IB Complex Analysis

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1 Complex Differentiation

Our goal in this course is to study the theory of complex-valued differentiable functions in one complex variable. Example include:

- Polynomials $p(z) = a_d z^d + \cdots + a_1 z + a_0$, with coefficients in $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$ or $\mathbb{C}$.
- The infinite series
$$\sum_{n=1}^{\infty} \frac{1}{n^z},$$
which we showed convergence for z having real part greater than 1.
- Harmonic functions $u(x, y) : \mathbb{R}^2 \to \mathbb{R}$, $u_{xx} + u_{yy} = 0$.

In this course, we make the convention that $\theta = \arg(z) \in [0, 2\pi)$.

1.1 Basic Notions

- $U \subset \mathbb{C}$ is *open* if for all $u \in U$, there exists $\varepsilon > 0$ such that
$$D(x, \varepsilon) = \{z \in \mathbb{C} \mid |z - u| < \varepsilon\} \subset U.$$
- A *path* in $U \subset \mathbb{C}$ is a continuous map $\gamma : [a, b] \to U$. We say the path is C^1 if γ' exists and is continuous (we take one-sided derivatives at the endpoints).

  γ is *simple* if it is injective.
- $U \subset \mathbb{C}$ is *path-connected* if for all $z, w \in U$, there exists a path in U with endpoints at z, w.

Remark. If U is open, and $z, w \in U$ are connected by a path γ in U, then there exists a path γ in U connected z, w consisting of finitely many horizontal and vertical segments.

Definition 1.1. A *domain* is a non-empty, open, path-connected subset of $\mathbb{C}$.

Definition 1.2.

(i) $f : U \to \mathbb{C}$ is *differentiable* at $u \in U$ if
$$f'(u) = \lim_{z \to u} \frac{f(z) - f(u)}{z - u}$$
exists.

(ii) $f : U \to \mathbb{C}$ is *holomorphic* at $u \in U$ if there exists $\varepsilon > 0$ such that f is differentiable at z, for all $z \in D(u, \varepsilon)$. We may also call such a function *analytic*.

(iii) $f : \mathbb{C} \to \mathbb{C}$ is *entire* if it is holomorphic everywhere.

Remark. All differentiation rules (sum, products, ...) in $\mathbb{R}$ hold, by the same proofs.

Identifying $\mathbb{C}$ with $\mathbb{R}^2$, we may write $f : U \to \mathbb{C}$ as $f(x+iy) = u(x,y) + iv(x,y)$, where u, v are the real and imaginary parts of f.

From analysis and topology, recall that $u : U \to \mathbb{R}$ as a function of two real variables if ($\mathbb{R}^2$) differentiable at $(c,d) \in \mathbb{R}^2$with $Du|_{(c,d)} = (\lambda, \mu)$ if

$$\frac{u(x,y) - u(c,d) - [\lambda(x-c) + \mu(y-d)]}{\sqrt{(x-c)^2 + (y-d)^2}} \to 0,$$

as $(x,y) \to (c,d)$. However, **this is a weaker condition** than differentiability over $\mathbb{C}$.

Proposition 1.1 (Cauchy-Riemann equations)**.** *Let $f : U \to \mathbb{C}$ on an open set $U \subset \mathbb{C}$. Then f is differentiable at $w = c+id \in U$ if and only if, writing $f = u+iv$, we have u, v are $\mathbb{R}^2$-differentiable at (c,d), and*

$$u_x = v_y, \qquad\qquad u_y = -v_x.$$

Proof: f is differentiable at w if and only if $f'(w) = p + iq$ exists, so

$$\lim_{z \to w} \frac{f(z) - f(w) - (z-w)(p+iq)}{|z-w| = 0}.$$

Writing $f = u + iv$ and considering the real and imaginary parts in the quotient above, this holds if and only if

$$\lim_{(x,y)\to(c,d)} \frac{u(x,y) - u(c,d) - [p(x-c) - q(y-d)]}{\sqrt{(x-c)^2 + (y-d)^2}} = 0,$$

and

$$\lim_{(x,y)\to(c,d)} \frac{v(x,y) - v(c,d) - [q(x-c) + p(y-d)]}{\sqrt{(x-c)^2 + (y-d)^2}} = 0.$$

This holds if and only if u, v are $\mathbb{R}^2$-differentiable at (c,d), and $u_x = v_y$, $u_y = -v_x$.

