IB Electromagnetism

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Contents

1 Introduction

1.1 Charges and Currents

Electric charge is a physical property of elementary particles. It is:

- Positive, negative or zero.
- Quantized (an integer multiple of the *elementary charge* e).
- Conserved (even if particles are created or destroyed).

By convention, the electron has charge $-e$, the proton has charge $+e$, and the neutron has charge 0.

On macroscopic scales, the number of particles is so large that charge can be considered to have continuous *electric charge density* $\rho(\mathbf{x}, t)$. The total charge in a volume V is then

$$Q = \int_V \rho \, \mathrm{d}V.$$

The *electric current density* $\mathbf{J}(\mathbf{x}, t)$ is the flux of electric charge per unit area. The current flowing through a surface S is

$$I = \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}.$$

Consider a time-independent volume V with boundary S. Since charge is conserved, we have

$$\begin{aligned} \frac{\mathrm{d}Q}{\mathrm{d}t} &= -I, \\ \frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho \, \mathrm{d}V + \int_S \mathbf{J} \cdot d\mathbf{S} &= 0, \\ \int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} \right) \mathrm{d}V &= 0. \end{aligned}$$

Since this is true for any V, we must have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

This *equation of charge conservation* has the typical form of a conservation law.

The discrete charge distribution of a single particle of charge q_i, and position vector $\mathbf{x}_i(t)$ is

$$\begin{aligned}\rho &= q_i \delta(\mathbf{x} - \mathbf{x}_i(t)),\\ \mathbf{J} &= q_i \dot{\mathbf{x}}_i \delta(\mathbf{x} - \mathbf{x}_i(t)).\end{aligned}$$

For N particles, it is

$$\begin{aligned}\rho &= \sum_{i=1}^{N} q_i \delta(\mathbf{x} - \mathbf{x}_i(t)),\\ \mathbf{J} &= \sum_{i=1}^{N} q_i \dot{\mathbf{x}}_i \delta(\mathbf{x} - \mathbf{x}_i(t)).\end{aligned}$$

We can verify that these distributions satisfy the charge conservation equation.

1.2 Fields and Forces

Electromagnetism is a *field theory*. Charged particles interact not directly, but by generating fields around them that are experienced by other charged particles.

In general, we have two time-dependent vector fields: the *electric field* $\mathbf{E}(\mathbf{x}, t)$, and the *magnetic field* $\mathbf{B}(\mathbf{x}, t)$.

The *Lorentz force* on a particle of charge q and velocity $\mathbf{v}$ is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

1.3 Maxwell's equations

In this course we will explore some consequences of *Maxwell's equations*

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0}, & \nabla \cdot \mathbf{B} &= 0,\\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, & \nabla \times \mathbf{B} &= \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right).\end{aligned}$$

Some properties of Maxwell's equations are:

- They are coupled linear PDE's in space and time.
- They involve two positive constants: ϵ_0 (vacuum permittivity), and μ_0 (vacuum permeability).
- Charges (ρ) and currents ($\mathbf{J}$) are the sources of the electromagnetic fields.

- Each equation has an equivalent integral form, related via the divergence theorem of Stokes' theorem.
- These are the vacuum equations that apply on microscopic scales (or in a vacuum). A related macroscopic version applies in media (for examples air).
- The equations are consistent with each other and with charge conservation. For example, $\nabla \cdot (M3) = \frac{\partial}{\partial t}(M2)$, and

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = \frac{\partial}{\partial t}(\epsilon_0 \nabla \cdot \mathbf{E}) + \nabla \cdot \left(-\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\mu_0} \nabla \times \mathbf{B} \right) = 0.$$

1.4 Units

The SI unit of electric charge is the coulomb (C). The elementary charge is (exactly)

$$e = 1.602\,176\,634 \times 10^{-19}\,\mathrm{C}.$$

The SI unit of electric current is the ampere, or amp (A), equal to $1\,\mathrm{C\,s^{-1}}$.

The SI base units needed in electromagnetism are:

second (s)

metre (m)

kilogram (kg)

ampere (A)

From the Lorentz force law, we can see that the units of $\mathbf{E}$ and $\mathbf{B}$ must be

$$\mathrm{kg\,m\,s^{-3}\,A^{-1}} \text{ and } \mathrm{kg\,s^{-2}\,A^{-1}}.$$

The latter is also called the tesla (T). From Maxwell's equations, we can work out the units of ϵ_0 and μ_0. The experimentally determined values are

$$\begin{aligned} \epsilon_0 &= 8.854 \cdots \times 10^{-12}\,\mathrm{kg^{-1}\,m^{-3}\,s^4\,A^2} \\ \mu_0 &= 1.256 \cdots \times 10^{-6}\,\mathrm{kg\,m\,s^{-2}\,A^{-2}} \\ &\approx 4\pi \times 10^{-7}\,\mathrm{kg\,m\,s^{-2}\,A^{-2}}. \end{aligned}$$

The speed of light is (exactly)

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 299\,792\,458\,\mathrm{m\,s^{-1}} \approx 3 \times 10^8\,\mathrm{m\,s^{-1}}.$$

2 Electrostatics

In a time-independent situation, Maxwell's equations reduce to

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \nabla \times \mathbf{E} = \mathbf{0},$$
$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

Since $\mathbf{E}$ and $\mathbf{B}$ are decoupled, we can study them separately.

Electrostatics is the study of the electric field generated by a stationary charge distribution

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{M1}$$
$$\nabla \times \mathbf{E} = \mathbf{0}. \tag{M3'}$$

2.1 Gauss' Law

Consider a closed surface S enclosing a volume V. Integrating (M1) over V and using the divergence theorem, we obtain *Gauss' law*

$$\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q}{\epsilon_0},$$

where $Q = \int_V \rho \, \mathrm{d}V$ is the total charge in V.

Gauss' law is the integral version of (M1) and is valid generally. This says that the electric flux of a closed surface is proportional to the total charge enclosed.

In special situations, we can use Gauss' law together with symmetry to deduce $\mathbf{E}$ from ρ. By choosing the *Gaussian surface* S appropriately.

2.1.1 Spherical Symmetry

Consider a spherically symmetric charge distribution, $\rho(r)$ in spherical polar coordinates, with total charge Q contained within an outer radius R.

To have spherical symmetry, the electric field should have the form

$$\mathbf{E} = E(r)\mathbf{e}_r.$$

This will satisfy (M3'), as required. To find $E(r)$, we apply Gauss' law to a sphere of radius r. If $r > R$, then

$$\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = E(r) \int_S \mathbf{e}_r \cdot \mathrm{d}\mathbf{S} = E(r) \int_S \mathrm{d}S = E(r)\, 4\pi r^2 = \frac{Q}{\epsilon_0}.$$

Thus, outside of the sphere of radius R,

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{e}_r.$$

So the external electric field of a spherically symmetric body depends only on the total charge.

The Lorentz force on a particle of charge q in $r > R$ is

$$\mathbf{F} = q\mathbf{E} = \frac{Qq}{4\pi\epsilon_0 r^2}\mathbf{e}_r.$$

This is the *Coulomb force* between charged particles. The force is repulsive if the charges have the same sign ($Qq > 0$) and attractive if they have opposite signs ($Qq < 0$).

If we take the limit $R \to 0$, we obtain the electric field of a *point charge* Q, corresponding to

$$\rho = Q\delta(\mathbf{x}).$$

There is a close analogy between the Coulomb force and the gravitational force between massive particles,

$$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{e}_r.$$

Both involve an inverse-square law, and the product of the charges/masses. However,

- While gravity is always attractive, electric forces can be repulsive or attractive.
- Gravity is very much weaker than the Coulomb force, e.g. for two protons the ratio of the electric to gravitational forces is

  $$\frac{e^2}{4\pi\epsilon_0 G m_p^2} \approx 10^{36}.$$

  On the atomic scale, gravity is irrelevant. But positive and negative charges balance so accurately that on the planetary scale, gravity is dominant.

2.1.2 Cylindrical Symmetry

Consider a cylindrically symmetric charge distribution $\rho(r)$ in cylindrical polar coordinates, with total charge λ per unit length, contained within an outer radius R.

To have cylindrical symmetry,

$$\mathbf{E} = E(r)\mathbf{e}_r.$$

To find $E(r)$ we apply Gauss' law to a cylinder of radius r and arbitrary length L. Again, we consider $r > R$. Then, since only the curved part of the cylinder contributes to the flux,

$$\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = E(r) \int_S \mathbf{e}_r \cdot \mathrm{d}\mathbf{S} = E(r) \int_S \mathrm{d}S = E(r) 2\pi r L = \frac{\lambda L}{\epsilon_0}.$$

Thus, we get

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 r}\mathbf{e_r}.$$

In the limit $R \to 0$, we obtain the electric field of a *line charge* λ per unit length, corresponding to

$$\rho = \lambda\delta(x)\delta(y).$$

2.1.3 Planar Symmetry

We consider a planar charge distribution $\rho(z)$ in Cartesian coordinates, with total charge σ per unit area, contained within a region $-d < z < d$ of thickness $2d$. We assume reflectional symmetry, so $\rho(z)$ is even.

