$$p(y|\theta) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\underbrace{\sum_{i=1}^n (y_i-\theta)^2}\right)$$

$$\sum_{i=1}^n (y_i-\theta)^2 = \sum_{i=1}^n (y_i^2 - 2y_i\theta + \theta^2)$$

$$= \sum_{i=1}^n y_i^2 - \sum_{i=1}^n 2y_i\theta + \sum_{i=1}^n \theta^2$$

$$= \sum_{i=1}^n y_i^2 - 2\theta \underbrace{\sum_{i=1}^n y_i}_{=n\bar{y}} + n\theta^2$$

$$= \sum_{i=1}^n y_i^2 - 2\theta n\bar{y} + n\theta^2$$

$g(\theta) = \exp\left(-\frac{n\theta^2}{2\sigma^2}\right)$, $\eta(\theta) = \frac{n\theta}{\sigma^2}$, σ^2 is known

conjugate prior of θ is

$$p(\theta) \propto \underline{g(\theta)^\nu \exp(\eta(\theta)\cdot\tau)}$$

$$= \exp\left(-\frac{\nu n\theta^2}{2\sigma^2} + \frac{\tau n\theta}{\sigma^2}\right)$$

$$= \exp(-c_1\theta^2 + c_2\theta)$$

$\uparrow$

this is the core part of a density

$\theta \in (-\infty, +\infty)$

$c_1 = \frac{\nu n}{2\sigma^2}$

$c_2 = \frac{\tau n}{\sigma^2}$

$N(\theta, \sigma^2)$ (θ, σ^2) (mean, var)

$(\omega = \theta + \sigma, \sigma^2)$

(ω, σ^2)

$$p(\theta) = \frac{1}{C}\exp(-c_1\theta^2 + c_2\theta), \text{ for } \theta \in (-\infty, +\infty)$$

$$\Rightarrow c_1 > 0, \quad c_2 \in (-\infty, +\infty) \qquad \left(\int_{-\infty}^{\infty} \exp(-c_1\theta^2 + c_2\theta)\, d\theta < \infty\right)$$

$$-c_1\theta^2 + c_2\theta = -(c_1\theta^2 - c_2\theta) = -c_1\left(\theta^2 - \frac{c_2}{c_1}\theta\right) = -c_1\cdot\left(\theta - \frac{c_2}{2c_1}\right)^2 + \frac{c_2^2}{4c_1}.$$

$$p(\theta) \propto \exp(-c_1\theta^2 + c_2\theta) = \exp\left(-c_1\left(\theta - \frac{c_2}{2c_1}\right)^2 + \underbrace{\frac{c_2^2}{4c_1}}\right)$$

$$\propto \exp\left(-c_1\left(\theta - \frac{c_2}{2c_1}\right)^2\right) \quad \Rightarrow p(\theta) \text{ is a normal density.}$$

$p(\theta)$ is the density of $N\left(\frac{c_2}{2c_1}, \frac{1}{2c_1}\right)$ $\left(\begin{array}{l} \frac{c_2}{2c_1} = \mu_0 \\ \frac{1}{2c_1} = \tau_0^2 \end{array} \right)$

$p(\theta)$ is the density of $N(\mu_0, \tau_0^2)$

↘ not important

likelihood $p(y|\theta) \propto \exp\left(-\frac{n\theta^2}{2\sigma^2} + \frac{n\theta\bar{y}}{\sigma^2}\right)$

$$p(\theta) = \frac{1}{\sqrt{2\pi\tau_0^2}} \exp\left(-\frac{1}{2\tau_0^2}(\theta-\mu_0)^2\right)$$

$$\propto \exp\left(\underbrace{-\frac{\theta^2}{2\tau_0^2} + \frac{\mu_0\theta}{\tau_0^2}}\right)$$

$$p(\theta|y) \propto p(y|\theta)\,p(\theta)$$

$$\propto \exp\left(-\frac{n}{2\sigma^2}\theta^2 + \frac{n\bar{y}}{\sigma^2}\theta - \frac{1}{2\tau_0^2}\theta^2 + \frac{\mu_0}{\tau_0^2}\theta\right)$$

$$= \exp\left[-\frac{1}{2}\left(\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}\right)\theta^2 + \left(\frac{n\bar{y}}{\sigma^2} + \frac{\mu_0}{\tau_0^2}\right)\theta\right]$$

$$= \exp\left[-\frac{1}{2}\left(\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}\right)\left\{\theta^2 - \frac{2\left(\frac{n\bar{y}}{\sigma^2} + \frac{\mu_0}{\tau_0^2}\right)}{\left(\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}\right)}\theta\right\}\right]$$

