CDCL SAT Solvers

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Theory and Practice of SAT Solving
Dagstuhl Workshop

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The Success of SAT

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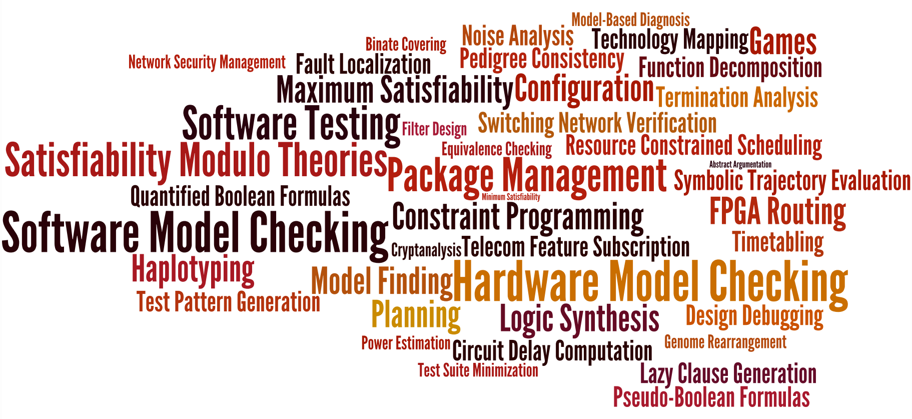
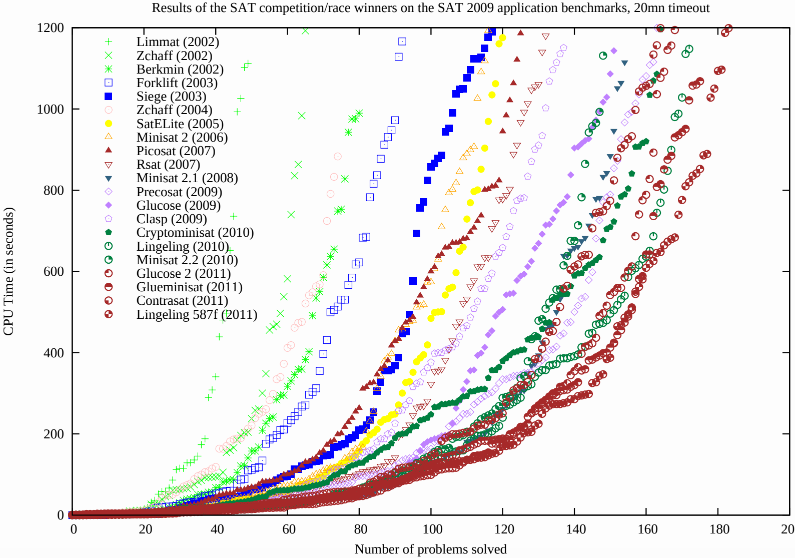
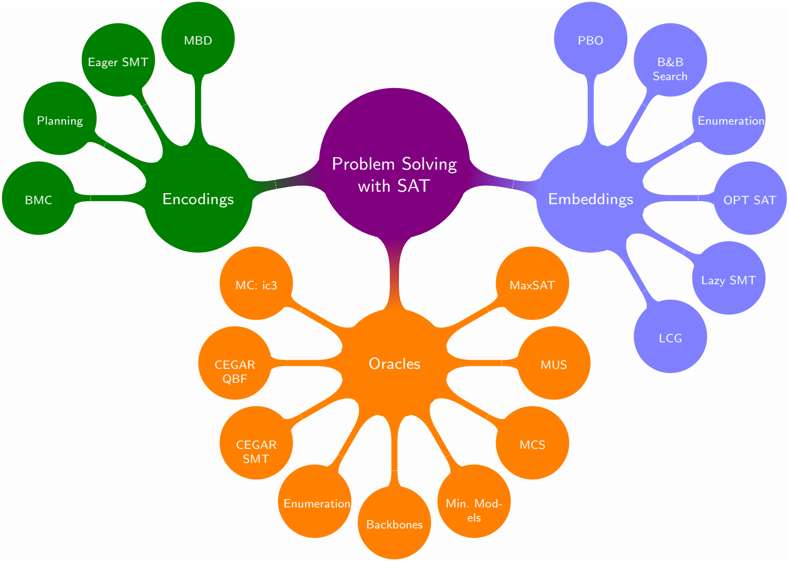


SAT Solver Improvement

[Source: Le Berre&Biere 2011]



SAT-Based Problem Solving



This Talk

- Review key ideas in implementing CDCL SAT solvers
  - Review standard concepts
    - Unit propagation
    - Resolution
    - Proof traces
    - ...
  - Outline organization of DPLL/CDCL SAT solvers
  - Survey most effective techniques
    - Clause learning & non-chronological bakctracking
    - UIPs
    - Clause minimization
    - Search restarts
    - Several heuristics
    - ...

Outline

Basic Definitions

DPLL Solvers

CDCL Solvers

What Next in CDCL Solvers?

CNF Encodings

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Preliminaries

- Variables: $w, x, y, z, a, b, c, \ldots$
- Literals: $w, \bar{x}, \bar{y}, a, \ldots$, but also $\neg w, \neg y, \ldots$
- Clauses: disjunction of literals or set of literals
- Formula: conjunction of clauses or set of clauses
- Model (satisfying assignment): partial/total mapping from variables to $\{0, 1\}$ that satisfies formula
- Formula can be SAT/UNSAT

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- Formula can be SAT/UNSAT
- Example:

$$\mathcal{F} \triangleq (r) \wedge (\bar{r} \vee s) \wedge (\bar{w} \vee a) \wedge (\bar{x} \vee b) \wedge (\bar{y} \vee \bar{z} \vee c) \wedge (\bar{b} \vee \bar{c} \vee d)$$

  - Example models:
    - $\{r, s, a, b, c, d\}$
    - $\{r, s, \bar{x}, y, \bar{w}, z, \bar{a}, b, c, d\}$

Resolution

- Resolution rule: [DP60,R65]

$$\frac{(\alpha \vee x) \qquad\qquad\qquad (\beta \vee \bar{x})}{(\alpha \vee \beta)}$$

  - Complete proof system for propositional logic

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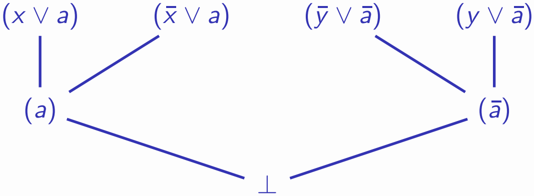

$(x \vee a)$ $(\bar{x} \vee a)$ $(\bar{y} \vee \bar{a})$ $(y \vee \bar{a})$

(a) $(\bar{a})$

$\perp$

  - Extensively used with (CDCL) SAT solvers

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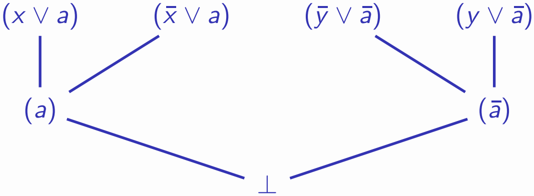

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  - Extensively used with (CDCL) SAT solvers

- Self-subsuming resolution (with $\alpha' \subseteq \alpha$): [e.g. SP04,EB05]

$$\frac{(\alpha \vee x) \qquad\qquad (\alpha' \vee \bar{x})}{(\alpha)}$$

  - (α) subsumes $(\alpha \vee x)$

Unit Propagation

$$\begin{aligned}\mathcal{F} \quad = \quad & (r) \wedge (\bar{r} \vee s) \wedge \\ & (\bar{w} \vee a) \wedge (\bar{x} \vee \bar{a} \vee b) \\ & (\bar{y} \vee \bar{z} \vee c) \wedge (\bar{b} \vee \bar{c} \vee d)\end{aligned}$$

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- Decisions / Variable Branchings:
  $w = 1, x = 1, y = 1, z = 1$

Unit Propagation

$$\begin{aligned}\mathcal{F} &= (r) \wedge (\bar{r} \vee s) \wedge \\ &\quad (\bar{w} \vee a) \wedge (\bar{x} \vee \bar{a} \vee b) \\ &\quad (\bar{y} \vee \bar{z} \vee c) \wedge (\bar{b} \vee \bar{c} \vee d)\end{aligned}$$

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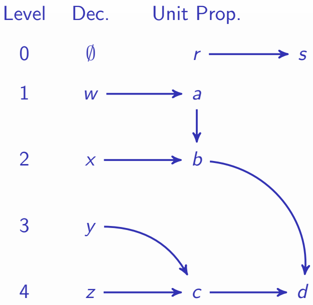

Unit Propagation

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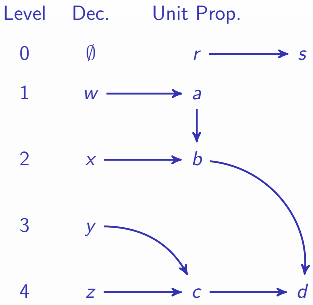


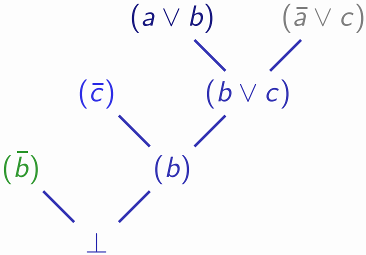
Level	Dec.	Unit Prop.
0	$\emptyset$	r s
1	w	a
2	x	b
3	y	
4	z	c d

