Compiling to Categories

mathematically-principled program transformation

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Haskell and Category Theory

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Categories

Categories

A *category* $\underline{\mathbf{C}}$ consists of

1. a class $\mathrm{Obj}(\underline{\mathbf{C}})$ of *objects*, and
2. for each pair of objects $A, B \in \mathrm{Obj}(\underline{\mathbf{C}})$, a set $\mathrm{Hom}_{\underline{\mathbf{C}}}(A, B)$ of *arrows* (or *morphisms*) from A to B, known as a *hom-set*.

$$A \overset{\mathrm{Hom}_{\underline{\mathbf{C}}}(A,B)}{\rightrightarrows} B$$

Many familiar parts of Haskell form a category $\underline{\mathbf{Hask}}$: objects are *types* (**Int**, **Char**, etc.), and arrows are *functions* between types (e.g. `ord :: Int -> Char`).

Category Laws

In a category $\underline{\mathbf{C}}$:

1. Given arrows $f\colon A \to B$ and $g\colon B \to C$ in $\underline{\mathbf{C}}$, the *composition* $g \circ f\colon A \to C$ ($=$ `g.f`) is also in $\underline{\mathbf{C}}$.
2. Given arrows $f\colon A \to B$, $g\colon B \to C$ and $h\colon C \to D$, $(h \circ g) \circ f = h \circ (g \circ f) = h \circ g \circ f$:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, dashed, "g\circ f"'] \arrow[drr, dashed, bend left=60, "h\circ g\circ f"] \arrow[drr, dashed, bend right=60, "h\circ g\circ f"'] & B \arrow[d, "g"] \arrow[dr, dashed, "h\circ g"] & \\
& C \arrow[r, "h"'] & D
\end{tikzcd}.$$

3. Every object $A \in \mathrm{Obj}(\underline{\mathbf{C}})$ is associated with an *identity arrow* $1_A\colon A \to A$ ($=$ $\mathbf{id}$). Given any arrow $f\colon A \to B$, we have

$$\begin{tikzcd}
A \arrow[r, dotted, "1_A"] \arrow[rr, bend right, "f"'] & A \arrow[r, "f"] \arrow[rr, bend left, "f"] & B \arrow[r, dotted, "1_B"'] & B
\end{tikzcd}.$$

Examples

	Set	**Hask**	**POrd**	**Cat**
Objects	sets	types	items	small cats
Morphisms	functions	functions	$a \leq b$	functors
Composition	$f \circ g$	`f.g`	transitivity	$F \circ G$
Identity	1_A	**id**	$a = a$	$1_{\mathbf{C}}$

Not everything in Haskell can be in **Hask** if we want it to be a category. Every type in the language contains a Bottom ($\perp$) or **undefined** value, but these 'values' cause mayhem with the category laws (in particular the **Identity** constraint). So when we talk about **Hask** we'll be talking about vanilla **Hask** without these abnormal values. (Haskell wiki page on **Hask**.)

Category Theory: Terminal Objects

A *terminal object* is a type 1 (a.k.a. T) in $\mathrm{Obj}(\underline{\mathbf{C}})$, such that there is only a single mapping from any other type A onto that type:

$$\forall A \in \mathrm{Obj}(\underline{\mathbf{C}}), \left|\mathrm{Hom}_{\underline{\mathbf{C}}}(A, 1)\right| = 1.$$

$$\begin{tikzcd}
A \arrow[dr, "\exists!"] & \\
B \arrow[r, dashed, "\exists!"] & 1 \\
C \arrow[ur, "\exists!"'] &
\end{tikzcd}$$

In $\underline{\mathbf{Hask}}$:

```haskell
() -- the type corresponding to 1, containing only itself
terminalMap :: t -> ()
terminalMap _ = ()
```

Examples

	Set	**Hask**	**POrd**	**Cat**
Objects	sets	types	items	small cats
Morphisms	functions	functions	$a \leq b$	functors
Composition	$f \circ g$	`f.g`	transitivity	$F \circ G$
Identity	1_A	**id**	$a = a$	$1_{\underline{\mathbf{C}}}$
Terminal obj.	$\{*\}$	`()`	upper bound	$\underline{\mathbf{1}}$

Products

Given objects A, B in $\underline{\mathbf{C}}$ there may be a *(pairwise) product* $A \sqcap B \in \mathrm{Obj}(\underline{\mathbf{C}})$ and *projection arrows* $\pi_A \colon A \sqcap B \to A$ and $\pi_B \colon A \sqcap B \to B$ such that for any object X in the same category and arrows $a \colon X \to A$ and $b \colon X \to B$ there is a *unique* arrow $x : X \to A \sqcap B$ such that $a = \pi_A \circ x$ and $b = \pi_B \circ x$:

$$
\begin{tikzcd}
A & A \sqcap B \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"] & B \\
& X \arrow[ul, "a"] \arrow[u, dotted, "\exists ! x"] \arrow[ur, "b"'] &
\end{tikzcd}
.
$$

In other words: Given a particular way of mapping X to A and to B, there's only *one* way of mapping X to $A \sqcap B$ such that everything's consistent.

