IB Fluid Dynamics

Ishan Nath, Lent 2023

Based on Lectures by Prof. John Lister

March 15, 2023

Contents

0 Introduction **3**

1 Kinematics **4**
1.1 Pathlines and Streamlines . . . 4
1.2 The Material Derivative . . . 5
1.3 Conservation of Mass . . . 5
1.4 Kinematic Boundary Condition . . . 7
1.5 Stream Function for 2D Incompressible Flow . . . 7

2 Dynamics of Inviscid Flow **10**
2.1 Surface and Volume Forces . . . 10
2.2 The Euler Momentum Equation . . . 11
2.3 Bernoulli's Equation for Steady Flow with Potential Forces . . . 12
2.4 Hydrostatic Pressure and Archimedes Principle . . . 17
2.5 Vorticity . . . 17
2.5.1 Interpretation of Vorticity . . . 18
2.5.2 Vorticity Equation . . . 19
2.5.3 Vorticity Stretching . . . 19
2.5.4 Circulation . . . 21

3 Introduction to Viscous Flow **22**
3.1 Plane Couette Flow and Viscosity . . . 22
3.2 2D Parallel Viscous Flow . . . 23
3.2.1 Boundary Conditions at an Interface . . . 26
3.3 Unsteady Parallel Viscous Flow and Viscous Diffusion . . . 27
3.4 The Navier-Stokes Equation . . . 30
3.4.1 The Reynolds Number . . . 30
3.5 Stagnation Point Flow . . . 31
3.6 Vorticity Equation for Viscous Flow . . . 33

4 Inviscid Irrotational Flow or Potential Flow **34**
4.1 The Velocity Potential . . . 34
4.2 Examples . . . 34
4.3 Pressure in Potential Flow with Potential Forces . . . 37
4.4 Force on Translating Sphere and Cylinder . . . 38
4.4.1 Steady, Translating Sphere . . . 38
4.5 Some Unsteady Potential Flows . . . 39
4.5.1 Free Oscillations in a U-Tube Manometer . . . 39
4.5.2 Oscillation, Expansion and Collapse of a Bubble . . . 40

5 Geophysical Flows **42**
5.1 Water Waves 42
5.1.1 Governing Equations 42
5.1.2 Linear Water Waves 42
5.1.3 Standing Waves in a Container 45
5.1.4 Rayleigh Instability 46
5.2 Rotating Fluid Dynamics 46
5.3 Euler Equations in a Rotating Frame 47
5.3.1 Rossay Number 47
5.3.2 Rotating Flow in a Shallow Layer 48
5.3.3 Geostrophic Balance 49
5.3.4 Potential Vorticity 50
5.3.5 Linearised Evolution 50

Index **52**

0 Introduction

This course focuses on *fluids*: we can think of these are liquids or gases, or more rigorously objects which are described well by the Navier-Stokes equations.

Studying fluids puts us in the realm of *continuum mechanics*, where materials are modelled with continuous mass; of course, we know in reality they are discrete particles. Areas of continuum mechanics include:

- Fluid dynamics, concerned with liquids and gases.
- Solid mechanics, where we look at solids and their properties, and fracture mechanics.
- Other materials, such as complex fluid, soft matter, biomechanics, porous media, granular flow, and so on.

To obtain the continuum hypothesis, we average over the irrelevant molecular detail to get a continuous description in terms of *fields*. Important examples are *velocity* $\mathbf{u}(\mathbf{x}, t)$, *pressure* $p(\mathbf{x}, t)$ and *density* $\rho(\mathbf{x}, t)$

In this course, we take ideas in physics: mass, momentum, Newton's laws, vector calculus, and other methods and combine them to find properties of the fields and accompanying forces.

This course starts looking at *kinematics*: properties of how fluids move, including velocities and trajectories. Then we delve into more serious *dynamics*, involving forces and equations of motions.

For simplicity, we consider *inviscid flow.* This gives a good approximation for water and air, apart from on small scales.

The obvious reason to study fluids is that fluids are everywhere: from our environment, in the oceans and atmosphere, to molecular biology; from aerosols to astrophysics.

1 Kinematics

1.1 Pathlines and Streamlines

There are two natural perspectives to view a fluid $\mathbf{u}(\mathbf{x}, t)$:

1. A passive approach; as a stationary observer watching the flow go past (the *Eulerian picture*).
2. An active approach; as a moving observer, travelling along with (same part of) the flow (the *Lagrangian picture*).

Taking the passive approach, one way to visualise the fluid is using *streamlines*. Streamlines are curves that are everywhere parallel to the flow at a given instant. These are given parametrically as

$$\mathbf{x} = \mathbf{x}(s; x_0, t_0), \text{ derived from } \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = \mathbf{u}(\mathbf{x}, t_0),$$

subject to initial condition $\mathbf{x} = \mathbf{x}_0$ at $s = 0$. These are similar to *characteristics* (which we will see later). At a given time, the set of streamlines shows the direction of flow at that instant - it shows all particles, at a fixed time. For $\mathbf{u} = (1, t)$, we get the streamlines seen in figure 1.

Figure 1: Streamlines of $\mathbf{u} = (1, t)$ for different t

(a) $t = 0$ (b) $t = 1$

In contrast, a *pathline* is when we look at the trajectory of a fluid 'particle' (which we can think of as a very small part of the fluid). The pathline $\mathbf{x} = \mathbf{x}(t; \mathbf{x}_0)$ of the particle that starts at $\mathbf{x}_0$ at $t = 0$ is found from

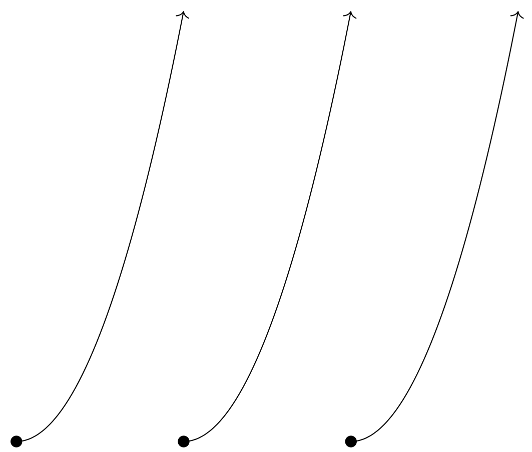
$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{u}(\mathbf{x}, t),$$

with initial conditions $\mathbf{x} = \mathbf{x}_0$ at $t = 0$. The set of pathlines give the flow of one particle, at all times. The pathline of $\mathbf{u} = (1, t)$ is graphed in figure 2.

We can consider many particles, such as all $\mathbf{x}_0$ in a given region, to see how the shape and position of a dyed patch of fluid evolves. This is useful for thinking about transport and mixing problems.

For *steady-state flows*, streamlines and pathlines are the same.

Figure 2: Pathlines of $\mathbf{u} = (1, t)$ for three initial points



1.2 The Material Derivative

Both streamlines and pathlines are taken from the Eulerian picture: we are sitting outside and watching the fluid go by. Things are different if we are drifting with the fluid.

In particular, for a field $F(\mathbf{x}, t)$, which may be density or velocity or something else we want to know, we can measure it with respect to some fixed position. However, if we are moving along with the fluid, then this field is changing: we can parametrize it as $F(\mathbf{x}(t), t)$. We can calculate

$$\begin{aligned} \delta F &= F(\mathbf{x} + \delta\mathbf{x}, t + \delta t) - F(\mathbf{x}, t) \\ &= \delta\mathbf{x} \cdot \nabla F + \delta t \frac{\partial F}{\partial t} + \mathcal{O}(\delta^2), \end{aligned}$$

and we know $\delta\mathbf{x} = \mathbf{u}(\mathbf{x}, t)\delta t + \mathcal{O}(\delta^2)$, so these equations simplify to

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \mathbf{u} \cdot \nabla F,$$

where we use new notation for the total time derivative. The additional $\mathbf{u} \cdot \nabla F$ corresponds to the movement due to the fluid. This is called the *material derivative*, as it describes the rate of change of a field, when moving through a fluid (or material).

1.3 Conservation of Mass

Consider a (rigid) tube with constant cross-section, with fluid coming in with speed $\mathbf{u} = 2$, and exiting with speed $\mathbf{u} = 1$.

For a fluid such as air, this could be conceivably possible: the air could be compressed in the tube.

However, with water this would be impossible, as the density of water ρ_{water} is relatively constant. Hence a difference in velocities means that mass is being destroyed.

This suggests that there must be a relationship between $\rho(\mathbf{x}, t)$ and $\mathbf{u}(\mathbf{x}, t)$ such that mass is never created or destroyed, so we go looking for such a relationship.

Consider an arbitrary volume V, fixed in space and bounded by a surface ∂V with outwards normal $\mathbf{n}$. Then the mass in V, which is

$$M = \int_V \rho \,\mathrm{d}V,$$

can only change if mass flows in or out at the surface ∂V. The amount of mass that escapes an area A over time t is $\rho\mathbf{u} \cdot \mathbf{n}\delta A\delta t$. We can think of $\rho\mathbf{u}$ as the *mass flux*. Integrating over V,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho \,\mathrm{d}V &= -\int_S \rho\mathbf{u} \cdot \mathrm{d}\mathbf{S}, \\ \iff \int_V \frac{\partial p}{\partial t} \,\mathrm{d}V &= -\int_V \nabla \cdot (\rho\mathbf{u}) \,\mathrm{d}V, \end{aligned}$$

where we have used the divergence theorem. Since this holds for all volumes V, we must have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{u}) = 0.$$

Using $\nabla \cdot (\rho\mathbf{u}) = \mathbf{u} \cdot \nabla\rho + \rho\nabla \cdot \mathbf{u}$, we can rewrite this as

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{u} = 0.$$

Physically, this means that if a fluid is flowing away from a point, the density should decrease, and vice versa.

If a fluid is *incompressible*, then the density of the fluid is constant, so $\dot{\rho} = \nabla\rho = 0$. Hence the velocity must satisfy

$$\nabla \cdot \mathbf{u} = 0$$

for incompressible flow. In this course, we make the assumption that density is constant and uniform. This assumption is good when the speed of the fluid is much less than the speed of sound, which is around $330\,\mathrm{m\,s^{-1}}$ in air, and $1500\,\mathrm{m\,s^{-1}}$ in water.

1.4 Kinematic Boundary Condition

Suppose that the material boundary of a body of fluid has velocity $\mathbf{U}(\mathbf{x}, t)$. Then at a point $\mathbf{x}$ on the boundary of the fluid, the velocity relative to the moving boundary is $\mathbf{u}(\mathbf{x}, t) - \mathbf{U}(\mathbf{x}, t)$.

The condition that there is no mass flux across the boundary can be written as $\rho(\mathbf{u} - \mathbf{U}) \cdot \mathbf{n}\, \delta A \delta t = 0$, or

$$\mathbf{n} \cdot \mathbf{u} = \mathbf{n} \cdot \mathbf{U}.$$

Example 1.1.

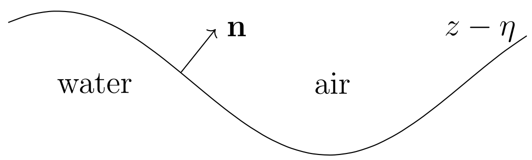
1. For a stationary rigid boundary, $\mathbf{U} = 0$, so we must have $\mathbf{u} \cdot \mathbf{n} = 0$.
2. Water waves have an air-water interface $z = \eta(x, y, t)$. We can think of the water surface as a contour of the function $F(x, y, z, t) = z - \eta(x, y, t)$.

   Then the normal $\mathbf{n}$ will be parallel to $\nabla F = (-\eta_x, \eta_y, 1)$. Since $\mathbf{U} = (0, 0, \eta_t)$, if we take $\mathbf{u} = (u, v, w)$, the boundary flux condition implies

   $$-u\eta_x - v\eta_y + w = \eta_t.$$

   This is equivalent to
   $$\frac{D}{Dt}(z - \eta) = 0,$$
   which means that particles on the surface of the wave will stay on the surface.

Figure 3: Surface of Water



1.5 Stream Function for 2D Incompressible Flow

Recall that a fluid is incompressible if $\nabla \cdot \mathbf{u} = 0$. However, this is equivalent to $\mathbf{u} = \nabla \times \mathbf{A}$ for some potential $\mathbf{A}$.

For two dimensional flows $\mathbf{u} = (u(x, y), v(x, y), 0)$, we can take

$$\mathbf{A} = (0, 0, \psi(x, y)) \implies \mathbf{u} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0\right).$$

Indeed, this implies

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

so $\mathbf{u}$ is divergence-free, as expected. We say that $\psi(x, y)$ is the *stream function*.

The stream function satisfies the following properties:

(i) The streamlines are given by $\psi = C$ (so $\mathbf{u}$ is parallel to the contours of ψ).

(ii) $|\mathbf{u}| = |\nabla\psi|$, so the flow is faster where the streamlines are closer together.

(iii) The volume that is crossing the line from $\mathbf{x}_0$ to $\mathbf{x}_1$ is

$$\int_{\mathbf{x}_0}^{\mathbf{x}_1} \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}l = \psi(\mathbf{x}_1) - \psi(\mathbf{x}_0).$$

(iv) ψ is constant on a stationary rigid boundary.

