THE JOURNAL OF SYMBOLIC LOGIC
VOLUME 22, Number 3, Sept. 1957

LINEAR REASONING. A NEW FORM OF THE HERBRAND-GENTZEN THEOREM.

WILLIAM CRAIG

1. Introduction. In Herbrand's Theorem [2] or Gentzen's Extended Hauptsatz [1], a certain relationship is asserted to hold between the structures of A and A′, whenever A *implies* A′ (i.e., A ⊃ A′ is valid) and moreover A is a conjunction and A′ an alternation of first-order formulas in prenex normal form. Unfortunately, the relationship is described in a roundabout way, by relating A and A′ to a quantifier-free tautology. One purpose of this paper is to provide a description which in certain respects is more direct. Roughly speaking, ascent to A ⊃ A′ from a quantifier-free level will be replaced by movement from A to A′ on the quantificational level. Each movement will be closely related to the ascent it replaces.

The new description makes use of a set L of rules of inference, the *L-rules*. L is *complete* in the sense that, if A is a conjunction and A′ an alternation of first-order formulas in prenex normal form, [1] and if A ⊃ A′ is valid, then A′ can be obtained from A by an *L-deduction*, i.e., by applications of L-rules only. The distinctive feature of L is that each L-rule possesses two characteristics which, especially in combination, are desirable. First, each L-rule yields only conclusions implied by the premisses. It thus resembles, e.g., the rule of modus ponens and differs, e.g., from certain rules for universally quantifying free variables. Second, each L-rule is a one-premiss rule. In contrast therefore to most formal deductions, which take the form of a "tree", an L-deduction *from* A *to* A′ is *linear*. It consists of a finite sequence $A_1, \ldots, A_r$, where $r \geq 2$, $A_1 = A$, $A_r = A'$, and each A_i yields A_{i+1} by one of the L-rules, so that $A_i \supset A_{i+1}$ is valid. One may therefore think of $A_2, \ldots, A_{r-1}$ as "intermediate" formulas, and of the entire L-deduction from A to A′ as an "analysis" of the implication A ⊃ A′ into a sequence of implications $A_i \supset A_{i+1}$ each of which is of a special and relatively simple kind.

Among the L-rules there is a fundamental distinction between the *equivalence rules*, which always transform a premiss into an equivalent conclusion, and the remaining or *implication rules*, which in general yield a conclusion that is weaker than the premiss. It will be shown that one can separate the use of the two kinds of rules by restricting the use of implication rules to a middle portion, and the use of equivalence rules to the two end portions, of L-deductions. Moreover, a certain symmetry with respect to the middle portion can be imposed.

Received January 21, 1957.

[1] This restriction on A and A′ can be removed by adding to L further rules for converting to and from prenex normal form.

This may help to bring about a closer accord between proof theory and model theory. It may also give further insight into the structure of the Lindenbaum algebra of first-order predicate calculus. For during the end portions of such L-deductions only the expressions are changed, never the set of models being expressed. Any changing of the set of models must occur during the middle portion and moreover must be a widening. Also, a regular pattern can be imposed on these widenings, since as it turns out there are only three implication rules, since the first and third of these are in a sense duals of each other, and since the use of the second and third rule can be forced to follow that of the first or second rule respectively.

2. *L*-deduction. The present section serves mainly as a preparation for § 4, where the principal results will be proved. It may also be of interest in its own right. For, in view of Theorem 2 below, the set of L-rules, to be described presently, when enlarged by further rules for converting to and from prenex normal form, may be regarded by itself as a complete formulation of first-order logic. It is a formulation without theorems, which nevertheless establishes in the form of sequences all valid implications. Extralogical or mathematical first-order systems can then be formulated by simply adding to this set of rules a conjunction of the appropriate extralogical axioms. The distinction between axioms and rules then corresponds exactly to the distinction between the assumptions and the apparatus for deriving their first-order consequences. A formulation of this kind seems therefore natural, if one regards logic as a tool and hence logical theory primarily as a study of valid implications rather than of what is "logically true". As the referee points out, a more conventional formulation of first-order logic is obtained by adding to this set of rules the single axiom $Fx \vee \neg Fx$.

The formulas to be considered for our purpose may be those of any system of first-order predicate calculus in which there are symbols $\vee$, $.$, $\exists$, and $\forall$, and in which vacuous quantification is allowed. In addition to predicate symbols of $n \geq 0$ arguments, formulas may contain individual constants and function symbols of $n \geq 1$ arguments. Each argument of a predicate or function symbol shall be an *individual term*. The symbols A, B, C, ... shall range over arbitrary formulas, M over matrices, i.e., quantifier-free formulas, (P) and (Q) over prefixes, including the empty prefix, t over individual terms, and x, y, z, ... over individual variables. Applying subscripts or superscripts to any of these symbols shall leave the range unchanged. In the list below, l ranges over the positive integers, and the part indicated by ... is the same in premiss and conclusion.

L-rules

L1a. Duplication.

$$\frac{(P)[(Q)^1M^1 \,.\, \dots \,.\, (Q)^kM^k \,.\, \dots \,.\, (Q)^lM^l]}{(P)[(Q)^1M^1 \,.\, \dots \,.\, (Q)^kM^k \,.\, (Q)^kM^k \,.\, \dots \,.\, (Q)^lM^l]}.$$

L1b. Simplication. [2]

$$\frac{(P)[(Q)^1M^1 \vee \dots \vee (Q)^kM^k \vee (Q)^kM^k \vee \dots \vee (Q)^lM^l]}{(P)[(Q)^1M^1 \vee \dots \vee (Q)^kM^k \vee \dots \vee (Q)^lM^l]}.$$

L2a. $\exists$-exportation.

$$\frac{(P)[(Q)^1M^1 \,.\, \dots \,.\, \exists x(Q)^kM^k(x) \,.\, \dots \,.\, (Q)^lM^l]}{(P)\exists y[(Q)^1M^1 \,.\, \dots \,.\, (Q)^kM^k(y) \,.\, \dots \,.\, (Q)^lM^l]}$$

L2b. $\forall$-importation.

$$\frac{(P)\forall y[(Q)^1M^1 \vee \dots \vee (Q)^kM^k(y) \vee \dots \vee (Q)^lM^l]}{(P)[(Q)^1M^1 \vee \dots \vee \forall x(Q)^kM^k(x) \vee \dots \vee (Q)^lM^l]}$$

where (α) $(Q)^kM^k(y)$ is the result of substituting y for the free occurrences of x in $(Q)^kM^k(x)$, (β) y is free at the free occurrences of x in $(Q)^kM^k(x)$, and (γ) y does not occur free in $(Q)^jM^j$, $j \neq k$, nor in $\exists x(Q)^kM^k(x)$ (nor in $\forall x(Q)^kM^k(x)$).

L3a. $\forall$-exportation.

Same as *L2a*, with $\forall x$ and $\forall y$ in place of $\exists x$ and $\exists y$ respectively.

L3b. $\exists$-importation.

Same as *L2b*, with $\exists x$ and $\exists y$ in place of $\forall x$ and $\forall y$ respectively.

L4a. $\forall$-vacuous-introduction.

$$\frac{(P)'(P)''M}{(P)'\forall y(P)''M}$$

L4b. $\exists$-vacuous-removal.

$$\frac{(P)'\exists y(P)''M}{(P)'(P)''M}$$

where y does not occur free in $(P)''M$.

