IB Complex Analysis

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Contents

1 Complex Differentiation

Our goal in this course is to study the theory of complex-valued differentiable functions in one complex variable. Example include:

- Polynomials $p(z) = a_d z^d + \cdots + a_1 z + a_0$, with coefficients in $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$ or $\mathbb{C}$.
- The infinite series
$$\sum_{n=1}^{\infty} \frac{1}{n^z},$$
which we showed convergence for z having real part greater than 1.
- Harmonic functions $u(x, y) : \mathbb{R}^2 \to \mathbb{R}$, $u_{xx} + u_{yy} = 0$.

In this course, we make the convention that $\theta = \arg(z) \in [0, 2\pi)$.

1.1 Basic Notions

- $U \subset \mathbb{C}$ is *open* if for all $u \in U$, there exists $\varepsilon > 0$ such that
$$D(x, \varepsilon) = \{z \in \mathbb{C} \mid |z - u| < \varepsilon\} \subset U.$$
- A *path* in $U \subset \mathbb{C}$ is a continuous map $\gamma : [a, b] \to U$. We say the path is C^1 if γ' exists and is continuous (we take one-sided derivatives at the endpoints).

  γ is *simple* if it is injective.
- $U \subset \mathbb{C}$ is *path-connected* if for all $z, w \in U$, there exists a path in U with endpoints at z, w.

Remark. If U is open, and $z, w \in U$ are connected by a path γ in U, then there exists a path γ in U connected z, w consisting of finitely many horizontal and vertical segments.

Definition 1.1. A *domain* is a non-empty, open, path-connected subset of $\mathbb{C}$.

Definition 1.2.

(i) $f : U \to \mathbb{C}$ is *differentiable* at $u \in U$ if
$$f'(u) = \lim_{z \to u} \frac{f(z) - f(u)}{z - u}$$
exists.

(ii) $f : U \to \mathbb{C}$ is *holomorphic* at $u \in U$ if there exists $\varepsilon > 0$ such that f is differentiable at z, for all $z \in D(u, \varepsilon)$. We may also call such a function *analytic*.

(iii) $f : \mathbb{C} \to \mathbb{C}$ is *entire* if it is holomorphic everywhere.

Remark. All differentiation rules (sum, products, ...) in $\mathbb{R}$ hold, by the same proofs.

Identifying $\mathbb{C}$ with $\mathbb{R}^2$, we may write $f : U \to \mathbb{C}$ as $f(x + iy) = u(x, y) + iv(x, y)$, where u, v are the real and imaginary parts of f.

From analysis and topology, recall that $u : U \to \mathbb{R}$ as a function of two real variables if ($\mathbb{R}^2$) differentiable at $(c, d) \in \mathbb{R}^2$with $Du|_{(c,d)} = (\lambda, \mu)$ if

$$\frac{u(x, y) - u(c, d) - [\lambda(x - c) + \mu(y - d)]}{\sqrt{(x - c)^2 + (y - d)^2}} \to 0,$$

as $(x, y) \to (c, d)$. However, **this is a weaker condition** than differentiability over $\mathbb{C}$.

Proposition 1.1 (Cauchy-Riemann equations)**.** *Let $f : U \to \mathbb{C}$ on an open set $U \subset \mathbb{C}$. Then f is differentiable at $w = c+id \in U$ if and only if, writing $f = u+iv$, we have u, v are $\mathbb{R}^2$-differentiable at (c, d), and*

$$u_x = v_y, \qquad\qquad u_y = -v_x.$$

Proof: f is differentiable at w if and only if $f'(w) = p + iq$ exists, so

$$\lim_{z\to w} \frac{f(z) - f(w) - (z - w)(p + iq)}{|z - w| = 0}.$$

Writing $f = u + iv$ and considering the real and imaginary parts in the quotient above, this holds if and only if

$$\lim_{(x,y)\to(c,d)} \frac{u(x, y) - u(c, d) - [p(x - c) - q(y - d)]}{\sqrt{(x - c)^2 + (y - d)^2}} = 0,$$

and

$$\lim_{(x,y)\to(c,d)} \frac{v(x, y) - v(c, d) - [q(x - c) + p(y - d)]}{\sqrt{(x - c)^2 + (y - d)^2}} = 0.$$

This holds if and only if u, v are $\mathbb{R}^2$-differentiable at (c, d), and $u_x = v_y$, $u_y = -v_x$.

Remark. If the partial u_x, u_y, v_x, v_y exist and are continuous on U, then u, v are differentiable on U. So it suffices to check the partials exist and are continuous, and the Cauchy-Riemann equations hold to deduce complex differentiability.

Example 1.1.

1. Take $f(z) = \overline{z}$. Then f has $u(x, y) = x$ and $v(x, y) = -y$, so $u_x = 1$, $v_y = -1$. So $f(z) = \overline{z}$ is not holomorphic or differentiable anywhere.
2. Any polynomial $p(z) = a_d z^d + \cdots + a_1 z + a_0$, with $a_i \in \mathbb{C}$ is entire.
3. Rational function, which are quotients of polynomials $\frac{p(z)}{q(z)}$ are holomorphic on the open set $\mathbb{C} \setminus \{\text{zeroes of } q\}$.

Note that $f = u + iv$ satisfying the Cauchy-Riemann equations at a point does not mean it is differentiable at that point.

Some proofs in regular analysis have natural extensions to complex analysis. For example, if $f : U \to \mathbb{C}$ on a domain U with $f'(z) = 0$ on U, then f is constant on U.

Now we ask: why are we interested in complex analysis?

- Unlike $\mathbb{R}^2$ differentiable functions, holomorphics functions are very constrained. For example, if f is entire and bounded (so $|f(z)| < M$ for all $z \in \mathbb{C}$), then f is constant. Contrast with sin, for example.
- We will see that f holomorphic on a domain U has holomorphic derivative on U. This implies that f is infinitely differentiable, as are u and v.

In particular, we can differentiate the Cauchy-Riemann equations to get

$$u_{xx} = v_{yx} = v_{xy} = -u_{yy},$$

so $u_{xx} + u_{yy} = 0$, and similarly $v_{xx} + v_{yy} = 0$. Hence the real and imaginary parts of a holomorphic function are harmonic.

Let $f : U \to \mathbb{C}$ be a holomorphic function on an open set U_1 and $w \in U$ with $f'f(w) \neq 0$. We want to look at the geometric behaviour of f at w.

In fact, we claim f is *conformal* at w. Let γ_1, γ_2 be C^1-paths through w, say $\gamma_1, \gamma_2 : [-1, 1] \to U_1$, such that $\gamma_1(0) = \gamma_2(0) = w$, and $\gamma_i'(0) \neq 0$. If we write $\gamma_j(t) = w + r_j(t) = e^{i\theta_j(t)}$, then we have

$$\arg(\gamma_j'(z)) = \theta_j(0),$$

and the argument of the image line is

$$\arg((f \circ \gamma_j)'(0)) = \arg(\gamma_j'(0) f'(\gamma_j(0))) = \arg(\gamma_j'(0)) + \arg(f'(w)) + 2\pi n,$$

where crucially we use $\gamma_j'(0)f'(\gamma_j(0)) \neq 0$, so the direction of γ_j at w under the application of f is rotated by $\arg(f'(w))$. This is independent of γ_j. Since the angle between γ_1 and γ_2 is the difference of the arguments f preserves the angle. This is what it means to be conformal.

Definition 1.3. Let U, V be domains in $\mathbb{C}$. A map $f : U \to V$ is a *conformal equivalence* of U and V if f is a bijective holomorphic map with $f'(z) \neq 0$, for all $z \in U$.

Remark.

1. Using the real inverse function theorem, one can show if $f : U \to V$ is a holomorphic bijection of open sets with $f'(z) \neq 0$ for all $z \in U$, then the inverse of f is also holomorphic, so also conformal by the chain rule. So conformally equivalent domains are equal from the perspective of the functions f.
2. We will later see than being injective and holomorphic on a domain implies $f'(z) \neq 0$ for all $z \in U$, so this requirement is redundant.

Example 1.2.

1. Any change of coordinates: on $\mathbb{C}$, take $f(z) = az + b$, for $a \neq 0$ and b, which is a conformal equivalence $\mathbb{C} \to \mathbb{C}$. More generally, a Möbius map
$$f(z) = \frac{az+b}{cz+d},$$
for $ad - bc \neq 0$, is a conformal equivalence from the Riemann sphere to itself. This can eb seen as adding a point at infinity to make a sphere $\mathbb{C}_\infty$ (or gluing two copies of the unit disc with coordinates z and $\frac{1}{z}$).

   If $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is continuous, then

   - if $f(\infty) = \infty$, then f is holomorphic at ∞ if and only if $g(z) = \frac{1}{f(\frac{1}{z})}$ is holomorphic at 0.
   - If $f(\infty) \neq \infty$, then f is homolorphic at ∞ if and only if $f(\frac{1}{z})$ is holomorphic at 0.
   - If $f(a) = \infty$ for $a \in \mathbb{C}$, then f is holomorphic at a if and only if $\frac{1}{f(z)}$ is holomorphic at a.

   We can then think of Möbius maps as change of coordinates for the sphere.

Choosing $z_1 \to 0$, $z_2 \to \infty$, $z_3 \to 1$ defined a Möbius map

$$f(z) = \frac{z - z_1}{z - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_1},$$

for distinct $z_1, z_2, z_3 \in \mathbb{C}$.

2. For $n \in \mathbb{N}$, $f(z) = z^n$ is a conformal equivalence from the sector $\{z \in \mathbb{C}^\times \mid 0 < \arg z < \frac{\pi}{n}\}$ to the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \Im z > 0\}$.
3. The Möbius map $f(z) = \frac{z-i}{z+i}$ is a conformal equivalence between $\mathbb{H}$ and $D(0,1)$. We can compute $f'(z) \neq 0$ on $\mathbb{H}$, and

$$z \in \mathbb{H} \iff |z - i| < |z + i| \iff |f(z)) < 1.$$

   Note that $f^{-1}(w) = -i\frac{w+1}{w-1}$.
4. We can use these examples to write down conformal equivalences. Let U_1 be the upper half semicircle, and U_2 the lower half plane. Considering $g(z) = \frac{z+1}{z-1}$, we know that sends $D(0,1)$ to the left half-plane, so it sends U_1 to the upper left quadrant.

   Then, the upper left quadrant if mapped by the squaring map to U_2. So $f(z) = (\frac{z+1}{z-1})^2$ is a conformal equivalence from $U_1 \to U_2$.

These are all examples of the deep *Riemann mapping theorem*:

Theorem 1.1 (Riemann mapping theorem)**.** *Let $U \subset \mathbb{C}$ be a proper domain which is simply connected. Then there exists a conformal equivalence between U and $D(0,1)$.*

Here, *simply connected* means a subset $U \subset \mathbb{C}$ which is path-connected, and contractible: any loop in U can be contracted to a point. So any continuous path $\gamma : S^1 \to U$ extends to a continuous map $\hat{\gamma} : D(0,1) \to U_1$ with $\hat{\gamma}|_{S_1} = \gamma$.

In fact any domain bounded by a simple closed curve is simply connected, so all of these are conformally equivalent to $D(0,1)$.

Example 1.3.

We look at a domains in the Riemann sphere, with bounded and connected complement. This is simply connected as a subset of $\mathbb{C}_\infty$.

Now, the Mandelbrot set is bounded and connected, so the complement of the Mandelbrot set is simply connected in $\mathbb{C}_\infty$.

Recall the following facts about functions defined by power series, or sequences of functions:

1. A sequence (f_n) of functions *converges uniformly* to a function f on some set S if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and for all $x \in S$,
$$|f_n(x) - f(x)| < \varepsilon.$$
2. The uniform limit of continuous functions is continuous.
3. The *Weierstrass M-test*: if there exists $M_n \in \mathbb{R}$ for all n such that $0 \leq |f_n(x)| \leq M_n$ for all $x \in S$, then
$$\sum_{n=1}^{\infty} M_n < \infty \implies \sum_{n=1}^{\infty} f_n(x) \text{ converges uniformly on } S \text{ as } N \to \infty.$$
4. Let (c_n) be complex numbers, and fix $a \in \mathbb{C}$. Then there exists unique $R \in [0, \infty]$ such that the function
$$z \mapsto \sum_{n=1}^{\infty} c_n (z-a)^n$$
converges absolutely if $|z - a| < R$, and diverges if $|z - a| > R$. If $0 < r < R$, then the series converges uniformly in $D(a, r)$. R is the *radius of convergence* of the series. We can compute
$$R = \sup\{r \geq 0 \mid |c_n| r^n \to 0\},$$
or
$$R = \frac{1}{\lambda}, \qquad \lambda = \limsup_{n\to\infty} |c_n|^{1/n}.$$

Theorem 1.2.
$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$
is a complex power series with radius of convergence R. Then,

(i) *f is holomorphic on $D(a, R)$.*

(ii) *f has derivative*
$$f'(z) = \sum_{n=1}^{\infty} n c_n (z-a)^{n-1},$$
with radius of convergence R about a.

(iii) *f has derivatives of all orders on $D(a, R)$, and $f^{(n)}(a) = n! c_n$.*

Proof: We can let $a = 0$ by change of variables $z \to z - a$. Consider the series
$$\sum_{n=1}^{\infty} nc_n z^{n-1}.$$
Since $|nc_n| \geq |c_n|$, the radius of convergence of this series is no larger than R. If $0 < R_1 < R$, then for $|z| < R_1$, we have
$$|nc_n z^{n-1}| = n|c_n|R_1^{n-1}\frac{|z|^{n-1}}{R_1^{n-1}},$$
and
$$n\left(\frac{|z|}{R_1}\right)^{n-1} \to 0.$$
Applying the M-test with $M_n = c_n R_1^{n-1}$, we have the convergence of the series. So the series has radius of convergence R.

