

Compiling to Categories

Our attempt to explain what Conal Elliott is up to

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November 12, 2017

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Categories

Categories

A category $\underline{\mathbf{C}}$ consists of

1. a class $\text{Obj}(\underline{\mathbf{C}})$ of *objects*, and
2. for each pair of objects $A, B \in \text{Obj}(\underline{\mathbf{C}})$, a set $\text{Hom}_{\underline{\mathbf{C}}}(A, B)$ of *arrows* (or *morphisms*) from A to B , known as a *hom-set*.

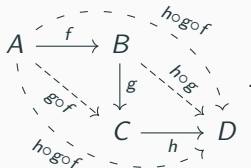
$$\begin{array}{ccc} & \text{Hom}_{\underline{\mathbf{C}}}(A, B) & \\ A & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & B \end{array}$$

Many familiar parts of Haskell form a category **Hask**: objects are *types* (**Int**, **Char**, etc.), and arrows are *functions* between types (e.g. **ord** :: **Int** \rightarrow **Char**).

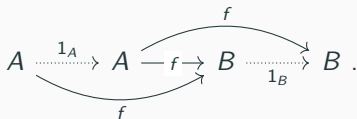
Category Laws

In a category $\underline{\mathbf{C}}$:

1. Given arrows $f: A \rightarrow B$ and $g: B \rightarrow C$ in $\underline{\mathbf{C}}$, the *composition* $g \circ f: A \rightarrow C$ ($= g.f$) is also in $\underline{\mathbf{C}}$.
2. Given arrows $f: A \rightarrow B$, $g: B \rightarrow C$ and $h: C \rightarrow D$,
 $(h \circ g) \circ f = h \circ (g \circ f) = h \circ g \circ f$.



3. Every object $A \in \text{Obj}(\underline{\mathbf{C}})$ is associated with an *identity arrow* $1_A: A \rightarrow A$ ($= \text{id}$). Given any arrow $f: A \rightarrow B$, we have



Examples

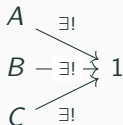
	<u>Set</u>	<u>Hask</u>	<u>POrd</u>	<u>Cat</u>
Objects	sets	types	items	small cats
Morphisms	functions	functions	$a \leq b$	functors
Composition	$f \circ g$	$f.g$	transitivity	$F \circ G$
Identity	1_A	id	$a = a$	$1_{\underline{C}}$

Not everything in Haskell can be in Hask if we want it to be a category. Every type in the language contains a **Bottom** (\perp) or **undefined** value, but these 'values' cause mayhem with the category laws (in particular the **Identity** constraint). So when we talk about Hask we'll be talking about vanilla Hask without these abnormal values. Haskell wiki page on Hask

Category Theory: Terminal Objects

A *terminal object* is a type 1 (a.k.a. T) in $\text{Obj}(\underline{\mathbf{C}})$, such that there is only a single mapping from any other type A onto that type:

$$\forall A \in \text{Obj}(\underline{\mathbf{C}}), |\text{Hom}_{\underline{\mathbf{C}}}(A, 1)| = 1.$$



In **Hask**:

```
1  () -- the type corresponding to 1, containing only itself
2  terminalMap :: t -> ()
3  terminalMap _ = ()
```


Global Elements

A *global element* of an object A in category $\underline{\mathbf{C}}$ with terminal object 1 is an arrow $a : 1 \rightarrow A$.

$$1 \xrightarrow{a} A$$

In **Hask**, if we have a value v in some type a , we can upgrade it to the global element by use of **const** v .

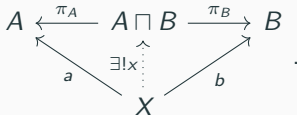
```
1  const :: a -> b -> a -- but for our purposes, choose b = ()  
2  const v = \ _ -> v
```

Examples

	<u>Set</u>	<u>Hask</u>	<u>POrd</u>	<u>Cat</u>
Objects	sets	types	items	small cats
Morphisms	functions	functions	$a \leq b$	functors
Composition	$f \circ g$	$f.g$	transitivity	$F \circ G$
Identity	1_A	id	$a = a$	$1_{\underline{c}}$
Terminal obj.	$\{*\}$	$()$	upper bound	$\underline{1}$

Products

Given objects A, B in $\underline{\mathbf{C}}$ there may be a (*pairwise*) *product* $A \sqcap B \in \text{Obj}(\underline{\mathbf{C}})$ and *projection arrows* $\pi_A: A \sqcap B \rightarrow A$ and $\pi_B: A \sqcap B \rightarrow B$ such that for any object X in the same category and arrows $a: X \rightarrow A$ and $b: X \rightarrow B$ there is a *unique* arrow $x: X \rightarrow A \sqcap B$ such that $a = \pi_A \circ x$ and $b = \pi_B \circ x$:



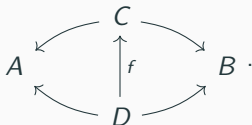
In other words: Given a particular way of mapping X to A and to B , there's only *one* way of mapping X to $A \sqcap B$ such that everything's consistent.

Products

Alternatively, the triplet $\langle A \sqcap B, \pi_A, \pi_B \rangle$ is a *terminal object* in the category whose objects are diagrams of the form

$$A \longleftarrow C \longrightarrow B ,$$

and whose arrows are (commutative) diagrams of the form



Products in Haskell

```
1  (a,b) -- the type containing pairs from types a and b ( $A \times B$ )
2  fst :: (a,b) -> a -- the projection function  $\pi_A$ 
3  fst (x,y) = x
4  snd :: (a,b) -> b -- the projection function  $\pi_B$ 
5  snd (x,y) = y
6  factorThroughProd :: (c -> a) -> (c -> b) -> (c -> (a,b))
7  factorThroughProd f g = \ x -> (f x,g x)
```

It should be obvious that

fst.(factorThroughProd f g) = f, and

snd.(factorThroughProd f g) = g.

