IB Linear Algebra

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Based on Lectures by Prof. Pierre Raphael

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Contents

1 Vector Spaces and Subspaces

Let F be an arbitrary field.

Definition 1.1 (F vector space)**.** A F vector space is an abelian group $(V, +)$ equipped with a function

$$\begin{aligned} F \times V &\to V \\ (\lambda, v) &\mapsto \lambda v \end{aligned}$$

such that

- $\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$,
- $(\lambda_1 + \lambda_2)v = \lambda_1 v + \lambda_2 v$,
- $\lambda(\mu v) = (\lambda\mu)v$,
- $1 \cdot v = v$.

We know how to

- Sum two vectors
- Multiply a vector $v \in V$ by a scalar $\lambda \in F$.

Example 1.1.

(i) Take $n \in \mathbb{N}$, then F^n is the set of column vectors of length n with elements in F. We have

$$v \in F^n, v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, x_i \in F,$$

$$v + w = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix},$$

$$\lambda v = \begin{pmatrix} \lambda v_1 \\ \vdots \\ \lambda v_n \end{pmatrix}.$$

Then F^n is a F vector space.

(ii) For any set X, take
$$\mathbb{R}^X = \{f : X \to \mathbb{R}\}.$$
Then $\mathbb{R}^X$ is an $\mathbb{R}$ vector space.

(iii) Take $M_{n,m}(F)$, the set of $n \times m$ F valued matrices. Then $M_{n,m}(F)$ is a F vector space.

Remark. The axiom of scalar multiplication implies that for all $v \in V$, $0 \cdot v = \mathbf{0}$.

Definition 1.2 (Subspace)**.** Let V be a vector space over F. A subset U of V is a vector subspace of V (denoted $U \leq V$) if

- $0 \in U$,
- $(u_1, u_2) \in U \times U$ implies $u_1 + u_2 \in U$,
- $(\lambda, u) \in F \times U$ implies $\lambda u \in U$.

Note if V is an F vector space, and $U \leq V$, then U is an F vector space.

Example 1.2.

(i) Take $V = \mathbb{R}^\mathbb{R}$, the space of functions $f : \mathbb{R} \to \mathbb{R}$. Let $\mathcal{C}(\mathbb{R})$ be the space of continuous function $f : \mathbb{R} \to \mathbb{R}$. Then $\mathcal{C}(\mathbb{R}) \leq \mathbb{R}^\mathbb{R}$.

(ii) Take the elements of $\mathbb{R}^3$ which sum up to t. This is a subspace if and only if $t = 0$.

Note that the union of two subspaces is generally not a subspace, as it is usually not closed under addition.

Proposition 1.1. *Let V be an F vector space, and $U, W \leq V$. Then $U \cap W \leq V$.*

Proof: Since $0 \in U, 0 \in W$, $0 \in U \cap W$. Now consider $(\lambda, \mu) \in F^2$, and $(v_1, v_2) \in (U \cap W)^2$. Take $\lambda_1 v_1 + \lambda_2 v_2$. Since $u_1, v_1 \in U$, this is in U. Similarly, it is in W. So it is in $U \cap W$, and $U \cap W \leq V$.

Definition 1.3 (Sum of subspaces)**.** Let V be an F vector space. Let $U, W \leq V$. Then the **sum** of U and W is the set

$$U + W = \{u + w \mid (u, w) \in U \times W\}.$$

Proof: Note $0 = 0 + 0 \in U + W$. Take $\lambda_1 f + \lambda_2 g$, where $f, g \in U + W$. Then we can write $f = f_1 + f_2, g = g_1 + g_2$, where $f_1, g_1 \in U$, $f_2, g_2 \in W$. Then

$$\lambda_1 f + \lambda_2 g = \lambda_1(f_1 + f_2) + \lambda_2(g_1 + g_2) = (\lambda_1 f_1 + \lambda_2 g_1) + (\lambda_1 f_2 + \lambda_2 g_2) \in U + W.$$

Remark. $U + W$ is the smallest subspace of V which contains both U and W.

