ST4234: Bayesian Statistics

Tutorial 6, AY 19/20

Instructions: For each group who were scheduled to present on Tuesday, 17 March, please upload **one** copy of your solutions to the folder "Tutorial submission" by **6pm Tuesday, 17 March**.

1. Find closed-form analytical solutions to the questions below.

   (a) Let y be an observation from Binomial(n, θ). Suppose that we assign a Beta(a_0, b_0) prior to θ. From Chapter 2 we know that the posterior distribution of θ is Beta(a_0+y, b_0+n-y).

   (i) Find the normal approximation to the posterior distribution of θ.

   (ii) Find the Laplace approximation to the posterior mean $\mathrm{E}(\theta|y)$, using the two approximation methods in Chapter 5, respectively.

   (b) Let $\boldsymbol{y} = \{y_1, \ldots, y_n\}$ be an i.i.d. sample from Poisson(θ). Suppose that we assign a Gamma(a_0, b_0) prior to θ. From Chapter 2 we know that the posterior distribution of θ is Gamma$(a_0 + t, b_0 + n)$, where $t = \sum_{i=1}^{n} y_i$.

   (i) Find the normal approximation to the posterior distribution of θ.

   (ii) Find the Laplace approximation to the posterior mean $\mathrm{E}(\theta|\boldsymbol{y})$, using the two approximation methods in Chapter 5, respectively.

2. Suppose $Y \sim$ Binomial(n, p) and we are interested in the log-odds $\theta = \log \dfrac{p}{1-p}$. Our prior for θ is $\theta \sim \mathrm{N}(\mu, \sigma^2)$. Show that the posterior density of θ is given by

$$p(\theta|y) \propto \frac{\exp(y\theta)}{[1 + \exp(\theta)]^n} \exp\left\{-\frac{(\theta-\mu)^2}{2\sigma^2}\right\}.$$

More concretely, suppose we are interested in learning about the probability that a special coin lands heads when tossed. Let θ denote the probability of landing heads. *A priori* we believe that the coin is fair, so we assign θ an $\mathrm{N}(0, .25^2)$ prior. We toss the coin $n = 5$ times and obtain $y = 5$ heads. Using a normal approximation to the posterior density, compute the probability that the coin is biased toward heads (i.e., that θ is positive).

3. Fifteen reciprocating pumps were tested for a prespecified time and one assumes that the failure times follow the two-parameter exponential distribution

$$p(y|\beta, \mu) = \frac{1}{\beta} \exp\left\{-\frac{y-\mu}{\beta}\right\}, \quad y \geq \mu.$$

Suppose one places a uniform prior on (μ, β). The posterior density is then given by

$$p(\beta, \mu|\text{data}) \propto \frac{1}{\beta^s} \exp\left\{-\frac{t-n\mu}{\beta}\right\}, \quad \mu \leq t_1,$$

where n is the number of items placed on test, t is the total time on test, t_1 is the smallest failure time, and s is the observed number of failures in a sample of size n. When $n = 15$ pumps were tested for a total time of $t = 15962989$, eight failures (s = 8) were observed and the smallest failure time was $t_1 = 237217$. Suppose one transforms the parameters to the real line using $\theta_1 = \log \beta$ and $\theta_2 = \log(t_1 - \mu)$.

(a) Write down the posterior density of (θ_1, θ_2) up to a proportionality constant.

(b) Construct an R function that computes the log posterior density of (θ_1, θ_2) up to an additive constant.

(c) Find a normal approximation to the posterior density.