[2] This name was suggested by H. B. Curry.

L5a. *∀-instantiation.*

$$\frac{(P)'\forall y(P)''M(y)}{(P)'(P)''M(t)}$$

L5b. *∃-generalization.*

$$\frac{(P)'(P)''M(t)}{(P)'\exists y(P)''M(y)}$$

where (α) (P)″M(t) is the result of substituting t for the free occurrences of y in (P)″M(y), and (β) t is free at the free occurrences of y in (P)″M(y).

L6. *Matrix change.*

$$\frac{(P)M}{(P)M'}$$

where $M \supset M'$ is tautologous.

Of these rules, *L1a* to *L4b* are equivalence rules, and *L5a*, *L5b*, and *L6* implication rules. Thus any weakening from premiss to conclusion in an *L*-deduction is due to either a change from "all" to "this particular", or a change from "this particular" to "at least one", or a matrix change. There is a certain duality between *a*-rules and *b*-rules. If A_i and A_{i+1} contain no $\supset$ or $\equiv$, and if A_i^* and A_{i+1}^* are the dual formulas of A_i and A_{i+1} respectively, obtained in the usual manner by interchanging ∃ and ∀ etc., then A_i yields A_{i+1} by an *a*-rule if and only if A_{i+1}^* yields A_i^* by the corresponding *b*-rule.

LEMMA 1. *If* A_i *yields* A_{i+1} *by an L-rule, then* $\exists yA_i$ *yields* $\exists yA_{i+1}$ *and* $\forall yA_i$ *yields* $\forall yA_{i+1}$ *by the same L-rule.*

LEMMA 2. *If* A_i *yields* A_{i+1} *by an L-rule other than ∃-generalization (∀-instantiation) or matrix change and if* y *occurs free in* A_i (*in* A_{i+1}), *then* y *occurs free in* A_{i+1} (*in* A_i).

There is sometimes more than one way of interpreting a step from A_i to A_{i+1}. For example, let $A_i = A_{i+1} = \exists u\forall v\exists wRuvw$. Then each of *L2a*, *L2b*, *L3a*, *L3b*, and *L6* yields A_{i+1} from A_i. Moreover, even if it is assumed, for example, that A_i yields A_{i+1} by an ∃-exportation, then one may still consider either ∃u or ∃w as the ∃x of the scheme. Again, in those cases of ∀-instantiation (∃-generalization) where y does not occur free in (P)″M(y), so that a vacuous quantifier is removed (introduced), t may be interpreted as any individual term satisfying (β). Also, in some cases of Duplication or Simplication the *k* of the scheme may be interpreted in several ways. This multiple interpretability gives rise to certain difficulties of exposition. To avoid these, we shall admit from now on for any step in an *L*-deduction only one interpretation of one scheme. For this purpose, an *L*-deduction will be regarded as a sequence of formulas that is supplemented when necessary by explanations,[3] although it does not always matter what the specific explanations are.

With this understanding, some further terminology can now be introduced. Any application of an L-rule shall be an *L-operation*. Any application of an L-rule other than *L1a*, *L1b*, and *L6* shall be a *quantifier change*. Any application of *L1a*, *L2a*, or *L3a* (*L1b*, *L2b*, or *L3b*) shall be an *assembling* (*disassembling*) *operation*. In the case of an assembling or disassembling operation, the possibly empty part which occurs to the left of the part indicated in the scheme by square brackets, and in the case of another L-operation the prefix itself, shall be the *main prefix* of the premiss or conclusion respectively. The remaining part shall be the *main conjunction* or the *main alternation* respectively, consisting of $l \geq 1$ *terms*. Each assembling or disassembling operation shall *operate on* the k-th term, as indicated by the scheme, of the main conjunction or alternation respectively of the premiss. *The quantifier occurrence associated with* a quantifier change shall be the occurrence of the quantifier $\exists$y or $\forall$y indicated by the scheme (there may be other occurrences), and $\exists$y or $\forall$y shall be *the quantifier of* the quantifier change. *The individual term of* an $\forall$-instantiation ($\exists$-generalization) shall be the term indicated by t in the scheme, and an *instantiating occurrence* (*occurrence being generalized*) shall be an occurrence in the conclusion (premiss) of t where the y of the scheme occurs in the premiss (conclusion).

Suppose A_i yields A_{i+1} by an equivalence L-rule. Then to any free occurrence of an individual term t in A_i there correspond in an obvious manner either one or two free occurrences of t in A_{i+1} in case of Duplication and exactly one free occurrence of t otherwise. Any free occurrence of t in A_{i+1} thus corresponds to either one or two free occurrences of t in A_i in case of Simplication and to exactly one free occurrence of t in A_i otherwise. If $\mathrm{A}_p, \ldots, \mathrm{A}_q$, $p \leq q$, is a segment of an L-deduction in which only equivalence L-rules are used, then a free occurrence of an individual term in A_p and one in A_q shall be *traces* of each other if and only if either $p = q$ and the two occurrences are identical or $p < q$ and the occurrence in A_q corresponds in the manner just described to a trace in A_{q-1} of the occurrence in A_p.

LEMMA 3. *Suppose* A_i *yields* A_{i+1} *by an equivalence L-rule, and suppose certain free occurrences of* t *in* A_i (*in* A_{i+1}) *are given. Then the traces in* A_{i+1} (*in* A_i) *of the given occurrences are free occurrences of* t. *Moreover, if* t′ *is free at the given occurrences of* t *in* A_i (*in* A_{i+1}), *then* t′ *is free at the traces in* A_{i+1} (*in* A_i) *of the given occurrences.*

---

[3] Alternatively, one may enlarge the symbolism of first-order predicate calculus to incorporate these explanations into the system. For example, one may always use a special set of brackets to mark off that part of a formula which is to the right of the intended main prefix (see below).

LEMMA 4. *Let* $A_p, \ldots, A_q$ *be an L-deduction in which only equivalence rules are used, suppose certain free occurrences of* t *in* A_p (*in* A_q) *are given, suppose that* t′ *is free at the given occurrences of* t *in* A_p (*in* A_q), *and let* $A_i(t')$ *and* $A_{i+1}(t')$ *be obtained by replacing in* A_i *and* A_{i+1} *respectively the traces of the given occurrences by* t′, $p \leq i < q$. *Then* $A_i(t')$ *yields* $A_{i+1}(t')$ *by the same L-rule by which* A_i *yields* A_{i+1}, $p \leq i < q$.

PROOF. The traces in A_i and A_{i+1} of the given occurrences of t in A_p (in A_q) are traces of each other. Moreover, by Lemma 3 and induction, these traces in A_i and A_{i+1} are free occurrences of t, and t′ is free at these traces. That $A_i(t')$ yields $A_{i+1}(t')$ by the same L-rule by which A_i yields A_{i+1} can then be verified for each of the eight equivalence rules.

Consider the L-deduction ∃x∀yRxy, ∃z∀yRzy, ∃z∀y(Rzy ∨ Fx). The first step is either an ∃-exportation with ∃z as the main prefix in the conclusion or an ∃-importation with an empty main prefix in the conclusion, while the second step is a matrix change with ∃z∀y as the main prefix of the premiss. In either case, therefore, the formula ∃z∀yRzy of the L-deduction, as conclusion, has one main prefix and, as premiss, has a different main prefix. We shall be concerned with L-deductions where this kind of situation does not arise. We shall say of an L-deduction that *the coupled main prefixes match*, if and only if, when a formula is both the conclusion of one step and the premiss of the next step in the L-deduction, then its main prefix in both cases is the same. [4] If the coupled main prefixes match, then one can talk about *the* main prefix of a formula A_i of an L-deduction, provided that a particular occurrence of A_i in the sequence constituting the L-deduction is understood.

