ST4234: Bayesian Statistics

Tutorial 4, AY 19/20

Instructions: For each group who were scheduled to present on Tuesday, 18 February, please upload **one** copy of your solutions to the folder "Tutorial submission" by **6pm Tuesday, 18 February**.

1. Answer the following questions:

   (a) Let $Y \sim \text{Binomial}(n, \theta)$ where $0 < \theta < 1$. Identify Jeffreys prior distribution $p_J(\theta)$ for this model.

   (b) Reparameterize the binomial sampling model with $\psi = \log \frac{\theta}{1-\theta}$ and show that $p(y|\psi) = \binom{n}{y} \mathrm{e}^{\psi y}(1+\mathrm{e}^{\psi})^{-n}$. Show that Jeffreys prior distribution for this model is

   $$p_J(\psi) \propto \frac{\mathrm{e}^{\psi/2}}{1+\mathrm{e}^{\psi}}, \quad \psi \in \mathbb{R}.$$

   (c) Take the prior distribution from (a) and apply the change of variables formula to obtain the induced prior density on ψ. This density should be the same as the one derived in (b). This consistency under reparametrization is the defining characteristic of Jeffreys prior.

   (d) Plot $p_J(\theta)$ and $p_J(\psi)$.

2. Let $Y \sim \text{Poisson}(\theta)$ where $\theta > 0$.

   (a) Obtain Jeffreys prior distribution for this model. Is this a proper prior for θ? In other words, can $\sqrt{I(\theta)}$ be proportional to an actual probability density for $\theta > 0$?

   (b) Obtain the form of the function $f(\theta, y) = \sqrt{\theta} \times p(y|\theta)$. What probability density for θ is $f(\theta, y)$ proportional to? Can we think of $f(\theta, y)/\int f(\theta, y)\, \mathrm{d}\theta$ as a posterior density of θ given $Y = y$?

3. Suppose that $Y \sim \mathrm{N}(\theta, v)$ where the mean θ and variance v are both unknown.

(a) Show that Jeffreys prior for this model is $p_J(\theta, v) \propto v^{-3/2}$. (Hint: You may refer to the derivation on page 58 of Chapter 3 notes, but with (θ, v) as the parameters instead of (θ, σ).)

(b) Let $\boldsymbol{y} = (y_1, \ldots, y_n)$ be the observed values of an i.i.d. sample from $\mathrm{N}(\theta, v)$.

(i) Find a probability density $p_J(\theta, v|\boldsymbol{y})$ such that $p_J(\theta, v|\boldsymbol{y}) \propto p_J(\theta, v)p(\boldsymbol{y}|\theta, v)$. It may be convenient to write this joint density as $p_J(\theta|v, \boldsymbol{y}) \times p_J(v|\boldsymbol{y})$. (Hint: You may refer to page 30 of Chapter 3 notes and the handwritten notes.)

(ii) Find the marginal posterior distribution of θ up to a proportionality constant. By making a change of variable, show that the marginal posterior for $t = \dfrac{\theta - \bar{y}}{s'/\sqrt{n}}$ is a Student's t distribution with n degrees of freedom, where $s'^2 = \sum_{i=1}^{n}(y_i - \bar{y})^2/n$.

(iii) How would your answers to (i) and (ii) change if instead of the Jeffreys prior, we adopt a "reference prior", $p(\theta, v) \propto 1/v$, which is the product of the prior $p(\theta) \propto 1$ for θ and $p(v) \propto 1/v$ for v.