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Hierarchical reasoning in local theory extensions and applications

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***Abstract*—We give an overview of results on hierarchical and modular reasoning in complex theories we established in previous work. We present a special type of extensions of a base theory, namely local theory extensions, for which hierarchic reasoning is possible (i.e. proof tasks in the extension can be hierarchically reduced to proof tasks w.r.t. the base theory). We show how such local theory extensions can be identified and under which conditions locality is preserved when combining theories. We briefly discuss the way we used the possibility of hierarchical reasoning in local theory extensions in various application areas such as automated reasoning in mathematics, verification of reactive, real time and hybrid systems, and description logics.**

***Keywords*-Automated Reasoning; Combinations of Theories; Deductive Verification**

I. Introduction

Many problems in mathematics and computer science can be reduced to proving the satisfiability of conjunctions of literals in a background theory (which can be the extension of a base theory with additional functions – e.g., free, monotone, or recursively defined – or a combination of theories). It is therefore very important to identify situations where reasoning in complex theories can be done efficiently and accurately. Efficiency can be achieved, on the one hand, by reducing the search space and, on the other hand, by exploiting possibilities for modular reasoning, i.e., delegating some proof tasks which refer to a specific theory to provers specialized in handling formulae of that theory. Identifying situations where reducing the search space and/or modular reasoning can be achieved without loss of completeness is extremely important, especially in applications where efficient algorithms (in space, but also in time) are essential.

In [13], [24] Givan and McAllester introduced the so-called "local inference systems" (for which validity of ground Horn clauses can be checked in polynomial time). A link between this proof theoretic notion of locality and algebraic arguments used for identifying classes of algebras with a word problem decidable in PTIME [3] was established in [10]. In [11], [27] these results were further extended to so-called *local extensions* of theories. Locality phenomena were also studied in the verification literature, mainly motivated by the necessity of devising methods for efficient reasoning in theories of pointer structures [25] and arrays [2]. In [18] we showed that these results are instances of a general concept of locality of a theory extension – parameterized by a closure operator on ground terms. In fact, many theories important for computer science or mathematics fall into this class (typical examples are theories of data structures, theories of free or monotone functions, but also functions occurring in mathematical analysis).

In this paper we give an overview of results on hierarchical and modular reasoning in (combinations) of local theory extensions we established in previous work. We show how such local theory extensions can be identified and under which conditions locality is preserved when combining theories, and we investigate possibilities of efficient reasoning in such theory extensions. We give examples of local theory extensions and mention application areas – e.g. mathematics, verification of reactive, real time and hybrid systems – where such theories occur in a natural way.

The paper is structured as follows: In Section II the notion of local theory extension is introduced, and a method for hierarchical reasoning in such extensions is presented. Section III presents possibilities of identifying local theory extensions; Section IV presents locality transfer results when (i) enriching the base theory or (ii) combining theories. Section V provides various examples of local theory extensions. In Section VI we present several application areas.

II. Local Theory Extensions

We assume standard definitions from first-order logic. In this paper, (logical) theories are simply sets of sentences.

Extensions of theories. Let $\Pi_0{=}(\Sigma_0, \mathsf{Pred})$ be a signature, and $\mathcal{T}_0$ be a "base" theory with signature Π_0. We consider extensions $\mathcal{T} := \mathcal{T}_0 \cup \mathcal{K}$ of $\mathcal{T}_0$ with new function symbols Σ (called *extension functions*) whose properties are axiomatized using a set $\mathcal{K}$ of formulae in the extended signature $\Pi = (\Sigma_0 \cup \Sigma, \mathsf{Pred})$ which contain function symbols in Σ.

- If $\mathcal{K}$ is a set of (universally closed) clauses, we say that $\mathcal{T}$ is an extension of $\mathcal{T}_0$ *with set of clauses* $\mathcal{K}$.
- If $\mathcal{K}$ consists of formulae of the form $\forall \bar{x}\, (\Phi(\bar{x}) \vee D(\bar{x}))$ where $\Phi(\bar{x})$ is an *arbitrary* Π_0*-formula* and $D(\bar{x})$ is a clause in the extended signature Π, which contains at least one function symbol of Σ, we say that $\mathcal{T}$ is an extension of $\mathcal{T}_0$ *with augmented clauses*. If for every formula $\forall \bar{x}\, (\Phi(\bar{x}) \vee D(\bar{x})) \in \mathcal{K}$, $\Phi(\bar{x})$ is universal we speak of *extension by universal augmented clauses*; if $\Phi(\bar{x})$ belongs to a certain class $\mathcal{F}$ of Π_0-formulae we speak of *extension by* $\mathcal{F}$*-augmented clauses*.

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Example 1 *Let $\mathcal{T}_0$ be the theory of Presburger arithmetic, with signature Π_0. Let $\Sigma = \{f\}$ where f is a new function symbol, and let $\mathcal{K}_f = \{\forall x, y, z\, (y \leq z \rightarrow f(x,y) \leq f(x,z))\}$ be an axiomatization for the monotonicity of f in its second argument. Then $\mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}_f$ is an extension of $\mathcal{T}_0$ with the set $\mathcal{K}$ of clauses and f is an extension function.*

Let $g \notin \Sigma_0 \cup \Sigma$ and $\mathcal{K}_g = \{\forall x, y([\forall z(z \neq y \rightarrow f(z,x) < f(y,x))] \rightarrow g(x) = f(x,y))\}$. Then $\mathcal{T}_1 \cup \mathcal{K}_g$ is an extension of $\mathcal{T}_1$ with a set $\mathcal{K}_g$ of augmented clauses (more precisely $\mathcal{F}$-augmented clauses where $\mathcal{F}$ is the $\exists$-fragment of $\mathcal{T}_1$).