Remark. If the partial u_x, u_y, v_x, v_y exist and are continuous on U, then u, v are differentiable on U. So it suffices to check the partials exist and are continuous, and the Cauchy-Riemann equations hold to deduce complex differentiability.

Example 1.1.

1. Take $f(z) = \overline{z}$. Then f has $u(x, y) = x$ and $v(x, y) = -y$, so $u_x = 1$, $v_y = -1$. So $f(z) = \overline{z}$ is not holomorphic or differentiable anywhere.
2. Any polynomial $p(z) = a_d z^d + \cdots + a_1 z + a_0$, with $a_i \in \mathbb{C}$ is entire.
3. Rational function, which are quotients of polynomials $\frac{p(z)}{q(z)}$ are holomorphic on the open set $\mathbb{C} \setminus \{\text{zeroes of } q\}$.

Note that $f = u + iv$ satisfying the Cauchy-Riemann equations at a point does not mean it is differentiable at that point.

Some proofs in regular analysis have natural extensions to complex analysis. For example, if $f : U \to \mathbb{C}$ on a domain U with $f'(z) = 0$ on U, then f is constant on U.

Now we ask: why are we interested in complex analysis?

- Unlike $\mathbb{R}^2$ differentiable functions, holomorphics functions are very constrained. For example, if f is entire and bounded (so $|f(z)| < M$ for all $z \in \mathbb{C}$), then f is constant. Contrast with sin, for example.
- We will see that f holomorphic on a domain U has holomorphic derivative on U. This implies that f is infinitely differentiable, as are u and v.

In particular, we can differentiate the Cauchy-Riemann equations to get

$$u_{xx} = v_{yx} = v_{xy} = -u_{yy},$$

so $u_{xx} + u_{yy} = 0$, and similarly $v_{xx} + v_{yy} = 0$. Hence the real and imaginary parts of a holomorphic function are harmonic.

Let $f : U \to \mathbb{C}$ be a holomorphic function on an open set U_1 and $w \in U$ with $f' f(w) \neq 0$. We want to look at the geometric behaviour of f at w.

In fact, we claim f is *conformal* at w. Let γ_1, γ_2 be C^1-paths through w, say $\gamma_1, \gamma_2 : [-1, 1] \to U_1$, such that $\gamma_1(0) = \gamma_2(0) = w$, and $\gamma_i'(0) \neq 0$. If we write $\gamma_j(t) = w + r_j(t) = e^{i\theta_j(t)}$, then we have

$$\arg(\gamma_j'(z)) = \theta_j(0),$$

and the argument of the image line is

$$\arg((f \circ \gamma_j)'(0)) = \arg(\gamma_j'(0) f'(\gamma_j(0))) = \arg(\gamma_j'(0)) + \arg(f'(w)) + 2\pi n,$$

where crucially we use $\gamma_j'(0)f'(\gamma_j(0)) \neq 0$, so the direction of γ_j at w under the application of f is rotated by $\arg(f'(w))$. This is independent of γ_j. Since the angle between γ_1 and γ_2 is the difference of the arguments f preserves the angle. This is what it means to be conformal.

Definition 1.3. Let U, V be domains in $\mathbb{C}$. A map $f : U \to V$ is a *conformal equivalence* of U and V if f is a bijective holomorphic map with $f'(z) \neq 0$, for all $z \in U$.

Remark.

1. Using the real inverse function theorem, one can show if $f : U \to V$ is a holomorphic bijection of open sets with $f'(z) \neq 0$ for all $z \in U$, then the inverse of f is also holomorphic, so also conformal by the chain rule. So conformally equivalent domains are equal from the perspective of the functions f.

2. We will later see than being injective and holomorphic on a domain implies $f'(z) \neq 0$ for all $z \in U$, so this requirement is redundant.

Example 1.2.

