Parametric Gröbner basis Computation and Elimination

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- Gröbner basis was first proposed as a special basis of polynomial ideals by Buchberger (1965), where he also gave an algorithm to construct it.
- Comprehensive Gröbner system(CGS) and comprehensive Gröbner basis(CGB) were proposed by Weispfenning(1992) for parametric polynomial systems.
- Parametric Gröbner basis was proposed independently by Kapur (1994).

Main Applications of Comprehensive Gröbner System

1. solving systems of parametric polynomials under various specializations.
2. grouping together structurally similar polynomial systems under various specializations.
3. study the solution structure, such as dimension, leading terms, of a parametric system.
4. automatic geometric theorem proving and geometric theorem discovery.
5. geometric reasoning in model based scene analysis and image understanding.
6. program analysis – automatic generation of loop invariants.
7. analysis of equalities appearing in a problem formulated in the theory of real closed field.

Motivation

Example

Let $F = \{ax^2 + by^2, cx^2 + y^2, ax - cy\}$ where $\{x, y\}$ are unknowns and $\{a, b, c\}$ are parameters. Solve

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- A specialization $\sigma_{(1,2,3)}$ deduced by $(1, 2, 3) \in \mathbb{Q}^3$ is to set $a = 1$, $b = 2$ and $c = 3$. $\sigma_{(1,2,3)}(ax^2 + by^2) = x^2 + 2y^2$.

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  for other polynomials, we get $3x^2 + y^2, x - 3y$.
- Different specializations lead to different systems.

What is a Gröbner Basis?

Given a polynomial ideal generated by a basis $\{f, g, h\}$, its Gröbner basis is a special basis with very nice properties:

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- Every polynomial simplifies to a unique normal form (also called the canonical form) of the polynomial based on the term ordering used.

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- Using lexicographic ordering on terms, a Gröbner basis has a nice structure from which common zeros of polynomials can be computed.
- If a polynomial system has no solution, its Gröbner basis consists of a nonzero constant.

Buchberger's Algorithm for Computing a Gröbner basis

- Define an admissible term ordering: $t \geq 1$, and $t_1 \geq t_2 \implies s * t_1 \geq s * t_2$ for any term s.

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- Upon termination algorithm gives a Gröbner basis.

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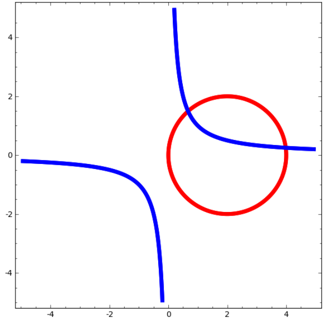

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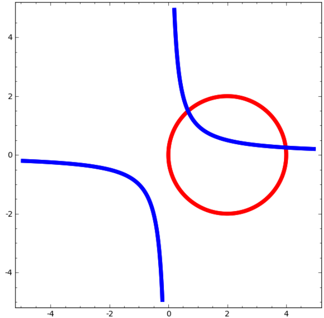

- Computing intersection points of the curves (which is the same as common solutions).

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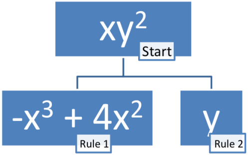

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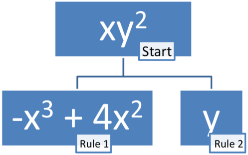

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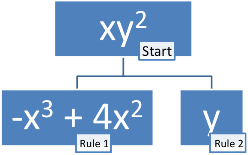

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- Simplifying the second polynomial: 4. $x^4 - 4x^3 + 1 = 0$.

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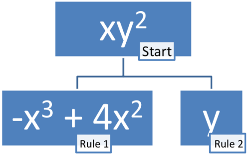

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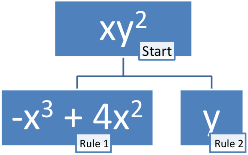

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- x and y coordinates of the four intersection points can be obtained from 4th and 3rd polynomials, resp.