Consider any L-deduction $A_1, \ldots, A_r$ with matching coupled main prefixes. Then to any quantifier occurrence in the main prefix of A_{i+1} (of A_i), $1 \leq i < r$, except to the occurrence associated with an exportation, ∀-vacuous-introduction, or ∃-generalization (importation, ∃-vacuous-removal, or ∀-instantiation), there corresponds in an obvious manner exactly one occurrence of the same quantifier in the main prefix of A_i (of A_{i+1}). Using this correspondence, we shall now define a certain class of sequences. Any quantifier occurrence in the main prefix of any A_i, $1 < i \leq r$, such that no quantifier occurrence in the main prefix of A_{i-1} corresponds to it in the manner just described, and also any quantifier occurrence in the main prefix of A_1, shall be the first term of a sequence of the class to be defined. Now suppose the n-th term of a sequence of this class is the occurrence of a quantifier in the main prefix of some A_j, $j \leq r$; then either there corresponds to it, in the manner just described, a quantifier occurrence in the main prefix of A_{j+1} and this occurrence shall be

[4] The formula is allowed to occur elsewhere in the L-deduction with a different main prefix.

the $(n+1)$-st term of the sequence, or else the sequence shall terminate. Any sequence of this class shall be a *main prefix chain* or, more briefly, a *chain*. The consecutive terms of a chain therefore are quantifier occurrences in the main prefix of consecutive formulas of the L-deduction. A segment $A_i, \ldots, A_j$ $(i \leq j)$ of an L-deduction and a main prefix chain shall be *coextensive* if and only if the first term of the chain is an occurrence in A_i and the last term an occurrence in A_j. Evidently any given chain consists of occurrences of only one quantifier, so that one can talk about *the* quantifier and *the* variable of the chain. Also evidently any given quantifier occurrence in the main prefix of any A_i, $1 \leq i \leq r$, belongs to exactly one chain, so that one can talk about *the* chain to which the occurrence belongs.

For example, consider the following L-deduction in which the coupled main prefixes are assumed to match:

$$A_1 = \forall x(\exists y Fy \,.\, \forall x Gx) \qquad A_4 = \exists z \forall x(Fz \,.\, Gx)$$

$$A_2 = \forall x \exists z(Fz \,.\, \forall x Gx) \qquad A_5 = \exists z \forall x Gx$$

$$A_3 = \forall x \exists z \forall x(Fz \,.\, Gx) \qquad A_6 = \exists z \forall y Gy.$$

Then the main prefix chains in $A_1, \ldots, A_6$ are a chain coextensive with A_1, A_2, A_3 whose quantifier is $\forall x$, a chain coextensive with $A_2, \ldots, A_6$ whose quantifier is $\exists z$, and a chain coextensive with A_3, A_4, A_5 whose quantifier is $\forall x$. Since the main prefix of A_5, as conclusion of a matrix change, is $\exists z \forall x$ and since coupled main prefixes are assumed to match, the main prefix of A_5 as a premiss is also $\exists z \forall x$. Then A_5 can yield A_6 only by $\forall$-importation, and not by $\forall$-exportation. Then $\forall y$ does not belong to the main prefix of A_6, and its occurrence in A_6 belongs to no chain.

LEMMA 5. *If A_i yields A_{i+1} by an L-rule, if A_i^- and A_{i+1}^- are obtained from A_i or A_{i+1} respectively by deleting from the main prefix one quantifier occurrence, if the two deleted occurrences belong to the same chain in A_i, A_{i+1}, and if the variable of this chain has no other occurrence in the main prefix of A_i or A_{i+1},*[5] *then A_i^- yields A_{i+1}^- by the same L-rule.*

A property of chains is the following. Consider any L-deduction $A_1, \ldots, A_r$ with matching coupled main prefixes, and consider any two distinct main prefix chains containing an occurrence both in the main prefix of A_i and in the main prefix of A_j. Then both occurrences belonging to one of the two chains are in the main prefix of A_i or A_j respectively to the left of the occurrences belonging to the other chain. More briefly, chains never "cross". If two chains contain occurrences in the main prefix of a common A_i, one may therefore talk of one *chain* as being *to the left* of the other, and the other as being *to the right* of the first. If and only if there is a common A_i, we shall call two chains *comparable*. In the above example, any two chains are

[5] This condition is required only for the case of *L4a* to *L5b*.

comparable, the chain coextensive with A_1, A_2, A_3 being to the left of the two others, and the chain coextensive with $A_2, \ldots, A_6$ being to the left of the chain coextensive with A_3, A_4, A_5.

Another property of chains is the following. In any L-deduction with matching coupled main prefixes the quantifier occurrence associated with an exportation, $\forall$-vacuous-introduction, or $\exists$-generalization (importation, $\exists$-vacuous-removal, or $\forall$-instantiation) is always the first (last) term of a chain. The quantifier change will be said to *initiate* (*terminate*) this chain. Conversely, unless the segment of the L-deduction coextensive with a chain includes the first (last) formula of the L-deduction, the chain is initiated (terminated) by an exportation, $\forall$-vacuous-introduction, or $\exists$-generalization (importation, $\exists$-vacuous-removal, or $\forall$-instantiation). A chain shall be *bracketed* if and only if it is both initiated and terminated by a quantifier change. The initiating and the terminating quantifier change of a bracketed chain shall be said to *complement* each other or also to form a *pair*. Evidently, for all chains in an L-deduction to be bracketed it is necessary and sufficient that the main prefix of the first and that of the last formula be empty. In that case, each quantifier change has a complementary change.

We now come to our three main concepts regarding L-deductions. An L-deduction shall be *Q-pure* if and only if no two chains in it have the same variable. An L-deduction shall be *Q-balanced*, if and only if the coupled main prefixes match, all chains are bracketed, and the following conditions are satisfied:

(i) Each $\exists$- or $\forall$-exportation ($\forall$- or $\exists$-importation) is complemented by an $\exists$-vacuous-removal or $\forall$-instantiation ($\forall$-vacuous-introduction or $\exists$-generalization) respectively, and vice versa.

(ii) If an $\forall$-exportation ($\exists$-importation) precedes another, then the complementary change of the first precedes the complementary change of the second.

An L-deduction shall be *symmetric* if and only if the order in which the different kinds of L-rules are applied satisfies the following conditions:

(iii) There is at least one matrix change.

(iv) Any $\forall$-instantiation ($\exists$-generalization) precedes (follows) any matrix change.

(v) Any $\forall$-vacuous-introduction ($\exists$-vacuous-removal) precedes (follows) any $\forall$-instantiation ($\exists$-generalization) and any matrix change.

(vi) Any assembling (disassembling) operation precedes (follows) any other operation.

For symmetric L-deductions, condition (ii) of Q-balance may be replaced by the condition that the $\forall$-instantiations ($\exists$-generalizations) proceed from left to right (right to left) in the following sense:

(ii)* The chain terminated (initiated) by an earlier $\forall$-instantiation ($\exists$-generalization) is to the left (right) of the chain terminated (initiated) by a later $\forall$-instantiation ($\exists$-generalization).

