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COMPLEXITY OF INTERPOLATION AND RELATED PROBLEMS IN POSITIVE CALCULI

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Abstract. We consider the problem of recognizing important properties of logical calculi and find complexity bounds for some decidable properties. For a given logical system L, a property P of logical calculi is called decidable over L if there is an algorithm which for any finite set Ax of new axiom schemes decides whether the calculus $L + Ax$ has the property P or not. In [11] the complexity of tabularity, pre-tabularity, and interpolation problems over the intuitionistic logic Int and over modal logic S4 was studied, also we found the complexity of amalgamation problems in varieties of Heyting algebras and closure algebras.

In the present paper we deal with positive calculi. We prove NP-completeness of tabularity, DP-hardness of pretabularity and PSPACE-completeness of interpolation problem over Int^+. In addition to above-mentioned properties, we consider Beth's definability properties. Also we improve some complexity bounds for properties of superintuitionistic calculi.

§1. Introduction. Complexity of provability and satisfiability problems in non-classical logics, for instance, in intuitionistic logic, various systems of modal logic, temporal and dynamic logics was investigated in many papers (see, for instance, [3, 4, 9, 19, 21, 22]). R. Ladner [9] proved that the provability problem is PSPACE-complete for modal logics K, T and S4 and coNP-complete for $S5$. R. Statman [22] proved that the problem of determining if an arbitrary implicational formula is intuitionistically valid is PSPACE-complete. We consider the problem of recognizing some important properties of logical calculi and find complexity bounds.

Logical calculi are usually defined by systems of axioms and rules of inference. Natural problems arising in general study of logical calculi, for example, the problem of equivalence or the problem of determining for an arbitrary calculus whether it is consistent or not, are, in general, undecidable. When we restrict ourselves by considering particular families of calculi, for instance, propositional calculi extending intuitionistic or some modal logic, many important properties of calculi appear to be decidable (see a survey in [1]). When the rules of inference are fixed, for any given finite system of additional axioms, one can effectively decide the consistency problem for normal modal calculi, tabularity and interpolation problems for extensions of the intuitionistic logic or of the modal system $S4$ and some other problems. Sets of new postulates must necessarily be finite because

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of Kuznetsov's statement: No non-trivial property of logics is decidable under recursive axiomatization (see [1]).

Let a propositional calculus L_0 be given. We consider arbitrary extensions of L_0 by adding finitely many new axiom schemes. We say that a property P of logical calculi is *decidable over* L_0 if there is an algorithm which for any finite system Ax of axiom schemes decides whether the system $L_0 + Ax$ has the property P or not.

We take as L_0 some standard calculus for intuitionistic propositional logic Int or its positive fragment Int^+ containing the cut rule or modus ponens among its postulates. We consider the families $E(\mathrm{Int})$ of all superintuitionistic logics and $E(\mathrm{Int}^+)$ of extensions of the positive fragment Int^+ of the intuitionistic logic. The language of positive logics contains $\&, \vee, \rightarrow$ and $\top$ as primitive, the language of Int has an additional constant $\bot$. As usual, $\neg A \rightleftharpoons A \rightarrow \bot$. If A and B are two formulas, we denote by $A \vee' B$ a disjunction $A \vee B'$, where B' is a result of renaming variables of B such that A and B' have no variables in common.

The *size* $|A|$ of a formula A is the number of occurrences of variables and logical symbols in A.

Each calculus determines its *logic*, i.e., the set of its theorems. Two calculi are *equivalent* if they determine the same logic. A *superintuitionistic logic* is a set of formulas containing the set Int of all intuitionistically valid formulas and closed under substitution and modus ponens. A *positive logic* is a set of positive formulas containing Int^+ and closed under the same rules. A positive logic is determined by some set of axiom schemes added to Int^+. It is clear that one can replace a finite set of axiom schemes with their conjunction. We denote by $L + A$ the extension of a logic L by an extra axiom scheme A. In particular,

$$
\begin{aligned}
\mathrm{For} &= \mathrm{Int} + \bot, & \mathrm{For}^+ &= \mathrm{Int}^+ + p,\\
\mathrm{Cl} &= \mathrm{Int} + (p \vee \neg p), & \mathrm{Cl}^+ &= \mathrm{Int}^+ + (p \vee (p \rightarrow q)),\\
\mathrm{LC} &= \mathrm{Int} + (p \rightarrow q) \vee (q \rightarrow p), & \mathrm{LC}^+ &= \mathrm{Int}^+ + (p \rightarrow q) \vee (q \rightarrow p),\\
\mathrm{KC} &= \mathrm{Int} + (\neg p \vee \neg\neg p). & &
\end{aligned}
$$

A logic L is *consistent* if it is different from the set of all formulas in the language of L. A logic is called *tabular* if it can be characterized by finitely many finite models; and *pretabular* if it is maximal among non-tabular logics. A logic L is called *locally tabular* if for any finite set P of propositional variables there exist only finitely many formulas of variables in P non-equivalent in L. All tabular logics are locally tabular.

