$$p(\theta) \xrightarrow{y_1} p(\theta|y_1) \propto p(y_1|\theta)\,p(\theta) \xrightarrow{y_2} p(\theta|y_1,y_2) \propto p(y_2|\theta)\cdot p(\theta|y_1)$$

prior

posterior 1

as a prior

Assumption: y_1, y_2 independent given θ $\quad (y_1 \perp y_2 | \theta)$

Joint = Conditional × marginal

$$p(\tilde{y},\theta|y) = \underbrace{p(\tilde{y}|\theta,y)}_{= p(\tilde{y}|\theta)} \cdot p(\theta|y)$$

because $\tilde{y} \perp y | \theta$

$p(\theta|y) \propto$ "$g(y,\theta)$"

$$p(\theta|y) = \frac{g(y,\theta)}{\int g(y,\theta)\,d\theta'}$$ ← normalization

$$\int p(\theta|y)\,d\theta = 1$$

$$p(\theta|y) = \frac{\theta^{a_0+y-1}(1-\theta)^{b_0+n-y-1}}{C}$$

$$C = B(a_0+y,\, b_0+n-y) = \int_0^1 \theta^{a_0+y-1}(1-\theta)^{b_0+n-y-1}\,d\theta$$

$$p(\theta|y) \propto \underbrace{\theta^{a_0+y-1}(1-\theta)^{b_0+n-y-1}}_{\text{① core part of a beta density}}, \quad \underline{0<\theta<1} \ ②$$

Example. $p(\theta|y) \propto e^{-10(\theta-\bar{y})^2}$ core, $\quad \underline{-\infty<\theta<+\infty}$ ②

$$\Rightarrow p(\theta|y) = \frac{e^{-10(\theta-\bar{y})^2}}{C}$$

$$\Rightarrow \theta|y \sim N\left(\bar{y}, \frac{1}{20}\right)$$

$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i.$

Normal density $f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \underbrace{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}_{\text{core}}$ constant

$-\infty < x < +\infty.$

$$\frac{1}{2\sigma^2} = 10 \Rightarrow \sigma^2 = \frac{1}{20}$$

$$E(\theta|y) = \frac{a_0+y}{a_0+b_0+n}$$

If we want "no information" from the prior, $E(\theta|y) = \frac{y}{n}$, we can let $\underline{a_0=0,\ b_0=0.}$

prior becomes $\quad p(\theta) = \frac{1}{B(0,0)?}\,\theta^{-1}(1-\theta)^{-1}$

$$\Rightarrow p(\theta) \propto \frac{1}{\theta(1-\theta)}$$ not a proper density "improper density"

$$\int_0^1 \frac{1}{\theta(1-\theta)}\,d\theta = +\infty$$ prior density doesn't exist.

$$E(\theta|y) = \frac{a_0+y}{a_0+b_0+n} \approx \frac{y}{n} \qquad n >> a_0+b_0, \quad y >> a_0$$

$n \to +\infty$

conjugate prior:

binomial model: $p(y|\theta) = \binom{n}{y}\,\theta^y(1-\theta)^{n-y}$

$$p(\theta) = \frac{1}{C}\,\theta^{a_0-1}(1-\theta)^{b_0-1}\cdot e^{-c_0\theta}, \quad 0<\theta<1 \;\Rightarrow\; P(a_0,b_0,c_0)$$

$$p(\theta|y) \propto p(y|\theta)\,p(\theta)$$

$$\propto \theta^y(1-\theta)^{n-y}\cdot\theta^{a_0-1}(1-\theta)^{b_0-1}e^{-c_0\theta}$$

$$\propto \theta^{y+a_0-1}(1-\theta)^{n-y+b_0-1}\cdot e^{-c_0\theta} \;\Rightarrow\; P(y+a_0-1,\, n-y+b_0-1,\, c_0)$$

If t is sufficient, then

"$y=t$"

$$p(y|t) = \frac{p(y|\theta)}{p(t|\theta)} = \frac{\theta^y(1-\theta)^{n-y}}{\binom{n}{y}\,\theta^y(1-\theta)^{n-y}} = \frac{1}{\binom{n}{y}}$$

$t = Y$ counts

$$p(x,y,z) = p(x,y|z)\,p(z)$$
$$= p(x|y,z)\cdot p(y|z)\cdot p(z)$$

predictive distribution:

$$p(\tilde{Y}=1|Y) = \int_0^1 p(\tilde{Y}=1,\theta|Y)\,d\theta = \int_0^1 \underbrace{p(\tilde{Y}=1|\theta,Y)}_{=p(\tilde{Y}=1|\theta)}\,p(\theta|Y)\,d\theta = \int_0^1 \overset{=\theta}{p(\tilde{Y}=1|\theta)}\,p(\theta|Y)\,d\theta$$

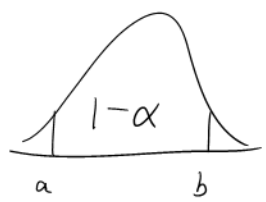
$$\tilde{Y} = \begin{cases} 1 & \text{w.p. } \theta \\ 0 & \text{w.p. } 1-\theta \end{cases}$$

$$= \int_0^1 \theta\,p(\theta|y)\,d\theta = E(\theta|y) = \frac{a_0+y}{a_0+b_0+n}$$

$\theta|y \sim \text{Beta}(a_0+y,\; b_0+n-y)$

Credible set $[a,b]$,

In most cases $\int_a^b p(\theta|y)\,d\theta = 1-\alpha$



$a=a(y)$, $b=b(y)$

Equal-tails CI