

Gröbner Fan, related ideas and an algorithm

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December 25, 2018

Abstract

In this technical report we discuss the Gröbner fan construction. We examine the connections between the Gröbner basis algorithm, monomial orderings, Euclidean geometry and linear programming used for the Gröbner fan construction. We also provide some examples and a detailed explanation of the algorithm by Mora and Robbiano [1].

Keywords— commutative algebra, computational algebraic geometry, linear programming

1 Introduction

The Gröbner fan of an ideal was introduced in [1]. This idea was motivated to study all possible reduced Gröbner basis for all monomial orders since the latter heavily depends on a particular monomial order [2] since the main component in many Gröbner basis algorithms relies on a division algorithm. A simple (and quite expected) observation notices that different monomial orders lead different Gröbner basis. Moreover, the performance of certain algorithms is faster using certain monomial orders. In addition, current complexity results [3] indicate instances where the runtime for ideals with polynomials of degree at most d is $\mathcal{O}(2^{2^d})$. Hence, it is interesting to obtain a characterization of how to transform one Gröbner basis to another with different monomial order due to the expensive nature of the algorithm. For the latter, some techniques which are based on the Gröbner fan, like Gröbner walk, provide a solution. More efficient methods might rely on the use of Universal Gröbner basis. However, the Gröbner fan has proven to be useful in other areas of mathematics like tropical algebra, auction design and optimization problems [4].

Among of the main outcomes of [1] are:

1. There is a one-to-one correspondance between reduced marked Gröbner bases with initial ideals.
2. The set of initial ideals is finite (hence the set of all reduced marked Gröbner bases is finite too).
3. For every initial ideals in $k[x_1, \dots, x_n]$ there is a corresponding positive vector in \mathbb{R}^n .

Similarly, in [5], the authors motivated the above points through three questions. An important fact mentioned in both [5, 6], which received little attention, was about the encoding of monomial orders. I consider this crucial since the main focus of our study is to characterize Gröbner bases independently of the monomial order, hence it is necessary to give a formal representation of the latter. For instance in [5] this was achieved by representing monomial orders using matrices, whereas [6] considered a recursive definition using *vectors* and the standard *dot product* in \mathbb{Z}^n assuming the existence of an arbitrary term order, which typically is the lexicographical order. In the end, both approaches are useful since the main property of these encodings is the relevance of the first row of the matrix (initial vector respectively) to define an implicit Gröbner basis, which is the so called *marked Gröbner basis*.

Given all these properties, the algorithm by Mora and Robbiano [1] for computing Gröbner fans exploits the fact that the Gröbner basis are finite and so a naive algorithm of three steps will always terminate. Their approach can be studied as follows:

1. Given a set of polynomials S , compute all possible marked Gröbner bases by enumerating all possible initial monomials and filter the ones that are feasible using linear programming techniques. The enumeration is a combinatorial process of choosing a initial monomial m for each polynomial f in S and making the correspondant inequalities with each non-initial monomial in f .

2. Extend the previous marked Gröbner bases using a Gröbner basis algorithm to find new marked Gröbner basis. This step is motivated by a result in [5] which states that the Gröbner fan cover the positive orthant of \mathbb{R}^n . Hence, if this step cannot find more extensions it means the set of marked Gröbner basis has already covered this section of \mathbb{R}^n .
3. Filter the previous marked Gröbner basis by computing the reduced Gröbner basis. This step give us additionally a complete geometrically characterization of the reduced marked Gröbner bases and it is important since many of the marked Gröbner bases computed in the previous step might be the equivalent with different monomial orders.

We will discuss several examples to illustrate the above algorithm in Section 4. In Section 2 we will discuss the relevance of the monomial orders, initial monomials, and its relevance with the geometric structure that entails the finiteness of the construction ¹. In Section 3 we will discuss an implementation of the Mora and Robbiano algorithm using Python will additional support of the computer algebra system *Sage* [8] and the linear programming library *scipy* [9].

2 The Gröbner Fan of an ideal: A mix of commutative algebra, combinatorics, and linear programming

For this discussion, we will choose to represent monomial orders as matrices with real entries. There are different conditions for particular entries in such matrices to define a monomial order [5].

In order to compare two terms it is just necessary to use their exponents. Let M be a matrix and α, β the exponents of two polynomials. A matrix order works as follows: $\alpha \prec_M \beta$ if there exists a row vector ω in M , say the i th row, such that $\alpha \cdot \omega < \beta \cdot \omega$ and all previous rows ω' in M we have that $\alpha \cdot \omega' = \beta \cdot \omega'$.

