IB Geometry

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Contents

1 Surfaces

Definition 1.1. A *topological surface* is a topological space Σ such that

(a) for all $p \in \Sigma$, there is an open neighbourhood $p \in U \subset \Sigma$ such that U is homeomorphic to $\mathbb{R}^2$, or a disc $D^2 \subset \mathbb{R}^2$, with its usual Euclidean topology.

(b) Σ is Hausdorff and second countable.

Remark. $\mathbb{R}^2 \simeq D(0,1) = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$.

1. A space X is *Hausdorff* if for $p \neq q$ in X, there exist disjoint open sets U, V with $p \in U, q \in V$.

   A space is *second countable* if it has a countable base, i.e there exist open sets $\{U_i\}_{i \in \mathbb{N}}$, such that every open set is a union of some of the U_i.

   The key point of defining surfaces is point (a), point (b) is for ruling out surfaces that are too weird.

2. If X is Hausdorff or second countable, then so are subspaces of X. Moreover Euclidean space has these properties (to show it is second countable, consider open balls $B(c,r)$ with $c \in \mathbb{Q}^n \subset \mathbb{R}^n$, and $r \in \mathbb{Q}_+ \subset \mathbb{R}_+$).

Example 1.1.

(i) The plane $\mathbb{R}^2$.

(ii) Any open set in $\mathbb{R}^2$ is a surface, i.e. $R^2 \setminus Z$ where Z is closed is a surface.

(iii) Graphs of functions. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous. Then the graph of f is

$$\Gamma_f = \{(x, y, f(x,y)) \mid (x,y) \in \mathbb{R}^2\}.$$

This is a subspace of $\mathbb{R}^3$, so we can endow it with the subspace topology. We claim it is a subspace homeomorphic to $\mathbb{R}^2$.

Recall that if X, Y are topological spaces, then the product topology $X \times Y$ has a basis of open sets $U \times V$, where $U \subset X$, $V \subset Y$ are open

A feature is that if $g : Z \to X \times Y$ is continuous if and only if $\Pi_x \circ g : Z \to X$ and $\Pi_y \circ g : Z \to Y$ are continuous, where Π_x, Π_y are the canonical projectors.

We can now show that if $f : X \to Y$ is continuous, then $\Gamma_f \subset X \times Y$ is homeomorphic to X, as $s(x) = (x, f(x))$ is a continuous function from X to Γ_f, $\Pi_x|_{\Gamma_f}$ and s are inverse homeomorphisms.

In particular, for our example $\Gamma_f \simeq \mathbb{R}^2$. So any $f : \mathbb{R}^2 \to \mathbb{R}$ continuous produces a surface Γ_f.

(iv) The sphere: $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ (with the subspace topology). To show this is a surface, we can consider the stereographic projection $\Pi_+ : S^2 \setminus \{(0, 0, 1)\} \to \mathbb{R}^2$:

$$(x, y, z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

Then Π_+ is continuous and has an inverse

$$(u, v) \mapsto \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right).$$

So Π_+ is a continuous bijection with continuous inverse, and hence a homeomorphism.

Similarly, taking a stereographic projection from the south pole $\Pi_- : S^2 \setminus \{(0, 0, -1)\} \to \mathbb{R}^2$, by

$$(x, y, z) \mapsto \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$$

is another homeomorphism. Hence S^2 is a topological surface, as the open sets $S^2 \setminus \{(0, 0, 1)\}$ and $S^2 \setminus \{(0, 0, -1)\}$ cover S^2, and it is Hausdorff and second countable as it is a subspace of $\mathbb{R}^3$.

(v) The *real projective plane*. The group $\mathbb{Z}_2$ acts on S^2 by homeomorphisms, via the antipodal map

$$a : S^2 \to S^2$$
$$a(x, y, z) \mapsto (-x, -y, -z)$$

Definition 1.2. The real projective plane is the quotient of S^2 by identifying every point with its antipodal image:

$$\mathbb{RP}^2 = S^2/\mathbb{Z}_2 = S^2/\sim .$$

Lemma 1.1. *As a set, $\mathbb{RP}^2$ is naturally in bijection with the set of straight lines through* 0.

This is because any straight line through $0 \in \mathbb{R}^3$ intersects S^2 in exactly a pair of antipodal points, and each such pair determines a straight line.

Lemma 1.2. $\mathbb{RP}^2$ *is a topological surface with the quotient topology.*

Recall the quotient topology: given the quotient map $q : X \to Y$, we say $V \subset Y$ is open if and only if $q^{-1}(V) \subset X$ is open in X.