Proof tasks. Our goal is to find an efficient way of checking whether a set G of ground clauses with additional (fresh) constants in a countable set C, i.e. in the signature $\Pi^C = (\Sigma_0 \cup \Sigma \cup C, \mathsf{Pred})$ is unsatisfiable in a local extension $\mathcal{T}_0 \cup \mathcal{K}$ of a theory $\mathcal{T}_0$ (written: $G \models_{\mathcal{T}_0 \cup \mathcal{K}} \perp$), provided we have decision procedures for checking satisfiability of a class of formulae w.r.t. the base theory $\mathcal{T}_0$.

For the case of extensions $\mathcal{T}_0 \cup \mathcal{K}$ by augmented clauses we also consider the more general task of checking satisfiability problems of the form $\Gamma \models_{\mathcal{T}_0 \cup \mathcal{K}} \perp$, where Γ is a conjunction of sentences of the form $\Phi_0 \vee D$, where D is a ground clause in the signature Π^C, and Φ_0 is a Π_0^C-sentence.

Locality conditions. Let $\mathcal{T}_0$ be an arbitrary theory with signature $\Pi_0 = (\Sigma_0, \mathsf{Pred})$, where the set of function symbols is Σ_0. Let $\Pi = (\Sigma_0 \cup \Sigma, \mathsf{Pred}) \supseteq \Pi_0$ be an extension by a non-empty set Σ of new function symbols and $\mathcal{K}$ be a set of (implicitly universally closed) clauses in the extended signature. Let C be a fixed countable set of fresh constants. We say that an extension $\mathcal{T}_0 \cup \mathcal{K}$ of $\mathcal{T}_0$ is *local* if it satisfies the following condition:

(Loc) For every set G of ground clauses in Π^C:
$\mathcal{T}_0 \cup \mathcal{K} \cup G \models \perp$ if and only if $\mathcal{T}_0 \cup \mathcal{K}[G] \cup G \models \perp$

where $\mathcal{K}[G]$ consists of those instances of $\mathcal{K}$ in which the terms starting with *extension functions* are in the set $\mathsf{est}(\mathcal{K}, G)$ of extension ground terms (i.e. terms starting with a function in Σ) which already occur in G or $\mathcal{K}$.

The notion of local theory extension generalizes the notion of *local theories* [13], [24], [14], [10]. In [18] and [20] we studied a generalization of condition (Loc) which considers operators on ground terms, and thus allows us to be more flexible w.r.t. the instances needed. With the notations above, let T be a set of ground terms in the signature Π^C. We denote by $\mathcal{K}[T]$ the set of all instances of $\mathcal{K}$ in which the terms starting with a function symbol in Σ are in T. Formally:

$\mathcal{K}[T] := \{\varphi\sigma \mid \forall \bar{x}\, \varphi(\bar{x}) \in \mathcal{K},$ where (i) if $f \in \Sigma$ and $t = f(t_1, ..., t_n)$ occurs in $\varphi\sigma$ then $t \in T$; (ii) if x is a variable that does not appear below some Σ-function in φ then $\sigma(x) = x\}$.

Let Ψ be a map associating with every set T of ground terms a set $\Psi(T)$ of ground terms. For any set G of (augmented) ground Π^C-clauses we write $\mathcal{K}[\Psi_{\mathcal{K}}(G)]$ for $\mathcal{K}[\Psi(\mathsf{est}(\mathcal{K}, G))]$.

Let $\mathcal{T}_0 \cup \mathcal{K}$ be an extension of $\mathcal{T}_0$ with clauses in $\mathcal{K}$. We consider the following locality condition:

(Loc^Ψ) For every set G of ground clauses in Π^C:
$\mathcal{T}_0 \cup \mathcal{K} \cup G \models \perp$ iff $\mathcal{T}_0 \cup \mathcal{K}[\Psi_{\mathcal{K}}(G)] \cup G \models \perp$.

Let $\mathcal{T}_0 \cup \mathcal{K}$ be an extension of $\mathcal{T}_0$ with augmented clauses in $\mathcal{K}$. We define:

(ELoc^Ψ) For every set of formulae $\Gamma = \Gamma_0 \cup G$, where Γ_0 is a Π_0-sentence and G a set of ground Π^C-clauses:
$\mathcal{T}_0 \cup \mathcal{K} \cup \Gamma \models \perp$ iff $\mathcal{T}_0 \cup \mathcal{K}[\Psi_{\mathcal{K}}(G)] \cup \Gamma \models \perp$.

Extensions satisfying condition (Loc^Ψ) are called Ψ-local; (ELoc^Ψ) is the *extended Ψ-locality* condition. *Finite* locality conditions ((E)Loc^Ψ_f) are defined restricting the locality conditions to hold for *finite* sets G of ground clauses.

Example 2 *Local theory extensions are Ψ-local, where Ψ is the identity. The order-local theories introduced in [1] satisfy a Ψ-locality condition, where for every set T of ground clauses $\Psi(T) = \{s \mid s$ ground term and $s \preceq t$ for some $t \in T\}$, where $\prec$ is the order on terms considered in [1].*

Hierarchical reasoning. Let $\mathcal{T}_0 \subseteq \mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ be a theory extension satisfying ((E)Loc^Ψ). To check the satisfiability w.r.t. $\mathcal{T}$ of a formula $\Gamma_0 \cup G$, where Γ_0 is a Π_0^C-sentence[1] and G is a set of ground Π^C-clauses, we proceed as follows:

Step 1: Instantiation. By locality, $\mathcal{T} \cup \Gamma_0 \cup G \models \perp$ iff $\mathcal{T}_0 \cup \mathcal{K}[\Psi_{\mathcal{K}}(G)] \cup \Gamma_0 \cup G \models \perp$.