Now for $|z|, |w| < R$, we need to consider
$$\frac{f(z) - f(w)}{z - w}.$$
Taking the partial sums,
$$\sum_{n=0}^{N} c_n \frac{z^n - w^n}{z - w} = \sum_{n=0}^{N} c_n \left(\sum_{j=0}^{n-1} z^j w^{n-1-j}\right).$$
For $|z|, |w| < \rho < R$, we have
$$\left| c_n \left(\sum_{j=0}^{n-1} z^j w^{n-1-j}\right)\right| \leq |c_n| n\rho^{n-1}.$$
Hence the partial sums converge uniformly on $\{(z, w) \mid |z|, |w| < \rho\}$. So the series converges to a continuous limit on $\{|z|, |w| < R\}$, say $g(z, w)$. When $z \neq w$, we know
$$g(z, w) = \frac{f(z) - f(w)}{z - w}.$$
When $z = w$, we have
$$g(w, w) = \sum_{n=0}^{\infty} nc_n w^{n-1}.$$
Hence by the continuity of g, this proves (i) and (ii). Then (iii) follows from a simple induction.

Corollary 1.1. *Suppose* $0 < \rho < R$*, where* R *is the radius of convergence of the complex power series*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n,$$

and $f(z) = 0$ *for all* $z \in D(a, \rho)$*. Then* $f \equiv 0$ *on* $D(a, R)$*.*

Proof: Since $f \equiv 0$ on $D(a, \rho)$, we have $f^{(n)}(a) = 0$ for all n. Hence $c_n = 0$ for all n, so $f \equiv 0$ on $D(a, R)$.

1.2 Exponential and Logarithm

We define the complex exponential

$$e^z = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

The complex exponential has the following properties:

1. It has radius of convergence ∞, so the function is entire, and we have $\frac{\mathrm{d}}{\mathrm{d}z} e^z = e^z$.
2. For all $z, w \in \mathbb{C}$, $e^{z+w} = e^z e^w$, and $e^z \neq 0$.

   This follows from setting $F(z) = e^{z+w} e^{-z}$, then taking the derivative,

   $$F'(z) = e^{z+w} e^{-z} - e^{z+w} e^{-z} = 0,$$

   so F is constant. Since $e^0 = 1$, $F(z) = e^w$, and $e^{z+w} = e^z e^w$. Since $e^z e^{-z} = e^0 = 1$, $e^z \neq 0$.
3. Let $z = x + iy$. Then $e^z = e^{x+iy} = e^x e^{iy}$. But $e^{iy} = \cos y + i \sin y$, and note that $|e^{iy}| = 1$, so

   $$e^z = e^x (\cos y + i \sin y),$$

   and $|e^z| = e^x$, so $e^z = 1$ if and only if $x = 0$ and $y = 2\pi k$ for $k \in \mathbb{Z}$. In fact, for all $w \in \mathbb{C}^\times$, there exist infinitely many $z \in \mathbb{C}$ such that $e^z = w$, differing by integer multiples of $2\pi i$.

Definition 1.4. Let $U \subset \mathbb{C}^\times$ be an open set. We say a continuous function $\lambda : U \to \mathbb{C}$ is a *branch of the logarithm* if for all $z \in U$, $\exp(\lambda(z)) = z$.

Example 1.4.

Let $U = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Define $\log : U \to \mathbb{C}$ by

$$\log(z) = \ln|z| + i\theta,$$

where $\theta = \arg(z)$, and $\theta \in (-\pi, \pi)$. This is the *principal branch of the logarithm*.

Proposition 1.2. $\log(z)$ *is holomorphic on* $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ *with derivative* $\frac{1}{z}$. *Moreover, if* $|z| < 1$, *then*

$$\log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}.$$

Proof: As an inverse to e^z and by the chain rule, we have $\log z$ is holomorphic with $\frac{\mathrm{d}}{\mathrm{d}z} \log z = \frac{1}{z}$, We have

$$\frac{\mathrm{d}}{\mathrm{d}z} \log(1+z) = \frac{1}{z+1} = 1 - z + z^2 - z^3 + z^4 - \cdots,$$

which is the derivative of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}.$$

So $\log(1+z)$ agrees with this series up to a constant. Since $\log(1) = 0$, the equality holds.

If $\alpha \in \mathbb{C}$, we can define $z^\alpha = \exp(\alpha \log z)$. This gives a definition of z^α on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. We can compute that $\frac{\mathrm{d}}{\mathrm{d}z} z^\alpha = \alpha z^{\alpha - 1}$.

It is not necessarily true that $z^\alpha w^\alpha = (zw)^\alpha$. Take $\alpha = \frac{1}{2}$, then

$$z^{1/2} = \exp\left(\frac{1}{2} \log z\right) = \exp\left(\frac{1}{2} \ln|z| + \frac{1}{2} i\theta\right),$$

for $\theta \in (-\pi, \pi)$. Hence the argument of $z^{1/2}$ is in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

1.3 Contour Integration

If $f : [a, b] \to \mathbb{C}$ is continuous, we define

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^b \Re(f(t))\,\mathrm{d}t + i \int_a^b \Im(f(t))\,\mathrm{d}t.$$

Proposition 1.3. *Let $f : [a, b] \to \mathbb{C}$ be continuous. Then,*

$$\left| \int_a^b f(t) \, \mathrm{d}t \right| \leq (b - a) \sup_{a \leq t \leq b} |f(t)|,$$

with equality if and only if f is constant.

Proof: Write $M = \sup_{a \leq t \leq b} |f(t)|$, and $\theta = \arg(\int_a^b f(t) \, \mathrm{d}t)$. Then

$$\begin{aligned} \left| \int_a^b f(t) \, \mathrm{d}t \right| &= e^{-i\theta} \int_a^b f(t) \, \mathrm{d}t = \int_a^b e^{-i\theta} f(t) \, \mathrm{d}t \\ &= \int_a^b \Re(e^{-i\theta} f(t)) \, \mathrm{d}t \\ &\leq \int_a^b |f(t)| \, \mathrm{d}t \leq M(b - a). \end{aligned}$$

If we have equality, then $|f(t)| = M$, and $\arg f(t) = \theta$, so f is constant.

Definition 1.5. Let $\gamma : [a, b] \to \mathbb{C}$ be a C^1-smooth curve. Then we define the arc-length of γ to be

$$\mathrm{length}(\gamma) = \int_{\mathrm{a}}^{\mathrm{b}} |\gamma'(\mathrm{t})| \, \mathrm{dt}.$$

We say γ is *simple* if $\gamma(t_1) = \gamma(t_2) \iff t_1 = t_2$ or $\{t_1, t_2\} = \{a, b\}$. If γ is simple, then $\mathrm{length}(\gamma)$ is the length of the image of γ.

Definition 1.6. Let $f : U \to \mathbb{C}$ be continuous, with U open, and $\gamma : [a, b] \to U$ be a C^1-smooth curve. Then the integral of f along γ is

$$\int_\gamma f(z) \, \mathrm{d}z = \int_a^b f(\gamma(t)) \gamma'(t) \, \mathrm{d}t.$$

This integral satisfies the following properties:

1. Linearity:
$$\int_\gamma c_1 f_1 + c_2 f_2 \, \mathrm{d}z = c_1 \int_\gamma f_1 \, \mathrm{d}z + c_2 \int_\gamma f_2 \, \mathrm{d}z.$$

2. Additivity: if $a < a' < b$, then
$$\int_{\gamma|_{[a,a']}} f(z) \, \mathrm{d}z + \int_{\gamma|_{[a',b]}} f(z) \, \mathrm{d}z = \int_\gamma f(z) \, \mathrm{d}z.$$

3. Inverse path: if $(-\gamma)(t) = \gamma(-t)$ on $[-b, -a]$, then

$$\int_{-\gamma} f(z)\,\mathrm{d}z = -\int_{\gamma} f(z)\,\mathrm{d}z.$$

4. Independence of parametrization: if $\phi : [a', b'] \to [a, b]$ is C^1-smooth with $\phi(a') = a$, $\phi(b') = b$ and $\delta = \gamma \circ \phi$, then

$$\int_{\delta} f(z)\,\mathrm{d}z = \int_{\gamma} f(z)\,\mathrm{d}z.$$

   This lets us assume that $\gamma : [0, 1] \to U$.

We can loosen the restriction that γ is C^1-smooth and allow it to be piecewise C^1-smooth, i.e. there exist $a = a_0 < a_1 < \cdots < a_n = b$ such that $\gamma_i = \gamma|_{[a_{i-1}, a_i]}$ is C^1-smooth. Define then

$$\int_{\gamma} f(z)\,\mathrm{d}z = \sum_{i=1}^{n} \int_{\gamma_i} f(z)\,\mathrm{d}z.$$

Remark. Any piecewise C^1-smooth curve can be reparametrized to be C^1: for such a γ as above, replace γ_i by $\gamma_i \circ h_i$ where h_i is monotonic C^1-smooth bijection with endpoint derivative 0.

So C^1-smooth paths can have corners, for example

$$\gamma(t) = \begin{cases} 1 + i\sin(\pi t) & t \in [0, \frac{1}{2}], \\ \sin(\pi t) + i & t \in [\frac{1}{2}, 1]. \end{cases}$$

We say a "curve" is a piecewise C^1-smooth path, and a "contour" is a simple *closed* piecewise C^1-smooth path, where closed means the endpoints are equal.

Proposition 1.4. *For any continuous $f : U \to \mathbb{C}$ with U open, and any curve $\gamma : [a, b] \to U$,*

$$\left| \int_{\gamma} f(z)\,\mathrm{d}z \right| \leq \mathrm{length}(\gamma) \sup_{\mathrm{z} \in \gamma} |\mathrm{f(z)}|.$$

Proof:

$$\begin{aligned}\left|\int_\gamma f(z)\,\mathrm{d}z\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t\right| \\ &\le \int_a^b |f(\gamma(t))\gamma'(t)|\,\mathrm{d}t \\ &\le \sup_{z\in\gamma}|f(z)|\text{length}(\gamma).\end{aligned}$$

Proposition 1.5. *If $f_n : U \to \mathbb{C}$ for $n \in \mathbb{N}$ and $f : U \to \mathbb{C}$ are continuous, and $\gamma : [a,b] \to U$ is a curve in U with $f_n \to f$ uniformly on γ, then*

$$\int_\gamma f_n(z)\,\mathrm{d}z \to \int_\gamma f(z)\,\mathrm{d}z,$$

as $n \to \infty$.

Proof: By uniform convergence, $\sup_{z\in\gamma}|f(z) - f_n(z)| \to 0$ as $n \to \infty$. So by the previous proposition,

$$\begin{aligned}\left|\int_\gamma f(z)\,\mathrm{d}z - \int_\gamma f_n(z)\,\mathrm{d}z\right| &\le \text{length}(\gamma)\sup_\gamma|\mathrm{f} - \mathrm{f_n}| \\ &\to 0\end{aligned}$$

as $n \to \infty$.

Example 1.5.

Let $f_n(z) = z^n$ for $n \in \mathbb{Z}$ on $C^\times = U$, and $\gamma : [0, 2\pi] \to U$ with $\gamma(t) = e^{it}$. Then,

$$\int_\gamma f_n(z)\,\mathrm{d}z = \int_0^{2\pi} e^{nit} i e^{it}\,\mathrm{d}t = i\int_0^{2\pi} e^{(n+1)it}\,\mathrm{d}t = \begin{cases} 2\pi i & n = -1, \\ 0 & n \neq -1. \end{cases}$$

Theorem 1.3 (Fundamental Theorem of Calculus)**.** *If $f : U \to \mathbb{C}$ is a continuous function on open $U \subset \mathbb{C}$ with $F' = f$ an antiderivative of f in U, then for any curve $\gamma : [a,b] \to U$,*

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)).$$

In particular, if γ is closed then $\int_\gamma f = 0$.

Proof:

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma'(t))\gamma'(t)\,\mathrm{d}t = \int_a^b (F\circ\gamma)'(t)\,\mathrm{d}t = F(\gamma(b)) - F(\gamma(a)).$$

Note that in the $z \mapsto z^{-1}$ integral computation, from the fundamental theorem of calculus there does not exist a branch of the logarithm on any neighbourhood around 0.

The surprising thing is that the converse of this is true.

Theorem 1.4. *Let $f : D \to C$ be continuous on a domain D. If $\int_\gamma f = 0$ for all closed curves γ in D, then there exists a holomorphic $F : D \to \mathbb{C}$ with $F' = f$.*

Proof: Fix $a \in D$. If $w \in D$, choose any curve $\gamma_w : [0,1] \to D$ with $\gamma_w(0) = a$, $\gamma_w(1) = w$. Define

$$F(w) = \int_{\gamma_w} f(z)\,\mathrm{d}z.$$

Find $r_w > 0$ such that $D(w, r_w) \subset D$. For $|h| < r$, let $\delta_h : [0,1] \to D$ be the line segment from w to $w + h$. Then,

$$F(w+h) = \int_{\gamma_{w+h}} f(z)\,\mathrm{d}z = \int_{\gamma_w + \delta_h} f(z)\,\mathrm{d}z.$$

So

$$F(w+h) = F(w) + \int_{\delta_h} f(z)\,\mathrm{d}z = F(w) + hf(w) + \int_{\delta_h} f(z) - f(w)\,\mathrm{d}z.$$

Hence

$$\begin{aligned}\left|\frac{F(w+h)-F(w)}{h} - f(w)\right| &= \left|\frac{1}{h}\int_{\delta_h} f(z) - f(w)\,\mathrm{d}z\right| \\ &\leq \frac{\mathrm{length}(\delta_h)}{|h|}\sup_{\delta_h}|f(z) - f(w)| \\ &\leq \sup_{z\in D(w,r_w)}|f(z) - f(w)| \to 0,\end{aligned}$$

as $r_w \to 0$. So $F'(w) = f(w)$.

Definition 1.7. An open subset $U \subset \mathbb{C}$ is *convex* if for all $a, b \in U$, the line segment between a and b is in U. U is *starlike* (or starshaped) if there exists $a \in U$ such that for all $b \in U$, the line segment from a to b is in U.