In an L-deduction which is symmetric and Q-balanced, the chains which are either initiated by an $\forall$-exportation or terminated by an $\exists$-importation shall be called *transient*, since they contain no occurrence in the main prefix of a premiss or conclusion of a matrix change.

In a symmetric L-deduction phases may be distinguished. The *assembling* (*disassembling*) *phase* shall be the largest initial (terminal) segment of the L-deduction such that any formula in the segment is obtained from its predecessor (yields its successor), if any, by an assembling (disassembling) operation.

3. *H*-deduction. The Herbrand-Gentzen Theorem can be stated by means of a set H of rules of inference, the *H-rules*. The conventions for the following list of H-rules shall be the same as for the earlier list of L-rules.

H-rules.

H1a. **A**-*simplication.*

$$\frac{[(Q)^1M^1 \,.\, \ldots \,.\, (Q)^kM^k \,.\, (Q)^kM^k \,.\, \ldots \,.\, (Q)^lM^l] \supset B'}{[(Q)^1M^1 \,.\, \ldots \,.\, (Q)^kM^k \,.\, \ldots \,.\, (Q)^lM^l] \supset B'.}$$

H1b. **C**-*simplication.*

$$\frac{B \supset [(Q)^1M^1 \vee \ldots \vee (Q)^kM^k \vee (Q)^kM^k \vee \ldots \vee (Q)^lM^l]}{B \supset [(Q)^1M^1 \vee \ldots \vee (Q)^kM^k \vee \ldots \vee (Q)^lM^l].}$$

H2a. $\exists$**A**-*introduction.*

$$\frac{[(Q)^1M^1 \,.\, \ldots \,.\, (Q)^kM^k(y) \,.\, \ldots \,.\, (Q)^lM^l] \supset B'}{[(Q)^1M^1 \,.\, \ldots \,.\, \exists x(Q)^kM^k(x) \,.\, \ldots \,.\, (Q)^lM^l] \supset B'}$$

H2b. $\forall$**C**-*introduction.*

$$\frac{B \supset [(Q)^1M^1 \vee \ldots \vee (Q)^kM^k(y) \vee \ldots \vee (Q)^lM^l]}{B \supset [(Q)^1M^1 \vee \ldots \vee \forall x(Q)^kM^k(x) \vee \ldots \vee (Q)^lM^l]}$$

where (α) $(Q)^kM^k(y)$ is the result of substituting y for the free occurrences of x in $(Q)^kM^k(x)$, (β) y is free at the free occurrences of x in $(Q)^kM^k(x)$, and (γ) y does not occur free in the conclusion.

H3a. $\forall$**A**-*introduction.*

$$\frac{[(Q)^1M^1 \,.\, \ldots \,.\, (Q)^kM^k(t) \,.\, \ldots \,.\, (Q)^lM^l] \supset B'}{[(Q)^1M^1 \,.\, \ldots \,.\, \forall x(Q)^kM^k(x) \,.\, \ldots \,.\, (Q)^lM^l] \supset B'}$$

H3b. $\exists$**C**-*introduction.*

$$\frac{B \supset [(Q)^1M^1 \vee \ldots \vee (Q)^kM^k(t) \vee \ldots \vee (Q)^lM^l]}{B \supset [(Q)^1M^1 \vee \ldots \vee \exists x(Q)^kM^k(x) \vee \ldots \vee (Q)^lM^l]}$$

where (α) $(Q)^k M^k(t)$ is the result of substituting t for the free occurrences of x in $(Q)^k M^k(x)$, and (β) t is free at the free occurrences of x in $(Q)^k M^k(x)$.

An *H-deduction* shall be any finite sequence of formulas in which the first formula is a tautologous matrix and in which each formula except the last yields its successor by an H-rule. It constitutes a derivation or H-deduction *of* the last formula. As in the case of an L-deduction, we shall assume that an H-deduction is supplemented by remarks, so that each step is an application of one scheme in one particular way. Remarks are necessary in the case where a vacuous quantifier is introduced and in some cases of **A**- or **C**-simplication.

With this understanding, some further terminology can be used. Any application of an H-rule shall be an *H-operation*. Any application of an *a*-rule (*b*-rule), which affects the antecedent (consequent) only, shall be an **A**-*operation* (**C**-*operation*). Any application of *H2a* to *H3b* shall be a *quantifier introduction*. Each H-operation shall *operate on* the k-th term of the antecedent (consequent) of the premiss, as indicated by the scheme. *The occurrence* of a quantifier *introduced* by a quantifier introduction shall be the occurrence of $\exists$x or $\forall$x indicated by the scheme, and *the quantifier introduced* shall be $\exists$x or $\forall$x respectively. *The individual variable* or *the individual term of* a quantifier introduction shall be the y or t respectively indicated by the scheme. Evidently, in an H-deduction no y is the variable or more than one non-vacuous $\exists$**A**- or $\forall$**C**-introduction. By assuming a suitable choice of the individual variables of vacuous $\exists$**A**- and $\forall$**C**-introductions, we shall assume from now on that each H-deduction satisfies the stronger condition that no y is the variable of more than one $\exists$**A**- or $\exists$**C**-introduction, non-vacuous or vacuous.

HERBRAND-GENTZEN THEOREM. *If* $A \supset A'$ *is a theorem of first-order predicate calculus and if* A *is a conjunction and* A′ *an alternation of prenex normal forms, then there is an H-deduction of* $A \supset A'$.

(This version of the Theorem can be obtained from Theorem 50 of [3] as follows: First, Theorem 50 can be strengthened by adding the assertion that no Thinning occurs between midsequent and endsequent, since any Thinning in this portion of a pure variable proof can be moved upward, and since any Thinning introducing a formula containing quantifiers can then be replaced by a Thinning introducing a formula without quantifiers, followed by quantifier-introductions. Second, use of the rule of interchange can be avoided, by disregarding the position of a formula in the antecedent or succedent. Third, as can be shown by induction, the two rules of Contraction can then be restricted again to apply only to adjacent formulas in the antecedent or succedent respectively. Fourth, the requirement that no variable occurs both free and bound in the end sequent can then be dropped, provided that the assertion that the proof is a pure variable proof is also dropped. Finally, the notation can be changed to that of the present version.)

4. Relationship between *H*- and *L*-deductions. We shall say that the *pattern* of a symmetric *L*-deduction *reflects* that of an *H*-deduction and conversely, if and only if between the assembling and disassembling operations of the *L*-deduction and the operations of the *H*-deduction there is a one-to-one correspondence satisfying the following conditions:

(vii) Each Duplication, ∃-exportation, and ∀-exportation (Simplication, ∀-importation, and ∃-importation) corresponds to an **A**-simplication, ∃**A**-introduction, or ∀**A**-introduction (**C**-simplication, ∀**C**-introduction, or ∃**C**-introduction) respectively, and vice versa.

(viii) If an assembling (disassembling) operation precedes (follows) another, then the **A**-operation (**C**-operation) corresponding to the first follows that corresponding to the second.

Evidently between an *L*-deduction and an *H*-deduction there is at most one correspondence satisfying these conditions, so that one can talk about *the* **A**-operation (**C**-operation) corresponding to an assembling (disassembling) operation, and vice versa. Between the assembling (disassembling) operations and the **A**-operations (**C**-operations) this correspondence is, according to (viii), *order-inverting* (*order-preserving*).