Products

Alternatively, the triplet $\langle A \sqcap B, \pi_A, \pi_B \rangle$ is a *terminal object* in the category whose objects are diagrams of the form

$$A \longleftarrow C \longrightarrow B \ ,$$

and whose arrows are (commutative) diagrams of the form

$$\begin{tikzcd} & C \arrow[dl, bend right] \arrow[dr, bend left] & \\ A & & B \\ & D \arrow[ul, bend left] \arrow[ur, bend right] \arrow[uu, "f"'] & \end{tikzcd} \ .$$

Products in Haskell

```haskell
(a,b) -- the type containing pairs from types a and b (A ⊓ B)
fst :: (a,b) -> a -- the projection function π_A
fst (x,y) = x
snd :: (a,b) -> b -- the projection function π_B
snd (x,y) = y
factorThroughProd :: (c -> a) -> (c -> b) -> (c -> (a,b))
factorThroughProd f g = \ x -> (f x,g x)
```

It should be obvious that

`fst .(factorThroughProd f g) = f`, and

`snd.(factorThroughProd f g) = g`.

Examples

	$\underline{\textbf{Set}}$	$\underline{\textbf{Hask}}$	$\underline{\textbf{POrd}}$	$\underline{\textbf{Cat}}$
Objects	sets	types	items	small cats
Morphisms	functions	functions	$a \leq b$	functors
Composition	$f \circ g$	`f.g`	transitivity	$F \circ G$
Identity	1_A	**id**	$a = a$	$1_{\underline{\mathbf{C}}}$
Terminal obj.	$\{*\}$	`()`	upper bound	$\underline{\mathbf{1}}$
Product	$A \times B$	`(a,b)`	$\min(a, b)$	$\underline{\mathbf{C}} \times \underline{\mathbf{D}}$

Exponential Objects

Given objects A and B in $\underline{\mathbf{C}}$, an *exponential object* B^A (also written $[A \to B]$) is an object with an arrow eval_B^A such that for any C and any arrow $f\colon C \sqcap A \to B$,

$$\begin{tikzcd} C \sqcap A \arrow[d, dotted, "\exists!"'] \arrow[dr, "f"] & \\ B^A \sqcap A \arrow[r, "\mathrm{eval}_B^A"'] & B \end{tikzcd} .$$

Alternatively, the pair $\langle B^A, \mathrm{eval}_B^A \rangle$ constitutes a terminal object in the category whose objects are diagrams of the form

$$C \sqcap A \longrightarrow B \ ,$$

and whose arrows are commutative diagrams of the form

$$\begin{tikzcd} D \sqcap A \arrow[d] \arrow[dr, bend left] & \\ & B \\ C \sqcap A \arrow[ur, bend right] & \end{tikzcd} .$$

Exponential Objects in Haskell

In **<u>Hask</u>**, the exponential object of two types `a` and `b` is the *function type* `(a -> b)` (it's akin to the *hom-set* of `a` and `b`). Let's see how this satisfies the above definition.

```
eval :: ((a -> b),a) -> b
eval (f,x) = f x
factoredArrow :: ((c,a) -> b) -> ((c,a) -> ((a -> b),a))
factoredArrow f = \ (y,x) -> ((\ x' -> f(y,x')),x)
```

(Spot the currying!)

It can be proven that `eval . (factoredArrow f) = f` — and that `factoredArrow` is the *only* arrow for which this is true.