Note that disks are a subset of convex sets, which are a subset of starlike sets, which are a subset of domains.

We can simplify the previous theorem as follows:

Lemma 1.1. *Suppose U is a starlike domain, and $f : U \to \mathbb{C}$ is continuous with $\int_{\partial T} f(z)\,\mathrm{d}z = 0$ for all triangles T in U. Then, f has an antiderivative in U.*

Proof: This is exactly the same as the previous proof, except we stipulate γ_w are straight lines from a basepoint a.

Theorem 1.5 (Cauchy's theorem for Triangles)**.** *If $f : U \to \mathbb{C}$ is holomorphic on open $U \subset \mathbb{C}$, and $T \subset U$ is a triangle in U, then*

$$\int_{\partial T} f(z)\,\mathrm{d}z = 0.$$

We adopt the notion that curves are oriented anticlockwise.

Proof: We can name

$$\left| \int_{\partial T} f(z)\,\mathrm{d}z \right| = I, \qquad L = \mathrm{length}(\partial T).$$

We subdivide T by bisecting the sides, to obtain T_1, T_2, T_3 and T_4. Hence, since

$$\partial T_1 + \partial T_2 + \partial T_3 = \partial T - \partial T_4,$$

we find

$$\int_{\partial T} f(z)\,\mathrm{d}z = \sum_{i=1}^{4} \int_{\partial T_i} f(z)\,\mathrm{d}z.$$

By the triangle inequality, there exists $i \in \{1, 2, 3, 4\}$ such that

$$\left| \int_{\partial T_i} f(z)\,\mathrm{d}z \right| \geq \frac{1}{4} I.$$

Call this triangle $T^{(1)}$ and $\mathrm{length}(\partial T^{(1)}) = \frac{L}{2}$.

Continuing this way, we get

$$T \supset T^{(1)} \supset T^{(2)} \supset T^{(3)} \supset \cdots$$

These triangles have length$(T^{(n)}) = \frac{L}{2^n} \to 0$, and

$$\left| \int_{\partial T^{(n)}} f(z) \, \mathrm{d}z \right| \geq \frac{1}{4^n} I.$$

Since the lengths tend to 0, we get

$$\bigcap_{n=1}^{\infty} T^{(n)} = \{w\},$$

a single point. Note that $z, 1$ have holomorphic derivatives. Hence we can bound

$$\frac{1}{4^n} I \leq \left| \int_{\partial T^{(n)}} f(z) \, \mathrm{d}z \right| = \left| \int_{\partial T^{(n)}} f(z) - f(w) - (z-w) f'(w) \, \mathrm{d}z \right|.$$

Since f is differentiable at w, there $\delta > 0$ such that for all $\varepsilon > 0$,

$$|w - z| < \delta \implies |f(z) - f(w) - (z-w) f'(w)| < \varepsilon |z - w|.$$

So for $n \gg 1$, we have

$$\left| \int_{\partial T^{(n)}} f(z) - f(w) - (z-w) f'(w) \, \mathrm{d}z \right| \leq \frac{L}{2^n} \sup_{z \in \partial T^{(n)}} |z - w| \cdot \varepsilon.$$

So

$$\frac{I}{4^n} \leq \frac{L}{2^n} \cdot \frac{L}{2^n} \varepsilon, \qquad I \leq L^2 \varepsilon.$$

Letting $\varepsilon \to 0$, we get $I = 0$.

Theorem 1.6. *Let $S \subset U$ be a finite set and $f : U \to \mathbb{C}$ be continuous on U and holomorphic on $U \setminus S$. Then $\int_{\partial T} f = 0$ for all triangles $T \in U$.*

Proof: Using the triangle subdivision, assume that $S = \{a\}$, for $a \in T$. If $a \in T' \subset T$ for another triangle T', then by the triangular subdivision and the previous theorem,

$$\int_{\partial T} f = \int_{\partial T'} f,$$

since f is holomorphic on $T \setminus T'$. Hence,

$$\begin{aligned}\left|\int_{\partial T} f(z)\,\mathrm{d}z\right| = \left|\int_{\partial T'} f(z)\,\mathrm{d}z\right| &\leq \operatorname{length}(T') \sup_{\partial T'} |f| \\ &\leq \operatorname{length}(T') \sup_{T} |f|,\end{aligned}$$

so letting $\operatorname{length}(T') \to 0$, we have $\int_{\partial T} f = 0$.

Theorem 1.7 (Cauchy's theorem in a Disk). *Let D be any disk (or any starlike domain), and $f : D \to \mathbb{C}$ a continuous function, holomorphic away from at most a finite set of points in D. Then, $\int_{\partial_\gamma} f = 0$ for any closed curve γ in D.*

Proof: By our previous theorem and the converse of FTC for starlike domains, there exists an antiderivative F for f in D. So by the fundamental theorem of calculus, Cauchy's theorem follows.

Theorem 1.8 (Cauchy's integral formula). *Let $U \subset \mathbb{C}$ be a domain, $f : U \to \mathbb{C}$ holomorphic, and $\overline{D(a,r)} \subset U$. Then for all $z \in D(a,r)$,*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D(a,r)} \frac{f(w)}{w-z}\,\mathrm{d}w.$$

Proof: Define an auxiliary function

$$g(w) = \begin{cases} \frac{f(w)-f(z)}{w-z} - f'(z) & w \neq z, \\ 0 & w = z. \end{cases}$$

Then g is continuous at z and homomorphic on $D(a,r)$, except possibly at z. Find $r_1 > 0$ such that $\overline{D(a,r)} \subset D(a,r_1) \subset U$. Applying Cauchy's theorem to g on $D(a,r_1)$ with curve $\gamma = \partial D(a,r)$, we get

$$\int_{\partial D(a,r)} g(w)\,\mathrm{d}w = 0 \iff \int_{\partial D(a,r)} \frac{f(w)}{w-z}\,\mathrm{d}w = \int_{\partial D(a,r)} \frac{f(z)}{w-z}\,\mathrm{d}w.$$

We can expand $\frac{1}{w-z}$ as

$$\frac{1}{w-z} = \frac{1}{(w-a)[1-\frac{z-a}{w-a}]} = \sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}}.$$

Hence we get

$$\int_{\partial D(a,r)} \frac{f(z)}{w-z}\,\mathrm{d}w = \sum_{n=0}^{\infty}\left[f(z)(z-a)^n \int_{\partial D(a,r)} \frac{1}{(w-a)^{n+1}}\,\mathrm{d}w\right].$$

The latter integral vanishes unless $n = 0$, which gives

$$\int_{\partial D(a,r)} \frac{f(w)}{w-z} = 2\pi i f(z).$$

Corollary 1.2 (Mean Value Property)**.** *If $f : U \to \mathbb{C}$ is holomorphic on a domain U, and $\overline{D(a,r)} \subset U$, then*

$$f(a) = \int_0^1 f(a + re^{2\pi i t})\,\mathrm{d}t.$$

Proof: We can apply Cauchy's integral formula, with $t \mapsto a + re^{2\pi i t}$ on $[0,1]$ for $\partial D(a,r)$.

We can use Cauchy's integral formula to obtain the following:

Corollary 1.3 (Local Maximum Principle)**.** *Let $f : D(a,r) \to \mathbb{C}$ be holomorphic. If $|f(z)| \le |f(a)|$ for all $z \in D(a,r)$, then f is constant.*

Proof: By the mean value property, for all $0 < \rho < r$,

$$|f(a)| = \left|\int_0^1 f(a + \rho e^{2\pi i t})\,\mathrm{d}t\right| \le \sup_{|z-a|=\rho} |f(z)| = |f(a)|.$$

Since we have equality, we have $|f(z)| = |f(a)|$ for all $|z - a| = \rho$. So $|f|$ is a constant function on $D(a,r)$, hence f is constant on $D(a,r)$.

Theorem 1.9 (Liouville's theorem)**.** *Every bounded entire function is constant.*

Proof: Say $|f(z)| \le M$ for f entire. Take $R \gg 1$, then for any $0 < |z| < \frac{R}{2}$

by Cauchy's integral formula,

$$\begin{aligned}
|f(z) - f(0)| &= \frac{1}{2\pi}\left|\int_{\partial D(0,r)} f(w)\left[\frac{1}{w-z} - \frac{1}{w}\right] \mathrm{d}w\right| \\
&= \frac{1}{2\pi}\left|\int_{\partial D(0,r)} f(w)\frac{z}{(w-z)w} \mathrm{d}w\right| \\
&\le \frac{1}{2\pi} \cdot 2\pi R \cdot \sup_{w\in\partial D(0,R)} |f(w)| \cdot |z| \cdot \frac{1}{R\cdot\frac{R}{2}} \\
&\le M|z|\frac{1}{R/2} \to 0
\end{aligned}$$

as $R \to \infty$, so $f(z) = f(0)$. Hence f is constant.

Corollary 1.4 (Fundamental Theorem of Algebra)**.** *Every non-constant polynomial with complex coefficients has a root in* $\mathbb{C}$*.*

Proof: If $p(z) = a_d z^d + a_{d-1}z^{d-1} + \cdots + a_0$ has no root in $\mathbb{C}$, then $f(z) = \frac{1}{p(z)}$ is entire.

As $p(z)$ is non-constant, we have $a_d \neq 0$ and $d \geq 1$. So

$$\frac{p(z)}{z^d} = a_d + a_{d-1}\cdot\frac{1}{z} + \cdots + a_n\frac{1}{z^d}$$

shows that $|p(z)| \to \infty$ as $|z| \to \infty$. Hence $|f(z)| \to 0$ as $|z| \to \infty$

Hence there exists $R > 0$ such that for all $z \notin D(0,R)$, $|f(z)| \leq 1$, but if $M = \max_{z\in\overline{D(0,R)}} |f(z)|$, $|f|$ is bounded by $\max\{1, M\}$, and so by Liouville's theorem is constant. Therefore p must be constant.

Theorem 1.10. *Let* $f : D(a,r) \to \mathbb{C}$ *be holomorphic. Then* f *is represented by convergent power series on* $D(a,r)$*:*

$$f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n,$$

with

$$c_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i}\int_{\partial D(a,\rho)} \frac{f(w)}{(w-a)^{n+1}} \mathrm{d}w,$$

for $0 < \rho < r$*.*

Proof: For $|z-a|<\rho<r$, Cauchy's integral formula gives

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\partial D(a,\rho)} \frac{f(w)}{w-z}\,\mathrm{d}w \\ &= \frac{1}{2\pi i}\int_{\partial D(a,\rho)} f(w)\sum_{n=0}^{\infty}\frac{(z-a)^n}{(w-a)^{n+1}}\,\mathrm{d}w \\ &= \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\int_{\partial D(a,\rho)} f(w)\frac{1}{(w-a)^{n+1}}\,\mathrm{d}w\right](z-a)^n, \end{aligned}$$

proving the theorem.

Remark.

1. Holomorphic functions therefore have derivatives of all orders, which are holomorphic themselves.
2. This shows holomorphic functions are exactly the analytic functions.

Corollary 1.5 (Morera's theorem)**.** *Let D be a disk and $f : D \to \mathbb{C}$ be continuous such that $\int_\gamma f = 0$ for all closed curves γ in D. Then f is holomorphic.*

Proof: By the converse of the fundamental theorem of calculus, there exists holomorphic F on D with $F' = f$. So f is holomorphic.

Corollary 1.6. *Let $f_u : U \to \mathbb{C}$ be holomorphic functions on a domain U, and $f_n \to f$ uniformly on U (note is is sufficient for uniform convergence on compact subsets on U). Then f is holomorphic on U, and*

$$f'(z) = \lim_{n\to\infty} f_n'(x).$$

Proof: Since U is a union of open disks, it suffices to work with $D(z,\varepsilon) \subset U$. Given γ, a closed curve in $D(z,\varepsilon)$, since $\int_\gamma f_n \to \int_\gamma f$, and we know $\int_\gamma f_n = 0$, we get $\int_\gamma f = 0$.

Since f is continuous on $D(z,\varepsilon)$, Morera's theorem applies, so f is holomorphic on $D(z,\varepsilon)$.

Recall the Taylor expansion computation for $0<\rho<\varepsilon$:

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial D(z,\rho)} \frac{f(\xi)}{(\xi - z)^{n+1}}\,\mathrm{d}\xi.$$

Hence we get

$$\begin{aligned}|f'(z) - f_n'(z)| &= \frac{1}{2\pi}\left|\int_{\partial D(x,\rho} \frac{f(\xi)}{(\xi - z)^2} - \frac{f_n(\xi)}{(\xi - z)^2}\,\mathrm{d}\xi\right| \\ &\le \rho \cdot \frac{1}{\rho^2} \sup_{\zeta \in \partial D(x,\rho)} |f(\xi) - f_n(\xi)| \to 0,\end{aligned}$$

as $n \to \infty$. Hence $f'(z) = \lim f_n'(z)$.

Remark. Note f need not be nonconstant, for example take $f_n(z) = z^n$ on $D(0, r)$, with $0 < r < 1$. Then $f_n \to 0$ uniformly.

Corollary 1.7. *If $f : U \to \mathbb{C}$ is continuous on a domain U, and holomorphic on $U \setminus S$ for some finite set S, then f is holomorphic on U.*

Proof: If $a \in S$, find $D(a, r) \subset U$ an open disk. Then by Cauchy's theorem on a disk, $\int_\gamma f = 0$ for any closed curve γ in $D(a, r)$. By Morera's theorem, f is holomorphic on $D(a, r)$.

Since this holds for all a, f is holomorphic on U.

Let $f : D(a, R) \to \mathbb{C}$ be holomorphic, so

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$$

on $D(a, R)$. If $f \not\equiv 9$, then some c_n is non-zero. Let

$$m = \min\{n \in \mathbb{N}_0 \mid c_n \neq 0\}.$$

If $m > 0$, then we say f has a zero of order m at a. In this case, we can write

$$f(z) = (z - a)^m g(z),$$

where $g(z)$ is holomorphic on $D(a, R)$, and $g(a) \neq 0$.