1. Any change of coordinates: on $\mathbb{C}$, take $f(z) = az + b$, for $a \neq 0$ and b, which is a conformal equivalence $\mathbb{C} \to \mathbb{C}$. More generally, a Möbius map
$$f(z) = \frac{az+b}{cz+d},$$
for $ad - bc \neq 0$, is a conformal equivalence from the Riemann sphere to itself. This can eb seen as adding a point at infinity to make a sphere $\mathbb{C}_\infty$ (or gluing two copies of the unit disc with coordinates z and $\frac{1}{z}$).

   If $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is continuous, then

   - if $f(\infty) = \infty$, then f is holomorphic at ∞ if and only if $g(z) = \frac{1}{f(\frac{1}{z})}$ is holomorphic at 0.
   - If $f(\infty) \neq \infty$, then f is homolorphic at ∞ if and only if $f(\frac{1}{z})$ is holomorphic at 0.
   - If $f(a) = \infty$ for $a \in \mathbb{C}$, then f is holomorphic at a if and only if $\frac{1}{f(z)}$ is holomorphic at a.

   We can then think of Möbius maps as change of coordinates for the sphere.

Choosing $z_1 \to 0$, $z_2 \to \infty$, $z_3 \to 1$ defined a Möbius map

$$f(z) = \frac{z - z_1}{z - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_1},$$

for distinct $z_1, z_2, z_3 \in \mathbb{C}$.

2. For $n \in \mathbb{N}$, $f(z) = z^n$ is a conformal equivalence from the sector $\{z \in \mathbb{C}^\times \mid 0 < \arg z < \frac{\pi}{n}\}$ to the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \Im z > 0\}$.
3. The Möbius map $f(z) = \frac{z-i}{z+i}$ is a conformal equivalence between $\mathbb{H}$ and $D(0,1)$. We can compute $f'(z) \neq 0$ on $\mathbb{H}$, and

$$z \in \mathbb{H} \iff |z - i| < |z + i| \iff |f(z)) < 1.$$

   Note that $f^{-1}(w) = -i\frac{w+1}{w-1}$.
4. We can use these examples to write down conformal equivalences. Let U_1 be the upper half semicircle, and U_2 the lower half plane. Considering $g(z) = \frac{z+1}{z-1}$, we know that sends $D(0,1)$ to the left half-plane, so it sends U_1 to the upper left quadrant.

   Then, the upper left quadrant if mapped by the squaring map to U_2. So $f(z) = (\frac{z+1}{z-1})^2$ is a conformal equivalence from $U_1 \to U_2$.

These are all examples of the deep *Riemann mapping theorem*:

Theorem 1.1 (Riemann mapping theorem)**.** *Let $U \subset \mathbb{C}$ be a proper domain which is simply connected. Then there exists a conformal equivalence between U and $D(0,1)$.*

Here, *simply connected* means a subset $U \subset \mathbb{C}$ which is path-connected, and contractible: any loop in U can be contracted to a point. So any continuous path $\gamma : S^1 \to U$ extends to a continuous map $\hat{\gamma} : D(0,1) \to U_1$ with $\hat{\gamma}|_{S_1} = \gamma$.

In fact any domain bounded by a simple closed curve is simply connected, so all of these are conformally equivalent to $D(0,1)$.

Example 1.3.

We look at a domains in the Riemann sphere, with bounded and connected complement. This is simply connected as a subset of $\mathbb{C}_\infty$.

Now, the Mandelbrot set is bounded and connected, so the complement of the Mandelbrot set is simply connected in $\mathbb{C}_\infty$.

Recall the following facts about functions defined by power series, or sequences of functions:

1. A sequence (f_n) of functions *converges uniformly* to a function f on some set S if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and for all $x \in S$,
$$|f_n(x) - f(x)| < \varepsilon.$$
2. The uniform limit of continuous functions is continuous.
3. The *Weierstrass M-test*: if there exists $M_n \in \mathbb{R}$ for all n such that $0 \leq |f_n(x)| \leq M_n$ for all $x \in S$, then
$$\sum_{n=1}^{\infty} M_n \infty \implies \sum_{n=1}^{\infty} f_n(x) \text{ converges uniformly on } S \text{ as } N \to \infty.$$
4. Let (c_n) be complex numbers, and fix $a \in \mathbb{C}$. Then there exists unique $R \in [0, \infty]$ such that the function
$$z \mapsto \sum_{n=1}^{\infty} c_n (z-a)^n$$
converges absolutely if $|z - a| < R$, and diverges if $|z - a| > R$. If $0 < r < R$, then the series converges uniformly in $D(a, r)$. R is the *radius of convergence* of the series. We can compute
$$R = \sup\{r \geq 0 \mid |c_n| r^n \to 0\},$$
or
$$R = \frac{1}{\lambda}, \qquad \lambda = \limsup_{n \to \infty} |c_n|^{1/n}.$$