Proof: First we show that $\mathbb{RP}^2$ is Hausdorff. If $[p] \neq [q] \in \mathbb{RP}^2$, then $\pm p$, $\pm q$ are distinct, antipodal pairs.

We take open discs centred on p and q and their antipodal images, such that no two discs intersect. The images of these discs give open images of $[p]$ and $[q]$ in $\mathbb{RP}^2$. Indeed, $q(B_\delta(p))$ is open since $q^{-1}(q(B_\delta(p))) = B_\delta(p) \cup (-B_\delta(p))$.

Now we show $\mathbb{RP}^2$ is second countable. Let U be a countable base of S^2, and let $\overline{U} = \{q(u) \mid u \in U\}$. Then $q(u)$ is open, as $q(u) = u \cup (-u)$, and $\overline{U}$ is clearly countable as U is.

Take $V \subset \mathbb{RP}^2$ open. By definition, $q^{-1}(V)$ is open, so let $q^{-1}(V) = \bigcup U_\alpha$, for $U_\alpha \in U$. Then

$$V = q(q^{-1}(V)) = q\left(\bigcup_\alpha U_\alpha\right) = \bigcup_\alpha q(U_\alpha).$$

Finally, let $p \in S^2$ and $[p] \in \mathbb{RP}^2$ be its image. Let $\overline{D}$ be a small closed disc neighbourhood of $p \in S^2$, so that $q|_{\overline{D}}$ is injective and continuous, and has image a Hausdorff space.

Now recall that a countinuous bijection from a compact space to a Hausdorff space is a homeomorphism.

So $q|_{\overline{D}} : \overline{D} \to q(\overline{D})$ is a homeomorphism. This induces a homeomorphism

$$q|_D : D \to q(D) \subset \mathbb{RP}^2,$$

where D is an open disc contained in $\overline{D}$. So $[p] \in q(D)$ has an open neighbourhood in $\mathbb{RP}^2$ homeomorphic to an open disc.

Example 1.2.

We continue looking at examples of surfaces.

(vi) Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Then the *torus* is $S^1 \times S^1$ with the subspace topology of $\mathbb{C}^2$ (this is the same as taking the product topology).

Lemma 1.3. *The torus is a topological surface.*

Proof: We consider the map

$$\begin{aligned} \mathbb{R}^2 &\xrightarrow{e} S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C} \\ (s,t) &\mapsto (e^{2\pi i s}, e^{2\pi i t}). \end{aligned}$$

We can view this map using the following diagram:

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "e"] \arrow[d, "q"] & S^1 \times S^1 \\ \mathbb{R}^2/\mathbb{Z}^2 \arrow[ur, dashed, "\hat{e}"] & \end{tikzcd}$$

There is an equivalence relation on $\mathbb{R}^2$ given by translating by $\mathbb{Z}^2$. Now consider the map

$$[0,1]^2 \hookrightarrow \mathbb{R}^2 \xrightarrow{q} \mathbb{R}^2/\mathbb{Z}^2$$

is onto, so $\mathbb{R}^2/\mathbb{Z}^2$ is compact. Now note that $\hat{e}$ is a continuous bijection, so since it is onto a Hausdorff space, it is a homeomorphism.

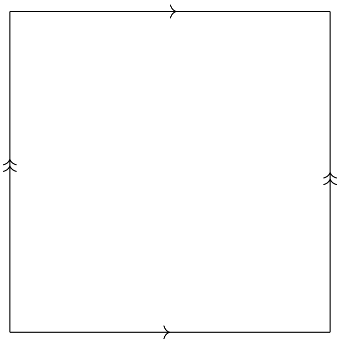
Similar to $\mathbb{RP}^2$, for $[p] \in q(p)$, take a small closed disc $\overline{D} \subset \mathbb{R}^2$ such that, for all $(m,n) \in \mathbb{Z}^2$, $\overline{D} \cap (\overline{D} + (m,n)) = \emptyset$.

Then $e|_{\overline{D}}$ and $q|_{\overline{D}}$ are injective. Now restricting to an open disc as before, we get an open disc as a neighbourhood of $[p]$, so $S^1 \times S^1$ is a topological surface.

Another viewpoint for a torus is by imposing on $[0,1]^2$ the equivalence relations

$$(x,0) \sim (x,1), \qquad (0,y) \sim (1,y).$$

Figure 1: Identification of a Torus



Example 1.3.

We look at yet another example of a surface.