Theorem 1.11 (Principle of Isolated Zeroes)**.** *If $f : D(a, R) \to \mathbb{C}$ is holomorphic, and not identically 0, then there exists $0 < r < R$ such that $f(z) \neq 0$ on $0 < |z - a| < r$.*

Proof: If $f(a) \neq 0$, then $f(z) \neq 0$ on $D(a, r)$ for some $0 < r < R$ by continuity of f.

If f has a zero of order m at a, write $f(z) = (z - a)^m g(z)$, where $g(a) \neq 0$ and g is holomorphic.

By the continuity of g, there exists $0 < r < R$ such that $g(z) \neq 0$ for all $z \in D(a, r)$. Hence $f(z) \neq 0$ for all $0 < |z - a| < r$.

Remark.

1. This says there is no accumulation point of the zero set of a holomorphic map inside its domain, unless it is everywhere 0.
2. It is possible for the zeroes of a holomorphic map to accumulate outside its domain: consider
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i},$$
which has zeroes at $z = n\pi$. Hence $\sin(\frac{1}{z})$ has zeroes accumulating at 0, on the boundary of its domain $\mathbb{C}^\times$.
3. Another application: since $\cos^2 z + \sin^2 z = 1$ holds for all $z \in \mathbb{R}$, then $\cos^2 z + \sin^2 z - 1$ is entire with $\mathbb{R}$ contained in its zero set. Hence $\cos^2 z + \sin^2 z = 1$ for all $z \in \mathbb{C}$.

Proposition 1.6 (Identity Theorem for holomorphic functions)**.** *Let $f, g : U \to \mathbb{C}$ be holomorphic on a domain U. Let $S = \{z \in U \mid f(z) = g(z)\}$. If S has a non-isolated point, then $f(z) = g(z)$ for all $z \in U$.*

Proof: Define $h(z) = f(z) - g(z)$. This is holomorphic on U. Now suppose w is non-isolated in S. Then for $\varepsilon > 0$ with $D(w, \varepsilon) \subset U$, by the principle of isolated zeroes, $h = 0$ on $D(w, \varepsilon)$.

Given $z \in U$, let $\gamma : [0, 1] \to U$ be a path with $\gamma(0) = w$, $\gamma(1) = z$. Consider the set
$$T = \{t \in [0, 1] \mid h^{(n)}(\gamma(t)) = 0 \text{ for all } n \geq 0\}.$$
Note that T is closed by definition. Moreover, since $h = 0$ on $D(w, \varepsilon)$, we get that T is non-empty, as $0 \in T$. Now define
$$t_0 = \sup\{t \in [0, 1] \mid [0, t] \subset T\}.$$
Then T is closed an non-empty, so $t_0 \in T$. Since $h^{(n)}(\gamma(t_0)) = 0$ for all $n \geq 0$, $h \equiv 0$ on a neighbourhood of $\gamma(t_0)$, contradicting the maximality if t_0, unless $t_0 = 1$.

Hence $h(\gamma(1)) = 0$, or $h(z) = 0$, as desired.

Definition 1.8. Let $U \subset V \subset \mathbb{C}$ be domains, and $f : U \to \mathbb{C}$ is holomorphic. $g : V \to \mathbb{C}$ is an *analytic continuation* of f if:

1. g is holomorphic on V, and
2. $g|_U = f$.

Example 1.6.

The series
$$\sum_{n\geq 1} = \frac{(-1)^{n+1}}{n} z^n.$$
Converges on $D(0,1)$, and takes the value $\log(1+z)$ on $D(0,1)$. So $\log(1+z)$ is an analytic continuation of this series, to the domain $\mathbb{C} \setminus (-\infty, -1]$.

Moreover the series
$$\sum_{n\geq 0} z^n$$
has radius of convergence 1 about $a = 0$, and on $D(0,1)$ we it takes the value $\frac{1}{1-z}$. Hence $\frac{1}{1-z}$ is an analytic continuation of the series to $\mathbb{C} \setminus \{1\}$.

Corollary 1.8 (Global maximum principle)**.** *Let $U \subset \mathbb{C}$ be a bounded domain, and let $\overline{U}$ be its closure. If $f : \overline{U} \to \mathbb{C}$ is continuous and f is holomorphic on U, then $|f|$ attains its maximum on $\overline{U} \setminus U$.*

Proof: As U is bounded, then $\overline{U}$ is bounded, hence as f is continuous $|f|$ attains a maximum on $\overline{U}$, say M.

If $|f(z_0)| = M$ for $z_0 \in U$, then by the local maximum principle, $f \equiv f(z_0)$ on any disk $D(z_0, r) \subset U$. By the identity theorem, $f \equiv f(z_0)$ on U, hence $f \equiv f(z_0)$ on $\overline{U}$.

Thus M is achieved by $|f|$ on $\overline{U} \setminus U$.

Our goal is to generalize the Cauchy integral formula by allowing more general closed curves for integration.

However, we cannot hope that Cauchy's integral formula works for any closed curve:

consider a curve γ' given by going around a closed disk γ twice. Then immediately

$$\int_{\gamma'} f \, \mathrm{d}z = 2 \int_{\gamma} f \, \mathrm{d}z.$$

Hence we need to deal with the notion of "winding around" a point more than once. Quantifying this notion, we will see this is the only issue to generalizing Cauchy's integral formula.

Our first hope is that we can count the crossing of some slit in the plane. However this doesn't work as we can cross infinitely often. But this cannot happen for every direction!

Theorem 1.12. *Let $\gamma : [a, b] \to \mathbb{C} \setminus \{w\}$ be a continuous curve. Then there exists a continuous function $\theta : [a, b] \to \mathbb{R}$ with $\gamma(t) = w + r(t)e^{i\theta(t)}$, with $r(t) = |\gamma(t) - w|$.*

Proof: By translation, we can translate to assume $w = 0$. Moreover, since

$$\arg \gamma(t) = \arg \frac{\gamma(t)}{|\gamma(t)|},$$

so dividing by the modulus of γ, we can assume $|\gamma(t)| = 1$ for all $t \in [a, b]$.

Notice that if $\gamma \subset \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, then $t \mapsto \arg(\gamma(t))$ (using the principal argument), gives a continuous choice of θ. More generally, if γ lies in any slit plane

$$\mathbb{C} \setminus \{z \mid z/e^{i\alpha} \subset \mathbb{R}_{\leq 0}\},$$

then $\theta(t) = \alpha + \arg(z/e^{i\alpha})$ will do.

Our strategy is to subdivide γ so that the pieces lie in slit planes, and so we can make θ continuous on the pieces.

Since γ is continuous on $[a, b]$, it is uniformly continuous, so there exists $\varepsilon > 0$ such that $|s - t| < \varepsilon$ implies $|\gamma(s) - \gamma(t)| < 2$. Subdividing $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$ with $a_{j+1} - a_j < 2\varepsilon$, then

$$\left| \gamma(t) - \gamma\left(\frac{a_{j+1} - a_j}{2}\right) \right| < 2,$$

for all $t \in [a_j, a_{j+1}]$. Hence $\gamma([a_{j-1}, a_j])$ lies in a slit plane, and we can define θ_j a continuous choice of argument for $\gamma|_{[a_{j-1}, a_j]}$, for all $j \in \{1, \ldots, n\}$. Then

$$\gamma(a_j) = e^{i\theta_j(a_j)} = e^{i\theta_{j+1}(a_j)}.$$

Since this is true, $\theta_{j+1}(a_j) = \theta_j(a_j) + 2\pi n_j$ for some $n_j \in \mathbb{C}$. Modifying each of θ_j for $j \geq 2$ by a suitable integer multiple of 2π ensures that the θ_j fit together to a continuous choice of θ on $[a, b]$.

Remark. θ is not unique, since $\theta(t) + 2\pi n$ is also valid for all $n \in \mathbb{Z}$. But if θ_1, θ_2 are two functions as in the theorem, then $\theta_1 - \theta_2$ is continuous, but takes values in the discrete set $2\pi\mathbb{Z}$, hence is constant.

Definition 1.9. Let $\gamma : [a, b] \to \mathbb{C}$ be a closed curve, and $w \notin \gamma$. The *winding number* or *index* of γ about w is

$$I(\gamma; w) = \frac{\theta(b) - \theta(a)}{2\pi},$$

where $\gamma(t) = w + r(t)e^{i\theta(t)}$ with θ continuous.

Lemma 1.2. *Let $\gamma : [a, b] \to \mathbb{C} \setminus \{w\}$ be a closed curve. Then*

$$I(\gamma; w) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}z}{z - w}.$$

Proof: Note γ is piecewise C^1, so $r(t)$ and $\theta(t)$ are piecewise C^1 as well, where $\gamma(t) = w + r(t)e^{i\theta(t)}$. So

$$\begin{aligned} \int_\gamma \frac{\mathrm{d}z}{z - w} &= \int_a^b \frac{\gamma'(t)}{\gamma(t) - w}\,\mathrm{d}t = \int_a^b \frac{r'(t)}{r(t)} + i\theta'(t)\,\mathrm{d}t \\ &= [\log r(t) + i\theta(t)]_{t=a}^{t=b} = 2\pi i I(\gamma, w), \end{aligned}$$

since γ is closed and $\theta(b) - \theta(a) = 2\pi I(\gamma, w)$.

Proposition 1.7. *If $\gamma : [0, 1] \to D(a, R)$ is a closed curve, then for all $w \notin D(a, R)$, $I(\gamma; w) = 0$.*

Proof: Consider the Möbius maps

$$z \mapsto \frac{z - w}{a - w}.$$

This takes $a \mapsto 1$, $w \mapsto 0$ so $D(a, R) \mapsto D(1, r)$ for some $r < 1$, as it is an affine map. So then $D(a, R)$ is contained in the slit plane

$$\mathbb{C} \setminus \{z \mid \frac{z - w}{a - w} \in \mathbb{R}_{\le 0}\}.$$

Hence there is a branch of $\arg(z - w)$ defined on $D(a, R)$, and so

$$I(\gamma; w) = \frac{\arg(\gamma(1) - w) - \arg(\gamma(0) - w)}{2\pi} = 0.$$

Definition 1.10. Let $U \subset \mathbb{C}$ be open. Then a closed curve γ in U is *homologous to zero* in U if, for all $w \notin U$, $I(\gamma; w) = 0$.

U is *simply connected* if every closed curve in U is homologous to zero.

Remark. For U open, this is equivalent to the homotopy definition of simply connected.

Example 1.7.

1. Any disk is simply connected, by the previous proposition.
2. Any punctured disk $D(a, R) \setminus \{a\}$ is not simply connected, since curves can wind around a.
3. Any annulus is not simply connected.

Theorem 1.13 (General Cauchy's Integral Formula)**.** *Let $f : U \to \mathbb{C}$ be holomorphic on a domain U, and γ in a closed curve homologous to zero in U. Then for all $w \in U \setminus \gamma$,*

$$I(\gamma; w) f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - w} \, \mathrm{d}z,$$

and in particular,

$$\int_\gamma f(z) \, \mathrm{d}z = 0.$$

Proof: Notice applying the first equality to $g(z) = f(z)(z - w)$ gives $\int_\gamma f = 0$. So it suffices to prove the first statement.

We have by the previous lemma that

$$I(\gamma; w) f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{z - w} \, \mathrm{d}z,$$

so we want to show that

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z) - f(w)}{z - w} \, \mathrm{d}z = 0,$$

for all $w \in U \setminus \gamma$. Consider the function

$$g(z, w) = \begin{cases} \frac{f(z) - f(w)}{z - w} & z \neq w, \\ f'(w) & z = w. \end{cases}$$

This is a continuous function on $U \times U$, and we wish to show that

$$\int_\gamma g(z,w)\,\mathrm{d}z = 0,$$

for all $w \in U \setminus \gamma$. Consider the auxiliary function h on $\mathbb{C}$,

$$h(w) = \begin{cases} \int_\gamma g(\zeta, w)\,\mathrm{d}\zeta & w \in U, \\ \int_\gamma \frac{f(\zeta)}{\zeta - w}\,\mathrm{d}\zeta & w \in C \setminus \gamma, I(\gamma; w) = 0. \end{cases}$$

Call the latter set V. If $w \in U \cap V$, then

$$\int_\gamma g(\zeta, w)\,\mathrm{d}\zeta = \int_\gamma \frac{(f(\zeta) - f(w)}{\zeta - w}\,\mathrm{d}\zeta = \int_\gamma \frac{f(\zeta)}{\zeta - w}\,\mathrm{d}\zeta,$$

so h is well-defined. For any disk $D(0,R)$ with $\gamma \subset D(0,R)$, we have that $I(\gamma; w) = 0$ for all $w \notin D(0,R)$. In fact, γ is homologous to zero in U, so $U \cup V = \mathbb{C}$. So we have

$$|h(w)| = \left| \int_\gamma \frac{f(\zeta)}{\zeta - w}\,\mathrm{d}\zeta \right| \leq \frac{\mathrm{length}(\gamma) \sup |f(\zeta)|}{|w| - R} \to 0,$$

as $|w| \to \infty$. Then we claim h is holomorphic on $\mathbb{C}$. If so, then h is bounded as $|h(w)| \to 0$. Hence it is constant by Liouville's theorem, taking the value 0 on the entirety of $\mathbb{C}$, which finishes the proof. We use the following:

Lemma 1.3. *Let $U \subset \mathbb{C}$ be open, and $\phi : U \times [a,b] \to \mathbb{C}$ be continuous with $z \mapsto \phi(z,s)$ holomorphic on U for every $s \in [a,b]$. Then,*

$$g(z) = \int_a^b \phi(z,s)\,\mathrm{d}s$$

is holomorphic on U.