- Decisions / Variable Branchings: $w = 1, x = 1, y = 1, z = 1$

- Additional definitions:
  - Antecedent (or reason) of an implied assignment
    - $(\bar{b} \vee \bar{c} \vee d)$ for d
  - Associate assignment with decision levels
    - $w = 1 @ 1,\ x = 1 @ 2,\ y = 1 @ 3,\ z = 1 @ 4$
    - $r = 1 @ 0,\ d = 1 @ 4,\ \ldots$

Resolution Proofs

- Refutation of unsatisfiable formula by iterated resolution operations produces resolution proof
- An example:
  $\mathcal{F} = (\bar{c}) \wedge (\bar{b}) \wedge (\bar{a} \vee c) \wedge (a \vee b) \wedge (a \vee \bar{d}) \wedge (\bar{a} \vee \bar{d})$
- Resolution proof:



$(a \vee b)$ $(\bar{a} \vee c)$

$(\bar{c})$ $(b \vee c)$

$(\bar{b})$ (b)

$\perp$

- A modern SAT solver can generate resolution proofs using clauses learned by the solver

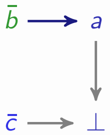
[ZM03]

Unsatisfiable Cores & Proof Traces

- CNF formula:

$$\mathcal{F} \;=\; (\bar{c}) \land (\bar{b}) \land (\bar{a} \lor c) \land (a \lor b) \land (a \lor \bar{d}) \land (\bar{a} \lor \bar{d})$$

Level	Dec.	Unit Prop.
0	$\emptyset$	



Implication graph with conflict

Unsatisfiable Cores & Proof Traces

- CNF formula:

$$\mathcal{F} \quad = \quad (\bar{c}) \wedge (\bar{b}) \wedge (\bar{a} \vee c) \wedge (a \vee b) \wedge (a \vee \bar{d}) \wedge (\bar{a} \vee \bar{d})$$

Level	Dec.	Unit Prop.
0	$\emptyset$	

$$\begin{tikzcd}
\bar{b} \arrow[r] & a \arrow[d] \\
\bar{c} \arrow[r] & \bot
\end{tikzcd}$$

Proof trace $\bot$: $(\bar{a} \vee c)$ $(a \vee b)$ $(\bar{c})$ $(\bar{b})$

Unsatisfiable Cores & Proof Traces

- CNF formula:

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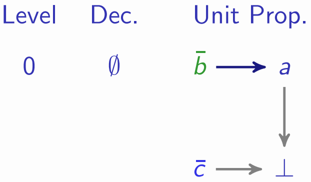
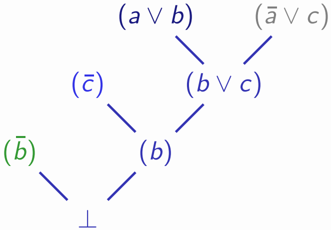


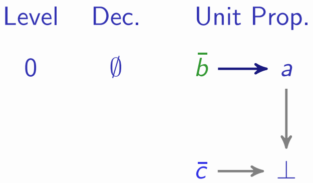


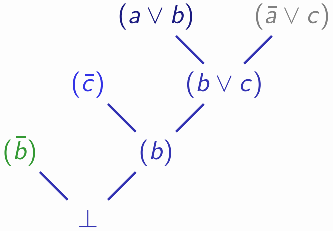
Resolution proof follows structure of conflicts

Unsatisfiable Cores & Proof Traces

- CNF formula:

$$\mathcal{F} \quad = \quad (\bar{c}) \wedge (\bar{b}) \wedge (\bar{a} \vee c) \wedge (a \vee b) \wedge (a \vee \bar{d}) \wedge (\bar{a} \vee \bar{d})$$





Unsatisfiable subformula (core): $(\bar{c}), (\bar{b}), (\bar{a} \vee c), (a \vee b)$

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DPLL Solvers

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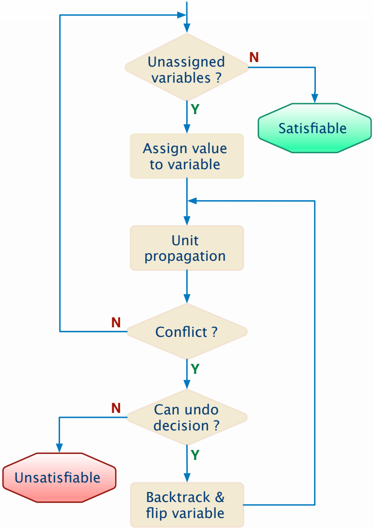
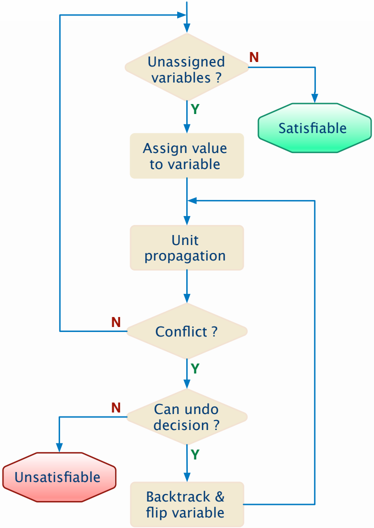
CNF Encodings

The DPLL Algorithm



- Optional: pure literal rule

The DPLL Algorithm



$$\mathcal{F} = (x \vee y) \wedge (a \vee b) \wedge (\bar{a} \vee b) \wedge (a \vee \bar{b}) \wedge (\bar{a} \vee \bar{b})$$

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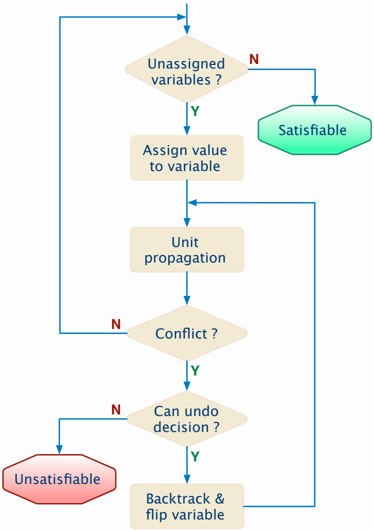

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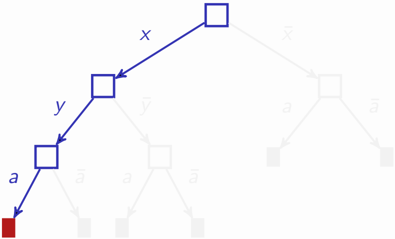
Level	Dec.	Unit Prop.
0	$\emptyset$	
1	x	
2	y	

$$\begin{tikzcd} 3 & a \arrow[r] \arrow[rr, bend right] & b \arrow[r] & \perp \end{tikzcd}$$



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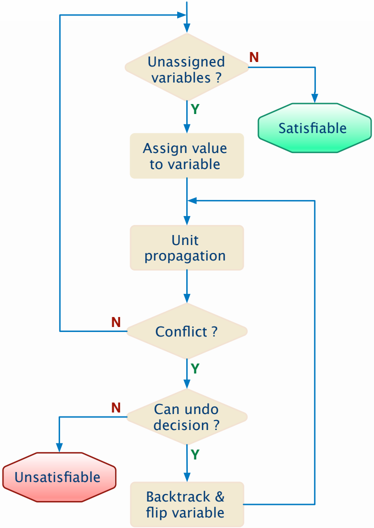


Unassigned variables ?

N

Satisfiable

Y

Assign value to variable

Unit propagation

Conflict ?

N

Y

Can undo decision ?

N

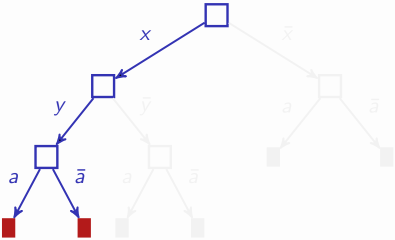
Unsatisfiable

Y

Backtrack & flip variable

$$\mathcal{F} = (x \vee y) \wedge (a \vee b) \wedge (\bar{a} \vee b) \wedge (a \vee \bar{b}) \wedge (\bar{a} \vee \bar{b})$$

Level	Dec.	Unit Prop.
0	$\emptyset$	
1	x	
2	y	
3	$\begin{tikzcd} \bar{a} \arrow[r] \arrow[rr, bend right] & \bar{b} \arrow[r] & \perp \end{tikzcd}$	



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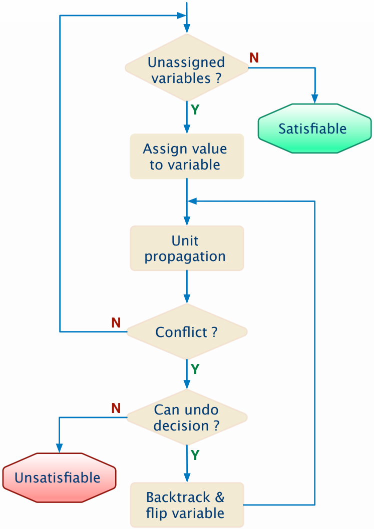

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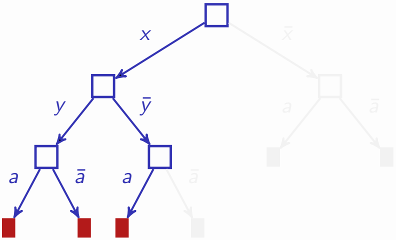
Level	Dec.	Unit Prop.
0	$\emptyset$	
1	x	
2	$\bar{y}$	
3	$a \longrightarrow$	$b \longrightarrow \bot$



