Efficient implementation of interpolation algorithms for the theories of quantifier-free equality with uninterpreted functions, unit two variable per inequality, and its combination

by

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B.Tech., Universidad de las Américas Puebla, 2015

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Dedication

To my family.

Acknowledgments

TODO.

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Abstract

This thesis discusses theoretical aspects and an implementation for the interpolation problem of the following theories: (quantifier-free) equalitiy with uninterpreted functions (EUF), unit two variable per inequality (UTVPI), and the combination of the two previous theories. The interpolation algorithms implemented in this thesis are originally proposed in [11].

Refutational proof-based solutions are the usual approach of many interpolation algorithms [8, 15, 14]. The approach taken in [11] relies on quantifier-elimination heuristics to construct an interpolant using one of the two formulas involved in the interpolation problem. The latter makes possible to study the complexity of the algorithms obtained. On the other hand, the combination method implemented in this thesis uses a Nelson-Oppen framework, thus we still require for this particular situation a refutational proof in order to guide the construction of the interpolant for the combined theory.

The implementation uses Z3 [4] for parsing purposes and satisfiability checking in the combination component of the thesis. Minor modifications were applied to the Z3's enode data structure in order to label and distinguish formulas efficiently (i.e. distinguish A-part, B-part). Thus, the project can easily be integrated to the Z3 solver to extend its functionality for verification purposes using the Z3 plug-in module.

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TODO.

Chapter 1

Introduction

TODO: motivation of the work

1.1 Backgroud

- General problem statement
- Redefinition (of the latter) for theories
- Applications in verification, i.e. invariant generation, refinement in model checking

1.2 Related work

- (Theoretical) approaches Proof-based Color-based Brutomesso Bonachina
- (Implementation) iZ3

TODO.

1.3 Outline of the thesis

- Chapter X is about Y ...
- and so on and so on ...

TODO.

Chapter 2

Preliminaries

This chapter discusses basic concepts from first-order logic that are used in the rest of this thesis. We will pay particular attention to their language and semantics since these are fundamental concepts necessary to understand the algorithmic constructions to compute interpolants. For a comprehensive treatment on the topic, the reader is suggested the following references [16, 7].

2.1 First-Order Predicate Logic

2.1.1 Language

A language is a collection of symbols of different sorts equipped with a rule of composition that effectively tells us how to recognize elements that belong to the language [23]. In particular, a first-order language is a language that expresses boolean combinations of predicates using terms (constant symbols and function applications). In mathematical terms,

Definition 2.1.1. *A first-order language (also denoted signature) is a triple* $\langle \mathfrak{C}, \mathfrak{P}, \mathfrak{F} \rangle$

of non-logical symbols where:

- $\mathfrak{C}$ *is a collection of constant symbols*
- $\mathfrak{P}$ *is a collection of n-place predicate symbols*
- $\mathfrak{F}$ *is a collection of n-place function symbols*

including logical symbols like quantifiers(universal ($\forall$)*, existential*($\exists$)*) logical symbols like parenthesis, propositional connectives (implication* ($\rightarrow$)*, conjunction* ($\wedge$)*, disjunction* ($\vee$)*, negation* ($\neg$)*), and a countably number of variables* Vars *(i.e.* $\text{Vars} = \{v_1, v_2, \dots\}$*).*

The rules of composition distinguish two objects, terms *and* formulas*, which are defined recursively as follows:*

- *Any variable symbol or constant symbol is a term.*
- *If* $t_1, \dots, t_n$ *are terms and* f *is an n-ary function symbol, then* $f(t_1, \dots, t_n)$ *is also a term.*
- *If* $t_1, \dots, t_n$ *are terms and* P *is an n-ary predicate symbol, then* $P(t_1, \dots, t_n)$ *is formula.*
- *If* x *is a variable and* ψ, φ *are formulas, then* $\neg\psi, \forall x.\psi, \exists x.\psi$ *and* $\psi \square \varphi$ *are formulas where* $\square \in \{\rightarrow, \wedge, \vee\}$

No other expression in the language can be considered terms nor formulas if such expressions are not obtained by the previous rules.

2.1.2 Semantics

In order to define a notion of truth in a first-order language is necessary to associate for each non-logical symbol (since logical symbols have established semantics from propositional logic) a denotation or mathematical object and an *assignment* to the collection of variables. The two previous components are part of an *structure* [7] (or interpretation [23]) for a first-order language.