Termination of Gröbner basis Algorithm

The process of generating additional rules always terminate because of a very elegant and simple combinatorial result from number theory:

Dickson's Lemma

Every infinite subset of N^k has at least two comparable k-tuples where comparison is done component-wise. $(a, b) \geq (c, d)$ iff $a \geq c \wedge b \geq d$.

Parametric Polynomial Systems

Example

Let $F = \langle ax^2 + by^2, cx^2 + y^2, ax - cy \rangle$ where $\{x, y\}$ are unknowns and $\{a, b, c\}$ are parameters.

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- Different specializations lead to different systems.
  - The Gröbner basis for $\sigma_{(0,1,0)}(F)$ = (y^2) is $\{y^2\}$.
  - The Gröbner basis for $\sigma_{(1,2,1)}(F)$ = $(x^2 + 2y^2, x^2 + y^2, x - y)$ is $\{x - y, y^2\}$.

Comprehensive (Parametric) Gröbner Basis

A finite set $CGB \subset \langle F \rangle$ is a comprehensive Gröbner basis iff for every specialization σ of parameters, $\sigma(CGB)$ is a Gröbner basis of $\sigma(\langle F \rangle)$.

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It is desirable to have $CGB \subset \langle F \rangle$.

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- Identify groups of specializations σ's that share the same Gröbner basis for the specialized ideal $\sigma(\langle F \rangle)$.

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- For each such set of specialization, compute the respective Gröbner basis.

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Comprehensive Gröbner System and Basis

- Comprehensive Gröbner system(CGS): $\{(A_1, G_1), \cdots, (A_l, G_l)\}$ for a finite basis F of a parametric ideal $\langle F \rangle$, where $\mathbb{C}^3 = A_1 \cup \cdots \cup A_l$ and $G_1, \cdots, G_l \subset \mathbb{Q}[a, b, c][x, y]$, s.t. $\sigma_\alpha(G_i)$ is a Gröbner basis for $\langle \sigma_\alpha(F) \rangle$ for $\forall \alpha \in A_i$..

  Each A_i is specified by a finite set of polynomial equality constraints and a finite set of polynomial disequality constraints over the parameters.

$$\{(A_i, G_i)\} = \begin{cases} \mathbb{C}^3 \setminus V(abc - a^2), & \{ax - cy, (bc - a)y^2\} \\ V(a) \setminus V(c), & \{cy, cx^2 + y^2\} \\ V(a, c), & \{y^2\} \\ V(bc - a) \setminus V(ab^2 + ac), & \{ax - cy, (b^2 + c)y^2\} \\ V(bc - a, b^2 + c, ab + c^2, c^3 + a^2) \setminus V(ac), & \{c^2 x + bcy\} \end{cases}$$

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- Comprehensive Gröbner basis(CGB): $G \subset \langle F\rangle \subset \mathbb{Q}[a, b, c][x, y]$, s.t. $\sigma_\alpha(G)$ is a Gröbner basis for $\langle\sigma_\alpha(F)\rangle$.

$$G = \{ax - cy, (bc - a)y^2, cx^2 + y^2, (bc - a)xy + (b^2 + c)y^2, abx - bcy\}.$$

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  - For the leading coefficients of the leading terms of these polynomials, perform case analysis assuming the leading coefficients are nonzero as well as zeros.
  - Such a case analysis gives possibly many branches.
- Repeat this process until all branches have been generated (i.e., all possible parametric specializations have been considered).

Key Idea of Kapur, Sun and Wang's algorithm (ISSAC 2010)

Among a set of polynomials which are candidates for inclusion in a Gröbner basis of a branch, it suffices to consider only those polynomials whose leading terms in **variables** are minimal and noncomparable to each others (since polynomials with leading comparable terms can be used to simplify each other).

Illustration

Example

Let $F := \{ax^2 + by^2, cx^2 + y^2, ax - cy\} \subset \mathbb{Q}[a, b, c][x, y]$ where $\{x, y\}$ are variables and $\{a, b, c\}$ are parameters.