1.1 Subspaces and Quotients

Definition 1.4 (Quotient)**.** Let V be an F vector space. Let $U \leq V$. The quotient space V/U is the abelian group V/U equipped with the scalar product multiplication

$$\begin{aligned} F \times V/U &\to V/U \\ (\lambda, v + U) &\mapsto \lambda v + U \end{aligned}$$

Proposition 1.2. *V/U is an F vector space.*

2 Spans, Linear Independence and the Steinitz Exchange Lemma

Definition 2.1 (Span of a family of vectors)**.** Let V be a F vector space. Let $S \subset B$ be a subset. We define

$$\begin{aligned}\langle S\rangle &= \{\text{finite linear combinations of elements of } S\} \\ &= \left\{\sum_{\delta\in J} \lambda_\delta v_\delta, v_\delta \in S, \lambda_\delta \in F, J \text{ finite}\right\}.\end{aligned}$$

By convention, we let $\langle\emptyset\rangle = \{0\}$.

Remark. $\langle S'\rangle$ is the smallest vector subspace which contains S.

Example 2.1. Take $V = \mathbb{R}^3$, and

$$S = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}, \begin{pmatrix}3\\-2\\4\end{pmatrix}\right\}.$$

Then we have

$$\langle S'\rangle = \left\{\begin{pmatrix}a\\b\\2b\end{pmatrix}, (a,b) \in \mathbb{R}^2\right\}.$$

Take $V = \mathbb{R}^n$, and let e_i be the i'th basis vector. Then $V = \langle e_1, \dots, e_n\rangle$.

Take X a set, and $V = \mathbb{R}^X$. Let $S_x : X \to \mathbb{R}$, such that $y \mapsto 1$ if $x = y$, otherwise $y \mapsto 0$. Then

$$\langle (S_x)_{x\in X}\rangle = \{f \in \mathbb{R}^X \mid f \text{ has finite support}\}.$$

Definition 2.2. Let V be a F vector space. Let S' be a subset of V. We may say that S **spans** V if $\langle S\rangle = V$.

Definition 2.3 (Finite dimension)**.** Let V be a F vector space. We say that V is **finite dimensional** if it is spanned by a finite set.

Example 2.2. Consider $P[x]$, the polynomials over $\mathbb{R}$, and $P_n[x]$, the polynomials over $\mathbb{R}$ with degree $\leq n$. Then since

$$\langle 1, x, \ldots, x^n \rangle = P_n[x],$$

$P_n[x]$ is finite dimensional, however $P[x]$ is not.

Definition 2.4 (Independence)**.** We say that $(v_1, \ldots, v_n)$, elements of V are **linearly independent** if

$$\sum_{i=1}^{n} \lambda_i v_i = 0 \implies \lambda_i = 0 \, \forall i.$$

Remark.

1. We also say that the family $(v_1, \ldots, v_n)$ is **free**.
2. Equivalently, $(v_1, \ldots, v_n)$ are not linearly independent if one of these vectors is a linear combination of the remaining $(n-1)$.
3. If (v_i) is free, then $v_i = 0$ for all i.

Definition 2.5 (Basis)**.** A subset S of V is a **basis** of V if and only if

(i) $\langle S' \rangle = V$,

(ii) S is linearly independent.

Remark. A subset S that generates V is a generating family, so a basis S is a free generating family.

Example 2.3. For $V = \mathbb{R}^n$, then (e_i) is a basis of V.

If $V = \mathbb{C}$, then for $F = \mathbb{C}$, $\{1\}$ is a basis.

If $V = P[x]$, then $S = \{x^n, n \geq 0\}$ is a basis for V.

Lemma 2.1. *V is a F vector space. Then $(v_1, \ldots, v_n)$ is a basis of V if and only if any vector $v \in V$ has a unique decomposition*

$$v = \sum_{i=1}^{n} \lambda_i v_i.$$

Remark. We call $(\lambda_1, \ldots, \lambda_n)$ the coordinates of v in the basis $(v_1, \ldots, v_n)$.

Proof: Since $\langle v_1, \dots, v_n \rangle = V$, we must have

$$v = \sum_{i=1}^{n} \lambda_i v_i$$

for some λ_i. Now assume

$$\begin{aligned} v = \sum_{i=1}^{n} \lambda_i v_i &= \sum_{i=1}^{n} \lambda_i' v_i, \\ \implies \sum_{i=1}^{n} (\lambda_i - \lambda_i') v_i &= 0. \end{aligned}$$

Since v_i are free, $\lambda_i = \lambda_i'$.