Example 1.2.

Take $\mathbf{u} = (y, x)$. Since $\nabla \cdot \mathbf{u} = 0$, a stream function exists. Integrating the above equations,

$$\begin{aligned} \frac{\partial \phi}{\partial y} &= y \implies \psi = \frac{1}{2}y^2 + f(x), \\ -\frac{\partial \psi}{\partial x} &= x \implies f(x) = -\frac{1}{2}x^2. \end{aligned}$$

Hence the streamlines are

$$y^2 - x^2 = C.$$

In polar coordinates, we have $\mathbf{u} = (u_r(r,\theta), u_\theta(r,\theta), 0)$. If $\mathbf{A} = (0, 0, \psi(r,\theta))$, then

$$\mathbf{u} = \nabla \times \mathbf{A} = \left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}, -\frac{\partial \psi}{\partial r}, 0\right).$$

We can verify that

$$\nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0.$$

Axisymmetric flow in spherical polar coordinates has

$$u_\phi = \frac{\partial}{\partial \phi} = 0.$$

If we have $\nabla \cdot \mathbf{u} = 0$, and we take

$$\mathbf{A} = \left(0, 0, \frac{\Psi(r,\theta)}{r\sin\theta}\right),$$

then

$$\mathbf{u} = \left(\frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta}, -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r}, 0\right)$$

is parallel to the contours of Ψ. Hence the contours of the Stokes stream $\Psi(r,\theta)$ form the stream tubes.

2 Dynamics of Inviscid Flow

2.1 Surface and Volume Forces

There are two types of force which act on a fluid:

(i) Those proportional to the volume, like gravity.

(ii) Those proportional to the surface area, like pressure or *viscous stress*, which is the friction between moving fluid and moving material, whether a boundary or another fluid.

We look at these individually.

(i) For *volume* or *body forces*, we denote the force on a small volume element as $\mathbf{f}(\mathbf{x}, t)\,\delta V$. Often, $\mathbf{f}$ is *conservative* with potential energy per unit volume χ, and hence $\mathbf{f} = -\nabla\chi$.

The most common case for us is gravity; here $\mathbf{f}\,\delta V = \rho\mathbf{g}\,\delta V$, with $\chi = \rho g z$, where $\mathbf{g} = (0, 0, g)$.

(ii) For a *surface force*, we consider a small element of area $\mathbf{n}\,\delta A$.

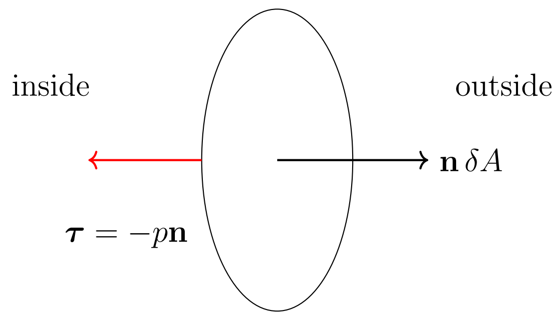
Denote the surface force exerted by the outside on the inside by $\boldsymbol{\tau}(\mathbf{x}, t; \mathbf{n})\,\delta A$, where $\boldsymbol{\tau}$ is the *stress* acting on the area element. Note that the stress $\boldsymbol{\tau}$ depends on the orientation $\mathbf{n}$.

By Newton's third law, the surface force exerted by the inside on the outside is

$$-\boldsymbol{\tau}(\mathbf{x}, t; \mathbf{n}) = \boldsymbol{\tau}(\mathbf{x}, t; \mathbf{n}).$$

We defer discussion of viscous stresses to chapter 3. In fact, in many phenomena the viscous stresses are negligible and the fluid behaves as if it is *inviscid* (frictionless).

Figure 4: Surface force on a small area δA



For inviscid fluids, the stress $\boldsymbol{\tau}$ acting across $\mathbf{n}\,\delta A$ has no tangential component, and has magnitude independent of the orientation:

$$\boldsymbol{\tau}(\mathbf{x}, t; \mathbf{n}) = -p(\mathbf{x}, t)\mathbf{n},$$

where p is the pressure. Note the sign: the outside pushes on the inside in the direction $-\mathbf{n}$.

2.2 The Euler Momentum Equation

Consider an arbitrary volume V, fixed in space, bounded by surface ∂V with an outward normal $\mathbf{n}$.

The momentum $\int_V p\mathbf{u}\,\mathrm{d}V$ inside V can change due to:

(i) Flow of momentum across the boundary ∂V,

(ii) Volume/body forces,

(iii) Surface Forces.

Recall the volume out across δA in time δt is $\mathbf{n}\,\delta A \cdot \mathbf{u}\,\delta t$, and the momentum out is $\rho\mathbf{u}(\mathbf{u} \cdot \mathbf{n}\,\delta A\delta t)$. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\mathbf{u}\,\mathrm{d}V = -\int_{\partial V} \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\,\mathrm{d}A + \int_{\partial V}(-p\mathbf{n})\,\mathrm{d}A + \int_V \mathbf{f}\,\mathrm{d}V.$$

Here $\rho u_i u_j$ is the momentum flux, a second order tensor. We can rewrite this in terms of components as

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho u_i\,\mathrm{d}V = -\int_{\partial V}(\rho u_i u_j)n_j\,\mathrm{d}A - \int_{\partial V} p n_i\,\mathrm{d}A + \int_V f_i\,\mathrm{d}V \\
\iff &\int_V \frac{\partial}{\partial t}(\rho u_i)\,\mathrm{d}V = -\int_V \frac{\partial}{\partial x_j}(\rho u_i u_j)\,\mathrm{d}V - \int_V \frac{\partial f}{\partial x_i}\,\mathrm{d}V + \int_V f_i\,\mathrm{d}V.
\end{aligned}$$

Since V is arbitrary, we obtain

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_h) = -\frac{\partial f}{\partial x_i} + f_i.$$

The left hand side can be expanded as

$$\begin{aligned}
LHS &= u_i\left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j)\right) + \rho\frac{\partial u_i}{\partial t} + \rho u_j\frac{\partial u_i}{\partial x_j} \\
&= \rho\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)u_i,
\end{aligned}$$

due to mass conservation. What we are left with is the material derivative, so we get

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{f}.$$

This is the *Euler momentum equation*.

The Euler momentum equation also has a dynamic boundary condition, which says that the stress $\boldsymbol{\tau}$ exerted on the fluid by the boundary is $-p\mathbf{n}$.

Example 2.1.

Consider a bent hose pipe with steady uniform flow U in and out, with constant cross-sectional area A. We neglect gravity (so $\mathbf{f} = 0$).

From the momentum integral equation, the momentum inside the volume is unchanging, and $\mathbf{f} = 0$, so this simplifies to

$$\int_{\text{walls}} [\rho \mathbf{u}(\mathbf{u} \cdot \mathbf{n}) + p\mathbf{n}] \, \mathrm{d}A + \int_{\text{ends}} [\rho \mathbf{u}(\mathbf{u} \cdot \mathbf{n}) + p\mathbf{n}] \, \mathrm{d}A = 0.$$

We know that

$$\int_{\text{walls}} [\rho \mathbf{u}(\mathbf{u} \cdot \mathbf{n}) + p\mathbf{n}] \, \mathrm{d}A = \int_{\text{walls}} p\mathbf{n} \, \mathrm{d}A,$$

as the walls are rigid, so $\mathbf{u} \cdot \mathbf{n} = 0$. This is simply the force by the fluid on the pipe. Moreover, on the ends

$$\int_{\text{ends}} [\rho \mathbf{u}(\mathbf{u} \cdot \mathbf{n}) + p\mathbf{n}] \, \mathrm{d}A = A[p_1 \mathbf{n}_1 + \rho(-U\mathbf{n}_1)(-U) + p_2 \mathbf{n}_2 + \rho(U\mathbf{n}_2)U].$$

In fact we get $p_1 = p_2$. Hence the force on the pipe is

$$-A(p + \rho U^2)(\mathbf{n}_1 + \mathbf{n}_2).$$

The ρU^2 contribution comes from the change in momentum flux: the fluid in changes direction.

2.3 Bernoulli's Equation for Steady Flow with Potential Forces

Recall the Euler equation

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \mathbf{f}.$$

For steady flows, the change in time will be 0. Moreover, we can rewrite the potential forces as $\mathbf{f} = -\nabla\chi$. Using the vector identity

$$\mathbf{u}\times(\nabla\times\mathbf{u}) = \nabla\left(\frac{1}{2}u^2\right) - (\mathbf{u}\cdot\nabla)\mathbf{u},$$

where $u = |\mathbf{u}|$, we can introduce the *vorticity*

$$\mathbf{w} = \nabla\times\mathbf{u}$$

to reduce the Euler equation to

$$\begin{aligned}
\rho\left(0 + \nabla\left(\frac{1}{2}u^2\right) - \mathbf{u}\times\mathbf{w}\right) &= -\nabla p - \nabla\chi \\
\iff \nabla\left(\frac{1}{2}\rho u^2 + p + \chi\right) &= \rho\mathbf{u}\times\mathbf{w} \\
\iff \mathbf{u}\cdot\nabla\left(\frac{1}{2}\rho u^2 + p + \chi\right) &= 0.
\end{aligned}$$

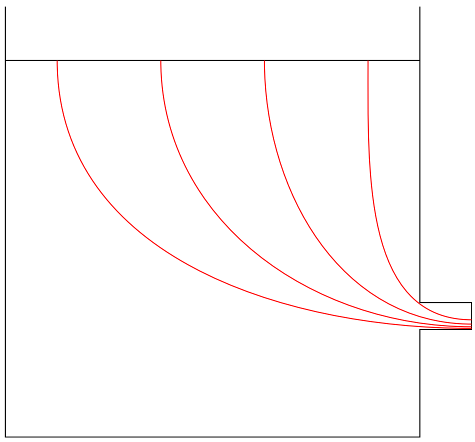
Hence

$$H = \frac{1}{2}\rho u^2 + p + \chi$$

is constant along streamlines. This is *Bernoulli's equation*.

Since H is constant, p is low where u is high, and vice versa.

Figure 5: Water Container

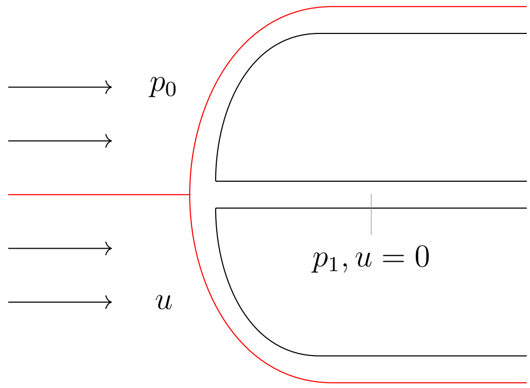


At the surface of the water, we have a large area, with $u \approx 0$, $\chi = 0$ and $p = p_a$.

At the small exit tube, $\chi = -\rho g h$, $p = p_a$ (as the water is next to the air), and so by Bernoulli, $u = \sqrt{2gh}$.

At a point in the water, $u \approx 0$, $\chi = -\rho g h$ and $p = p_a + \rho g h$.

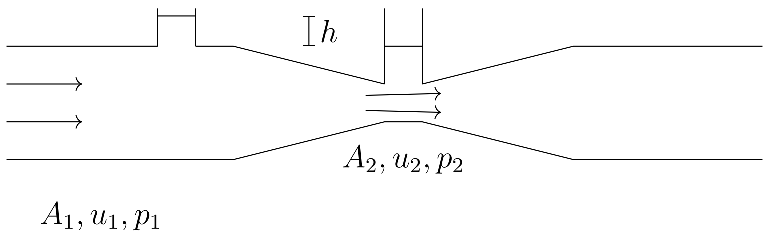
Figure 6: Pitot Tube



Using Bernoulli along the stagnation line,

$$\frac{1}{2}\rho_{\text{air}} u^2 + p_0 = 0 + p_1 \implies u = \left[\frac{2(p_1 - p_0)}{\rho_{\text{air}}}\right]^{1/2}.$$

Figure 7: Venturi Meter



Assume steady flow, so it is uniform across a cross section. This is alright for gentle variation of A. Due to mass conservation,

$$A_1 u_1 = A_2 u_2 = Q.$$

Due to Bernoulli,

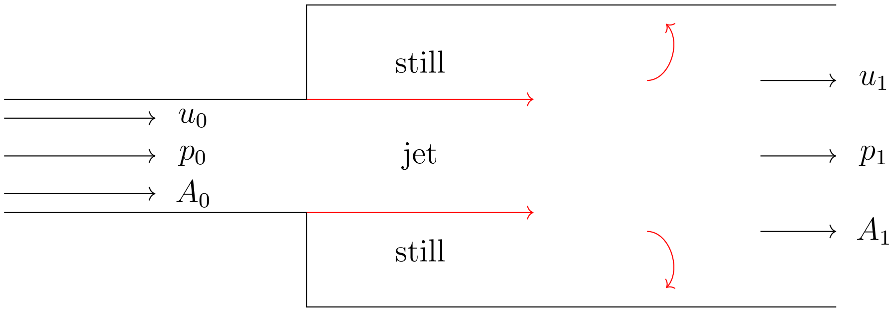
$$\frac{1}{2}\rho u_1^2 + p_1 = \frac{1}{2}\rho u_2^2 + p_2 \implies p_1 - p_2 = \frac{1}{2}\rho u_1^2\left(\frac{A_1^2}{A_2^2} - 1\right) > 0.$$

We can measure h to get $\rho g h = p_1 - p_2 = u_1$, so

$$Q = \sqrt{2gh}\frac{A_1 A_2}{\sqrt{A_1^2 - A_1^2}}.$$

We cannot use Bernoulli for a sudden enlargement of a pipe, as there will be complex unsteady flow with energy loss.

