ST4234: Bayesian Statistics

Tutorial 7, AY 19/20

Instructions: For each group who were scheduled to present on Tuesday, 24 March, please upload **one** copy of your solutions to the folder "Tutorial submission" by **6pm Tuesday, 24 March**.

1. Consider the genetic linkage model in Chapter 5. 197 animals are distributed into four categories as follows (Rao 1997):

$$\boldsymbol{y} = (y_1, y_2, y_3, y_4) = (125, 18, 20, 34)$$

with cell probabilities

$$\left(\frac{1}{2} + \frac{\theta}{4},\ \frac{1}{4}(1-\theta),\ \frac{1}{4}(1-\theta),\ \frac{\theta}{4}\right),$$

where $\theta \in [0, 1]$ is the parameter. Suppose that θ is assigned a uniform prior on $[0, 1]$.

(a) If θ is transformed to the real-valued logit $\eta = \log \frac{\theta}{1-\theta}$, show that the posterior density of η can be written as

$$p(\eta|\boldsymbol{y}) \propto \left(2 + \frac{\mathrm{e}^\eta}{1+\mathrm{e}^\eta}\right)^{125} \frac{1}{(1+\mathrm{e}^\eta)^{39}} \left(\frac{\mathrm{e}^\eta}{1+\mathrm{e}^\eta}\right)^{35}, \quad \eta \in \mathbb{R}.$$

(b) Use a normal approximation to find a 95% confidence interval for η. Transform this interval to obtain a 95% confidence interval for the original parameter of interest θ.

(c) Design a rejection sampling algorithm for simulating from the posterior density of η. Use t_4 as the proposal density with mean and scale parameters given by the normal approximation. Use rejection sampling to draw 10^4 samples from the posterior distribution of η. What is the acceptance rate of your algorithm? Using simulated draws from your algorithm, find a 95% confidence interval for η. Transform the simulated samples to obtain a 95% confidence interval θ.

2. Still consider the genetic linkage model in Chapter 5. n ($n = y_1 + y_2 + y_3 + y_4$) animals are distributed into four categories as follows:

$$\boldsymbol{y} = (y_1, y_2, y_3, y_4)$$

with cell probabilities

$$\left(\frac{1}{2} + \frac{\theta}{4}, \frac{1}{4}(1-\theta), \frac{1}{4}(1-\theta), \frac{\theta}{4}\right),$$

where $\theta \in [0,1]$ is the parameter. Suppose that θ is assigned a uniform prior on $[0,1]$. Consider the importance sampling estimation of the posterior mean $\mathrm{E}(\theta|\boldsymbol{y})$. We consider two possible sets of $\boldsymbol{y}$ values: $\boldsymbol{y} = (125, 18, 20, 34)$ and $\boldsymbol{y} = (14, 0, 1, 5)$.

(a) Suppose that we use the normal approximation density as the proposal density ($g(\theta)$ in Chapter 6). For each of the two $\boldsymbol{y}$ vectors: (i) Draw 10^4 samples of θ from the proposal and plot a histogram of the normalized weights ($W(\theta)$ in Chapter 6); (ii) Calculate the importance sampling estimates of $\mathrm{E}(\theta|\boldsymbol{y})$ and the standard error; (iii) Compare the importance sampling estimates to the Laplace approximation in Chapter 5 notes.

(b) Suppose that we use a Beta(a,b) density as the proposal density instead. For each of the two $\boldsymbol{y}$ vectors: (i) Find a suitable combination (a,b), such that the mode of Beta(a,b) approximately matches with the mode of $p(\theta|\boldsymbol{y})$, and the variance of Beta(a,b) approximately matches with the variance from the normal approximation. (ii) Draw 10^4 samples of θ from the beta proposal, plot a histogram of the normalized weights ($W(\theta)$ in Chapter 6), and compare with Part (a); (iii) Calculate the importance sampling estimates of $\mathrm{E}(\theta|\boldsymbol{y})$ and the standard error; (iv) Compare the importance sampling estimates to the estimates from Part (a) and the Laplace approximation in Chapter 5 notes.

3. (Continued from Tutorial 6 Question 2) Suppose $Y \sim$ Binomial(n,p) and we are interested in the log-odds $\theta = \log \frac{p}{1-p}$. Our prior for θ is $\theta \sim \mathrm{N}(\mu, \sigma^2)$. The posterior density of θ is given by

$$p(\theta|y) \propto \frac{\exp(y\theta)}{[1+\exp(\theta)]^n} \exp\left\{-\frac{(\theta-\mu)^2}{2\sigma^2}\right\}.$$

More concretely, suppose we are interested in learning about the probability that a special coin lands heads when tossed. Let θ denote the probability of landing heads. *A priori* we believe that the coin is fair, so we assign θ an $\mathrm{N}(0, .25^2)$ prior. We toss the coin $n = 5$ times and obtain $y = 5$ heads.

Using the prior density as a proposal density, design a rejection algorithm to sample 10^4 draws from the posterior distribution of θ. Using simulated draws from your algorithm, approximate the probability that the coin is biased toward heads. What is the acceptance rate of this algorithm?