(vii) Let P be a planar Euclidean polyon. Assume that the edges are oriented and paired, and for simplicity assume the Euclidean lengths of e and $\hat{e}$ are equal if $\{e, \hat{e}\}$ are paired.

Label by letters, and describe the orientation by a sign of $\pm$ relative to the clockwise orientation in $\mathbb{R}^2$.

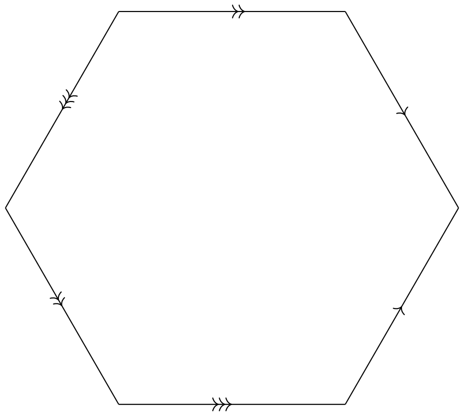
More precisely, if $\{e, \hat{e}\}$ are paired edges, there is a unique isometry from e to $\hat{e}$ respecting their orientations, say

$$f_{e\hat{e}} : e \to \hat{e}.$$

These maps generate an equivalence relation on P, where we identify $x \in \partial P$ with $f_{e\hat{e}}(x)$ whenever $x \in e$.

Lemma 1.4. *$P/\sim$ (with the quotient topology) is a topological surface.*

Figure 2: Orientation of Edges of a Hexagon



Proof: We begin by looking at a special case of the torus T^2 as $[0,1]^2/\sim$. Then if p is an interior point, we pick $\delta > 0$ small such that $\overline{B_\delta(p)}$ lies in the interior of the polygon P. Now we argue as before: the quotient map is injective on $\overline{B_\delta(p)}$ and is a homeomorphism on its interior.

Now suppose p is on an edge of P, but not a vertex. The idea is to take the two points in $q^{-1}(p)$, take half discs around them, and join them up to form a disc.

Say $p = (0, y_0) \sim (1, y_0) = p'$. Take δ small enough so the half discs of radius δ do not meet the vertices and don't intersect. Let U be the half disc around p and V the half disc around p'.

Define a map as follows:

$$U : (x, y) \xrightarrow{f_u} (x, y - y_0),$$
$$V : (x, y) \xrightarrow{f_v} (x - 1, y - y_0).$$

We want to show these maps glue well together. To do this, we use the following fact:

If $X = A \cup B$, A and B are closed, and $f : A \to Y$ and $g : B \to Y$ are continuous and $f|_{A\cap B} = g|_{A\cap B}$, then they define a continuous map on X.

Now f_u and f_v are continuous on $U, V \subset [0,1]^2$, so they induce continuous maps on $q(U)$ and $q(V)$.

In T^2, the intersection of the discs overlap on the paired edges, but our maps agree, so they are compatible with the equivalence relation. Hence f_u and f_v give a continuous map on an open image of $[p] \in T^2$ to $\mathbb{R}^2$. By the usual argument, we can show if $[p] \in T^2$ lies on an edge of P it has a neighbourhood homeomorphic to a disc.

Finally, we look at a vertex of $[0,1]^2$. In the image, there is really only one vertex. To find a homeomorphism to the open disc, we can take four quarter circles at each corner, and glue them appropriately.

For a general polygon, it is a similar idea. Interior and edge points are done analogously to T^2. For vertices, it is a bit different. We have different equivalence classes of vertices caused by orienting the edges in different ways.

If v is a vertex of P with k vertices in its equivalence class, then we have k sectors in P. Any sector can be identified with out favourite sector in $\mathbb{R}^2$, i.e. $(r, \theta) \in \mathbb{R}^2$ with $0 \le r < \delta$ and $\theta \in [0, 2\pi/k]$. Gluing these together, we get an open disc as a neighbourhood of v.

This works unless $k = 1$, in which case we have two paired edges coming into or out of a vertex in P. But this is homeomorphic to a cone, which is homeomorphic to a disc.

These neighbourhoods of points in $P/\sim$ show that P is locally homeomorphic to a disc, and we can easily check that $P/\sim$ is Hausdorff and second countable.

Example 1.4.

One more example now.

(viii) We now consider connecting surfaces. Given topological surfaces Σ_1 and Σ_2, we can remove an open disc from each, and glue the resulting boundary circles.

Explicitly, we take $\Sigma_1 \setminus D_1 \cup \Sigma_2 \setminus D_2$ as a disjoint union, and impose the quotient relation

$$\theta \in \partial D_1 \sim \theta \in \partial D_2,$$

where θ parametrizes $S^1 = \partial D_i$.