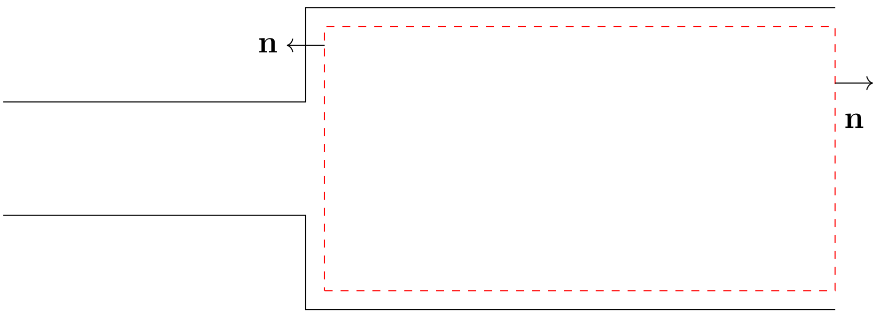
Figure 8: Large Change in Pipe



Assuming that $p_{\text{jet}} = p_{\text{still}}$, so there is no sideways acceleration, and $p_{\text{jet}} = p_0$, applying the momentum integral equation to the box, with $\mathbf{f} = 0$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int \rho \mathbf{u} \, \mathrm{d}V\right) \approx 0.$$

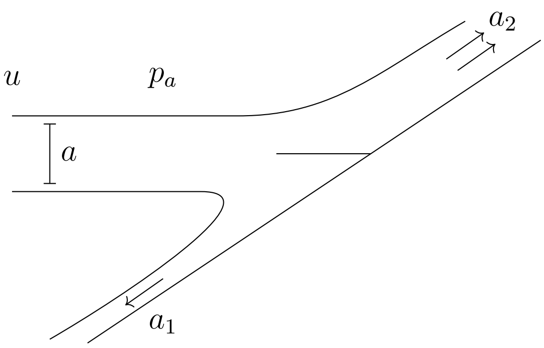
Figure 9: Momentum Integral for Large Change



Then we can show that

$$p_1 = p_0 + \rho u_1^2 \left(\frac{A_1}{A_0} - 1\right)\left(\frac{A_1}{A_0}\right).$$

Figure 10: Oblique Wall



Now consider a water jet hitting an oblique wall. We will consider a two-dimensional case, where we neglect gravity. Assume the cross section of the incoming fluid is a, and has speed U, and the area going up is a_2, and going down is a_1.

At the contact point of the stream and the wall, there will be pressure $p > p_a$, implying there is a force.

Using Bernoulli on the surface streamline where $p = p_a$ is constant, then the speed is constant along the streamline. Hence far from the impact where flow is uniform and $p = p_a$, we have $u = U$, so by mass conservation, $aU = a_1U + a_2U$.

The momentum integral equation applied to this gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\mathbf{u}\,\mathrm{d}V = -\int_{\partial V} \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n}) + p\mathbf{n}\,\mathrm{d}A + \int_V \mathbf{f}\,\mathrm{d}V.$$

The time derivative is 0, as the flow is steady, and we neglect gravity. Hence we get

$$\int_{\partial V} \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n}) + p\mathbf{n}\,\mathrm{d}A.$$

Note that $p = p_a$ apart from near the impact, and $\mathbf{u}\cdot\mathbf{n} = 0$ apart from the ends. Hence, the component parallel to the plane going upwards is

$$-\rho U^2 a\cos\beta + pU^2 a_2 - pU^2 a_1,$$

which we can solve (using conservation of mass) to get

$$a_2 = a\frac{1+\cos\beta}{2}, \qquad a_1 = a\frac{1-\cos\beta}{2}.$$

The component perpendicular to the plane is

$$\frac{1}{2}\rho U^2 a\sin\beta = \int_{\partial V} p\mathbf{n}\,\mathrm{d}A = \int_{\partial V} (p-p_a)\mathbf{n}\,\mathrm{d}A,$$

which is the force on the wall.

2.4 Hydrostatic Pressure and Archimedes Principle

If $\mathbf{u} = 0$, then by Euler,

$$0 = -\nabla p + \rho \mathbf{g} = -\nabla(p + \chi),$$

$$\implies p + \chi = \text{constant} \implies p = p_0 - pgz.$$

This is the *hydrostatic pressure*.

We can also calculate the pressure force on a submerged body with $\mathbf{u} = 0$, in a fluid with density ρ_f:

$$\mathbf{F} = -\int p\mathbf{n}\,\mathrm{d}A = -\int (p_0 - \rho g z)\mathbf{n}\,\mathrm{d}A = -\int \nabla(p_0 - \rho g z)\,\mathrm{d}V = \rho_f g \hat{\mathbf{z}} V.$$

Hence *Archimedes principle* says the upthrust or buoyancy is the weight of the fluid displaced.

For $\mathbf{u} \neq 0$ and constant ρ everywhere, we can write $p = p_0 - \rho g z + p'$, and get

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p'.$$

Here p' is the *dynamic* or *modified* pressure. This is the variation due to the motion after subtracting the hydrostatic balance.

We can ignore gravity if there are no density variations and no free surfaces. If there is a free surface between air and water, then since $\rho_{\text{water}} \neq \rho_{\text{air}}$, gravity has an effect, leading to waves.

2.5 Vorticity

Vorticity is defined by

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}.$$

This is another way to talk about the angular momentum of the fluid.

Example 2.2.

1. If $\mathbf{u} = \boldsymbol{\Omega} \times \mathbf{x}$, this is solid-body rotation, and gives $\boldsymbol{\omega} = 2\boldsymbol{\Omega}$.
2. If $\mathbf{u} = (0, \gamma_x, 0)$, this is a simple shear, and gives $\boldsymbol{\omega} = (0, 0, \gamma)$.
3. If $\mathbf{u} = (0, \frac{\kappa}{2\pi r}, 0)$ in cylindrical polars, this is known as a line vortex with $\boldsymbol{\omega} = \mathbf{0}$ except at $r = 0$. Indeed, doing the line integral around the

singularity gives

$$\oint_{r=a} \mathbf{u} \cdot \mathrm{d}\mathbf{l} = \int_0^{2\pi} \frac{R}{2\pi r} r \, \mathrm{d}\theta = R,$$

but using Stokes' theorem gives

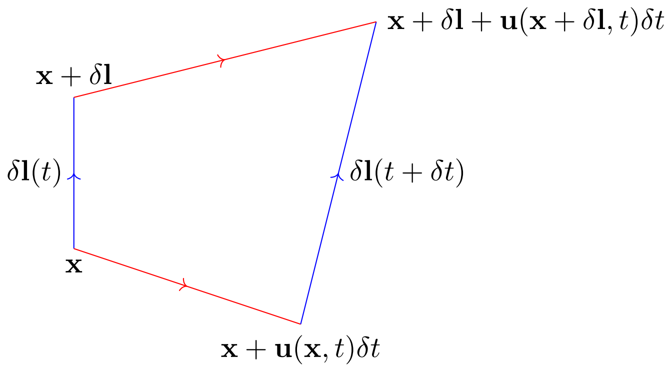
$$\oint_{r=a} \mathbf{u} \cdot \mathrm{d}\mathbf{l} = \int_{r<a} \boldsymbol{w} \cdot \hat{\mathbf{z}} \, \mathrm{d}A,$$

hence we get $\boldsymbol{\omega} = (0, 0, \kappa\delta(r))$.

2.5.1 Interpretation of Vorticity

Consider a material line element $\delta\mathbf{l}$, moving with the fluid.

Figure 11: Line Element



Then $\delta\mathbf{l} \to \delta\mathbf{l} + (\delta\mathbf{l} \cdot \nabla)\mathbf{u}\delta t$, or

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\mathbf{l} = (\delta\mathbf{l} \cdot \nabla)\mathbf{u}.$$

Hence the tensor $\partial u_i/\partial x_j$ determines the rate of change of $\delta\mathbf{l}$. We can write

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = e_{ij} + \frac{1}{2}\varepsilon_{jik}\omega_k.$$

The local rotation of line elements due to the second term is

$$\frac{1}{2}\varepsilon_{jik}\omega_k\delta l_j = \frac{1}{2}(\boldsymbol{w} \times \delta\mathbf{l}).$$

The local motion due to the first term is called the *strain rate*, which gives zero angular velocity averaged over all orientation $\delta\mathbf{l}$. $\mathbf{e}$ is symmetric and traceless when $\nabla \cdot \mathbf{u} = 0$.

Note that $\frac{1}{2}\boldsymbol{\omega}$ gives the rate of rotation of blobs, not whether the blobs are going in circles.

2.5.2 Vorticity Equation

Start with the identity

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \mathbf{f}.$$

Assuming that ρ is constant and $\mathbf{f}$ is conservative, if we take the earlier vector identity

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\left(\frac{1}{2}u^2\right) - \mathbf{u}\times\boldsymbol{\omega},$$

hence taking the curls

$$\rho\left(\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla\times(\mathbf{u}\times\boldsymbol{\omega})\right) = 0.$$

The second vector equation gives

$$\nabla\times(\mathbf{u}\times\boldsymbol{\omega}) = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} - \mathbf{u}\cdot\nabla\boldsymbol{\omega}.$$

Hence we get

$$\frac{D\boldsymbol{\omega}}{Dt} = \frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u}.$$

Hence moving with the fluid, $\boldsymbol{\omega}$ changes if $\mathbf{u}$ changes in the direction of $\boldsymbol{\omega}$.

2.5.3 Vorticity Stretching

Compare the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\mathbf{l} = (\delta\mathbf{l}\cdot\nabla)\mathbf{u}, \qquad \frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u}.$$

Moving with the fluid, $\boldsymbol{\omega}$ changes just like a material line element $\delta\mathbf{l}$ initially aligned with $\boldsymbol{\omega}$. In particular, if $\delta\mathbf{l}$ gets longer, then $\boldsymbol{\omega}$ gets bigger. This is just conservation of angular momentum.

For example, consider a uniformly rotating fluid cylinder. If over time the fluid stretches, then from mass conservation and angular momentum,

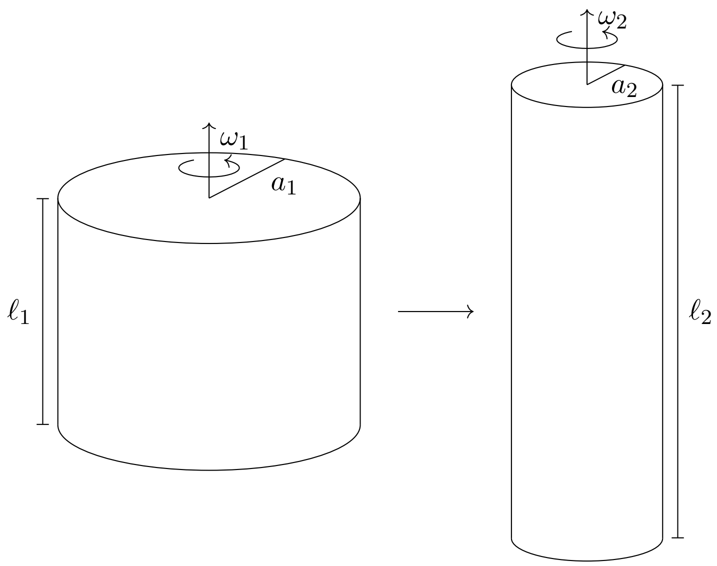
$$a_1^2 \ell_1 = a_2^2 \ell_2 \qquad\qquad a_1^6 \ell_1 \omega_1 = a_2^6 \ell_2 \omega_2,$$

Rearranging, we get

$$\omega_2 = \omega_1 \frac{\ell_2}{\ell_1}.$$

So $\boldsymbol{\omega}$ increases as fluid stretches in the direction of existing vorticity. This is called *vortex stretching*, or the ballerina effect.

Figure 12: Vortex Stretching



Example 2.3.

If we have

$$\mathbf{u} = \left(-\frac{1}{2}\beta_x, -\frac{1}{2}\beta_y, \beta_z\right) + \Omega(t)(y, -x, 0),$$

then we can check

$$\boldsymbol{\omega} = (0, 0, 2\Omega).$$

Hence from

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u},$$

looking at the z-component, we get

$$2\dot{\Omega} = \Omega\beta,$$

or $\Omega \propto e^{\beta t}$.

