CSC236 - Problem Set 1

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Problem 1

For a graph G with vertices that belong in V and edges that belong in E. We denote this graph as $G =< V, E >$

Define $P(n)$: for all non-empty, undirected, simple graphs G with n vertices, if $\exists u \in V$ s.t. $d_G(u) = |V| - 1$ then G is connected.

WTP: $\forall n \in N, P(n)$.

Base case:

Let $|V| = 0$, then the graph is empty so $P(0)$ holds vacuously.

Let $n = 1$. There is only one simple graph with one vertex, and it does not have any edges, so the degree of the one vertex is $0 = |V| - 1$. This graph is also connected, so $P(1)$ holds.

Inductive step:

Let n be an arbitrary natural number. Assume $P(n)$, so for any (non-empty, undirected and simple) graph $G =< V, E >$ with $|V| = n$, assume that $\exists u \in V$ s.t. $d_G(u) = n - 1$. and that G is connected.

Let be $H =< V', E' >$ be a arbitrary non-empty, undirected and simple graph with $n + 1$ vertices, where $\exists u \in V'$ s.t. $d_H(u) = |V'| - 1 = n$. Since $d_H(u) = |V'| - 1$ and in a simple graph there can only be one edge between any two vertices, u is adjacent to every vertex in H.

Let s be any vertex in H other than u, let $G =< V, E >$ be the graph created by removing s and all edges connected to it from H.

Since every vertex in H except for s is also in G and has the same adjacencies, u is also in G and is adjacent to every vertex in G, so it has degree $|V| - 1 = n - 1$. Thus, G is a graph with n vertices and a vertex with degree $n - 1$, so by the IH, G is connected.

Since G is connected, we know that there exists a path from u to any other vertex in G, therefore there is a path from u to any vertex in H other than s. Since s is adjacent to u, we know that there exists a path from s to u in H. Therefore we can make a path from s to any vertex v in H by making a path from s to u and then from u to v. Furthermore, any vertex v other than s can reach s by following its path to u and then reach s. Thus, every pair of vertices in H is connected, which means H is connected.

Since H was arbitrary, $P(n + 1)$ holds.

So by the Principle of Induction, $\forall n \in \mathbb{N}, P(n)$.

$\square$

Problem 2

For a game with players $P = \{p_1, p_2, ...p_n\}$:

Define $Wet : P \to \mathbb{N}$ where $Wet(p)$ is the number of water balloons thrown at p.

Define $d : P^2 \to R$ such that $d(p_i, p_j)$ is the distance between p_i and p_j.

First, a lemma: Every game of more than 2 players has a pair of players that hit each other with water balloons.

Suppose we have a game with a set of $n \geq 2$ players $P = \{p_1, p_2, ...p_n\}$.

Define $S = \{d(p_1, p_2) : p_1$ and p_2 are players in the game$\}$.

Since S is a finite set of real numbers, it has a least element, and therefore $\exists p_1, p_2 \in P, \forall p_3, p_4 \in P$ s.t. $d(p_1, p_1) \leq d(p_3, p_4)$

Since the distance between p_1 and p_2 is the least of all distances, p_1's closest neighbour is p_2 and p_2's closest neighbour is p_1, so p_1 and p_2 will hit each other.

Since n and the game were arbitrary, we know that for any game with $n \geq 2$ players, there exists a pair of players that hit each other with water balloons.

Let $P(n)$ be the predicate "if n is odd, then any game with n players will have at least one survivor" for $n \in \mathbb{N}$.

Base case:

Let $n = 0$. Since 0 is even, $P(0)$ holds vacuously.

Let $n = 1$. Since there is only one player in the game, that player will not be thrown at by anyone. Thus, the single player will be the only survivor, so $P(1)$.

Inductive step:

Suppose $P(k)$ for all $k \in \mathbb{N}, k \leq n$ for some n.

If $n + 1$ is even, then $P(n + 1)$ holds vacuously.

Suppose $n + 1$ is odd, and let G be a game with $n + 1$ players.

Since we know there exists a pair of players that hit each other with water balloons. Let p_1 and p_2 be those players, clearly $Wet(p_1) \geq 1$ and $Wet(p_2) \geq 1$.

Case 1: Assume $Wet(p_1) = 1$ and $Wet(p_2) = 1$.

Since these two players are not the target of water balloons other than each other, we can construct a new game G' without p_1, p_2 but the same positions and number of survivors.

Since this is a game of $n - 1$ players and $n - 1$ is odd, we can apply the IH and conclude that there is at least one person who does not get wet. Since G has the same survivors as G', so $P(n + 1)$.

Case 2. Assume that $Wet(p_1) > 1$ or $Wet(p_2) > 1$.

Assume without loss of generality that $Wet(p_1) > 1$.