Definition 2.1.2. *Given a first-order language $\mathfrak{L}$, an interpretation $\mathfrak{I}$ is a pair $(\mathfrak{A}, \mathfrak{J})$, where $\mathfrak{A}$ is a non-empty domain (set of elements) and $\mathfrak{J}$ is a map that associates to the non-logical symbols from $\mathfrak{L}$ the following elements:*

- $c^{\mathfrak{J}} \in \mathfrak{A}$ *for each* $c \in \mathfrak{C}$
- $f^{\mathfrak{J}} \in \{\mathfrak{A}^n \to \mathfrak{A}\}$ *for each n-ary function symbol* $f \in \mathfrak{F}$
- $P^{\mathfrak{J}} \in \{\mathfrak{A}^n\}$ *for each n-ary predicate symbol* $P \in \mathfrak{P}$

An assignment $s : Vars \to \mathfrak{A}$ is a map between Vars to elements from the domain of the interpretation.

With the definition of interpretation $\mathfrak{I}$ and assignment s, we can recursively define a notion of *satisfiability* (denoted by the symbol $\mathfrak{I} \models_s$) as a free extension from atomic predicates (function application of prediates) to general formulas as described in [7]. For the latter, we need to extend the assignment function to all terms in the language.

Definition 2.1.3. *Let $\mathfrak{I} = (\mathfrak{A}, \mathfrak{J})$ be an interpretation and s an assignment for a given language, Let $\bar{s} : Terms \to A$ be defined recursively as follows:*

- $\bar{s}(c) = c^{\mathfrak{J}}$
- $\bar{s}(f(t_1, \ldots, t_n)) = f^{\mathfrak{J}}(\bar{s}(t_1), \ldots, \bar{s}(t_n))$

Notice that the extension of s depends on the interpretation used.

Definition 2.1.4. *Given an interpretation $\mathfrak{I} = (\mathfrak{A}, \mathfrak{J})$, an assignment s, and ψ a formula, we define $\mathfrak{I} \models_s \psi$ (read ψ is satisfiable under interpretation $\mathfrak{I}$ and assignment s) recursively as follows:*

- $\models_{\mathfrak{I},s} P(t_1, \ldots, t_n)$ *if and only if* $P^{\mathfrak{J}}(\bar{s}(t_1), \ldots, \bar{s}(t_n))$
- $\models_{\mathfrak{I},s} \neg\psi$ *if and only if it is not the case that* $\models_{\mathfrak{I},s} \psi$
- $\models_{\mathfrak{I},s} \psi \wedge \varphi$ *if and only if* $\models_{\mathfrak{I},s} \psi$ *and* $\models_{\mathfrak{I},s} \varphi$
- $\models_{\mathfrak{I},s} \psi \vee \varphi$ *if and only if* $\models_{\mathfrak{I},s} \psi$ *or* $\models_{\mathfrak{I},s} \varphi$
- $\models_{\mathfrak{I},s} \psi \to \varphi$ *if and only if* $\models_{\mathfrak{I},s} \neg\psi$ *or* $\models_{\mathfrak{I},s} \varphi$
- $\models_{\mathfrak{I},s} \forall x.\psi$ *if and only if for every* $d \in \mathfrak{A}, \models_{\mathfrak{I},s_{x\mapsto d}} \psi$, *where* $s_{x\mapsto d} : Vars \to \mathfrak{A}$ *is reduct of* s *under* $Vars \setminus \{x\}$ *and* $s_{x\mapsto d}(x) = d$
- $\models_{\mathfrak{I},s} \exists x.\psi$ *if and only if exits* $d \in \mathfrak{A}, \models_{\mathfrak{I},s_{x\mapsto d}} \psi$, *where* $s_{x\mapsto d}$ *is defined as in the previous item.*

If an interpretation and assignment satisfies a formula, then we say that the interpretation and the assignment are a model for the respective formula. A collection of formulas are satified by an interpretation and assignment if these model each formula in the collection.

Additionally, if all the models $(\mathfrak{I}, s)$ in a language of a collection of formulas Γ satisfy a formula ψ, then we say that Γ logically implies ψ (written $\Gamma \models \psi$).