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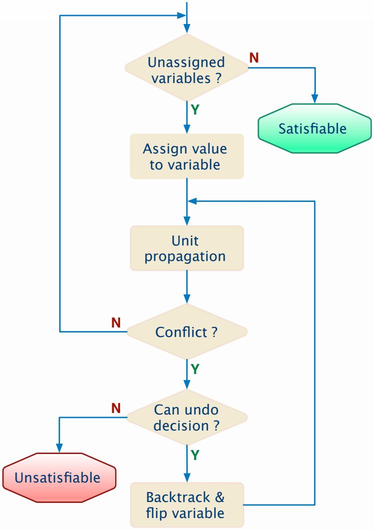

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<table>
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<tr><td>0</td><td>$\emptyset$</td><td></td></tr>
<tr><td>1</td><td>x</td><td></td></tr>
<tr><td>2</td><td>$\bar{y}$</td><td></td></tr>
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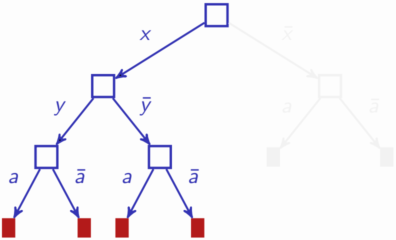

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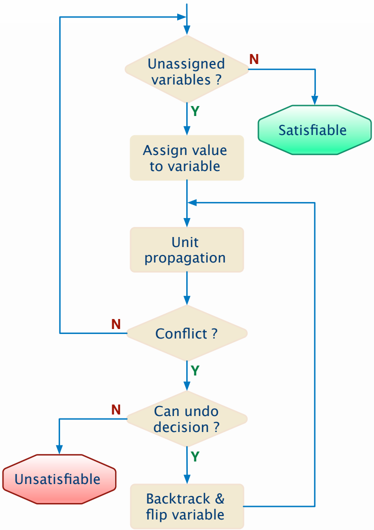

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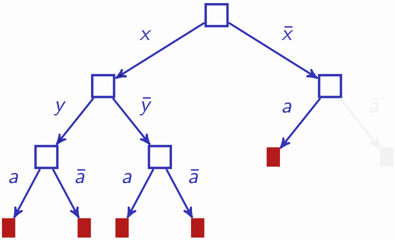
Level	Dec.	Unit Prop.
0	$\emptyset$	
1	$\bar{x}$	y
2	a	b $\perp$

$$\begin{tikzcd}
\bar{x} \arrow[r] & y & \\
a \arrow[r] \arrow[rr, bend right] & b \arrow[r] & \perp
\end{tikzcd}$$



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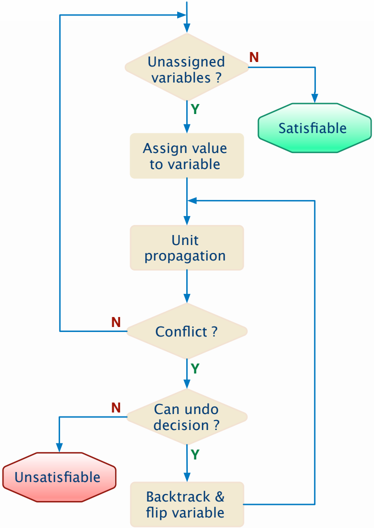

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2 & \bar{a} \arrow[r] \arrow[rr, bend right] & \bar{b} \arrow[r] & \perp &
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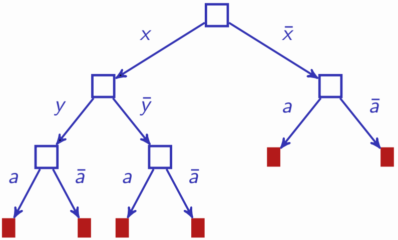

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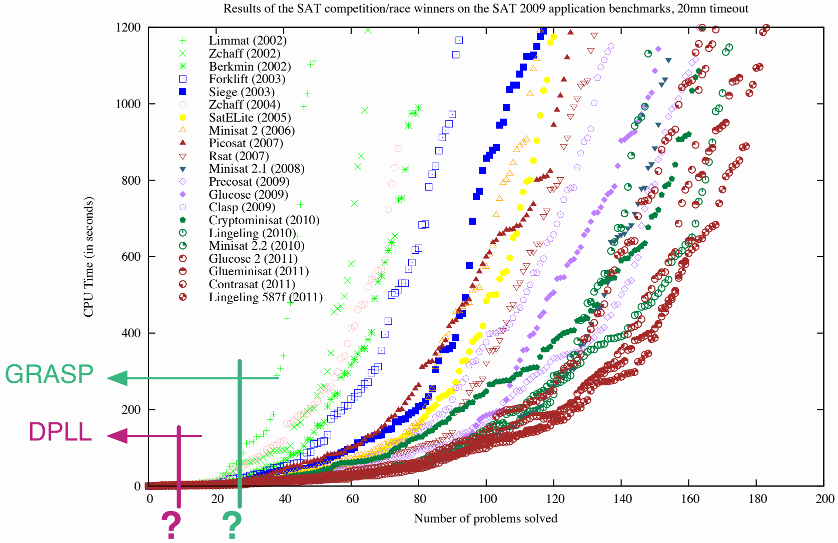
What is a CDCL SAT Solver?

- Extend DPLL SAT solver with: [DP60,DLL62]
  - Clause learning & non-chronological backtracking [MSS96,BS97,Z97]
  - Search restarts [GSK98,BMS00,H07,B08]
  - Lazy data structures
  - Conflict-guided branching
  - ...

What is a CDCL SAT Solver?

- Extend DPLL SAT solver with: [DP60,DLL62]
  - Clause learning & non-chronological backtracking [MSS96,BS97,Z97]
    - Exploit UIPs [MSS96,SSS12]
    - Minimize learned clauses [SB09,VG09]
    - Opportunistically delete clauses [MSS96,MSS99,GN02]
  - Search restarts [GSK98,BMS00,H07,B08]
  - Lazy data structures
    - Watched literals [MMZZM01]
  - Conflict-guided branching
    - Lightweight branching heuristics [MMZZM01]
    - Phase saving [PD07]
  - ...

How Significant are CDCL SAT Solvers?



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- Clause Learning, UIPs & Minimization
- Search Restarts & Lazy Data Structures

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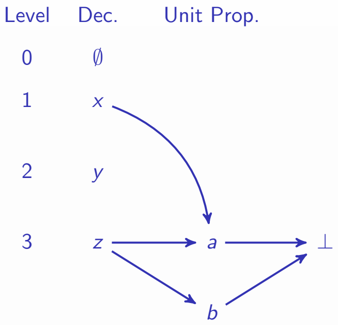
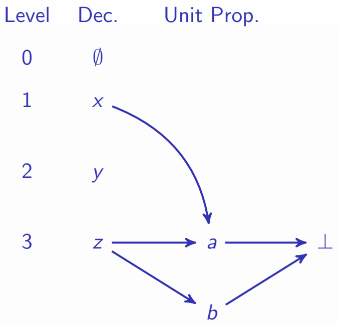
Clause Learning



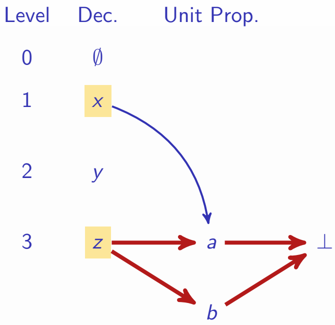
Level	Dec.	Unit Prop.	
0	$\emptyset$		
1	x		
2	y		
3	z	a	$\perp$
		b	

Clause Learning



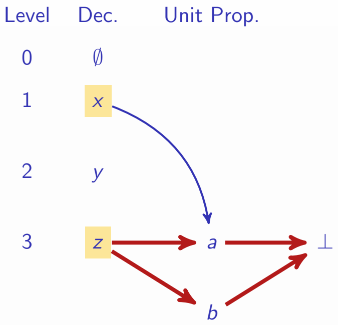
- Analyze conflict

Clause Learning



- Analyze conflict
  - Reasons: x and z
    - Decision variable & literals assigned at lower decision levels

Clause Learning



Level	Dec.	Unit Prop.
0	$\emptyset$	
1	x	
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3	z	a $\perp$
		b

- Analyze conflict
  - Reasons: x and z
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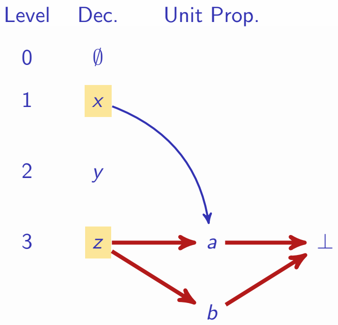
  - Create **new** clause: $(\bar{x} \vee \bar{z})$

Clause Learning



$(\bar{a} \vee \bar{b})$ $(\bar{z} \vee b)$ $(\bar{x} \vee \bar{z} \vee a)$

- Analyze conflict
  - Reasons: x and z
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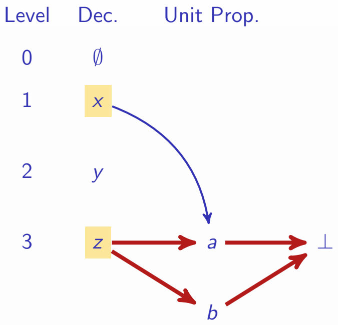
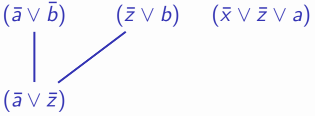
  - Create **new** clause: $(\bar{x} \vee \bar{z})$
- Can relate clause learning with resolution

Clause Learning





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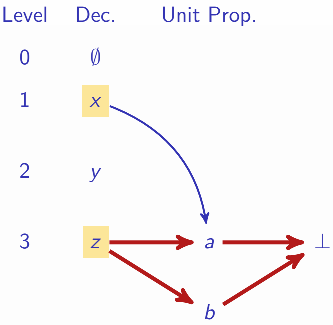
Clause Learning



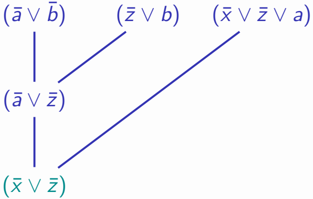
Level Dec. Unit Prop.