2.5.4 Circulation

The *circulation* around a closed curve Γ is defined by

$$C(\Gamma) = \oint_\Gamma \mathbf{u} \cdot \mathrm{d}\mathbf{l} = \int_S \boldsymbol{\omega} \cdot \mathrm{d}\mathbf{S},$$

by Stokes' theorem. For example, a line vortex

$$\mathbf{u} = \left(0, \frac{\kappa}{2\pi r}, 0\right)$$

has circulation κ for any circle $r = a$.

One further result about circulation is Kelvin's circulation theorem. This says if $\Gamma(t)$ is a material curve, i.e. one that is moving with the fluid, then

$$\frac{\mathrm{d}}{\mathrm{d}t}[C(\Gamma)] = \oint \left(\frac{D\mathbf{u}}{Dt}\,\mathrm{d}\mathbf{l} + \mathbf{u} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\delta\mathbf{l}\right) = 0.$$

This can be seen by rewriting Du/Dt using the Euler momentum equation, and noticing $\mathrm{d}\delta\mathbf{l}/\,\mathrm{d}t = (\mathbf{u} \cdot \nabla)\delta\mathbf{l}$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \oint_\Gamma \mathbf{u} \cdot \mathrm{d}\mathbf{l} &= \oint_\Gamma \frac{Du}{Dt} \cdot \delta\mathbf{l} + \mathbf{u} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\delta\mathbf{l} \\
&= \oint_\Gamma -\nabla\left(\frac{p+\chi}{p}\right) \cdot \mathrm{d}\mathbf{l} + \mathbf{u} \cdot (\delta\mathbf{l} \cdot \nabla)\mathbf{u} \\
&= \left[-\frac{p+\chi}{p} + \frac{1}{2}u^2\right]_\Gamma = 0.
\end{aligned}$$

Hence above, $C = \pi a^2(2\Omega)$ is constant.

3 Introduction to Viscous Flow

Chapter 2 was all about *inviscid* flow.

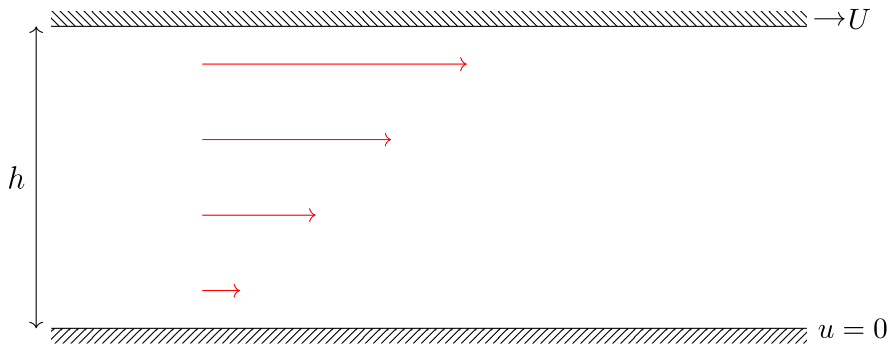
We neglected friction (viscous stresses) between layers of fluid or boundaries. Inviscid fluid exerted only a normal stress $-p\mathbf{n}$. The forces $-\nabla p + \mathbf{f}$ were balanced by $\rho \cdot D\mathbf{u}/Dt$. If $\mathbf{f} = -\nabla \chi$, we also found that the energy and angular momentum are conserved.

With viscous flow, we will find:

Velocity gradients give rise to viscous stresses (friction), and fluids also exert tangential (shear) stresses on boundaries. In the momentum equation, we will have another term, and there is dissipation of energy and diffusion of vorticity.

3.1 Plane Couette Flow and Viscosity

Figure 13: Couette Flow



Couette flow is steady flow between parallel plates driven only by the motion of the top plate. We can find experimentally that for simple (Newtonian) fluids, that

(i) Fluid velocity is U on the top plate and 0 on the bottom plate.

(ii) Velocity varies linearly between these values.

(iii) The tangential force per unit area τ_s required to move the top plate is proportional to $\frac{U}{h}$.

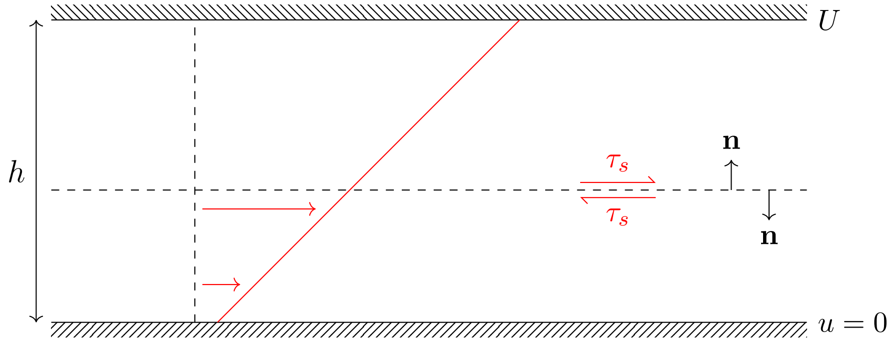
Write $\tau_s = \mu\frac{U}{h}$, where μ is the *dynamic viscosity* of the fluid. Later we will also use the *kinematic viscosity* $\nu = \frac{\mu}{\rho}$.

By considering a slab of fluid $a < y < b$ as a Couette flow, we can deduce that the tangential shear stress exerted by the positive side on the negative side of a surface

$y =$ constant is

$$\tau_s = \mu \frac{\partial u}{\partial n}.$$

Figure 14: Partial Couette Flow

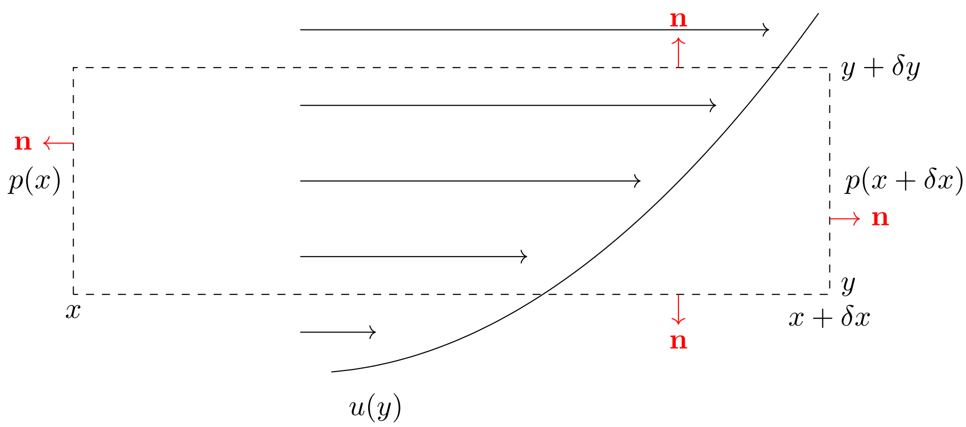


Thus the fluid exerts a shear stress $\mu \frac{\partial u}{\partial y}$ on the bottom plate and $-\mu \frac{\partial u}{\partial y}$ on the top plate.

3.2 2D Parallel Viscous Flow

We look at the steady case with $\mathbf{f} = 0$. Since it is steady, there is no acceleration, and hence the forces exerted by the surrounding fluid on the dashed rectangle must balance.

Figure 15: 2D Parallel Viscous Flow



In the x-direction,

$$p(x)\delta y - p(x+\delta x)\delta y + \mu \frac{\partial u}{\partial y}\bigg|_{y+\delta y} \delta x - \mu \frac{\partial u}{\partial y}\bigg|_{y} \delta x = 0.$$

Dividing by $\delta x \delta y$, we can take the limit to get

$$-\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} = 0.$$

Similarly, in the y-direction,

$$-\frac{\partial p}{\partial y} = 0.$$

For 2D unsteady parallel viscous flow with $\mathbf{u} = (u(y,t), 0, 0)$ and body force $\mathbf{f} = (f_x, f_y, 0)$, we get

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} + f_x,$$
$$0 = -\frac{\partial p}{\partial y} + f_y.$$

Note our independence of $\mathbf{u}$ on x means $\nabla \cdot \mathbf{u} = 0$

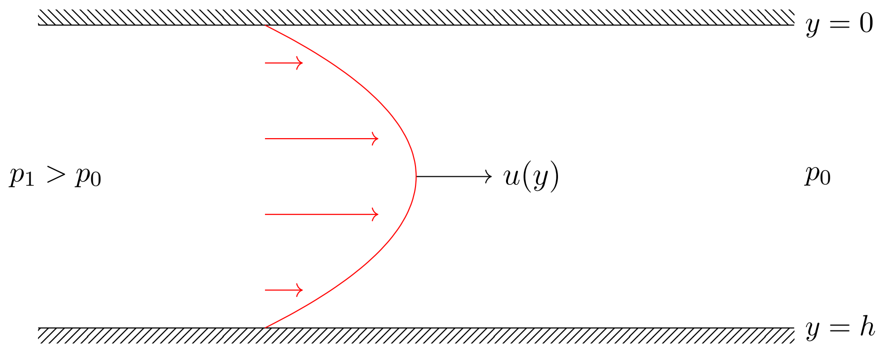
It has been verified experimentally (down to molecular scales) that at a rigid boundary, viscous fluids satisfy the *no-slip boundary condition*

the tangential component of the fluid velocity must equal that of the boundary.

We can combine this with the mass conserving kinematic boundary condition

$$\mathbf{u} = \mathbf{U}.$$

Figure 16: Poiséuille Flow



Example 3.1. (Poiséuille Flow)

Consider steady flow in a channel driven by a pressure gradient. We know $\frac{\partial p}{\partial y} = 0$, so $p = p(x)$. As it is steady,

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\mathrm{d}p}{\mathrm{d}x} = -G,$$

and by no-slip, $u(0) = u(h) = 0$. Hence we can determine

$$u = \frac{G}{2\mu} y(h-y).$$

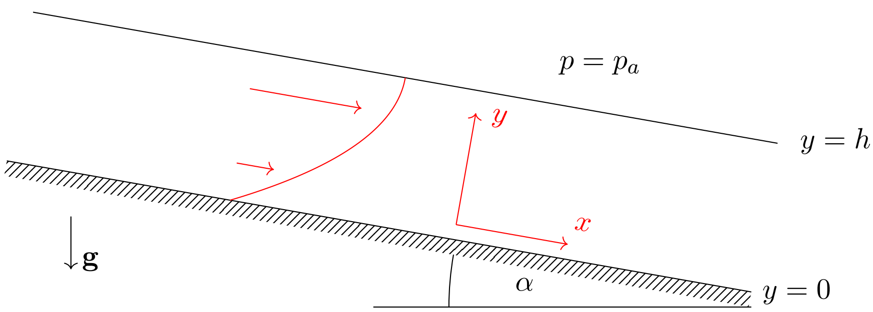
This determines a parabolic flow profile. The flux is

$$q = \int_0^h u\,\mathrm{d}y = \frac{Gh^3}{12\mu}.$$

The overall force balance is

$$Gh - \mu \frac{\partial u}{\partial u}\bigg|_0 + \mu \frac{\partial u}{\partial y}\bigg|_L = 0.$$

Figure 17: Viscous Flow down a Slope



Example 3.2.

We consider viscous flow down a slope, driven by gravity. Assume that p_{air} is uniform. Then the shear stress exerted by the air is negligible.

Take coordinates parallel and perpendicular to the plane. Then the force is

$\mathbf{f} = (\rho g \sin\alpha, -\rho g \cos\alpha)$. For the perpendicular force,

$$\frac{\partial p}{\partial y} = \rho g \cos\alpha,$$

which when combined with $p = p_a$ at $y = h$ gives

$$p = p_a + \rho g \cos\alpha (h - y),$$

hence $\frac{\partial p}{\partial h} = 0$. Now the parallel force gives

$$\mu \frac{\partial^2 u}{\partial y^2} = -\rho g \sin\alpha.$$

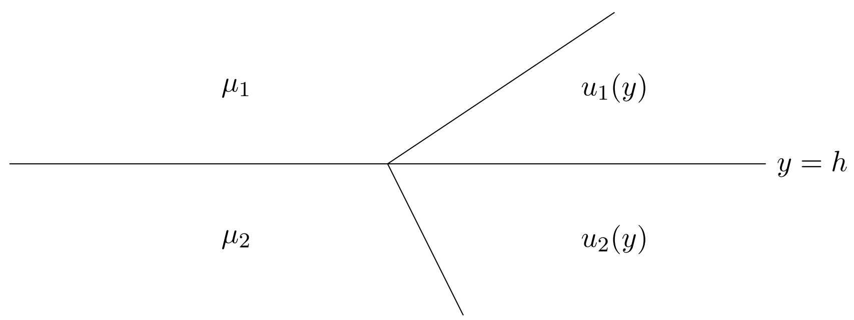
Then taking $u(0) = 0$ due to no-slip and $\mu \frac{\partial u}{\partial y}|_h = 0$ as there is no shear stress, we get

$$u = \frac{\rho g \sin\alpha}{2\mu} y(2h - y).$$

3.2.1 Boundary Conditions at an Interface

Consider two fluid in parallel viscous flow: At $y = h$, the boundary of their interface,

Figure 18: Fluids in Parallel Viscous Flow



by the no-slip condition we have $u_1 = u_2$. Hence,

$$\mu_1 \frac{\partial u_1}{\partial y} = \mu_2 \frac{\partial u_2}{\partial y}, \qquad p_1 = p_2.$$

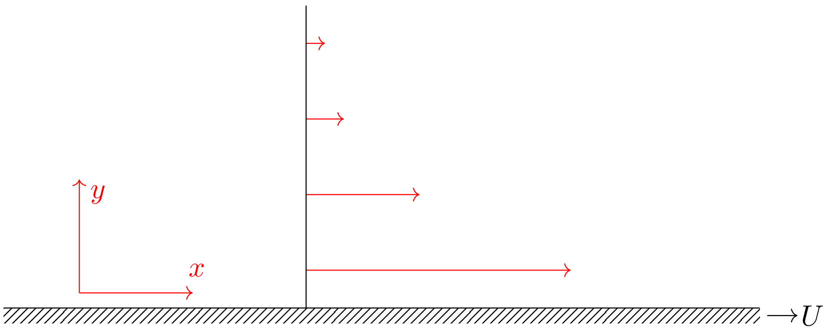
These are the *continuity of stress* conditions.