Step 2: Purification. We purify $\mathcal{K}[\Psi_{\mathcal{K}}(G)] \cup G$ (by introducing, in a bottom-up manner, new constants c_t for subterms $t = f(g_1, \dots, g_n)$ with $f \in \Sigma$, g_i ground Π_0^C-terms, and corresponding definitions $c_t \approx t$) and obtain the set of formulae $\mathcal{K}_0 \cup G_0 \cup \Gamma_0 \cup D$, where D consists of definitions $f(g_1, \dots, g_n) \approx c$, where $f \in \Sigma$, c is a constant, $g_1, \dots, g_n$ are ground Π_0^C-terms, and $\mathcal{K}_0, G_0, \Gamma_0$ are Π_0^C-formulae.

Step 3: Reduction to testing satisfiability in $\mathcal{T}_0$. We reduce the problem to testing satisfiability in $\mathcal{T}_0$ by replacing D with the following set of clauses:

$$\mathsf{Con}_0 = \{\bigwedge_{i=1}^{n} c_i \approx d_i \rightarrow c \approx d \mid \begin{array}{l} f(c_1, \dots, c_n) \approx c \in D \\ f(d_1, \dots, d_n) \approx d \in D \end{array}\}.$$

This yields a sound and complete hierarchical reduction to a satisfiability problem in the base theory $\mathcal{T}_0$.

Theorem 1 ([18]) *Let $\mathcal{K}$ and $\Gamma_0 \wedge G$ be as specified above. Assume that $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ satisfies condition ((E)Loc^Ψ). Let $\mathcal{K}_0 \cup G_0 \cup \Gamma_0 \cup \mathsf{Con}_0$ be obtained as described above. The following are equivalent:*

(1) $\mathcal{T}_0 \cup \mathcal{K} \cup \Gamma_0 \cup G \models \perp$.
(2) $\mathcal{T}_0 \cup \mathcal{K}_0 \cup G_0 \cup \Gamma_0 \cup \mathsf{Con}_0 \models \perp$.

[1] In the case of condition (Loc^Ψ), $\Gamma_0 = \top$.

Thus, satisfiability of goals $\Gamma_0 \cup G$ as above w.r.t. $\mathcal{T}$ is decidable provided $\mathcal{K}[\Psi_{\mathcal{K}}(G)]$ is finite and $\mathcal{K}_0 \cup G_0 \cup \Gamma_0 \cup \mathsf{Con}_0$ belongs to a decidable fragment of $\mathcal{T}_0$.

Implementation. This method is implemented in the program H-PILoT (Hierarchical Proving by Instantiation in Local Theory Extensions) ([19], [17]). H-PILoT carries out a hierarchical reduction to $\mathcal{T}_0$ step-by-step if the user specifies different levels for the extension functions in a chain of theory extensions. Standard SMT provers or specialized provers can be used for testing the satisfiability of the formulae obtained after the reduction.

III. Recognizing Ψ-local Theory Extensions

1. Semantic criteria. In [27], [20] we proved that local extensions can be recognized by showing that certain partial models embed into total ones. We define the concepts used.

1.1 Partial structures. Let $\Pi = (\Sigma, \mathsf{Pred})$ be a first-order signature with set of function symbols Σ and set of predicate symbols Pred. A *partial* Π*-structure* is a structure $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f\in\Sigma}, \{P_{\mathcal{A}}\}_{P\in\mathsf{Pred}})$, where A is a non-empty set, for every $f \in \Sigma$ with arity n, $f_{\mathcal{A}}$ is a partial function from A^n to A, and for every $P \in \mathsf{Pred}$, $P_{\mathcal{A}} \subseteq A^n$. We consider constants (0-ary functions) to be always defined. $\mathcal{A}$ is a *total structure* if the functions $f_{\mathcal{A}}$ are all total. Given a (total or partial) Π-structure $\mathcal{A}$ and $\Pi_0 \subseteq \Pi$ we denote the reduct of $\mathcal{A}$ to Π_0 by $\mathcal{A}|_{\Pi_0}$. The notion of evaluating a term t with variables X w.r.t. an assignment $\beta : X \to A$ for its variables in a partial structure $\mathcal{A}$ is the same as for total algebras, except that the evaluation is undefined if $t = f(t_1, \dots, t_n)$ and at least one of $\beta(t_i)$ is undefined, or else $(\beta(t_1), \dots, \beta(t_n))$ is not in the domain of $f_{\mathcal{A}}$. Let L be a literal. We say that $(\mathcal{A}, \beta) \models_w L$ if either (i) at least one term occurring in L is undefined in $(\mathcal{A}, \beta)$ or (ii) all terms occurring in L are defined in $(\mathcal{A}, \beta)$, and $(\mathcal{A}, \beta)(L) = 1$ (computed as for total structures). $(\mathcal{A}, \beta)$ weakly satisfies a clause C (notation: $(\mathcal{A}, \beta) \models_w C$) if it satisfies at least one literal in C. $\mathcal{A}$ is a *weak partial model* of a set of clauses $\mathcal{K}$ if $(\mathcal{A}, \beta) \models_w C$ for every valuation β and every $C \in \mathcal{K}$.

If $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ is an extension of a Π_0-theory $\mathcal{T}_0$ with new function symbols in Σ and (augmented) clauses $\mathcal{K}$, we denote by $\mathsf{PMod}_w(\Sigma, \mathcal{T})$ the set of weak partial models of $\mathcal{T}$ whose Σ_0-functions are total.

1.2 Embeddings. For total Π-structures $\mathcal{A}$ and $\mathcal{B}$, $\varphi : \mathcal{A} \to \mathcal{B}$ is an embedding iff it is an injective homomorphism and has the property that for every $P \in \mathsf{Pred}$ with arity n and all $(a_1, \dots, a_n) \in A^n$, $(a_1, \dots, a_n) \in P_{\mathcal{A}}$ iff $(\varphi(a_1), \dots, \varphi(a_n)) \in P_{\mathcal{B}}$. In particular, an embedding preserves the truth of all literals. A map $\varphi : \mathcal{A} \to \mathcal{B}$ is an *elementary embedding* iff it preserves and reflects *all* formulae, i.e., for every formula $F(x_1, \dots, x_n)$ with free variables $x_1, \dots, x_n$ and all elements $a_1, \dots, a_n$ from A,

$$\mathcal{A} \models F(a_1, \dots, a_n) \text{ iff } \mathcal{B} \models F(\varphi(a_1), \dots, \varphi(a_n)).$$

Two structures $\mathcal{A}, \mathcal{B}$ are *elementarily equivalent* (notation: $\mathcal{A} \equiv \mathcal{B}$) if they satisfy the same sentences. Note that if there is an elementary embedding between two structures, then they are elementarily equivalent in particular.