The proof of this is using Morera's. Without loss of generality, U is a disk. Then for any closed curve $\gamma : [0,1] \to U$, then

$$\begin{aligned} \int_\gamma g(z)\,\mathrm{d}z &= \int_0^1 \left[\int_a^b \phi(\gamma(t), s)\,\mathrm{d}s \right] \gamma'(t)\,\mathrm{d}t \\ &= \int_a^b \left[\int_0^1 \phi(\gamma(t), s)\gamma'(t)\,\mathrm{d}t \right] \mathrm{d}s, \end{aligned}$$

where we swap the order of integration by Fubini's theorem: suppose $f : [a,b] \times [c,d] \to \mathbb{C}$ is a continuous function. Then we have

$$\int_a^b \left(\int_c^d f(x,y)\,\mathrm{d}y \right) \mathrm{d}x = \int_c^d \left(\int_a^b f(x,y)\,\mathrm{d}x \right) \mathrm{d}y.$$

This clearly holds if f is constant, so it also holds when f is a step function. Since $[a,b] \times [c \times d]$ is closed and bounded, f is uniformly continuous. So f is a uniform limit of step functions, and we can exchange the order as claimed.

Now, back to our proof of the lemma. We have

$$\int_\gamma g(z)\,\mathrm{d}z = \int_a^b \left[\int_\gamma \phi(z,s)\,\mathrm{d}z \right] \mathrm{d}s.$$

Since $z \mapsto \Phi(z,s)$ is holomorphic, this is 0 by Cauchy's theorem on a disk. So,

$$\int_\gamma g(z)\,\mathrm{d}z = 0,$$

and by Morera's, g is holomorphic as claimed.

Therefore, proving this lemma, we get h is holomorphic as claimed and the generalized Cauchy integral formula follows.

Corollary 1.9 (Cauchy's theorem for Simply Connected Domains)**.** *Let $f : U \to \mathbb{C}$ be holomorphic on a simply connected domain U. Then for all closed curves γ in U,*

$$\int_\gamma f = 0$$

In fact, if $U \subset \mathbb{C}$ is open, then U is simply connected if and only if the complement of U in $\mathbb{C}_\infty$ is connected.

Example 1.8.

1. $D(a,R) \subset \mathbb{C}$ has disk complement in C_∞, so is simply connected.
2. Convex and starlike sets are simply connected.
3. The annulus is not simply connected.

1.4 Isolated Singularities of Holomorphic Maps

Definition 1.11. A point $a \in \mathbb{C}$ is an *isolated singularity* of $f : U \to \mathbb{C}$ holomorphic, if there exists $r > 0$ such that f is holomorphic on $D(a, r) \setminus \{a\}$, denoted $D(a, r)^\times$.

Example 1.9.

1. Take $a = 0$ and $f(z) = \frac{\sin z}{z}$. Using the identity theorem or expansion of e^z, we get
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$
about 0. So
$$f(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$
about 0. Thus f is a restriction of a holomorphic function on $\mathbb{C}$, say f, and $f(0) = 1$.
2. Take $a = 0$ and $g = \frac{1}{z^6}$. Then g is holomorphic on $\mathbb{C}^\times$, and $|g(z)| \to \infty$ as $z \to 0$, so there is no continuous extension at 0.
3. Recall the action $w \mapsto e^w = e^{\Re w} e^{i \Im w}$.

   The map $h(z) = e^{1/z}$ maps any $D(0, \varepsilon)^\times$ to all of $\mathbb{C}^\times$.

Theorem 1.14 (Laurent Expansion)**.** *Let f be holomorphic on an annulus $A = \{z \in \mathbb{C} \mid r < |z - a| < R\}$, where $0 \leq r < R \leq \infty$. Then,*

(i) *f has a (unique) convergent expansion on A:*
$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a)^n,$$
known as the "Laurent series".

(ii) *For any $r < \rho < R$, we have*
$$c_n = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{(z - a)^{n+1}} \, \mathrm{d}z.$$

(iii) *If $r < \rho' \leq \rho < R$, the Laurent series converges uniformly on $\{z \in \mathbb{C} \mid \rho' \leq |z - a| \leq \rho\}$.*

Proof: Fix $w \in A$, and choose $r < \rho_1 < |w-a| < \rho_2 < R$.

Define two closed curves γ_1, γ_2 by cutting along a diameter of the sub-annulus, labelled such that $I(\gamma_1; w) = 1$ and $I(\gamma_2; w) = 0$.

Then γ_1, γ_2 are both homologous to zero in A, so by the generalized Cauchy's integral formula, we have

$$f(w) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(z)}{z-w}\,\mathrm{d}z = \frac{1}{2\pi i}\int_{\gamma_1+\gamma_2} \frac{f(z)}{z-w}\,\mathrm{d}z.$$

Travelling around $\gamma_1 + \gamma_2$ is the same as travelling $\partial D(a,\rho_2) - \partial D(a,\rho_1)$. So,

$$f(w) = \frac{1}{2\pi i}\int_{|z-a|=\rho_2} \frac{f(z)}{z-w}\,\mathrm{d}z - \frac{1}{2\pi i}\int_{|z-a|=\rho_1} \frac{f(w)}{z-w}\,\mathrm{d}z.$$

Label the first integral as I_2, and the second as I_1. Using the geometric series for $(1-\frac{w-a}{z-a})^{-1}$ to compute I_2 as a Taylor series gives

$$I_2 = \sum_{n=0}^{\infty} c_n (w-a)^n,$$

where the coefficients are

$$c_n = \frac{1}{2\pi i}\int_{|z-a|=\rho_2} \frac{f(z)}{(z-a)^{n+1}}\,\mathrm{d}z,$$

for $n \geq 0$. For I_1, since $|z-a| < |w-a|$, using the expansion

$$-\frac{1}{z-w} = \frac{\frac{1}{w-a}}{1-\frac{z-a}{w-a}} = \sum_{m=1}^{\infty} \frac{(z-a)^{m+1}}{(w-a)^m},$$

we get that

$$I_1 = \sum_{m=1}^{\infty} d_m (w-a)^{-m},$$

where

$$d_m = \frac{1}{2\pi i}\int_{|z-a|=\rho_1} \frac{f(z)}{(z-a)^{-m+1}}\,\mathrm{d}z.$$

Reindexing with $n = -m$, we obtain the Laurent expansion for f.

To show part (ii) and (iii), suppose that

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

on A, and let $r < \rho' \leq \rho < R$. The non-negative power series

$$\sum_{n=0}^{\infty} c_n (z-a)^n$$

has radius of convergence greater than or equal to R, so it converges uniformly on $D(a, \rho)$. Similarly, if $u = \frac{1}{z-a}$, then the negative part of the Laurent expansion,

$$\sum_{n=1}^{\infty} c_{-n} u^n,$$

has radius of convergence greater than or equal to r^{-1}, so it converges uniformly on $\{|z-a| \geq \rho'\}$. So the Laurent series converges uniformly on $\rho' \leq |z-a| \leq \rho$.

So we can integrate term-by-term to get

$$\frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{(z-a)^{m+1}} \,\mathrm{d}z = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} c_n \int_{\partial D(a,\rho)} (z-a)^{n-m-1} \,\mathrm{d}z = c_m,$$

since this integral is 0 unless $n = m$, in which case it is $2\pi i$.

Remark. This shows that $f = f_1 + f_2$, where f_1 is holomorphic on $D(a, R)$, and f_2 is holomorphic on $|z-a| > r$.

Applying to the case $r = 0$, we have three possibilities on a punctured disk domain, i.e. an isolated singularity at a.

1. $c_n = 0$ for all $n < 0$. Then f is the restriction to $D(a, R)^\times$ of a function holomorphic on $D(a, R)$. We say f has a *removable singularity* at a.

   An example is
   $$f(z) = \frac{\sin z}{z},$$
   at $a = 0$.

2. There exists $k < 0$ such that $c_k \neq 0$, but $c_n = 0$ for all $n < k$. Then $(z-a)^{-k} f(z)$ is holomorphic and non-zero at a. We say f has a *pole* of order $|k|$ at a.

   An example is
   $$g(z) = \frac{1}{z^6},$$
   at $a = 0$, which has a pole of order 6.

3. $c_n \neq 0$ for infinitely many $n < 0$. Then we say f has an *essential singularity* at a.

   An example is
   $$h(z) = \exp\left(\frac{1}{z}\right),$$
   at $a = 0$.

We now look at the local behaviour around each of these kinds of singularities.

Proposition 1.8. *An isolated singularity at $z = a$ for f is removable if and only if*
$$\lim_{z \to a} (z - a) f(z) = 0.$$

Proof: If f is indeed holomorphic, then f is bounded as it is continuous on a neighbourhood of a, hence the limit approaches 0.

If the limit converges to 0, consider
$$g(z) = \begin{cases} (z-a)^2 f(z) & z \neq a, \\ 0 & z = a. \end{cases}$$

Then by definition,
$$g'(a) = \lim_{z \to a} (z - a) f(z) = 0,$$
so g is holomorphic at a, with $g(a) = 0$. Therefore
$$g(z) = \sum_{n=2}^{\infty} c_n (z - a)^n,$$
so
$$f(z) = \sum_{n=0}^{\infty} c_{n+2} (z - a)^n,$$
proving f is holomorphic at a.

Proposition 1.9. *An isolated singularity at $z = a$ for f is a pole if and only if $|f(z)| \to \infty$ as $z \to a$.*

Moreover, the following are equivalent:

(i) *f has a pole of order k at $z = a$.*

(ii) *$f(z) = (z - a)^{-k} g(z)$, where g is holomorphic at non-zero at a.*

(iii) $f(z) = \frac{1}{h(z)}$, *where h is holomorphic at a with a zero of order k at a.*

Proof: Immediately, (i) is equivalent to (ii) using the Laurent expansion, and (ii) is equivalent to (iii) as since g is holomorphic at a and non-zero, $\frac{1}{g}$ is holomorphic at a.

Now if f has a pole of order k at $z = a$, then $f(z) = (z-a)^{-k} g(z)$, so $|f(z)| \to \infty$ as $z \to a$.

Conversely, if $|f(z)| \to \infty$ as $z \to a$, then there exists $r > 0$ such that $f(z) \neq 0$ for all $0 < |z - a| < r$. So $\frac{1}{f}$ is holomorphic on $D(a,r)^\times$, and moreover $\frac{1}{f} \to 0$ as $z \to a$, so the singularity at 0 for $\frac{1}{f}$ is removable, and hence

$$\frac{1}{f(z)} = h(z),$$

where h is holomorphic on $D(a,r)$. Now h has a zero of order k for some $k \geq 1$, so $h(z) = (z-a)^k l(z)$ for l holomorphic and non-zero at a, so

$$f(z) = (z-a)^{-k} g(z),$$

meaning f has a pole of order k at $z = a$.

Corollary 1.10. *An isolated singularity at $z = a$ is essential if and only if $|f|$ does not approach a limit in $\mathbb{R} \cup \{\infty\}$ as $z \to a$.*

In fact, essential singularities are even more interesting.

Theorem 1.15 (Casorati-Weierstrass)**.** *Consider $f : D(a,R)^* \to \mathbb{C}$ with an essential singularity at $z = a$. Then f has a dense image on any neighbourhood of a. That is,*

$$\forall w \in \mathbb{C},\ \forall \varepsilon > 0,\ \forall \delta > 0, \exists z \in D(a,\delta)^* \text{ with } |f(z) - w| < \varepsilon.$$

This is proven in the second example sheet. In fact, we can prove the stronger (and much more difficult) result:

Theorem 1.16 (Great Picard Theorem)**.** *If $z = a$ is an essential singularity of f, then there exists $b \in \mathbb{C}$ such that for all $\varepsilon > 0$,*

$$\mathbb{C} \setminus \{b\} \subseteq f(D(a,\varepsilon)^*).$$

For example, $e^{1/z}$ has an essential singularity at 0, and takes every non-zero value in every neighbourhood of 0.

Remark. Using the Riemann sphere perspective, if $f : D(a, R)^* \to \mathbb{C}$ has a pole at $z = a$, we can view f as a continuous map $f : D(a, R) \to \mathbb{C}_\infty$ with $f(a) = \infty$.

Then f is holomorphic at a in the $\mathbb{C}_\infty$ sense since $\frac{1}{f}$ is holomorphic on a neighbourhood of a, with a zero of the same order as the pole of f.

Definition 1.12. Suppose D is a domain. A function f is *meromorphic* on D if $f : D \setminus S \to \mathbb{C}$ is holomorphic, where S is a set of isolated singularities for f which are removable or are poles.

Definition 1.13. Let $f : D(a, R)^* \to \mathbb{C}$ be holomorphic with Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n.$$

The *residue* of f at $z = a$ is

$$\operatorname{Res}_{z=a} f(z) = c_{-1} \in \mathbb{C}.$$

The *principal part* of f at $z = a$ is

$$\sum_{n=-\infty}^{-1} c_n (z-a)^n.$$

Proposition 1.10. *Let γ be a closed curve in $D(a, R)^*$. Then,*

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i I(\gamma; a) \operatorname{Res}_{z=a} f(z).$$

Proof: Using the uniform convergence of the Laurent expansion of f, we have that

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{n=-\infty}^{\infty} c_n \left[\int_\gamma (z-a)^n\,\mathrm{d}z\right].$$

Since we also have

$$\int_\gamma (z-a)^n\,\mathrm{d}z = \begin{cases} 0 & n \neq -1, \\ 2\pi i I(\gamma; a) & n = -1, \end{cases}$$

this completes the proof.

If f is meromorphic on a domain D, and $z = a$ is a pole of f in D, then its principal part at $z = a$ is of the form

$$\frac{c_{-k}}{(z-a)^k} + \frac{c_{-k+1}}{(z-a)^{k-1}} + \cdots + \frac{c_{-1}}{z-a},$$

a polynomial in $(z-a)^{-1}$, and hence it can be written as $p(z)(z-a)^{-k}$, for some polynomial p. So the principal part of f at $z = a$ is holomorphic on $\mathbb{C} \setminus \{a\}$.