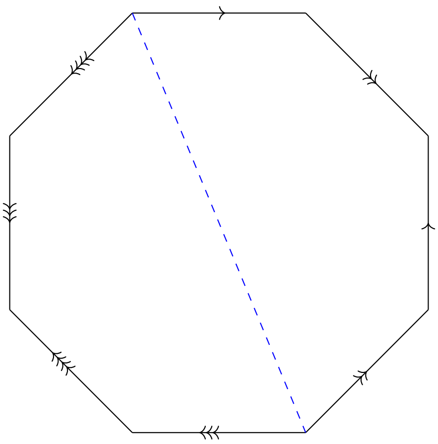
The result $\Sigma_1 \sharp \Sigma_2$ is called the *connected sum* of Σ_1 and Σ_2.

In principle, this depends on the choices of discs, and it takes some effort to prove that it is well-defined.

Lemma 1.5. *The connected sum $\Sigma_1 \sharp \Sigma_2$ is a topological surface.*

We will not prove this lemma in this course.

Figure 3: Octagon



As another example the octagon is homeomorphic to a double torus: cutting along the blue line reveals two copies of a torus, which are joined together.

Similarly, we can find $\mathbb{RP}^2$ as the quotient of a square: this can be seem by morphing it into a circle with antipodes identified, which is then homeomorphic to $\mathbb{RP}^2$, seen by 'squishing down' $\mathbb{RP}^2$ or projecting it onto a plane.

Figure 4: Identification of $\mathbb{RP}^2$

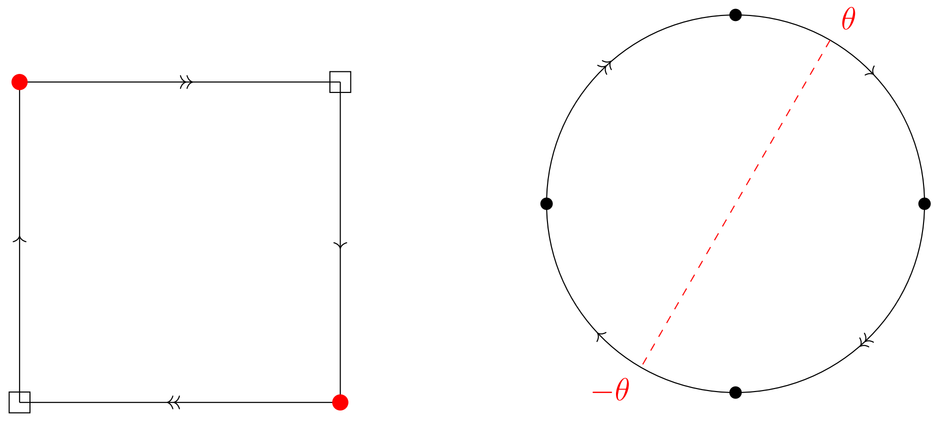
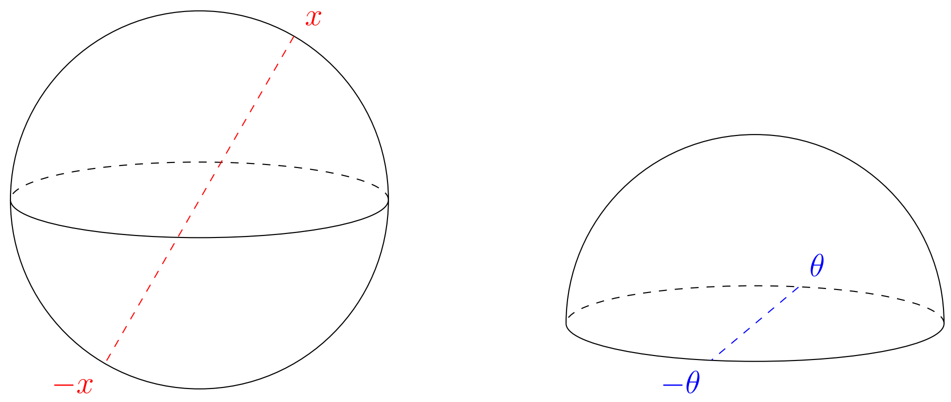


Figure 5: Squishing down $\mathbb{RP}^2$



1.1 Triangulation and Euler Characteristic

Definition 1.3. A *subdivision* of a compact topological surface Σ comprises of:

(i) a finite set V of *vertices*,

(ii) a finite collection of edges $E + \{e_i : [0,1] \to \Sigma\}$ such that

- for all i, e_i is a continuous injection on its interior and $e_i^{-1}(V) = \{0,1\}$,
- e_i and e_j have disjoint images except perhaps at their endpoints in V.