- Gröbner basis for $\langle F \rangle \subset \mathbb{Q}[x, y, a, b, c]$ w.r.t. $x, y \gg a, b, c$.

$$G = \{ax - cy, (bc - a)y^2, (ab + c^2)y^2, (c^3 + a^2)y^2,$$

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Theorem 4.3 in (Kapur et al. 2010) shows $\sigma_\alpha(G_1)$ is a Gröbner basis for $\langle \sigma_\alpha(F) \rangle$ for $\forall \alpha \in A_1 = \mathbb{C}^3 \setminus V(abc - a^2)$, which is the same as $a \neq 0 \wedge (bc - a) \neq 0$.

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- For other branches of CGS, making leading coefficients of some of these polynomials in G_1 to be nonzero, one by one.
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  Gröbner basis for $\langle F \cup \{a\}\rangle$ is:

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- Similarly, to generate additional branches, make $c = 0$, and so on.

Faithfulness of Comprehensive Gröbner basis

Comprehensive Gröbner system(CGS):

$$\{(A_i, G_i)\} = \begin{cases} \mathbb{C}^3 \setminus V(a(abc-a)), & \{ax-cy, (bc-a)y^2\} \\ V(a) \setminus V(c), & \{cy, cx^2+y^2\} \\ V(a,c), & \{y^2\} \\ V(bc-a) \setminus V(a(b^2+c)), & \{ax-cy, (b^2+c)y^2\} \\ V(bc-a, b^2+c, ab+c^2, c^3+a^2) \setminus V(ac), & \{c^2x+bcy\} \end{cases}$$

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- Split a polynomial into two parts, nonzero part and zero part under a specialization or more precisely, along a branch component.

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New technique

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1. $f = p + \bar{p}, \qquad f \longmapsto (p, \bar{p})$ *s.t.*
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Modified KSW's Algorithm and CGB Computation

- The new technique is applied to the algorithm in (Kapur et al. 2010) to compute comprehensive Gröbner bases.

Exa.	heuristics(Reduce)	Suzuki-Sato(Risa/Asir)	New(Singular)
F6	0.590	*error*	0.310
F8	$>$ 1h	0.6708	0.650
S1	$>$ 1h	*error*	0.120
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- ANY algorithm for computing CGS can use this new technique to compute CGB.

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  - CGB is made monic in $K[U, X]$, i.e. $LC_U(CGB)$ are monic.

Example 1

Example

Given an ideal
$I = \langle -2ux^2 - vxy, -4vy^2 + (-2u + 2v) \subseteq \rangle K[u, v][x, y]$ and a lexicographical term order $>$ with $x > y \gg u > v$.

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By the KSW algorithm, a CGS and CGB of I w.r.t. $>$ are:

	branch	basis	LT
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2	$v = 0 \wedge u \neq 0$	$\{vy^2 + (\frac{1}{2}u - \frac{1}{2}v)\}$	$\{1\}$
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A branch is represented by $(E = \{e_1, \ldots, e_k\}, N = \{n\})$, which defines the set of specializations $V(E) - V(N)$.

GB vs. CGB

	Gröbner Basis	**Comprehensive Gröbner**
minimal	Proper subsets not GB	Proper subsets not CGB.
monic	$LC(g) = 1$	$LC(g)$ is monic.
reduced	polynomials in normal form	
canonical	GB is reduced.	

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  - KSWCGB (KSW, 2011) – neither minimal nor unique.
- Starting with KSWCGB, compute a minimal CGB (MCGB), with a goal of achieving the canonical CGB (CGB) which is both minimal and reduced.

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  - Simplify essential polynomials while maintaining the CGBness.
- Result: a CGB of I which contains only essential polynomials, i.e. a minimal CGB of I.

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  - $f \in CGB$ is **essential** w.r.t. CGB, iff there is subset B of the parameter space, such that f has no covering by $CGB - \{f\}$ in B.

Example 2

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Give an ideal $I = \langle f = uy + x, g = vz + x + 1\rangle \subseteq K[v,u][z,y,x]$ and a lexicographical term order with $z > y > x \gg v > u$.