More generally, if f is meromorphic on D, and $\{a_1, \ldots, a_m\}$ are a subset of the poles of f in D, with $p_i(z)$ the principal part of f at $z = a_i$, then the function

$$g(z) = f(z) - \sum_{i=1}^{m} p_i(z)$$

is meromorphic on D, with removable singularities at $a_1, \ldots, a_m$.

Theorem 1.17 (Residue theorem)**.** *Let f be meromorphic on a domain D, and g a closed curve which is homologous to zero in D. Suppose γ does not contain any pole of f, and f has only finitely many poles in D with non-zero winding number, say $\{a_1, \ldots, a_m\}$. Then,*

$$\int_\gamma f(z)\mathrm{d}z = 2\pi i \sum_{i=1}^{m} I(\gamma; a_i) \operatorname{Res}_{z=a_i} f(z).$$

Proof: Let p_i denote the principal part of f at $z = a_i$, and write $g = f - \sum p_i$.

By Cauchy's theorem, we have

$$\int_\gamma g = 0, \implies \int_\gamma f = \sum_{i=1}^{m} \int_\gamma p_i.$$

Each p_i is holomorphic on $\mathbb{C} \setminus \{a_i\}$ as we have shown, so by the previous proposition we have

$$\int_\gamma p_i = 2\pi i I(\gamma, a_i) \operatorname{Res}_{z=a_i} p_i(z).$$

By definition, $\operatorname{Res}_{z=a_i} p_i(z) = \operatorname{Res}_{z=a_i} f(z)$, so

$$\int_\gamma f = 2\pi i \sum_{i=1}^{m} I(\gamma; a_i) \operatorname{Res}_{z=a_i} f(z).$$

Remark.

1. If γ is homologous to 0 in a domain D, then $\{z \in \mathbb{C} : I(\gamma; z) \neq 0\}$ is closed and bounded. Note that the winding number is a continuous function on $\mathbb{C} \setminus \gamma$, taking values in a discrete set. Hence
$$\{z \in \mathbb{C} \setminus \gamma : I(\gamma; z) = 0\}$$
is open, so the complement is closed. Since the poles are isolated, this closed and bounded set contains only finitely many of them.
2. If f is holomorphic on D, then this is simply Cauchy's theorem.
3. If $f(z) = \frac{g(z)}{z-a}$ for holomorphic g, then $\mathrm{Res}_{z=a} f(z) = g(a)$. Hence the residue theorem implies Cauchy's integral formula.
4. We say that a closed curve γ *bounds a domain* U if
$$I(\gamma; z) = \begin{cases} 1 & z \in U, \\ 0 & z \notin U. \end{cases}$$
If γ is a closed curve in a domain D which bounds a domain U, and f is holomorphic on D, then
$$\int_\gamma f = 0,$$
and for all $w \in U$,
$$\int_\gamma \frac{f(z)}{z-w}\,\mathrm{d}z = 2\pi i f(w).$$
If f is meromorphic on D with no poles on γ, then
$$\int_\gamma f = 2\pi i \sum_{w \text{ a pole}} \mathrm{Res}_{z=w} f(z).$$

From II Algebraic Topology, we can prove the following:

Theorem 1.18 (Jordan Curve Theorem)**.** *Every simple closed continuous curve in the plane separates* $\mathbb{C}$ *into two connected components, one bounded and one unbounded.*

This ensures all curves we are considering are indeed 'nice'.

Let us see how we can compute residues.

(i) If f has a simple pole at $z = a$, then the Laurent expansion at a is

$$f(z) = \frac{c_{-1}}{z-a} + c_0 + c_1(z-a) + c_2(z_a)^2 + \cdots$$

Hence we can find the residue as

$$\mathrm{Res}_{z=a}(f(z)) = \lim_{z\to a}(z-a)f(z).$$

(ii) If $f = \frac{g(z)}{h(z)}$, where g is holomorphic and non-zero at $z = a$, and h is holomorphic with a simple zero at $z = a$, then

$$g(z) = g(a) + (z-a)\tilde{g}(z),$$

and

$$h(z) = h'(a)(z-a)\tilde{h}(z),$$

where $\tilde{h}(a) = 1$ at $z = a$ and is holomorphic at a. Hence

$$\frac{g(z)}{h(z)} = \frac{g(a)}{h'(a)(z-a)\tilde{h}(z)} + \frac{\tilde{g}(z)}{h'(a)\tilde{h}(z)}.$$

The latter part is holomorphic at a, as $z = a$ is a simple pole of h. Hence applying part (i) to the first part, we see that

$$\mathrm{Res}_{z=a} f(z) = \frac{g(a)}{h'(a)}.$$

(iii) If $f(z) = \frac{g(z)}{(z-a)^k}$, where g is holomorphic at a, then $\mathrm{Res}_{z=a} f(z)$ is the coefficient of $(z-a)^{k-1}$ in the expansion of g, i.e.

$$\frac{g^{(k-1)}(a)}{(k-1)!}.$$

Example 1.10.

1. Take $f(z) = \frac{1}{1+z^2}$ at $z = i$. Then since

$$(z-i)f(z) = \frac{1}{z+i},$$

we have

$$\text{Res}_{z=i} f(z) = \frac{1}{2i}.$$

2. If $f(z) = \frac{e^z}{z^2+1}$, then using fact (ii) we get

$$\text{Res}_{z=i} f(z) = \frac{e^i}{2i}.$$

Now we look at applying this to real integrals.

Example 1.11.

1. Consider the integral

$$\int_0^\infty \frac{1}{1+x^4} \,\mathrm{d}x.$$

This integral has two properties: it is even, and also as $|x| \gg 1$, $|\frac{1}{1+x^2}| \ll 1$.

We consider the integral around the semicircle centred at the origin with radius R, with diameter along the real axis and going into the upper half plane.

Notice that $1 + x^4$ has four simple zeroes: $e^{\pi i/4}$, $e^{3\pi i/4}$, $e^{5\pi i/4}$ and $e^{7\pi i/4}$. Hence the curve γ_R has winding number 1 around $e^{\pi i/4}$ and $e^{3\pi i/4}$, and 0 around the other two poles. Now

$$\text{Res}_{z=e^{\pi i/4}} \frac{1}{1+z^4} = \left.\frac{1}{4z^3}\right|_{z=e^{\pi i/4}} = \frac{1}{4e^{3\pi i/4}}.$$

Similarly,

$$\text{Res}_{z=e^{3\pi i/4}} = \frac{1}{4e^{\pi i/4}}.$$

The integral can be broken up into two pieces:

$$\int_{\gamma_R} \frac{1}{1+z^4} \,\mathrm{d}z = \int_{C_R} \frac{1}{1+z^4} \,\mathrm{d}z + \int_{-R}^{R} \frac{1}{1+z^4} \,\mathrm{d}z.$$

Here C_R is the semicircular arc. For the first integral, we can parametrise $z = Re^{i\theta}$ to get

$$|I_1| = \left| \int_0^\pi \frac{1}{1+R^4 e^{4i\theta}} iRe^{i\theta} \,\mathrm{d}\theta \right| \le \frac{\pi R}{R^4 - 1} \to 0,$$

as $R \to \infty$. Hence

$$\int_{-R}^{R} \frac{1}{1+z^4}\,\mathrm{d}z = \int_{\gamma_R} \frac{1}{1+z^4}\,\mathrm{d}z - \int_{C_R} \frac{1}{1+z^4}\,\mathrm{d}z \to 2\pi i\left[\frac{1}{4e^{3\pi i/4}} + \frac{1}{4e^{\pi i/4}}\right],$$

as $R \to \infty$. Hence this integral I_2 tends to

$$\begin{aligned} I_2 &\to \frac{1}{2}\pi i\left(e^{-3\pi i/4} + e^{-\pi i/4}\right) = \frac{1}{2}\pi i\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) \\ &= \frac{1}{2}\pi i(-\sqrt{2}i) = \frac{\pi}{\sqrt{2}}. \end{aligned}$$

Therefore,

$$\int_0^\infty \frac{1}{1+x^4}\,\mathrm{d}x = \frac{\pi}{2\sqrt{2}}.$$

2. We compute the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x+x^2}\,\mathrm{d}x.$$

Note that

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{ix-y} + e^{-ix+y}}{2}.$$

This is large as $y \to \pm\infty$, hence we cannot use the semicircular arc as before. However if we instead consider the integral of

$$\frac{e^{ix}}{1+x+x^2},$$

then as $e^{ix} = \cos x + i \sin x$, the real part of this integral will be what we want. Notice then that $e^{i(x+iy)} = e^{ix-y}$ will be bounded above by 1 in modules for $y \geq 0$.

Take the same semicircle γ_R as before, with semicircular arc C_R. Now since the roots of $1+x+x^2$ are $e^{2\pi i/3}$, $e^{4\pi i/3}$, the winding number is 1 around the first, and 0 around the second. Again, we can write

$$\int_{\gamma_R} \frac{e^{iz}}{1+z+z^2}\,\mathrm{d}z = \int_{C_R} \frac{e^{iz}}{1+z+z^2}\,\mathrm{d}z + \int_{-R}^{R} \frac{e^{iz}}{1+z+z^2}\,\mathrm{d}z.$$

Now the first integral I_1 has a contribution of

$$|I_1| \leq \mathrm{length}(C_R)\frac{1}{R^2-R-1} = \frac{\pi R}{R^2-R-1} \to 0,$$

as $R \to \infty$. Now we know

$$\operatorname{Res}_{z=e^{2\pi i/3}} \frac{e^{iz}}{1+z+z^2} = \frac{e^{ie^{2\pi i/3}}}{1+2e^{2\pi i/3}},$$

hence the integral converges to

$$I_2 \to 2\pi i \left[\frac{e^{ie^{2\pi i/3}}}{1+2e^{2\pi i/3}}\right] - 0.$$

Now as

$$e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i,$$

we get $1 + 2e^{2\pi i/3} = \sqrt{3}i$. Hence also

$$e^{ie^{2\pi i/3}} = e^{i(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)} = e^{-i/2}e^{-\sqrt{3}/2},$$

so

$$I_2 \to 2\pi i \left[\frac{e^{-i/2}e^{-\sqrt{3}/2}}{\sqrt{3}i}\right] = \frac{2\pi}{\sqrt{3}}e^{-\sqrt{3}/2}e^{-i/2}.$$

Hence

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x+x^2}\,\mathrm{d}x = \Re\left(\frac{2\pi}{\sqrt{3}}e^{-\sqrt{3}/2}e^{-i/2}\right) = \frac{2\pi}{3}e^{-\sqrt{3}/2}\cos\left(-\frac{1}{2}\right).$$

The integral around the semicircle is sufficiently useful so that we prove the following result about it.

Lemma 1.4 (Jordans' lemma)**.** *Suppose $f(z)$ is holomorphic on $\{|z| > r\}$ for some $r > 0$, and $zf(z)$ is bounded. Then for all $\alpha > 0$, we have*

$$\int_{C_R} f(z)e^{i\alpha z}\,\mathrm{d}z \to 0$$

as $R \to \infty$, where $C_R : [0, \pi] \to \mathbb{C}$ is given by $C_R(t) = Re^{it}$.

Proof: We have for $z = Re^{it}$, that

$$|e^{i\alpha z}| = e^{-\alpha R \sin t},$$

and so using the basic estimate $\frac{\sin t}{t} \geq \frac{2}{\pi}$, we have

$$|e^{i\alpha z}| \leq \begin{cases} e^{-\alpha R \frac{2t}{\pi}}, & t \in [0, \frac{\pi}{2}], \\ e^{-\alpha R \frac{2t'}{\pi}}, & t' = \pi - t \in [0, \frac{\pi}{2}]. \end{cases}$$

By the hypothesis, there exists $M \in \mathbb{R}$ such that $|zf(z)| \leq M$. Putting these together, let $\tilde{C}_R$ be C_R for $[0, \frac{\pi}{2}]$. Then

$$\begin{aligned} \left| \int_{\tilde{C}_R} f(z) e^{i\alpha z} \, \mathrm{d}z \right| &\leq \int_0^{\pi/2} M e^{-\alpha R \frac{2t}{\pi}} \, \mathrm{d}t \\ &= M \left(\frac{1}{-aR\frac{2}{\pi}} \right) e^{-aR\frac{2t}{\pi}} \Bigg|_{t=0}^{t=\pi/2} \\ &= \frac{(1 - e^{-\alpha R}) \pi M}{2R\alpha} \to 0, \end{aligned}$$

as $R \to \infty$. Similarly, the second half of the integral goes to 0, so the entire integral goes to 0 as $R \to \infty$.

Now we will see how we can use Jordan's lemma.

Example 1.12.