3.3 Unsteady Parallel Viscous Flow and Viscous Diffusion

Consider a semi-infinite fluid domain $y > 0$, initially at rest, with no applied pressure gradient and $\mathbf{f} = 0$.

At $t = 0$, the boundary of $(y, 0)$ starts to move with velocity $(U, 0)$.

Figure 19: Impulsively Started Flat Plate



Then the change in velocity is subject to the following equations:

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} + f_x,$$

as well as initial conditions $u = 0$ at $t = 0$, far-field condition $u = 0$ as $y \to \infty$, and no-slip condition $u = U$ at $y = 0$, $t > 0$.

In the above expression, the first and last term cancel, and we find that the velocity satisfies the *diffusion equation*

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}.$$

Here we are using the kinematic viscosity $\nu = \frac{\mu}{\rho}$, which can be thought of as a measure of diffusivity for momentum (or for viscosity, as we will see later).

From the Methods course, we have many ways of attacking the diffusion equation (e.g. separation of variables, Fourier transforms, Green's functions), which we will use in the example sheet.

Here there is not externally imposed length scale for y, since $0 < y < \infty$. However to find a solution we need a length scale! The equation

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}$$

suggests that $y \sim (\nu t)^{1/2}$, hence letting $u = Uf(\eta)$, where

$$\eta = \frac{y}{(\nu t)^{1/2}} \implies \frac{\partial \eta}{\partial t} = -\frac{1}{2}\frac{\partial}{\partial t},$$

we can find that f satisfies the equation

$$-\frac{1}{2}\frac{\eta}{t}Uf' = \nu\frac{1}{\nu t}Uf'',$$

which we can then solve as

$$\begin{aligned} f'' = -\frac{1}{2}\eta f' &\implies -\frac{1}{2}\eta = \frac{f''}{f'} \\ &\implies -\frac{1}{4}\eta^2 = \log f' + \text{constant} \\ &\implies f = A + B\int_\eta^\infty e^{-\frac{1}{4}\eta'^2}\,\mathrm{d}\eta'. \end{aligned}$$

Since $f(\infty) = 0$, we get $A = 0$, and $f(0) = 1$ gives $B = \pi^{-1/2}$, which simplifies to

$$u = U\,\mathrm{erf}\left(\frac{y}{2\sqrt{vt}}\right).$$

Hence at all times, the shape of the flow is similar, but with growing width as $(vt)^{1/2} \to \infty$.

From dimensional analysis, we could also conclude that $\frac{y}{ut}$ and $\frac{yu}{\nu}$ are dimensionless, but the equations show that these are not relevant. Instead the scaling argument is better!

Below are the kinematic and dynamic viscosities of some fluids.

	ρ	μ	ν
Water	$1 \times 10^3\,\mathrm{kg\,m^{-3}}$	$1 \times 10^{-3}\,\mathrm{Pa\,s}$	$1 \times 10^{-6}\,\mathrm{m^2\,s^{-1}}$
Air	$1\,\mathrm{kg\,m^{-3}}$	$2 \times 10^{-5}\,\mathrm{Pa\,s}$	$2 \times 10^{-5}\,\mathrm{m^2\,s^{-1}}$
Golden Syrup	$1.4 \times 10^3\,\mathrm{kg\,m^{-3}}$	$200\,\mathrm{Pa\,s}$	$0.14\,\mathrm{m^2\,s^{-1}}$

Since $\nu_{\text{air}} \approx 10\nu_{\text{water}}$, momentum spreads faster and further in air.

Moreover since the shear stress exerted by a fluid (as in the plate) is

$$\tau_s = \mu\frac{\partial u}{\partial y} = -\mu\frac{u}{(vt)^{1/2}}f'(0) = -\frac{\mu U}{\sqrt{\pi\nu t}},$$

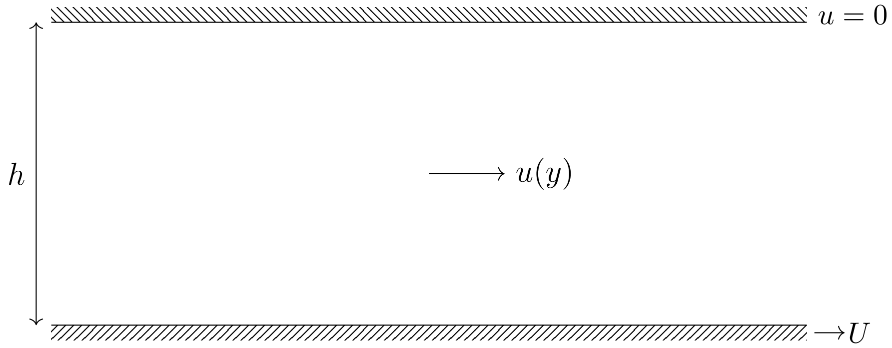
and we have

$$\left.\frac{\mu}{\sqrt{\nu}}\right|_{\text{water}} \approx 100\left.\frac{\mu}{\sqrt{\nu}}\right|_{\text{air}},$$

water exerts a much bigger stress.

If we add a stationary boundary at $y = h$, then we have two relevant length scales: h and $(\nu t)^{1/2}$.

Figure 20: Started and Steady Plates



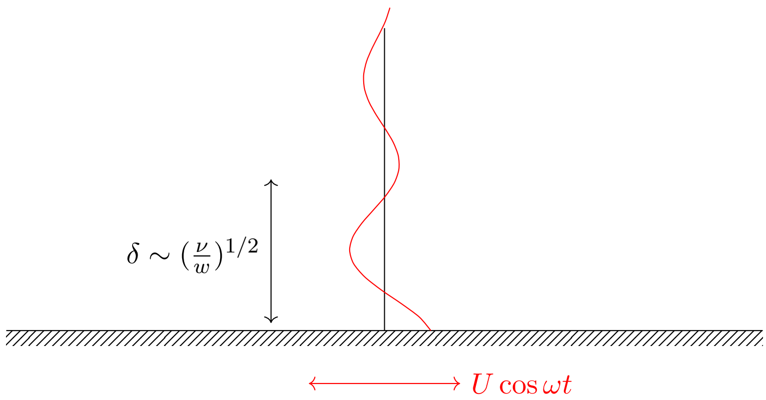
If we have $h \gg (\nu t)^{1/2}$, then we expect the boundary to have little (exponentially small) effect on the previous solution.

However as $t \to \infty$, we expect to approach steady Couette flow, with a linear profile. Deviations from this steady state decay like e^{-kt}, with $k \propto \frac{\nu}{h^2}$, and are small for $\nu t \gg h^2$.

The characteristic timescale is $T = \frac{h^2}{\nu}$ for diffusion across the cell. If $t \ll T$, then the effects of viscosity are confined to the bounding layers with $\delta \sim (vt)^{1/2}$.

If we have a single oscillating boundary, then there is an imposed timescale $\frac{1}{\omega}$ during which velocity variations can diffuse with $\delta \sim (\frac{\nu}{\omega})^{1/2}$.

Figure 21: Oscillating Boundary



Adding a stationary boundary to the oscillating boundary gives two timescales $\frac{\nu}{h^2}$ and $\frac{1}{\omega}$, and two length scales $(\frac{\nu}{\omega})^{1/2}$ and h. Hence we can determine a dimensionless parameter

$$S = \frac{\omega h^2}{\nu},$$

which is the *Stokes number*.

$S \gg 1$ looks like example the oscillating boundary with $\delta \approx (\nu/w)^{1/2} \ll h$. $S \ll 1$ looks like Couette flow (linear profile) with amplitude $U(t)$.

These examples illustrate that the inviscid solution ($\nu \approx 0$, so $u = 0$ and slipping at boundary) is not uniformly the same as the limit of a "small" viscosity (where smallness means $\nu \ll \frac{h^2}{t}$ or ωh^2), but the viscous effects are only felt in the boundary layers $\delta \ll h$.

3.4 The Navier-Stokes Equation

In part II, it is shown that

$$\boldsymbol{\tau}(\mathbf{x}, t; \mathbf{n}) = -p\mathbf{n} + \mu[(\mathbf{n} \cdot \nabla)\mathbf{u} + (\nabla \mathbf{u}) \cdot \mathbf{n}],$$

and hence the flow satisfies the *Navier-Stokes equation*

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f},$$

and $\nabla \cdot \mathbf{u} = 0$. This reduces to the Euler equation if $\mu = 0$, and is parallel to the viscous flow equations if $\mathbf{u} = (u(y,t), 0, 0)$.

3.4.1 The Reynolds Number

Suppose a flow has a characteristic lengthscale L and velocity scale U

Assume the characteristic timescale T is $\frac{L}{U}$ (or the flow is steady) and let the characteristic scale of the pressure differences be denoted by P.

We can estimate the scale of the terms in the Navier-Stokes equation as

$$\begin{array}{ccccc}
\dfrac{\partial \mathbf{u}}{\partial t} & + \mathbf{u} \cdot \nabla \mathbf{u} & = -\dfrac{1}{\rho} \nabla p & + \nu \nabla^2 \mathbf{u} & \\
\dfrac{U}{L/U} & \dfrac{U^2}{L} & \dfrac{p}{\rho L} & \dfrac{\nu U}{L^2} & \\
1 \quad : & 1 \quad : & \dfrac{p}{\rho U^2} \quad : & \dfrac{\nu}{LU} & \equiv \dfrac{1}{R_e}
\end{array}$$

The *Reynolds number* $\mathrm{Re} = \frac{UL}{\nu}$ is a dimensionless parameter describing the relative importance of inertia and viscosity:

$$\frac{pD\mathbf{u}/Dt}{\mu\nabla^2\mathbf{u}} \sim \mathrm{Re}\,.$$

- If $\mathrm{Re} \ll 1$, then we expect

  $$\rho\frac{D\mathbf{u}}{Dt} \ll \mu\nabla^2\mathbf{u},$$

  so inertia is negligible, and we can approximate the Navier-Stokes equation by the Stokes equations

  $$0 = -\nabla p + \mu\nabla^2\mathbf{u} + \mathbf{f}, \qquad \nabla\cdot\mathbf{u} = 0.$$

  Scaling for pressure is $p \sim \frac{\mu U}{L}$, like the viscous stress.

- If $\mathrm{Re} \gg 1$, we expect

  $$\mu\nabla^2\mathbf{u} \ll \rho\frac{D\mathbf{u}}{Dt},$$

  and hence viscosity is negligible (except perhaps in thin boundary layers at rigid boundaries). We can approximate the Navier-Stokes equation by the Euler equation

  $$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{f},$$

  outside the boundary layer. Pressure scaling is $p \sim \rho U^2$, like Bernoulli.

For large Reynolds numbers, and on timescale L/U, viscous diffusion affects the velocity over a distance

$$\delta \sim \left(v\frac{L}{U}\right)^{1/2}, \qquad \frac{\delta}{L} \sim \left(\frac{v}{LU}\right)^{1/2} = \frac{1}{\mathrm{Re}^{1/2}}.$$

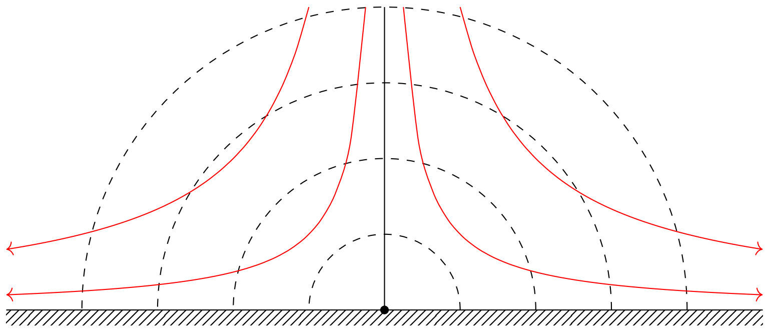
3.5 Stagnation Point Flow

Consider the 2D inviscid incompressible flow with $\mathbf{u} = (Ex, -Ey)$ for $y > 0$. Then $\psi = Exy$, and by Bernoulli,

$$p = p_0 - \frac{1}{2}\rho E^2(x^2 + y^2).$$

This solves $\rho(\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p$, with $v = 0$ on $y = 0$.