Theorem 1.2.
$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$
is a complex power series with radius of convergence R. Then,

(i) *f is holomorphic on $D(a, R)$.*

(ii) *f has derivative*
$$f'(z) = \sum_{n=1}^{\infty} n c_n (z-a)^{n-1},$$
with radius of convergence R about a.

(iii) *f has derivatives of all orders on $D(a, R)$, and $f^{(n)}(a) = n! c_n$.*

Proof: We can let $a = 0$ by change of variables $z \to z - a$. Consider the series
$$\sum_{n=1}^{\infty} nc_n z^{n-1}.$$
Since $|nc_n| \geq |c_n|$, the radius of convergence of this series is no larger than R. If $0 < R_1 < R$, then for $|z| < R_1$, we have
$$|nc_n z^{n-1}| = n|c_n|R_1^{n-1}\frac{|z|^{n-1}}{R_1^{n-1}},$$
and
$$n\left(\frac{|z|}{R_1}\right)^{n-1} \to 0.$$
Applying the M-test with $M_n = c_n R_1^{n-1}$, we have the convergence of the series. So the series has radius of convergence R.

Now for $|z|, |w| < R$, we need to consider
$$\frac{f(z) - f(w)}{z - w}.$$
Taking the partial sums,
$$\sum_{n=0}^{N} c_n \frac{z^n - w^n}{z - w} = \sum_{n=0}^{N} c_n \left(\sum_{j=0}^{n-1} z^j w^{n-1-j}\right).$$
For $|z|, |w| < \rho < R$, we have
$$\left| c_n \left(\sum_{j=0}^{n-1} z^j w^{n-1-j}\right)\right| \leq |c_n| n\rho^{n-1}.$$
Hence the partial sums converge uniformly on $\{(z, w) \mid |z|, |w| < \rho\}$. So the series converges to a continuous limit on $\{|z|, |w| < R\}$, say $g(z, w)$. When $z \neq w$, we know
$$g(z, w) = \frac{f(z) - f(w)}{z - w}.$$
When $z = w$, we have
$$g(w, w) = \sum_{n=0}^{\infty} nc_n w^{n-1}.$$
Hence by the continuity of g, this proves (i) and (ii). Then (iii) follows from a simple induction.

Corollary 1.1. *Suppose* $0 < \rho < R$, *where* R *is the radius of convergence of the complex power series*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n,$$

and $f(z) = 0$ *for all* $z \in D(a, \rho)$. *Then* $f \equiv 0$ *on* $D(a, R)$.

Proof: Since $f \equiv 0$ on $D(a, \rho)$, we have $f^{(n)}(a) = 0$ for all n. Hence $c_n = 0$ for all n, so $f \equiv 0$ on $D(a, R)$.

1.2 Exponential and Logarithm

We define the complex exponential

$$e^z = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

The complex exponential has the following properties:

1. It has radius of convergence ∞, so the function is entire, and we have $\frac{\mathrm{d}}{\mathrm{d}z} e^z = e^z$.
2. For all $z, w \in \mathbb{C}$, $e^{z+w} = e^z e^w$, and $e^z \neq 0$.