1. Consider the integral
$$\int_{-\infty}^{\infty} \frac{\cos mx}{1 + x^2} \, \mathrm{d}x,$$
for $m \in \mathbb{R}$. As $\cos z$ is large for large imaginary z, we instead use $\cos mx = \Re(\exp(imx))$, so
$$I = \Re\left(\int_{-\infty}^{\infty} \frac{e^{imx}}{1 + x^2} \, \mathrm{d}x \right).$$
We then use out usual contour γ_R and C_R. If $m > 0$, then using Jordan's lemma,
$$\int_{C_R} \frac{e^{imz}}{1 + z^2} \, \mathrm{d}z \to 0.$$
Also from the residue theorem,
$$\int_{\gamma_R} \frac{e^{imz}}{1 + z^2} \, \mathrm{d}z = 2\pi i \operatorname{Res}_{z=i} \frac{e^{imz}}{1 + z^2} = 2\pi i \frac{e^{im(i)}}{2i} = \pi e^{-m}.$$

So,

$$\pi e^{-m} = \int_{C_R} \frac{e^{imz}}{1+z^2}\,\mathrm{d}z + \int_{-R}^{R} \frac{e^{imz}}{1+z^2}\,\mathrm{d}z \implies I = \frac{\pi}{e^m}.$$

If $m < 0$, then $\cos(mx) = \cos(-mx)$, so from the above

$$I = \frac{\pi}{e^{-m}}.$$

If $m = 0$, then we have

$$\left|\int_{C_R} \frac{1}{1+z^2}\,\mathrm{d}z\right| \leq \frac{\pi R}{R^2-1} \to 0,$$

as $R \to \infty$, so with a similar computation

$$\mathrm{Res}_{z=i} \frac{1}{z^2+1} = \frac{1}{2i},$$

we get

$$I = \frac{\pi}{e^0} = \pi.$$

Hence in all cases we get

$$I = \pi e^{-|m|}.$$

2. Now consider the integral

$$\int_0^{2\pi} \frac{1}{5+4\cos\theta}\,\mathrm{d}\theta.$$

Let us use

$$\cos\theta = \frac{1}{2}\left[e^{i\theta} + e^{-i\theta}\right],$$

so if $z = e^{i\theta}$, we have $\cos\theta = \frac{1}{2}(z + z^{-1})$. Hence, as $\mathrm{d}z = ie^{i\theta}\,\mathrm{d}\theta = iz\,\mathrm{d}\theta$, we get

$$\begin{aligned}\int_0^{2\pi} \frac{1}{5+4\cos\theta}\,\mathrm{d}\theta &= \int_{|z|=1} \frac{1}{5+2(z+z^{-1})}\frac{\mathrm{d}z}{iz} = \frac{1}{i}\int_{|z|=1} \frac{1}{2z^2+5z+2}\,\mathrm{d}z \\ &= \frac{1}{i}\int_{|z|=1} \frac{1}{(2z+1)(z+2)}\,\mathrm{d}z.\end{aligned}$$

This function has poles at $z = -\frac{1}{2}, -2$ but our contour only winds around the first. Hence Cauchy's integral formula applied to $\frac{1}{2(z+2)}$ gives

$$\frac{1}{2(-\frac{1}{2}+z)} = \frac{1}{2\pi i}\int_{|z|=1} \frac{1}{2(z+2)(z+\frac{1}{2})}\,\mathrm{d}z,$$

so

$$\frac{2\pi i}{3} = \int_{|z|=1} \frac{1}{2z^2+5z+2}\,\mathrm{d}z = i\int_0^{2\pi} \frac{1}{5+4\cos\theta}\,\mathrm{d}\theta.$$

This gives

$$\int_0^{2\pi} \frac{1}{5+4\cos\theta}\,\mathrm{d}\theta = \frac{2\pi}{3}.$$

3. We also look at

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x.$$

Consider

$$\begin{aligned}\frac{1}{2i}\int_0^\infty \frac{e^{ix}-e^{-ix}}{x}\,\mathrm{d}x &= \frac{1}{2i}\int_0^\infty \frac{e^{ix}}{x}\,\mathrm{d}x - \frac{1}{2i}\int_0^{-\infty} \frac{e^{it}}{-t} - \mathrm{d}t \\ &= \frac{1}{2i}\int_{-\infty}^\infty \frac{e^{ix}}{x}\,\mathrm{d}x.\end{aligned}$$

Our usual contour γ_R doesn't work, as it passes through that pole 0. Instead, we consider the $\gamma_{R,\varepsilon}$ contour, which also consists of a semicircular arc of radius $\varepsilon > 0$ around 0.

Then from Cauchy's theorem, we get

$$\int_{\gamma_{R,\varepsilon}} \frac{e^{iz}}{z}\,\mathrm{d}z = 0.$$

We can use Jordan's lemma to get

$$\int_{C_R} \frac{e^{iz}}{z}\,\mathrm{d}z \to 0,$$

as $R \to \infty$. On C_ε, we can parametrize $z = \varepsilon e^{i\theta}$, and $\mathrm{d}z = iz\,\mathrm{d}\theta$, so

$$\int_{C_\varepsilon} \frac{e^{iz}}{z}\,\mathrm{d}z = \int_0^\pi e^{i\varepsilon e^{i\theta}} i\,\mathrm{d}\theta \to i\int_0^\pi 1\,\mathrm{d}\theta = \pi i,$$

as $\varepsilon \to 0$. Hence

$$\int_{\gamma_{R,\varepsilon}} \frac{e^{iz}}{z}\,\mathrm{d}z = \int_{C_R} \frac{e^{iz}}{z}\,\mathrm{d}z + \int_{-R}^{-\varepsilon} \frac{e^{iz}}{z}\,\mathrm{d}z - \int_{C_\varepsilon} \frac{e^{iz}}{z}\,\mathrm{d}z + \int_{\varepsilon}^{R} \frac{e^{iz}}{z}\,\mathrm{d}z.$$

As $\varepsilon \to 0$, $R \to \infty$, we obtain

$$0 = \int_{-\infty}^{\infty} \frac{e^{iz}}{z}\,\mathrm{d}z - \pi i,$$

so we get

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{1}{2i}\pi i = \frac{\pi}{2}.$$

4. We also look at the integral

$$\int_0^\infty \frac{x^\alpha}{1+x^2}\,\mathrm{d}x,$$

for $\alpha \in (0,1)$. Note that we can extend z^α holomorphically as $z^\alpha = \exp(\alpha \log z)$, for a suitable branch of the logarithm.

If we choose $\log z = \ln|z| + i \arg z$, for $\arg z \in (-\frac{\pi}{2}, \frac{3\pi}{2})$, then for $x > 0$ we get $(-x)^\alpha = (-1)^\alpha x^\alpha$.

Indeed, $\log(-x) = \ln|x| + \pi i = \ln x + \pi i$. In particular, $\log(-1) = \pi i$. So

$$\log x + \log(-1) = \ln x + \pi i = \log(-x).$$

Hence we have

$$\exp(\alpha \log x)\exp(\alpha \log(-1)) = \exp(\alpha \log(-x)).$$

We again take the contour $\gamma_{R,\varepsilon}$, which avoids the branch cut of the logarithm.

We can show that the integrals along C_R and $C_\varepsilon \to 0$ as $R \to \infty$ and $\varepsilon \to 0$. As we have a pole at $z = i$, we compute

$$\mathrm{Res}_{z=i} \frac{\exp(\alpha \log z)}{(z+i)(z-i)} = \frac{i^\alpha}{2i}.$$

Hence we get

$$\begin{aligned}2\pi i \operatorname{Res}_{z=i} \frac{\exp(\alpha \log z)}{(z+i)(z-i)} &= \int_{\gamma_{R,\varepsilon}} \frac{z^\alpha}{1+z^2}\,\mathrm{d}z \\ &= \int_{C_R} \frac{z^\alpha}{1+z^2}\,\mathrm{d}z - \int_{C_\varepsilon} \frac{z^\alpha}{1+z^2}\,\mathrm{d}z \\ &\quad + \int_{-R}^{-\varepsilon} \frac{z^\alpha}{1+z^2}\,\mathrm{d}z + \int_{\varepsilon}^{R} \frac{z^\alpha}{1+z^2}\,\mathrm{d}z.\end{aligned}$$

By the substitution $t = -z$, we have

$$\int_{-R}^{-\varepsilon} \frac{z^\alpha}{1+z^2}\,\mathrm{d}z = (-1)^\alpha \int_{\varepsilon}^{R} \frac{z^\alpha}{1+z^2}\,\mathrm{d}z.$$

Hence taking $\varepsilon \to 0$, $R \to \infty$, we have

$$2\pi i \frac{i^\alpha}{2i} = 0 - 0 + [(-1)^\alpha + 1] \int_0^\infty \frac{x^\alpha}{1+x^2}\,\mathrm{d}x,$$

so

$$\int_0^\infty \frac{x^\alpha}{1+x^2}\,\mathrm{d}x = \frac{\pi i^\alpha}{1+(-1)^\alpha}.$$

5. We consider

$$\int_0^\infty \frac{x^{1/3}}{(x+2)^2}\,\mathrm{d}x.$$

Let's define $\log z = \ln|z| + i \arg z$, where $\arg z \in (0, 2\pi)$. Hence we cut away the positive real axis.

We then define the 'keyhole contour', which is the integral around the branch cut, with two straight lines L_1, L_2 of $\arg z = \delta$ and $\arg z = 2\pi - \delta$ respectively, as well as connecting curves C_1, an arc of radius R, and C_2, an arc of radius ε.

On C_1, we have

$$\left| \int_{C_1} \frac{z^{1/3}}{(z+2)^2}\,\mathrm{d}z \right| \leq (2\pi - 2\delta) R \frac{R^{1/3}}{(R-2)^2} \to 0,$$

and on C_2 we have

$$\left| \int_{C_2} \frac{z^{1/3}}{(z+2)^2}\,\mathrm{d}z \right| \leq (2\pi - 2\delta) \varepsilon \frac{\varepsilon^{1/3}}{(2-\varepsilon)^2} \to 0.$$

On L_1, $z = te^{i\delta}$, for $t \in [\varepsilon, R]$. Then $\mathrm{d}z = e^{i\delta}\,\mathrm{d}t$. We get that

$$\int_\varepsilon^R \frac{t^{1/3}e^{i\delta/3}}{(te^{i\delta}+2)^2} e^{i\delta}\,\mathrm{d}t \to \int_\varepsilon^R \frac{t^{1/3}}{(t+2)^2}\,\mathrm{d}t,$$

as $\delta \to 0$. Now on L_2, $z = te^{i(2\pi-\delta)}$, so the integral is

$$\int_\varepsilon^R \frac{t^{1/3}e^{i\frac{2\pi-\delta}{3}}}{(te^{i(2\pi-\delta)}+2)^2} e^{i(2\pi-\delta)}\,\mathrm{d}t \to e^{2\pi i/3}\int_\varepsilon^R \frac{t^{1/3}}{(t+2)^2}\,\mathrm{d}t.$$

Hence by residue theorem,

$$\begin{aligned}\mathcal{O}\left(\frac{1}{R^{2/3}}\right) - e^{2\pi i/3}\int_\varepsilon^R \frac{t^{1/3}}{(t+2)^2}\,\mathrm{d}t - \mathcal{O}(\varepsilon^{4/3}) + \int_\varepsilon^R \frac{t^{1/3}}{(t+2)^2}\,\mathrm{d}t \\ = 2\pi i \operatorname{Res}_{z=-2} \frac{z^{1/3}}{(z+2)^2}.\end{aligned}$$

We can find this residue as

$$\left.\frac{\mathrm{d}}{\mathrm{d}z}\right|_{z=-2} z^{1/3} = \left.\frac{\mathrm{d}}{\mathrm{d}z}\right|_{z=-2} \exp(\log z/3) = \frac{1}{3z}\exp(\log z/3)\bigg|_{z=-2},$$

so

$$\operatorname{Res}_{z=-2}\frac{z^{1/3}}{(z+2)^2} = -\frac{1}{6}\sqrt[3]{2}e^{\pi i/3}.$$

We can also compute

$$\frac{e^{\pi i/3}}{(1-e^{2\pi i/3})} = \frac{i}{\sqrt{3}},$$

so putting this all together we get

$$\int_0^\infty \frac{t^{1/3}}{(t+2)^2}\,\mathrm{d}t = \frac{\pi}{3\sqrt{3}}\sqrt[3]{2}.$$

Proposition 1.11. *Let f have a zero (resp. pole) of order $k > 0$ at $z = a$. Then $\frac{f'(z)}{f(z)}$ has a simple pole at $z = a$, of residue k (resp. $-k$).*

Remark. By example sheet two, if $f : U \to \mathbb{C}$ is such that $f(U)$ is contained in a simply connected set which omits 0, then there exists a homomorphic function $g(z) = \log f(z)$ on U, so $\frac{f'(z)}{f(z)}$ has a holomorphic antiderivative $\log f$ on U.

Hence $\frac{f'}{f}$ is the 'logarithmic derivative' of f.

Proof: Suppose $f(z) = (z-a)^k g(z)$ near a, with $g(a) \neq 0$, then $f'(z) = k(z-a)^{k-1}g(z) + (z-a)^k g'(z)$, so

$$\frac{f'(z)}{f(z)} = \frac{k}{z-a} + \frac{g'(z)}{g(z)}.$$

Then as $g(a) \neq 0$, $\frac{g'}{g}$ is holomorphic at a. Hence $\mathrm{Res}_{z=a} \frac{f'}{f} = k$ (resp. $-k$).

Theorem 1.19 (Argument Principle)**.** *Let γ be a closed curve bounding a domain D, and f a function meromorphic on an open neighbourhood of $D \cup \gamma$. If f has no zeroes or poles on γ, then*

$$I(f \circ \gamma; 0) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, \mathrm{d}z = |\{\textit{zeroes of } f \textit{ in } D\}| - |\{\textit{poles of } f \textit{ in } D\}|,$$

where zeroes and poles are counted with multiplicity.

Proof: We have

$$I(f \circ \gamma; 0) = \frac{1}{2\pi i} \int_{f \circ \gamma} \frac{\mathrm{d}w}{w} = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, \mathrm{d}z,$$

by letting $w = f(z)$. Then by residue theorem, this equals

$$\sum_{a \text{ a pole in } D} \mathrm{Res}_{z=a} \frac{f'}{f}.$$

But this exactly equals what we want, from our previous proposition.

Remark.

1. Recall that γ is compact, so $f \circ \gamma$ is also a closed curve (and compact).
2. Morally, this says that $2\pi \times$sum of poles) is the change in $\arg f(z)$ as z travels γ.

The argument principle has important consequences for local behaviour of f.

Definition 1.14. If f is holomorphic and non-constant near $z = a$, then the *local degree* of $f(z)$ at $z = a$ is $\deg_{z=a} f(z)$, the order of the zero of $f(z) - f(a)$ at $z = a$.

If f is non-constant, then we can write $f(z) - f(a) = (z-a)^k g(z)$, where g is holomorphic and non-zero at a, and the zero at $z = a$ of $f(z) - f(a)$ is isolated. Hence for $|z - a|$ sufficiently small, $f(z) - f(a) \neq 0$.