Figure 22: Stagnation Point



This also satisfies the Navier-Stokes equations, but not the viscous no-slip boundary condition because $u = 0$ if $y = 0$ is a rigid boundary. Fortunately, we can find an exact solution. Guessing

$$\psi = Exf(y) \implies \mathbf{u} = (E_x f', -Ef),$$

we want $f(0) = f'(0) = 0$ and $f(y) \approx y$ as $y \to \infty$. Substituting into

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u},$$

the x-component gives

$$\left(E_x f' \frac{\partial}{\partial x} - Ef \frac{\partial}{\partial y}\right) E_x f' = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(E_x f')$$
$$\implies E^2 x(f'^2 - ff'') = -\frac{1}{\rho}\frac{\partial p}{\partial u} + \nu E_x f'''.$$

Similarly, the y-component gives

$$E^2 f' f = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \nu E f''.$$

Taking the x derivative of this equation gives

$$\frac{\partial^2 p}{\partial x \partial y} = 0 \implies p = X(x) + Y(y).$$

Then from the first equation,

$$\frac{\partial p}{\partial x} \propto x, \qquad f' \to 1 \implies X(x) = p_0 - \frac{1}{2}\rho E^2 x^2.$$

Hence

$$f'^2 - ff'' = 1 + \frac{\nu}{E} f'''.$$

Rescaling, $f(y) = \delta F(\eta)$, where $\eta = \frac{y}{\delta}$ and $\delta = (\frac{\nu}{E})^{1/2}$. So,

$$F''' = F'^2 - FF'' - 1, \qquad F(0) = F'(0) = 0, \qquad F'(1) \overset{y\to\infty}{\to} 1.$$

We can solve this numerically.

3.6 Vorticity Equation for Viscous Flow

Recall that

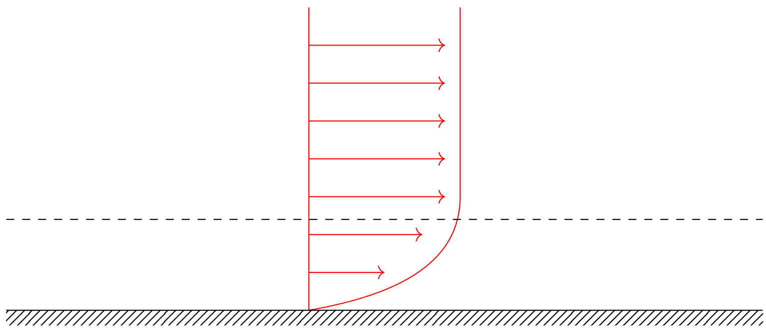
$$\nabla \times (\nu \nabla^2 \mathbf{u}) = \nu \nabla^2 \boldsymbol{\omega}.$$

Taking the curl of Navier-Stokes, and using the previous derivation, we obtain

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}.$$

For Re $\gg 1$, away from the rigid boundaries, vorticity is dominantly advected and amplified by vortex stretching, and the final diffusion term is small. At rigid boundaries, vorticity is generated by the no-slip boundary condition and diffuses a short distance to form a boundary layer.

Figure 23: Vorticity for Viscous Flow



For parallel flow, if $\mathbf{u} = (u(y,t), 0, 0)$, then $\boldsymbol{\omega} = (0, 0, \omega(y,t))$, where

$$\omega = -\frac{\partial u}{\partial y}, \qquad \frac{\partial \omega}{\partial t} = \nu \frac{\partial^2 \omega}{\partial y^2}.$$

Hence ω satisfies the diffusion equation.

4 Inviscid Irrotational Flow or Potential Flow

4.1 The Velocity Potential

In inviscid flow, if $\boldsymbol{\omega} = \nabla \times \mathbf{u} = 0$ at $t = 0$, i.e. we have *irrotational flow*, then

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} \implies \nabla \times \mathbf{u} = 0,$$

for all times $t \geq 0$. Hence irrotational flow remains irrotational in inviscid flow. Since $\nabla \times \mathbf{u} = 0$, there exists a *velocity potential* $\phi(\mathbf{x}, t)$ with

$$\mathbf{u} = \nabla\phi.$$

Remark.

1. Note the positive sign of the velocity potential, compared to $\mathbf{f} = -\nabla\chi$.
2. We can add any function $f(t)$ to ϕ without changing $\mathbf{u}$.

From incompressibility, we have $\nabla \cdot \mathbf{u} = 0$, i.e. $\nabla^2\phi = 0$, so ϕ satisfies Laplace's equation.

The kinematic boundary conditions transform from $\mathbf{u} \cdot \mathbf{n} = \mathbf{U} \cdot \mathbf{n}$ to $\mathbf{n} \cdot \nabla\phi = \mathbf{U} \cdot \mathbf{n}$.

Hence solving the Euler equation for an irrotational flow reduces to solving the more familiar, and linear, Laplace equation with Neumann boundary conditions. The solution is non-zero becuase of the boundary conditions.

Important examples of irrotational flow are flows starting from rest or with uniform flow upstream.

4.2 Examples

For simple boundaries (spheres, cylinders, rectangular channels, half-spaces), we can build solution using separable solutions to $\nabla^2\phi = 0$ in suitable coordinate systems.

Cartesians In uniform flow, with $\mathbf{u} = \mathbf{U}$, then $\phi = \mathbf{U} \cdot \mathbf{x} = U_1 x + U_2 y + U_3 z$, so

$$\phi = (e^{\pm kz} \text{ or } \cosh kz \text{ or } \sinh kz) \times (e^{\pm ikx} \text{ or } \cos kx \text{ or } \sin kx) \implies \nabla^2\phi = 0.$$

The corresponding flow is periodic in the x-direction.

Spherical Geometry The general axisymmetric solution to $\nabla^2\phi = 0$ in spherical polar coordinates is

$$\phi = \sum_{n=0}^{\infty}(A_n r^n + B_n r^{-n-1})P_n(\cos\theta),$$

where P_n are the Legendre polynomials (see Methods). We will only need the first few simple modes. When $\phi = A_0$, then $\mathbf{u} = \mathbf{0}$. If $\phi = \frac{B}{r}$, then

$$\mathbf{u} = \nabla\phi = -\frac{B}{r^2}\mathbf{e}_r.$$

This is radial flow proportional to $\frac{1}{r^2}$. The volume flux across any sphere $r = a$ is

$$q = \int_{r=a} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}A = 4\pi a^2\left(\frac{-B}{a^2}\right) = -4\pi B,$$

independent of a. Hence

$$\phi = -\frac{q}{4\pi r}$$

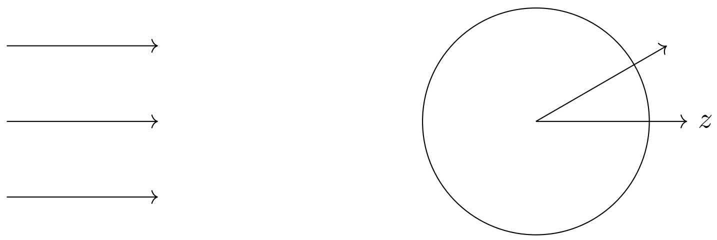
gives a *point source* of strength (volume flux) q. If $q < 0$, we get a *point sink*.

If $\phi = Ur\cos\theta = Uz$, then $\mathbf{u} = \nabla\phi = U\mathbf{e}_z$, which gives uniform flow in the z-direction.

If $\phi = \frac{U\cos\theta}{r^2}$, then $\mathbf{u} = \nabla\phi = -\frac{2U\cos\theta}{r^3}\mathbf{e}_r$. The flow is given by a *source dipole*.

Consider uniform flow past a stationary sphere.

Figure 24: Flow past a Stationary sphere



The flow satisfies the equations:

$$\begin{aligned} \nabla^2\phi &= 0 && \text{when } r > a,\\ \phi &\to Ur\cos\theta && \text{as } r \to \infty,\\ \mathbf{u}\cdot\mathbf{n} = \frac{\partial\phi}{\partial r} &= 0 && \text{on } r = a. \end{aligned}$$

These follow from the assumptions the flow is irritation/incompressible, the far-field flow being stationary, and the kinematic boundary condition.

This is a linear problem, with forcing proportional to $\cos\theta = P_1(\cos\theta)$. We try

$$\phi = U\cos\theta\left(r + \frac{B}{r^2}\right).$$

Then

$$\begin{aligned}
\left.\frac{\partial\phi}{\partial r}\right|_{r=a} &= U\cos\theta\left(1 - \frac{2B}{r^3}\right)\bigg|_{r=a} \implies B = \frac{a^3}{2} \\
&\implies \phi = U\cos\theta\left(r + \frac{a^3}{2r^2}\right) \implies \mathbf{u} = \nabla\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, 0\right) \\
&= \left(U\cos\theta\left(1 - \frac{a^3}{r^3}\right), -U\sin\theta\left(1 + \frac{a^3}{2r^3}\right), 0\right).
\end{aligned}$$

Cylindrical Geometry The general solution to $\nabla^2\phi = 0$ is

$$\phi = C_0 + A_0\log r + B_n\theta + \sum_{n=1}^{\infty}(A_n r^n + B_n r^{-n})(C_n\cos n\theta + D_n\sin n\theta).$$

Again, we will only need a few modes.

We first consider

$$\phi = \frac{q}{2\pi}\log r \implies \mathbf{u} = \frac{q}{2\pi r}\mathbf{e}_r.$$

This is radial flow with $u_r \propto \frac{1}{r}$, and the flow across any circle with $r = a$ is $2\pi a u_r = q$, independent of a.

The corresponding point is called a *2D point source* of strength q. This is also a *line source* in three dimensions.

We can also have

$$\phi = \frac{K}{2\pi}\theta \implies \mathbf{u} = \nabla\phi = \frac{K}{2\pi r}\mathbf{e}_\theta,$$

which is circular flow $\propto \frac{1}{r}$.

The circulation $\int \mathbf{u}\cdot \mathrm{d}l$ around a circle $r = a$ is

$$2\pi a\frac{K}{2\pi a} = K,$$

again independent of a. This is called a *point vortex* of circulation (or strength) K.

If $\phi = Ur\cos\theta$, we again have uniform flow.

If $\phi = \frac{U\cos\theta}{r}$, we get a 2D dipole.

Now consider uniform flow around a cylinder. We have the conditions

$$\begin{aligned} \nabla^2\phi &= 0 && r > a, \\ \phi &\to Ur\cos\theta && r \to \infty, \\ \frac{\partial\phi}{\partial r} &= 0 && r = a, \end{aligned}$$

and moreover we need the condition

$$\oint \mathbf{u}\cdot \mathrm{d}l = [\phi]_{r=a} = K.$$

This is needed to get a unique solution, which is then

$$\phi = U\cos\theta\left(r + \frac{a^2}{r}\right) + \frac{K}{2\pi}\theta,$$

which, solving for the velocity gives

$$\begin{aligned} u_r &= U\left(1 - \frac{a^2}{r^2}\right)\cos\theta, \\ u_\theta &= -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{K}{2\pi r}. \end{aligned}$$

4.3 Pressure in Potential Flow with Potential Forces

The non-linear equation of motion has been reduced to the linear Laplace equation for the kinematics of the system. However, we still have non-linearity in the dynamic boundary.

The momentum equation gives

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \mathbf{f}.$$

Assuming potential forces $\mathbf{f} = -\nabla\chi$, and the previous vector identity

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\left(\frac{1}{2}u^2\right) - \mathbf{u}\times\boldsymbol{\omega}.$$

Moreover, from the potential flow,

$$\frac{\partial\mathbf{u}}{\partial t} = \nabla\frac{\partial\phi}{\partial t}.$$

Thus the momentum equation reduces to

$$\nabla\left(\rho\frac{\partial\phi}{\partial t}+\frac{1}{2}\rho u^2+p+\chi\right)=0,$$

or analogously,

$$\rho\frac{\partial\phi}{\partial t}+\frac{1}{2}\rho u^2+p+\chi=f(t),$$

a function independent of space. This is *unsteady Bernoulli*.

Remark.

1. $f(t)$ being a function of time is irrelevant, since we can add an arbitrary $g(t)$ to ϕ. It is the independence of $\mathbf{x}$ that is more important.
2. This is not the same result as Bernoulli's.
3. For steady, irrotational flow, we may combine the two results to get H is constant everywhere.

4.4 Force on Translating Sphere and Cylinder

4.4.1 Steady, Translating Sphere

For steady motion, it is more convenient to use the frame of reference of moving with the sphere.

From our previous section, the uniform flow past a stationary sphere is given by

$$\phi=U\cos\theta\left(r+\frac{a^3}{2r^2}\right),$$

and on $r=0$, $u_r=0$ and $u_\theta=-\frac{3}{2}\sin\theta$. Applying either form of Bernoulli, with $\partial\phi/\partial t=0$ and $\chi=0$, we obtain

$$\frac{1}{2}\rho\left(-\frac{3}{2}U\sin\theta\right)^2+p(a,\theta)=\frac{1}{2}\rho U^2+p_\infty,$$

or similarly,

$$p(a,\theta)=p_\infty+\frac{1}{2}\rho U^2\left(1-\frac{9}{4}\sin^2\theta\right).$$

The pressure distribution is symmetric fore-aft and round the equator. Hence the net force on the sphere is zero.