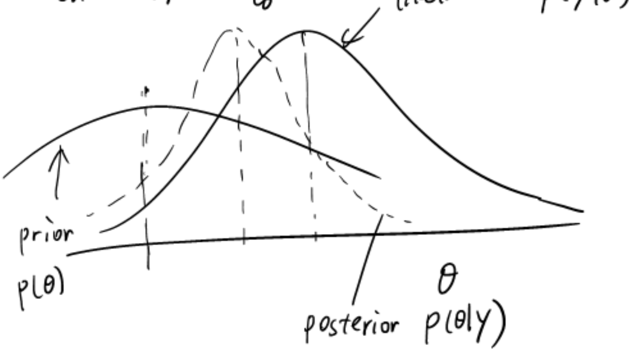
$$\propto \exp\left[-\frac{1}{2}\left(\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}\right)\left(\theta - \frac{\frac{n\bar{y}}{\sigma^2} + \frac{\mu_0}{\tau_0^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}}\right)^2\right] \rightsquigarrow \exp\left(-\frac{(\theta - \boxed{?})^2}{2\boxed{?}}\right)$$

$$\underline{x}^2 - 2ax + \underline{a}^2 = (x-a)^2$$

$$a = \frac{\frac{n\bar{y}}{\sigma^2} + \frac{\mu_0}{\tau_0^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}}$$

$\Rightarrow$ $p(\theta|y)$ is the density of $N\left(\underbrace{\frac{\frac{n\bar{y}}{\sigma^2} + \frac{\mu_0}{\tau_0^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}}}_{= \mu_n}, \underbrace{\frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}}}_{= \tau_n^2}\right)$.

$$\frac{1}{\tau_n^2} = \frac{1}{\sigma^2/n} + \frac{1}{\tau_0^2}$$



$$\mu_n = \underbrace{\frac{\frac{n}{\sigma^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}}}_{\to 1} \cdot \bar{y} + \underbrace{\frac{\frac{1}{\tau_0^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}}}_{\to 0} \mu_0$$

$n \to \infty$,

$\underline{\mu_n \approx \bar{y}}$, $\tau_n^2 \approx \frac{1}{\frac{n}{\sigma^2}} = \frac{\sigma^2}{n}$

↑ sample mean ↗ sample variance

$p(\theta|y) \propto p(y|\theta)\,p(\theta)$ $\qquad$ $\bar{y}$ is a sufficient statistic

$\quad\propto p(\bar{y}|\theta)\,p(\theta)$ $\qquad$ $\bar{y}|\theta \sim N\left(\theta, \frac{\sigma^2}{n}\right)$

$\Rightarrow$ posterior $\propto \underbrace{N\left(\theta, \frac{\sigma^2}{n}\right)} \times N(\mu_0, \tau_0^2) \Rightarrow N(\mu_n, \tau_n^2)$

$n \to \infty$

$N\left(\bar{y} ; \frac{\sigma^2}{n}\right)$

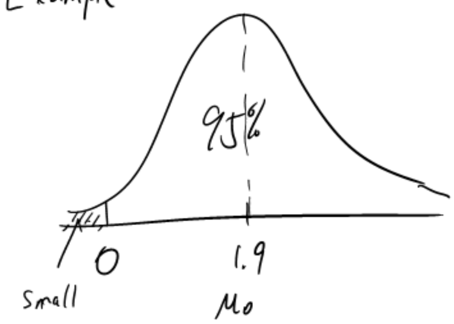
$$\Downarrow$$

$$\frac{1}{\sqrt{2\pi\frac{\sigma^2}{n}}} \exp\left\{-\frac{(\bar{y}-\theta)^2}{2\frac{\sigma^2}{n}}\right\}$$

$$\|$$

$$\frac{1}{\sqrt{2\pi\frac{\sigma^2}{n}}} \exp\left\{-\frac{(\theta-\bar{y})^2}{2\frac{\sigma^2}{n}}\right\}$$

Example



$$\sum_{i=1}^n (y_i-\theta)^2 = \sum_{i=1}^n (\underline{y_i - \bar{y}} + \underline{\bar{y} - \theta})^2$$

$$= \sum_{i=1}^n \left[(y_i-\bar{y})^2 + 2(y_i-\bar{y})(\bar{y}-\theta) + (\bar{y}-\theta)^2\right]$$

$$= \sum_{i=1}^n (y_i-\bar{y})^2 + \underline{2\sum_{i=1}^n (y_i-\bar{y})(\bar{y}-\theta)} + n(\bar{y}-\theta)^2$$

$$= 2(\bar{y}-\theta)\sum_{i=1}^n (y_i-\bar{y})$$

$$= 2(\bar{y}-\theta)\left(\sum_{i=1}^n y_i - n\bar{y}\right) = 0$$

Q2 last question

$$p(y_{n+1}|y_1,\cdots,y_n) = \boxed{\cdot ? \cdot}\,\frac{y_{n+1}^{2a-1}}{\left(y_{n+1}^2 + \boxed{\cdot ? \cdot}\right)^{\cdots}} \quad \text{for } y_{n+1} > 0.$$

$$z = \boxed{?}\,y_{n+1}^2, \quad p(z|y_1,\cdots,y_n) = \boxed{\cdots}\,\frac{z^{\cdots}}{(z + \cdots)^{\cdots}}, \quad \text{for } z > 0.$$

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Q3 $\quad p(\theta) = \kappa(\cdot, \cdots)\, g(\theta)^{\cdots} \exp(\eta(\theta) x \cdots)$