Functors

Functors

A *functor* is a mapping $F\colon \underline{\mathbf{C}} \to \underline{\mathbf{D}}$ that takes objects in $\underline{\mathbf{C}}$ to objects in $\underline{\mathbf{D}}$ and arrows in $\underline{\mathbf{C}}$ to arrows in $\underline{\mathbf{D}}$, in such a way that

1. for any $A \in \mathrm{Obj}(\underline{\mathbf{C}})$, $F(1_A) = 1_{F(A)}$:

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, mapsto] & A \arrow[d, mapsto] \\ F(A) \arrow[r, "1_{F(A)}"'] & F(A) \end{tikzcd} ;$$

2. for any $f\colon A \to B$ and $g\colon B \to C$ in $\underline{\mathbf{C}}$, $F(g \circ f) = F(g) \circ F(f)$:

$$\begin{tikzcd} & B \arrow[dr, "g"] \arrow[dd, mapsto] & \\ A \arrow[ur, "f"] \arrow[rr, "g \circ f"'] \arrow[dd, mapsto] & & C \arrow[dd, mapsto] \\ & F(B) \arrow[dr, "F(g)"] & \\ F(A) \arrow[ur, "F(f)"] \arrow[rr, "F(g) \circ F(f)"'] & & F(C) \end{tikzcd} .$$

Functors in Haskell

In Haskell, functors are *type constructors*: they take a type (a) and produce another type (F a); and via fmap, they take an arrow between two types (a -> b) and produce an arrow between the images of those two types (F a -> F b).

E.g. the list constructor:

```haskell
data [] a = [] | a : [a] -- "[]" is the type constructor for lists
fmap f [] = [] -- mapping f over an empty list does nothing
fmap f (x : xs) = (f x) : (fmap f xs)
-- to turn f into a list function, apply f to the head of the list,
-- apply the list version of f to the tail of the list, and construct
```

You can verify the functor laws in **Hask**:

fmap **id** (x : xs) = (**id** x) : (fmap **id** xs) = **id** (x : xs), and that

fmap f (fmap g (x : xs)) = fmap f ((g x) : (fmap g xs))

= (f g x) : (fmap f (fmap g xs)) = fmap f g (x : xs).

Examples

	Set	**Hask**	**POrd**	**Cat**
Objects	sets	types	items	small cats
Morphisms	functions	functions	$a \leq b$	functors
Composition	$f \circ g$	`f . g`	transitivity	$F \circ G$
Identity	1_A	**id**	$a = a$	$1_{\underline{\mathbf{C}}}$
Terminal obj.	$\{*\}$	`()`	upper bound	$\underline{\mathbf{1}}$
Product	$A \times B$	`(a,b)`	$\min(a, b)$	$\underline{\mathbf{C}} \times \underline{\mathbf{D}}$
Endofunctors	functors	type const.	OPTs	nat. trans.

Cartesian-Closed Categories

Cartesian-Closed Categories (CCC)

There is a terminal object 1.

There are binary products $\sqcap$.

There is a two-argument functor taking $A \sqcap B$ onto B^A, obeying the following rules:

$$A \cong 1 \sqcap A \cong A^1$$

$$\mathrm{Hom}_{\underline{\mathbf{C}}}(A \sqcap B, C) \cong \mathrm{Hom}_{\underline{\mathbf{C}}}(A, C^B) \tag{3.1}$$

The latter relation is called the *Howard-Curry isomorphism*, or *currying*.

Cartesian-Closed Categories

<u>Set</u> the singleton set, pairs, sets of functions

<u>Hask</u> `()`, `(a,b)`, `a -> b`

There are more examples, but they're pretty complicated.

CCC Constructions and the λ-Calculus

CCC Constructions in the λ-Calculus

We can give a λ-calculus expression which corresponds to each construction in the CCC.

But the reverse is also true.

We can map any λ-calculululus expression onto a construction in a CCC. The computation resulting from that construction just depends on what that CCC happens to be.

Category Definition

- identity $\texttt{id} = \lambda x \mapsto x$,
- composition $g \circ f = \lambda x \mapsto g(f(x))$.

The Product

- fork $f \,\Delta\, g = \lambda\, x \mapsto (f\ x, g\ x)$,
- extract-left $\texttt{exl} = \lambda\,(a, b) \mapsto a$,
- extract-right $\texttt{exr} = \lambda\,(a, b) \mapsto b$.

A Terminal Object

- terminal 1 is the terminal object in the category,
- terminal arrow $\mathtt{it} = \lambda\, a \mapsto ()$.
- unitarrow $\mathtt{unitarrow}\, b = \lambda\, () \mapsto b$.
- constants $\mathtt{const}\, b = (\mathtt{unitarrow}\, b) \circ \mathtt{it}$

Exponential Objects

- apply $\texttt{apply}\,(f, x) = f\,x$
- curry $\texttt{curry}\,f = \lambda\,ab \mapsto f\,(a, b)$
- uncurry $\texttt{uncurry}\,f = \lambda\,(a, b) \mapsto f\,a\,b$
- constant functions $\texttt{constFun}\,f = \texttt{curry}(f \circ exr) = \lambda\,x \mapsto f$ ignores x, returns a function

From λ-Calculus to CCCs

From λ-Calculus to CCCs

This direction is simpler.