(iii) We require that each connected component of

$$\Sigma \setminus \left(\bigcup_i e_i([0,1]) \cup V \right)$$

is homeomorphic to an open disc, called a *face*.

Hence the closure of a face $\overline{F} \setminus F$ has boundary lying in

$$\bigcup_i e_i([0,1]) \cup V.$$

A subdivision is a *triangulation* if every closed face (closure of a face) contains exactly three edges, and two closed faces are disjoint, meet in exactly one edge or just one vertex.

Example 1.5.

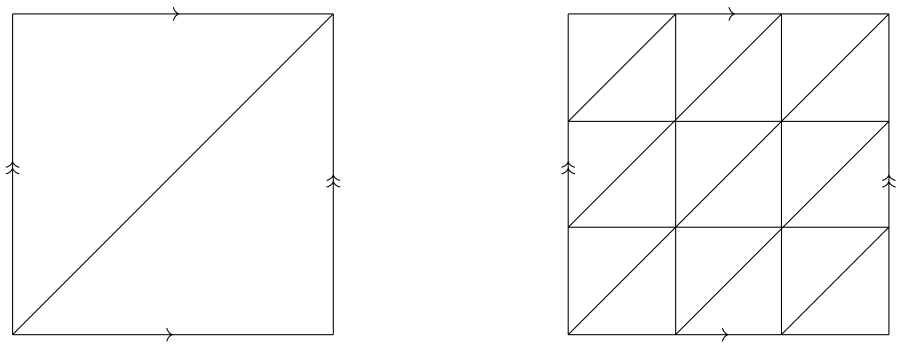
A cube displays a subdivision of S^2, and a tetrahedron displays a triangulation of S^2.

Moreover figure 1 displays a subdivision of T^2, with one vertex, two edges and one face.

In figure 6, only the right triangulation is a valid triangulation: in the left figure, the two triangles share more than one edge.

As well, figure 7 is a degenerate subdivision of the sphere, with one vertex, no edges and one face.

Figure 6: Triangulations of the Torus



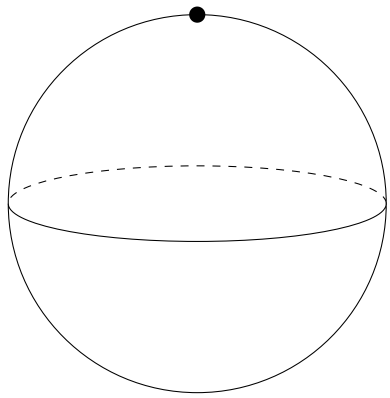
(a) Invalid (b) Valid

Definition 1.4. The *Euler characteristic* of a subdivision is

$$|V| - |E| + |F|.$$

Theorem 1.1.

Figure 7: Subdivision of S^2



(i) *Every compact topological surface admits subdivisions and triangulations.*

(ii) *The Euler characteristic, denoted $\chi(\Sigma)$, does not depend on the subdivision and defined a topological invariant of the surface.*

Remark. This is hard to prove, particularly (ii). There are cleaner approaches to this (seen in algebraic topology).

Example 1.6.

1. $\chi(S^2) = 2$.
2. $\chi(T^2) = 0$.
3. Let Σ_1, Σ_2 be compact topological spaces, and we form $\Sigma_1 \sharp \Sigma_2$. We remove open discs $D_i \subset \Sigma_i$ which is a face of a triangulation in each surface. Hence,

$$\chi(\Sigma_1 \sharp \Sigma_2) = \chi(\Sigma_1) + \chi(\Sigma_2) - 2.$$

In particular if Σ_g is a surface with g holes, i.e.

$$\Sigma_g = \sharp_{i=1}^{g} T^2,$$

then $\chi(\Sigma_g) = 2 - 2g$. g is called the *genus*.

2 Abstract Smooth Surfaces

Definition 2.1. A pair (U, φ) where $U \subset \Sigma$ is open and $\varphi : U \to V \subset \mathbb{R}^2$ is called a *chart*.

The inverse $\sigma = \varphi^{-1} : V \to U \subset \Sigma$ is called a *local parametrization* of Σ.

Definition 2.2. A collection of charts

$$\{(U_i, \varphi_i)_{i \in I}\}$$

such that

$$\bigcup_{i \in I} U_i = \Sigma$$

is called an *atlas* of Σ.