2.2 Mathematical Theories

A theory $\mathcal{T}$ is a collection of formulas that are closed under logical implication, i.e. if $\mathcal{T} \models \psi$ then $\psi \in \mathcal{T}$. This concept is quite relevant for our thesis work since we will focus on two theories, the quantifier-free fragment of the theory of equality with uninterpreted functions (EUF), and the theory of unit two variable per inequality (UTVPI).

For some theories it is enough to provide a collection of formulas (known as the axioms of the theory). For the case of the theories of interest for the thesis, the axiomatization if the following:

2.2.1 Equality with uninterpreted functions

Definition 2.2.1. *Let $\mathfrak{L}_{EUF} = \{\{\}, \{=\}, \{f_1, \ldots, f_n\}\}$ be the language of EUF. The axioms of the theory are:*

- *(Reflexivity)* $\forall x.x = x$
- *(Symmetry)* $\forall x.\forall y.x = y \rightarrow y = x$
- *(Transitivity)* $\forall x.\forall y.\forall z.(x = y \wedge y = z) \rightarrow x = z$
- *(Congruence)* $\forall x_1 \ldots \forall x_n.\forall y_1 \ldots \forall y_n.(x_1 = y_1 \wedge \cdots \wedge x_n = y_n) \rightarrow f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$

We notice that the congruence axiom is not a first-order logic axiom, but rather an axiom-scheme since it is necessary to *instantiate* such axiom for every arity of the function symbols in a given language.

2.2.2 Ordered commutative rings

In order to describe the UTVPI theory we will first introduce the language and theory of an ordered commutative ring.

Definition 2.2.2. *Let* $\mathfrak{L}_{Ord-R} = \{, \{0, 1\}, \{=, \leq\}, \{+, -, *, /\}, \}$ *be the language of an ordered ring R. The axioms of the theory are:*

- $\forall x.\forall y.\forall z.x + (y + z) = (x + y) + z$
- $\forall x.\forall y.x + y = y + x$
- $\forall x.x + 0 = x$
- $\forall x.x + (-x) = 0$
- $\forall x.\forall y.\forall z.x * (y * z) = (x * y) * z$
- $\forall x.x * 1 = x$
- $\forall x.\forall y.\forall z.x * (y + z) = x * y + x * z$
- $\forall x.\forall y.\forall z.(y + z) * x = y * x + z * x$
- $\forall x.\forall y.\forall z.x \leq y \rightarrow x + z \leq y + z$
- $\forall x.\forall y.(0 \leq x \wedge 0 \leq y) \rightarrow 0 \leq x * y.$
- $0 \neq 1 \wedge 0 \leq 1$

In the next section we will provide further computability aspects for the theories of interest that are relevant for verification.

2.3 Verification

Given a theory $\mathcal{T}$ in a formula ψ in the language of the theory, how can we know $\models_{\mathcal{T}} \psi$. This general question is known as the verification problem for the theory $\mathcal{T}$.

This question has been studied extensible for many theories interest. Problem

2.3.1 Satisfiability and Satisfiability Modulo Theories

TODO.

2.3.2 Congruence Closure

TODO.

2.3.3 Satisfiability of Horn clauses of propositions and grounded equations

TODO.

2.3.4 Nelson-Oppen framework for theory combination

TODO.

2.4 Interpolants

TODO.

2.4.1 Craig interpolation theorem

Let α, β, γ be logical formulas in a given theory. If $\vdash \alpha \rightarrow \beta$, we say that γ is an interpolant for the pair (α, β) if the following conditions are met:

- $\vdash \alpha \rightarrow \gamma$
- $\vdash \gamma \rightarrow \beta$
- Every non-logical symbol in γ occurs both in α and β.

The *interpolantion problem* can be stated naturally as follows: given two logical formulas α, β such that $\vdash \alpha \rightarrow \beta$, find the interpolant for the pair (α, β).

In his celebrated work [3], Craig proved that for every pair (α, β) of first order formulas such that $\vdash \alpha \rightarrow \beta$, an interpolation formula exists.

Usually, we see the interpolation problem defined differently in the literature, where we consider β' to be $\neg\beta$ and the problem requires that the pair (α, β') is mutually contradictory (unsatisfiable). This definition was popularized by McMillan [14]. This shift of attention explains partially the further development in interpolation generation algorithms since many of these relied on SMT solvers that provided refutation proofs in order to (re)construct interpolants for different theories (and their combination) [12, 2, 15].