0 $\emptyset$

1 x

2 y

3 z a b $\perp$



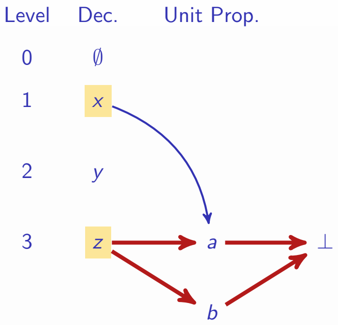
$(\bar{a} \vee \bar{b})$ $(\bar{z} \vee b)$ $(\bar{x} \vee \bar{z} \vee a)$

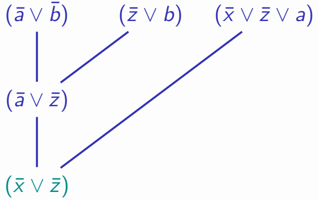
$(\bar{a} \vee \bar{z})$

$(\bar{x} \vee \bar{z})$

- Analyze conflict
  - Reasons: x and z
    - Decision variable & literals assigned at lower decision levels
  - Create **new** clause: $(\bar{x} \vee \bar{z})$
- Can relate clause learning with resolution

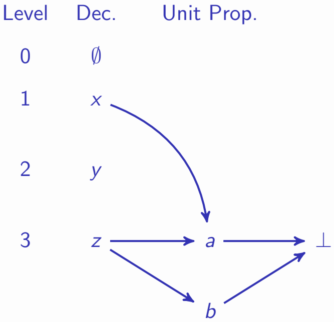
Clause Learning





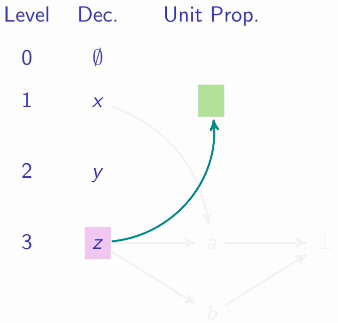
- Analyze conflict
  - Reasons: x and z
    - Decision variable & literals assigned at lower decision levels
  - Create **new** clause: $(\bar{x} \vee \bar{z})$
- Can relate clause learning with resolution
  - Learned clauses result from (**selected**) resolution operations

Clause Learning – After Backtracking



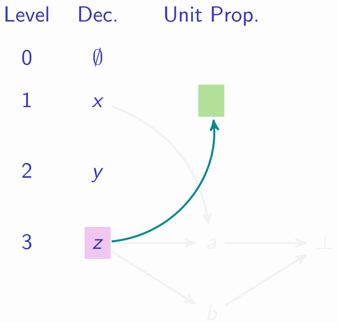
Level	Dec.	Unit Prop.	
0	$\emptyset$		
1	x		
2	y		
3	z	a	$\bot$
		b	

Clause Learning – After Backtracking



- Clause $(\bar{x} \vee \bar{z})$ is asserting at decision level 1

Clause Learning – After Backtracking



Level Dec. Unit Prop.

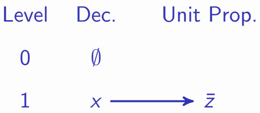
0 $\emptyset$

1 x

2 y

3 z a $\perp$

b



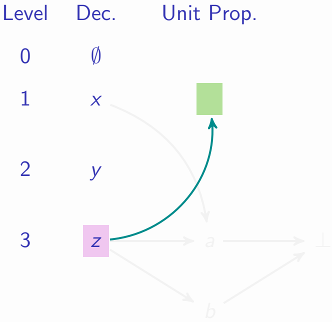
Level Dec. Unit Prop.

0 $\emptyset$

1 $x \longrightarrow \bar{z}$

- Clause $(\bar{x} \vee \bar{z})$ is asserting at decision level 1

Clause Learning – After Backtracking



Level Dec. Unit Prop.

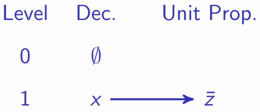
0 $\emptyset$

1 x

2 y

3 z a $\perp$

b



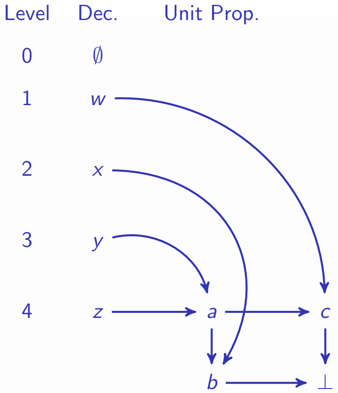
Level Dec. Unit Prop.

0 $\emptyset$

1 $x \longrightarrow \bar{z}$

- Clause $(\bar{x} \vee \bar{z})$ is asserting at decision level 1
- Learned clauses are always asserting [MSS96,MSS99]
- Backtracking differs from plain DPLL:
  - Always bactrack after a conflict [MMZZM01]

Unique Implication Points (UIPs)



Level Dec. Unit Prop.

0 $\emptyset$

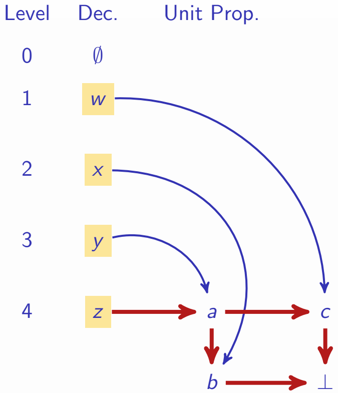
1 w

2 x

3 y

4 z

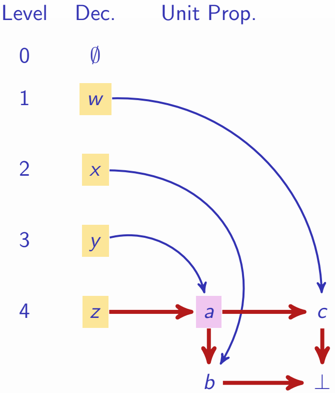
Unique Implication Points (UIPs)

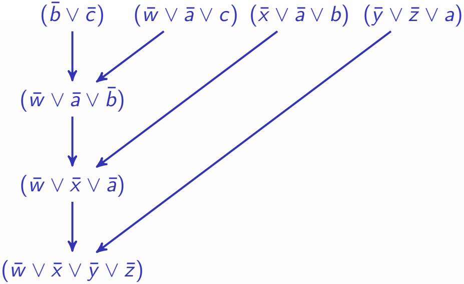


$$\begin{tikzcd}
(\bar{b} \vee \bar{c}) \arrow[d] & (\bar{w} \vee \bar{a} \vee c) \arrow[dl] & (\bar{x} \vee \bar{a} \vee b) \arrow[ddll] & (\bar{y} \vee \bar{z} \vee a) \arrow[dddlll] \\
(\bar{w} \vee \bar{a} \vee \bar{b}) \arrow[d] & & & \\
(\bar{w} \vee \bar{x} \vee \bar{a}) \arrow[d] & & & \\
(\bar{w} \vee \bar{x} \vee \bar{y} \vee \bar{z}) & & &
\end{tikzcd}$$

- Learn clause $(\bar{w} \vee \bar{x} \vee \bar{y} \vee \bar{z})$

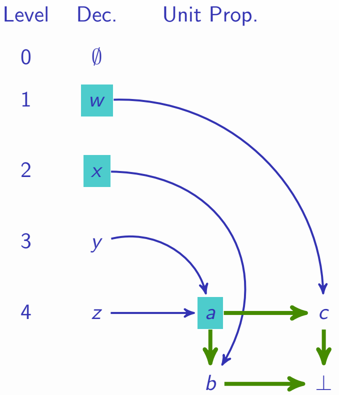
Unique Implication Points (UIPs)

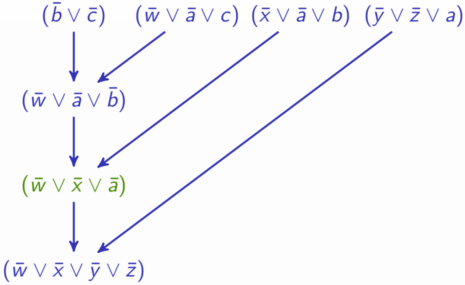



$(\bar{b} \vee \bar{c})$
$(\bar{w} \vee \bar{a} \vee c)$
$(\bar{x} \vee \bar{a} \vee b)$
$(\bar{y} \vee \bar{z} \vee a)$
$(\bar{w} \vee \bar{a} \vee \bar{b})$
$(\bar{w} \vee \bar{x} \vee \bar{a})$
$(\bar{w} \vee \bar{x} \vee \bar{y} \vee \bar{z})$

- Learn clause $(\bar{w} \vee \bar{x} \vee \bar{y} \vee \bar{z})$
- But a is an UIP
  - Dominator in DAG for level 4

Unique Implication Points (UIPs)