Example 2.1.

1. If $Z \subset \mathbb{R}^2$ is closed, then $\mathbb{R}^2 \setminus Z$ is a topological surface with an atlas with one chart: $(\mathbb{R}^2 \setminus Z, \varphi = \text{id})$.
2. For S^2 we have an atlas with 2 charts: the two stereographic projections.

Definition 2.3. Let (U_i, φ_i) for $i = 1, 2$ be two charts containing $p \in \Sigma$. The map

$$\varphi_2 \circ \varphi_1^{-1}|_{\varphi_1(U_1 \cap U_2)}$$

is called the *transition map* between charts.

Note that

$$\varphi_1(U_1 \cap U_2) \overset{\varphi_2 \circ \varphi_1^{-1}}{\to} \phi_2(U_1 \cap U_2)$$

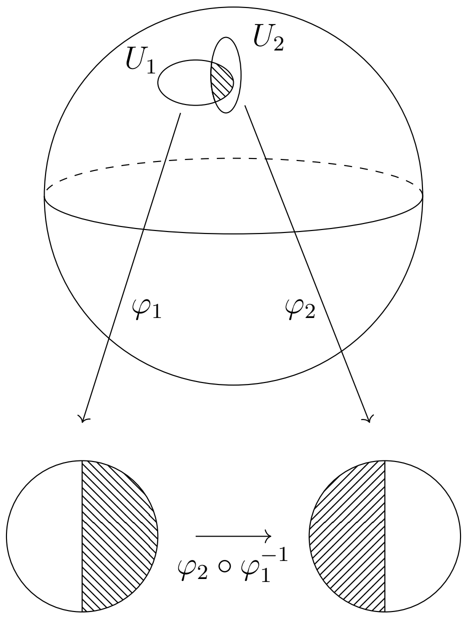
is a *homeomorphism*.

Recall if $V \subset \mathbb{R}^n$ and $V' \subset \mathbb{R}^m$ are open, a map $f : V \to V'$ is called *smooth* if it is infinitely differentiable, so it has continuous partial derivatives of all orders.

A homeomorphism $f : V \to V'$ is called a *diffeomorphism* if it is smooth and it has a smooth inverse.

Definition 2.4. An *abstract smooth surface* Σ is a topological surface with an atlas of charts $\{U_i, \varphi_i)\}$ such that all transition maps are diffeomorphisms.

Figure 8: Transition Map on S^2



Example 2.2.

1. The atlas of two charts with stereographic projections gives S^2 the structure of an abstract smooth surface.
2. The torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ is an abstract smooth surface. Recall that we obtained charts from (the inverses of) the projection restricted to small discs in $\mathbb{R}^2$. In particular, consider the atlas

$$\{(e(D_\varepsilon(x,y)), e^{-1} \text{ on its image})\},$$

where $\varepsilon < 1/3$. Here the transition maps are translations, so T^2 inherits the structure of a smooth surface.

Definition 2.5. Let Σ be an abstract smooth surface and $f : \Sigma \to \mathbb{R}^n$ a map. We say that f is *smooth* at $p \in \Sigma$ if whenever (U, φ) is a chart at p belonging to the smooth atlas of Σ, then the map

$$f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^n$$

is smooth at $\phi(p) \in \mathbb{R}^2$.

Note if this holds for one chart at p, then it holds for all charts at p, as

$$f \circ \varphi^{-1} = f \circ \varphi_2^1 \circ (\varphi_2 \circ \varphi_1^{-1}),$$

and $(\varphi_2 \circ \varphi_1^{-1})$ is a diffeomorphism.

Related, if Σ_1, Σ_2 are abstract smooth surfaces, then a map $f : \Sigma_1 \to \Sigma_2$ is *smooth* if it is smooth at the local charts: there are charts (U, φ) at p and (V, ψ) at $f(p)$ with $f(U) \subset V$ such that $\psi \circ f \circ \varphi^{-1}$ is smooth at $\varphi(p)$.

Again, if f is smooth at p, then the smoothness of the local representation of f at p will hold for all charts at p and $f(p)$ in the given atlases.

Definition 2.6. Σ_1 and Σ_2 are *diffeomorphic* if there exists $f : \Sigma_1 \to \Sigma_2$ that is smooth with smooth inverse.

Definition 2.7. If $Z \subset \mathbb{R}^n$ is an arbitrary subset, we say that $f : Z \to \mathbb{R}^m$ is smooth near $p \in Z$ if there exists open B with $p\, inB \subset \mathbb{R}^n$ and smooth $F : B \to \mathbb{R}^m$ such that

$$F|_{B\cap Z} = f|_{B\cap Z}.$$

So f is locally the restriction of a smooth map defined on an open set.