A logic L is said to have *Craig's interpolation property* (*CIP*), if for every formula $(A \rightarrow B) \in L$ there exists a formula C such that (i) both $A \rightarrow C$ and $C \rightarrow B$ belong to L, and (ii) every variable of C occurs in both A and B. A logic L is said to have the *projective Beth property* (*PBP*), if $L \vdash A(P, Q, X) \,\&\, A(P, Q', Y) \rightarrow (X \leftrightarrow Y)$ implies that there exists a formula $C(P)$ such that $L \vdash A(P, Q, X) \rightarrow (X \leftrightarrow C(P))$, where P, Q, Q' are disjoint lists of variables not containing variables X and Y; the *Beth property BP* is a special case of PBP, when Q and Q' are empty.

It follows from [7] that all logics in $E(\mathrm{Int}^+)$ and in $E(\mathrm{Int})$ possess the Beth property BP. In the same way as in [2], one can derive PBP from CIP in all these logics.

Let L be Int or Int$^+$. By the *tabularity* (*pretabularity, etc.*) *problem over* L we mean the problem of determining for arbitrary A, whether $L + A$ is tabular (pretabular, etc.), and consider its complexity with respect to the size of A over the intuitionistic logic Int and over the positive logic Int$^+$.

One can find necessary definitions from Complexity Theory in [6], [18]. We bring them out in Section 3.

In [11] we proved that (i) the tabularity problems over both Int and S4 are NP-complete, (ii) the pretabularity problems over both Int and S4 are in Δ_2^p and NP-hard, and coNP-hard, (iii) the local tabularity problem over S4 is NP-complete.

It is not yet known if the problem of local tabularity over Int or over Int$^+$ is decidable.

Also we stated that (i) the interpolation problem over Int is PSPACE-complete, (ii) the problem of determining whether Int $+ A$ is a tabular logic with CIP is NP-complete, (iii) the interpolation problem over S4 is in coNEXP and PSPACE-hard, (iv) the problems of determining whether S4 $+ A$ is a tabular (pretabular or locally tabular) logic with CIP is in Δ_2^p, and NP-hard, and coNP-hard.

In Sections 4 and 5 we prove analogous results for logics without negation. Moreover, we state that pretabularity problems over both Int and Int$^+$ are DP-hard, and find DP-complete problems over Int or Int$^+$. In addition, in Section 5 we consider the projective Beth property and find complexity bounds for PBP problem over Int. Also we prove that PBP problem over Int$^+$ is PSPACE-complete. In order to prove these results, we use an exhaustive description of positive extensions of Int$^+$ with PBP or CIP found in [12].

It is well known that there is a duality between $E(\text{Int})$ and the family of varieties of Heyting algebras. The analogous duality holds between positive logics and varieties of relatively pseudo-complemented lattices [20]. We proved in [11] that the amalgamation problem for finitely based varieties of Heyting algebras is PSPACE-complete. Rewriting the present results on positive logics in the algebraic language, we get that both amalgamation and PBP problems for finitely based varieties of relatively pseudo-complemented lattices are PSPACE-complete.

§2. Some reducibilities.

In our research the problem of equivalence between calculi is of great importance. In general, the equivalence problem is undecidable. When we restrict ourselves to considering particular families of logics, for instance, superintuitionistic logics then the problem of equivalence to a particular logic L may be undecidable too. On the other hand, this problem is decidable if we take, for instance, one of the logics Cl, KC or LC as L.

For arbitrary fixed L_1 in $E(\text{Int})$ we consider two inclusions Int $+ A \subseteq L_1$ and Int $+ A \supseteq L_1$. The former inclusion is the same as provability of A in L_1, and the latter is the same as provability of extra axioms of L_1 in Int $+ A$. It is evident that for any logic L_1 in $E(\text{Int})$, Int $+ A$ is equivalent to L_1 if and only if both inclusions Int $+ A \subseteq L_1$ and Int $+ A \supseteq L_1$ hold. On the other hand, we have

PROPOSITION 2.1. *Let L_0 be* Int *or* Int$^+$, *L_1 a finitely axiomatizable extension of L_0. Then each of inclusion problems $L_0 + A \subseteq L_1$ and $L_0 + A \supseteq L_1$ is polynomially reducible to equivalence to L_1 over L_0.*

PROOF. Take, for determiness, $L_0 = \mathrm{Int}$. Then A is provable in $L_1 = \mathrm{Int} + B$ if and only if $\mathrm{Int} + (A \& B)$ is equivalent to L_1, so provability problem of L_1 is linearly reducible to equivalence to L over Int. Further, for arbitrary A we have

$$\mathrm{Int} + A \supseteq L_1 \quad \text{if and only if} \quad \mathrm{Int} + (B \vee' A) \text{ is equivalent to } L_1. \qquad \dashv$$

If S is some set of logics extending a given logic L_0, we can consider the problem of membership in S (over L_0), i.e., the problem of recognizing whether $L_0 + A$ is in S.