Extended definitions are given to the interpolation problem when dealing with specific theories [25] in a way that interpreted function can be also part of the interpolant. The latter is justify since otherwise, many interpolation formulas might not exists in different theories (for example, lisp programs).

We also see different approaches to interpolation generation for particular theories that exploits aspects of the underlying theory [20, 22].

TODO.

2.4.2 Interpolants in verification (iZ3)

TODO.

2.5 Algorithms

TODO.

2.5.1 DPLL, DPLL(T) and extensions

TODO.

2.5.2 Congruence Closure algorithms (DST, NO, with explanations)

In [19], the authors introduced a Union-Find data structure that supports the Explanation operation. This operation receives as input an equation between constants. If the input equation is a consequence of the current equivalence relation defined in the Union Find data structure, the Explanation operation returns the minimal sequence of equations used to build such equivalence relation, otherwise it returns 'Not provable'. A proper implementation of this algorithm extends the traditional Union-Find data structure with a *proof-forest*, which consists of an additional representation of the underlying equivalence relation that does not compress paths whenever a call to the Find operation is made. For efficient reasons, the Find operation uses the path compression and weighted union.

The main observation in [19] is that, in order to recover an explanation between two terms, by traversing the path between the two nodes in the proof tree, the last edge in the path guarantees to be part of the explanation. Intuitively, this follows because only the last Union operation was responsible of merging the two classes into one. Hence, we can recursively recover the rest of the explanation by recursively traversing the subpaths found.

Additionally, the authors in [19] extended the Congruence Closure algorithm [18] using the above data structure to provide Explanations for EUF theory. The congruence closure algorithm is a simplification of the congruence closure algorithm in [6]. The latter combines the traditional *pending* and *combine* list into one single list, hence removing the initial *combination* loop in the algorithm in [6].

TODO.

2.5.3 Gallier

In [10] it was proposed an algorithm for testing the unsatisfiability of ground Horn clauses with equality. The main idea was to interleave two algorithms: *implicational propagation* (propositional satisfiability of Horn clauses) that updates the truth value of equations in the antecedent of the input Horn clauses [5]; and *equational propagation* (congruence closure for grounded equations) to update the state of a Union-Find data structure [9] that keeps the minimal equivalence relation defined by grounded equations in the input Horn clauses.

The author in [10] defined two variations of his algorithms by adapting the Congruence Closure algorithms in [6, 17]. Additionally, modifications in the data structures used by the original algorithms were needed to make the interleaving mechanism more efficient.

TODO.

2.6 Available tools

TODO.

2.6.1 Z3 and my minor modifications

TODO.

2.6.2 zChaff and my minor modifications

TODO.

2.6.3 DIMACS

TODO.

Chapter 3

The Theory of EUF

Interpolation algorithms for the theory of equality with uninterpreted functions are relevant as the core component of verification algorithms. Many useful techniques in software engineering like bounded/unbounded model checking and invariant generation benefit directly from this technique. In [1], the authors introduced a methodology to debug/verify the control logic of pipelined microprocessors by encoding its specification and a logical formula denoting the implementation of the circuit into a EUF solver.

Previous work addressing the interpolation problem for EUF has involved techniques ranging from interpolant-extraction from refutation proof trees [14, 15, 24], and colored congruence closure graphs [8]. Kapur's algorithm uses a different approach by using quantifier-elimination, a procedure that given a formula, it produces a logically equivalent formula without a variable in particular [7].

TODO.

3.1 Algorithm

Kapur's interpolation algorithm for the EUF theory uses quantifier-elimination techniques to remove symbols in the first formula of an interpolation problem instance that are not common with the second formula of the latter. Hence, the input for this algorithm is a conjunction of equalities in the EUF theory and a set of symbols to eliminate, also unknown as uncommon symbols.

The steps/ideas in Kapur's algorithm for interpolant generation for the EUF theory are the following:

- **Elimination of uncommon terms using Congruence Closure**. This step builds an equivalence relation using the input of the algorithm such that the representatives are common terms whenever possible. Let *answer* be a variable denoting the conjunction of all the input formulas which uncommon subterms are replaced by their representatives. If all the representatives in the equivalence relation are common terms, return *answer* as the interpolant. Otherwise, continue with the following step.

