ST4234: Bayesian Statistics

Tutorial 4 Solution, AY 19/20

Solutions

1. (a) Let $\ell_\theta = \log p(y|\theta) = \log \binom{n}{y} + y \log \theta + (n-y)\log(1-\theta)$.

$$\frac{\partial \ell_\theta}{\partial \theta} = \frac{y}{\theta} - \frac{n-y}{1-\theta}, \quad \frac{\partial^2 \ell_\theta}{\partial \theta^2} = -\frac{y}{\theta^2} - \frac{n-y}{(1-\theta)^2}.$$

$$I(\theta) = -\mathrm{E}_Y\left[\frac{\partial^2 \ell_\theta}{\partial \theta^2}\right] = \mathrm{E}_Y\left[\frac{y}{\theta^2} + \frac{n-y}{(1-\theta)^2}\right] = \frac{n\theta}{\theta^2} + \frac{n-n\theta}{(1-\theta)^2} = \frac{n}{\theta(1-\theta)}.$$

$$\therefore \quad p_J(\theta) \propto \sqrt{I(\theta)} \propto \sqrt{\frac{n}{\theta(1-\theta)}} \propto \theta^{1/2-1}(1-\theta)^{1/2-1}.$$

Hence we can recognize from the mathematical form that $p_J(\theta)$ is a Beta(0.5,0.5) density.

(b) Let $\psi = \log \dfrac{\theta}{1-\theta}$, then $\theta = \dfrac{\mathrm{e}^\psi}{1+\mathrm{e}^\psi}$ and $1-\theta = \dfrac{1}{1+\mathrm{e}^\psi}$. Reparameterizing the binomial sampling model,

$$p(y|\psi) = \binom{n}{y}\left(\frac{\mathrm{e}^\psi}{1+\mathrm{e}^\psi}\right)^y \left(\frac{1}{1+\mathrm{e}^\psi}\right)^{n-y} = \binom{n}{y}\frac{\mathrm{e}^{\psi y}}{(1+\mathrm{e}^\psi)^n}.$$

Let $\ell_\psi = \log p(y|\psi) = \log \binom{n}{y} + y\psi - n\log(1+\mathrm{e}^\psi)$.

$$\frac{\partial \ell_\psi}{\partial \psi} = y - \frac{n\mathrm{e}^\psi}{1+\mathrm{e}^\psi}, \quad \frac{\partial^2 \ell_\psi}{\partial \psi^2} = -\frac{n\mathrm{e}^\psi}{(1+\mathrm{e}^\psi)^2}.$$

$$I(\psi) = -\mathrm{E}_Y\left[\frac{\partial^2 \ell_\psi}{\partial \psi^2}\right] = \frac{n\mathrm{e}^\psi}{(1+\mathrm{e}^\psi)^2}.$$

$$\therefore \quad p_J(\psi) \propto \sqrt{I(\psi)} \propto \sqrt{\frac{n\mathrm{e}^\psi}{(1+\mathrm{e}^\psi)^2}} \propto \frac{\mathrm{e}^{\psi/2}}{1+\mathrm{e}^\psi}.$$

(c) Let $f(\theta) = \theta^{-0.5}(1-\theta)^{-0.5}/B(0.5,0.5)$ denote the pdf of Beta(0.5,0.5). Note that $B(0.5,0.5) = \frac{\Gamma(0.5)\Gamma(0.5)}{\Gamma(1)} = \frac{\sqrt{\pi}\sqrt{\pi}}{1} = \pi$. Since $\theta = \dfrac{\mathrm{e}^\psi}{1+\mathrm{e}^\psi} \Rightarrow \dfrac{d\theta}{d\psi} = \dfrac{\mathrm{e}^\psi}{(1+\mathrm{e}^\psi)^2}$, applying the change of variables formula,

