Question 1. [7 MARKS]

Solution:
$P(n) : 3^n < n!$

For a contradiction, assume that there is some $j \in \mathbb{N}, j \geq 7$, such that $P(j)$ does not hold.
Let S be the set of all natural numbers greater than or equal to 7, for which the predicate does not hold.
By our assumption S is not empty. By definition, S is a subset of $\mathbb{N}$. By PWO, S contains a least element a.

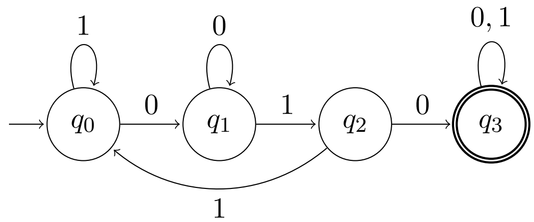
a must be strictly greater than 7 since $3^7 = 2187 < 5040 = 7!$ meaning that $P(7)$ holds. So $a - 1$ is greater than or equal to 7.
Moreover, since a is the min elements of S, $a - 1$ is not in S, and so $P(a - 1)$ holds.
Since $a - 1 \geq 7$, we have $3^{(a-1)} < (a - 1)!$. Then we have

$$\begin{aligned}
3^a &= 3 \times 3^{(a-1)} \\
&< 3 \times (a-1)! && \text{\# since } 3^{(a-1)} < (a-1)! \\
&< a \times (a-1)! && \text{\# since } a \geq 7 > 3 \\
&= a!
\end{aligned}$$

That is, $P(a)$ holds!
This, however, contradicts our assumption that a is an element of S because by definition elements of S do not satisfy the predicate. Thus we conclude that our initial assumption is false. That is, for all $n \in \mathbb{N}, n \geq 7, P(n)$ holds.

Question 2. [11 MARKS]

Solution:



Here are the state invariants

$$\delta^*(q_0, x) = \begin{cases} q_0 \text{ iff } x = \epsilon \text{ or } x = 1 \text{ or (010 is not a substring of } x \text{ and } x \text{ ends with 11);} \\ q_1 \text{ iff 010 is not a substring of } x \text{ and } x \text{ ends with 0;} \\ q_2 \text{ iff 010 is not a substring of } x \text{ and } x \text{ ends with 01;} \\ q_3 \text{ iff 010 is a substring of } x; \end{cases}$$

Question 3. [12 MARKS]

Solution:
$P(f)$: There exists g such that $g \in L$ and g and f are logically equivalent.

Base Case: For $f = x$, where x is a propositional variable.
Then $c_{not}(f) = 0 = c_{and}(f)$. Therefore $P(f)$ holds.

Induction Step: Assume $f_1, f_2 \in G$. Suppose $P(f_1)$ and $P(f_2)$. **[IH]**
That is, there exist g_1, g_2 such that $g_1, g_2 \in L$ and g_1 is logically equivalent to f_1, and g_2 is logically equivalent to f_2.

Case (A): Let $f = \neg f_1$, and $g = \neg(g_1 \wedge g_1)$.

Then $c_{not}(g) = 1 + 2 \times c_{not}(g_1)$ and $c_{and}(g) = 1 + 2 \times c_{and}(g_1)$.
By IH, $c_{not}(g_1) = c_{and}(g_1)$.
So, $c_{not}(g) = c_{and}(g)$.

By the IH, $\neg(g_1 \wedge g_1)$ is logically equivalent to $\neg(f_1 \wedge f_1)$, which is logically equivalent to $\neg f_1$, which by assumption is f.
Therefore g and f are logically equivalent, and so $P(f)$.

Case (B): Let $f = (f_1 \wedge f_2)$, and $g = (\neg\neg g_1 \wedge (g_2 \wedge g_2))$.

Then $c_{not}(g) = 2 + c_{not}(g_1) + 2 \times c_{not}(g_2)$ and $c_{and}(g) = 2 + c_{and}(g_1) + 2 \times c_{and}(g_2)$.
By IH, $c_{not}(g_1) = c_{and}(g_1)$ and $c_{not}(g_2) = c_{and}(g_2)$.
So, $c_{not}(g) = c_{and}(g)$.

By the IH, $(\neg\neg g_1 \wedge (g_2 \wedge g_2))$ is logically equivalent to $(\neg\neg f_1 \wedge (f_2 \wedge f_2))$, which is logically equivalent to $(f_1 \wedge f_2)$, which by assumption is f.
Therefore g and f are logically equivalent, and so $P(f)$.

Question 4. [10 MARKS]

Solution:
$P(n)$: for all $x \in \mathbb{R}$, if n is even, then $T(x, n) = x(2^{n+2} - 1)$

We prove that for all $n \in \mathbb{N}$, $P(n)$ holds.

Let $x \in \mathbb{R}$ be arbitrary.
Base Case: Let $k = 0$. Then $T(x, 0) = 3x = x(2^2 - 1) = x(2^{0+2} - 1)$. So $P(0)$ holds.

Induction Step: Suppose $k \in \mathbb{N}$, $k \geq 2$. Assume for all $j \in \mathbb{N}, 0 \leq j < k, P(j)$ holds. **[IH]**

Suppose k is even (otherwise $P(k)$ is vacuously true).

Then, by the induction hypothesis,

$$
\begin{aligned}
T(x,k) &= 2^3 T\left(\frac{x}{2}, k-2\right) + 3x && \text{\# By definition of } T(x,k) \\
&= 2^3 \left[\frac{x}{2}(2^{k-2+2} - 1)\right] + 3x && \text{\# By IH} \\
&= 2^3 \times 2^k \times \frac{x}{2} - 2^3 \times \frac{x}{2} + 3x \\
&= 2^2 \times 2^k \times x - 4x + 3x \\
&= 2^{k+2}x - x \\
&= x(2^{k+2} - 1)
\end{aligned}
$$

So for all $x \in \mathbb{R}, T(x, k) = x(2^{k+2} - 1)$, i.e. $P(k)$.
By induction, for all $n \in \mathbb{N}, P(n)$.