To have planar symmetry, we need

$$\mathbf{E} = E(z)\mathbf{e}_z,$$

which will satisfy (M3'). Reflectional symmetry implies $E(-z) = -E(z)$. To find $E(z)$ for $z > 0$, apply Gauss' law to a "Gaussian pillbox" of height $2z$ and arbitrary area A. If $z > d$, then

$$\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = E(z)A - E(-z)A = 2E(z)A = \frac{\sigma A}{\epsilon_0}.$$

Thus,

$$\mathbf{E} = \begin{cases} \frac{\sigma}{2\epsilon_0}\mathbf{e}_z & z > d, \\ -\frac{\sigma}{2\epsilon_0}\mathbf{e}_z & \text{z ¡ -d.} \end{cases}$$

In the limit $d \to 0$, we obtain the electric field of a *surface charge* σ per unit area, corresponding to

$$\rho = \sigma\delta(z).$$

2.1.4 Surface Charge and Discontinuity

Let $\mathbf{n}$ be a unit vector normal to the charged surface, pointing from region 1 to region 2. In our example, $\mathbf{n} = \mathbf{e}_z$.

The discontinuity in $\mathbf{E}$ is given by

$$[\mathbf{n} \cdot \mathbf{E}] = \frac{\sigma}{\epsilon_0},$$

where σ is the surface charge density, and

$$[X] = X_2 - X_1$$

denotes a discontinuity. The tangential components are continuous (they are both 0), so

$$[\mathbf{n} \times \mathbf{E}] = \mathbf{0}.$$

These equation apply to any surface charge (even if the surface is curved an non-uniform).

The first comes from applying Gauss' law to an infinitesimal Gaussian pillbox on the surface.

The second comes from considering an infinitesimal circuit that goes through the surface: in the limit, and by taking all orientations of loops, we can use Stokes' theorem to get the required result.

2.2 The Electrostatic Potential

For general $\rho(\mathbf{x})$, we cannot determine $\mathbf{E}(\mathbf{x})$ using Gauss' law alone.

Since $\nabla \times \mathbf{E} = \mathbf{0}$, we know that $\mathbf{E}$ can be written in terms of an *electrostatic potential* (or electric potential) $\Phi(\mathbf{x})$

$$\mathbf{E} = -\nabla \Phi.$$

The *potential difference* (or *voltage*) between two points $\mathbf{x}_1$ and $\mathbf{x}_2$ is

$$\Phi(\mathbf{x}_2) - \Phi(\mathbf{x}_1) = \int \mathrm{d}\Phi = -\int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{E}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x},$$

and is path-independent because $\nabla \times \mathbf{E} = \mathbf{0}$.

The electric force on a particle of charge q is

$$\mathbf{F} = q\mathbf{E} = -q\nabla\Phi$$

is a conservative force associated with the potential energy

$$U(\mathbf{x}) = q\Phi(\mathbf{x}).$$

(M1) implies that Φ satisfies *Poisson's equation*

$$-\nabla^2\Phi = \frac{\rho}{\epsilon_0}.$$

The solution can be written as an integral (over all space, assuming decay at infinity)

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}^3\mathbf{x}'.$$

This is the convolution of $\rho(\mathbf{x})$ with the potential of a unit point charge $\frac{1}{4\pi\epsilon_0|\mathbf{x}|}$, which is the solution of

$$-\nabla^2\Phi = \frac{\delta(\mathbf{x})}{\epsilon_0},$$

satisfying $\Phi \to 0$ as $|\mathbf{x}| \to \infty$.

Note that $\mathbf{E}$ is unaffected if we add an arbitrary constant to Φ. We usually choose this constant such that $\Phi \to 0$ as $|\mathbf{x}| \to \infty$. However if $\rho(\mathbf{x})$ does not decay sufficiently rapidly, this may not be possible. For example, a line charge has $E_r \propto \frac{1}{r}$, so $\Phi \propto \log r$, which does not decay.

2.2.1 Point Charge

The potential due to a point charge q at the origin is

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\epsilon_0|\mathbf{x}|} = \frac{q}{4\pi\epsilon_0 r}.$$

2.2.2 Electric Dipole

This consists of two equal and opposite charge at difference positions. Without loss of generality, consider charges $-q$ at $\mathbf{x} = \mathbf{0}$ and $+q$ and $\mathbf{x} = \mathbf{d}$.

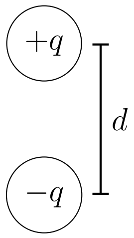
The potential due to the dipole will be

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\epsilon_0}\left(-\frac{1}{|\mathbf{x}|} + \frac{1}{|\mathbf{x}-\mathbf{d}|}\right).$$

Applying Taylor's theorem to a scalar field, we get

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + (\mathbf{h}\cdot\nabla)f(\mathbf{x}) + \frac{1}{2}(\mathbf{h}\cdot\nabla)^2 f(\mathbf{x}) + \mathcal{O}(|\mathbf{h}|^3),$$

Figure 1: Electric Dipole



so applying this to our potential (and letting $|\mathbf{x}| = r$,)

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{q}{4\pi\epsilon_0}\left(-\frac{1}{r} + \frac{1}{r} - (\mathbf{d}\cdot\nabla)\frac{1}{r} + \mathcal{O}(|\mathbf{d}|^2)\right)\\ &= \frac{q}{4\pi\epsilon_0}\frac{\mathbf{d}\cdot\mathbf{x}}{|\mathbf{x}|^3} + \mathcal{O}(|\mathbf{d}|^2).\end{aligned}$$

In the limit $|\mathbf{d}| \to 0$ with $q\mathbf{d}$ finite, we obtain a *point dipole* with *electric dipole moment*

$$\mathbf{p} = q\mathbf{d},$$

with potential

$$\Phi(\mathbf{x}) = \frac{\mathbf{x}\cdot\mathbf{x}}{4\pi\epsilon_0|\mathbf{x}|^3}.$$

The electric field can be found as

$$\mathbf{E} = -\nabla\Phi = \frac{3(\mathbf{p}\cdot\mathbf{x})\mathbf{x} - |\mathbf{x}|^3\mathbf{p}}{4\pi\epsilon_0|\mathbf{x}|^5}.$$

In spherical polar coordinates aligned with $\mathbf{p} = p\mathbf{e}_z$,

$$\begin{aligned}\Phi &= \frac{p\cos\theta}{4\pi\epsilon_0 r^2},\\ E_r &= -\frac{\partial\Phi}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3},\\ E_\theta &= -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3},\\ E_\phi &= 0.\end{aligned}$$

Note that

- Φ and $\mathbf{E}$ are not spherically symmetric.
- They decrease more rapidly with r than for a point charge.

A point dipole $\mathbf{p}$ at the origin corresponds to

$$\rho(\mathbf{x}) = -\mathbf{p} \cdot \nabla\delta(\mathbf{x}),$$
$$\Phi(\mathbf{x}) = \mathbf{p} \cdot \nabla\left(\frac{1}{4\pi\epsilon_0|\mathbf{x}|}\right).$$

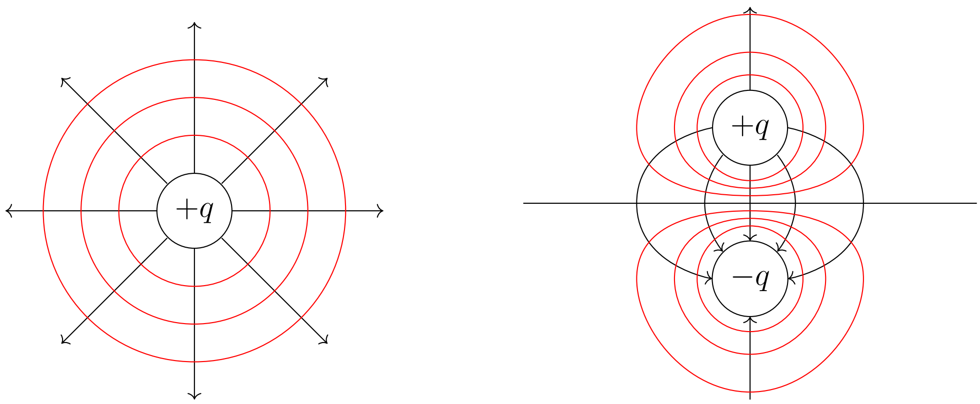
2.2.3 Field Lines and Equipotentials

Electric field lines are the integral curves of $\mathbf{E}$, being tangent to $\mathbf{E}$ everywhere.

