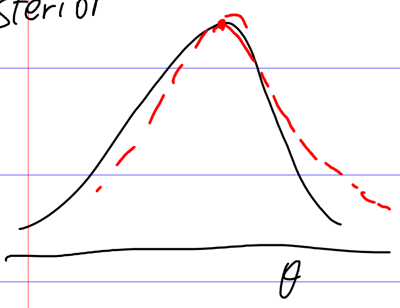
posterior



If the posterior is normal:

$$p(\theta|y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\theta-\mu)^2}{2\sigma^2}} \qquad (\mu, \sigma^2 \text{ depend on } y)$$

$$\underbrace{\log p(\theta|y)} = \underbrace{-\log\sqrt{2\pi} - \frac{1}{2}\log\sigma^2}_{const} - \boxed{\frac{(\theta-\mu)^2}{2\sigma^2}}$$

$p(\theta|y) \propto \exp(-A_1\theta^2 + A_2\theta - \dots)$

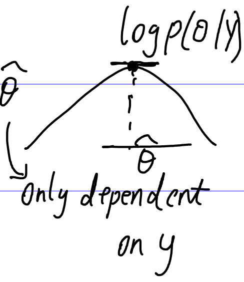
Taylor series expansion:

$f(\theta)$ expand around $\theta = \hat\theta$ ($\hat\theta$ — mode of $p(\theta|y)$)

$f(x)$ around $x = x_0$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \underbrace{\dots\dots}_{o((x-x_0)^3)}$$

$\ell(\theta) = (\log p(\theta|y)) \ (+C)$ expand at $\theta = \hat\theta$



If θ is 1-dimensional,

$$\ell(\theta) \approx \ell(\hat\theta) + \underbrace{\ell'(\hat\theta)(\theta-\hat\theta)}_{=0} + \frac{1}{2}\ell''(\hat\theta)(\theta-\hat\theta)^2 \quad (*)$$

$\ell'(\hat\theta) = 0$, $\hat\theta = \underset{\theta}{\operatorname{argmax}}\, \ell(\theta)$

$\ell(\theta) = \log p(\theta|y) + C$ ✓

$\ell'(\theta) = \frac{d}{d\theta}\log p(\theta|y)$, $\ell''(\theta) = \frac{d^2}{d\theta^2}\log p(\theta|y)$

$$p(\theta|y) = \exp(\log p(\theta|y)) = \exp(\ell(\theta) - C)$$

$$\overset{(*)}{\approx} \exp\Big(\underbrace{\ell(\hat\theta)}_{const} + \frac{1}{2}\ell''(\hat\theta)(\theta-\hat\theta)^2 - \underbrace{C}_{const}\Big)$$

$$\propto \exp\Big(\underbrace{\frac{1}{2}\ell''(\hat\theta)}_{<0}(\theta-\hat\theta)^2\Big)$$

$\ell''(\hat\theta)$

Laplace approximation

$$I = \int f(\theta) \underbrace{\exp(-n h(\theta))}_{= p(y|\theta)p(\theta)} d\theta$$

$$\exp(-nh(\theta)) = p(y|\theta)p(\theta)$$

$$-nh(\theta) = \log p(y|\theta) + \log p(\theta)$$

$$h(\theta) = -\frac{1}{n}[\underbrace{\log p(y|\theta) + \log p(\theta)}_{=\ell(\theta)}]$$

$$h'(\theta) = -\frac{1}{n}\ell'(\theta)$$

$$h''(\theta) = -\frac{1}{n}\ell''(\theta)$$

$$\int f(\theta)\exp(-nh(\theta))\,d\theta \approx f(\hat\theta)\sqrt{\frac{2\pi}{n}}\,\hat\sigma\cdot\exp(-nh(\hat\theta))$$

$$= f(\hat\theta)\sqrt{\frac{2\pi}{n}}\,\frac{1}{\sqrt{h''(\hat\theta)}}\cdot\exp(-nh(\hat\theta))$$

$$\Rightarrow \int g(\theta)\,p(y|\theta)p(\theta)\,d\theta \approx g(\hat\theta)\sqrt{\frac{2\pi}{n}}\,\frac{1}{\sqrt{-\frac{1}{n}\ell''(\hat\theta)}}\cdot p(y|\hat\theta)p(\hat\theta)$$

$$\approx g(\hat\theta)\underbrace{\sqrt{\frac{2\pi}{-l''(\hat\theta)}}}_{\sqrt{2\pi\cdot \text{Variance}}}\cdot p(y|\hat\theta)p(\hat\theta)$$

Normal approximation

$$\theta|y \overset{\text{approx}}{\sim} N(\hat\theta, \underbrace{[-l''(\hat\theta)]^{-1}}_{\text{Variance}})$$

Laplace approximation

$$p(\theta|y) \approx \frac{p(y|\theta)p(\theta)}{\hat I} = \frac{p(y|\theta)\,p(\theta)}{\sqrt{\frac{2\pi}{-l''(\hat\theta)}}\;p(y|\hat\theta)\,p(\hat\theta)}$$

$l''(\theta)$ – 2nd deriv of

$\ell(\theta) = \log p(y|\theta) + \log p(\theta)$.