$(\bar{b} \vee \bar{c})$
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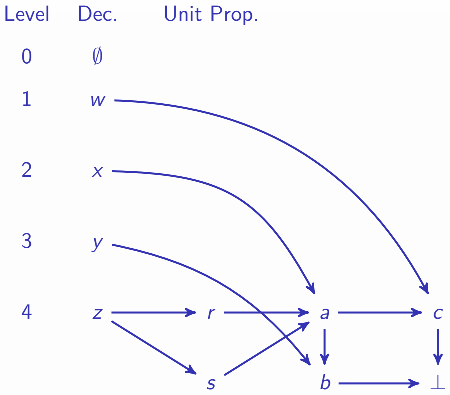
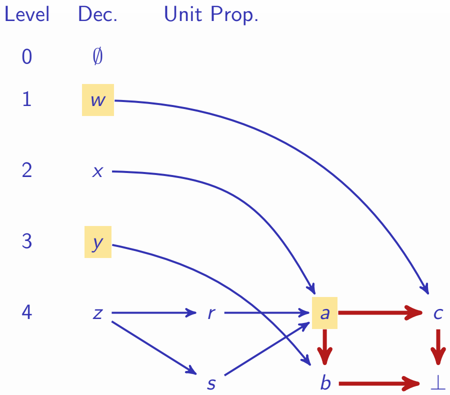
$(\bar{x} \vee \bar{a} \vee b)$
$(\bar{y} \vee \bar{z} \vee a)$
$(\bar{w} \vee \bar{a} \vee \bar{b})$
$(\bar{w} \vee \bar{x} \vee \bar{a})$
$(\bar{w} \vee \bar{x} \vee \bar{y} \vee \bar{z})$

- ~~Learn clause $(\bar{w} \vee \bar{x} \vee \bar{y} \vee \bar{z})$~~
- But a is an UIP
  - Dominator in DAG for level 4
- Learn clause $(\bar{w} \vee \bar{x} \vee \bar{a})$

Multiple UIPs

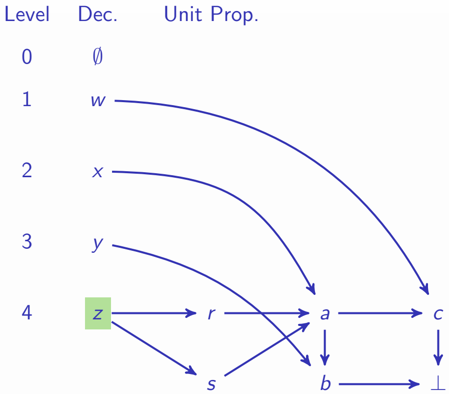


Multiple UIPs



- First UIP:
  - Learn clause $(\bar{w} \vee \bar{x} \vee \bar{a})$

Multiple UIPs



- First UIP:
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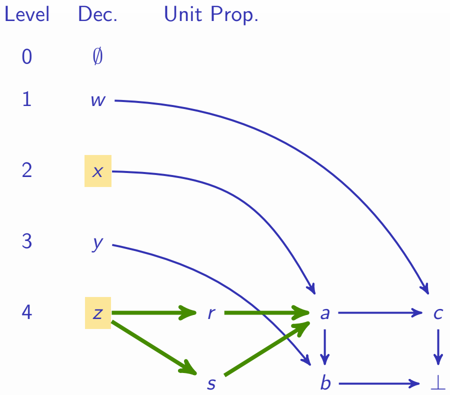
- But there can be more than 1 UIP

Multiple UIPs



- First UIP:
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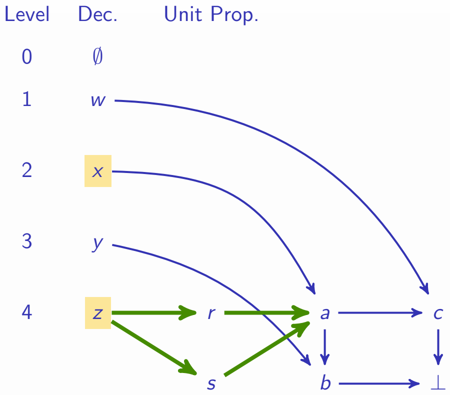
- But there can be more than 1 UIP
- Second UIP:
  - Learn clause $(\bar{x} \vee \bar{z} \vee a)$

Multiple UIPs



- First UIP:
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- But there can be more than 1 UIP
- Second UIP:
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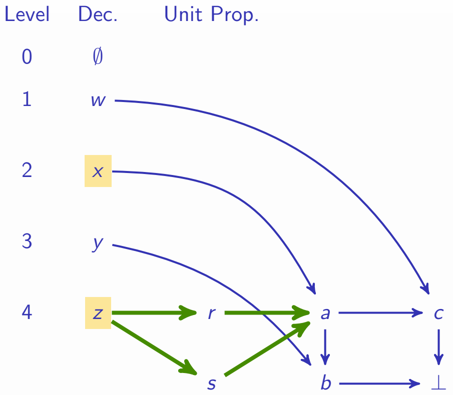
- In practice smaller clauses more effective
  - Compare with $(\bar{w} \vee \bar{x} \vee \bar{y} \vee \bar{z})$

Multiple UIPs



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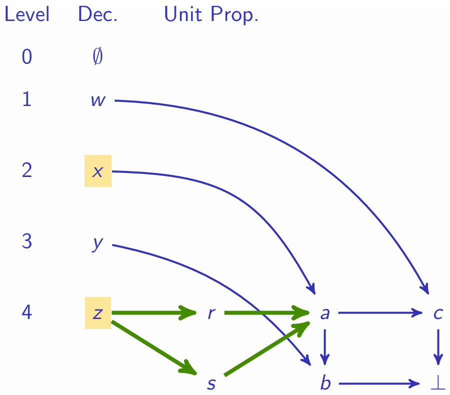
- Multiple UIPs proposed in GRASP [MSS96]
  - First UIP learning proposed in Chaff [MMZZM01]
- Not used in recent state of the art CDCL SAT solvers

Multiple UIPs



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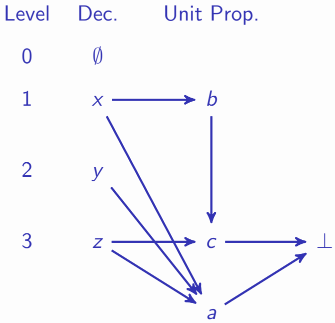
- Multiple UIPs proposed in GRASP [MSS96]
  - First UIP learning proposed in Chaff [MMZZM01]
- Not used in recent state of the art CDCL SAT solvers
- Recent results show it can be beneficial on current instances [SSS12]

Clause Minimization I



Level Dec. Unit Prop.

0 $\emptyset$

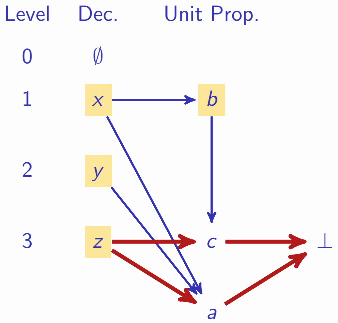
1 x b

2 y

3 z c $\perp$

a

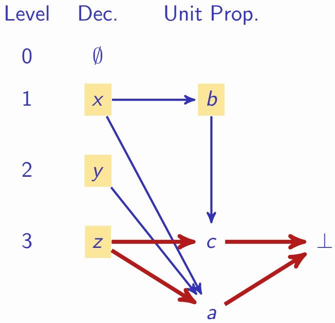
Clause Minimization I

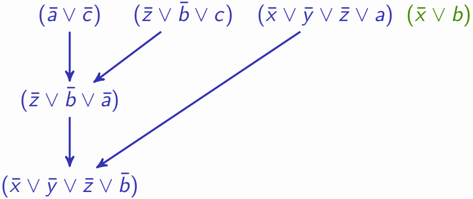


$$\begin{tikzcd}
(\bar{a} \vee \bar{c}) \arrow[d] & (\bar{z} \vee \bar{b} \vee c) \arrow[dl] & (\bar{x} \vee \bar{y} \vee \bar{z} \vee a) \arrow[ddll] \\
(\bar{z} \vee \bar{b} \vee \bar{a}) \arrow[d] & & \\
(\bar{x} \vee \bar{y} \vee \bar{z} \vee \bar{b}) & &
\end{tikzcd}$$

- Learn clause $(\bar{x} \vee \bar{y} \vee \bar{z} \vee \bar{b})$

Clause Minimization I

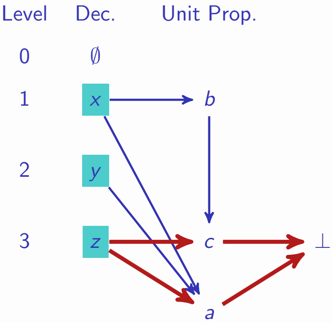


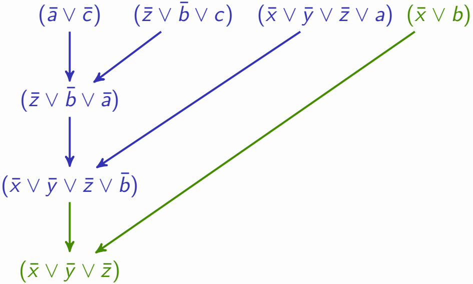


$(\bar{a} \vee \bar{c})$
$(\bar{z} \vee \bar{b} \vee c)$
$(\bar{x} \vee \bar{y} \vee \bar{z} \vee a)$
$(\bar{x} \vee b)$
$(\bar{z} \vee \bar{b} \vee \bar{a})$
$(\bar{x} \vee \bar{y} \vee \bar{z} \vee \bar{b})$

- Learn clause $(\bar{x} \vee \bar{y} \vee \bar{z} \vee \bar{b})$
- Apply self-subsuming resolution (i.e. local minimization) [SB09]