Since $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$, the field lines begin at positive charges and end on negative charges.

Furthermore, in electrostatics $\mathbf{E} = -\nabla\Phi$, so the field lines are perpendicular to the equipotential surface $\Phi =$ constant.

Figure 2: Electric Field Lines



(a) Point charge (b) Dipole

2.2.4 Dipole in an External Field

Consider a dipole $\mathbf{p}$ in an external electric field $\mathbf{E} = -\nabla\Phi$ generated by distinct charges. If the dipole has charge $-q$ at $\mathbf{x}$ and $+q$ at $\mathbf{x} + \mathbf{d}$, then the potential energy of the dipole due to the external field is

$$U = -q\Phi(\mathbf{x}) + q\Phi(\mathbf{x} + \mathbf{d}) = q(\mathbf{d} \cdot \nabla)\Phi(\mathbf{x}) + \mathcal{O}(|\mathbf{d}|^2).$$

In the limit of a point dipole,

$$U = \mathbf{p} \cdot \nabla\Phi = -\mathbf{p} \cdot \mathbf{E}.$$

This is minimized when $\mathbf{p}$ is aligned with $\mathbf{E}$.

2.2.5 Multipole Expansion

For a general charge distribution $\rho(\mathbf{x})$ confined to a ball $\{V \mid |\mathbf{x}| < \ell\}$, then

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}^3\mathbf{x}'.$$

For an external potential with $|\mathbf{x}| > R$, we can expand

$$\begin{aligned}\frac{1}{|\mathbf{x}-\mathbf{x}'|} &= \frac{1}{r} - (\mathbf{x}'\cdot\nabla)\frac{1}{r} + \frac{1}{2}(\mathbf{x}'\cdot\nabla)^2\frac{1}{r} + \mathcal{O}(|\mathbf{x}'|^3)\\ &= \frac{1}{r}\left[1 + \frac{\mathbf{x}'\cdot\mathbf{x}}{r^2} + \frac{3(\mathbf{x}'\cdot\mathbf{x})^2 - |\mathbf{x}'|^2|\mathbf{x}|^2}{2r^4} + \mathcal{O}\left(\frac{R^3}{r^3}\right)\right].\end{aligned}$$

This leads to the *multipole expansion* of the potential

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{\mathbf{p}\cdot\mathbf{x}}{r^3} + \frac{1}{2}\frac{Q_{ij}x_ix_j}{r^5} + \cdots\right).$$

The first three multipole moments are the:

- total charge (or monopole moment) - a scalar, where
$$Q = \int_V \rho(\mathbf{x})\,\mathrm{d}^3\mathbf{x}.$$
- electric dipole moment - a vector, where
$$\mathbf{p} = \int_V \mathbf{x}\rho(\mathbf{x})\,\mathrm{d}^3\mathbf{x}.$$
- electric quadrupole moment - a traceless, symmetric second order tensor
$$Q_{ij} = \int_V (3x_ix_j - |\mathbf{x}|^2\delta_{ij})\rho(\mathbf{x})\,\mathrm{d}^3\mathbf{x}$$

For $r \gg R$, Φ and $\mathbf{E}$ look increasingly like those of a point charge Q unless $Q = 0$, in which case they look like those of a point dipole, unless $\mathbf{p} = 0$, etc.

2.3 Electrostatic Energy

The work done against the electric force $\mathbf{F} = q\mathbf{E}$ in bringing a particle of charge q from infinity (where we assume $\Phi = 0$) to $\mathbf{x}$ is

$$-\int_\infty^{\mathbf{x}} \mathbf{F}\cdot\mathrm{d}\mathbf{x} = +q\int_\infty^{\mathbf{x}} \nabla\Phi\cdot\mathrm{d}\mathbf{x} = q\Phi(\mathbf{x}).$$

Consider assembling a configuration of N point charges one by one. Particle i of charge q_i is brought from ∞ to $\mathbf{x}_i$, while the previous particles remain fixed.

Particle 1. There is no work involved, so $W_1 = 0$.

Particle 2.

$$W_1 = q_2\left(\frac{q}{4\pi\epsilon_0|\mathbf{x}_2 - \mathbf{x}_1|}\right).$$

Particle 3.

$$W_3 = q_3\left(\frac{q_1}{4\pi\epsilon_0|\mathbf{x}_3 - \mathbf{x}_1|} + \frac{q_2}{4\pi\epsilon_0|\mathbf{x}_3 - \mathbf{x}_2|}\right),$$

and so on. The total work done is

$$U = \sum_{i=1}^{N} W_i = \sum_{i=2}^{N}\sum_{j=1}^{i-1} \frac{q_i q_j}{4\pi\epsilon_0|\mathbf{x}_i - \mathbf{x}_j|}.$$

This can be rewritten as

$$U = \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{q_i q_j}{4\pi\epsilon_0|\mathbf{x}_i - \mathbf{x}_j|},$$

or

$$U = \frac{1}{2}\sum_{i=1}^{N} q_i \Phi(\mathbf{x}_i).$$

Generalizing to a continuous charge distribution $\rho(\mathbf{x})$, occupying a finite volume V,

$$U = \frac{1}{2}\int_V \rho(\mathbf{x})\Phi(\mathbf{x})\,\mathrm{d}^3\mathbf{x} = \frac{1}{2}\int_V \rho\Phi\,\mathrm{d}V.$$

Using (M1) we have

$$\begin{aligned} U &= \frac{1}{2}\int_V (\epsilon_0\nabla\cdot\mathbf{E})\Phi\,\mathrm{d}V = \frac{\epsilon_0}{2}\int_V (\nabla\cdot(\Phi\mathbf{E}) - \mathbf{E}\cdot\nabla\Phi)\,\mathrm{d}V \\ &= \frac{\epsilon_0}{2}\int_S \phi\mathbf{E}\cdot\mathrm{d}\mathbf{S} + \int_V \frac{\epsilon_0|\mathbf{E}|^2}{2}\,\mathrm{d}V. \end{aligned}$$

Let $S = \partial V$ be a sphere of radius $R \to \infty$. Then $\Phi = \mathcal{O}(R^{-1})$, and $\mathbf{E} = \mathcal{O}(R^{-2})$ on S, while the area of S is $\mathcal{O}(R^2)$, so the area integral is $\mathcal{O}(R^{-1})$ and goes to zero as $R \to \infty$. Thus,

$$U = \int \frac{\epsilon_0|\mathbf{E}|^2}{2}\,\mathrm{d}V,$$

integrated over all space.

This implies that energy is stored in the electric field, even in a vacuum.

Any of the expression for U suggest that the self-energy of a point charge is infinite. We can discard this as it is unchanging and causes no force.

2.4 Conductors

In an *conductor* such as a metal, some charges (usually electrons) can move freely. In electrostatics we require

$$\mathbf{E} = \mathbf{0}, \qquad \Phi = \text{constant}$$

inside a conductor, hence $\rho = 0$. Otherwise free charges would move in response to the electric force and a current would flow.

A surface charge density ρ can exist on the surface of a conductor, which is an equipotential.

Taking a normal $\mathbf{n}$ to the point of the conductor, the condition

$$[\mathbf{n} \cdot \mathbf{E}] = \frac{\sigma}{\epsilon_0} \implies \mathbf{n} \cdot \mathbf{E} = \frac{\sigma}{\epsilon_0}$$

immediately outside the conductor.

The constant potential of a conductor can be set by connecting it to a battery or another conductor. An *earthed* (or *grounded*) conductor is connected to the ground, usually taken as $\Phi = 0$.

To find $\Phi(\mathbf{x})$ and $\mathbf{E}(\mathbf{x})$ due to a charge distribution $\rho(\mathbf{x})$ in the presence of conductors with surfaces S_i and potentials Φ_i, we solve Poisson's equation

$$-\nabla^2 \Phi = \frac{\rho}{\epsilon_0},$$

with Dirichlet boundary conditions $\Phi = \Phi_i$ on S_i. The solution depends linearly on ρ and $\{\Phi_i\}$.

Example 2.1.

Consider a point charge q at position $(0, 0, h)$ in a half-space $z > 0$, bounded by an earthed conducting wall ($\Phi = 0$ on $z = 0$).

By the method of images, the solution in $z > 0$, is identical to that of a dipole, with image charge $-q$ at $(0, 0, -h)$.