There are only 5 main cases we need to deal with.

The mapping operation is symbolised as $\mathfrak{K}$.

Each transformation either reduces the size of the body of λ-expression, or eliminates a λ. Consequently, the transformation process must terminate.

1. Expression Body is a Single Variable

$\mathfrak{K}(\lambda\, x \mapsto x) = \texttt{id}$

2. Expression Body is an Application

$$\mathfrak{K}(\lambda\, x \mapsto U\, V) = \texttt{apply} \circ (\mathfrak{K}(\lambda\, x \mapsto U) \,\Delta\, \mathfrak{K}(\lambda\, x \mapsto V))$$

3. Lambda Abstraction

$$\mathfrak{K}(\lambda\, x \mapsto \lambda\, y \mapsto U) = \texttt{curry}\, \mathfrak{K}(\lambda\, (x, y) \mapsto U)$$

4. Case Expressions

(more complexity than we wish to cover here)

5a. Simple Constants

$$\mathfrak{K}(\lambda\, x \mapsto c) = \texttt{const}\, c$$

5b. Constant Functions

$$\mathfrak{K}(\lambda\, x \mapsto f) = \texttt{constFun}\, \mathfrak{K}(f)$$

f may need to be *Curried* to reduce its argument dimensionality.

From Haskell to CCC

Haskell to CCC Constructions

- `ghc` compiles haskell code to lambda-calculus
- `simplifier` reduces the lambda-calculus size where possible
- `concat` intervenes in the simplifier and converts the lambda calculus to CCC constructions

Looking at GHC Intermediate Stages

Following the stackoverflow answer:
`https://stackoverflow.com/questions/27635111`.

- use the GHC module
- functions compileToCoreModule or compileToCoreSimplified to compile a file
- the code has been reproduced as processor.hs in the repository with today's talk. You need to compile it with

```
$ ghc -package ghc -package ghc-paths processor.hs
```

Haskell to λ-Calculus

```haskell
example :: Int -> Int -> Int
example x y = x + y
```

```haskell
example = \ (x :: Int) (y :: Int) -> + @ Int $fNumInt x y
```

Haskell to λ-Calculus

```haskell
example :: Int -> Int -> Int
example x y = x + y
```

```haskell
example = \ (x :: Int) (y :: Int) -> + @ Int $fNumInt x y
```

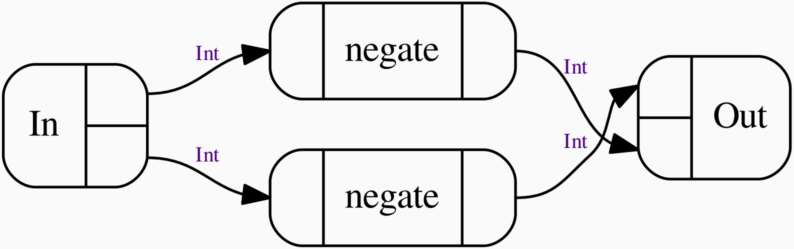
λ-Calculus to CCC Constructions

Example: Syntactic Analysis

Example: Interval Analysis

Negation

```
instance (Iv a ~ (a :* a), Num a, Ord a) => NumCat IF a where
  negateC = pack (\ (al,ah) -> (-ah, -al))
  ..
  {-# INLINE negateC #-}
  ..
```



Addition

```
instance (Iv a ~ (a :* a), Num a, Ord a) => NumCat IF a where
  ..
  addC = pack (\ ((al,ah),(bl,bh)) -> (al+bl,ah+bh))
  ..
  {-# INLINE addC #-}
  ..
```