We say that a logic L has the *Hallden Property HP* if for any formulas A and B without common variables, $L \vdash A \vee B$ implies $L \vdash A$ or $L \vdash B$.

PROPOSITION 2.2. *Let L_0 be* Int *or* Int^+, *S some family of extensions of L_0 containing some finitely axiomatizable $L_1 = L_0 + A$ satisfying the conditions*:

(1) $(\exists L' \supset L_1)\,(\forall L)\,((L_1 \subseteq L \subseteq L'$ *and* $L \in S) \Rightarrow L = L_1)$,

(2) *L_1 has the Hallden property.*

Then the problem of L_1-provability is polynomially reducible to the problem of membership in S over L_0.

PROOF. Assume $L_1 = \mathrm{Int} + A$, A' any formula in $L' - L_1$, B an arbitrary formula. Then we have $L_1 \vdash B$ if and only if $(\mathrm{Int} + A \,\&\, (A' \vee' B)) \in S$.

By the well-known Glivenko theorem one can reduce provability in the classical logic Cl to the provability in Int as follows: for each formula φ

$$\mathrm{Cl} \vdash \varphi \Leftrightarrow \mathrm{Int} \vdash \neg\neg\varphi.$$

Moreover,

$$\mathrm{Cl} \vdash \varphi \Rightarrow \mathrm{Int} + \neg\neg\varphi = \mathrm{Int}, \qquad \mathrm{Cl} \nvdash \varphi \Rightarrow \mathrm{Int} + \neg\neg\varphi = \mathrm{For}. \qquad \dashv$$

Also we can reduce Cl to Int^+.

PROPOSITION 2.3. *Let φ be a formula built from variables $p_1, \ldots, p_n$ and their negations with help of & and $\vee$, and q is different from $p_1, \ldots, p_n$. Denote*

$$g(\varphi) \rightleftharpoons \bigwedge (p_i \vee (p_i \leftrightarrow q)) \to \varphi^*,$$

where φ^* is the result of replacing all occurrences of $\neg p_i$ by $p_i \to q$. Then

$$\mathrm{Cl} \vdash \varphi \Leftrightarrow \mathrm{Int}^+ \vdash g(\varphi).$$

Moreover,

$$\mathrm{Cl} \vdash \varphi \Rightarrow \mathrm{Int}^+ + g(\varphi) = \mathrm{Int}^+,$$
$$\mathrm{Cl} \nvdash \varphi \Rightarrow \mathrm{Int}^+ + g(\varphi) = \mathrm{For}^+.$$

§3. Lower bounds for complexity. In this section we bring out necessary notions of Complexity Theory [6, 18] and find some lower bounds for complexity.

With any set X of formulas one can associate a decision problem: for arbitrary formula A to determine whether A is in X or not. Complexity classes P, NP, EXP, NEXP consist of sets X such that the problem of membership in X can be decided on Turing machines in polynomial time (with respect to the size of formula A), non-deterministic polynomial time, exponential time and non-deterministic exponential time respectively. For sets in PSPACE this problem can be decided in polynomial space. A set X is in coNP or in coNEXP if and only if its complement is in NP or in NEXP respectively. The class DP consists of all intersections $X \cap Y$ such that $X \in \mathsf{NP}$ and $Y \in \mathsf{coNP}$. The class Δ_2^p contains all sets decidable by polynomial-time oracle machines with an oracle in NP. It is known that

$$\mathsf{P} \neq \mathsf{EXP},$$
$$\mathsf{P} \subseteq \mathsf{NP} \cap \mathsf{coNP},$$
$$\mathsf{NP} \cup \mathsf{coNP} \subseteq \mathsf{DP} \subseteq \Delta_2^p \subseteq \mathsf{PSPACE} \subseteq \mathsf{EXP} \subseteq \mathsf{NEXP} \cap \mathsf{coNEXP}.$$

Each of the classes NP, coNP, Δ_2^p, PSPACE, EXP, NEXP, coNEXP is closed under finite unions and intersections of sets, and Δ_2^p contains all boolean combinations of sets in NP. In addition, the classes P, Δ_2^p and PSPACE are closed under complements.

Let $\mathscr{C}$ be a complexity class. A decision problem of X is *$\mathscr{C}$-hard* if any set Y in $\mathscr{C}$ is polynomially reducible to X. The problem is *$\mathscr{C}$-complete* if it is in $\mathscr{C}$ and $\mathscr{C}$-hard. To prove that a decision problem of X is $\mathscr{C}$-hard, it is sufficient to show that some $\mathscr{C}$-hard problem is polynomially reducible to X. To prove that X is in $\mathscr{C}$, it is sufficient to reduce X by a polynomial to some $\mathscr{C}$-complete problem.