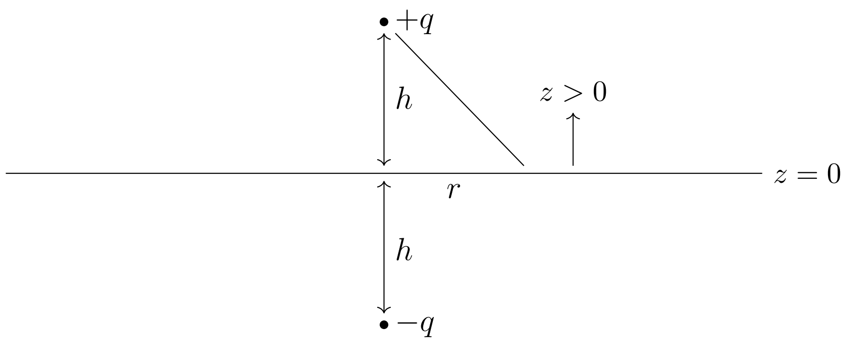
This is as the wall coincides with an equipotential of the dipole. The induced surface charge density on the wall can be worked out from

$$\frac{\sigma}{\epsilon_0} = \mathbf{n} \cdot \mathbf{E} = E_z = -\frac{qh}{4\pi\epsilon_0 (r^2 + h^2)^{3/2}},$$

where $r = \sqrt{x^2 + y^2}$. The total induced surface charge is

$$\int_0^\infty \sigma 2\pi r \,\mathrm{d}r = -qh \int_0^\infty \frac{r\,\mathrm{d}r}{(r^2 + h^2)^{3/2}} = -q.$$

Figure 3: Point Charge and Wall



A simple *capacitor* consists of two separated conductors carrying charges $\pm Q$.

If the potential difference (voltage) between them is V, then the capacitance is defined by

$$C = \frac{Q}{V},$$

and depends only on the geometry, because Φ depends linearly on Q.

Example 2.2.

Consider two infinite parallel plates separated by d. Let the plate surfaces be at $z = 0$, $z = d$, and have surface charge densities $\pm\sigma$. Then, $\mathbf{E} = E\mathbf{e}_z$ with $E = \sigma/\epsilon_0$ constant for $0 < z < d$.

Then $\Phi = -Ez + \text{constant}$ and $V = Ed$.

The same solution holds approximately for parallel plates of area $A \gg d^2$ if end-effects are neglected. So,

$$C = \frac{Q}{V} \approx \frac{\sigma A}{Ed} \approx \frac{\epsilon_0 A}{d}.$$

The electrostatic energy stored in the capacitor is

$$U = \int \frac{\epsilon_0 |\mathbf{E}|^2}{2}\,\mathrm{d}V \approx \frac{\epsilon_0 E^2}{2} A d \approx \frac{1}{2} C V^2.$$

In general,

$$U = \frac{1}{2}CV^2 = \frac{Q^2}{2C}.$$

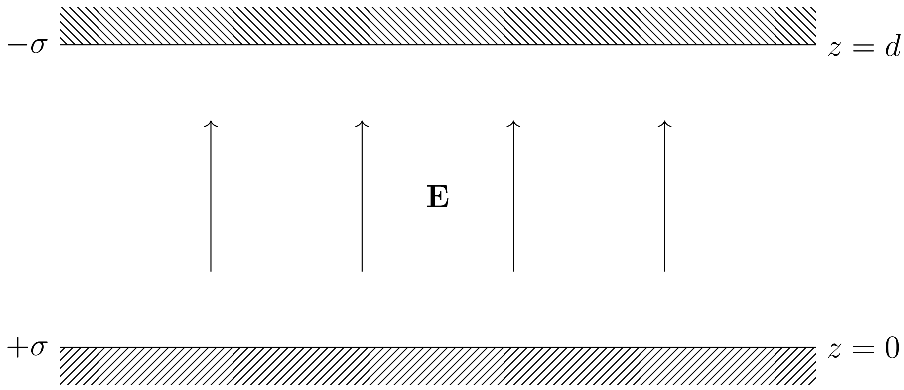
The work done in moving an element of charge δQ from one plate to another is $\delta W = V\delta Q$. So the total work done is

$$\int_0^Q \frac{Q'}{C}\,\mathrm{d}Q' = \frac{Q^2}{2C}.$$

Or we can use

$$U = \frac{1}{2}\int \rho\Phi\,\mathrm{d}V = \frac{1}{2}Q\Phi_+ - \frac{1}{2}Q\Phi_- = \frac{1}{2}QV.$$

Figure 4: Capacitors



3 Magnetostatics

Magnetostatics is the study of the magnetic field generated by a stationary current distribution:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{M4'}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{M2}$$

From (M4'), we get $\nabla \cdot \mathbf{J} = 0$, the time-independent equation of charge conservation.

3.1 Ampère's Law

Consider a closed curve C that is the boundary of an open surface S. Integrate (M4') over S and applying Stokes' theorem, we obtain *Ampère's law*

$$\int_C \mathbf{B} \cdot \mathrm{d}\mathbf{x} = \mu_0 I,$$

where

$$I = \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}$$

is the total current through S.

Since $\nabla \cdot \mathbf{J} = 0$, the same current I flows through any open surface S such that $\partial S = C$.

Ampère's law is the integral version of (M4') and is valid provided $\mathbf{E}$ is constant through time. In words, it says:

The circulation of magnetic field around a loop is proportional to the total current through the loop.

In special situations, we can use Ampère's law, together with symmetry to deduce $\mathbf{B}$ from $\mathbf{J}$.

A cylindrically symmetric situation could involve:

- An axial current distribution $J_z(r)\mathbf{e}_z$,
- An azimuthal current distribution $J_\phi(r)\mathbf{e}_\phi$,

or a combination. Since $\nabla \cdot \mathbf{J} = 0$, we have no radial component.

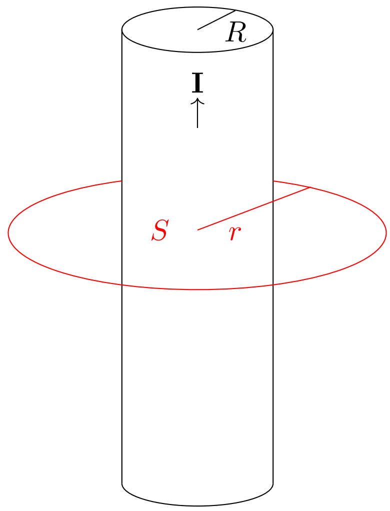
The same applies to $\mathbf{B}$. Hence the curl in Maxwell's equations implies B_ϕ is linearly related to J_z, and B_z is linearly related to J_ϕ.

3.1.1 Long Straight Wire

A cylindrical wire of radius R carries a total current I parallel to its axis.

To find $B_\phi(r)$ generated by $J_z(r)$, we apply Ampère's law to a circle C of radius r. Here S is a disc.

Figure 5: Long Straight Wire



If $r > R$, then

$$\begin{aligned}\int_C \mathbf{B} \cdot \mathrm{d}\mathbf{x} + B_\phi(r) \int_C \mathbf{e}_\phi \cdot \mathrm{d}\mathbf{x} &= B_\phi(r) \int_C \mathrm{d}\ell \\ &= B_\phi(r) 2\pi r = \mu_0 I.\end{aligned}$$

Therefore, outside the wire,

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r} \mathbf{e}_\phi.$$

3.1.2 Solenoid

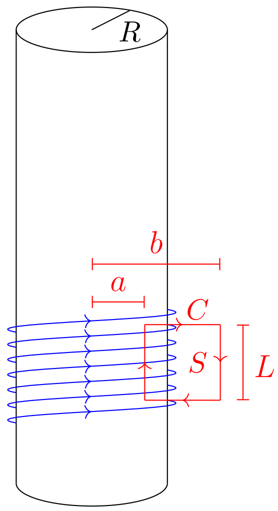
A thin wire is coiled arund a cylindrical tube of radius R. An *ideal solenoid* is infinitely long and tightly wound, having cylindrical geometry and purely azimuthal current.

The wire carries current I and has N turns per unit length of the tube.

To find $B_z(r)$ generated by $J_\phi(r)$, we apply Ampère's law to a rectangular loop C. Taking $a < b < R$ or $R < a < b$ gives

$$L(B_z(a) - B_z(b)) = 0.$$

Figure 6: Solenoid



Taking $a < R < b$ gives

$$L(B_z(a) - B_z(b)) = \mu_0 NLI.$$

Assuming that $B_z(r) \to 0$ as $r \to \infty$, we deduce that

$$B_z(r) = \begin{cases} \mu_0 NI & r < R, \\ 0 & r > R. \end{cases}$$

The ideal solenoid is an example of a *surface current*. Here it is of the form

$$J_\phi(r) = K_\phi \delta(r - R), .$$

where $K_\phi = NI$. Generally, a *surface current density* $\mathbf{K}$ produces a discontinuity in the tangential magnetic field:

$$[\mathbf{n} \times \mathbf{B}] = \mu_0 \mathbf{K}.$$

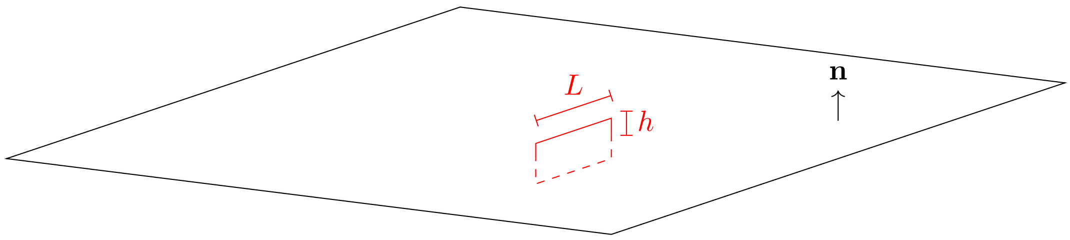
This follows from Ampere's law applied to a loop through a surface, where we take $L, h \to 0$.