A *weak* Π*-embedding* between partial Π-structures $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f\in\Sigma}, \{P_{\mathcal{A}}\}_{P\in\mathsf{Pred}})$ and $\mathcal{B} = (B, \{f_{\mathcal{B}}\}_{f\in\Sigma}, \{P_{\mathcal{B}}\}_{P\in\mathsf{Pred}})$ is a total map $\varphi : A \to B$ such that for all $f \in \Sigma$ and $P \in \mathsf{Pred}$ the following hold:

(1) whenever $f_{\mathcal{A}}(a_1, \dots, a_n)$ is defined (in $\mathcal{A}$), then $f_{\mathcal{B}}(\varphi(a_1), \dots, \varphi(a_n))$ is defined (in $\mathcal{B}$) and $\varphi(f_{\mathcal{A}}(a_1, \dots, a_n)) = f_{\mathcal{B}}(\varphi(a_1), \dots, \varphi(a_n))$;
(2) if the arity of $P \in \mathsf{Pred} \cup \{\approx\}$ is n then for all $a_1, \dots, a_n \in \mathcal{A}$, $(a_1, \dots, a_n) \in P_{\mathcal{A}}$ if and only if $(\varphi(a_1), \dots, \varphi(a_n)) \in P_{\mathcal{B}}$.

1.3 Locality and embeddability. In [27] we proved that if all weak partial models of an extension $\mathcal{T}_0 \cup \mathcal{K}$ of a base theory $\mathcal{T}_0$ with total base functions can be embedded into a total model of the extension, then the extension is local. In [18], [20] we lifted these results to Ψ-locality and obtained semantical *characterizations* of various types of Ψ-locality, which we present here.

In what follows, let $\mathcal{T}_0$ be a Π_0-theory, and $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K} = \mathcal{T}$ a theory extension with functions in Σ and (augmented) clauses $\mathcal{K}$ and let Ψ be as in Section II.

Notation. Let $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f\in\Sigma_0\cup\Sigma}, \{P_{\mathcal{A}}\}_{P\in\mathsf{Pred}})$ be a partial Π^C-structure with total Σ_0-functions. We denote by Π^A the extension of the signature Π with constants from A. We denote by $\mathcal{D}(\mathcal{A})$ the following set of ground Π^A-terms:

$$\begin{aligned}\mathcal{D}(\mathcal{A}) := \ & \{f(a_1, ..., a_n) \mid f \in \Sigma, a_i \in A, i = 1, \dots, n, \\ & f_{\mathcal{A}}(a_1, ..., a_n) \text{ is defined } \}.\end{aligned}$$

Let $\mathsf{PMod}_w^{\Psi}(\Sigma, \mathcal{T})$ be the class of weak partial models $\mathcal{A}$ of $\mathcal{T}_0 \cup \mathcal{K}$ in which the Σ-functions are partial, all other functions are total and all terms in $\Psi(\mathsf{est}(\mathcal{K}, \mathcal{D}(\mathcal{A})))$ are defined in the extended Π^A-structure $\mathcal{A}^A$. We consider the following embeddability properties:

(Emb_w^{Ψ}) Every $\mathcal{A} \in \mathsf{PMod}_w^{\Psi}(\Sigma, \mathcal{T})$ weakly embeds into a total model of $\mathcal{T}$.

(EEmb_w^{Ψ}) Every $\mathcal{A} \in \mathsf{PMod}_w^{\Psi}(\Sigma, \mathcal{T})$ weakly embeds into a total model $\mathcal{B}$ of $\mathcal{T}$ such that $\mathcal{A}|_{\Pi_0}$ and $\mathcal{B}|_{\Pi_0}$ are elementarily equivalent.

Variants ($\mathsf{EEmb}_{w,f}^{\Psi}$) and ($\mathsf{Emb}_{w,f}^{\Psi}$) can be obtained by requiring embeddability only for extension functions with a finite domain of definition.

A *model complete* theory is one that has the property that all embeddings between its models are elementary. So if $\mathcal{T}_0$ is model complete then (EEmb_w) and (Emb_w) coincide.

Example 3 ([20]) *The following theories have quantifier-elimination and are therefore model complete: (1) Presburger arithmetic with congruence mod. n ($\equiv_n$), $n = 2, 3, ...$*

([7], p.197); (2) Rational linear arithmetic in the signature $\{+, 0, \leq\}$ *([38]); (3) Real closed ordered fields ([15], 7.4.4), e.g., the real numbers; (4) Algebraically closed fields ([4], Ex. 3.5.2; Rem. p.204; [15], Ch. 7.4, Ex. 2); (5) Finite fields ([15], Ch. 7.4, Example 2); (6) The theory of acyclic lists in the signature* $\{\mathrm{car}, \mathrm{cdr}, \mathrm{cons}\}$ *([23], [12]).*

When establishing links between locality and embeddability we require that the extension clauses in $\mathcal{K}$ are *flat* (*quasi-flat*) and *linear* w.r.t. Σ-functions. We distinguish between ground and non-ground clauses.

Non-ground clauses: An extension clause D is *quasi-flat* when all symbols below a Σ-function symbol in D are variables or ground Π_0-terms. D is *flat* when all symbols below a Σ-function symbol in D are variables. D is *linear* if whenever a variable occurs in two terms of D which start with Σ-functions, the terms are identical, and no such term contains two occurrences of a variable.