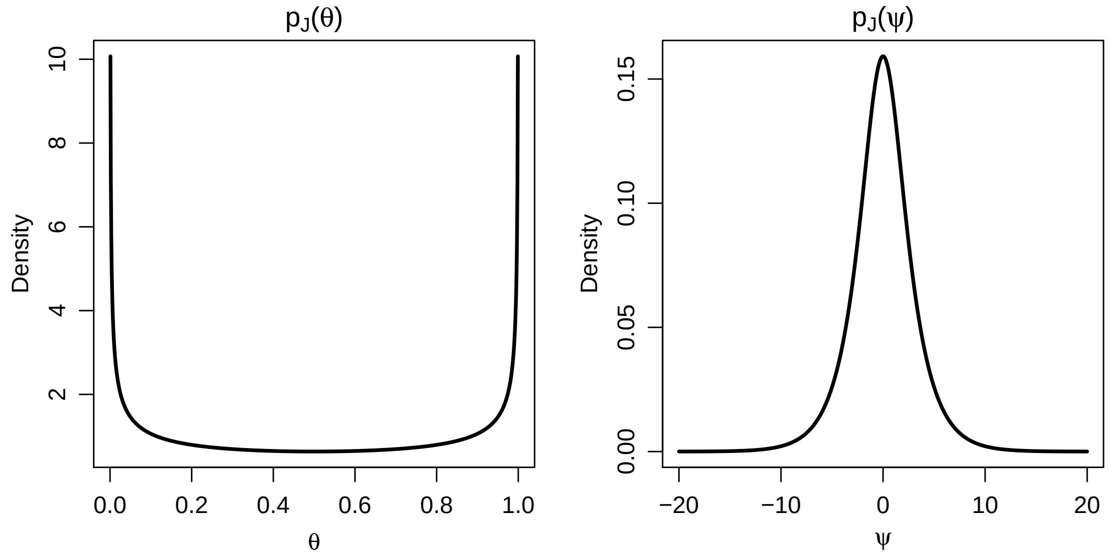
$$
\begin{aligned}
p_J(\psi) &= f\left(\frac{\mathrm{e}^{\psi}}{1+\mathrm{e}^{\psi}}\right)\left|\frac{d\theta}{d\psi}\right| \\
&= \frac{1}{\pi}\left(\frac{\mathrm{e}^{\psi}}{1+\mathrm{e}^{\psi}}\right)^{-0.5}\left(\frac{1}{1+\mathrm{e}^{\psi}}\right)^{-0.5}\frac{\mathrm{e}^{\psi}}{(1+\mathrm{e}^{\psi})^2} \\
&= \frac{1}{\pi}\frac{\mathrm{e}^{\psi/2}}{(1+\mathrm{e}^{\psi})},
\end{aligned}
$$

which is the same as the one derived in (b).

(d) The plots of $p_J(\theta)$ and $p_J(\psi)$ are shown below.

```
theta <- seq(from=0,to=1,by=0.001)
psi <- seq(from=-20,to=20,by=0.01)
Jpriorpsi <- function(psi){
    exp(psi/2)/(1+exp(psi))/pi
}

par(mfrow=c(1,2))
par(mar=c(3.5,3.5,1.5,1))
par(mgp=c(2.1,0.8,0))
plot(theta, dbeta(theta,0.5,0.5), type="l", lwd=2.5,
    xlab=expression(theta), ylab="Density",
    main=expression(paste(p[J],"(", theta, ")")))
plot(psi, Jpriorpsi(psi), type="l", lwd=2.5,
    xlab=expression(psi), ylab="Density",
    main=expression(paste(p[J],"(", psi, ")")))
```



2. (a) Let $\ell = \log p(y|\theta) = -\theta + y\log\theta - \log(y!)$.