A symmetric and Q-balanced *L*-deduction and an *H*-deduction shall be respectively an *L-transform* and an *H-transform* of the other, if and only if, when the *L*-deduction is from A to A′, then the last formula of the *H*-deduction is A ⊃ A′, the pattern of the *L*-deduction reflects that of the *H*-deduction, and the following conditions hold:

(ix) If an assembling (disassembling) operation operates on the k-th term of the main conjunction (main alternation), then the corresponding **A**-operation (**C**-operation) operates on the k-th term of the antecedent (consequent).

(xa) The variable of the chain initiated (terminated) by an ∃-exportation (∀-importation) is the variable of the corresponding ∃**A**-introduction (∀**C**-introduction).

(xb) The individual term of an ∀-instantiation (∃-generalization) is the individual term of the ∀**A**-introduction (∃**C**-introduction) which corresponds to the complementary ∀-exportation (∃-importation).

(xi) Of two chains which are comparable, that one is to the left of the other whose exportation initiating or importation terminating it corresponds to a later *H*-operation. [6]

The following remarks may help in the understanding of this notion. Consider any symmetric, Q-balanced, and Q-pure *L*-deduction and any *H*-deduction such that the two are transforms of each other. Then by

[6] This condition is already implied by the others except in cases where one of the two chains is initiated (terminated) by an ∃-exportation (∀-importation) and the other terminated (initiated) by an importation (exportation).

(vii) to (xa) and induction, starting with the first (last) formula of the L-deduction, the following holds:

(ix)* The main conjunction (main alternation) of the premiss or conclusion (conclusion or premiss) respectively of an assembling (disassembling) operation has the same number of terms as the antecedent (consequent) of the conclusion or premiss respectively of the corresponding **A**-operation (**C**-operation); moreover, for each m, the m-th term of this conjunction (alternation) yields the m-th term of this antecedent (consequent) by replacing all free occurrences of the variable of any chain initiated (terminated) by an $\forall$-exportation ($\exists$-importation) by the individual term of the complementary $\forall$-instantiation ($\exists$-generalization). [7]

Now by (xb), the $\forall$-instantiations (the inverses of the $\exists$-generalizations) provide precisely the replacements needed to obtain from the main conjunction (main alternation) of the conclusion (premiss) of the last (first) assembling (disassembling) operation the antecedent (consequent) of the premiss of the corresponding first **A**-operation (**C**-operation). Hence if this antecedent (consequent) is M (M′), so that the initial tautologous matrix of the H-deduction is $\mathrm{M} \supset \mathrm{M}'$, then the premiss of the first and the conclusion of the last matrix change are respectively (P)M and (P)M′ for some (P).

It seems natural then to think of each assembling or disassembling operation, *together* with its complementary operation if it is a quantifier change, as *counterpart* of the corresponding H-operation, and vice versa. The matrix changes from (P)M to (P)M′ may then be regarded as counterpart of the initial tautology $\mathrm{M} \supset \mathrm{M}'$.

THEOREM 1. *For any H-deduction one can construct a symmetric, Q-balanced, and Q-pure L-transform.*

PROOF. Let $\mathrm{B}_0 \supset \mathrm{B}_0', \ldots, \mathrm{B}_s \supset \mathrm{B}_s'$ be any H-deduction. Then the sequence $\mathrm{B}_0, \mathrm{B}_0'$ is an L-deduction, since B_0 yields B_0' by a matrix change. The L-deduction is trivially symmetric, Q-balanced, and Q-pure. Also, it is trivially an L-transform of the H-deduction $\mathrm{B}_0 \supset \mathrm{B}_0'$. Furthermore it trivially satisfies:

(xii) The variable of any transient chain is distinct from that of any other and from the variable of any $\exists$**A**- or $\forall$**C**-introduction in $\mathrm{B}_0 \supset \mathrm{B}_0', \ldots, \mathrm{B}_s \supset \mathrm{B}_s'$.

Now consider any $m < s$ and assume as inductive hypothesis that an L-transform $\mathrm{A}_1 = \mathrm{B}_m, \mathrm{A}_2, \ldots, \mathrm{A}_r = \mathrm{B}_m'$ of $\mathrm{B}_0 \supset \mathrm{B}_0', \ldots, \mathrm{B}_m \supset \mathrm{B}_m'$ has been constructed such that $\mathrm{A}_1, \ldots, \mathrm{A}_r$ satisfies (xii) and also is symmetric and Q-balanced. We shall now construct an L-transform of $\mathrm{B}_0 \supset \mathrm{B}_0', \ldots,$

[7] This condition could replace Condition (ix) and the condition that when the L-deduction is from A to A′ then the last formula of the H-deduction is $\mathrm{A} \supset \mathrm{A}'$.

$B_{m+1} \supset B'_{m+1}$ which satisfies (xii) and is symmetric and Q-balanced.[8] Since all variables of distinct $\exists$A- and $\forall$C-introductions in an H-deduction are assumed to be distinct, it follows from (xii) and (xa) that the L-transform thus constructed is Q-pure.

CASE 1a: $B_m \supset B'_m$ yields $B_{m+1} \supset B'_{m+1}$ by *HIa*, **A**-simplication. Then the sequence $B_{m+1}, A_1, \ldots, A_r = B'_m = B'_{m+1}$ has the desired properties. By the inductive hypothesis, it is an L-deduction since B_{m+1} yields $B_m = A_1$ by Duplication. Also, adjoining a Duplication as a first step to an L-deduction does not destroy symmetry or Q-balance.[9] Further, the pattern of $B_{m+1}, A_1, \ldots, A_r = B'_{m+1}$ reflects that of $B_0 \supset B'_0, \ldots, B_{m+1} \supset B'_{m+1}$, with the Duplication from B_{m+1} to A_1 corresponding to the **A**-simplication from $B_m \supset B'_m$ to $B_{m+1} \supset B'_{m+1}$. Finally, conditions (x), (xi) and (xii) are preserved, and (ix) holds for a suitable interpretation of the Duplication from B_{m+1} to A_1.

CASE 1b: $B_m \supset B'_m$ yields $B_{m+1} \supset B'_{m+1}$ by **C**-simplication. Then the sequence $B_{m+1} = B_m = A_1, \ldots, A_r, B'_{m+1}$ has the desired properties. The proof is similar.

CASE 2a: $B_m \supset B'_m$ yields $B_{m+1} \supset B'_{m+1}$ by $\exists$**A**-introduction. Let y be the variable of the $\exists$**A**-introduction and let A_q be the first formula in $A_1, \ldots, A_r$ belonging to the disassembling phase. It will now be shown that the sequence $B_{m+1}, \exists y A_1, \ldots, \exists y A_q, A_q, \ldots, A_r = B'_m = B'_{m+1}$ has the desired properties.