Applying the same reasoning with (M2), we get

$$[\mathbf{n} \cdot \mathbf{B}] = 0,$$

so the normal component is continuous.

Figure 7: Surface Current



3.2 Magnetic Vector Potential

(M2) implies that $\mathbf{B}$ can be written in terms of a *magnetic vector potential* $\mathbf{A}(\mathbf{x})$:

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

$\mathbf{A}$ is not unique. If we make a *gauge transformation*, replacing $\mathbf{A}$ with

$$\mathbf{A}' = \mathbf{A} + \nabla\chi,$$

where $\chi(\mathbf{x})$ is an arbitrary scalar field, then $\mathbf{B}$ is unchanged, as

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times \mathbf{A}'.$$

A convenient gauge for many calculation is the *Coulomb gauge* in which $\nabla \cdot \mathbf{A} = 0$.

We can assume this condition without loss of generality. If $\nabla \cdot \mathbf{A} \neq 0$, then we can make a gauge transformation $\nabla \cdot \mathbf{A}' = 0$ by choosing χ to be the solution of Poisson's equation

$$-\nabla^2\chi = \nabla \cdot \mathbf{A}.$$

In terms of $\mathbf{A}$, (M4') becomes

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0\mathbf{J}.$$

Using the identity

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A},$$

and assuming a Coulomb gauge, we obtain Poisson's equation in vector form:

$$-\nabla^2\mathbf{A} = \mu_0\mathbf{J}.$$

3.3 The Biot-Savart Law

The solution of Poisson's equation is

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\,\mathrm{d}^3\mathbf{x}'.$$

We should check that the solution satisfies the assumed Coulomb gauge condition:

$$\begin{aligned}\nabla\cdot\mathbf{A}(\mathbf{x}) &= \frac{\mu_0}{4\pi}\int_V \nabla\cdot\left(\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right)\mathrm{d}^3\mathbf{x}'\\ &= \frac{\mu_0}{4\pi}\int_V \mathbf{J}(\mathbf{x}')\cdot\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\mathrm{d}^3\mathbf{x}'\\ &= -\frac{\mu_0}{4\pi}\int_V \mathbf{J}(\mathbf{x}')\cdot\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\mathrm{d}^3\mathbf{x}'\\ &= -\frac{\mu_0}{4\pi}\int_V \nabla'\cdot\left(\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right)\mathrm{d}^3\mathbf{x}'\\ &= -\frac{\mu_0}{4\pi}\int_{\partial V}\frac{\mathbf{J}(\mathbf{x}')\cdot\mathrm{d}\mathbf{S}'}{|\mathbf{x}-\mathbf{x}'|}.\end{aligned}$$

This is 0, as assumed, if the current is contained in some finite volume and we take V to be at least as large, or if $\mathbf{J}$ decays sufficiently as $|\mathbf{x}|\to\infty$.

To find the magnetic field, derive $\mathbf{B}=\nabla\times\mathbf{A}$ to get

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\,\mathrm{d}^3\mathbf{x}'.$$

This is the *Biot-Savart law*, giving the magnetic field generated by a stationary current distribution.

A special case is when the current is restricted to a thin wire in the form of a curve C. Then the courrent element $\mathbf{J}\,\mathrm{d}^3\mathbf{x}$ can be replaced by $I\,\mathrm{d}\mathbf{x}$. Charge conservation means that I is constant along the wire, so

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi}\int_C\frac{\mathrm{d}\mathbf{x}'\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}.$$

Another way to derive this is using delta functions. The thin wire current density can be represented as

$$\mathbf{J}(\mathbf{x}) = I\int_C\delta(\mathbf{x}-\mathbf{x}')\,\mathrm{d}\mathbf{x}'.$$

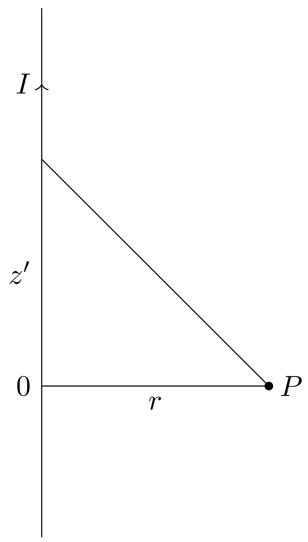
Substituting this into the Biot-Savart law, gives the same result. Note that charge conservation takes the form

$$\begin{aligned}\nabla\cdot\mathbf{J}(\mathbf{x}) &= I\int_C\nabla\delta(\mathbf{x}-\mathbf{x}')\cdot\mathrm{d}\mathbf{x}'\\ &= -I\int_C\nabla'\delta(\mathbf{x}-\mathbf{x}')\cdot\mathrm{d}\mathbf{x}'\\ &= -I[\delta(\mathbf{x}-\mathbf{x}_2)-\delta(\mathbf{x}-\mathbf{x}_1)],\end{aligned}$$

where C runs from $\mathbf{x}_1$ to $\mathbf{x}_2$. If C is closed then $\mathbf{x}_2 = \mathbf{x}_1$, and $\nabla \cdot \mathbf{J} = 0$ as expected. If C is infinite, then $\nabla \cdot \mathbf{J} = 0$ for any finite $\mathbf{x}$.

We can check that the Biot-Savart law gives the same result as Ampère's law for a long straight thin wire:

Figure 8: Thin Wire Magnetic Field



We have $\mathbf{x} = r\mathbf{e}_r$, taking $z = 0$ by translation symmetry, and $\mathbf{x}' = z'\mathbf{e}_z$. Hence $\mathbf{x} - \mathbf{x}' = r\mathbf{e}_r - z'\mathbf{e}_z$, and $\mathrm{d}\mathbf{x}' = \mathrm{d}z'\mathbf{e}_z$, giving

$$\begin{aligned}
\mathbf{B}(\mathbf{x}) &= \frac{\mu_0 I}{4\pi}\mathbf{e}_\phi \int_{-\infty}^{\infty} \frac{r\,\mathrm{d}z'}{(r^2 + z'^2)^{3/2}} \\
&= \frac{\mu_0 I}{4\pi}\mathbf{e}_\phi \left[\frac{z'}{r(r^2 + z'^2)^{1/2}}\right]_{-\infty}^{\infty} \\
&= \frac{\mu_0 I}{2\pi r}\mathbf{e}_\phi.
\end{aligned}$$

3.4 Magnetic Dipole

For a general current distribution $\mathbf{J}(\mathbf{x})$ confined to a ball $\{V \mid |\mathbf{x}| < R\}$,

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\,\mathrm{d}^3\mathbf{x}'.$$

The external field for $|\mathbf{x}| = r > R$ can be evaluated by expanding

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{r}\left(1 + \frac{\mathbf{x}' \cdot \mathbf{x}}{r^2} + \mathcal{O}\left(\frac{R^2}{r^2}\right)\right),$$

leading to a multipole expansion, as for the electric field. To do this, we need to calculate the moments of the current distribution.