Hence for small $\varepsilon > 0$, the circle $\gamma(t) = a + \varepsilon e^{it}$, $t \in [0, 2\pi]$, about a gives

$$\begin{aligned} I(f \circ \gamma; f(a)) &= I(f(\gamma(t)) - f(a); 0) = \sum \text{zeroes in } D(a,\varepsilon) \text{ of } f(z) - f(a) \\ &= \deg_{z=a} f(z). \end{aligned}$$

Consider the local behaviour of $f(z) = z^k$ at $z = 0$ for $k > 0$. We have $\deg_{z=0} f(z) = k$. Note that for all $w \in D(0, \varepsilon)$, w has k preimages under f in $D(0, \varepsilon^{1/k})$. We will generalize this to all functions.

Theorem 1.20 (Local mapping degree). *Let $f : D(a, R) \to \mathbb{C}$ be holomorphic and non-constant, with local degree $k > 0$. Then for $r > 0$ sufficiently small, there exists $\varepsilon > 0$ such that if $|w - f(a)| < \varepsilon$, then $f(z) = w$ has exactly k (simple) roots in $D(a, r)$.*

Proof: Choose $r > 0$ such that $f(z) - f(a)$ and $f'(z)$ have no zeroes in $0 < |z - a| \leq r$. Then r exists by the identity principle. Let γ be the circle of radius r about a.

Then $f \circ \gamma$ doesn't contain $f(a)$, so there exists $\varepsilon > 0$ such that $D(f(a), \varepsilon) \cap (f \circ \gamma) = \emptyset$. For $w \in D(f(a), \varepsilon)$, the number of zeroes of $f(z) - w$ in $D(a, r)$ is $I(f \circ \gamma, w)$. But this is exactly $I(f \circ \gamma; f(a)) = k$.

Now since $f(z) - w$ has nonzero derivative in $D(a, r)^\times$, the zeroes are simple. Hence there are k preimages.

Note $I(f \circ \gamma; w) = I(f \circ \gamma; f(a))$ because the winding number is constant on connected components of $\mathbb{C} \setminus f \circ \gamma$.

Corollary 1.11 (Open mapping theorem). *A constant holomorphic function maps open sets to open sets.*

Proof: We want to show that if $f : D \to \mathbb{C}$, then for all $a \in D$, and for all $r > 0$ sufficiently small, there exists $\varepsilon > 0$ with $f(D(a, r)) \supseteq D(f(a), \varepsilon)$.

By the previous theorem, if r, ε are sufficiently small, then for all $w \in D(f(a), \varepsilon)$, we have the number of zeroes of $f(z) - w$ in $D(a, r)$ equals $\deg_{z=a} f(z) > 0$.

Theorem 1.21 (Rouche's theorem)**.** *Suppose γ bounds a domain D, and f, g are holomorphic on a neighbourhood of $D \cup \gamma$. If $|f(z)| > |g(z)|$ for all $z \in \gamma$, then f and $f + g$ have the same number of zeroes in D.*

Proof: Define $h(z) = \frac{f(z)+g(z)}{f(z)} = 1 + \frac{g(z)}{f(z)}$. Then h is meromorphic on a neighbourhood of $D \cup \gamma$.

Since $|f(z)| > |g(z)|$ for all $z \in \gamma$, $f + g$ and f are non-zero on γ, so h has no zeroes or poles on γ. By the argument principle, we have

$$|\{\text{zeroes of } f + g \text{ on } D\}| - |\{\text{zeroes of } f \text{ on } D\}| = I(h \circ \gamma; 0).$$

By hypothesis, $h \circ \gamma \subset D(1, 1)$. But this shows $I(h \circ \gamma; 0) = 0$, as required.

Now we can use Rouche's.

Example 1.13.

1. Consider $p(z) = z^4 + 6z + 3$. If $|z| \geq 2$, then

   $$\left| z^3 + 6 + \frac{3}{z} \right| \geq |z|^3 - 6 - \frac{3}{|z|} > 0,$$

   so $p(z) \neq 0$.

   We could instead apply Rouche's with $\gamma : |z| = 2$, $f(z) = z^4$ and $g(z) = 6z + 3$. Now

   $$|z|^4 = 16 > 15 = 6|z| + 3 \geq |6z + 3|.$$

   By Rouche's, $p(z)$ has 4 zeroes inside $D(0, 2)$.

   Now for $|z| = 1$, $|6z| = 6$ and $|z^4 + 3| \leq 4$. So using $\gamma : |z| = 1$, $f(z) = 6z$ and $g(z) = z^4 + 3$, we see that $p(z)$ has one zero inside $D(0, 1)$.

   Note that this implies $p(z)$ has a real root, since roots come in complex conjugate pairs for polynomials in $\mathbb{R}$.

2. We can see how Rouche's implies the open mapping theorem. If $f : D \to \mathbb{C}$ is holomorphic and nonconstant, then we can choose $r > 0$ such that $D(a, 2r)^\times$ has no zeroes of $f(z) - f(a)$.

Let γ be $|z-a|=r$, and let $0<\varepsilon<\min_{z\in\gamma}|f(z)-f(a)|$. Then for $w\in D(f(a),\varepsilon)$, since $f(z)-w=f(a)-w+f(z)-f(a)$, and the fact

$$|f(a)-w|<\varepsilon<|f(z)-f(a)|,$$

by Rouche's the zeroes in $D(a,r)$ of $f(z)-w$ equals the number of zeroes in $D(a,r)$ of $f(z)-f(a)$, which is greater than 0. So $f(D(a,r))\supseteq D(f(a),\varepsilon)$.

2 Uniform Limits of Holomorphic Functions

Definition 2.1. Let $U \subset \mathbb{C}$ be open, and $f_N : U \to \mathbb{C}$ a sequence of functions. Then $f_n \to f$ converges *locally uniformly* on U if for all $a \in U$, there exists $D(a, r) \subset U$ on which $f_n \to f$ uniformly.

Example 2.1.

Consider $f_n(z) = z^n$ on $U = D(0, 1)$. As $n \to \infty$, $f_n \to 0$ pointwise. For $a \in D(0, 1)$, we can pick

$$\overline{D\left(a, \frac{1 - |a|}{2}\right)} \subset D(0, 1),$$

and $f_n \to 0$ uniformly on this disc. So $f_n \to 0$ locally uniformly on $D(0, 1)$.

However, for any $\varepsilon > 0$,

$$|f_n(z)| < \varepsilon \iff |z|^n < \varepsilon \iff |z| < \varepsilon^{1/n},$$

so no uniform bound can holds for all $|z| < 1$.

Proposition 2.1. $\{f_n\} : U \to \mathbb{C}$ *is locally uniformly convergent on* U *if and only if* $\{f_n\}$ *converges uniformly on any compact subset of* U.

Recall that a subset in $\mathbb{C}$ is compact if and only if it is closed and bounded. However we will use the fact that every open cover has a finite subcover.

Proof: If $f_n \to f$ locally uniformly on U, and $K \subset U$ is compact, then for all $a \in K$, there exists $r_a > 0$ such that $\{f_n\}$ converges uniformly on $D(a, r_a)$. Now notice

$$\bigcup_{a \in K} D(a, r_a)$$

is an open cover if K, so by compactness there exists $\{a_1, \ldots, a_\ell\}$ such that

$$K \subset D(a_1, r_{a_1}) \cup \cdots \cup D(a_\ell, r_{a_\ell}).$$

Taking the maximum of the constants of uniform convergence on these disks, $f_n \to f$ uniformly on K.

Now if $f_n \to f$ uniformly on compact subsets of U, then for $a \in U$, we can find a closed disk $\overline{D(a, r)} \subset U$. Then $f_n \to f$ converges uniformly on $D(a, r)$.

Theorem 2.1. *Let $\{f_n\}$ be a sequence of analytic functions on U, converging locally uniformly to f. Then f is holomorphic, with $f_n' \to f'$ locally uniformly.*

Proof: Fix $a \in U$ and $\overline{D(a,r)} \subset U$. For r sufficiently small $f_n \to f$ uniformly on $\overline{D(a,r)}$. So

$$|f(z) - f(w)| = |f(z) - f_n(z) + f_n(z) - f_n(w) + f_n(w) - f(w)|,$$

so uniform convergence implies that f is continuous on $\overline{D(a,r)}$. Given γ a closed curve in $D(a,r)$, we have

$$\int_\gamma = \lim_{n\to\infty} \int_\gamma f_n = 0,$$

by Cauchy's theorem. So from Morera's theorem, f is holomorphic on $D(a,r)$. By Cauchy's integral formula, we have

$$|f'(w) - f_n'(w)| = \frac{1}{2\pi} \left| \int_{|z-a|=r} \frac{f(z) - f_n(z)}{(z-w)^2} \,\mathrm{d}z \right|.$$

For $|w-a| \le \frac{r}{2}$, we have

$$|f'(w) - f_n'(w)| \le r \frac{1}{(r/2)^2} \sup_{|z-a|=r} |f(z) - f_n(z)| \to 0,$$

as $n \to \infty$. Hence $f_n \to f$ uniformly on $\overline{D(a,r)}$ implies $|f_n' - f'|$ converges uniformly on $D(a, \frac{r}{2})$.

Remark. The assumption of locally uniform convergence is necessary. A construction with a non-holomorphic limit can be done via Runge's theorem (see Topics in Analysis).

We look at some applications.

2.1 Newton's Method and Complex Dynamics

Recall that Newton's method is an iterative root-finding algorithm, that takes a polynomial $p(z)$ and an initial guess z_0 for a root of $p(z)$. Then we compute a sequence $z_1, z_2, \ldots, z_n = f^n(z_0), \ldots$, where

$$f(z) = z - \frac{p(z)}{p'(z)}.$$

Then in some circumstances this sequence limits to a root of p.

Example 2.2.

Take $p(z) = z^3 - 1$, then

$$f(z) = \frac{2z^3 + 1}{3z^2}.$$

Then some values of z_0 lead to iterates which converge to the poles. However some, such as $z_0 = 0$, do not work.

This is because $f^n(z)$ is a sequence of meromorphic functions, so if $f^n(z_0)$ approaches a limit for some region U of initial guesses, then $f^n|_U$ has a holomorphic limit.

Definition 2.2. A family $\mathcal{F} = \{f_i\}_{i \in I}$ of holomorphic functions on a domain D is *normal* if every sequence in F has a locally uniformly convergent subsequence.

This definition allows convergence to ∞.

A deep theorem which shows normality is as follows:

Theorem 2.2 (Montel's theorem). *If there exist $a, b, c \in \mathbb{C}_\infty$ such that for all $f \in \mathcal{F}$, $f(D) \cap \{a, b, c\} = \emptyset$, then $\mathcal{F}$ is a normal family.*

Definition 2.3. The *Fatou set* of a rational map f is

$$F(f) = \{z \in \mathbb{C}_\infty \mid \exists \text{ nbhd } U \text{ of } z \text{ such that } \{f^n|_U\} \text{ forms a normal family}\}.$$

Then Newton's method works if we are in the Fatou set of the map f. For $p(z) = z^3 - 1$, this works for almost all real numbers, but for, say the polynomial $z^4 - 2z^2 + 9$ without real roots, Newton's method is guaranteed to fail.

2.2 Riemann Mapping Theorem

Theorem 2.3 (Riemann Mapping theorem). *Let $U \subset \mathbb{C}$ be a non-empty, proper, open, simply connected subset of $\mathbb{C}$. Then there exists a conformal isomorphism $f : U \to \mathbb{D} = D(0, 1)$.*

We begin a sketch of the proof.

Proof: Fix $z_0 \in U$, and consider

$$\mathcal{F} = \{f : U \to \mathbb{D}, f \text{ holomorphic, injective, } f(z_0) = 0\}.$$

Then we proceed in three steps:

1. We show $\mathcal{F}$ is non-empty.
2. We show there exists $g \in \mathcal{F}$ such that $|g'(z_0)|$ is finite and maximal among the elements of $\mathcal{F}$.
3. Then we can prove g is a conformal isomorphism.

To construct an element of $\mathcal{F}$, first notice $U \neq \mathbb{C}$ as it is proper, and so there exists $a \in \mathbb{C} \setminus U$. Hence there exists a holomorphic branch of the logarithm $\log(z-a)$ on U. Hence there exists a holomorphic branch of $h(z) = \sqrt{z-a}$ on U.

We can show that h is injective on U, and $h(U) \cap -h(U) = \emptyset$. By the open mapping theorem, $h(U)$ contains some $D(h(z_0), \varepsilon)$, so $|h(z) + h(z_0)| > \varepsilon$ for all $z \in U$. We can then check that

$$f_0(z) = \frac{\varepsilon}{4} \frac{|h'(z_0)|}{|h(z_0)|^2} \frac{h(z_0)}{h'(z_0)} \frac{h(z) - h(z_0)}{h(z) + h(z_0)} \in \mathcal{F}.$$

To show there is a maximal element, let $A = \sup_{f \in \mathcal{F}} |f'(z_0)|$, and choose $\{f_n\}$ such that $f_n'(z_0) \to A$.

By Montel's, $\mathcal{F}$ is a normal family, so there exists f_{n_k} converging locally uniformly to some g, holomorphic. We can then show g is in this family (where we show injectivity using the argument principle).

Now if g is not surjective, we can construct and element of $\mathcal{F}$ violating the maximality of g. If $c \in D(0,1) \setminus g(U)$, then choose a holomorphic branch

$$k(z) = \frac{\sqrt{g(z) - c}}{1 - cg(z)}.$$

Then

$$F(z) = \frac{e^{i\theta}(k(z) - k(z_0))}{1 - k(z_0)k(z)}, \qquad \frac{k'(z_0)}{|k'(z_0)|} = e^{-i\theta}$$

is in $\mathcal{F}$, with $|F'(z_0)| > |g'(z_0)|$.

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