The step from B_{m+1} to $\exists y A_1 = \exists y B_m$ is an $\exists$-exportation, conditions (α) to (γ) of this $\exists$-exportation being implied by conditions (α) to (γ) of the $\exists$**A**-introduction.[10] By Lemma 1, each $\exists y A_i$ yields $\exists y A_{i+1}$ by the same L-rule by which A_i yields A_{i+1}. By (γ) of $\exists$**A**-introduction, y does not occur free in $A_r = B'_m = B'_{m+1}$ and hence, by Lemma 2, does not occur free in A_q. Hence $\exists y A_q$ yields A_q by $\exists$-vacuous-removal. It follows from the inductive hypothesis that the sequence $B_{m+1}, \exists y A_1, \ldots, \exists y A_q, A_q, \ldots, A_r$ is an L-deduction. By the choice of q it is symmetric. Also it is Q-balanced, with the $\exists$-exportation from B_{m+1} to $\exists y A_1$ and the $\exists$-vacuous-removal from $\exists y A_q$ to A_q forming the only new pair of quantifier changes. The pattern of this L-deduction reflects that of $B_0 \supset B'_0, \ldots, B_{m+1} \supset B'_{m+1}$, with the $\exists$-exportation from B_{m+1} to $\exists y B_m$ corresponding to the $\exists$**A**-introduction from $B_m \supset B'_m$ to $B_{m+1} \supset B'_{m+1}$ and with each assembling (disassembling) operation from $\exists y A_i$ to $\exists y A_{i+1}$ (from A_i to A_{i+1}) corresponding to the same **A**-operation (**C**-operation) to which the operation from A_i to A_{i+1} in the L-deduction $A_1, \ldots, A_r$ corresponds. Finally, (ix), (xb),

[8] Variations in this construction and, with these, variations of the notion of L-transform and of the L-rules are possible.

[9] Here, and in many places later in the proof, the proviso "for a suitable interpretation of the L-deduction" should be added.

[10] If B_{m+1} is in prenex normal form, its main prefix is regarded as empty.

and (xii) hold, (xi) is also satisfied by the new chain, and (xa) is satisfied by the choice of y.

CASE 2b: $B_m \supset B'_m$ yields $B_{m+1} \supset B'_{m+1}$ by $\forall$C-introduction. Let y be the variable of the $\forall$C-introduction and let A_p be the last formula in $A_1, \ldots, A_r$ belonging to the assembling phase. Then the sequence $B_{m+1} = B_m = A_1, \ldots, A_p, \forall y A_p, \ldots, \forall y A_r, B'_{m+1}$ has the desired properties. The proof is similar to that for Case *2a*.

CASE 3a: $B_m \supset B'_m$ yields $B_{m+1} \supset B'_{m+1}$ by $\forall$A-introduction. Let t be the individual term of the $\forall$A-introduction, let the k-th term of B_m be the term operated on, let this term be the formula $\forall x(Q)^k M^k$, and select those free occurrences of t in A_1, if any, which occur in the k-th term and which take the place of free occurrences of x in $(Q)^k M(x)$. Let A_p be the first formula in $A_1, \ldots, A_r$ which yields its successor either by $\forall$-instantiation or a matrix change, let z be an individual variable which does not occur in $A_1, \ldots, A_r$ and also is distinct from the variable of any vacuous $\exists$A- or $\forall$C-introduction, and let $A_i(z)$, $1 \leq i \leq p$, be the result of replacing in A_i the traces of the selected occurrences of t in A_1 by z. It will now be shown that the sequence $B_{m+1}, \forall z A_1(z), \ldots, \forall z A_p(z), A_p, \ldots, A_r = B'_m = B'_{m+1}$ has the desired properties.

The step from B_{m+1} to $\forall z A_1(z)$ is an $\forall$-exportation, since all occurrences of t in the k-th term of A_1 taking the place of free occurrences of x in $(Q)^k M(x)$ are free occurrences in A_1 and since therefore the k-th term of $A_1(z)$ is the result of substituting z for the free occurrences of x in $(Q)^k M^k(x)$, and since (β) and (γ) of $\forall$-exportation are satisfied by the choice of z. [11] By (β) of $\forall$A-introduction, any selected occurrence of t in A_1 is a free occurrence in $(Q)^k M^k(t)$ and therefore in A_1. By the choice of z, z is free at any occurrence of t in A_1. It follows by Lemma 4 that $A_i(z)$ yields $A_{i+1}(z)$ by the same L-rule by which A_i yields A_{i+1}, $1 \leq i < p$. Hence, by Lemma 1, $\forall z A_i(z)$ yields $\forall z A_{i+1}(z)$ by the same L-rule by which A_i yields A_{i+1}, $1 \leq i < p$. Finally, $\forall z A_p(z)$ yields A_p by $\forall$-instantiation with $\forall z$ the quantifier and t the individual term of the $\forall$-instantiation, since, by Lemma 3, t is free at the occurrences of z in $A_p(z)$ and since, by the choice of z, A_p is the result of substituting t for the free occurrences of z in $A_p(z)$. It follows from the inductive hypothesis that the sequence $B_{m+1}, \forall z A_1(z), \ldots, \forall z A_p(z), A_p, \ldots, A_r$ is an L-deduction. By the choice of p, it is symmetric. Also it is Q-balanced, with the $\forall$-exportation from B_{m+1} to $\forall z A_1(z)$ and the $\forall$-instantiation from $\forall z A_p(z)$ to A_p forming the only new pair of quantifier changes, the p having been chosen so that the new $\forall$-exportation satisfies (ii). The pattern of this L-deduction reflects that of $B_0 \supset B'_0, \ldots, B_{m+1} \supset B'_{m+1}$, with the $\forall$-exportation from B_{m+1} to $\forall z A_1(z)$ corresponding to the $\forall$A-introduction from $B_m \supset B'_m$ to $B_{m+1} \supset B'_{m+1}$ and with each

[11] If B_{m+1} is in prenex normal form, its main prefix is regarded as empty.

assembling (disassembling) operation from $\forall z A_i(z)$ to $\forall z A_{i+1}(z)$ (from A_i to A_{i+1}) corresponding to the same **A**-operation (**C**-operation) to which the operation from A_i to A_{i+1} in the L-deduction $A_1, \ldots, A_r$ corresponds. Finally, (ix) and (xa) hold, (xi) is also satisfied by the new chain, (xb) is satisfied by the choice of t, and (xii) is preserved by the choice of z.

CASE 3b. $B_m \supset B'_m$ yields $B_{m+1} \supset B'_{m+1}$ by $\exists$**C**-introduction. Then a desired L-transform can be constructed in a manner analogous to that of Case 3a. Q.E.D.

Our construction yields an L-deduction in which all transient chains are such that the variable of the chain occurs in no formula of the L-deduction outside of the segment coextensive with the chain. If in the given H-deduction no variable of an $\exists$**A**- or $\forall$**C**-introduction occurs in the last formula, then the remaining chains also have this property.

An immediate consequence of Theorem 1 and the Herbrand-Gentzen Theorem is:

THEOREM 2. *If* $A \supset A'$ *is a theorem of the first-order predicate calculus, and if* A *is a conjunction and* A′ *an alternation of prenex normal forms, then there is a symmetric, Q-balanced, and Q-pure L-deduction from* A *to* A′.

A consequence of Theorem 2 and of the completeness of first-order predicate calculus is that L is complete in the sense described earlier, viz. if $A \supset A'$ is valid and if A is a conjunction and A′ an alternation of first-order prenex normal forms, then A yields A′ by applications of L-rules only. Indeed, since the L-deductions can be required to be Q-pure, L remains complete in this sense even if each L-rule is weakened by adding the condition that *no variable occurs more than once in the main prefix of the premiss or conclusion.*

Theorem 2 may be regarded as a new form of the Herbrand-Gentzen Theorem, since conversely the Herbrand-Gentzen Theorem is an immediate consequence of Theorem 2 and the following theorem.