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- The CGB by Weispfenning's algorithm (2003) is

$$CGB = \{f, g, h, -h\}$$

where $h = \boxed{g - f} = vz - uy + 1$.

$$RGB = \{f, g\}$$

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  - For $\sigma : u = v = 0$, $\sigma(I) = \langle x, x + 1 \rangle$, implying $1 \in \sigma(I)$. Neither $\sigma(f)$ nor $\sigma(g)$ can reduce it to 0, implying that $h = g - f$ is essential ($\sigma(h) = 1$).

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  - If f has a covering in each of such branches, then f is non-essential (redundant).

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First, compute CGS and CGB using the KSW algorithm:

	branch	basis	LT
1	$3u - 2v \neq 0 \wedge v \neq 0$	$\{g_1, g_2\}$	$\{y, x^2\}$
2	$v = 0 \wedge u \neq 0$	$\{g_2, g_5\}$	$\{y, x^2\}$
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$$\begin{aligned} CGB = \{g_1 &= (u - (2/3)v)y + (-(4/3)u^2 - (1/2)uv + (4/3)v^2)z, \\ g_2 &= vx^2 - 3y + (4u + (11/2)v)z, \\ g_4 &= (u - (40/49)v)x^2 + (22/49)y + ((36/49)u - (24/49)v)z, \\ g_5 &= (u - (2/3)v)x^2 + ((4/3)u + (1/3)v)z\} \end{aligned}$$

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$$

- g_5 appears in A_2 with LT x^2: g_4 covers it with LT x^2;
- g_5 appears in A_4 with LT z: g_1 covers it with LT z.

When a Polynomial Is Non-Essential

- $f \in CGB$ is non-essential w.r.t. CGB:
  - Update CGS by substituting f's covering for each occurrence of f;
  - Remove f from CGB.
- In Example 3, g_5 is non-essential w.r.t. CGB:

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- g_4 appears in A_2 with LT x^2: $CGB - G_2 = \{g_1\}$ doesn't cover g_4 in A_2.

Example 3 (result)

Finally, we achieve a minimal CGB of I w.r.t. $>$:

$$
\begin{aligned}
M = \{g_1 &= (u - (2/3)v)y + (-(4/3)u^2 - (1/2)uv + (4/3)v^2)z,\\
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GB vs. CGB

	Gröbner Basis	**Comprehensive Gröbner**
minimal	Proper subsets not GB	Proper subsets not CGB
monic	$LC(g) = 1$	$LC(g)$ is monic.
reduced	polynomials in normal form	simplification violates CGBness
canonical	*GB* is reduced.	*CGB* is minimal and reduced

Towards a Canonical CGB (CCGB)

- There can be multiple MCGBs of the same parametric ideal.
  - In Example 3, besides M, it's easy to check that both

    $$M_1 = \{g_1, g_2, g_4\}$$

    and

    $$M_2 = \{g_1, g_2, g_5\}$$

    are MCGBs of I.
- These MCGBs are comparable using the set ordering w.r.t. $>$:

  $$M < M_1 < M_2,$$

  since $g_1 < g_2 < g_3 < g_4 < g_5$.
- The least MCGB under the set ordering w.r.t. $>$ since this is a total ordering.

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- A term ordering is a well-ordering implying a polynomial ordering is a well-ordering leading to a well-ordering on finite subsets of polynomials. Hence all CGB's can be ordered and the least element is a canonical CGB.

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$$S = \bigcup_{\sigma} \{\text{the least } p \in I \mid LT(\sigma(p)) \text{ is minimal} \in LT(\sigma(I))$$

Lemma: *S is finite and is a CGB.*

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Theorem (Kapur, JSSC 2017) *Let $C \subseteq S$ be a subset of polynomials after removing from S all redundant polynomials in the descending order. C is the least CGB among all CGBs of I and is unique.*

A completion Procedure for cgb MCGB directly from a parametric basis

See Kapur and Yang in ISSAC 2016,

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