   This follows from setting $F(z) = e^{z+w} e^{-z}$, then taking the derivative,

   $$F'(z) = e^{z+w} e^{-z} - e^{z+w} e^{-z} = 0,$$

   so F is constant. Since $e^0 = 1$, $F(z) = e^w$, and $e^{z+w} = e^z e^w$. Since $e^z e^{-z} = e^0 = 1$, $e^z \neq 0$.
3. Let $z = x + iy$. Then $e^z = e^{x+iy} = e^x e^{iy}$. But $e^{iy} = \cos y + i \sin y$, and note that $|e^{iy}| = 1$, so

   $$e^z = e^x (\cos y + i \sin y),$$

   and $|e^z| = e^x$, so $e^z = 1$ if and only if $x = 0$ and $y = 2\pi k$ for $k \in \mathbb{Z}$. In fact, for all $w \in \mathbb{C}^\times$, there exist infinitely many $z \in \mathbb{C}$ such that $e^z = w$, differing by integer multiples of $2\pi i$.

Definition 1.4. Let $U \subset \mathbb{C}^\times$ be an open set. We say a continuous function $\lambda : U \to \mathbb{C}$ is a *branch of the logarithm* if for all $z \in U$, $\exp(\lambda(z)) = z$.

Example 1.4.

Let $U = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Define $\log : U \to \mathbb{C}$ by

$$\log(z) = \ln|z| + i\theta,$$

where $\theta = \arg(z)$, and $\theta \in (-\pi, \pi)$. This is the *principal branch of the logarithm*.

Proposition 1.2. *$\log(z)$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ with derivative $\frac{1}{z}$. Moreover, if $|z| < 1$, then*

$$\log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}.$$

Proof: As an inverse to e^z and by the chain rule, we have $\log z$ is holomorphic with $\frac{\mathrm{d}}{\mathrm{d}z} \log z = \frac{1}{z}$, We have

$$\frac{\mathrm{d}}{\mathrm{d}z} \log(1+z) = \frac{1}{z+1} = 1 - z + z^2 - z^3 + z^4 - \cdots,$$

which is the derivative of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}.$$

So $\log(1+z)$ agrees with this series up to a constant. Since $\log(1) = 0$, the equality holds.

If $\alpha \in \mathbb{C}$, we can define $z^\alpha = \exp(\alpha \log z)$. This gives a definition of z^α on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. We can compute that $\frac{\mathrm{d}}{\mathrm{d}z} z^\alpha = \alpha z^{\alpha - 1}$.

It is not necessarily true that $z^\alpha w^\alpha = (zw)^\alpha$. Take $\alpha = \frac{1}{2}$, then

$$z^{1/2} = \exp\left(\frac{1}{2} \log z\right) = \exp\left(\frac{1}{2} \ln|z| + \frac{1}{2} i\theta\right),$$

for $\theta \in (-\pi, \pi)$. Hence the argument of $z^{1/2}$ is in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

1.3 Contour Integration

If $f : [a, b] \to \mathbb{C}$ is continuous, we define

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^b \Re(f(t))\,\mathrm{d}t + i \int_a^b \Im(f(t))\,\mathrm{d}t.$$

Proposition 1.3. *Let $f : [a, b] \to \mathbb{C}$ be continuous. Then,*

$$\left|\int_a^b f(t)\,\mathrm{d}t\right| \leq (b-a) \sup_{a \leq t \leq b} |f(t)|,$$

with equality if and only if f is constant.

Proof: Write $M = \sup_{a \leq t \leq b} |f(t)|$, and $\theta = \arg(\int_a^b f(t)\,\mathrm{d}t)$. Then

$$\begin{aligned}\left|\int_a^b f(t)\,\mathrm{d}t\right| &= e^{-i\theta} \int_a^b f(t)\,\mathrm{d}t = \int_a^b e^{-i\theta} f(t)\,\mathrm{d}t \\ &= \int_a^b \Re(e^{-i\theta} f(t))\,\mathrm{d}t \\ &\leq \int_a^b |f(t)|\,\mathrm{d}t \leq M(b-a).\end{aligned}$$

If we have equality, then $|f(t)| = M$, and $\arg f(t) = \theta$, so f is constant.

Definition 1.5. Let $\gamma : [a, b] \to \mathbb{C}$ be a C^1-smooth curve. Then we define the arc-length of γ to be

$$\text{length}(\gamma) = \int_{\mathrm{a}}^{\mathrm{b}} |\gamma'(\mathrm{t})|\,\mathrm{dt}.$$

We say γ is *simple* if $\gamma(t_1) = \gamma(t_2) \iff t_1 = t_2$ or $\{t_1, t_2\} = \{a, b\}$. If γ is simple, then length(γ) is the length of the image of γ.