Lemma 2.2. *If $(v_1, \dots, v_n)$ spans V, then some subset of this family is a basis of V.*

Proof: If $(v_1, \dots, v_n)$ are linearly independent, we are done. Otherwise assume they are not independent, then by possibly reordering the vectors, we have

$$v_n \in \langle v_1, \dots, v_{n-1} \rangle.$$

Then we have $V = \langle v_1, \dots, v_n \rangle = \langle v_1, \dots, v_{n-1} \rangle$. By iterating, we must eventually get to an independent set.

Theorem 2.1 (Steinitz Exchange Lemma)**.** *Let V be a finite dimensional vector space over F. Take*

(i) $(v_1, \dots, v_m)$ *free,*

(ii) $(w_1, \dots, w_n)$ *generating.*

Then $m \leq n$, and up to reordering, $(v_1, \dots, v_m, w_{m+1}, \dots, w_n)$ spans V.

Proof: Induction. Suppose that we have replaced l of the w_i, reordering if necessary, so

$$\langle v_1, \dots, v_l, w_{l+1}, \dots, w_n \rangle = V.$$

If $m = l$, we are done. Otherwise, $l < m$. Then since these vectors span V, we have

$$v_{l+1} = \sum_{i \leq l} a_i v_i + \sum_{i>l} \beta_i w_i.$$

Since $(v_1, \ldots, v_{l+1})$ is free, some of the β_i are non-zero. Upon reordering, we may let $\beta_{l+1} \neq 0$. Then,

$$w_{l+1} = \frac{1}{\beta_{l+1}} \left[v_{l+1} - \sum_{i \leq l} \alpha_i v_i - \sum_{i>l+1} \beta_i w_i \right].$$

Hence, $V = \langle v_1, \ldots, v_l, w_{l+1}, \ldots, w_n \rangle = \langle v_1, \ldots, v_l, v_{l+1}, w_{l+1}, \ldots, w_n \rangle = \langle v_1, \ldots, v_{l+1}, w_{l+2}, \ldots, w_n \rangle$. Iterating this process, we eventually get $l = m$, which then proves $m \leq n$.

3 Basis, Dimension and Direct Sums

Corollary 3.1. *Let V be a finite dimensional vector space over F. Then any two bases of V have the same number of vectors, called the* ***dimension*** *of V.*

Proof: take $(v_1, \ldots, v_n), (w_1, \ldots, w_m)$ bases of V.

(i) As (v_i) is free and (w_i) is generating, $n \leq m$.

(ii) As (w_i) is free and (v_i) is generating, $m \leq n$.

So $m = n$.

Corollary 3.2. *Let V be a vector space over F with dimension $n \in \mathbb{N}$.*

(i) *Any set of independent vectors has at most n elements, with equality if and only if it is a basis.*

(ii) *Any spanning set of vectors has at least n elements, with equality if and only if it is a basis.*

Proof: Exercise (fill this in).

Proposition 3.1. *Let U, W be finite dimensional subspaces of V. If U and W are finite dimensional, then so is $U + W$, and*

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

Proof: Pick $(v_1, \ldots, v_l)$ a basis of $U \cap W$. Extend to a basis $(v_1, \ldots, v_l, u_1, \ldots, u_m)$ of U, and a basis $(v_1, \ldots, v_l, w_1, \ldots, w_n)$ of W. Then we show $(v_1, \ldots, v_l, u_1, \ldots, u_m, w_1, \ldots, w_n)$ is a basis of $U + W$.

It is clearly a generating family, so we will show it is free. Suppose

$$\sum_{i=1}^{l} \alpha_i v_i + \sum_{=1}^{m} \beta_i u_i + \sum_{i=1}^{n} \gamma_i w_i = 0.$$

Then we get

$$\sum_{i=1}^{n} \gamma_i w_i \in U \cap W,$$

implying that

$$\sum_{i=1}^{l} s_i v_i = \sum_{i=1}^{n} \gamma_i w_i.$$

But since $(v_1, \ldots, w_n)$ is a basis of W, we get $\gamma_i = 0$. Similarly, $\beta_i = 0$. Thus,

$$\sum_{i=1}^{l} \alpha_i v_i = 0.$$

Since (v_i) is a basis of $U \cap W$, $\alpha_i = 0$.