Since $\mathbf{J} = \mathbf{0}$ on ∂V and $\nabla \cdot \mathbf{J} = 0$, the divergence theorem implies

$$\begin{aligned} 0 &= \int_{\partial V} x_i J_j \, \mathrm{d}S_j = \int_V \partial_j (x_i J_j) \, \mathrm{d}3\mathbf{x} \\ &= \int_V (\delta_{ij} J_j + x_j \partial_j J_j) \, \mathrm{d}^3\mathbf{x} \\ &= \int_V J_i \, \mathrm{d}^3\mathbf{x}. \end{aligned}$$

So the zeroth moment vanishes. Similarly,

$$\begin{aligned} 0 &= \int_{\partial V} x_i x_j J_k \, \mathrm{d}S_k = \int_V \partial_k (x_i x_j J_k) \, \mathrm{d}^3\mathbf{x} \\ &= \int_V (\delta_{ik} x_j J_k + x_j \delta_{jk} J_k + x_i x_j \partial_k J_k) \, \mathrm{d}^3\mathbf{x} \\ &= \int_V x_j J_i \, \mathrm{d}^3\mathbf{x} + \int_V x_i J_j \, \mathrm{d}^3\mathbf{x}. \end{aligned}$$

The first moment is an antisymmetric matrix. The *magnetic dipole moment* is

$$\mathbf{m} = \frac{1}{2} \int_V \mathbf{x} \times \mathbf{J} \, \mathrm{d}^3\mathbf{x},$$

so

$$m_i = \frac{1}{2} \epsilon_{ijk} \int_V x_j J_k \, \mathrm{d}^3\mathbf{x}.$$

This is a vector related to the antisymmetric matrix by

$$\int_V x_i J_j \, \mathrm{d}^3\mathbf{x} = \epsilon_{ijk} m_k.$$

Returning to the multipole expansion for $\mathbf{A}$, we have

$$\begin{aligned} A_i(\mathbf{x}) &= \frac{\mu_0}{4\pi|\mathbf{x}|} \left(\int_V J_i(\mathbf{x}') \, \mathrm{d}^3\mathbf{x}' + \frac{x_j}{|\mathbf{x}|^3} \int_V x'_j J_i(\mathbf{x}') \, \mathrm{d}^3\mathbf{x}' + \cdots \right) \\ &= \frac{\mu_0}{4\pi|\mathbf{x}|} \left(0 + \frac{x_j \epsilon_{jik} m_k}{|\mathbf{x}|^3} + \cdot \right). \end{aligned}$$

The leading approximation is therefore

$$\mathbf{A}(\mathbf{x}) = \mathbf{A}_{\mathrm{dipole}}(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{x}}{|\mathbf{x}|^3}.$$

which is the vector potential due to a point dipole $\mathbf{m}$ at the origin. The corresponding magnetic field is

$$\mathbf{B}_{\text{dipole}} = \nabla \times \mathbf{A}_{\text{dipole}} = \frac{\mu_0}{4\pi}\left(\frac{3(\mathbf{x}\cdot\mathbf{x})\mathbf{x} - |\mathbf{x}|^3\mathbf{m}}{|\mathbf{x}|^5}\right).$$

A point dipole $\mathbf{m}$ at the origin corresponds to the current density and vector potential

$$\mathbf{J} = \nabla \times (\mathbf{m}\delta(\mathbf{x})), \qquad \mathbf{A} = \nabla \times \left(\frac{\mu_0 \mathbf{m}}{4\pi|\mathbf{x}|}\right).$$

The magnetic dipole moment of a thin wire carrying current I around a closed curve C is

$$\mathbf{m} = \frac{I}{2}\int_C \mathbf{x} \times \mathrm{d}\mathbf{x}.$$

To evaluate this, let $\mathbf{a}$ be any constant vector. Then by Stokes' theorem,

$$\begin{aligned}\mathbf{a}\cdot\mathbf{m} &= \frac{I}{2}\int_C \mathbf{a}\cdot(\mathbf{x}\cdot\mathrm{d}\mathbf{x}) = \frac{I}{2}\int_C (\mathbf{a}\times\mathbf{x})\cdot\mathrm{d}\mathbf{x}\\ &= \frac{I}{2}\int_S (\nabla\times(\mathbf{a}\times\mathbf{x}))\cdot\mathrm{d}\mathbf{S} = I\int_S \mathbf{a}\cdot\mathrm{d}\mathbf{S},\end{aligned}$$

where S is an open surface with boundary C, and we use

$$\begin{aligned}\nabla\times(\mathbf{a}\times\mathbf{x}) &= \mathbf{x}\cdot\nabla\mathbf{a} - \mathbf{a}\cdot\nabla\mathbf{x} + (\nabla\times\mathbf{x})\mathbf{a} - (\nabla\times\mathbf{a})\mathbf{x}\\ &= \mathbf{0} - \mathbf{a} + 3\mathbf{a} - \mathbf{0} = 2\mathbf{a}.\end{aligned}$$

Since $\mathbf{a}$ is arbitrary, we obtain

$$\mathbf{m} = I\mathbf{S},$$

where

$$\mathbf{S} = \int_S \mathrm{d}\mathbf{S}$$

is the vector area of S, which depends only on C, not on the choice of S.

Example 3.1.

Consider a circular loop with $x^2 + y^2 = a^2$, $\tau = 0$, for which $\mathbf{m} = I\pi a^2\mathbf{e}_z$.

On the z-axis, the dipole approximation gives

$$B_z = \frac{\mu_0}{4\pi}\left(\frac{3m_z z^2 - z^2 m_z}{|z|^5}\right) = \frac{\mu_0 I a^2}{2|z|^3}.$$

The exact solution is

$$B_z = \frac{\mu I a^2}{2(z^2 + a^2)^{3/2}}.$$

3.5 Permanent Magnets

A bar magnet has north and south poles and a dipole moment. This comes from the superposition of aligned dipoles on the atomic scale. Atoms contain electrons, which are spinning charged particles, with magnetic dipole moment.

A classical model of a particle is a spinning charged sphere, which is a current loop with a magnetic dipole moment proportional to its charge and spin.

As far as we know, there are no magnetic charges (monopoles).

The Earth may also be viewed as a magnet. The liquid iron outer core of the Earth is a conducting fluid in convective motion and supports electric currents that generate a magnetic field. At the Earth's surface, this resembles a dipole field.

3.6 Magnetic Forces

The *Lorentz force* on a particle of charge q at position $\mathbf{x}_i(t)$ is

$$q(\mathbf{E} + \dot{\mathbf{x}}_i \times \mathbf{B}).$$

In the limit of continuous charge and current densities, the Lorentz force per unit volume is then

$$\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}.$$

We can recover the discrete version by substituting

$$\begin{aligned}
\rho &= \sum_i q_i \delta(\mathbf{x} - \mathbf{x}_i(t)), \\
\mathbf{J} &= \sum_i q - i\dot{\mathbf{x}}_i(t)\delta(\mathbf{x} - \mathbf{x}_i(t)).
\end{aligned}$$

Consider two or more think wires with currents I_i along curves C_i. The total magnetic field $\mathbf{B} = \sum_i \mathbf{B}_i$, where

$$\mathbf{B}_i(\mathbf{x}) = \frac{\mu_0 I_i}{4\pi} \int_{C_i} \frac{\mathrm{d}\mathbf{x}_i \times (\mathbf{x} - \mathbf{x}_i)}{|\mathbf{x} - \mathbf{x}_i|^3}$$

is the magnetic field due to wire i. The current density is $\mathbf{J} = \sum_i \mathbf{J}_i$, where

$$\mathbf{J}_i(\mathbf{x}) = I_i \int_{C_i} \delta(\mathbf{x} - \mathbf{x}_i)\,\mathrm{d}\mathbf{x}_i.$$

The total magnetic force acting on a volume V is

$$\mathbf{F} = \int_V \mathbf{J} \times \mathbf{B}\,\mathrm{d}V.$$

The force acting on wire i is

$$\mathbf{F} - i = \int \mathbf{J}_i(\mathbf{x}) \times \mathbf{B}(\mathbf{x})\,\mathrm{d}^3\mathbf{x} = I_i \int_{C_i} \mathrm{d}\mathbf{x}_i \times \mathbf{B}(\mathbf{x}_i).$$

Since $\mathbf{B} = \sum_i \mathbf{B}_i$, we have

$$\mathbf{F}_i = \sum_j \mathbf{F}_{ij},$$

where

$$\mathbf{F}_{ij} = I_i \int_{C_i} \mathrm{d}\mathbf{x}_i \times \mathbf{B}_j(\mathbf{x}_i)$$

is the force on wire i due to wire j. Using the Biot-Savart law,

$$\mathbf{F}_{ij} = \frac{\mu_0 I_i I_j}{4\pi} \int_{C_i} \int_{C_j} \mathrm{d}\mathbf{x}_i \times \left(\frac{\mathrm{d}\mathbf{x}_j \times (\mathbf{x}_i - \mathbf{x}_j)}{|\mathbf{x}_i - \mathbf{x}_j|^3} \right).$$

This can be rewritten in a manifestly antisymmetric way that shows that

$$\mathbf{F}_{ji} = -\mathbf{F}_{ij},$$

as expected from Newton's third law. The self force $\mathbf{F}_{ii}$ vanishes, although the thin-wire integral is singular, and it is better to treat the case of thick wires.

Consider two infinitely long, parallel, thin wires separated by a distance r. Use cylindrical polars centred on wire two, we have

$$\mathbf{B}_2 = \frac{\mu_0 I_2}{2\pi r}\mathbf{e}_\phi, \qquad \mathbf{F}_{12} = I_u \int_{-\infty}^{\infty} \mathrm{d}z \mathbf{e}_2 \times \mathbf{B}_2.$$

The total force is infinite. The force per unit length is

$$I\mathbf{e}_z \times \mathbf{B}_z = -\frac{\mu_0 I_1 I_2}{2\pi r}\mathbf{e}_r.$$

This is directed towards wire two if $I_1 I_2 > 0$. So the force is attractive if the currents are aligned, and repulsive otherwise.