Ground clauses: A ground clause D is *flat* if all symbols below a Σ-function in D are constants. A ground clause D is *linear* if whenever a constant occurs in two terms in D whose root symbol is in Σ, the two terms are identical, and if no term which starts with a Σ-function contains two occurrences of the same constant.

Let Ψ be a map associating with $\mathcal{K}$ and a set of Π^C-ground terms T a set $\Psi_{\mathcal{K}}(T)$ of Π^C-ground terms. We call $\Psi_{\mathcal{K}}$ a *term closure operator* if the following holds for all sets of ground terms T, T':

(1) $\mathrm{est}(\mathcal{K}, T) \subseteq \Psi_{\mathcal{K}}(T)$,
(2) $T \subseteq T' \Rightarrow \Psi_{\mathcal{K}}(T) \subseteq \Psi_{\mathcal{K}}(T')$,
(3) $\Psi_{\mathcal{K}}(\Psi_{\mathcal{K}}(T)) \subseteq \Psi_{\mathcal{K}}(T)$,
(4) for any map $h : C \to C$, $\bar{h}(\Psi_{\mathcal{K}}(T)) = \Psi_{\bar{h}\mathcal{K}}(\bar{h}(T))$, where $\bar{h}$ is the canonical extension of h to extension ground terms.

In [27], [28], [20] we established links between embeddability and (Ψ-)locality, if Ψ is a term closure operator:

Theorem 2 ([27], [28], [20])

(1) *Assume that $\mathcal{K}$ is a family of Σ-flat Π-clauses.*
 (1.1) *If $\mathcal{T}_0$ is a first-order theory and the extension $\mathcal{T}_0 \subseteq \mathcal{T}{=}\mathcal{T}_0{\cup}\mathcal{K}$ satisfies (Loc^Ψ) then every model in $\mathsf{PMod}_w^\Psi(\Sigma, \mathcal{T})$ weakly embeds into a total model of $\mathcal{T}$.*
 (1.2) *If $\mathcal{T}_0 \subseteq \mathcal{T}{=}\mathcal{T}_0{\cup}\mathcal{K}$ satisfies (ELoc^Ψ) then every $\mathcal{A} \in \mathsf{PMod}_w^\Psi(\Sigma, \mathcal{T})$ weakly embeds into a total model $\mathcal{B}$ of $\mathcal{T}$ such that restriction of this embedding to the reducts to Π_0 of $\mathcal{A}$, $\mathcal{B}$ preserves the truth of all Π_0-sentences.*

(2) *Let $\mathcal{T}_0$ be a Π_0-theory, $\Pi = (\Sigma_0 \cup \Sigma, \mathsf{Pred})$ and let $\mathcal{K}$ be a set of universally closed, linear and quasi-flat clauses in the signature Π and let $\Psi_{\mathcal{K}}$ be a term closure operator such that for every flat set of ground terms T, $\Psi(T)$ is flat.*
 (2.1) *If the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ satisfies (Emb_w^Ψ) then it satisfies (Loc^Ψ).*
 (2.2) *If the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ satisfies (EEmb_w^Ψ) then it satisfies (ELoc^Ψ).*

2. Locality and saturation. In [1], Ganzinger and Basin established a link between saturation and order locality, and used these results for automated complexity analysis, and for obtaining local axiomatizations from non-local ones.

Theorem 3 ([1]) *Let $\prec_{\mathsf{Terms}}$ be a well-founded term ordering and $\prec$ a compatible and total atom ordering. Let $\mathcal{K}$ be a set of clauses without equality which is reductive w.r.t. $\prec_{\mathsf{Terms}}$ and $\prec$ (i.e. for every ground instance C of a clause in $\mathcal{K}$, if A is a maximal atom in C then for every term occurring in C there exists a larger or equal term in A). If $\mathcal{K}$ is saturated w.r.t. $\prec$-ordered resolution, then $\mathcal{K}$ is order local w.r.t. $\prec$.*

Theorem 3 is used in [10] for proving (by saturation) the locality of the presentation Int:

(1) $p(x) \approx y \to s(y) \approx x$ (3) $p(x) \approx p(y) \to y \approx x$
(2) $s(x) \approx y \to p(y) \approx x$ (4) $s(x) \approx s(y) \to y \approx x$

together with an explicit axiomatization of the predicate $\approx$ by congruence axioms. However, using saturation for detecting locality or for generating local presentations from non-local ones has the following drawbacks:

(i) Equality cannot be used as a built-in predicate: If the clauses contain the equality predicate then the congruence axioms have to be added explicitly.
(ii) The size of the saturated sets of clauses can be very large. Often infinitely many clauses are generated.

Example 4 ([16]) *By saturation under ordered resolution of the clause set: $N = \{x \leq x, x \leq y \wedge y \leq z \to x \leq z,\ f(x) \leq f(s(x))\}$ we obtain an infinite set containing all clauses of the form $f(x) \leq f(s^n(x))$, where $n{\geq}0$, thus a usual resolution-based theorem prover will not be able to detect saturation.*

To reduce the size of a representation, in [16] we used constrained clauses and showed that Thm. 3 has a counterpart for constrained clauses in which equality is used in a controlled way. The advantage of using constrained clauses is that in many cases it allows us to obtain a finite symbolic representation for possibly infinite sets of clauses.

Example 5 ([16]) *This approach allows us to obtain a finite representation for the theory in Example 4 by using constrained clauses of the form $[y \approx s^*(x)] || f(x) \leq f(y)$.*

IV. Locality Transfer

We present locality transfer results established in [20], [28].