Proposition 3.2. *Let V be a finite dimensional vector space over F. Let $U \leq V$. Then U and V/U are both finite dimensional and*

$$\dim V = \dim U + \dim(V/U).$$

Proof: Let $(u_1, \ldots, u_l)$ be a basis of U. Extend to a basis $(u_1, \ldots, u_l, w_{l+1}, \ldots, w_n)$ of V. Then we show that $(w_{l+1} + U, \ldots, w_n + U)$ is a basis of V/U. (Fill this in).

Remark. If $U \leq V$, then we say U is proper if $U \neq V$. Then for finite dimensions, U proper implies $\dim U < \dim V$, as $\dim(V/U) > 0$.

Definition 3.1 (Direct sum)**.** Let V be a vector space over F, and $U, W \leq V$. We say $V = U \oplus W$ if and only if any element of $v \in V$ can be uniquely decomposed as $v = u + w$ for $u \in U, w \in W$.

Remark. If $V = U \oplus W$, we say that W is a complement of U in V. There is no uniqueness of such a complement.

In the sequel, we use the following notation. Let $\mathcal{B}_1 = \{u_1, \dots, u_l\}$ and $\mathcal{B}_2 = \{w_1, \dots, w_m\}$ be collections of vectors. Then

$$\mathcal{B}_1 \cup \mathcal{B}_2 = \{u_1, \dots, u_l, w_1, \dots, w_m\}$$

with the convention that $\{v\} \cup \{v\} = \{v, v\}$.

Lemma 3.1. *Let $U, W \leq V$. Then the following are equivalent:*

(i) $V = U \oplus W$*;*

(ii) $V = U + W$ *and* $U \cap W = \{0\}$*;*

(iii) *For any basis $\mathcal{B}_1$ of U, $\mathcal{B}_2$ of W, the union $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ is a basis of V.*

Proof: We show (ii) implies (i). Let $V = U + W$, then clearly U, W generate V. We only need to show uniqueness. Suppose $u_1 + w_1 = u_2 + w_2$. Then

$$u_1 - u_2 = w_2 - w_1 \in U \cap W = \{0\}.$$

Hence $u_1 = u_2$ and $w_1 = w_2$, as required.

Now we show (i) implies (iii). Let $\mathcal{B}_1$ be a basis of U, and $\mathcal{B}_2$ a basis of W. Then $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ generates $U + W = V$, and $\mathcal{B}$ is free, as if $\sum \lambda_i v_i = u + w = 0$, then $0 = 0 + 0$ uniquely, so $u = 0, w = 0$, giving $\lambda_i = 0$ for all i.

Finally, we show (iii) implies (ii). Let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$. Then since $\mathcal{B}$ is a basis of V,

$$v = \sum_{u_i \in \mathcal{B}_1} \lambda_i u_i + \sum_{w_i \in \mathcal{B}_2} \lambda_i w_i = u + w.$$

Now if $v \in U \cap W$,

$$v = \sum_{u \in \mathcal{B}_1} \lambda_u u = \sum_{w \in \mathcal{B}_2} \lambda_w w.$$

This gives

$$\sum_{u \in \mathcal{B}_1} \lambda_u u - \sum_{w \in \mathcal{B}_2} \lambda_w w = 0.$$

Since $\mathcal{B}_1 \cup \mathcal{B}_2$ is free, we get $\lambda_u = \lambda_w = 0$, so $U \cap W = \{0\}$.

Definition 3.2. Let V be a vector space over F, and $V_1, \dots, V_l \leq V$. Then

(i) The sum of the subspaces is

$$\sum_{i=1}^{l} V_i = \{v_1 + \cdots + v_l \mid v_j \in V_J, 1 \leq j \leq l\}.$$

(ii) The sum is direct:

$$\sum_{i=1}^{l} V_i = \bigoplus_{i=1}^{l} V_i$$

if and only if

$$v_1 + \cdots + v_l = v_1' + \cdots + v_l' \implies v_1 = v_1', \ldots, v_l = v_l'.$$

Proof: Exercise.