This surprising result is called d'Alemberts paradox.

In fact, there is no drag (which is net force parallel to motion) on any steadily moving body in unbounded potential flow. This is as kinetic energy is constant, as we are neglecting viscosity/friction.

For viscous flow on a translating sphere, we find experimentally that

$$F = 0.4\frac{1}{2}\rho U^2(\pi a^2).$$

Using expansion for pressure in potential flow, we deduce

$$\frac{1}{2}\rho\left(-2U\sin\theta + \frac{\kappa}{2\pi a}\right)^2 + p(a,\theta) = \frac{1}{2}\rho U^2 + p_\infty.$$

The fluid force on the cylinder (per unit length) is

$$\begin{aligned}
-\int_{r=a} p(a,\theta)\mathbf{n}\,\mathrm{d}A = &-\int_0^{2\pi}\Bigg[p_\infty + \frac{1}{2}\rho U^2 \\
&-\frac{1}{2}\rho\left(4U^2\sin^2\theta - \frac{2U\kappa}{\pi a}\Bigg|_0^{\sin\theta} + \frac{\kappa^2}{4\pi^2a^2}\right)\Bigg](\cos\theta,\sin\theta)a\,\mathrm{d}\theta \\
&= \left(0, -\frac{1}{2}\rho\frac{2U\kappa}{\pi a}\frac{1}{2}2\pi a\right) = (0, -\rho U\kappa).
\end{aligned}$$

Therefore, the lift force (perpendicular to $\mathbf{U}$) is down for $\kappa > 0$, and up for $\kappa < 0$.

We can show that uniform potential flow past any two dimensional body with circulation κ produces lift $-\rho U\kappa$.

4.5 Some Unsteady Potential Flows

4.5.1 Free Oscillations in a U-Tube Manometer

Starting from rest, the displacement is $\zeta(0) \neq 0$. Since the flow is rotational, we get a potential flow.

Assume that we have long tubes of equal areas and a short, wide junction. Then,

$$\nabla\phi = \int_L^R \mathbf{u}\cdot\mathrm{d}\mathbf{x} \approx 0$$

along the junction, as $\mathrm{d}\mathbf{x}$ is small as the junction is short, and $\mathbf{u}$ is small as the junction is wide. Hence we can take $\phi = 0$ at the base of both tubes. By mass conservation, u is uniform and equals $\dot{\zeta}$. Hence, on the right hand side,

$$\phi = uz, \qquad \frac{\partial \phi}{\partial t} = \dot{u}z, \qquad \left.\frac{\partial \phi}{\partial t}\right|_{h+\zeta} = \ddot{\zeta}(h+\zeta),$$
$$\phi = -uz, \qquad \frac{\partial \phi}{\partial t} = -\dot{u}z, \qquad \left.\frac{\partial \phi}{\partial t}\right|_{h-\zeta} = -\ddot{\zeta}(h-\zeta).$$

Applying unsteady Bernoulli,

$$\rho\ddot{\zeta}(h+\zeta) + \frac{1}{2}\rho\dot{\zeta}^2 + p_a + \rho g(h+\zeta) = -\rho\ddot{\zeta}(h-\zeta) + \frac{1}{2}\rho\dot{\zeta}^2 + p_a + \rho g(h-\zeta),$$

which simplifies to

$$\rho\ddot{\zeta}(2h) = \rho g(2\zeta),$$

which is simple harmonic motion with frequency $\sqrt{g/h} = \omega$. However, this is non-linear if the tubes have different areas.

4.5.2 Oscillation, Expansion and Collapse of a Bubble

Air is 15 000 times more compressible than water. Consider incompressible flow of fluid outside of a bubble of radius $a(t)$. Neglecting gravity, and using the spherical symmetry, we get radial flow proportional to r^{-2} by mass conservation. Hence, if $u = \dot{a}$ on $r = a$,

$$\mathbf{u} = \frac{a^2\dot{a}}{r^2}\mathbf{e}_r = \nabla\phi,$$

with

$$\phi = -\frac{a^2\dot{a}}{r},$$

is source flow. Now,

$$\frac{\partial \phi}{\partial t} = -\frac{a^2\ddot{2} + 2a\dot{a}^2}{r}, \qquad \left.\frac{\partial \phi}{\partial t}\right|_{r=a} = -a\ddot{a} - 2\dot{a}^2,$$

so using Bernoulli's equation with $\chi = 0$, at $r = a$, we obtain

$$-\rho a\ddot{a} - \frac{3}{2}\rho\dot{a}^2 = p_\infty - p(a(t), t).$$

Multiplying by $a^2\dot{a}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\rho a^3\dot{a}^2\right) = a^2\dot{a}(p(a) - p(\infty)).$$

This can be interpreted as the change in kinetic energy, is the rate of working by the pressure at a and ∞. If $p(a,t)$ is given as $F(a)$, then it is known from the interior of the bubble and we can integrate. For an underwater explosion, $p_a \gg p_\infty$, and for the collapse of a void, $p_a \ll p_\infty$.

For small oscillations of a gas bubble, $a = a_0 + \eta(t)$, where $|\eta| \ll a_0$. Linearising, we get

$$\rho a_0 \ddot{\eta} = p(a) - p_\infty.$$

Assume p_∞ is constant, and the gas in the bubble obeys pV^γ is constant (the adiabatic variation in an ideal gas is $\gamma = \frac{c_1}{c_v} = 1.4$ for air). Then,

$$\begin{aligned} \frac{\delta p}{p} &= -\gamma \frac{\delta V}{V} = -3\gamma \frac{\partial a}{a} \\ &\implies \rho a_0 \ddot{\eta} - p_0(-3\gamma)\frac{\eta}{a_0}. \end{aligned}$$

This is simple harmonic motion with

$$\omega = \left(\frac{3\gamma p_0}{\rho a_0^2} \right)^{1/2}.$$

5 Geophysical Flows

5.1 Water Waves

These are a particularly successful application of potential flow. Consider the waves.

5.1.1 Governing Equations

Assume water is inviscid, and motion starts from rest. Hence flow is, and remains, irrotational. So $\mathbf{u} = \nabla\phi$ and $\nabla^2\phi = 0$ in the region $-h < z < \zeta(x, y, t)$.

The kinematic boundary conditions: at the rigid bottom,

$$\frac{\partial \phi}{\partial z} = 0,$$

at $z = -h$. At the air-water interface, often called the free surface as it is free to move, we have

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = w,$$

at $z = \zeta$. Moreover the dynamic boundary condition means $p_{\text{water}} = p_{\text{air}}$ at the air-water interface.

As $\rho_{\text{air}} \ll \rho_{\text{water}}$, we can assume that the pressure variations in the air are must less than those in the water. Hence $p_{\text{air}} = \text{constant} = p_0$. Then Bernoulli for potential flow gives

$$\rho\frac{\partial \phi}{\partial t} + \frac{1}{2}\rho u^2 + p + \chi = f(t),$$

independent of x. Applying to the surface and using $p = p_0 = \text{constant}$,

$$\rho\frac{\partial \phi}{\partial t} + \frac{1}{2}\rho|\nabla\phi|^2 + \rho g\zeta = f(t),$$

at $z = \zeta(x, y, t)$. These equations give the full non-linear problem. While the Laplace equation is linear, it is on an unknown domain, and the boundary conditions and Bernoulli equations are complicated non-linear boundary conditions to be applied at an unknown interfacial position ζ.

5.1.2 Linear Water Waves

For small amplitudes, $\zeta \ll h$ and $\frac{\partial \zeta}{\partial x}, \frac{\partial \zeta}{\partial y} \ll 1$, we can linearise the problem about a state of rest. We:

(a) Ignore the quadratic terms in the disturbance equation, such as $u\frac{\partial \zeta}{\partial x}$ and ρu^2,

(b) Use Taylor series to expand the boundary conditions at $z = \zeta$ in terms of information at $z = 0$, e.g.

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=\zeta} = \left.\frac{\partial \phi}{\partial z}\right|_{z=0} + \zeta \left.\frac{\partial^2 \phi}{\partial z^2}\right|_{z=0} + \cdots$$

The linearised problem is:

$$\begin{aligned}
\nabla^2 \phi &= 0 && \text{in } -h < z < 0, && (1)\\
\frac{\partial \phi}{\partial z} &= 0 && \text{at } z = -h, && (2)\\
\frac{\partial \zeta}{\partial t} &= \frac{\partial \phi}{\partial z} && \text{at } z = 0, && (3)\\
\rho \frac{\partial \phi}{\partial t} + \rho g \zeta &= f(t) && \text{independent of } x, y \text{ at } z = 0. && (4)
\end{aligned}$$

From Methods, we can either look for separable solutions, of the form

$$\phi(x, y, z, t) = \Phi(z) X(x) Y(y) T(t),$$

and find

$$X'' = -k^2 X, \qquad Y'' = -l^2 Y, \qquad T'' = -\omega^2 T,$$

or, on an infinite domain, take Fourier transforms with respect to x, y and t. In 2D, we end up looking for a solution

$$\phi = e^{i(kx - \omega t)} \Phi(z), \qquad \zeta = \zeta_0 e^{i(kx - \omega t)}.$$

Then from (1), $\nabla^2 \phi = 0$, $\Phi'' = k^2 \Phi \neq 0$. We thus have

$$\Phi = A \cosh(k(z + h)) + B \sinh(k(z + h)),$$

and due to boundary condition (2), $B = 0$.

In condition (4), the left-hand side is proportional to e^{ikx}, hence by independence of x, $f(t) = 0$ and the left-hand side is 0. Hence, (3) and (4) become

$$-i\omega \zeta_0 = Ak \sinh kh, \qquad -i\omega A \cosh kh + g\zeta_0 = 0.$$

These are linear, homogeneous equations with non-zero solutions if and only if

$$\begin{vmatrix} -k \sinh kh & -i\omega \\ -i\omega \cosh kh & g \end{vmatrix} = 0.$$

Hence we get $\omega^2 \cosh kh = gk \sinh kh$, or in other words, solutions must satisfy the *dispersion relationship*:

$$\omega^2 = gk \tanh kh.$$

The wavecrest moves with *phase speed*

$$c = \frac{\omega}{k}.$$

As a result, long waves propagate faster.

Unlike light and sound, water waves are dispersive: waves of different frequencies propagate at different speeds and thus disperse as they propagate away from a local disturbance.

We can show that a wave packet moves with *group velocity*

$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}.$$

Limiting Cases:

1. Deep-water limit: We have $kh \gg 1$, so $\tanh kh \approx 1$. Thus $\omega = \sqrt{gk}$, $c = \sqrt{\frac{g}{k}}$, and $c_g = \frac{1}{2}c$ independent of h.

Example 5.1.

Atlantic storms produces swell waves with period $15\,\mathrm{s}$, hence $\omega = \frac{2\pi}{15}\mathrm{s}^{-1} \approx 0.4\,\mathrm{s}^{-1}$.

Therefore $k^{-1} = \frac{g}{\omega^2} = 60\,\mathrm{m}$, which is much less than the ocean depth of $4000\,\mathrm{m}$ (on average).

However the phase speed is $c = \frac{\omega}{k} = 15\,\mathrm{m\,s^{-1}}$, approximately 1000 kilometres per day. Therefore the waves arrive before the storm.

2. Shallow water limit: We have $kh \ll 1$, so $\tanh kh \approx kh$. Hence $\omega \approx \sqrt{gh}k$, $c \approx \sqrt{gh}$ and $c_g = c$, independent of k.

Example 5.2.

Hood waves on a river have $h = 2\,\mathrm{m}$, so $c = \sqrt{gh} = 4\,\mathrm{m\,s^{-1}}$.

A tsunami from the ocean will have $h = 4000\,\mathrm{m}$ on average, so $c = \sqrt{gh} = 200\,\mathrm{m\,s^{-1}}$, roughly the speed of a commercial plane. The wavelength is $\lambda \approx 500\,000\,\mathrm{m}$, as in Boxing Day 2004.

Now we look at the velocities. We have

$$\zeta = \zeta_0 \exp(i(kx - \omega t)), \qquad \phi = \frac{-i\omega\zeta_0}{k} \frac{\cosh k(z+h)}{\sinh kh} \exp(i(kx - \omega t)),$$

hence

$$\mathbf{u} = \nabla\phi = (\cosh k(z+h), -i \sinh k(z+h)) \frac{\zeta_0 \omega}{\sinh kh} \exp(i(kx - \omega t)).$$

We have u_x and ζ are in phase.

1. Deep water limit: We have $kh \gg 1$, so cosh, sinh $\approx$ exp, hence

   $$\mathbf{u} \approx (1, -i) \exp(kz) \zeta_0 \omega \exp(i(kx - \omega t)).$$

   This is circular motion, smaller at depth.

2. Shallow water limit: We have $kh \ll 1$, so

   $$\mathbf{u} \approx \left(\frac{1}{kh}, -i\frac{z+h}{h} \right) \zeta_0 \omega \exp(i(kx - \omega t)).$$

   Note kh is small, so horizontal velcoity is much larger than vertical velocity.