One can find lists of $\mathscr{C}$-complete problems for known complexity classes in [6, 18]. Satisfiability problem of the classical propositional logic is a standard example of NP-complete problem, and validity and non-satisfiability in Cl are typical examples of coNP-complete problems. The best known example of DP-complete problem is SAT-UNSAT: Given two boolean formulas φ and ψ, to determine whether it is true that φ is satisfiable and ψ is not [18].

It is known that consistency of L over Int or over Int^+ is equivalent to the validity of all axioms of L in Cl. It immediately implies

PROPOSITION 3.1. *Consistency problems over both* Int *and* Int^+ *are* coNP-*complete.*

Now we find some lower bounds for complexity.

THEOREM 3.2. *Let P be a property of logics non-trivial on the class of finitely axiomatizable superintuitionistic (or positive) logics. Then the problem of determining for arbitrary formula A whether* $\mathrm{Int}+A$ (*resp.* $\mathrm{Int}^+ + A$) *has the property P is* NP-*hard or* coNP-*hard.*

Below we give more details. Theorem 1 immediately follows from

PROPOSITION 3.3. *Let L_0 be* Int *or* Int^+, $S \subseteq E(L_0)$, *L_1 and L_2 two finitely axiomatizable extensions of L_0 such that $L_1 \subset L_2$.*

(a) *If $L_1 \notin S$ and $L_2 \in S$ then the problem of membership in S over L_0 is* NP-*hard.*
(b) *If $L_1 \in S$ and $L_2 \notin S$ then the problem of membership in S over L_0 is* coNP-*hard.*

PROOF. Take, for definiteness, $L_0 = \mathrm{Int}^+$, $L_1 = \mathrm{Int}^+ + A_1$, $L_2 = \mathrm{Int}^+ + A_2$. By Proposition 2.3 one can prove that for arbitrary boolean formula φ

$$\mathrm{Cl} \vdash \varphi \Rightarrow \mathrm{Int}^+ + (A_1 \,\&\, (A_2 \vee' g(\varphi))) = L_1,$$
$$\mathrm{Cl} \nvdash \varphi \Rightarrow \mathrm{Int}^+ + (A_1 \,\&\, (A_2 \vee' g(\varphi))) = L_2. \qquad \dashv$$

PROPOSITION 3.4. *Let L_0 be* Int *or* Int^+, $S \subseteq E(L_0)$. *If there exist finitely axiomatizable extensions L, L' and L'' of L_0 such that $L' \subset L \subset L''$, $L \in S$ and L', $L'' \notin S$ then the problem of membership in S over L_0 is* DP-*hard.*

PROOF. Assume $L_0 = \mathrm{Int}^+$, $L = \mathrm{Int}^+ + A$, $L' = \mathrm{Int}^+ + A'$, $L'' = \mathrm{Int}^+ + A''$. For each pair φ, ψ of boolean formulas we define

$$f(\varphi,\psi) = A' \,\&\, (A'' \vee' g(\varphi) \vee' (A \,\&\, g(\psi))),$$

where g was defined in Proposition 2.3. One can prove that

$$(\mathrm{Cl} \nvdash \varphi \text{ and } \mathrm{Cl} \vdash \psi) \quad \text{if and only if} \quad (\mathrm{Int}^+ + f(\varphi,\psi)) \in S.$$

The proof for extensions of Int is analogous. We only re-define

$$f(\varphi,\psi) = A' \,\&\, (A'' \vee' \neg\neg\varphi \vee' (A \,\&\, \neg\neg\psi)). \qquad \dashv$$

THEOREM 3.5. *Let L_0 be* Int *or* Int^+, *L any finitely axiomatizable extension of L_0.*

(a) *If L is consistent then the problem of provability in L is* coNP*-hard and the problem of non-provability* NP*-hard.*
(b) *If $L \neq L_0$ then the problem of equivalence to L over L_0 is* NP*-hard.*
(c) *If L is consistent then the problem of equivalence to L over L_0 is* coNP*-hard.*
(d) *If $L \neq L_0$ and L is consistent then the problem of equivalence to L over L_0 is* DP*-hard.*

PROOF.
(a) It follows from Proposition 2.3 that $\mathrm{Cl} \vdash \varphi$ if and only if $L \vdash g(\varphi)$.
(b) and (c) follow from Proposition 3.3 by $S = \{L\}$.
(d) In Proposition 3.4 we take $S = \{L\}$, $L' = L_0$, $L'' = \mathrm{For}$. $\dashv$

§4. **Tabularity and pretabularity.** It was stated in [11] that the tabularity problem over Int is NP-complete, and also bounds of complexity for pretabularity problem over Int were found. Here we prove NP-completeness of the tabularity problem over Int^+ and DP-hardness of pretabularity problems over both Int and Int^+. Moreover, the problem of equivalence to L (over both Int and Int^+) is DP-complete for any fixed pretabular or consistent tabular logic L.