1. Enrichment of theories. We first identify conditions under which locality results lift if we enrich the base theory.

Theorem 4 ([20]) *Let $\Pi_0 = (\Sigma_0, \mathsf{Pred})$ be a signature, $\mathcal{T}_0$ a theory in Π_0, Σ_1 and Σ_2 two disjoint sets of new function symbols, $\Pi_i := (\Sigma_0 \cup \Sigma_i, \mathsf{Pred})$, $i = 1, 2$. Assume that $\mathcal{T}_2$ is a Π_2-theory with $\mathcal{T}_0 \subseteq \mathcal{T}_2$, and $\mathcal{K}$ is a set of universally closed Π_1-clauses. If the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ enjoys (EEmb_w) then so does the extension $\mathcal{T}_2 \subseteq \mathcal{T}_2 \cup \mathcal{K}$. In particular, if $\mathcal{K}$ is (quasi)-flat and linear then extension $\mathcal{T}_2 \subseteq \mathcal{T}_2 \cup \mathcal{K}$ satisfies condition (ELoc).*

We illustrate the applicability of this result on one example.

Example 6 ([20]) *Let Lat be the theory of lattices and $\mathcal{T}_1 = \mathsf{Lat} \cup \mathsf{Mon}_f$, where $\mathsf{Mon}_f = \{\forall x, y \ (x \leq y \rightarrow f(x) \leq f(y))\}$ is the monotonicity of a new function symbol f. Using techniques similar to the ones used in [36] we can prove that the extension $\mathsf{Lat} \subseteq \mathsf{Lat} \cup \mathsf{Mon}_f$ satisfies condition $(\mathsf{EEmb}_{w,f})$. Let $\mathcal{T}$ be any extension of the theory of lattices (this can be the theory of distributive lattices, Heyting algebras, Boolean algebras, any theory with a total order – e.g. the (ordered) theory of integers or of reals, etc.). By Theorem 4, $\mathcal{T} \subseteq \mathcal{T} \cup \mathsf{Mon}_f$ satisfies condition $(\mathsf{EEmb}_{w,f})$, hence the extended locality condition (ELoc_f).*

Theorem 5 ([20]) *Let $\mathcal{T}_0$ be a theory. Assume that $\mathcal{T}_0$ has a model completion $\mathcal{T}_0^*$ such that $\mathcal{T}_0 \subseteq \mathcal{T}_0^*$. Let $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ be an extension of $\mathcal{T}_0$ with new function symbols Σ whose properties are axiomatized by a set of flat and linear clauses $\mathcal{K}$ (all of which contain symbols in Σ).*

1. *Assume that: (i) Every model of $\mathcal{T}_0 \cup \mathcal{K}$ embeds[2] into a model of $\mathcal{T}_0^* \cup \mathcal{K}$; and (ii) $\mathcal{T}_0 \cup \mathcal{K}$ is a local extension of $\mathcal{T}_0$. Then $\mathcal{T}_0^* \subseteq \mathcal{T}_0^* \cup \mathcal{K}$ satisfies condition (EEmb_w), hence if $\mathcal{K}$ is a set of quasi-flat and linear augmented clauses also condition (ELoc) as extension of $\mathcal{T}_0^*$.*
2. *If all variables in $\mathcal{K}$ occur below an extension function and $\mathcal{T}_0^* \cup \mathcal{K}$ is a local extension of $\mathcal{T}_0^*$ then $\mathcal{T}_0 \cup \mathcal{K}$ is a local extension of $\mathcal{T}_0$.*

The results extend in a natural way to Ψ-locality and to finite versions of embeddability and locality.

Example 7 ([20]) *To show that the extension of the theory TOrd of total orderings with a function f satisfying:*

$$\mathsf{SMon}_f \quad \forall x, y (x < y \rightarrow f(x) < f(y))$$

satisfies condition (Loc_f) we can proceed as follows: Note that the model completion TOrd^* of TOrd is the theory of dense total orderings without endpoints. It can be checked that the extension $\mathsf{TOrd}^* \subseteq \mathsf{TOrd}^* \cup \mathsf{SMon}_f$ satisfies condition (ELoc_f). As all variables in SMon_f occur below an extension function, $\mathsf{TOrd} \subseteq \mathsf{TOrd} \cup \mathsf{SMon}_f$ is local.

2. Combinations of local theory extensions We now identify situations in which the union of two local extensions of a common base theory is again a local extension of the base theory. We studied these preservation properties (at various levels of generality) in [29], [28] and [20].

Theorem 6 ([28]) *Let $\mathcal{T}_0$ be a theory in the signature Π_0, Σ_1 and Σ_2 two disjoint sets of new function symbols, and $\Pi_i = (\Sigma_0 \cup \Sigma_i, \mathsf{Pred})$. Let $\mathcal{K}_i$ be a set of Π_i-clauses for $i = 1, 2$, and $\mathcal{T}_i := \mathcal{T}_0 \cup \mathcal{K}_i$, $i = 1, 2$.*

1. *If both extensions $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_i$, $i = 1, 2$, satisfy condition (EEmb_w) then so does the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$. If $\mathcal{K}_1 \cup \mathcal{K}_2$ is (quasi-)flat and linear then the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$ is local.*
2. *Assume that: (i) $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies (EEmb_w); (ii) $\mathcal{T}_0 \subseteq \mathcal{T}_2$ satisfies (Emb_w); and (iii) $\mathcal{K}_1$ is Σ_1-flat in which all variables occur below some Σ_1-function.*
   Then the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$ satisfies (Emb_w). If $\mathcal{K}_1 \cup \mathcal{K}_2$ is (quasi-)flat and linear then the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$ is local.
3. *Assume that (i) $\mathcal{T}_0$ is a $\forall\exists$ theory; (ii) $\mathcal{K}_i$ is Σ_i-flat and $\mathcal{T}_0 \subseteq \mathcal{T}_i$ satisfies condition (Emb_w) for $i = 1, 2$; and (iii) all variables occur below an extension function in $\mathcal{K}_i$, for $i = 1, 2$.*
   Then $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$ has (Emb_w). If $\mathcal{K}_1 \cup \mathcal{K}_2$ is (quasi-)flat and linear then the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$ is local.