3.7 Force and Torque on a Magnetic Dipole

Consider a localized current distribution confined to a ball $\{V \mid |\mathbf{x}| < R\}$. Place this in an external magnetic field $\mathbf{B}(\mathbf{x})$ that varies slowly over the length scale R.

The magnetic torque (about the origin) on the current loop is

$$\begin{aligned}\boldsymbol{\tau} &= \int_V \mathbf{x} \times (\mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x}))\, \mathrm{d}^3\mathbf{x} \\ &= \int_V ((\mathbf{x} \cdot \mathbf{B}(\mathbf{x}))\mathbf{J}(\mathbf{x}) - (\mathbf{x} \cdot \mathbf{J}(\mathbf{x}))\mathbf{B}(\mathbf{x}))\, \mathrm{d}^3\mathbf{x}.\end{aligned}$$

Within V, $\mathbf{B}(\mathbf{x})$ can be expressed as a Taylor series

$$B_i(\mathbf{x}) = B_i(\mathbf{0}) + x_j \partial_j B_i(\mathbf{0}) + \cdots$$

Retaining only the zeroth-order term, we have

$$\tau_i \approx B_j(\mathbf{0}) \int_V x_j J_i \, \mathrm{d}^3\mathbf{x} - B_i(\mathbf{0}) \int_V x_j J_j \, \mathrm{d}^3\mathbf{x}.$$

Recall the first moments of the current distribution

$$\int_V x_i J_j \, \mathrm{d}^3\mathbf{x} = \epsilon_{ijk} m_k.$$

Thus $\tau_i \approx B_j(\mathbf{0})\epsilon_{jik} m_k$. In general,

$$\boldsymbol{\tau} \approx \mathbf{m} \times \mathbf{B}.$$

For the force, we need to go to the first order of the Taylor expansion of $\mathbf{B}$:

$$\begin{aligned}\mathbf{F} &= \int_V \mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x})\, \mathrm{d}^3\mathbf{x}, \\ F_i &\approx \int_V \epsilon_{ijk} J_j(\mathbf{x})(B_k(\mathbf{0}) + x_l \partial_l B_k(\mathbf{0}))\, \mathrm{d}^3\mathbf{x} \\ &= \epsilon_{ijk} B_k(\mathbf{0}) \int_V J_j \, \mathrm{d}^3\mathbf{x} + \epsilon_{ijk}\partial_l B_k(\mathbf{0}) \int_V x_l J_j \, \mathrm{d}^3\mathbf{x} \\ &= 0 + \epsilon_{ijk}\partial_l B_k(\mathbf{0})\epsilon_{ljn} m_n \\ &= \partial_i B_k(\mathbf{0}) m_k - \partial_k B_k(\mathbf{0}) m_i \\ &= \partial_i (m_k B_k)(\mathbf{0}),\end{aligned}$$

since $\nabla \times \mathbf{B} = 0$. In general, $\mathbf{F} \approx \nabla(\mathbf{m} \cdot \mathbf{B})$. This can also be written as $\mathbf{F} = -\nabla U$, where $U = -\mathbf{m} \cdot \mathbf{B}$ is the potential energy of a magnetic dipole in an external field.

As in the electric case, this in minimized when $\mathbf{m}$ is aligned with $\mathbf{B}$.

4 Electrodynamics

4.1 Faraday's Law of Induction

Maxwell's third equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{M3}$$

implies that a time-dependent magnetic field must be accompanied by an electric field. This can induce a current to flow in a conductor - a process known as *electromagnetic induction*.

Consider a closed curve C that is the boundary of a time-independent open surface S. Integrating (M3) over S and using Stokes' theorem,

$$\int_C \mathbf{E} \cdot \mathrm{d}\mathbf{x} = -\int_S \frac{\partial B}{\partial t} \cdot \mathrm{d}\mathbf{S} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S}.$$

This is *Faraday's law of induction* for a static current:

$$\mathcal{E} = -\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t},$$

where

$$\mathcal{E} = \int_C \mathbf{E} \cdot \mathrm{d}\mathbf{x}$$

is the *electromotive force* (or emf) around C, and

$$\mathcal{F} = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S}$$

is the *magnetic flux* through S.

Since $\nabla \cdot \mathbf{B} = 0$, the flux $\mathcal{F}$ is the same for any S such that $\partial S = C$, so it can be regarded as the magnetic flux through C.

Using $\mathbf{B} = \nabla \times \mathbf{A}$ and Stokes' theorem, we can write the magnetic flux as

$$\mathcal{F} = \int_C \mathbf{A} \cdot \mathrm{d}\mathbf{x},$$

which is invariant under a gauge transformation

$$\mathbf{A}' = \mathbf{A} + \nabla\chi.$$

The electromotive force is not actually a force; it is the line integral of the Lorentz force on a particle of unit charge confined to C:

$$\mathcal{E} = \frac{1}{q}\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_C (\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{x} = \int_C \mathbf{E} \cdot \mathrm{d}\mathbf{x},$$

since $\dot{\mathbf{x}}$ is tangent to C for a particle confined to a time-independent curve C.

We will see later that if C coincides with a thin wire of resistance R, then the current induced in the wire is $I = \mathcal{E}/R$.

There are several ways in which the magnetic flux through C could change in time:

- a magnet is moved near C.
- a current-carrying circuit is moved near C.
- the current in a nearby circuit is changed.

All these will induce an electromotive force around C and cause a current to flow.

Moreover, we can also generalize Faraday's law for a moving circuit. Let $C(t)$ be a time-dependent closed curve that is the boundary of an open surface $S(t)$. We want to look at how the magnetic flux through S,

$$\mathcal{F} = \int_S \mathcal{B} \cdot \mathrm{d}\mathbf{S}$$

changes through time. We have

$$\begin{aligned}
\mathcal{F}(t+\delta t) - \mathcal{F}(t) &= \int_{S(t+\delta t)} \mathbf{B}(\mathbf{x}, t+\delta t) \cdot \mathrm{d}\mathbf{S} - \int_{S(t)} \mathcal{B}(\mathbf{x}, t) \cdot \mathrm{d}\mathbf{S} \\
&= \int_{S(t+\delta t)} \left(\mathbf{B}(\mathbf{x}, t) + \frac{\partial \mathbf{B}}{\partial t} \delta t + \mathcal{O}(\delta t^2) \right) \cdot \mathrm{d}\mathbf{S} - \int_{S(t)} \mathbf{B}(\mathbf{x}, t) \cdot \mathrm{d}\mathbf{S} \\
&= \int_{S(t+\delta t) - S(t)} \mathbf{B}(\mathbf{x}, t) \cdot \mathrm{d}\mathbf{S} + \int_{S(t)} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S}\, \delta t + \mathcal{O}(\delta t^2).
\end{aligned}$$

Let δV be the volume swept out by $S(t)$ in the time interval δt. Its boundary is the closed surface $S(t+\delta t) - S(t) + \Sigma$, where Σ is the surface swept out by $C(t)$ in time δt.

By (M2) and the divergence theorem,

$$0 = \int_{\partial V} (\nabla \cdot \mathbf{B})\, \mathrm{d}V = \int_{S(t+\delta t) - S(t)} \mathbf{B} \cdot \mathrm{d}\mathbf{S} + \int_{\Sigma} \mathbf{B} \cdot \mathrm{d}\mathbf{S}.$$

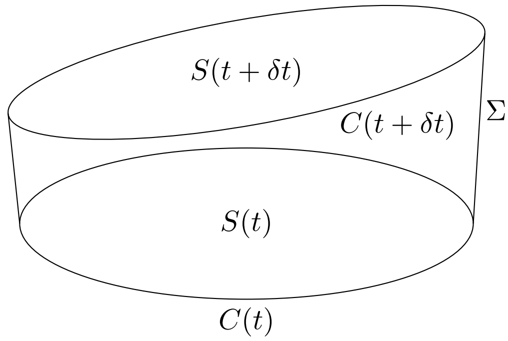
To evaluate the last term, parametrize C as $\mathbf{x} = \mathbf{x}(\lambda, t)$, where λ is a parameter around C an element of C is

$$\mathrm{d}\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \lambda}\, \mathrm{d}\lambda,$$

and has velocity

$$\mathbf{v} = \frac{\partial \mathbf{x}}{\partial t}.$$

Figure 9: Change in Magnetic Flux



In time δt, it sweeps out the vector area element

$$\mathrm{d}\mathbf{S} = \mathrm{d}\mathbf{x} \times (\mathbf{v}\delta t).$$

Thus, we get

$$\int_\Sigma \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \int_C \mathbf{B} \cdot (\mathrm{d}\mathbf{x} \times \mathbf{v})\delta t + \mathcal{O}(\delta t^2) = \int_C (\mathbf{v} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{x}\,\delta t + \mathcal{O}(\delta t^2).$$

Hence we get

$$\mathcal{F}(t+\delta t) - \mathcal{F}(t) = -\int_C (\mathbf{v} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{x}\,\delta t + \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S}\,\delta t + \mathcal{O}(\delta t^2).$$

This gives the first derivative

$$\begin{aligned}\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} &= -\int_C (\mathbf{v} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{x} + \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S} \\ &= -\int_C (\mathbf{v} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{x} - \int_S (\nabla \times \mathbf{E}) \cdot \mathrm{d}\mathbf{S} \\ &= -\int_C (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{x}.\end{aligned}$$

We recover Faraday's law

$$\mathcal{E} = -\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t},$$

with the redefined electromotive force

$$\mathcal{E} = \int_C (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{x}.$$

This $\mathcal{E}$ is again the line integral around C of the Lorentz force on a particle of unit charge confined to C (for which the perpendicular components of $\dot{\mathbf{x}}$ must agree with those of the curve velocity $\mathbf{v}$).