For calculation of complexity of tabularity and pretabularity problems we need some computational characteristics of particular logics and their models. Although there exist logics in $E(\mathrm{Int})$ and $E(\mathrm{Int}^+)$ which are not Kripke-complete, our main results can be proved in terms of Kripke models. We remind some definitions.

An *intuitionistic Kripke model* $\mathbf{M} = (W, \leq, \models)$ is a non-empty set W partially ordered by $\leq$, where truth-relation $\models$ satisfies the *monotonicity condition*:

$$(x \models p \text{ and } x \leq y) \Rightarrow y \models p$$

for each variable p and, moreover,

$$x \models A \to B \quad \text{if and only if} \quad \forall y\, (x \leq y \Rightarrow (y \models A \Rightarrow y \models B)),$$
$$x \models \top; \qquad x \not\models \bot;$$

$x \models (A\&B)$ and $x \models (A \vee B)$ are defined as usual.

We get a definition of model for positive logic Int^+ by deleting $\bot$.

If $\mathbf{M} = (W, \leq, \models)$ is an intuitionistic model, a formula A is called *true in* $\mathbf{M}$ if $x \models A$ for all $x \in W$; A is *satisfiable in* $\mathbf{M}$ if $x \models A$ for some $x \in W$; A is *refutable* in $\mathbf{M}$ if $x \not\models A$ for some $x \in W$. A formula A is *valid in a frame* $\mathbf{W} = (W, \leq)$, and $\mathbf{W}$ *validates* A if A is true in any model $\mathbf{M} = (W, \leq, \models)$ based on $\mathbf{W}$; A is *refutable in* $\mathbf{W}$ if it is not valid in $\mathbf{W}$.

We bring out a well-known

LEMMA 4.1. *For each finite frame* $\mathbf{W}$, *refutability in* $\mathbf{W}$ *is* NP-*complete and validity in* $\mathbf{W}$ *is* coNP-*complete.*

If L is a logic, a frame $\mathbf{W}$ is called an *L-frame* if all formulas provable in L are true in each model based on $\mathbf{W}$. We say that a formula A is *L-valid* if A is valid in all L-frames; A is *L-refutable* if A is not L-valid.

A logic L is called *Kripke-complete* if provability in L is equivalent to L-validity. A logic L is said to have *finite model property (FMP)* if provability in L is equivalent to validity in all finite L-frames. A logic L is *polynomially approximable* if any formula A non-provable in L is refutable in some L-frame whose cardinality is a polynomial function of the size of A.

LEMMA 4.2. *If a logic L in $E(\mathrm{Int})$ or in $E(\mathrm{Int}^+)$ is polynomially approximable by a class of frames definable by finitely many first order formulas then L-refutability problem is* NP-*complete and L-provability problem is* coNP-*complete.*

Let us define the following sequences of frames for $n \geq 1$:

- S_n is the set $\{1, \dots, n\}$ with the natural ordering relation;
- U_{n+1} is the set $\{0, 1, \dots, n+1\}$, where $0 < x < (n+1)$ for $1 \leq x \leq n$;
- V_n is the subframe of U_{n+1} obtained by deleting $(n+1)$,
- $V_n' = V_n \cup \{a\}$, where $a < 0 < n$ for all n.

In order to find the complexity of tabularity and pretabularity problems, we recall [13] that there are exactly three pretabular extensions of Int, namely, the logics

$$\mathrm{LC} = \mathrm{Int} + (p \to q) \vee (q \to p),$$
$$LP_2 = \mathrm{Int} + (p \vee (p \to q \vee \neg q)),$$
$$LQ_3 = \mathrm{Int} + (r \vee (r \to (p \vee (p \to q \vee \neg q))) + \neg p \vee \neg\neg p.$$

The logic LC is characterized by finite linearly ordered intuitionistic frames, LP_2 by finite frames satisfying the condition

$$\forall x \, \forall y \, \forall z \, (x \leq y \leq z \Rightarrow (x = y \text{ or } y = z)).$$

At last, LQ_3 is characterized by finite frames with the least and the greatest elements and with the chains of length not more than 3.

The logic Int^+ has exactly two pretabular extensions, namely, LC^+ and LP_2^+ which can be characterized by the same frames as LC and LP_2 respectively.