Similar results can be established for Ψ-locality (the details are presented in [20]). Thm. 6 allows us to identify situations in which decidability of ground satisfiability is preserved when combining non-disjoint theories for which ground satisfiability is decidable. These results are orthogonal to other results on combinations of decision procedures existing in the literature, in which the component theories are required to have disjoint signatures and be stably infinite [26], or the component theories need to fulfill strong compatibility conditions with the shared theory cf. e.g. [37], [12].

V. Examples of Local Theory Extensions

The results presented in Section III and IV allowed us to identify many examples of Ψ-local theory extensions.

1. Mathematical analysis. We first present some classes of theory extensions from mathematical analysis.

1.1 Monotonicity, boundedness for monotone functions. Any extension of a theory for which $\leq$ is a partial order with functions satisfying[3] Mon_f^σ and/or Bound_f^t is local (cf. e.g. [36], [18]). The extensions satisfy condition (ELoc) if e.g. in $\mathcal{T}_0$ all finite and empty infima (or suprema) exist.

$$\mathsf{Mon}_f^\sigma \quad \bigwedge_{i \in I} x_i \leq_i^{\sigma_i} y_i \wedge \bigwedge_{i \notin I} x_i \approx y_i \rightarrow f(x_1, .., x_n) \leq f(y_1, .., y_n)$$
$$\mathsf{Bound}_f^{s,t} \quad s(x_1, .., x_n) \leq f(x_1, .., x_n) \leq t(x_1, .., x_n)$$

[2] If $\mathcal{T}_0$ is universal, this is the notion of compatibility defined in [12].

[3] For $i \in I$, $\sigma_i \in \{-, +\}$, and for $i \notin I$, $\sigma_i = 0$; $\leq^+ = \leq$, $\leq^- = \geq$.

where the free variables are universally quantified; s, t are new functions or Π_0-terms with variables among $x_1, \dots, x_n$ and: (i) s, t have the same monotonicity as f in any model and (ii) $\forall x_1, \dots, x_n \; s(x_1, \dots, x_n) \leq t(x_1, \dots, x_n)$.

1.2 Convexity/concavity [30]. Let $\mathcal{T}_0$ be $\mathbb{R}$, the theory of real numbers (or the many-sorted combination of the theories of real numbers (sort real) and integers (sort int)).

Let f be a unary function symbol with arity real$\to$real (or resp. int$\to$real). Then the theory extensions $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathsf{Conv}_f$ and $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathsf{Conc}_f$ satisfy condition (ELoc) where:

$$\mathsf{Conv}_f \quad \forall x, y, z \left(x < z \leq y \rightarrow \frac{f(z)-f(x)}{z-x} \leq \frac{f(y)-f(x)}{y-x} \right)$$

Conc_f is the convexity condition for $-f$.

1.3 Linear combinations of functions [30], [33]. Let $f_1, \dots, f_n, f, g$ be unary function symbols. The extension $\mathbb{R} \subseteq \mathbb{R} \cup \mathsf{BS}$ satisfies condition (ELoc), where BS contains conjunctions of axioms of type:

$$\forall t (a \leq \sum_{i=1}^{n} a_i f_i(t) \leq b), \; \forall t (g(t) \leq \sum_{i=1}^{n} a_i f_i(t) \leq f(t)),$$

or $$\forall t, t' (t < t' \rightarrow a \leq \sum_{i=1}^{n} a_i \frac{f_i(t') - f_i(t)}{t' - t} \leq b),$$

where $a, b \in \mathbb{R}$ and (i) g is convex and f concave and (ii) either g and f satisfy the condition $\forall t (g(t) \leq f(t))$ or correspond to Π_0-terms and $\models_{\mathcal{T}_0} \forall t \; g(t) \leq f(t)$. Using arguments similar to those used in [30] it can be proved that if we additionally require the functions to be continuous locality is still preserved.

If I is an interval of the form $(-\infty, a], [a, b]$ or $[a, \infty)$ then we can define versions of monotonicity/boundedness, convexity/concavity and boundedness axioms for linear combinations of functions and of their slopes relative to the interval I (then conditions (i) and (ii) for f and g are relative to the interval I). Similar locality results can be proved also for such axiomatizations.

2. Theories of data structures. We now present some theories of data structures proved to be local in [18], [32]

2.1 The array property fragment. In [18] we showed that a decidability result for the array property fragment in [2] is due to the Ψ-locality (for a certain Ψ) of the corresponding extensions of the many-sorted combination of Presburger arithmetic (for indices) with the given theory of elements.

2.2 Theories of pointer structures. In [25], McPeak and Necula investigate reasoning in pointer data structures. The language used has sorts p (pointer) and s (scalar). Sets Σ_p and Σ_s of pointer resp. scalar fields are modeled by functions of sort $\mathsf{p} \to \mathsf{p}$ and $\mathsf{p} \to \mathsf{s}$, respectively. A constant null of sort p exists. The only predicate of sort p is equality; predicates of scalar sort can have any arity. The axioms considered in [25] are of the form

$$\forall p \quad \mathcal{E} \vee \mathcal{C} \qquad (1)$$

where $\mathcal{E}$ contains disjunctions of pointer equalities and $\mathcal{C}$ contains scalar constraints (sets of both positive and negative literals). It is assumed that for all terms $f_1(f_2(\dots f_n(p)))$ occurring in the body of an axiom, the axiom also contains the disjunction $p = \mathsf{null} \vee f_n(p) = \mathsf{null} \vee \dots \vee f_2(\dots f_n(p))) = \mathsf{null}$.[4] In [18] we gave an alternative proof of the decidability result established in [25] by showing that sets of axioms of the form in (1) define local theory extensions of the disjoint, many-sorted combination of the theory of scalars (sort s) and the pure theory of equality (of sort p).