4.1.1 Lenz's Law

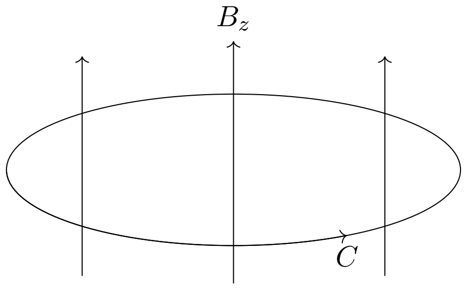
Lenz's law says that the direction of the induced current is always such as to produce a magnetic field that opposes the change in flux that cause the emf.

Example 4.1.

Consider a circular wire in the xy-plane. If B_z inside the loop increases in time, then $\mathcal{E} = -\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} < 0$. This induces a clockwise current ($I < 0$), that generates a magnetic field with $B_z < 0$ inside the loop.

Hence the minus sign in Faraday's law. This avoids an unstable situation in which the flux grows indefinitely.

Figure 10: Lenz's Law



4.1.2 Inductance and Magnetic Energy

If a current I around a circuit C generates a magnetic field with flux $\mathcal{F}$, then the *inductance* of the circuit is defined by

$$L = \frac{\mathcal{F}}{I},$$

and depends only on the geometry of the circuit.

Example 4.2.

Consider an ideal solenoid with cross-sectional area A and N turns per unit length. The uniform field $B = \mu_0 N I$ inside the solenoid produces a flux BA per turn, so the inductance per unit length of the solenoid is $\mu_0 N^2 A$.

It can be shown that the magnetic flux through a thin wire C_i due to a current I_j around another thin wire C_j is $\mathscr{F}_{ij} = L_{ij} I_j$, where the *mutual inductance* is

$$L_{ij} = \frac{\mu_0}{4\pi} \int_{C_i} \int_{C_j} \frac{\mathrm{d}\mathbf{x}_i \cdot \mathrm{d}\mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|} = L_{ji}.$$

When the current I around a circuit C is varied, an emf

$$\mathcal{E} = -\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} = -L\frac{\mathrm{d}I}{\mathrm{d}t}$$

is induced. In a small time interval δt, a charge $\delta Q = I\delta t$ flows around C and the work done on it by the Lorentz force is

$$\delta W = \mathcal{E}\delta Q = -LI\frac{\mathrm{d}I}{\mathrm{d}t}\delta t.$$

So the rate at which work is done by the current on the electromagnetic field is

$$-\frac{\mathrm{d}W}{\mathrm{d}t} = LI\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}LI^2\right).$$

Consider reaching a magnetostatic state by building up the current from 0 to I. The energy stored is

$$\begin{aligned} U &= \frac{1}{2}LI^2 = \frac{1}{2}I\mathcal{F} = \frac{1}{2}I\int_C \mathbf{A}\cdot\mathrm{d}\mathbf{x} \\ &= \frac{1}{2}\int \mathbf{J}\cdot\mathbf{A}\,\mathrm{d}V, \end{aligned}$$

analogous to

$$U = \frac{1}{2}\int \rho\Phi\,\mathrm{d}V$$

that appears in electrostatics.

Now, using (M4'), we have

$$U = \frac{1}{2\mu_0}\int (\nabla\times\mathbf{B})\cdot\mathbf{A}\,\mathrm{d}V,$$

and since $(\nabla\times\mathbf{B})\cdot\mathbf{A} = \nabla\cdot(\mathbf{B}\times\mathbf{A}) - \mathbf{B}\cdot(\nabla\times\mathbf{A})$, if we take the integral over all space, then the first term gives zero by the divergence theorem, as

$$|\mathbf{B}| = \mathcal{O}\left(\frac{1}{|\mathbf{x}|^3}\right), \qquad |\mathbf{A}| = \mathcal{O}\left(\frac{1}{|\mathbf{x}|^2}\right),$$

as $|\mathbf{x}|\to\infty$ for a finite current distribution, leaving

$$U = \int \frac{|\mathbf{B}|^2}{2\mu_0}\,\mathrm{d}V$$

as the energy stored in the magnetic field.

4.2 Ohm's Law

In a stationary conductor,

$$\mathbf{J} = \sigma \mathbf{E},$$

where σ is the *electrical conductivity*. This is not a fundamental physical law, but a constitutive relation, a macroscopic property of a material.

The inverse relation gives

$$\mathbf{E} = \sigma^{-1} \mathbf{J},$$

where σ^{-1} is the *resistivity*. It is usually denoted as ρ, but both σ and ρ conflict with notation for charge densities.

A *perfect conductor* corresponds to the limit $\sigma \to \infty$, so ($\mathbf{E} = 0$), and a *perfect insulator* to $\sigma \to 0$ (so $\mathbf{J} = 0$).

Example 4.3.

Consider a straight wire of length L in the direction of the unit vector $\mathbf{n}$, and with uniform cross-sectional area A and conductivity σ. If the electric field is $\mathbf{E} = E\mathbf{n}$, where E is constant, then $\mathbf{J} = \sigma E \mathbf{n}$, and the total current is $I = \sigma E A$.

The potential difference (voltage) along the wire is

$$V = \int \mathbf{E} \cdot \mathrm{d}\mathbf{x} = EL = \frac{IL}{\sigma A} = IR,$$

where $R = \frac{L}{\sigma A}$ is the resistance of the wire.

Accompanying the resistance of a wire is *Joule heating* (or *Ohmic heating*), conversion of electromagnetic energy into heat at the rate $I^2 R$.

If the voltage V is maintained by a battery, then $VI = I^2 R$ is the rate at which the emf of the battery ($\mathcal{E} = V$) does work to maintain the current I.

4.3 Time-dependent Electric Fields

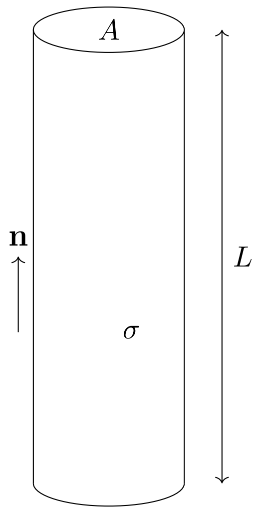
Due to time dependence, in electrodynamics we can no longer write $\mathbf{E} = -\nabla\Phi$. But (M2) still allows us to write

$$\mathbf{B} = \nabla \times \mathbf{A},$$

and using (M3) then gives

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0.$$

Figure 11: Ohm's Law in a Wire



This allows us to write

$$\mathbf{E} = -\nabla\Phi - \frac{\partial A}{\partial t},$$

generalizing the electrostatic expression.

Under a time-dependent gauge transformation

$$\mathbf{A}' = \mathbf{A} + \nabla\chi, \qquad \Phi' = \Phi - \frac{\partial \chi}{\partial t},$$

where $\chi(\mathbf{x}, t)$ is any scalar field, then both $\mathbf{E}$ and $\mathbf{B}$ are unchanged.

4.3.1 The Displacement Current

In magnetostatics we used Ampere's law

$$\int_C \mathbf{B} \cdot \mathrm{d}\mathbf{x} = \mu_0 \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = \mu_0 I,$$

or its differential form (M4')

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

For time-dependent situation, Maxwell's fourth equation,

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right), \tag{M4}$$

contains an extra term, the *displacement current*.

This is needed, otherwise we would have $\nabla \times \mathbf{J} = 0$, which describes charge conservation in a situation where ρ is constrained to remain constant.

But suppose we place free particles of positive charge in some localized region. Repulsive coulomb forces cause the particles to separate, implying $\nabla \times \mathbf{J} > 0$.

We have seen that the correct form for charge conservation is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

This follows from Maxwell's equations, including the displacement current.

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