Clause Minimization I

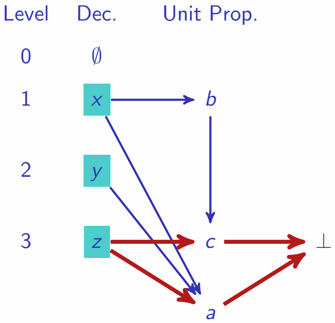


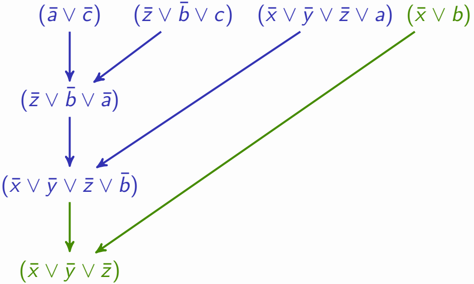


$(\bar a \vee \bar c)$
$(\bar z \vee \bar b \vee c)$
$(\bar x \vee \bar y \vee \bar z \vee a)$
$(\bar x \vee b)$
$(\bar z \vee \bar b \vee \bar a)$
$(\bar x \vee \bar y \vee \bar z \vee \bar b)$
$(\bar x \vee \bar y \vee \bar z)$

- ~~Learn clause $(\bar x \vee \bar y \vee \bar z \vee \bar b)$~~
- Apply self-subsuming resolution (i.e. local minimization)

Clause Minimization I

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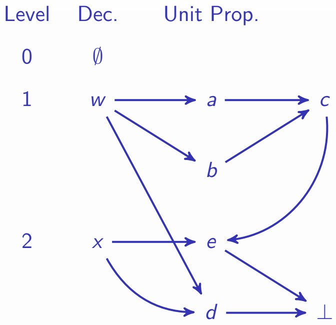
- Learn clause $(\bar{x} \vee \bar{y} \vee \bar{z})$

Clause Minimization II



Level Dec. Unit Prop.

0 $\emptyset$

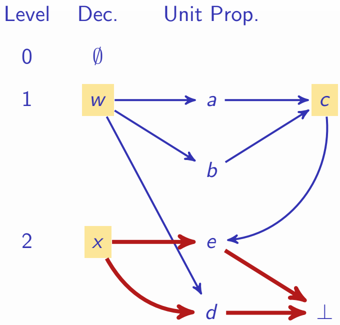
1 w a c

b

2 x e

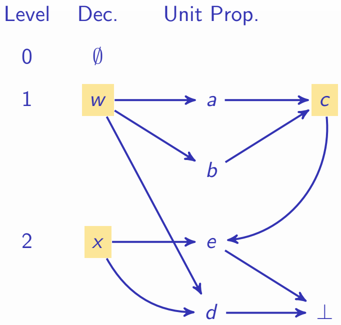
d $\perp$

Clause Minimization II



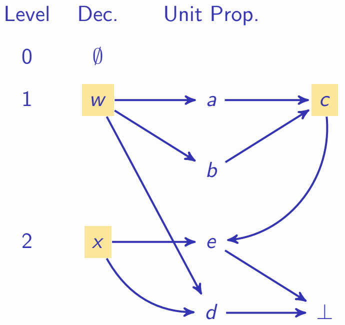
- Learn clause $(\bar{w} \vee \bar{x} \vee \bar{c})$

Clause Minimization II



- Learn clause $(\bar{w} \vee \bar{x} \vee \bar{c})$
- Cannot apply self-subsuming resolution
  - Resolving with reason of c yields $(\bar{w} \vee \bar{x} \vee \bar{a} \vee \bar{b})$

Clause Minimization II



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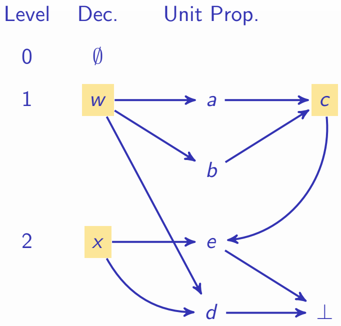
- Can apply recursive minimization

Clause Minimization II

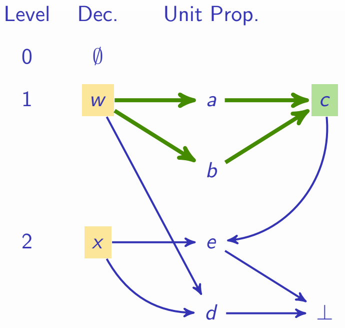


- ~~Learn clause ($\bar{w} \vee \bar{x} \vee \bar{c}$)~~
- Cannot apply self-subsuming resolution
  - Resolving with reason of c yields ($\bar{w} \vee \bar{x} \vee \bar{a} \vee \bar{b}$)
- Can apply recursive minimization

- Marked nodes: literals in learned clause

[SB09]

Clause Minimization II



- ~~Learn clause $(\bar{w} \vee \bar{x} \vee \bar{c})$~~
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  - Resolving with reason of c yields $(\bar{w} \vee \bar{x} \vee \bar{a} \vee \bar{b})$
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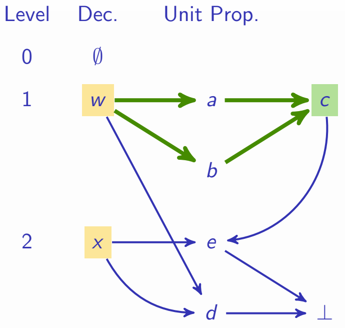
- Marked nodes: literals in learned clause [SB09]
- Trace back from c until marked nodes or new decision nodes
  - Drop literal c if only marked nodes visited

Clause Minimization II



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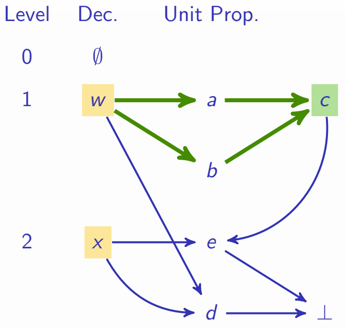
Clause Minimization II



- ~~Learn clause $(\bar{w} \vee \bar{x} \vee \bar{c})$~~
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  - Resolving with reason of c yields $(\bar{w} \vee \bar{x} \vee \bar{a} \vee \bar{b})$
- Can apply recursive minimization
- Learn clause $(\bar{w} \vee \bar{x})$

- Marked nodes: literals in learned clause [SB09]
- Trace back from c until marked nodes or new decision nodes
  - Drop literal c if only marked nodes visited
- Complexity of recursive minimization?

Outline

Basic Definitions

DPLL Solvers

CDCL Solvers

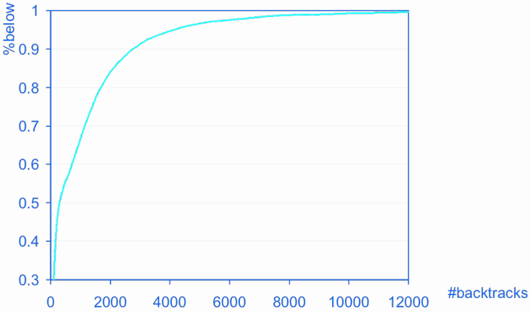
- Clause Learning, UIPs & Minimization
- Search Restarts & Lazy Data Structures

What Next in CDCL Solvers?

CNF Encodings

Search Restarts I

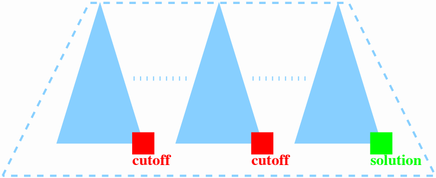
- Heavy-tail behavior: [GSK98]



  - 10000 runs, branching randomization on industrial instance
    - Use rapid randomized restarts (search restarts)

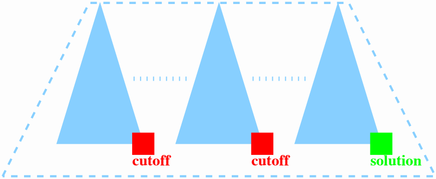
Search Restarts II

- Restart search after a number of conflicts



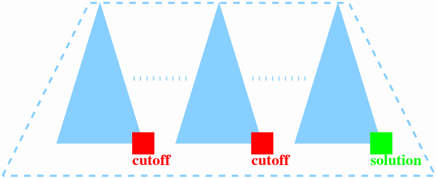
Search Restarts II

- Restart search after a number of conflicts
- Increase cutoff after each restart
  - Guarantees completeness
  - Different policies exist (see refs)



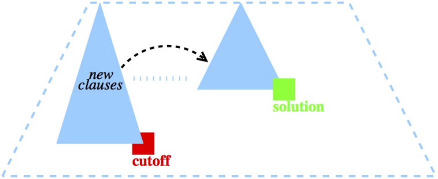
Search Restarts II

- Restart search after a number of conflicts
- Increase cutoff after each restart
  - Guarantees completeness
  - Different policies exist (see refs)
- Works for SAT & UNSAT instances. Why?