- **Horn clause generation by exposure**. This step uses Ackermann's reduction [13] to produce Horn clauses to eliminate uncommon terms identifying two cases:

  - The term is uncommon because the function symbol is uncommon.
  - The term is uncommon because at least one of its arguments is an uncommon term.

  Conjunct to *answer* the conjunction of all these Horn Clauses. If all the Horn Clauses above are common, return *answer* as the interpolant. Otherwise, continue with the following step.

- **Conditional elimination**. Since some of the Horn clauses produced by the previous step are not common, we identify the Horn clauses that have *common antecedents* and head equation with at least one uncommon term. For each of these Horn clauses, replace the uncommon term in its head by the common term in its head in the rest of the Horn Clauses, and include its antecedent to the antecedent of such Horn clauses. We can see that this step reduces the number of uncommon terms in the equalities of the Horn Clauses. We repeat this step until it cannot be performed. At the end, we take only the set of Horn Clauses that have common antecedents and have one uncommon term in its head. The call these Horn clauses *useful Horn clauses*. Continue with the next step.

- **Conditional replacement**. Using the *useful Horn clauses* generated in the previous step, update *answer* to be the formula resulting after *conditionally replacing* uncommon terms in each equation of *answer* by an appropriate common term in the head of a *useful Horn clauses*. To be more precise, let $\bigwedge_i a_i = b_i \rightarrow u \mapsto c$ be a useful Horn clause, where the antecedent is a conjunction of common grounded equations, u is an uncommon term, and c is a common term. Then for every instance of u in each equation of *answer*, conditionally replace u by c under $\bigwedge_i a_i = b_i$. We notice that equations in *answer* of the previous step will become Horn Clauses with less uncommon terms. For completeness, we perform these replacements zero or more times (up to the maximal number of instances per equation) in order to leave space for other *useful Horn Clauses* to replace the uncommon term in their heads as well. Remove all the literals in the current *answer* that contain uncommon terms and return this as the interpolant.

If the user is not interested in an explicit interpolant, we can present the *usable Horn clauses* in a proper order such that the replacements can be done without

exponentially increasing the size of the interpolant. This representation is useful because it provides a more compact representation of the interpolant that the user might be able quicker to obtain. Additionally, the user might be just interested in a particular subformula of the interpolant, so the latter representation offers such feature.

This algorithm allows a flexible implementation which can lead several optimizations based on the nature and applications of the interpolant.

3.1.1 Contributions

First, we explain a high level ideal on how we improve the *conditional elimination* step in Kapur's algorithm. We notice that this step *propagates equationally* the head equations of grounded Horn clauses with common antecedents. Initially we employ the unsatisfiability algorithm for Horn clauses to achieve such propagation. However, the original algorithm will not be enough because it will only propagate the head equation when all the antecedents have truth value equal to true. To fix that problem, we modify two steps in Gallier's algorithms:

- When we build the data structure *numargs* that keeps track of the number of unproven equations in the antecedent of each Horn clause, we change this number by the number of unproven uncommon equations in the antecedent of each Horn clause. This will be useful because we only introduce head equations intro the queue data structure in Gallier's algorithm when all the antecedents are true. With this modification, our algorithm introduces head equations when all the antecedent equations are common. Additionally the algorithm can still update correctly the truth value of common equations, but these are not relevant for our propagation purposes.

- To guarantee that *numargs* keeps the right number of uncommon equations yet to be proven, we also modify the update mechanism for *numargs* in the main while loop of the algorithm. The original algorithm reduces by one the corresponding entry in *numargs* whenever a recently popped element from the queue matches the antecedent of a Horn clause. We only decrease this value if such popped equation is uncommon. This prevents the algorithm from accidentally reducing the number of uncommon equations yet to be proven, which can cause that we propagate the uncommon head equation when the antecedent of a Horn clause only consists of common equations.

At the end of this algorithm we can identify *usable Horn clauses* by checking the Horn clauses with *numargs* entries equal to 0. Nonetheless, these Horn clauses are not the desired *usable Horn clauses* because the unsatisfiability testing algorithm did not update the antecedents of the Horn clauses. The main difficulty to design a data structure for the latter to work inside the unsatisfiability testing algorithm was the queue data structure only adds grounded equation whenever the truth value of the literal changes to true, which happens during *equational propagation* or during the *implicational propagation* steps. For the *implicational propagation* the task is easy because we can know the clause where the just new proven ground equation comes, but it cannot be the same situation for the *equational propagation* since this step relies on the Congruence Closure.