Examples

	<u>Set</u>	<u>Hask</u>	<u>POrd</u>	<u>Cat</u>
Objects	sets	types	items	small cats
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Identity	1_A	id	$a = a$	$1_{\underline{C}}$
Terminal obj.	$\{*\}$	$()$	upper bound	$\underline{1}$
Product	$A \times B$	(a,b)	$\min(a, b)$	$\underline{C} \times \underline{D}$

Exponential Objects

Given objects A and B in $\underline{\mathbf{C}}$, an *exponential object* B^A (also written $[A \rightarrow B]$) is an object with an arrow eval_B^A such that for any C and any arrow $f: C \sqcap A \rightarrow B$,

$$\begin{array}{ccc} C \sqcap A & & \\ \downarrow \exists! & \searrow f & \\ B^A \sqcap A & \xrightarrow{\text{eval}_B^A} & B \end{array} .$$

Alternatively, the pair $\langle B^A, \text{eval}_B^A \rangle$ constitutes a terminal object in the category whose objects are diagrams of the form

$$C \sqcap A \longrightarrow B ,$$

and whose arrows are commutative diagrams of the form

$$\begin{array}{ccc} D \sqcap A & & \\ \downarrow & \searrow & \\ C \sqcap A & \searrow & B \end{array} .$$

Exponential Objects in Haskell

In Hask, the exponential object of two types `a` and `b` is the *function type* `(a -> b)` (it's akin to the *hom-set* of `a` and `b`). Let's see how this satisfies the above definition.

```
1  eval :: ((a -> b),a) -> b
2  eval (f,x) = f x
3  factoredArrow :: ((c,a) -> b) -> ((c,a) -> ((a -> b),a))
4  factoredArrow f = \ (y,x) -> ((\ x' -> f(y,x')),x)
```

(Spot the currying!)

It can be proven that `eval . (factoredArrow f) = f` — and that `factoredArrow` is the *only* arrow for which this is true.

Functors

Functors

A *functor* is a mapping $F: \underline{\mathbf{C}} \rightarrow \underline{\mathbf{D}}$ that takes objects in $\underline{\mathbf{C}}$ to objects in $\underline{\mathbf{D}}$ and arrows in $\underline{\mathbf{C}}$ to arrows in $\underline{\mathbf{D}}$, in such a way that

1. for any $A \in \text{Obj}(\underline{\mathbf{C}})$, $F(1_A) = 1_{F(A)}$:

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ \downarrow & & \downarrow \\ F(A) & \xrightarrow{1_{F(A)}} & F(A) \end{array} ;$$

2. for any $f: A \rightarrow B$ and $g: B \rightarrow C$ in $\underline{\mathbf{C}}$, $F(g \circ f) = F(g) \circ F(f)$:

$$\begin{array}{ccccc} & & B & & \\ & f \nearrow & \downarrow & \nwarrow g & \\ A & \xrightarrow{\quad} & & \xrightarrow{\quad} & C \\ \downarrow & & \downarrow g \circ f & & \downarrow \\ & & F(B) & & \\ \downarrow & F(f) \nearrow & & \nwarrow F(g) & \\ F(A) & \xrightarrow{\quad} & & \xrightarrow{\quad} & F(C) \\ & & F(g) \circ F(f) & & \end{array} .$$

Functors in Haskell

In Haskell, functors are *type constructors*: they take a type (a) and produce another type ($F\ a$); and via `fmap`, they take an arrow between two types ($a \rightarrow b$) and produce an arrow between the images of those two types ($F\ a \rightarrow F\ b$).

E.g. the list constructor:

```
1 data [] a = [] | a : [a] -- "[]" is the type constructor for lists
2 fmap f [] = [] -- mapping f over an empty list does nothing
3 fmap f (x : xs) = (f x) : (fmap f xs)
4 -- to turn f into a list function, apply f to the head of the list ,
5 -- apply the list version of f to the tail of the list , and construct
```

You can verify the functor laws in Hask:

`fmap id (x : xs) = (id x) : (fmap id xs) = id (x : xs)`, and that
`fmap f (fmap g (x : xs)) = fmap f ((g x) : (fmap g xs))`
`= (f g x) : (fmap f (fmap g xs)) = fmap f g (x : xs)`.

Examples

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Product	$A \times B$	(a,b)	$\min(a, b)$	$\underline{C} \times \underline{D}$
Endofunctors	functors	type const.	OPTs	nat. trans.

Cartesian-Closed Categories

Cartesian-Closed Categories (CCC)

There is a terminal object 1 .

There are binary products \sqcap .

There is a two-argument functor taking $A \sqcap B$ onto B^A , obeying the following rules:

$$A \cong 1 \sqcap A \cong A^1$$

$$\mathrm{Hom}_{\underline{\mathbf{C}}}(A \sqcap B, C) \cong \mathrm{Hom}_{\underline{\mathbf{C}}}(A, C^B) \quad (3.1)$$

The latter relation is called the *Howard-Curry isomorphism*, or *currying*.

Cartesian-Closed Categories

Set the singleton set, pairs, sets of functions

Hask $()$, (a,b) , $a \rightarrow b$

There are more examples, but they're pretty complicated.

Further Reading

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