constraints by G and M.
- Psi is a map (dictionary structure in Python) that associates an element in G with an element in M.
- B keeps track of the pair of elements in G that have not reduced using Buchberger's criterion in order to avoid unnecessary computations.

In order to test if a set of inequalities define a monomial order (lines 29, 71) we used the linear programming library 'scipy.optimize.linprog' to find solution to the set of inequalities. We realize that the library uses non-strict inequalities and the lack of attention of the latter produced bugs in the program since the linear solver was accepting set of inequalities that do not define monomial orders. The latter was fixed by including a small epsilon value to the non-strict inequalities to convert them into strict inequalities.

Additionally, we used a theorem in [2], to be more precise a corollary about two monomial ideals being equivalent, to code line 80 in the previous implementation.

4 Some examples

First we will compute some examples using knowledge previously discussed about the Gröbner fan.

Example 4.0.1. *Lol*

5 Conclusion

Blah Blahhhhh

References

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