

Compiling to Categories

mathematically-principled program transformation

T. Mark Ellison and Siva Kalyan

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Australian National University

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Haskell and Category Theory

Haskell	Category Theory
Category	Category
Type	Object
Function	Morphism
<u>Hask</u>	<u>Set</u>
...	Terminal Objects
Tuple	Product
Currying, Function Application	Cartesian Closure
Type Constructor, Functor	Functor

Categories

Categories

A category $\underline{\mathbf{C}}$ consists of

1. a class $\text{Obj}(\underline{\mathbf{C}})$ of *objects*, and
2. for each pair of objects $A, B \in \text{Obj}(\underline{\mathbf{C}})$, a set $\text{Hom}_{\underline{\mathbf{C}}}(A, B)$ of *arrows* (or *morphisms*) from A to B , known as a *hom-set*.

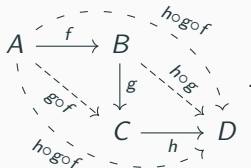
$$\begin{array}{ccc} & \text{Hom}_{\underline{\mathbf{C}}}(A, B) & \\ A & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & B \end{array}$$

Many familiar parts of Haskell form a category **Hask**: objects are *types* (**Int**, **Char**, etc.), and arrows are *functions* between types (e.g. **ord** :: **Int** -> **Char**).

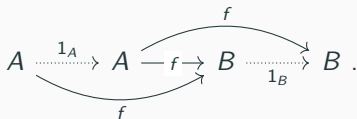
Category Laws

In a category $\underline{\mathbf{C}}$:

1. Given arrows $f: A \rightarrow B$ and $g: B \rightarrow C$ in $\underline{\mathbf{C}}$, the *composition* $g \circ f: A \rightarrow C$ ($= g.f$) is also in $\underline{\mathbf{C}}$.
2. Given arrows $f: A \rightarrow B$, $g: B \rightarrow C$ and $h: C \rightarrow D$,
 $(h \circ g) \circ f = h \circ (g \circ f) = h \circ g \circ f$.



3. Every object $A \in \text{Obj}(\underline{\mathbf{C}})$ is associated with an *identity arrow* $1_A: A \rightarrow A$ ($= \text{id}$). Given any arrow $f: A \rightarrow B$, we have



Examples

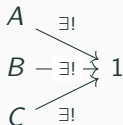
	<u>Set</u>	<u>Hask</u>	<u>POrd</u>	<u>Cat</u>
Objects	sets	types	items	small cats
Morphisms	functions	functions	$a \leq b$	functors
Composition	$f \circ g$	$f.g$	transitivity	$F \circ G$
Identity	1_A	id	$a = a$	$1_{\underline{C}}$

Not everything in Haskell can be in Hask if we want it to be a category. Every type in the language contains a **Bottom** (\perp) or **undefined** value, but these 'values' cause mayhem with the category laws (in particular the **Identity** constraint). So when we talk about Hask we'll be talking about vanilla Hask without these abnormal values. (Haskell wiki page on Hask.)

Category Theory: Terminal Objects

A *terminal object* is a type 1 (a.k.a. T) in $\text{Obj}(\underline{\mathbf{C}})$, such that there is only a single mapping from any other type A onto that type:

$$\forall A \in \text{Obj}(\underline{\mathbf{C}}), |\text{Hom}_{\underline{\mathbf{C}}}(A, 1)| = 1.$$



In **Hask**:

```
1  () -- the type corresponding to 1, containing only itself
2  terminalMap :: t -> ()
3  terminalMap _ = ()
```


Examples

	<u>Set</u>	<u>Hask</u>	<u>POrd</u>	<u>Cat</u>
Objects	sets	types	items	small cats
Morphisms	functions	functions	$a \leq b$	functors
Composition	$f \circ g$	$f.g$	transitivity	$F \circ G$
Identity	1_A	id	$a = a$	$1_{\underline{c}}$
Terminal obj.	$\{*\}$	$()$	upper bound	$\underline{1}$

Products

Given objects A, B in $\underline{\mathbf{C}}$ there may be a (*pairwise*) *product* $A \sqcap B \in \text{Obj}(\underline{\mathbf{C}})$ and *projection arrows* $\pi_A: A \sqcap B \rightarrow A$ and $\pi_B: A \sqcap B \rightarrow B$ such that for any object X in the same category and arrows $a: X \rightarrow A$ and $b: X \rightarrow B$ there is a *unique* arrow $x: X \rightarrow A \sqcap B$ such that $a = \pi_A \circ x$ and $b = \pi_B \circ x$:

$$\begin{array}{ccccc} A & \xleftarrow{\pi_A} & A \sqcap B & \xrightarrow{\pi_B} & B \\ & \swarrow a & \uparrow \exists! x & \searrow b & \\ & & X & & \end{array} .$$

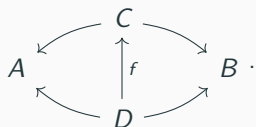
In other words: Given a particular way of mapping X to A and to B , there's only *one* way of mapping X to $A \sqcap B$ such that everything's consistent.