Proposition 3.3. *The following are equivalent:*

(i)

$$\sum_{i=1}^{l} V_i = \bigoplus_{i=1}^{l} V_i,$$

(ii)

$$\forall i, V_i \cap \left(\sum_{j<i} V_i \right) = \{0\},$$

(iii) *For any basis $\mathcal{B}_i$ of V_i,*

$$\mathcal{B} = \bigcup_{i=1}^{l} \mathcal{B}_i \text{ is a basis of } \sum_{i=1}^{l} V_l.$$

4 Linear maps, Isomorphism and the Rank-Nullity Theorem

Definition 4.1 (Linear map)**.** Let V, W be vector spaces over F. A map $\alpha : V \to W$ is **linear** if and only if for all $\lambda_1, \lambda_2 \in F$ and $v_1, v_2 \in V$, we have

$$\alpha(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \alpha(v_1) + \lambda_2 \alpha(v_2).$$

Example 4.1.

(i) Take an $m \times n$ matrix M, Then we can take the linear map $\alpha : \mathbb{R}^m \to \mathbb{R}^n$ defined by $X \mapsto MX$.

(ii) Take the linear map $\alpha : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ by

$$f \mapsto \alpha(f)(x) = \int_0^x f(t)\,\mathrm{d}t.$$

(iii) Fix $x \in [a, b]$. Then we can take a linear map $\mathcal{C}[a,b] \to \mathbb{R}$ by $f \mapsto f(x)$.

Remark. Let U, V, W be F-vector spaces.

(i) The identity map $\mathrm{id}_V : V \to V$ by $x \mapsto x$ is a linear map.

(ii) If $U \to V$ is β linear, and $V \to W$ is α linear, then $U \to W$ is linear by $\alpha \circ \beta$.

Lemma 4.1. *Let V, W be F-vector spaces, and $\mathcal{B}$ a basis of V. Let $\alpha_0 : \mathcal{B} \to W$ be any map, then there is a unique linear map $\alpha : V \to W$ extending α_0.*

Proof: For $v \in V$, we can write

$$v = \sum_{i=1}^n \lambda_i v_i,$$

where $\mathcal{B} = (v_1, \dots, v_n)$. Then by linearity, we must have

$$\alpha(v) = \alpha\left(\sum_{i=1}^n \lambda_i v_i\right) = \sum_{i=1} \lambda_i \alpha_0(v_i).$$

This is unique as $\mathcal{B}$ is a basis.

Remark. This is true in infinite dimensions as well.

Often, to define a linear map, we define its value on a basis and extend by linearity. As a corollary, if $\alpha_1, \alpha_2 : V \to W$ are linear and agree on a basis of V, they are equal.

Definition 4.2 (Isomorphism)**.** Let V, W be vector spaces over F. A map $\alpha : V \to W$ is called an **isomorphism** if and only if α is linear and bijective. If such an α exists, we say $V \cong W$.

Remark. If $\alpha : V \to W$ is an isomorphism, then $\alpha^{-1} : W \to V$ is linear. Indeed, for $w_1, w_2 \in W \times W$, let $w_1 = \alpha(v_1), w_2 = \alpha(v_2)$. Then,

$$\begin{aligned}\alpha^{-1}(\lambda_1 w_1 + \lambda_2 w_2) &= \alpha^{-1}(\lambda_1 \alpha(v_1) + \lambda_2 \alpha(v_2)) \\ &= \alpha^{-1}(\alpha(\lambda_1 v_1 + \lambda_2 v_2)) \\ &= \lambda_1 v_1 + \lambda_2 v_2 \\ &= \lambda_1 \alpha^{-1}(v_1) + \lambda_2 \alpha^{-1}(v_2).\end{aligned}$$

Lemma 4.2. *Congruence is an equivalence relation on the class of all vector spaces of F:*

(i) $\mathrm{id}_V : V \to V$ *is an isomorphism.*

(ii) $\alpha : V \to W$ *is an isomorphism implies* $\alpha^{-1} : W \to V$ *is an isomorphism.*

(iii) *If* $\alpha : U \to V$ *is an isomorphism,* $\beta : V \to W$ *is an isomorphism, then* $\beta \circ \alpha : U \to W$ *is an isomorphism.*

Proof: Exercise.