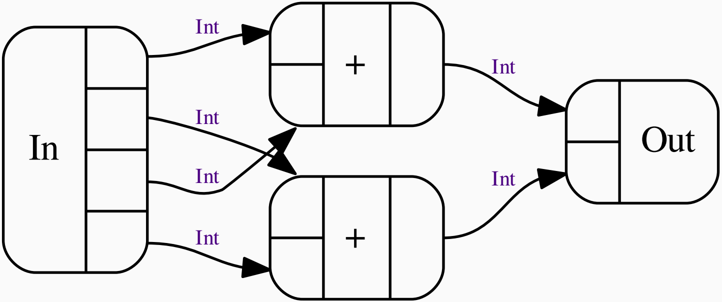
Addition

```
runSynME "add" $ toCcc $ ivFun $ uncurry ((+) @Int)
```

```
uncurry (curry (apply . (exl &&& exr))) .
(curry
 (
  (add . (exl  .  exl  &&& exl . exr)
   &&&
   add . (exr  .  exl  &&& exr . exr)
  )  .  exr
 ) &&& id
)
```

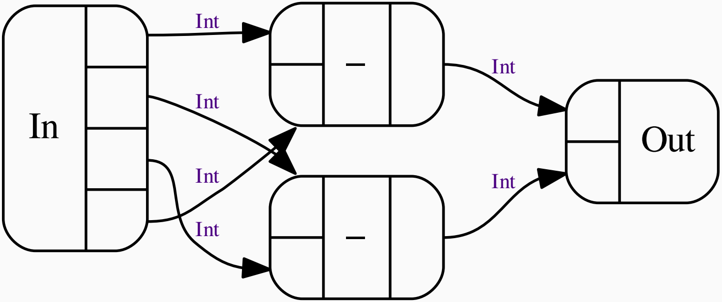
Addition

```
runSynCirc "add" $ toCcc $ ivFun $ uncurry ((+) @Int)
```



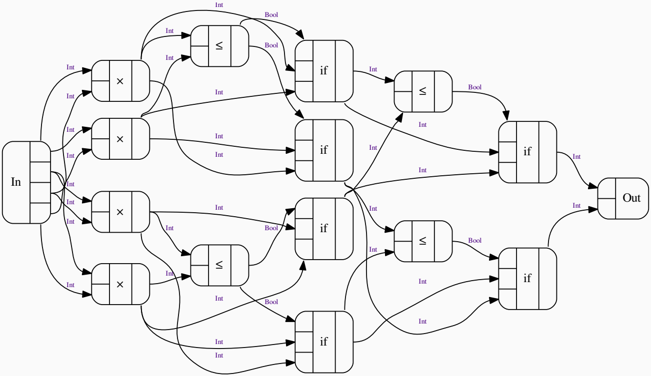
Subtraction

```
instance (Iv a ~ (a :* a), Num a, Ord a) => NumCat IF a where
  ..
  subC = addC . second negateC
  ..
  {-# INLINE subC #-}
  ..
```



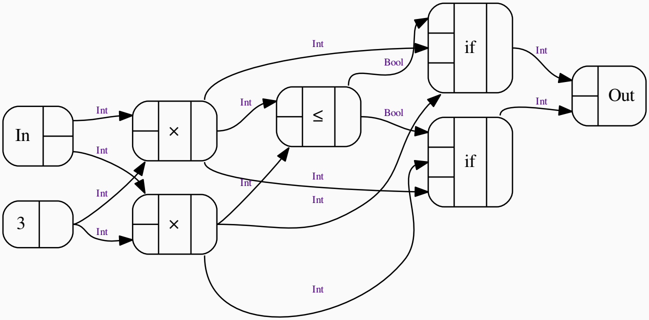
Multiplication

```
instance (Iv a ~ (a :* a), Num a, Ord a) => NumCat IF a where
  mulC = pack (\ ((al,ah),(bl,bh)) ->
              let cs = ((al*bl, al*bh),(ah*bl,ah*bh)) in
                (min4 cs, max4 cs))
  ..
  {-# INLINE mulC #-}
```



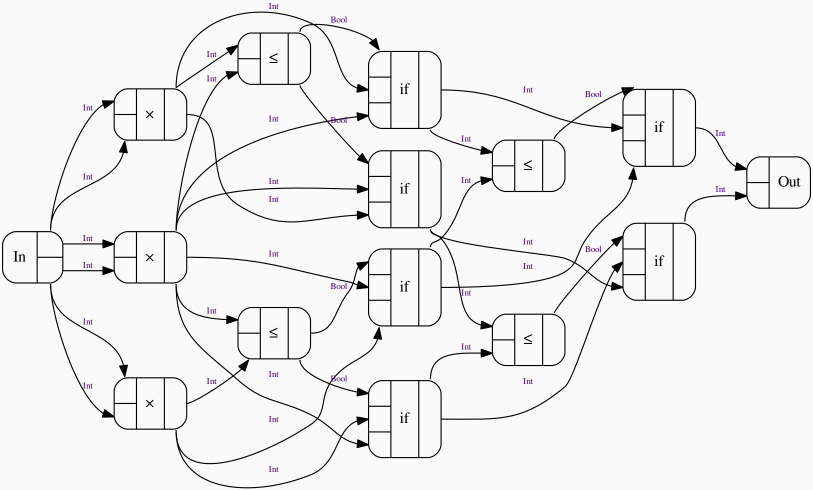
Thrice

```
runSynCirc "thrice-iv" $ toCcc $ ivFun $ \ x -> 3 * x :: Int
```