Search Restarts II

- Restart search after a number of conflicts
- Increase cutoff after each restart
  - Guarantees completeness
  - Different policies exist (see refs)
- Works for SAT & UNSAT instances. Why?
- Learned clauses effective after restart(s)



Data Structures Basics

- Each literal l should access clauses containing l
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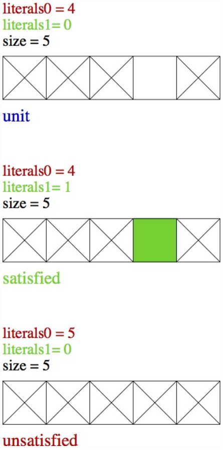
- Clause learning to be effective requires a more efficient representation: **Watched Literals**
  - Watched literals are one example of lazy data structures
    - But there are others

Watched Literals

[MMZZM01]

- Important states of a clause



literals0 = 4
literals1= 0
size = 5

unit

literals0 = 4
literals1= 1
size = 5

satisfied

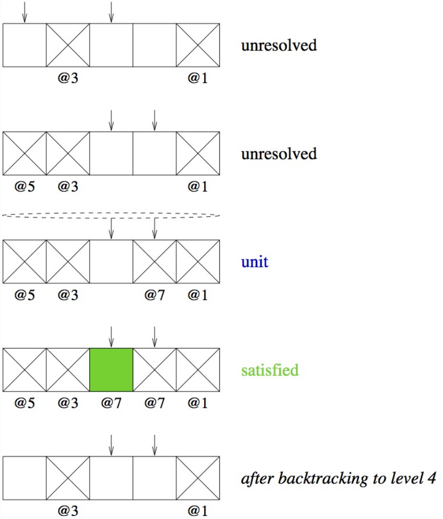
literals0 = 5
literals1= 0
size = 5

unsatisfied

Watched Literals

[MMZZM01]

- Important states of a clause
- Associate 2 references with each clause



Watched Literals

[MMZZM01]

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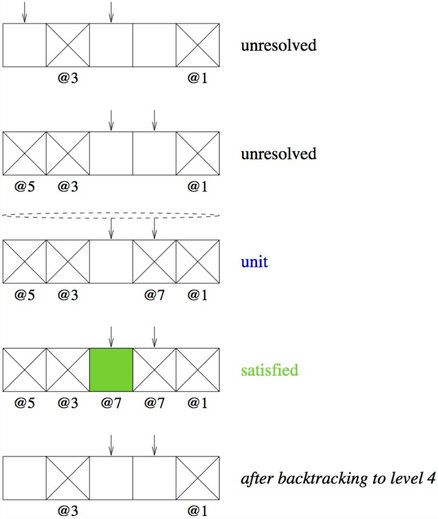
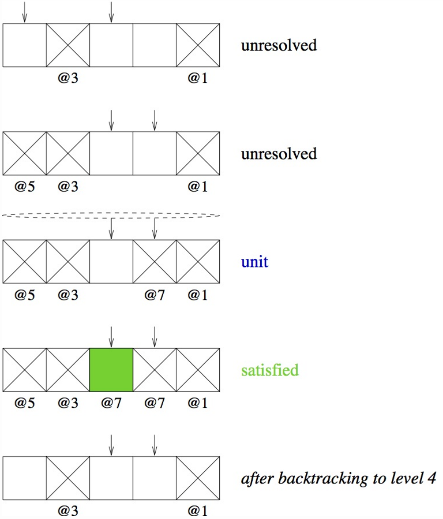


Watched Literals

[MMZZM01]

- Important states of a clause
- Associate **2** references with each clause
- Deciding unit requires traversing all literals
- References **unchanged** when backtracking



Watched Literals, in Practice

[ES03,G13]

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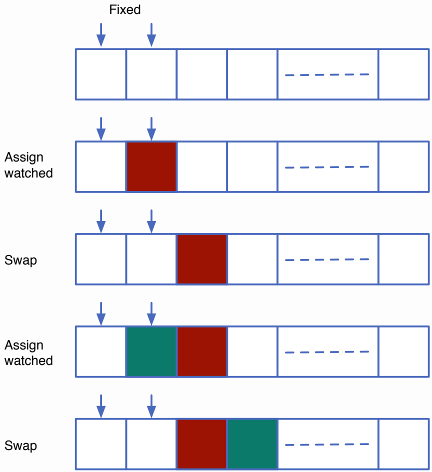

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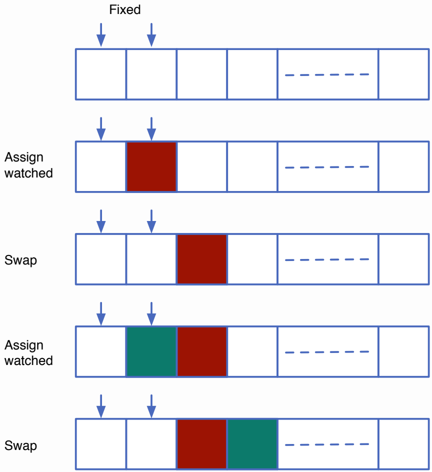

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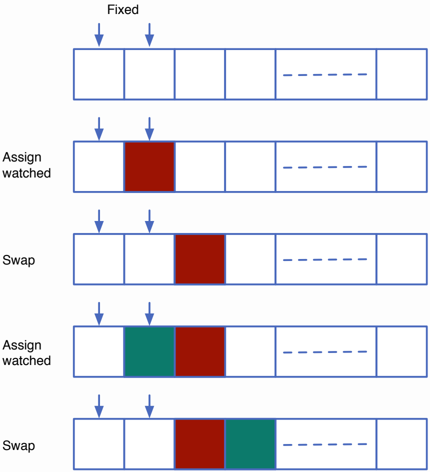

Watched Literals, in Practice

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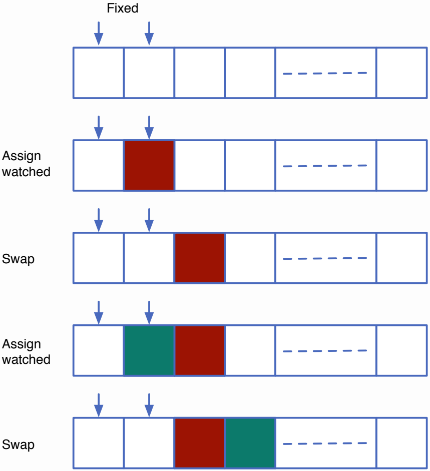
- In practice, first two positions of clause are watched
- May require to traverse already assigned literals, multiple times
- Worst-case time of unit propagation is **quadratic** on clause size and so on number of literals
- In practice, no gains observed from considering alternative implementations (see previous slide)



Additional Key Techniques

- Lightweight branching [e.g. MMZZM01]
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  - Delete larger clauses [e.g. MSS96]
  - Delete less used clauses [e.g. GN02,ES03]
- Proven recent techniques:
  - Phase saving [PD07]
  - Luby restarts [H07]
  - Literal blocks distance [AS09]

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Basic Definitions

DPLL Solvers

CDCL Solvers

What Next in CDCL Solvers?

CNF Encodings

CDCL – A Glimpse of the Future

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  - Clause learning is the key technique in CDCL SAT solvers
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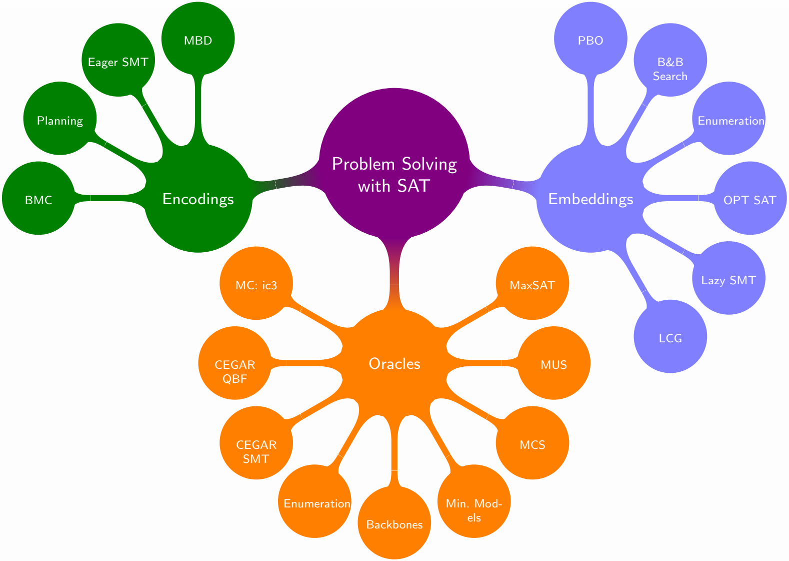
  - Essential in some applications

- Application-driven improvements
  - Incremental SAT
    - Handling of assumptions due to MUS extractors [LB13]
    - Changing SAT solvers to better suit applications [AS13]

But Also, SAT-Based Problem Solving



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CNF Encodings

Encoding to CNF

- What to encode?
  - Boolean formulas
    - Tseitin's encoding
    - Plaisted&Greenbaum's encoding
    - ...
  - Cardinality constraints
  - Pseudo-Boolean (PB) constraints
  - Can also translate to SAT:
    - Constraint Satisfaction Problems (CSPs)
    - Answer Set Programming (ASP)
    - Model Finding
    - ...

- Key issues:
  - Encoding size
  - Arc-consistency?