Theorem 4.1. *If V is a vector space over F of dimension n, then $V \cong F^n$.*

Proof: Let $\mathcal{B} = (v_1, \ldots, v_n)$ be a basis of V. Then take

$$\begin{aligned}\alpha : V &\to F^n \\ v = \sum_{i=1}^{n} \lambda_i v_i &\mapsto \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}\end{aligned}$$

as an isomorphism.

Remark. In this way, choosing a basis of V is like choosing an isomorphism from V to F^n.

Theorem 4.2. *Let V, W be vector spaces over F with finite dimension. Then $V \cong W$ if and only if* $\dim V = \dim W$.

Proof: If $\dim V = \dim W$, then $V \cong F^n \cong W$, so $V \cong W$.

Otherwise, let $\alpha : V \to W$ be an isomorphism, and $\mathcal{B}$ a basis of V. Then we show $\alpha(\mathcal{B})$ is a basis of W.

- $\alpha(\mathcal{B})$ spans V from the surjectivity of α.
- $\alpha(\mathcal{B})$ is free from the injectivity of α.

Hence $\dim V = \dim W$.

Definition 4.3 (Kernal and Image)**.** Let V, W be vector spaces over F. Let $\alpha : V \to W$ be a linear map. We define

(i) $\operatorname{Ker} \alpha = \{v \in V \mid \alpha(v) = 0\}$, the kernel of α.

(ii) $\operatorname{Im}(\alpha = \{w \in W \mid \exists v \in V, \alpha(v) = w\}$, the image of α.

Lemma 4.3. $\operatorname{Ker} \alpha$ *is a subspace of V, and* $\operatorname{Im} \alpha$ *is a subspace of W.*

Proof: Let $\lambda_1, \lambda_2 \in F$, and $v_1, v_2 \in \operatorname{Ker} \alpha$. Then

$$\alpha(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \alpha(v_1) + \lambda_2 \alpha(v_2) = 0.$$

So $\lambda_1 v_1 + \lambda_2 v_2 \in \operatorname{Ker} \alpha$.

Now if $w_1 = \alpha(v_1), w_2 = \alpha(v_2)$, then

$$\lambda_1 w_1 + \lambda_2 w_2 = \lambda_1 \alpha(v_1) + \lambda_2 \alpha(v_2) = \alpha(\lambda_1 v_1 + \lambda_2 v_2).$$

Hence $\lambda_1 w_1 + \lambda_2 w_2 \in \operatorname{Im} \alpha$.

Example 4.2. Consider $\alpha : \mathcal{C}^\infty(\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R})$, given by

$$f \mapsto \alpha(f) = f'' + f.$$

Then α is linear, and

$$\operatorname{Ker} \alpha = \{f \in \mathcal{C}^\infty(\mathbb{R}) \mid f'' + f = 0\} = \langle \sin t, \cos t \rangle.$$

Remark. If $\alpha : V \to W$ is linear, then α is injective if and only if $\operatorname{Ker}\alpha = \{0\}$, as

$$\alpha(v_1) = \alpha(v_2) \iff \alpha(v_1 - v_2) = 0.$$

Theorem 4.3. *Let V, W be vector spaces over F, and $\alpha : V \to W$ linear. Then*

$$\begin{aligned} V/\operatorname{Ker}\alpha &\to \operatorname{Im}\alpha \\ \bar{\alpha}(v + \operatorname{Ker}\alpha) &\mapsto \alpha(v) \end{aligned}$$

is an isomorphism.

Proof: We proceed in steps.

- $\bar{\alpha}$ is well defined: Note if $v + \operatorname{Ker}\alpha = v' + \operatorname{Ker}\alpha$, then $v - v' \in \operatorname{Ker}\alpha$, so $\alpha(v - v') = 0$. Hence $\alpha(v) = \alpha(v')$.
- $\bar{\alpha}$ is linear: This follows from linearity of α.
- $\bar{\alpha}$ is a bijection: First, if $\bar{\alpha}(v + \operatorname{Ker}\alpha) = 0$, then $\alpha(v) = 0$, so $v \in \operatorname{Ker}\alpha$, hence $v + \operatorname{Ker}\alpha = 0 + \operatorname{Ker}\alpha$, so α is injective. Then $\bar{\alpha}$ is surjective from the definition of the image.

Index