Since $Wet(p_1)$ water balloons have been thrown at p_1, so there are $n - Wet(p_1)$ water balloons left to be thrown at the rest of the players. However, there are $n - 1$ other players, so there are not enough water balloons to get everyone wet, so there is at least one survivor, so $P(n + 1)$.

Therefore, $P(n)$ holds for all natural numbers.

□

Problem 3

We define the predicate $P(n)$: "Every tournament with n teams that has a cycle has a cycle of length 3", where $n \in \mathbb{N}$.

Base case:

Suppose G is a tournament with 0, 1, or 2 teams. Then G does not have enough teams to form a cycle, so $P(0), P(1), P(2)$ all hold vacuously.

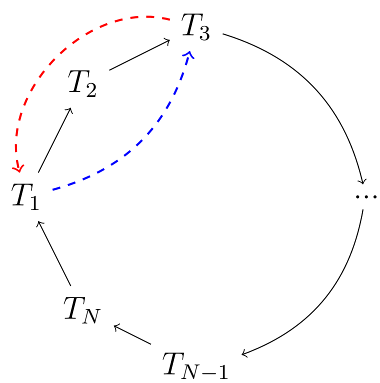
Suppose G is a tournament with 3 teams, T_1, T_2, T_3 which has a cycle. Then since a cycle requires at least 3 teams and the tournament only has 3 teams, there is a cycle of length 3, so $P(3)$ holds.

Inductive step:

Let $S = \{n \in \mathbb{N} : \neg P(n)\}$ so $S \subseteq \mathbb{N}$. Suppose S is non-empty, then by the Well-Ordering Principle, S has a least element which we will call N. We know that $N > 3$ because we showed above that $P(0), P(1), P(2), P(3)$ all hold.

Let G be a tournament with N teams with a cycle, so G has no cycle of length 3.

Suppose the cycle has length $3 \leq m < N$, then let G' be the tournament with the same outcomes as G (aka the same teams win/lose against each other), however with only the teams in the cycle. Then G' is a tournament with m teams which has a cycle. Since $m < N$, by assumption $P(m)$, so there is a cycle of length 3 in G'. Since all of the teams in G' are in G with the same outcomes, the same cycle of length 3 must also be in G. This is a contradiction.



Suppose the cycle has length N. Let $T_1, T_2, ...T_n$ be all the teams in the cycle where $T_1 \to T_2, T_2 \to T_3, T_3 \to T_4, ...T_{N-1} \to T_N, T_N \to T_1$. Then since every team plays every other team and there are no ties, either $T_3 \to T_1$ or $T_1 \to T_3$.

Suppose $T_3 \to T_1$, then $T_1 \to T_2, T_2 \to T_3, T_3 \to T_1$, so there is a cycle of length 3, so we have reached a contradiction.

Suppose $T_1 \to T_3$, then $T_1 \to T_3, T_3 \to T_4, ...T_{N-1} \to T_N, T_n \to T_1$, so there is a cycle of length $N-1$ which is less than N. Then as shown earlier, there must be a cycle of length 3, so we have reached another contradiction.

Since we have reached a contradiction in every case, we must conclude that our assumption that S is non-empty must have been incorrect, so $\forall n \in \mathbb{N}, P(n)$.

$\square$

Problem 4

Base case:

Let $k = 0$, then $(2^{k+1} + 1, 2^k + 1) = (2^1 + 1, 1 + 1) = (3, 2)$, so $(3, 2) \in R$.

Inductive step:

Suppose for some $(x, y) \in S$, $(x, y) \in R$, so $\exists k \in \mathbb{N}, (x, y) = (2^{k+1} + 1, 2^k + 1)$. Since $(x, y) \in S$, we know that $(3x - 2y, x) \in S$. Then if $k' = k + 1$,

$$\begin{aligned}
(3x - 2y, x) &= (3(2^{k+1} + 1) - 2(2^k + 1), 2^{k+1} + 1) \\
&= (3 \cdot 2^{k+1} + 3 - 2 \cdot 2^k - 2, 2^{k+1} + 1) \\
&= ((2 + 1)2^{k+1} - \cdot 2^{k+1} + 1, 2^{k+1} + 1) \\
&= (2^{k+2} + 2^{k+1} - \cdot 2^{k+1} + 1, 2^{k+1} + 1) \\
&= (2^{k+2} + 1, 2^{k+1} + 1) \\
&= (2^{k'+1} + 1, 2^{k'} + 1)
\end{aligned}$$

So there exists a natural number k' so that $(3x - 2y, x) = (2^{k'+1} + 1, 2^{k'} + 1)$, so $(3x - 2y, x) \in R$. So by the principle of structural induction, $(x, y) \in S \implies (x, y) \in R$, so $S \subseteq R$.

$\square$