One can show that all pretabular superintuitionistic and positive logics are linearly approximable. By Lemma 4.2 we get

LEMMA 4.3. *For each pretabular logic L in $E(\mathrm{Int})$ or in $E(\mathrm{Int}^+)$, refutability problem of L is* NP-*complete, and validity problem of L is* coNP-*complete.*

We state

THEOREM 4.4. *Let L_0 be* Int *or* Int^+ *and L its pretabular or consistent tabular extension. Then the problem of equivalence to L over L_0 is* DP-*complete.*

PROOF. We restrict ourselves to pretabular superintuitionistic logics. Remind that $L_0 + A$ is equivalent to L if and only if A is valid in L and $L_0 + A \supseteq L$. The problem of L-validity is coNP-complete by Lemma 4.3. For the inclusion problem we use the following well-known criteria [5, 15]:

$$\begin{array}{lll} \mathrm{Int} + A \supseteq \mathrm{LC} & \text{if and only if} & A \text{ is refutable in both } V_2 \text{ and } U_3, \\ \mathrm{Int} + A \supseteq LP_2 & \text{if and only if} & A \text{ is refutable in } S_3, \\ \mathrm{Int} + A \supseteq LQ_3 & \text{if and only if} & A \text{ is refutable in both } V_2 \text{ and } S_4. \end{array}$$

We see that for every pretabular logic L we only should verify refutability of A in finitely many fixed finite frames, so our inclusion problem for L is in NP by Lemma 4.1. Therefore, the problem of equivalence to L over L_0 is in DP, so it is DP-complete by Theorem 3.5 (d). ⊣

Now we are in position to prove

THEOREM 4.5.

(i) *The tabularity problem over* Int^+ *is* NP-*complete.*
(ii) *The pretabularity problems over both* Int *and* Int^+ *are in* Δ_2^p, *and* DP-*hard.*

PROOF.
(i) It was proved by A. V. Kuznetsov [8] that a logic is tabular if and only if it is not contained in L for each pretabular logic L. So a logic $\mathrm{Int}^+ + A$ is tabular if and only if A is refutable in each of LC^+ and LP_2^+. Thus the tabularity problem over Int^+ is in NP by Lemma 4.3. On the other hand, this problem is NP-hard by Proposition 3.3 (a) (one can take $L_1 = \mathrm{Int}^+$ and $L_2 = \mathrm{For}^+$).
(ii) The pretabularity problem over Int^+ is the problem of membership in the set $\{\mathrm{LC}^+, LP_2^+\}$ which is in Δ_2^p by Theorem 4.4. Actually, this problem is in BH, the *Boolean hierarchy* [6]. On the other hand, it is DP-hard by Proposition 3.4. The proof for Int is analogous. ⊣

§5. Interpolation and projective Beth property. In this section we find complexity bounds for PBP problem in superintuitionistic calculi. Also we prove PSPACE-completeness of the interpolation and PBP problems over Int^+.

In [15] we described all superintuitionistic logics with CIP and proved decidability of the interpolation problem over Int. PSPACE-completeness of CIP over Int was stated in [11]. All extensions of Int with PBP were found in [10]; it appeared that there are exactly sixteen logics with PBP in $E(\mathrm{Int})$. All of them are finitely axiomatizable and have both finite model property and Hallden property. Decidability of PBP over Int was proved in [16].

THEOREM 5.1.

(i) *PBP problem over* Int *is in* coNEXP *and* PSPACE-*hard.*
(ii) *The problem of determining whether* $\mathrm{Int} + A$ *is a tabular logic with PBP* (*or CIP*) *is* NP-*complete.*
(iii) *The problem of determining whether* $\mathrm{Int} + A$ *is a locally tabular* (*or pretabular*) *logic with PBP is in* Δ_2^p *and* DP-*hard.*

PROOF. In order to prove this theorem, we study the problem of equivalence to L over Int for each of sixteen logics with PBP. Their list contains For, twelve locally tabular logics (among them all pretabular logics) and also Int, KC and $\mathrm{Int} + (q \vee (q \to (\neg p \vee \neg\neg p)))$ which are not locally tabular. Then (i) follows from Propositions 5.4, 5.3 and 2.2. Further, (iii) follows from Propositions 5.4 and 3.4.

At last, $L = \mathrm{Int} + A$ is a tabular logic with PBP if and only if L contains one of two tabular logics with PBP, so we get (ii) from Proposition 5.2. ⊣

REMARK. One can show that PBP problem over Int is of the same complexity as provability in $\mathrm{Int} + (q \vee (q \to (\neg p \vee \neg\neg p)))$.