THEOREM 3. *For any symmetric, Q-balanced, and Q-pure L-deduction one can construct an H-transform.*

PROOF. By induction on the number of assembling and disassembling operations in an L-deduction. Let A be a conjunction and A′ an alternation of prenex normal forms and consider any Q-balanced L-deduction from A to A′ in which there occur no assembling or disassembling operations. Then, by the Q-balance, there also occur no other quantifier changes, and hence A yields A′ by a sequence of one or more matrix changes. Then A (A′) can be written as (P)M (as (P)′M′) and also is the last (first) formula of the assembling (disassembling) phase. Since all chains are bracketed, (P) and (P)′ are empty, so that $A = M$ and $A' = M'$.

Since M yields M′ by matrix changes, $M \supset M'$ is an H-deduction. It is trivially an H-transform of the L-deduction from $A = M$ to $A' = M'$.

Now assume as inductive hypothesis that for any L-deduction from a conjunction to an alternation of prenex normal forms an H-transform can be constructed provided that the L-deduction is symmetric, Q-balanced, and Q-pure, and that there occur in it at most $m \geq 0$ assembling or disassembling operations. Let A be a conjunction and A′ an alternation of prenex normal forms and consider any symmetric, Q-balanced, and Q-pure L-deduction $A = A_0, \ldots, A_r = A'$ in which there occur $m+1$ assembling or disassembling operations. We shall now construct an H-transform of $A_0, \ldots, A_r$. Since $A_0, \ldots, A_r$ is symmetric and since there occurs at least one assembling or disassembling operation, at least one of the following cases must hold.

CASE 1a: A_0 yields A_1 by Duplication. Then $A_1, \ldots, A_r$ is a symmetric, Q-balanced, and Q-pure L-deduction in which there occur only m assembling or disassembling operations. By the inductive hypothesis, one can construct an H-transform of $A_1, \ldots, A_r$. By adjoining to the last formula $A_1 \supset A_r$ of this H-transform the further formula $A_0 \supset A_r$, obtainable from $A_1 \supset A_r$ by **A**-simplication, one obtains an H-transform of $A_0, \ldots, A_r$.

CASE 1b: A_{r-1} yields A_r by Simplication. The proof is similar.

CASE 2a: A_0 yields A_1 by $\exists$-exportation. Let $A_1, \ldots, A_{q-1}$ be the segment of $A_0, \ldots, A_r$ which is coextensive with the chain initiated by this $\exists$-exportation, let $\exists$y be the quantifier of this chain, and let $A_1^-, \ldots, A_{q-1}^-$ be obtained from $A_1, \ldots, A_{q-1}$, respectively, by deleting that quantifier occurrence which belongs to the chain. Since $A_0, \ldots, A_r$ is Q-balanced, A_{q-1} yields A_q by $\exists$-vacuous-removal, so that A_{q-1}^- is A_q. Since $A_0, \ldots, A_r$ is Q-pure, the variable of the quantifier deleted from $A_1, \ldots, A_{q-1}$ has no other occurrence in the main prefix of $A_1, \ldots, A_{q-1}$. Hence by Lemma 5, each A_i^- yields A_{i+1}^- by the same L-rule by which A_i yields A_{i+1}, $1 \leq i < q-1$. It follows that $A_1^-, \ldots, A_{q-1}^-, A_{q+1}, \ldots, A_r$ is an L-deduction which is symmetric, Q-balanced, and Q-pure. By the inductive hypothesis, one can construct for it an H-transform $B_0 \supset B_0', \ldots, B_s \supset B_s' = A_1^- \supset A_r$. Since $\exists$y is the quantifier of the $\exists$-vacuous-removal from A_{q-1} to A_q and since y therefore has no free occurrence in A_q and since moreover $A_0, \ldots, A_r$ is symmetric, it follows from Lemma 2 that y has no free occurrence in A_r. Hence the step from $A_1^- \supset A_r$ to $A_0 \supset A_r$ is an $\exists$**A**-introduction. It is then easily verified that $B_0 \supset B_0', \ldots, B_s \supset B_s' = A_1^- \supset A_r, A_0 \supset A_r$ is an H-transform of $A_0, \ldots, A_r$.

CASE 2b: A_{r-1} yields A_r by $\forall$-importation. The proof is similar to that for Case 2a.

CASE 3: A_0 yields A_1 by an $\forall$-exportation and A_{r-1} yields A_r by an $\exists$-importation. Since $A_0, \ldots, A_r$ is symmetric and Q-balanced, all chains which are either initiated by an $\forall$-vacuous-introduction or terminated by an $\exists$-vacuous-removal are comparable, since they all contain an occurrence in each premiss and conclusion of each $\forall$-instantiation, matrix change, and $\exists$-generalization. Hence at least one of the following two subcases must hold.

SUBCASE 3a: Among all those chains, if any, which are either initiated by an $\forall$-vacuous-introduction or terminated by an $\exists$-vacuous-removal the one furthest to the left is terminated by an $\exists$-vacuous-removal. Since this chain is initiated by an $\exists$-exportation it contains, by symmetry, in each premiss or conclusion of an $\forall$-vacuous-introduction an occurrence to the right of that belonging to the chain initiated by the earlier $\forall$-exportation from A_0 to A_1. Then any chain initiated by an $\forall$-vacuous-introduction contains in the conclusion of this $\forall$-vacuous-introduction an occurrence to the right of that belonging to the chain initiated by the $\forall$-exportation from A_0 to A_1.

Then the segment $A_1, \ldots, A_{p-1}$ of $A_0, \ldots, A_r$ which is coextensive with the chain initiated by the $\forall$-exportation from A_0 to A_1 is of the form $\forall z A_1^-, \ldots, \forall z A_{p-1}^-$ where the occurrence in each A_i belonging to the chain is indicated by $\forall z$. Since $A_0, \ldots, A_r$ is Q-pure, z does not occur in the main prefix of $A_1, \ldots, A_{p-1}$ except in the first quantifier. Hence by Lemma 5, each A_i^- yields A_{i+1}^- by the same L-rule by which A_i yields A_{i+1}, $1 \leq i < p-1$. By condition (ii) of Q-balance the step from A_{p-1} to A_p is the first $\forall$-instantiation in $A_0, \ldots, A_r$. Hence each A_i^- yields A_{i+1}^- by an a-rule other than $\forall$-instantiation, $1 \leq i < p-1$.

Now let t′ be the individual term of the $\forall$-instantiation from $A_{p-1} = \forall z A_{p-1}^-$ to A_p, so that (α) A_p is the result of substituting t′ for the free occurrences of z in A_{p-1}^-, and (β) t′ is free at the free occurrences of z in A_{p-1}^-. Let $A_i^-(t')$, $1 \leq i < p$, be the result of replacing in A_i^- the traces of the free occurrences of z in A_{p-1}^- by t′. Then by (β) and Lemma 4, $A_i(t')$ yields $A_{i+1}(t')$ by the same L-rule by which A_i yields A_{i+1}, $1 \leq i < p-1$. By (α), $A_{p-1}^-(t')$ is A_p. It follows that $A_1^-(t'), \ldots, A_{p-1}^-(t'), A_{p+1}, \ldots, A_r$ is an L-deduction which is symmetric, Q-balanced, and Q-pure. By the inductive hypothesis, one can construct for it an H-transform $B_0 \supset B_0', \ldots, B_s \supset B_s' = A_1^-(t') \supset A_r$.

