Gröbner Fan, related ideas and an algorithm

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December 24, 2018

Abstract

In this technical report we discuss the Gröbner fan construction. We examine the connections between the Gröbner basis algorithm, monomial orderings, Euclidean geometry and linear programming used for the Gröbner fan construction. We also provide some examples and a detailed explanation of the algorithm by Mora and Robbiano [1].

Keywords— commutative algebra, computational algebraic geometry, linear programming

1 Introduction

The Gröbner fan of an ideal was introduced in [1]. This idea was motivated to study all possible reduced Gröbner basis for all monomial orderings since the latter heavily depends on a particular monomial ordering [2] since the main component in many Gröbner basis algorithms relies on a division algorithm. A simple (and quite expected) observation notices that different monomial orderings lead different Gröbner basis. Moreover, the performance of certain algorithms is faster using certain monomial orderings. In addition, current complexity results [3] indicate instances where th runtime for ideals with polynomials of degree at most d is $\mathcal{O}(2^{2^d})$. Hence, it is interesting to obtain a characterization of how to transform one Gröbner basis to another with different monomial ordering due to the expensive nature of the algorithm. For ther latter, some techniques which are based on the Gröbner fan, like Gröbner walk, provide a solution. More efficient methods might rely on the use of Universal Gröbner basis. However, the Gröbner fan has proven to be useful in other areas of mathematics like tropical algebra, auction design and optimization problems [4].

Among of the main outcomes of [1] are:

- There is a one-to-one correspondance between reduced marked Gröbner bases with initial ideals.
- The set of initial ideals is finite (hence the set of all reduced marked Gröbner bases is finite too).
- For every initial ideals in $k[x_1, \ldots, x_n]$ there is a corresponding positive vector in $\mathbb{R}^n$.

Similarly, in [5], the authours motivated the above points throught three questions. However,

2 Gröbner Fan: A mix of commutative algebra, combinatorics, and linear programming

We will like to study Gröbner basis without *explicitly* specifying the monomial ordering, the idea of a *marked Gröbner basis* was introduced.

3 Mora and Robianno Algorithm: Discussion and Implementation in Sage

Blah blah

4 Some examples

Blah blah

5 Conclusion

Blah Blahhhhh

References

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