Definition 1.6. Let $f : U \to \mathbb{C}$ be continuous, with U open, and $\gamma : [a, b] \to U$ be a C^1-smooth curve. Then the integral of f along γ is

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t.$$

This integral satisfies the following properties:

1. Linearity:
$$\int_\gamma c_1 f_1 + c_2 f_2\,\mathrm{d}z = c_1 \int_\gamma f_1\,\mathrm{d}z + c_2 \int_\gamma f_2\,\mathrm{d}z.$$

2. Additivity: if $a < a' < b$, then
$$\int_{\gamma|_{[a,a']}} f(z)\,\mathrm{d}z + \int_{\gamma|_{[a',b]}} f(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z.$$

3. Inverse path: if $(-\gamma)(t) = \gamma(-t)$ on $[-b, -a]$, then

$$\int_{-\gamma} f(z)\,\mathrm{d}z = -\int_{\gamma} f(z)\,\mathrm{d}z.$$

4. Independence of parametrization: if $\phi : [a', b'] \to [a, b]$ is C^1-smooth with $\phi(a') = a$, $\phi(b') = b$ and $\delta = \gamma \circ \phi$, then

$$\int_{\delta} f(z)\,\mathrm{d}z = \int_{\gamma} f(z)\,\mathrm{d}z.$$

   This lets us assume that $\gamma : [0, 1] \to U$.

We can loosen the restriction that γ is C^1-smooth and allow it to be piecewise C^1-smooth, i.e. there exist $a = a_0 < a_1 < \cdots < a_n = b$ such that $\gamma_i = \gamma|_{[a_{i-1}, a_i]}$ is C^1-smooth. Define then

$$\int_{\gamma} f(z)\,\mathrm{d}z = \sum_{i=1}^{n} \int_{\gamma_i} f(z)\,\mathrm{d}z.$$

Remark. Any piecewise C^1-smooth curve can be reparametrized to be C^1: for such a γ as above, replace γ_i by $\gamma_i \circ h_i$ where h_i is monotonic C^1-smooth bijection with endpoint derivative 0.

So C^1-smooth paths can have corners, for example

$$\gamma(t) = \begin{cases} 1 + i\sin(\pi t) & t \in [0, \frac{1}{2}], \\ \sin(\pi t) + i & t \in [\frac{1}{2}, 1]. \end{cases}$$

We say a "curve" is a piecewise C^1-smooth path, and a "contour" is a simple *closed* piecewise C^1-smooth path, where closed means the endpoints are equal.

Proposition 1.4. *For any continuous $f : U \to \mathbb{C}$ with U open, and any curve $\gamma : [a, b] \to U$,*

$$\left| \int_{\gamma} f(z)\,\mathrm{d}z \right| \leq \mathrm{length}(\gamma) \sup_{\mathrm{z} \in \gamma} |\mathrm{f(z)}|.$$

Proof:

$$\begin{aligned}\left|\int_\gamma f(z)\,\mathrm{d}z\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t\right| \\ &\le \int_a^b |f(\gamma(t))\gamma'(t)|\,\mathrm{d}t \\ &\le \sup_{z\in\gamma}|f(z)|\mathrm{length}(\gamma).\end{aligned}$$

Proposition 1.5. *If $f_n : U \to \mathbb{C}$ for $n \in \mathbb{N}$ and $f : U \to \mathbb{C}$ are continuous, and $\gamma : [a, b] \to U$ is a curve in U with $f_n \to f$ uniformly on γ, then*

$$\int_\gamma f_n(z)\,\mathrm{d}z \to \int_\gamma f(z)\,\mathrm{d}z,$$

as $n \to \infty$.

Proof: By uniform convergence, $\sup_{z\in\gamma}|f(z) - f_n(z)| \to 0$ as $n \to \infty$. So by the previous proposition,

$$\begin{aligned}\left|\int_\gamma f(z)\,\mathrm{d}z - \int_\gamma f_n(z)\,\mathrm{d}z\right| &\le \mathrm{length}(\gamma)\sup_\gamma|\mathrm{f} - \mathrm{f_n}| \\ &\to 0\end{aligned}$$

as $n \to \infty$.