By Lemma 3 and induction, the traces in A_1^- of the free occurrences of z in A_{p-1}^- are the free occurrences of z in A_1^-. Since A_0 yields $\forall z A_1^-$ by $\forall$-exportation, with $\forall z$ the quantifier exported into the main prefix, it follows that $A_1^-(t')$ differs from A_0 only in containing a term $(Q)^k M^k(t')$ where A_0 contains a term $\forall x (Q)^k M^k(x)$, such that $(Q)^k M^k(t')$ is the result of substituting t′ for the free occurrences of x in $(Q)^k M^k(x)$. By (β), Lemma 3, and induction, t′ is free in $A_1^-(t')$ at these occurrences. Hence $A_1^-(t') \supset A_r$ yields $A_0 \supset A_r$ by $\forall$A-introduction. It is then easily verified that $B_0 \supset B_0', \ldots, B_s \supset B_s' = A_1^-(t') \supset A_r, A_0 \supset A_r$ is an H-transform of $A_0, \ldots, A_r$.

Subcase 3b: Among all those chains, if any, which are either initiated by an $\forall$-vacuous-introduction or terminated by an $\exists$-vacuous-removal the one furthest to the left is initiated by an $\forall$-vacuous-introduction. The proof is similar to that for Subcase 3a. Q.E.D.

The differences between two symmetric and Q-balanced L-transforms of the same H-deduction are inessential, as can be seen by:

THEOREM 4. *Two symmetric and Q-balanced L-transforms of the same H-deduction differ from each other at most in these two respects: The order of the different* $\forall$*-vacuous-introductions or* $\exists$*-vacuous-removals (but not the relative position of the chains thus initiated or terminated* [12]*), and the variables of the transient chains.*

PROOF. By induction, proceeding inward from the first and last formulas of the two *L*-deductions.

The method provided by Theorem 1 of constructing symmetric and *Q*-balanced *L*-transforms is inductive, i.e., proceeds by means of *L*-transforms of shorter *H*-deductions. With the aid of Theorem 4 this can now be replaced by a more direct method. According to Theorem 4, at most one construction, barring inessential differences, proceeding inward from the first and last formula may produce a desired *L*-transform. According to Theorem 1, it will in fact produce it.

In contrast to Theorem 4, different *H*-deductions may be *H*-transforms of the same *L*-deduction. This is already apparent in the proof of Theorem 3 where different cases are not mutually exclusive. An example are the two *H*-deductions $Fz \supset Fz$, $\forall xFx \supset Fz$, $\forall xFx \supset \exists xFx$ and $Fz \supset Fz$, $Fz \supset \exists xFx$, $\forall xFx \supset \exists xFx$ which are both *H*-transforms of $\forall xFx$, Fz, Fz, $\exists xFx$. Differences in *H*-transforms of the same *L*-deduction are due to the fact that in *H*-deductions the **A**-operations and **C**-operations are in general intertwined and variations in their relative order may occur. In an *L*-transform of these *H*-deductions the counterparts of **A**-operations and **C**-operations are, with the exception of the $\forall$-vacuous-introductions and $\exists$-vacuous-removals, extricated from each other so that these relatively unimportant variations are not reflected.

In view of Theorem 2, some questions concerning the syntactical or prooftheoretic "role" of certain symbols can be restated as questions concerning their "history" during *L*-deductions. The following theorem is concerned with the "history" of certain predicate symbols. The proof uses Theorem 2 to reduce the problem on the quantificatonal level to one on the quantifier-free level. For convenience, derivability in first-order predicate calculus will be denoted by $\vdash$.

THEOREM 5. *If* $\vdash A \supset A'$ *and if* A *and* A′ *have at least one predicate symbol in common, then there is an "intermediate" formula* B *such that* $\vdash A \supset B$, $\vdash B \supset A'$, *and all predicate symbols occurring in* B *also occur both in* A *and in* A′. *Also, if* $\vdash A \supset A'$ *and if* A *and* A′ *have no predicate symbol in common,* [13] *then either* $\vdash \neg A$ *or* $\vdash A'$.

[12] This is the reason for including Condition (xi) in the notion of transform.

[13] This case was called to my attention by P. C. Gilmore. For the special case where moreover no individual variable occurs free in either A or A′ he has found a simpler argument in terms of satisfiability.

PROOF. We change A and A′ to prenex normal forms A_1 and A_r respectively, containing the same predicate symbols as A or A′ respectively. By Theorem 2, there exists a symmetric L-deduction from A_1 to A_r. Let (P)M be the premiss of the first and (P)M′ the conclusion of the last matrix change in this L-deduction, so that $\vdash A_1 \supset (P)M$, $\vdash M \supset M'$, and $\vdash (P)M' \supset A_r$. Then M contains the same predicate symbols as A_1 and M′ the same as A_r.

Now consider first the case where M and M′ have at least one predicate symbol in common. Suppose first that some assignment of truth-values satisfies M. Then M can be rewritten as a special ("ausgezeichnete") disjunctive normal form M_0 such that each term of the alternation M_0 is a conjunction in which the predicate symbol of at least one atomic formula occurs both in M and in M′ and therefore both in A and in A′. Now delete from each conjunction that is a term of M_0 all those terms whose predicate symbol does not occur both in A and in A′, and let M* be the alternation of the resulting conjunctions. Then the predicate symbols in M* occur both in A and in A′. Also $\vdash M \supset M^*$ and therefore $\vdash A \supset (P)M^*$. Furthermore, $\vdash M^* \supset M'$ and hence $\vdash (P)M^* \supset A'$, so that $B = (P)M^*$ satisfies the theorem. To see that $\vdash M^* \supset M'$, suppose there were some assignment of truth-values to the atomic formulas in M* and M′ for which M* is true while M′ is false. Then by adding suitable truth-values for those atomic formulas which occur in M_0 but not in M* and therefore not in M′, one could extend this to a truth-value assignment making M_0 true while keeping M′ false. But this is incompatible with the fact that $\vdash M_0 \supset M'$. Suppose now that no assignment of truth-values satisfies M, so that $\vdash \neg M$. Then $B = (P)M^*$ satisfies the theorem for any M* such that $\vdash \neg M^*$ and such that the predicate symbols in M* occur both in A and in A′.

Consider now the case where M and M′ have no predicate symbol in common. Then, by an argument in terms of truth-value assignments similar to the one just used, either $\vdash \neg M$ and therefore $\vdash \neg (P)M$, or $\vdash M'$ and therefore $\vdash (P)M'$. Then also either $\vdash \neg A$ or $\vdash A'$. Q.E.D.

BIBLIOGRAPHY

[1] G. GENTZEN, *Untersuchungen über das logische Schliessen*, **Mathematische Zeitschrift,** vol. 39 (1934–5), pp. 176–210, 405–431.

[2] J. HERBRAND, ***Recherches sur la théorie de la démonstration,*** Travaux de la Société des Sciences et Lettres de Varsovie, Classe III sciences mathématiques et physiques, no. 33, 128 pp.

[3] S. C. KLEENE, ***Introduction to metamathematics,*** Amsterdam (North Holland), Groningen (Noordhoff), New York and Toronto (van Nostrand), 1952, X+550 pp.

THE PENNSYLVANIA STATE UNIVERSITY