Products

Alternatively, the triplet $\langle A \sqcap B, \pi_A, \pi_B \rangle$ is a *terminal object* in the category whose objects are diagrams of the form

$$A \longleftarrow C \longrightarrow B ,$$

and whose arrows are (commutative) diagrams of the form



Products in Haskell

```
1  (a,b) -- the type containing pairs from types a and b ( $A \times B$ )
2  fst :: (a,b) -> a -- the projection function  $\pi_A$ 
3  fst (x,y) = x
4  snd :: (a,b) -> b -- the projection function  $\pi_B$ 
5  snd (x,y) = y
6  factorThroughProd :: (c -> a) -> (c -> b) -> (c -> (a,b))
7  factorThroughProd f g = \ x -> (f x, g x)
```

It should be obvious that

fst.(factorThroughProd f g) = f, and

snd.(factorThroughProd f g) = g.

Examples

	<u>Set</u>	<u>Hask</u>	<u>POrd</u>	<u>Cat</u>
Objects	sets	types	items	small cats
Morphisms	functions	functions	$a \leq b$	functors
Composition	$f \circ g$	$f.g$	transitivity	$F \circ G$
Identity	1_A	id	$a = a$	$1_{\underline{C}}$
Terminal obj.	$\{*\}$	$()$	upper bound	$\underline{1}$
Product	$A \times B$	(a,b)	$\min(a, b)$	$\underline{C} \times \underline{D}$

Exponential Objects

Given objects A and B in $\underline{\mathbf{C}}$, an *exponential object* B^A (also written $[A \rightarrow B]$) is an object with an arrow eval_B^A such that for any C and any arrow $f: C \sqcap A \rightarrow B$,

$$\begin{array}{ccc} C \sqcap A & & \\ \downarrow \exists! & \searrow f & \\ B^A \sqcap A & \xrightarrow{\text{eval}_B^A} & B \end{array} .$$

Alternatively, the pair $\langle B^A, \text{eval}_B^A \rangle$ constitutes a terminal object in the category whose objects are diagrams of the form

$$C \sqcap A \longrightarrow B ,$$

and whose arrows are commutative diagrams of the form

$$\begin{array}{ccc} D \sqcap A & & \\ \downarrow & \searrow & \\ C \sqcap A & \searrow & B \end{array} .$$

Exponential Objects in Haskell

In Hask, the exponential object of two types `a` and `b` is the *function type* `(a -> b)` (it's akin to the *hom-set* of `a` and `b`). Let's see how this satisfies the above definition.

```
1  eval :: ((a -> b),a) -> b
2  eval (f,x) = f x
3  factoredArrow :: ((c,a) -> b) -> ((c,a) -> ((a -> b),a))
4  factoredArrow f = \ (y,x) -> ((\ x' -> f(y,x')),x)
```

(Spot the currying!)

It can be proven that `eval . (factoredArrow f) = f` — and that `factoredArrow` is the *only* arrow for which this is true.

Functors

Functors

A *functor* is a mapping $F: \underline{\mathbf{C}} \rightarrow \underline{\mathbf{D}}$ that takes objects in $\underline{\mathbf{C}}$ to objects in $\underline{\mathbf{D}}$ and arrows in $\underline{\mathbf{C}}$ to arrows in $\underline{\mathbf{D}}$, in such a way that

1. for any $A \in \text{Obj}(\underline{\mathbf{C}})$, $F(1_A) = 1_{F(A)}$:

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ \downarrow & & \downarrow \\ F(A) & \xrightarrow{1_{F(A)}} & F(A) \end{array} ;$$

2. for any $f: A \rightarrow B$ and $g: B \rightarrow C$ in $\underline{\mathbf{C}}$, $F(g \circ f) = F(g) \circ F(f)$:

$$\begin{array}{ccccc} & & B & & \\ & f \nearrow & \downarrow & \searrow g & \\ A & \xrightarrow{\quad} & & \xrightarrow{g \circ f} & C \\ \downarrow & & \downarrow & & \downarrow \\ F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{F(g)} & F(C) \\ & \searrow F(g \circ f) & & & \end{array}$$

Functors in Haskell

In Haskell, functors are *type constructors*: they take a type (a) and produce another type ($F\ a$); and via `fmap`, they take an arrow between two types ($a \rightarrow b$) and produce an arrow between the images of those two types ($F\ a \rightarrow F\ b$).