Definition 2.8. If $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ are subsets, we say that X and Y are *diffeomorphic* if there exists $f : X \to Y$, smooth with smooth inverse.

Definition 2.9. A *smooth surface* in $\mathbb{R}^3$ is a subset $\Sigma \subset \mathbb{R}^3$ such that for all $p \in \Sigma$, there exists an open set $p \in U \subset \Sigma$ such that U is diffeomorphic to an open set in $\mathbb{R}^2$.

In other words, for all $p \in \Sigma$, there exists an open ball B such that $p \in B \subset \mathbb{R}^3$ and $F : B \to V \subset \mathbb{R}^2$ smooth, with

$$F|_{B\cap\Sigma} : B \cap \Sigma \to V$$

a homeomorphism with inverse $V \to B \cap \Sigma$ smooth.

Hence we have two notions of a smooth surface: one abstract, and one taking advantage of the ambient space $\mathbb{R}^3$.

Theorem 2.1. *For a subset $\Sigma \subset \mathbb{R}^3$, the following are equivalent:*

(a) *Σ is a smooth surface in $\mathbb{R}^3$.*

(b) *Σ is locally the graph of a smooth function over one of the coordinate planes, so for all $p \in \Sigma$, there exists open $p \in B \subset \mathbb{R}^3$ and open $V \subset \mathbb{R}^2$ such that*

$$\Sigma \cap B = \{(x, y, g(x, y) \mid g : V \to \mathbb{R}\},$$

with g smooth.

(c) *Σ is locally cut out by a smooth function with non-zero derivative, so for all $p \in \Sigma$, there open exists $p \in B \subset \mathbb{R}^n$ and $f : B \to \mathbb{R}$ such that*

$$\Sigma \cap B = f^{-1}(0), \qquad Df|_x \neq 0,$$

for all $x \in B$.

(d) *Σ is locally the image of an* allowable parametrization*, so if $p \in \Sigma$, there exists open $p \in U \subset \Sigma$ and smooth $\sigma : V \to U$, such that σ is a homeomorphism and $D\sigma|_x$ has rank 2 for all $x \in V$.*

Remark. (b) says that if Σ is a smooth surface in $\mathbb{R}^3$, then each $p \in \Sigma$ belongs to a chart (U, φ) where φ is the restriction of $\pi_{xy}, \pi_{yz}, \pi_{xz}$ from $\mathbb{R}^3$ to $\mathbb{R}^2$.

2.1 Inverse Function Theorem

Theorem 2.2. *Let $U \subset \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}^n$ be a continuously differentiable map. Let $p \in U$, $f(p) = q$, and suppose $Df|_p$ is invertible. Then there is an open neighbourhood V of q and a differentiable map $g : V \to \mathbb{R}^n$ with $g(q) = p$, with image an open neighbourhood $U' \subset U$ of p, such that*

$$f \circ g = \mathrm{id}_V, \qquad g \circ f = \mathrm{id}_{u'}.$$

If f is smooth, then so is g.

Remark. $(Dg|_q) = (Df|_p)^{-1}$ by the chain rule.

If we have a map $f : \mathbb{R}^n \to \mathbb{R}^m$ where $n > m$, then

$$Df|_p = \left(\frac{\partial f_i}{\partial x_j}\right)_{m \times n}$$

having full rank means that, permuting coordinates if necessary, we can assume that the first m columns are linearly independent.

Theorem 2.3 (Implicit Function theorem)**.** *Let $p = (x_0, y_0) \in U$, where $U \subset \mathbb{R}^k \times \mathbb{R}^\ell$ is open, and $f : U \to \mathbb{R}^\ell$ be a continuously differentiable map with $f(p) = 0$, and*

$$\left(\frac{\partial f_i}{\partial y_j}\right)_{\ell \times \ell} \text{ is an isomorphism at } p.$$

Then there exists an open neighbourhood $x_0 \in V \subset \mathbb{R}^k$ and a continuously differentiable map $g : V \to \mathbb{R}^\ell$ taking x_0 to y_0, such that if $(x, y) \in U \cap (V \times \mathbb{R}^\ell)$, then

$$f(x, y) = 0 \iff y = g(x).$$

Proof: Introduce $F : U \to \mathbb{R}^k \times \mathbb{R}^\ell$, where $(x, y) \mapsto (x, f(x, y))$. Then

$$DF = \begin{pmatrix} I & * \\ 0 & \frac{\partial f_i}{\partial y_j} \end{pmatrix}.$$

So $DF|_{(x_0,y_0)}$ is an isomorphism. The inverse function says that F is locally invertible near $F(x_0, y_0) = (x_0, f(x_0, y_0)) = (x_0, 0)$.