To remedy this issue, we equip our Congruence Closure algorithm with the Explanation operator, so we can recover the grounded equations needed to entail any particular grounded equation. Additionally, this will require a data structure to maintain the Horn clauses for each grounded equation that it is the head equation of. With the latter we can recover the Horn Clauses where each grounded equation came from to update the antecedents and obtain *usable Horn clauses*.

3.1.2 New optimized conditional elimination step in Kapur's algorithm

The algorithm appears below in pseudo-code notation:

3.1.3 Ground Horn Clauses with Explanations

We notice that, by removing our changes to the unsatisfiability testing for grounded Horn clauses regarding uncommon symbols, we effectively combine the congruence closure with explanations to the original unsatisfiability testing algorithm. With the latter, we can query the membership of a Horn clauses in a given user-defined theory and additionally obtain a proof of the latter. This approach works by introducing the antecedent equations of a grounded Horn clause as part of the user-defined theory in order to prove its head equation. By the Deduction Theorem [16], we can recover a proof of the original queried Horn clause by removing the antecedent equations appearing the proof given by the Explain operation.

TODO.

Theorem 3.1.1. *Let t be an uncommon term and let H be a collection of Horn equations. Assume $antecedent(H) \neq \{\}$. If $consequent(t) = \{\}$, then we cannot conditionality eliminate t from any Horn equation $h \in H$.*

Proof. Suppose, by contradiction, that we can conditionally eliminate the term t from H. Then, there exists $h^{'} \in H$ such that t appears in the consequent of $h^{'}$. But $consequent(t) = \{\}$, contradiction. □

Corollary 3.1.1.1. *We cannot eliminate an uncommont term t unconditionally once $consequent(t) = \{\}$.*

3.2 Implementation

TODO.

3.3 Evaluation

TODO.

Algorithm 1 Modified Unsatisfiability Testing for Ground Horn Clauses

```
 1: procedure SATISFIABLE(var H : Hornclause; var queue, combine: queuetype;
    var GT(H) : Graph; var consistent : boolean)
 2:     while queue not empty and consistent do
 3:         node := pop(queue);
 4:         for clause1 in H[node].clauselist do
 5:             if ¬ clause1.isCommon() then
 6:                 numargs[clause1] := numargs[clause1] - 1
 7:             end if
 8:             if numargs[clause1] = 0 then
 9:                 nextnode := poslitlist[clause1];
10:                 if ¬ H[nextnode].val then
11:                     if nextnode ≠ ⊥ then
12:                         queue := push(nextnode, queue);
13:                         H[nextnode].val := true;
14:                         u := left(H[nextnode].atom); v := right(H[nextnode].atom);
15:                         if FIND(R, u) ≠ FIND(R, v) then
16:                             combine := push((u, v), combine);
17:                         end if
18:                     else
19:                         consistent := false;
20:                     end if
21:                 end if
22:             end if
23:         end for
24:         if queue is empty and consistent then
25:             closure(combine, queue, R);
26:         end if
27:     end while
28: end procedure

29: procedure CLOSURE(var combine, queue : queuetype; var R : partition)
30:     while combine is not empty do
31:         (u, v) = pop(combine)
32:         MERGE(R, u, v, queue)
33:     end while
34: end procedure
```

Algorithm 2 Modified Congruence Closure with Explanation Algorithms - Merge

```
    procedure MERGE(R : partition, u, v : node; queue, combine : queuetype)
 2:     if u then and v are constants a and b
            add a = b to Pending; Propagate();
 4:     else                                          ▷ u=v is of the form f(a1, a2)=a
            if L thenookup(Representative(a1), Representative(a2)) is some f(b1,
    b2)=b
 6:             add (f(a1, a2)=a, f(b1, b2) = b) to Pending; Propagate();
            else
 8:             set Lookup(Representative(a1), Representative(a2)) to f(a1, a2)=a;
                add f(a1, a2)=a to UseList(Representative(a1)) and to
    UseList(Representative(a2));
10:         end if
        end if
12: end procedure
```