2.3 Theories of absolutely free data structures. Consider the axiomatization for absolutely free data structures with constructors in a set Σ_0:

$$\mathsf{AbsFree}_{\Sigma_0} = \Big(\bigcup_{c \in \Sigma_0} (\mathsf{Inj}_c) \cup (\mathsf{Acyclic}_c) \Big) \cup \bigcup_{c,d \in \Sigma, c \neq d} \mathsf{Disj}_{c,d},$$

where:

Inj_c	$c(x_1, \dots, x_n) \approx c(y_1, \dots, y_n) \rightarrow x_j \approx y_j$
$\mathsf{Disj}_{c,d}$	$c(x_1, \dots, x_n) \not\approx d(y_1, \dots, y_m)$
$\mathsf{Acyclic}_c$	$c(t_1, \dots, t_n) \not\approx x$ if x occurs in some t_i

In [32] we proved that the theories $\mathsf{AbsFree}_{\Sigma_0}$ and $\mathcal{T}_{cs} = \mathsf{AbsFree}_{\Sigma_0} \cup \mathsf{Sel}(\Sigma_0)$ are local – where $\mathsf{Sel}(\Sigma_0) = \bigcup_{c \in \Sigma_0} \bigcup_{i=1}^{ac} \mathsf{Sel}(\mathsf{sel}_i^c, c)$ axiomatizes a family of selectors $\mathsf{sel}_1^c, \dots, \mathsf{sel}_n^c$, where ac is the arity of c, corresponding to constructors $c \in \Sigma_0$, where:

$$\mathsf{Sel}(s_i^c, c) \quad \forall x, x_1, \dots, x_{ac} \;\; x \approx c(x_1, \dots, x_{ac}) \rightarrow \mathsf{sel}_i^c(x) \approx x_i$$

In [32] we also proved locality results for extensions of such theories with recursively defined functions.

3. Updates. Large classes of update rules in transition constraint systems define local theory extensions. Theorem 7 shows that updating functions in a set Σ by case distinction – according to a partition of their domain of definition in such a way that the new values $f'(x)$ satisfy certain conditions (expressed by formulae in the base theory) – defines a local extension.

Theorem 7 ([21], [18]) *Let $\mathcal{T}_0$ be a base theory with signature $\Pi_0 = (\Sigma_0, \mathsf{Pred})$, where $\Sigma_0 = \Sigma_b \cup \Sigma$ and let $\Sigma' = \{f' \mid f \in \Sigma\}$ be a copy of Σ containing new function symbols. Consider a family* $\mathsf{Update}(\Sigma, \Sigma')$ *of update axioms of the form:*

$$\forall \overline{x} (\phi_i(\overline{x}) \rightarrow F_i(f'(\overline{x}), \overline{x})) \qquad i \in I, f \in \Sigma. \qquad (2)$$

which describe how the values of the Σ-functions change, depending on a finite set $\{\phi_i \mid i \in I\}$ of conditions ϕ_i, expressed as formulae over the base signature which are mutually exclusive, i.e.:

- *$\phi_i(\overline{x}) \wedge \phi_j(\overline{x}) \models_{\mathcal{T}_0} \perp$ for $i \neq j, i, j \in I$.*

and using Π_0-formulae $F_i(y, \overline{x})$ with the property:

- *$\mathcal{T}_0 \models \forall \overline{x} (\phi_i(\overline{x}) \rightarrow \exists y \; F_i(y, \overline{x}))$ for all $i \in I$.*

Then the extension of $\mathcal{T}_0$ with axioms $\mathsf{Update}(\Sigma, \Sigma')$ *satisfies condition* (ELoc).

[4]This has the rôle of excluding null pointer errors.

VI. Applications

We used the concept of local theory extensions for efficient reasoning in various areas (mathematics, verification, cryptography, description logics). We present the main ideas; for details we refer to the cited papers.

Mathematics. The locality results for functions over numerical domains described in Section V.1, satisfying axioms such as monotonicity, boundedness, or the locality of extensions of the theory of real numbers with families of functions satisfying conditions like convexity, boundedness for linear combinations for the functions in the family, boundedness for the slopes of these functions, possibly combined with continuity, are very useful in theorem proving in mathematics. We presented such results in [30].

Verification. We used the locality of the theories presented in Section V for verification. We considered parametric systems, with various forms of parametricity: parametric data (including parametric constants of systems but also parametric change and environment) – specified using functions with certain properties – and parametric topology, specified using data structures.

The first type of problems we studied was to check whether a safety property is an inductive invariant, or holds for paths of bounded length starting in an initial state, for given instances of the parameters or under the assumption that certain constraints on parameters hold.

The second type of problems was to derive constraints on the parameters which guarantee that a certain safety property is an inductive invariant of the system (or holds for paths of bounded length). We showed that sound and complete hierarchical reduction for SMT checking in local extensions allows to reduce the problem of checking that certain formulae are invariants to testing the satisfiability of certain formulae w.r.t. a standard theory. For generating constraints on the parameters of the system (be they data or functions) which guarantee safety we used quantifier elimination. These constraints on the parameters may also be used to solve optimization problems (maximize/minimize some of the parameters) such that safety is guaranteed.

We used these ideas for program verification [33], for the verification of real time systems (typically systems of trains with a parametric topology; for representing such topologies we used theories of arrays [22], [9], and pointers [8]), and for the verification of increasingly more complex parametric hybrid systems, possibly with a parametric number of components [33], [5], [6], [34].

Description logics. We also proved that the algebraic theories associated with tractable description logics which are extensions of the description logic $\mathcal{EL}$ are local, resp. define local theory extensions and thus obtained alternative ways of giving PTIME decision procedures for these description logics and of extending them [31], [35].

VII. Conclusion

We gave an overview of our work on local theory extensions and on efficient hierarchical and modular reasoning in such theories. Many theories important for computer science or mathematics fall into this class (theories of data structures, theories of free or monotone functions, but also theories defining properties of functions occurring in mathematical analysis). We presented criteria for recognizing local theory extensions (semantic, proof theoretic, or criteria which use locality transfer results) and briefly discussed the way we used these results in automated theorem proving in mathematics, in verification and in description logics.

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