Take a product of open neighbourhoods $(x_0, 0) \in V \times V'$, where $V \subset \mathbb{R}^k$, $V' \subset \mathbb{R}^\ell$ are open. Then there is some continuously differentiable inverse $G : V \times V' \to U' \subset U$ such that $F \circ G = \mathrm{id}_{V \times V'}$

Write $G(x, y) = (\varphi(x, y), \psi(x, y))$. Then,

$$F \circ G(x, y) = (\varphi(x, y) f(\varphi(x, y), \psi(x, y))) = (x, y).$$

Hence $\varphi(x, y) = x$, $f(x, \psi(x, y)) = y$. Thus, $f(x, y) = 0 \iff y = \psi(x, 0)$.

Define $g : V \to \mathbb{R}^\ell$ by $g(x) = \psi(x, 0)$. Then $g(x_0) = y_0$, and this is the required function g.

Example 2.3.

1. Take $f : \mathbb{R}^2 \to \mathbb{R}$ smooth, and $f(x_0, y_0) = 0$. Assume

   $$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \neq 0.$$

   Then there exists smooth $g : (x_0 - \varepsilon, x_0 + \varepsilon) \to \mathbb{R}$. Such that $g(x_0) = y_0$ and $f(x, y) = 0 \iff y = g(x)$.

   Since $f(x, g(x)) = 0$ by chain rule

   $$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} g'(x) = 0 \implies g'(x) = -\frac{\partial f/\partial x}{\partial f/\partial y}.$$

2. Let $f : \mathbb{R}^3 \to \mathbb{R}$ smooth and $f(x_0, y_0, z_0) = 0$, and assume

   $$Df|_{(x_0,y_0,z_0)} \neq 0.$$

   Permuting coordinates if necessary, we may assume that

   $$\frac{\partial f}{\partial z}|_{(x_0,y_0,z_0)} \neq 0.$$

Then there exists an open neighbourhood $(x_0, y_0) \in V \subset \mathbb{R}^2$ and a smooth $g : V \to \mathbb{R}$, with $g(x_0, y_0) = z_0$, such that for an open set $(x_0, y_0, z_0) \in U$,

$$f^{-1}(0) \cap U = \{(x, y, g(x, y)) \mid (x, y) \in V\}.$$

We return to theorem 2.1, which we can now prove.

Proof: Note (b) implies all other statements. If Σ is locally $\{(x, y, g(x, y)) \mid (x, y) \in V\}$, then we get a chart from the projection Π_{xy}, which is smooth and defined on an open neighbourhood of Σ, hence (b) implies (a).

Also, it is cut out by

$$f(x, y, z) = z - g(x, y).$$

Clearly $\frac{\partial f}{\partial z} = 1 \neq 0$, so (b) implies (c).

Also, $\sigma(x, y) = (x, y, g(x, y))$ is allowable and smooth, with

$$\sigma_x = (1, 0, g_x), \qquad \sigma_y = (0, 1, g_y)$$

linearly independent. So (b) implies (d).

Now (a) implies (d), as this is part of the definition of being a smooth surface in $\mathbb{R}^3$.

Moreover, (c) implies (b) from the above example of the implicit function theorem.

We finally show that (d) implies (b). Let $p \in \Sigma$, and $\sigma : V \to U \subset \Sigma$ with $\sigma(0) = p \in U$, and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$. Then

$$D\sigma = \begin{pmatrix} \partial\sigma_1/\partial u & \partial\sigma_1/\partial v \\ \partial\sigma_2/\partial u & \partial\sigma_2/\partial v \\ \partial\sigma_3/\partial u & \partial\sigma_3/\partial v \end{pmatrix}.$$

So there exists two rows defining an invertible matrix, as $D\sigma$ has rank two. Suppose the first two rows are. Then $\Pi_{xy} \circ \sigma : V \to \mathbb{R}^2$ satisfies $D(\Pi_{xy} \circ \sigma)|_0$ is an isomorphism.

By the inverse function theorem, this is locally invertible, so if we let $\phi = \Pi_{xy} \circ \sigma$, then Σ is the graph of $(x, y, \sigma_3(\phi^{-1}(x, y)))$.

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