Example 1.5.

Let $f_n(z) = z^n$ for $n \in \mathbb{Z}$ on $C^\times = U$, and $\gamma : [0, 2\pi] \to U$ with $\gamma(t) = e^{it}$. Then,

$$\int_\gamma f_n(z)\,\mathrm{d}z = \int_0^{2\pi} e^{nit}ie^{it}\,\mathrm{d}t = i\int_0^{2\pi} e^{(n+1)it}\,\mathrm{d}t = \begin{cases}2\pi i & n = -1,\\ 0 & n \neq -1.\end{cases}$$

Theorem 1.3 (Fundamental Theorem of Calculus)**.** *If $f : U \to \mathbb{C}$ is a continuous function on open $U \subset \mathbb{C}$ with $F' = f$ an antiderivative of f in U, then for any curve $\gamma : [a, b] \to U$,*

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)).$$

In particular, if γ is closed then $\int_\gamma f = 0$.

Proof:

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma'(t))\gamma'(t)\,\mathrm{d}t = \int_a^b (F\circ\gamma)'(t)\,\mathrm{d}t = F(\gamma(b)) - F(\gamma(a)).$$

Note that in the $z \mapsto z^{-1}$ integral computation, from the fundamental theorem of calculus there does not exist a branch of the logarithm on any neighbourhood around 0.

The surprising thing is that the converse of this is true.

Theorem 1.4. *Let $f : D \to C$ be continuous on a domain D. If $\int_\gamma f = 0$ for all closed curves γ in D, then there exists a holomorphic $F : D \to \mathbb{C}$ with $F' = f$.*

Proof: Fix $a \in D$. If $w \in D$, choose any curve $\gamma_w : [0,1] \to D$ with $\gamma_w(0) = a$, $\gamma_w(1) = w$. Define

$$F(w) = \int_{\gamma_w} f(z)\,\mathrm{d}z.$$

Find $r_w > 0$ such that $D(w, r_w) \subset D$. For $|h| < r$, let $\delta_h : [0,1] \to D$ be the line segment from w to $w+h$. Then,

$$F(w+h) = \int_{\gamma_{w+h}} f(z)\,\mathrm{d}z = \int_{\gamma_w+\delta_h} f(z)\,\mathrm{d}z.$$

So

$$F(w+h) = F(w) + \int_{\delta_h} f(z)\,\mathrm{d}z = F(w) + hf(w) + \int_{\delta_h} f(z) - f(w)\,\mathrm{d}z.$$

Hence

$$\begin{aligned}\left|\frac{F(w+h)-F(w)}{h} - f(w)\right| &= \left|\frac{1}{h}\int_{\delta_h} f(z)-f(w)\,\mathrm{d}z\right| \\ &\le \frac{\mathrm{length}(\delta_h)}{|h|}\sup_{\delta_h}|f(z)-f(w)| \\ &\le \sup_{z\in D(w,r_w)} |f(z)-f(w)| \to 0,\end{aligned}$$

as $r_w \to 0$. So $F'(w) = f(w)$.

Definition 1.7. An open subset $U \subset \mathbb{C}$ is *convex* if for all $a, b \in U$, the line segment between a and b is in U. U is *starlike* (or starshaped) if there exists $a \in U$ such that for all $b \in U$, the line segment from a to b is in U.

Note that disks are a subset of convex sets, which are a subset of starlike sets, which are a subset of domains.

We can simplify the previous theorem as follows:

Lemma 1.1. *Suppose U is a starlike domain, and $f : U \to \mathbb{C}$ is continuous with $\int_{\partial T} f(z)\,\mathrm{d}z = 0$ for all triangles T in U. Then, f has an antiderivative in U.*

Proof: This is exactly the same as the previous proof, except we stipulate γ_w are straight lines from a basepoint a.

Theorem 1.5 (Cauchy's theorem for Triangles)**.** *If $f : U \to \mathbb{C}$ is holomorphic on open $U \subset \mathbb{C}$, and $T \subset U$ is a triangle in U, then*

$$\int_{\partial T} f(z)\,\mathrm{d}z = 0.$$

We adopt the notion that curves are oriented anticlockwise.