Example 2.0.1. Let us consider the following matrix $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. This matrix encodes the graded

lexicographical order $x > y > z$ for the polynomial ring $k[x, y, z]$. We can see that the first row essentially compares the total grade in the exponent of terms and the rest of the rows break tie using the lexicographic order.

Example 2.0.2. On the other hand, not all matrices define a monomial order. For instance, let us

consider the matrix $M' = \begin{pmatrix} 1 & 2 & -1 & -2 \\ 2 & -1 & -2 & 1 \\ -1 & -2 & 1 & 2 \\ -2 & 1 & 2 & -1 \end{pmatrix}$. We notice that multiplying M' with any vector with

the same elements in each entry will give us a zero vector. Hence M' cannot distinguish these set of vectors, which is problematic according to the definition of a monomial order. In general it should be desirable that the kernel of these matrices intersecting the correspondent \mathbb{N}^n is empty. The latter will provide injectivity to the matrix in order to distinguish different vectors. The latter property is common and shared among different monomial order in different domains (rational, real) of matrices.

We will like to study Gröbner bases without *explicitly* specifying the monomial ordering. For the latter the idea of a *marked Gröbner basis* was introduced.

Definition 2.1. [5] A (reduced) marked Gröbner basis is a set of pairs of polynomials and monomials $\{(f_i, g_i) | i = 1, \dots, r\}$ such that $\{f_i | i = 1, \dots, r\}$ is a (reduced) Gröbner basis and $g_i = LT_{\prec}(f_i)$ for some monomial order \prec .

The latter definition helps providing an implicit monomial ordering, i.e. the monomial order used to compute the Gröbner basis is not available. Is this information enough to uniquely determine the

¹It is worth mentioning the set of monomial ideals is finite for the commutative case. However, for the non commutative case this construction is not finite as noted by Weispfenning in [7]

original monomial order used to compute the Gröbner basis? As shown in [5], having the information of the leading monomial we can produce a set of inequalities that, if satisfiable, the solution of such system of inequalities corresponds to the first row of matrix order.

Example 2.1.1. *We will highlight the leading monomial of a polynomial using parenthesis. Let us consider the marked Gröbner basis $\{(xy) - z^2, (wyz) - x^3, (wy^2) - x^2z, (wz^3) - x^4\}$. The set of inequalities entailed by the marked Gröbner basis is:*

- $(0, 1, 1, 0) \cdot (a, b, c, d) \geq (0, 0, 0, 2) \cdot (a, b, c, d)$
- $(1, 1, 1, 0) \cdot (a, b, c, d) \geq (0, 3, 0, 0) \cdot (a, b, c, d)$
- $(1, 0, 2, 0) \cdot (a, b, c, d) \geq (0, 2, 0, 1) \cdot (a, b, c, d)$
- $(1, 0, 0, 3) \cdot (a, b, c, d) \geq (0, 4, 0, 0) \cdot (a, b, c, d)$

Using a linear solver we compute a solution for the above system of inequalities:

```
>>> linprog(np.array([1, 1, 1, 1]), A_ub = np.array([[0, -1, -1, 2], [-1, 2, -1, 0], [-1, 2, -2, 1], [-1, 4, 0, -3]]), b_ub = np.array([0, 0, 0, 0]), bounds = (0.01, None))
fun: 0.040000000000000008
message: 'Optimization terminated successfully.'
nit: 8
slack: array([ 0.00000000e+00,  0.00000000e+00,  2.31296463e-18,
               2.31296463e-18,  0.00000000e+00,  2.89120579e-19,
               5.78241159e-19,  0.00000000e+00])
status: 0
success: True
x: array([ 0.01,  0.01,  0.01,  0.01])
```

Figure 1: The solution found by `scipy.optimize.linprog` [9] is $x = [0.01, 0.01, 0.01, 0.01]$

We notice then that the first row for the previous matrix order are all the same elements. Thus we can suspect an equi-graded order (i.e. an order which doesn't prefer an indeterminate while computing the total grade) is used. This phenomenon happens quite frequently, specifically with equi-graded orders. For example, consider the graded lexicographical order $x > y > z$ and the graded lexicographical order $y > z > x$. The first row of their matrix order will be the same, but the rest of their matrices will be different.