E.g. the list constructor:

```
1 data [] a = [] | a : [a] -- "[]" is the type constructor for lists
2 fmap f [] = [] -- mapping f over an empty list does nothing
3 fmap f (x : xs) = (f x) : (fmap f xs)
4 -- to turn f into a list function, apply f to the head of the list ,
5 -- apply the list version of f to the tail of the list , and construct
```

You can verify the functor laws in Hask:

`fmap id (x : xs) = (id x) : (fmap id xs) = id (x : xs)`, and that
`fmap f (fmap g (x : xs)) = fmap f ((g x) : (fmap g xs))`
`= (f g x) : (fmap f (fmap g xs)) = fmap f g (x : xs)`.

Examples

	<u>Set</u>	<u>Hask</u>	<u>POrd</u>	<u>Cat</u>
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Terminal obj.	$\{*\}$	$()$	upper bound	$\underline{1}$
Product	$A \times B$	(a,b)	$\min(a, b)$	$\underline{C} \times \underline{D}$
Endofunctors	functors	type const.	OPTs	nat. trans.

Cartesian-Closed Categories

Cartesian-Closed Categories (CCC)

There is a terminal object 1 .

There are binary products \sqcap .

There is a two-argument functor taking $A \sqcap B$ onto B^A , obeying the following rules:

$$A \cong 1 \sqcap A \cong A^1$$

$$\mathrm{Hom}_{\underline{\mathbf{C}}}(A \sqcap B, C) \cong \mathrm{Hom}_{\underline{\mathbf{C}}}(A, C^B) \quad (3.1)$$

The latter relation is called the *Howard-Curry isomorphism*, or *currying*.

Cartesian-Closed Categories

Set the singleton set, pairs, sets of functions

Hask $()$, (a,b) , $a \rightarrow b$

There are more examples, but they're pretty complicated.

From Haskell to CCC

Example: Syntactic Analysis

Example: Interval Analysis

Example: Category Products

Example: Linear maps

Linear maps as a category

A **linear map** is a function $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$ such that $f(x + y) = f(x) + f(y)$ and $f(cx) = cf(x)$. It can also be thought of as an $n \times m$ matrix (where the columns tell you what the basis vectors of the domain space map to).

Linear maps form a category, because:

1. Given $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$ and $g: \mathbb{R}^n \rightarrow \mathbb{R}^p$, we can define the composition $g \circ f: \mathbb{R}^m \rightarrow \mathbb{R}^p$, which is also a linear map.
2. Composition of linear maps (alternatively: matrix multiplication) is associative.
3. For any vector space \mathbb{R}^n , the identity function $1_{\mathbb{R}^n}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear map, and has the properties we expect of an identity.

Implementation of linear maps in Haskell

Help!! I don't get this! What on earth is V s a ?

Example: Automatic differentiation

Types of differentiation

Symbolic differentiation What you learn when you learn calculus; cumbersome for computers.

Numeric differentiation Evaluate the function at two nearby points and compute the slope of the resulting line; easy for computers, not so useful for humans.

Automatic differentiation Tell the computer how to compute the derivatives of simple functions, and it will give you the analytic derivative of any composition of those functions. Easy for a computer, useful for humans.

The chain rule

$$(g \circ f)' = (g' \circ f)(f')$$

In higher dimensions, the derivative of a function is a vector or a matrix, and the derivative of a composition of two functions is the product of the two matrices that give the derivatives of the individual functions.

Here's another way of thinking about it: The derivative of a function is a linear map (a line, plane, linear subspace). In the category of linear maps, composition is multiplication (of matrices, which reduces to scalar multiplication for 1×1 matrices). Differentiation is just a functor $D: \underline{\mathbf{Set}} \rightarrow \underline{\mathbf{LMap}}$, with the property that

$$D(g \circ f) = (Dg) \circ (Df).$$

Thus, if we know how to apply D to all our atomic functions, then we know how to apply D to all compositions of these functions, by just doing matrix multiplication.

Implementation of automatic differentiation in Haskell

Help!! I'm not getting this! I guess you represent each function $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$ as a function from \mathbb{R}^m to $\mathbb{R}^n \times (\mathbb{R}^n)^{\mathbb{R}^m}$, i.e. a function that takes a value in the domain of f and gives you not only the value of f at that point, but also a linear map representing the derivative of f at that point. Beyond that, I'm not following it.

Example: Compiling to hardware

Hardware: a preview

The basic idea is that we have a language for describing circuit diagrams, and so we can compile a CCC into a graph, and compile this graph into the circuit-description language. But I don't get how the graph category is implemented.

Future Work

Conclusions

Further Reading

Further Reading