$$\frac{\partial \ell}{\partial \theta} = -1 + \frac{y}{\theta}, \quad \frac{\partial^2 \ell}{\partial \theta^2} = -\frac{y}{\theta^2}.$$

$$I(\theta) = -\mathrm{E}_Y\left[\frac{\partial^2 \ell}{\partial \theta^2}\right] = \mathrm{E}_Y\left[\frac{y}{\theta^2}\right] = \frac{1}{\theta}.$$

$$\therefore \; p_J(\theta) \propto \sqrt{I(\theta)} \propto \sqrt{\frac{1}{\theta}} \propto \theta^{-1/2}$$

$$\Rightarrow p_J(\theta) = C\theta^{-1/2}.$$

for some $C > 0$. However $\int_0^\infty p(\theta)\ \mathrm{d}\theta = C\int_0^\infty \theta^{-1/2}\ \mathrm{d}\theta = 2C[\theta^{1/2}]_0^\infty = \infty \neq 1$ for any $C > 0$. Therefore Jeffrey's procedure does not produce an actual probability density for θ.

(b) $f(\theta, y) = \sqrt{\theta} \times p(y|\theta) = \theta^{1/2}\mathrm{e}^{-\theta}\theta^y/y! = \mathrm{e}^{-\theta}\theta^{y+1/2}/y! \propto \mathrm{e}^{-\theta}\theta^{y+3/2-1}$ which is proportional to the pdf of a Gamma$(y + 3/2, 1)$ density.

Yes, we can think of $f(\theta, y)/\int f(\theta, y)\ \mathrm{d}\theta$ as a posterior density of θ given $Y = y$ with the prior $p(\theta) \propto \sqrt{\theta}$. Even though the prior $p(\theta) \propto \sqrt{\theta}$ is improper, it still produced a proper posterior density.

3. (a) Let $\ell = \log p(y|\theta, v) = -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log(v) - \dfrac{(y-\theta)^2}{2v}$. We have

$$\frac{\partial \ell}{\partial \theta} = \frac{y-\theta}{v}, \quad \frac{\partial^2 \ell}{\partial \theta^2} = \frac{-1}{v}, \quad \frac{\partial^2 \ell}{\partial \theta \partial v} = -\frac{y-\theta}{v^2},$$
$$\frac{\partial \ell}{\partial v} = -\frac{1}{2v} + \frac{(y-\theta)^2}{2v^2}, \quad \frac{\partial^2 \ell}{\partial v^2} = \frac{1}{2v^2} - \frac{(y-\theta)^2}{v^3}.$$

Therefore,

$$I(\theta, v) = -\mathrm{E}_Y \begin{bmatrix} \dfrac{\partial^2 \ell}{\partial \theta^2} & \dfrac{\partial^2 \ell}{\partial \theta \partial v} \\ \dfrac{\partial^2 \ell}{\partial v \partial \theta} & \dfrac{\partial^2 \ell}{\partial v^2} \end{bmatrix} = \mathrm{E}_Y \begin{bmatrix} \dfrac{1}{v} & \dfrac{y-\theta}{v^2} \\ \dfrac{y-\theta}{v^2} & -\dfrac{1}{2v^2} + \dfrac{(y-\theta)^2}{v^3} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{v} & 0 \\ 0 & \dfrac{1}{2v^2} \end{bmatrix}.$$

$$\therefore \quad p_J(\theta, v) \propto \sqrt{|I(\theta, v)|} \propto \sqrt{\frac{1}{2v^3}} \propto v^{-3/2}.$$

(b) (i) Let $S = \sum_{i=1}^n (y_i - \bar{y})^2$. The posterior of (θ, v) is

$$\begin{aligned} p_J(\theta, v|\boldsymbol{y}) &\propto p_J(\theta, v) p(\boldsymbol{y}|\theta, v) \\ &\propto v^{-3/2} \cdot \prod_{i=1}^n p(y_i|\theta, v) \\ &\propto v^{-3/2} v^{-n/2} \exp\left\{-\frac{S}{2v} - \frac{n(\bar{y}-\theta)^2}{2v}\right\} \quad \text{[from page 24 of Chapter 3]} \\ &\propto v^{-1/2} \exp\left\{-\frac{n(\bar{y}-\theta)^2}{2v}\right\} \cdot v^{-n/2-1} \exp\left\{-\frac{S}{2v}\right\}. \end{aligned}$$