PROPOSITION 5.2. *For each logic L with PBP in $E(\mathrm{Int}) - \{\mathrm{Int}\}$, the inclusion problem* $\mathrm{Int} + A \supseteq L$ *is* NP*-complete.*

PROOF. It is shown in [16] that for each of mentioned logics the inclusion $\mathrm{Int}+A \supseteq L$ is equivalent to refutability of A in finitely many finite frames. ⊣

PROPOSITION 5.3. *Let L be a consistent logic with PBP in $E(\mathrm{Int})$. Then the problem of L-provability is*

(i) coNP*-complete whenever L is locally tabular,*
(ii) *in* coNEXP *and* PSPACE*-hard otherwise.*

We omit the proof. From Theorem 3.5 and Propositions 5.2 and 5.3 we immediately get

PROPOSITION 5.4. *Let L be a logic with PBP in $E(\mathrm{Int})$. Then the problem of equivalence to L over* Int *is*

(i) NP*-complete for $L =$* For,
(ii) DP*-complete whenever L is consistent and locally tabular,*
(iii) *in* coNEXP *and* PSPACE*-hard otherwise.*

Now we consider positive logics.

THEOREM 5.5.

(i) *Both CIP and PBP problems over* Int^+ *are* PSPACE*-complete.*
(ii) *The problem of determining whether* $\mathrm{Int}^+ + A$ *is a tabular logic with CIP (or PBP) is* NP*-complete,*
(iii) *The problem of determining whether* $\mathrm{Int}^+ + A$ *is a pretabular logic with CIP is* DP*-complete.*

PROOF is similar to that of Theorem 5.1 and follows from Propositions 5.7–5.9 below.

In [14] we found all logics with CIP in $E(\mathrm{Int}^+)$. In the analogous way we described in [12] all positive logics with PBP. First we proved

LEMMA 5.6. *Let L be a positive logic in $E(\mathrm{Int}^+)$, and*

$$L' = \mathrm{Int} + (\neg p \vee \neg\neg p) + \{ A' \mid A \in L \},$$

where A' is obtained from A by substituting $(x \vee \neg x)$ for each variable x of A. Then

(i) *L has CIP if and only if L' has CIP,*
(ii) *L has PBP if and only if L' has PBP.*

We note that the transfer from L to L' in general does not preserve finite axiomatizability. Thus Lemma 5.6 does not give any immediate reduction of CIP and PBP problems over Int^+ to the corresponding problems over Int. Using the detailed description of superintuitionistic logics with PBP found in [10] and Lemma 5.6, in [12] we obtained

PROPOSITION 5.7. *There exist exactly seven positive logics with PBP in $E(\mathrm{Int}^+)$. They are*

(1) Int^+, *and its extensions by axiom schemes*
(2) $(p \to q) \vee (q \to p)$,
(3) $p \vee (p \to q)$,
(4) p,
(5) $r \vee (r \to (p \to q) \vee (q \to p))$,
(6) $r \vee (r \to p \vee (p \to q))$,
(7) $p \vee (p \to q) \vee (q \to r)$.

The logics (1)–(4) *possess CIP and the others do not possess CIP.*

It is well known that the logic $\mathrm{LC}^+ = \mathrm{Int}^+ + (2)$ is characterized by all finite chains, and $\mathrm{Cl}^+ = \mathrm{Int}^+ + (3)$ by the one-element frame; $\mathrm{Int}^+ + (4) = \mathrm{For}^+$ is inconsistent, so it has no Kripke model. One can show that $\mathrm{Int}^+ + (5)$ is determined by frames satisfying the condition

$$(x < y \leq u \text{ and } x < y \leq v) \Rightarrow (u \leq v \text{ or } v \leq u);$$

in addition, $\mathrm{Int}^+ + (6)$ is characterized by frames whose chains contain not more than two elements, and $\mathrm{Int}^+ + (7)$ by a two-element chain.

Further, all logics (2)–(7) are locally tabular, and (3), (4) and (7) are tabular; (2) and (6) are all pretabular logics in $E(\mathrm{Int}^+)$.

It is known [22] that the provability problem is PSPACE-complete for Int^+. We can prove that all logics (2)–(7) are polynomially approximable. By Lemma 4.2 we get

PROPOSITION 5.8. *The provability problem is* coNP-*complete for any consistent logic with PBP in* $E(\mathrm{Int}^+) - \{\mathrm{Int}^+\}$.

An analog of Proposition 5.2 also holds for positive logics. Namely, for any positive formula A we have ([12]):

$$\begin{aligned}
\mathrm{Int}^+ + A \supseteq \mathrm{Int}^+ + (2) &\iff V_2 \not\models A,\\
\mathrm{Int}^+ + A \supseteq \mathrm{Int}^+ + (3) &\iff S_2 \not\models A,\\
\mathrm{Int}^+ + A \supseteq \mathrm{Int}^+ + (4) &\iff S_1 \not\models A,\\
\mathrm{Int}^+ + A \supseteq \mathrm{Int}^+ + (5) &\iff V_2' \not\models A,\\
\mathrm{Int}^+ + A \supseteq \mathrm{Int}^+ + (6) &\iff S_3 \not\models A,\\
\mathrm{Int}^+ + A \supseteq \mathrm{Int}^+ + (7) &\iff (S_3 \not\models A \text{ and } V_2 \not\models A).
\end{aligned}$$

So we obtain

PROPOSITION 5.9. *Let L be a logic with PBP in* $E(\mathrm{Int}^+)$. *Then the problem of equivalence to L over* Int^+ *is decidable. More exactly, it is*

(i) NP-*complete for* $L = \mathrm{For}^+ = \mathrm{Int}^+ + (4)$,
(ii) DP-*complete for* $L = \mathrm{Int}^+ + (n)$, *where* $n = 2, 3, 5, 6, 7$,
(iii) PSPACE-*complete for* $L = \mathrm{Int}^+$.