From the previous observation, if we take all the vectors in the \mathbb{R}^n such that they satisfy the set of inequalities demanded by a matrix order we will obtain a *cone* in \mathbb{R}^n . In [5] it is shown that the geometry properties of these cones form a fan, which is a structure such that the intersection of cones (known as faces) belong to the structure as well as each of the faces of the cones belong to the structures. In hindsight, the faces of this structure correspond to elements of the cones such that the the dot product cannot distinguish between exponent vectors, hence it is important to rely on the interior points of these cones to fully distinguish/classify exponent vectors.

3 Mora and Robianno Algorithm: Discussion and Implementation in Sage

Here is an implementation of the Mora and Robbiano algorithm:

```
# 'inputBasis' is an array of polynomials representing
# the Groebner basis
def groebnerFan(inputBasis):
    # Initialization
    L = ([], [], [], {}, [])
    Lnew = [L]
    for polynomial in inputBasis:
        Lold = Lnew
        Lnew = []
        for (G, M, E, Psi, B) in Lold:
            for leadingMonomial in polynomial.monomials():
                Gnew = G[:]
```

```

Gnew.append(polynomial)
Mnew = M[:]
Mnew.append(leadingMonomial)
Enew = E[:]
for nonLeadingMonomial in polynomial.monomials():
    if (nonLeadingMonomial != leadingMonomial):
        # We subtract the Leading Monomial to the
        # Non Leading Monomial because the LP solver
        # has <= as default inequalities
        Enew.append(subtractExponents(nonLeadingMonomial,
                                       leadingMonomial))

Psinew = copy.deepcopy(Psi)
Psinew[polynomial] = leadingMonomial
Bnew = B[:]
for g in G:
    Bnew.append((g, polynomial))
if isEmptyTO(Enew):
    L = (Gnew, Mnew, Enew, Psinew, Bnew)
    Lnew.append(L)
# Computation of the Groebner Bases
Lwork = Lnew
Lpartial = []
while (Lwork != []):
    G, M, E, Psi, B = Lwork.pop()
    f, g = B.pop()
    T = lcm(Psi[f], Psi[g])
    gCoeffPsiG = g.monomial_coefficient(Psi[g])
    fCoeffPsiF = f.monomial_coefficient(Psi[f])
    h = gCoeffPsiG*T*f//Psi[f] - fCoeffPsiF*T*g//Psi[g]
    check, subtract = minimalPolynomialCheck(G, Psi, h)
    while check:
        h = h - subtract
        check, subtract = minimalPolynomialCheck(G, Psi, h)
    if h == 0 :
        if (B == []):
            Lpartial.append((G, M, E, Psi))
        else:
            Lwork.append((G, M, E, Psi, B))
    else:
        for leadingMonomial in h.monomials():
            Gnew = G[:]
            Gnew.append(h)
            Mnew = M[:]
            Mnew.append(leadingMonomial)
            Enew = E[:]
            for nonLeadingMonomial in h.monomials():
                if (nonLeadingMonomial != leadingMonomial):
                    # We subtract the Leading Monomial to the
                    # Non Leading Monomial because the LP solver
                    # has <= as default inequalities
                    Enew.append(subtractExponents(nonLeadingMonomial,
                                                    leadingMonomial))

            Psinew = copy.deepcopy(Psi)
            Psinew[h] = leadingMonomial
            Bnew = B[:]

```

```

        for g in G:
            Bnew.append((g, h))
        if isEmptyTO(Enew):
            Lwork.append((Gnew, Mnew, Enew, Psinew, Bnew))
# Computation of the Reduced Groebner Bases
# and of the Groebner Region
Loutput = []
Mon = []
while (Lpartial != []):
    G, M, E, Psi = Lpartial.pop()
    if (not membershipIdealArrayTest(M, Mon)):
        polynomial, check = reducibilityCheck(G, M, Psi)
        while check:
            G.remove(polynomial)
            M.remove(Psi[polynomial])
            del Psi[polynomial]
            polynomial, check = reducibilityCheck(G, M, Psi)
        for g in G:
            G.remove(g)
            M.remove(Psi[g])
            gnew = red(G, M, g)
            coeffGNew = gnew.monomial_coefficient(Psi[g])
            gnew = 1/coeffGNew*gnew
            G.append(gnew)
            M.append(Psi[g])
            tempPsiG = Psi[g]
            del Psi[g]
            Psi[gnew] = tempPsiG
    E = []
    for g in G:
        for monomial in g.monomials():
            if (monomial != Psi[g]):
                E.append(subtractExponents(Psi[g],
                                            monomial))

    Loutput.append((G, M, E, Psi))
    Mon.append(M)
return Loutput

```

4 Some examples

Blah blah

5 Conclusion

Blah Blahhhhh

References

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