Algorithm 3 Modified Congruence Closure with Explanation Algorithms - Propagate

```
    procedure PROPAGATE( )
 2:     while P doending is non-empty
            Remove E of the form a=b or (f(a1, a2) = a, f(b1, b2) = b) from Pending
 4:         if thenRepresentative(a) ≠ Representative(b) and w.l.o.g.
    |ClassList(Representative(a))| ≤ |ClassList(Representative(b))|
                oldReprA := Representative(a);
 6:             Insert edge a → b labelled with E into the proof forest;
                for e doach c in ClassList(oldReprA)
 8:                 set Representative(c) to Representative(b)
                    move c from ClassList(oldReprA) to ClassList(Representative(b))
10:                 for e doach pointer L in ClassList(u)
                        if H then[L].val = false
12:                         set the field H[L].lclass or H[L].rclass pointed to by p to
    Representative(b)
                            if H then[L].lclass = H[L].rclass
14:                             queue := push(L, queue);
                                H[L].val := true
16:                         end if
                        end if
18:                 end for
                end for
20:             for e doach f(c1, c2) = c in UseList(oldReprA)
                    if L thenookup(Representative(c1), Representative(c2)) is some
    f(d1, d2) = d
22:                     add (f(c1, c2) = c, f(d1, d2) = d) to Pending;
                        remove f(c1, c2) = c from UseList(oldReprA);
24:                 else
                        set Lookup(Representative(c1), Representative(c2)) to f(c1, c2)
    = c;
26:                     move f(c1, c2) = c from UseList(oldReprA) to
    UseList(Representative(b));
                    end if
28:             end for
            end if
30:     end while
    end procedure
```

Chapter 4

The Theory of Octagonal Formulas

TODO.

4.1 Algorithm

TODO.

Theorem 4.1.1. *Given a mutually contradictory pair (α, β), where α, β are finite conjunctions of octagonal atoms, the above algorithm terminates with an interpolant I_α that is a finite conjunction of octagonal atoms and is equivalent to $\exists \vec{x}.\alpha$, where $\vec{x}$ is the symbols in α which are not in β. Further I_α is he strongest interpolant for (α, β).*

Proof. First we will prove that $\models \alpha \rightarrow I_\alpha$. The latter follows since the algorithm produces a conjunction of octagonal formulas using the Elim rule, which is a truth-preserving rule of inference, eliminating conjuncts with uncommon symbols.

Now, we will prove that $\not\models I_\alpha \wedge \beta$. We will prove the latter by induction on the number of variables to eliminate (k).

- Base case: $k = 0$. Then the algorithm outputs α. Then the statement holds since (α, β) is unsatisfiable.

- Inductive case: $k = n + 1$. Since the set S of variables to eliminate is non-empty, we just take any variable $x \in S$ and apply the above algorithm to eliminate such variable. Let X be the set of octagonal inequalities of (α, β) and X' be the set of octagonal inequalities (α', β) where α' is the conjunct obtained after removing the variable x. We know X and X' are equisatisfiable using a similar argument as in the Fourier-Motzkin elimination method [21]. Let us suppose (α', β) is satisfiable, hence (α, β) is unsatisfiable as well. But the latter entails a contradiction since (α, β) is assumed to be unsatisfiable. Hence, (α', β) is unsatisfiable. Since (α', β) is an unsatisfiable formula with n variables to eliminate, using the Inductive Hypothesis we conclude that (I_α, β) is unsatisfiable as well.

□

4.2 Implementation

TODO.

4.3 Evaluation

TODO.

Chapter 5

Combining Theories: EUF and Octagonal Formulas

A relevant comment from the paper A combination method for generating interpolants.

The question is how to split it inot two formulas, A' and B'. The condition for splitting is that the common symbols for ¡A', B'¿ should be (a subset of) AB-common symbols, because we would like to use the resultant interpolant for ¡A', B'¿ as a part of an interpolant for the original A and B.

Suppose that Eq contains only AB-pure equalities ...

The latter captures the idea behind equality-interpolating theories. The core definition guarantees that the Nelson-Oppen combination framework is able to propagate AB-pure equalities so the splitting part is not a concern for the algorithm.

TODO.

5.1 Algorithm

TODO.

5.2 Implementation

TODO.

5.3 Evaluation

TODO.

Chapter 6

Future Work

TODO.

Appendices

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