Proof: We can name

$$\left| \int_{\partial T} f(z)\,\mathrm{d}z \right| = I, \qquad L = \mathrm{length}(\partial T).$$

We subdivide T by bisecting the sides, to obtain T_1, T_2, T_3 and T_4. Hence, since

$$\partial T_1 + \partial T_2 + \partial T_3 = \partial T - \partial T_4,$$

we find

$$\int_{\partial T} f(z)\,\mathrm{d}z = \sum_{i=1}^{4} \int_{\partial T_i} f(z)\,\mathrm{d}z.$$

By the triangle inequality, there exists $i \in \{1, 2, 3, 4\}$ such that

$$\left| \int_{\partial T_i} f(z)\,\mathrm{d}z \right| \geq \frac{1}{4} I.$$

Call this triangle $T^{(1)}$ and $\mathrm{length}(\partial T^{(1)}) = \frac{L}{2}$.

Continuing this way, we get

$$T \supset T^{(1)} \supset T^{(2)} \supset T^{(3)} \supset \cdots$$

These triangles have length$(T^{(n)}) = \frac{L}{2^n} \to 0$, and

$$\left|\int_{\partial T^{(n)}} f(z)\,\mathrm{d}z\right| \geq \frac{1}{4^n} I.$$

Since the lengths tend to 0, we get

$$\bigcap_{n=1}^{\infty} T^{(n)} = \{w\},$$

a single point. Note that $z, 1$ have holomorphic derivatives. Hence we can bound

$$\frac{1}{4^n} I \leq \left|\int_{\partial T^{(n)}} f(z)\,\mathrm{d}z\right| = \left|\int_{\partial T^{(n)}} f(z) - f(w) - (z-w)f'(w)\,\mathrm{d}z\right|.$$

Since f is differentiable at w, there $\delta > 0$ such that for all $\varepsilon > 0$,

$$|w - z| < \delta \implies |f(z) - f(w) - (z-w)f'(w)| < \varepsilon |z - w|.$$

So for $n \gg 1$, we have

$$\left|\int_{\partial T^{(n)}} f(z) - f(w) - (z-w)f'(w)\,\mathrm{d}z\right| \leq \frac{L}{2^n} \sup_{z \in \partial T^{(n)}} |z - w| \cdot \varepsilon.$$

So

$$\frac{I}{4^n} \leq \frac{L}{2^n} \cdot \frac{L}{2^n} \varepsilon, \qquad I \leq L^2 \varepsilon.$$

Letting $\varepsilon \to 0$, we get $I = 0$.

Theorem 1.6. *Let $S \subset U$ be a finite set and $f : U \to \mathbb{C}$ be continuous on U and holomorphic on $U \setminus S$. Then $\int_{\partial T} f = 0$ for all triangles $T \in U$.*

Proof: Using the triangle subdivision, assume that $S = \{a\}$, for $a \in T$. If $a \in T' \subset T$ for another triangle T', then by the triangular subdivision and the previous theorem,

$$\int_{\partial T} f = \int_{\partial T'} f,$$

since f is holomorphic on $T \setminus T'$. Hence,

$$\begin{aligned}\left|\int_{\partial T} f(z)\,\mathrm{d}z\right| = \left|\int_{\partial T'} f(z)\,\mathrm{d}z\right| &\leq \mathrm{length}(T')\sup_{\partial T'}|f| \\ &\leq \mathrm{length}(T')\sup_{T}|f|,\end{aligned}$$

so letting $\mathrm{length}(T') \to 0$, we have $\int_{\partial T} f = 0$.

Theorem 1.7 (Cauchy's theorem in a Disk)**.** *Let D be any disk (or any starlike domain), and $f : D \to \mathbb{C}$ a continuous function, holomorphic away from at most a finite set of points in D. Then, $\partial_\gamma f = 0$ for any closed curve γ in D.*

Proof: By our previous theorem and the converse of FTC for starlike domains, there exists an antiderivative F for f in D. So by the fundamental theorem of calculus, Cauchy's theorem follows.

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