Thus $p_J(\theta, v|\boldsymbol{y}) = p_J(\theta|v, \boldsymbol{y}) \times p_J(v|\boldsymbol{y})$, where

$$\theta|v, \boldsymbol{y} \sim \mathrm{N}(\bar{y}, v/n), \quad v|\boldsymbol{y} \sim \text{Inv-Gamma}(n/2, S/2).$$

(ii) The marginal posterior distribution of θ is

$$
\begin{aligned}
p(\theta|\boldsymbol{y}) &= \int p(\theta, v|\boldsymbol{y}) \, \mathrm{d}v \\
&\propto \int v^{-(n+1)/2-1} \exp\left\{-\frac{S + n(\bar{y}-\theta)^2}{2v}\right\} \mathrm{d}v \\
&\propto \left[\frac{S + n(\bar{y}-\theta)^2}{2}\right]^{-(n+1)/2} \\
&\propto \left[1 + \frac{n(\bar{y}-\theta)^2}{S}\right]^{-(n+1)/2}. \\
&\propto \left[1 + \frac{(\bar{y}-\theta)^2}{s'^2}\right]^{-(n+1)/2}.
\end{aligned}
$$

When using the proportionality sign here, note that the LHS depends on θ and the integral depends on v. Hence, we cannot discard any factors that involve either θ or v.

For $t = \dfrac{\theta - \bar{y}}{s'/\sqrt{n}}$, since $\frac{d\theta}{dt} = \frac{s'}{\sqrt{n}}$, applying the change of variables formula,

$$
p(t|\boldsymbol{y}) \propto \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2},
$$

which is proportional to the pdf of a Student's t distribution with degrees of freedom n. Therefore $t|\boldsymbol{y} \sim t_n$.

(iii) If we adopt a "reference prior", $p(\theta, v) \propto 1/v$, we only need to modify the power of v in the joint posterior distribution in (i), and it would become $p(\theta, v|\boldsymbol{y}) = p(\theta|v, \boldsymbol{y}) \times p(v|\boldsymbol{y})$, where

$$
\theta|v, \boldsymbol{y} \sim \mathrm{N}(\bar{y}, v/n), \quad v|\boldsymbol{y} \sim \text{Inv-Gamma}((n-1)/2, S/2),
$$

with $S = \sum_{i=1}^{n} (y_i - \bar{y})^2$.

The marginal posterior distribution of θ is

$$\begin{aligned}
p(\theta|\boldsymbol{y}) &= \int p_J(\theta, v|\boldsymbol{y})\, \mathrm{d}v \\
&\propto \int v^{-n/2-1} \exp\left\{-\frac{S+n(\bar{y}-\theta)^2}{2v}\right\} \mathrm{d}v \\
&\propto \left[\frac{S+n(\bar{y}-\theta)^2}{2}\right]^{-n/2} \\
&\propto \left[1+\frac{n(\bar{y}-\theta)^2}{S}\right]^{-n/2} \\
&\propto \left[1+\frac{n(\bar{y}-\theta)^2}{s^2(n-1)}\right]^{-(n-1+1)/2},
\end{aligned}$$

where $s^2 = (n-1)^{-1}\sum_{i=1}^{n}(y_i-\bar{y})^2 = (n-1)^{-1}S$ is the sample variance. Let $u = \frac{(\bar{y}-\theta)}{s/\sqrt{n}}$. Then $\frac{d\theta}{du} = \frac{s}{\sqrt{n}}$. Applying the change of variable formula,

$$p(u|\boldsymbol{y}) \propto \left(1+\frac{u^2}{n-1}\right)^{-(n-1+1)/2},$$

which is proportional to the pdf of a Student's t distribution with degrees of freedom $n-1$. Therefore $u|\boldsymbol{y} \sim t_{n-1}$.