5.1.3 Standing Waves in a Container

Consider water in a deep rectangular box:

$$0 < x < a, \qquad 0 < y < b, \qquad z < 0.$$

We look for linearised water waves with surface at $z = \zeta(x, y, t)$, with $|\zeta| \ll a, b$, and $|\nabla\zeta| \ll 1$. Hence we want to solve:

$$\begin{aligned}
\nabla^2 \phi &= 0 && z < 0, \\
\frac{\partial \phi}{\partial A} &= 0 && \text{on the sides}, \\
\nabla \phi &\to 0 && z \to -\infty, \\
\frac{\partial \zeta}{\partial t} &= \frac{\partial \phi}{\partial z} && z = 0, \\
\rho \frac{\partial \phi}{\partial t} &+ \rho g \zeta \text{ independent of } x, y && z = 0.
\end{aligned}$$

We try separation of variables:

$$\phi(x, y, z, t) = A \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{kz} e^{i\omega t},$$

where Laplace's equation gives

$$-\frac{m^2\pi^2}{a^2}-\frac{n^2\pi^2}{b^2}+k^2=0.$$

The kinematic boundary conditions imply

$$\frac{\partial\zeta}{\partial t}=\frac{\partial\phi}{\partial z}\implies\zeta=\frac{iAk}{\omega}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{-i\omega t}.$$

The dynamic boundary condition gives

$$\frac{\partial\phi}{\partial t}+g\zeta=\text{ constant }\implies -i\omega A+g\frac{iAk}{\omega}=0\implies\omega^2=gk,$$

the same as for deep-water waves. The only difference is k is quantized by the side walls:

$$k_{mn}=\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)^{1/2}.$$

These are standing waves/normal modes of the form $\zeta=f_{mn}(x,y)\cos(\omega_{mn}t)$.

5.1.4 Rayleigh Instability

Consider an upside down container. There is an obvious instability. Then what is the growth rate of the instability?

Just let $g\to -g$, which implies $\omega^2=-gk$. Hence

$$\omega=\pm i\sqrt{gk}\implies e^{-i\omega t}=e^{\pm\sqrt{gk}t}.$$

5.2 Rotating Fluid Dynamics

We live on a rotating Earth, with

$$\Omega=\frac{2\pi}{86\,400\,\mathrm{s}}\approx 10^{-4}\,\mathrm{s}^{-1},$$

and the rotation has a great effect on the length-scale motion of the oceans and atmosphere, hence the climate. Moreover, the ocean and atmosphere are thin (about 4000 m and 10 000 m respectively) compared to the Earth's radius (which is around 6×10^7 m).

The rotation of an Earth leads to a speed $R\Omega$ at the equator, around $500\,\mathrm{m\,s^{-1}}$, creating winds relative to the Earth. Hence it is sensible to work in a rotating frame.

5.3 Euler Equations in a Rotating Frame

Assume ρ uniform and $\nabla \cdot \mathbf{u} = 0$; this is a good approximation for the ocean, but we need a small fix to make it work for the atmosphere.

The acceleration of a fluid particle in a rotating frame is

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{u} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x}).$$

Now as

$$\rho\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x}) = \nabla\left(\frac{1}{2}\rho|\boldsymbol{\Omega} \times \mathbf{x}|^2\right),$$

we can combine this with $\rho\mathbf{g} = -\nabla\chi$, and treat this as an effective $\mathbf{g}$ with an effective definition. This force is normal to the surface of the spheroidal Earth. Hence

$$\rho\frac{D\mathbf{u}}{Dt} + 2\rho\boldsymbol{\Omega} \times \mathbf{u} = -\nabla p + \rho\mathbf{g}.$$

Moreover,

$$\nabla \times (2\boldsymbol{\Omega} \times \mathbf{u}) = 2\boldsymbol{\Omega}\nabla \cdot \mathbf{u} = 2\boldsymbol{\Omega} \cdot \nabla\mathbf{u},$$

hence the vorticity equation becomes

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \nabla\mathbf{u}.$$

The vortex stretching term now involves the *total vorticity*

$$\nabla \times (\mathbf{u} + \boldsymbol{\Omega} \times \mathbf{x}) = \boldsymbol{\omega} + 2\boldsymbol{\Omega}.$$

5.3.1 Rossay Number

Suppose we have a flow with characteristic lengthscale L, velocity lengthscale U and timescale $T = \frac{L}{U}$. Then,

$$\rho(\mathbf{u} \cdot \nabla)\mathbf{u} \sim \frac{\rho U^2}{L}, \qquad 2\rho\boldsymbol{\Omega} \times \mathbf{u} \sim \rho\Omega U.$$

The ratio

$$\frac{\rho U^2/L}{\rho U\Omega} = \frac{U}{\Omega L} \equiv R_0$$

is the *Rossay number*, which describes the relative importance of inertia and the Coriolis effect.

Example 5.3.

For a weather system with $L \sim 10^7\,\mathrm{m}$, $U = 10\,\mathrm{m\,s^{-1}}$ and $T \sim 10^5\,\mathrm{s}$, $R_0 \sim 10^{-1}$, and hence the Coriolis terms dominate.

When emptying a bathtub, near the plughole $L \sim 0.3\,\mathrm{m}$, $U \sim 0.03\,\mathrm{m\,s^{-1}}$, and $T \sim 10\,\mathrm{s}$. Hence $R_0 \sim 10^3$, and the Coriolis effects are just a tiny perturbation.

When $R_0 \ll 1$, we have $\boldsymbol{\omega} < \frac{U}{L} \ll \boldsymbol{\Omega}$, so the relative vorticity is much less than the planetary vorticity.

5.3.2 Rotating Flow in a Shallow Layer

Consider flows with horizontal lengthscale L such that $H \ll L \ll R$. Take local Cartesian coordinates, where x moves Eastwards, y moves Northwards and z moves upwards, and let $\mathbf{u} = (u, v, w)$.

If $\bar{\theta}$ is the latitude and $\boldsymbol{\Omega} = (0, \Omega\cos\bar{\theta}, \Omega\sin\bar{\theta})$, then

$$2\boldsymbol{\Omega} \times \mathbf{u} = 2\Omega(w\cos\bar{\theta} - v\sin\bar{\theta}, u\sin\bar{\theta}, -u\cos\bar{\theta}).$$

We make three good approximations.

1. For $|\mathbf{u}| \sim 10\,\mathrm{m\,s^{-1}}$,
$$\frac{\Omega u}{g} \sim \frac{10^{-4}10}{10} = 10^{-4} \ll 1,$$
so we can neglect $-u\cos\bar{\theta}$ in the vertical equation.

2. For $L \gg H$,
$$\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
suggest that
$$\frac{w}{H} \sim \frac{(u,v)}{L} \implies w \sim \frac{H}{L}(u,v) \ll (u,v),$$
so the flow is dominantly horizontal (if the velocities are in shallow water waves). Hence, we can neglect $w\cos\bar{\theta}$ in the x-direction (except near the equator), and neglect the vertical acceleration $\rho\frac{Dw}{Dt}$, so the vertical force balance is hydrostatic.

Hence we can simplify to

$$\begin{aligned}
\frac{Du}{Dt} - fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x},\\
\frac{Dv}{Dt} + fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y},\\
-\frac{1}{\rho}\frac{\partial p}{\partial z} - g &= 0,\\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0.
\end{aligned}$$

Here $f = 2\Omega \sin\bar{\theta}$ is the *Coriolis parameter*. Also,

$$\frac{\partial w}{\partial z} \sim \frac{w}{H} \gg \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \sim \frac{w}{L},$$

so the vertical component of the vorticity equation becomes

$$\frac{D\zeta}{Dt} = (\zeta + f)\frac{\partial w}{\partial z} = (-\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right),$$

where $\zeta = \omega_z$.

Consider flow in a shallow layer of thickness $h(x, y, t)$. Then

$$\frac{\partial p}{\partial z} = \rho g \implies p = p_0 + \rho g(h - z) \implies \nabla_H p = \rho g \nabla_H h$$

is independent of z, where $\nabla_H = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$.

5.3.3 Geostrophic Balance

If $R_0 \equiv \frac{U}{\Omega L} \ll 1$ and we have a steady flow, then the shallow water equations becomes

$$\begin{aligned}
-fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x} = -g\frac{\partial L}{\partial x},\\
+fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y} = -g\frac{\partial L}{\partial y}.
\end{aligned}$$

Henece flow is given by the two-dimensional streamfunction

$$\psi = -\frac{p}{\rho f} = -\frac{gh}{f},$$

i.e. the pressure and height contours are streamlines. Therefore, winds blow parallel to isobars, and not perpendicular.

The equation

$$\rho(f\mathbf{e}_z) \times \mathbf{u} = -\nabla_H p$$

is called *geostrophic balance*. Note, if given $h(x, y)$ we can calculate the steady flow, but there is no such equation for h or $\partial L/\partial t$.

5.3.4 Potential Vorticity

Recall that:

$$\frac{D}{Dt}(\zeta + f) = (-\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right),$$
$$\frac{D}{Dt}h = -h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right).$$

Hence we get

$$\frac{D}{Dt}\left(\frac{\zeta + f}{h}\right) = 0.$$

The quantity $(\zeta + f)/h$ is called the *potential vorticity*.

Potential vorticity is conserved for a material cylinder; the total vorticity $\zeta + f$ is proportional to the height of a cylinder, as vertical stretching leads to a smaller radius, which leads to faster spinning.

Potential vorticity is useful because, taking the curl of the momentum equation, it eliminates the largest terms ∇p and $2\rho\mathbf{\Omega} \times \mathbf{u}$ of nearly geostrophic balance, and reveals how the smaller terms give the evaluation of h with time.

5.3.5 Linearised Evolution

Suppose $h = h_0 + \eta(x, y, t)$, where $|\eta| \ll h_0$ and $|\nabla\eta| \ll \frac{H}{L} \ll 1$. The linearised equations are:

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x},$$
$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y},$$
$$\frac{\partial \eta}{\partial t} + h_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0,$$
$$\frac{\partial}{\partial t}\left(\frac{f}{h_0}\frac{1 + \zeta/f}{1 + \eta/h_0}\right) = 0,$$

where the last equation implies

$$\frac{\zeta}{f} - \frac{\eta}{h_0} \text{ is constant,}$$

i.e. $\zeta h_0 - \eta f$ equals the initial value, $\zeta_0 h_0 - \eta_0 f$. Now taking

$$h_0\left(\frac{\partial(1)}{\partial x} + \frac{\partial(2)}{\partial y}\right) - \frac{\partial(M)}{\partial t} \implies -\frac{\partial^2\eta}{\partial t^2} + gh_0\nabla^2\eta - h_0 f\zeta = 0.$$

Combining with the above equation, we get

$$\frac{\partial^2\eta}{\partial t^2} + c^2\nabla^2\eta + f^2\eta = f^2\eta_0 - h_0 f\zeta_0,$$

where $c = \sqrt{gh_0}$ is the shallow-water wave speed.

Example 5.4.

If $\eta_0 = \varepsilon\,\mathrm{sgn}(x)$, and $\mathbf{u}_0 = 0$, then $\zeta_0 = 0$. Without rotation, so if $f = 0$, then we just have

$$\frac{\partial^2\eta}{\partial t^2} = c^2\frac{\partial^2\eta}{\partial x^2},$$

with d'Alembert solution.

With rotation, the final steady state solves

$$\frac{\partial^2\eta_\infty}{\partial x^2} - \frac{f^2}{c^2}\eta_\infty = -\frac{f^2}{c^2}\varepsilon\,\mathrm{sgn}(x),$$

which solves to give $\eta_\infty = \varepsilon\,\mathrm{sgn}(x)(1 - e^{-|x|f/c})$. Hence

$$u_\infty = -\frac{g}{f}\frac{\partial\eta_\infty}{\partial y} = 0, \qquad v_\infty = \frac{g}{f}\frac{\partial\eta_\infty}{\partial x} = \varepsilon\frac{g}{c}e^{-|x|f/c}.$$

Here $a = \frac{c}{f} = \frac{\sqrt{gh_0}}{f}$ is the Rossby radius of deformation.

Index

Archimedes principle, 17

Bernoulli's equation, 13
body force, 10

circulation, 21
continuity of stress, 26
continuum mechanics, 3
Coriolis parameter, 49
Couette flow, 22

diffusion equation, 27
dispersion relationship, 44
dynamic pressure, 17
dynamic viscosity, 22

Euler momentum equation, 12

fields, 3

geostrophic balance, 50
group velocity, 44

hydrostatic pressure, 17

incompressible, 6
inviscid, 10
irrotational flow, 34

Kelvin's circulation theorem, 21
kinematic viscosity, 22

line source, 36

mass flux, 6
material derivative, 5

Navier-Stokes equation, 30
no-slip boundary condition, 24

pathline, 4
phase speed, 44
point sink, 35
point source, 35, 36
point vortex, 36
potential vorticity, 50

Rossay number, 47
Rossby radius, 51

source dipole, 35
steady-state flows, 4
Stokes equations, 31
Stokes number, 30
stream function, 8
streamlines, 4
stress, 10
surface force, 10

total vorticity, 47

unsteady Bernoulli, 38

velocity potential, 34
volume force, 10
vortex stretching, 20
vorticity, 13, 17