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CNF Encodings

- Boolean Formulas
- Cardinality Constraints
- Pseudo-Boolean Constraints
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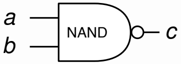
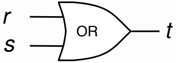
Representing Boolean Formulas / Circuits I

- Satisfiability problems can be defined on Boolean circuits/formulas
- Can represent circuits/formulas as CNF formulas [T68,PG86]
  - For each (simple) gate, CNF formula encodes the consistent assignments to the gate's inputs and output
    - Given $z = \mathrm{OP}(x, y)$, represent in CNF $z \leftrightarrow \mathrm{OP}(x, y)$
  - CNF formula for the circuit is the conjunction of CNF formula for each gate

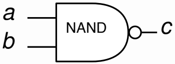
$$\mathcal{F}_c = (a \vee c) \wedge (b \vee c) \wedge (\bar{a} \vee \bar{b} \vee \bar{c})$$

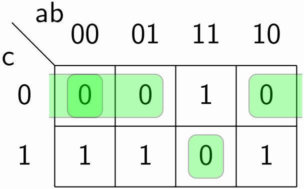


$$\mathcal{F}_t = (\bar{r} \vee t) \wedge (\bar{s} \vee t) \wedge (r \vee s \vee \bar{t})$$



Representing Boolean Formulas / Circuits II





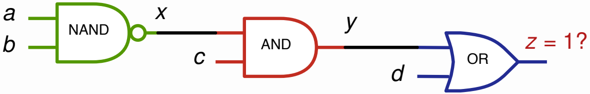
c \ ab	00	01	11	10
0	0	0	1	0
1	1	1	0	1

a	b	c	$\mathcal{F}_c$(a,b,c)
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

$$\mathcal{F}_c = (a \vee c) \wedge (b \vee c) \wedge (\bar{a} \vee \bar{b} \vee \bar{c})$$

Representing Boolean Formulas / Circuits III

- CNF formula for the circuit is the conjunction of the CNF formula for each gate
  - Can specify objectives with additional clauses



$$\begin{aligned}\mathcal{F} \quad = \quad & (a \vee x) \wedge (b \vee x) \wedge (\bar{a} \vee \bar{b} \vee \bar{x}) \wedge \\ & (x \vee \bar{y}) \wedge (c \vee \bar{y}) \wedge (\bar{x} \vee \bar{c} \vee y) \wedge \\ & (\bar{y} \vee z) \wedge (\bar{d} \vee z) \wedge (y \vee d \vee \bar{z}) \wedge (z)\end{aligned}$$

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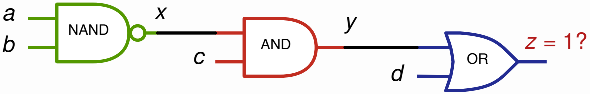

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- Note: $z = d \vee (c \wedge (\neg(a \wedge b)))$
  - **No** distinction between Boolean circuits and formulas

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What Next in CDCL Solvers?

CNF Encodings

- Boolean Formulas
- Cardinality Constraints
- Pseudo-Boolean Constraints
- Encoding CSPs

Cardinality Constraints

- How to handle cardinality constraints, $\sum_{j=1}^{n} x_j \leq k$?
  - How to handle AtMost1 constraints, $\sum_{j=1}^{n} x_j \leq 1$?
  - General form: $\sum_{j=1}^{n} x_j \bowtie k$, with $\bowtie \in \{<, \leq, =, \geq, >\}$

- Solution #1:
  - Use native PB solver, e.g. BSOLO, PBS, Galena, Pueblo, etc.
  - Difficult to keep up with advances in SAT technology
  - For SAT/UNSAT, best solvers already encode to CNF
    - E.g. Minisat+, WBO, etc.

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- Solution #2:
  - Encode cardinality constraints to CNF
  - Use SAT solver

Equals1, AtLeast1 & AtMost1 Constraints

- $\sum_{j=1}^{n} x_j = 1$: encode with $(\sum_{j=1}^{n} x_j \leq 1) \wedge (\sum_{j=1}^{n} x_j \geq 1)$
- $\sum_{j=1}^{n} x_j \geq 1$: encode with $(x_1 \vee x_2 \vee \ldots \vee x_n)$
- $\sum_{j=1}^{n} x_j \leq 1$ encode with:
  - Pairwise encoding
    - Clauses: $\mathcal{O}(n^2)$; No auxiliary variables
  - Sequential counter [S05]
    - Clauses: $\mathcal{O}(n)$; Auxiliary variables: $\mathcal{O}(n)$
  - Bitwise encoding [P07,FP01]
    - Clauses: $\mathcal{O}(n \log n)$; Auxiliary variables: $\mathcal{O}(\log n)$
  - ...

Bitwise Encoding

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- An example: $x_1 + x_2 + x_3 \leq 1$

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  - Auxiliary variables $v_0, \ldots, v_{r-1}$; $r = \lceil \log n \rceil$ (with $n > 1$)
  - If $x_j = 1$, then $v_0 \ldots v_{r-1} = b_0 \ldots b_{r-1}$, the binary encoding of $j - 1$

    $x_j \rightarrow (v_0 = b_0) \wedge \ldots \wedge (v_{r-1} = b_{r-1}) \Leftrightarrow (\bar{x}_j \vee (v_0 = b_0) \wedge \ldots \wedge (v_{r-1} = b_{r-1}))$

- An example: $x_1 + x_2 + x_3 \leq 1$

	$j-1$	$v_1 v_0$
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x_2	1	01
x_3	2	10

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    - $l_i \equiv v_i$, if $b_i = 1$
    - $l_i \equiv \bar{v}_i$, otherwise
  - If $x_j = 1$, assignment to v_i variables must encode $j - 1$
    - All other x variables must take value 0
  - If all $x_j = 0$, any assignment to v_i variables is consistent
  - $\mathcal{O}(n \log n)$ clauses ; $\mathcal{O}(\log n)$ auxiliary variables

- An example: $x_1 + x_2 + x_3 \leq 1$

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x_1	0	00
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x_3	2	10

$$(\bar{x}_1 \vee \bar{v}_1) \wedge (\bar{x}_1 \vee \bar{v}_0)$$
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$$(\bar{x}_3 \vee v_1) \wedge (\bar{x}_3 \vee \bar{v}_0)$$

General Cardinality Constraints

- General form: $\sum_{j=1}^{n} x_j \leq k$ (or $\sum_{j=1}^{n} x_j \geq k$)
  - Sequential counters [S05]
    - Clauses/Variables: $\mathcal{O}(n\,k)$
  - BDDs [ES06]
    - Clauses/Variables: $\mathcal{O}(n\,k)$
  - Sorting networks [ES06]
    - Clauses/Variables: $\mathcal{O}(n \log^2 n)$
  - Cardinality Networks: [ANORC09,ANORC11a]
    - Clauses/Variables: $\mathcal{O}(n \log^2 k)$
  - Pairwise Cardinality Networks: [CZI10]
  - ...

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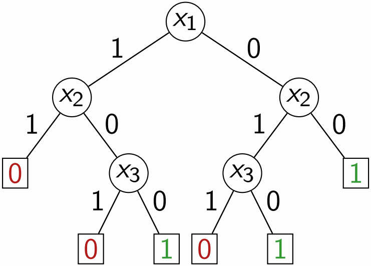
Pseudo-Boolean Constraints

- General form: $\sum_{j=1}^{n} a_j \, x_j \leq b$
  - Operational encoding [W98]
    - Clauses/Variables: $\mathcal{O}(n)$
    - Does not guarantee arc-consistency
  - BDDs [ES06]
    - Worst-case exponential number of clauses
  - Polynomial watchdog encoding [BBR09]
    - Let $\nu(n) = \log(n) \log(a_{max})$
    - Clauses: $\mathcal{O}(n^3 \nu(n))$; Aux variables: $\mathcal{O}(n^2 \nu(n))$
  - Improved polynomial watchdog encoding [ANORC11b]
    - Clauses & aux variables: $\mathcal{O}(n^3 \log(a_{max}))$
  - ...

Encoding PB Constraints with BDDs I

- Encode $3x_1 + 3x_2 + x_3 \leq 3$
- Construct BDD
  - E.g. analyze variables by decreasing coefficients
- Extract ITE-based circuit from BDD



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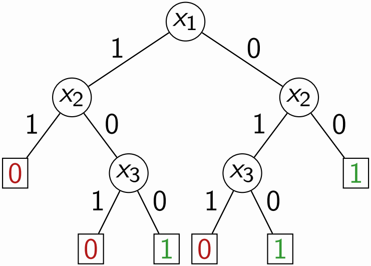
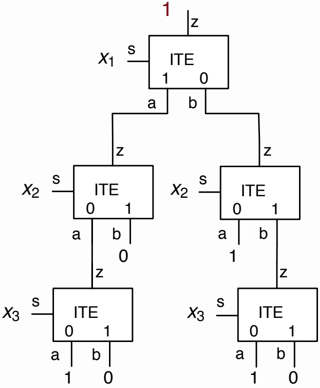
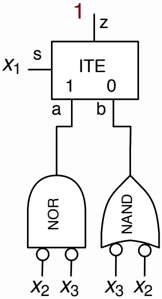




Encoding PB Constraints with BDDs II

- Encode $3x_1 + 3x_2 + x_3 \leq 3$
- Extract ITE-based circuit from BDD
- Simplify and create final circuit:



More on PB Constraints

- How about $\sum_{j=1}^{n} a_j \, x_j = k$?

More on PB Constraints

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  - Either constraint can still be satisfied, but **not** both

Outline

Basic Definitions

DPLL Solvers

CDCL Solvers

What Next in CDCL Solvers?

CNF Encodings

- Boolean Formulas
- Cardinality Constraints
- Pseudo-Boolean Constraints
- Encoding CSPs

CSP Constraints

- Many possible encodings:
  - Direct encoding [dK89,GJ96,W00]
  - Log encoding [W00]
  - Support encoding [K90,G02]
  - Log-Support encoding [G07]
  - Order encoding for finite linear CSPs [TTKB09]
  - ...

Direct Encoding for CSP w/ Binary Constraints

- Variable x_i with domain D_i, with $m_i = |D_i|$
- Represent values of x_i with Boolean variables $x_{i,1}, \ldots, x_{i,m_i}$
- Require $\sum_{k=1}^{m_i} x_{i,k} = 1$
  - Suffices to require $\sum_{k=1}^{m_i} x_{i,k} \geq 1$ [W00]
- If the pair of assignments $x_i = v_i \wedge x_j = v_j$ is not allowed, add binary clause $(\bar{x}_{i,v_i} \vee \bar{x}_{j,v_j})$

Thanks!