§6. Algebraisation. It is well known that there is a duality between $E(\mathrm{Int})$ and the family of varieties of Heyting (or pseudo-boolean) algebras (see, for instance, [1]), and also between $E(\mathrm{Int}^+)$ and the family of varieties of relatively pseudo-complemented lattices [20]. If Ax is a set of formulas and $L = \mathrm{Int} + Ax$ then $\{A = \top \mid A \in Ax\}$ forms a base of identities for variety $V(L)$ associated with

L. The definitions for $L \in E(\mathrm{Int}^+)$ are analogous. On the other hand, with any identity $A = B$ one can associate a formula $A \leftrightarrow B$ which is valid in some algebra if and only if the identity $A = B$ holds in this algebra.

We denote the variety of all Heyting algebras by $\mathbf{H}$; $\mathbf{H}^+$ stands for the variety of all relatively pseudo-complemented lattices. Adding new identities to the identities determining $\mathbf{H}$, we define new varieties of Heyting algebras. If P is some property of varieties, a natural P-problem arises: for any given finite system of identities, to decide whether the variety of Heyting algebras defined by this system has the property P or not. For a number of properties, we find the complexity of P-problem with respect to the sum of the lengths of identities.

A variety V is *tabular* if it is generated by finitely many finite algebras, and *pretabular* if it is minimal among non-tabular varieties; V is *locally tabular* (or *locally finite*) if any finitely generated algebra in V is finite. For L in $E(\mathrm{Int})$ or in $E(\mathrm{Int}^+)$, L is tabular (pretabular or locally tabular) if and only if such is its associated variety $V(L)$.

The Beth properties and the interpolation property have natural analogs in the algebraic language. Let us fix a variety V and consider some system of equations on V.

We say that a system of equations *implicitly defines* its unknowns if its solution is unique (if there is any). A parameter of this system is *unessential* if implicit definability is preserved by varying this parameter. Then the *Beth property* BP means that implicit definability of the unknown implies the existence of an explicit term which gives a solution for the system. The *projective Beth property* PBP says that, in addition, this explicit term is independent of unessential parameters. The *interpolation property* IP may be formulated as follows:

If an equation ρ is derivable from a system Γ of equations then ρ is derivable from a system Γ' whose each equation is implied by Γ and contains only those variables which occur in both Γ and ρ.

One can easily prove that a logic L in $E(\mathrm{Int})$ or in $E(\mathrm{Int}^+)$ has CIP, BP or PBP if and only if $V(L)$ has IP, BP or PBP respectively. So each subvariety of $\mathbf{H}$ or of $\mathbf{H}^+$ possesses the Beth property BP [7].

We add one more property which is rather popular. Say that a variety V has *Amalgamation Property AP* if it satisfies the following condition for all algebras $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in V:

> If $\mathbf{A}$ is a common subalgebra of both $\mathbf{B}$ and $\mathbf{C}$ then there exist an algebra $\mathbf{D}$ in V and embeddings $\gamma\colon \mathbf{B} \to \mathbf{D}$, $\delta\colon \mathbf{C} \to \mathbf{D}$ which coincide on $\mathbf{A}$.

The following Proposition is an immediate corollary of Theorem 1 in [15].

PROPOSITION 6.1. *Let L be in $E(\mathrm{Int})$ or in $E(\mathrm{Int}^+)$. Then*

L has CIP if and only if $V(L)$ has Amalgamation Property.

Now we re-write the main results of Sections 4 and 5 in the algebraic language.

THEOREM 6.2.

(i) *Tabularity problem for $\mathbf{H}^+$ is* NP-*complete.*
(ii) *Pretabularity problem for $\mathbf{H}^+$ is in Δ_2^p and* DP-*hard.*
(iii) *Interpolation, Amalgamation and PBP problems for $\mathbf{H}^+$ are* PSPACE-*complete.*
(iv) *PBP problem for $\mathbf{H}$ is in* coNEXP *and* PSPACE-*hard.*

Remind [11] that the complexity of tabularity and pretabularity problems for $\mathbf{H}$ is the same as for $\mathbf{H}^+$, and Interpolation and Amalgamation problems for $\mathbf{H}$ are PSPACE-complete.

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