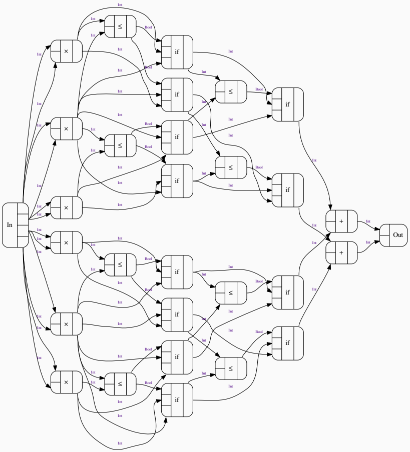
Square

```
runSynCirc "sqr-iv"   $ toCcc $ ivFun $ sqr @Int
```



Magic Square

```
runSynCirc "magSqr−iv" $ toCcc $ ivFun $ magSqr @Int
```



Example: Category Products

Example: Linear maps

Linear maps as a category

A **linear map** is a function $f\colon \mathbb{R}^m \to \mathbb{R}^n$ such that $f(x+y) = f(x) + f(y)$ and $f(cx) = c\,f(x)$. It can also be thought of as an $n \times m$ matrix (where the columns tell you what the basis vectors of the domain space map to).

Linear maps form a category, because:

1. Given $f\colon \mathbb{R}^m \to \mathbb{R}^n$ and $g\colon \mathbb{R}^n \to \mathbb{R}^p$, we can define the composition $g \circ f\colon \mathbb{R}^m \to \mathbb{R}^p$, which is also a linear map.
2. Composition of linear maps (alternatively: matrix multiplication) is associative.
3. For any vector space $\mathbb{R}^n$, the identity function $1_{\mathbb{R}^n}\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear map, and has the properties we expect of an identity.

Implementation of linear maps in Haskell

Help!! I don't get this! What on earth is $V\ s\ a$?

Example: Automatic differentiation

Types of differentiation

Symbolic differentiation What you learn when you learn calculus; cumbersome for computers.

Numeric differentiation Evaluate the function at two nearby points and compute the slope of the resulting line; easy for computers, not so useful for humans.

Automatic differentiation Tell the computer how to compute the derivatives of simple functions, and it will give you the analytic derivative of any composition of those functions. Easy for a computer, useful for humans.

The chain rule

$$(g \circ f)' = (g' \circ f)(f')$$

In higher dimensions, the derivative of a function is a vector or a matrix, and the derivative of a composition of two functions is the product of the two matrices that give the derivatives of the individual functions.

Here's another way of thinking about it: The derivative of a function is a lnear map (a line, plane, linear subspace). In the category of linear maps, composition is multiplication (of matrices, which reduces to scalar multiplication for 1×1 matrices). Differentiation is just a functor $D\colon \underline{\textbf{Set}} \to \underline{\textbf{LMap}}$, with the property that

$$D(g \circ f) = (Dg) \circ (Df).$$

Thus, if we know how to apply D to all our atomic functions, then we know how to apply D to all compositions of these functions, by just doing matrix multiplication.

Implementation of automatic differentiation in Haskell

Help!! I'm not getting this! I guess you represent each function $f : \mathbb{R}^m \to \mathbb{R}^n$ as a function from $\mathbb{R}^m$ to $\mathbb{R}^n \times (\mathbb{R}^n)^{\mathbb{R}^m}$, i.e. a function that takes a value in the domain of f and gives you not only the value of f at that point, but also a linear map representing the derivative of f at that point. Beyond that, I'm not following it.

Example: Compiling to hardware

Hardware: a preview

The basic idea is that we have a language for describing circuit diagrams, and so we can compile a CCC into a graph, and compile this graph into the circuit-description language. But I don't get how the graph category is implemented.

Future Work

Conclusions

Further Reading

Further Reading