IB Analysis & Topology

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Contents

Part I

Generalizing Continuity and Convergence

1 Three Examples of Convergence

1.1 Convergence in $\mathbb{R}$

Let (x_n) be a sequence in $\mathbb{R}$ and $x \in \mathbb{R}$. We say (x_n) **converges** to x, and write $x_n \to x$, if for all $\varepsilon > 0$, there exists N such that for all $n \geq N$, $|x_n - x| < \varepsilon$.

In $\mathbb{R}$, one useful fact is the **triangle inequality** – $|a + b| \leq |a| + |b|$. We also have two key theorems:

Theorem 1.1 (Bolzano-Weierstrass Theorem)**.**

A bounded sequence in $\mathbb{R}$ must have a convergent subsequence.

Recall that a sequence (x_n) in $\mathbb{R}$ is **Cauchy** if for all $\varepsilon > 0$, there exists N, such that for all $m, n \geq N$, $|x_m - x_n| < \varepsilon$. It is easy to show every convergent sequence is Cauchy. We also have the following:

Theorem 1.2 (General Principle of Convergence)**.**

Any Cauchy sequence in $\mathbb{R}$ converges.

This can be proven by Bolzano-Weierstrass theorem.

1.2 Convergence in $\mathbb{R}^2$

Let (z_n) be a sequence in $\mathbb{R}^2$, and $z \in \mathbb{R}^2$. We wish to define $(z_n) \to z$.

In $\mathbb{R}$, we used the norm $|x|$. In $\mathbb{R}^2$, if we have $z = (x, y)$, then we can say $\|z\| = \sqrt{x^2 + y^2}$. This also satisfies the triangle inequality – $\|a + b\| \leq \|a\| + \|b\|$.

Definition 1.1. Let (z_n) be a sequence in $\mathbb{R}^2$, and $z \in \mathbb{R}^2$. We say that (z_n) **converges** to z, and write $z_n \to z$, if for all $\varepsilon > 0$, there exists N such that for all $n \geq N$, $\|z_n - z\| < \varepsilon$.

Equivalently, $z_n \to z$ if and only if $\|z_n - z\| \to 0$.

Lemma 1.1. *If (z_n), (w_n) are sequences in $\mathbb{R}^2$ with $z_n \to z$, $w_n \to w$. Then $z_n + w_n \to z + w$.*

Proof:

$$\|(z_n + w_n) - (z + w)\| \leq \|z_n - z\| + \|w_n - w\| \to 0 + 0 = 0.$$

In fact, given convergence in $\mathbb{R}$, convergence in $\mathbb{R}^2$ is easy.

Proposition 1.1. *Let (z_n) be a sequence in $\mathbb{R}^2$ and let $z \in \mathbb{R}^2$. Write $z_n = (x_n, y_n)$ and $z = (x, y)$. Then $z_n \to z$ if and only if $x_n \to x$ and $y_n \to y$.*

Proof:

First, note $|x_n - x|, |y_n - y| \leq \|z_n - z\|$, so $\|z_n - z\| \to 0$ implies $|x_n - x|, |y_n - y| \to 0$.

Now, if $|x_n - x|, |y_n - y| \to 0$, then $\|z_n - z\| = \sqrt{|x_n - x|^2 + |y_n - y|^2} \to 0$.

Definition 1.2. A sequence (z_n) in $\mathbb{R}^2$ is **bounded** if there exists $M \in \mathbb{R}$ such that for all n, $\|z_n\| \leq M$.

Theorem 1.3 (Bolzano-Weierstrass in $\mathbb{R}^2$)**.**

A bounded sequence in $\mathbb{R}^2$ must have a convergent subsequence.

Proof: Let (z_n) be a bounded subsequence in $\mathbb{R}^2$. Write $z_n = (x_n, y_n)$. Now $|x_n|, |y_n| \leq \|z_n\|$, so x_n, y_n are bounded in $\mathbb{R}$.

By Bolzano-Weierstrass, x_n has a convergent subsequence, say $x_{n_j} \to x \in \mathbb{R}$. Similarly (y_{n_j}) is bounded, so it has a convergent subsequence $y_{n_{j_k}} \to y$. Since we know $x_{n_{j_k}} \to x$, $z_{n_{j_k}} \to z = (x, y)$.

Definition 1.3. A sequence $(z_n) \in \mathbb{R}^2$ is **Cauchy** if for all $\varepsilon > 0$, there exists N such that for all $m, n \geq N$, $\|z_m - z_n\| < \varepsilon$.

It is easy to show a convergent sequence in $\mathbb{R}^2$ is Cauchy.

Theorem 1.4 (General Principle of Convergence for $\mathbb{R}^2$)**.**

Any Cauchy sequence in $\mathbb{R}^2$ converges.

Proof: Let (z_n) be a Cauchy sequence in $\mathbb{R}^2$. Write $z_n = (x_n, y_n)$. For all m, n, $|x_m - x_n| \leq \|z_m - z_n\|$, so (x_n) is a Cauchy sequence in $\mathbb{R}$, thus it converges in $\mathbb{R}$. Similarly, (y_n) converges in $\mathbb{R}$, so (z_n) converges.

1.3 Convergence of Functions

Let $X \subset \mathbb{R}$. Let $f_n : X \to \mathbb{R}$, and let $f : X \to \mathbb{R}$. What does it mean for (f_n) to converge to f?

Definition 1.4. Say (f_n) **converges pointwise** to f, and we write $f_n \to f$ pointwise, if for all $x \in X$, $f_n(x) \to f(x)$ as $n \to \infty$.

Although this is simple and easy to check, it doesn't preserve some 'nice' properties that we want.

Example 1.1. In all three examples, $X = [0, 1]$, and $f_n \to f$ pointwise.

1. We will construct f_n continuous, but f not. Take

$$f_n(x) = \begin{cases} nx & x \leq \frac{1}{n}, \\ 1 & x \geq \frac{1}{n}. \end{cases}, f = \begin{cases} 0 & x = 0, \\ 1 & x > 0. \end{cases}$$

   Then $(f_n) \to f$ pointwise, but f is not continuous.

2. We will construct f_n Riemann integrable, but f not. Take the function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q}, \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

   Enumerate the rationals in $[0, 1]$ as $q_1, q_2, \ldots$. For $n \geq 1$, set

$$f_n(x) = \begin{cases} 1 & x = q_1, \ldots, q_n, \\ 0 & \text{otherwise.} \end{cases}$$

3. We will construct f_n Riemann integrable, f Riemann integrable, but the integrals do not converge. Take $f(x) = 0$ for all x. We construct f_n with integral 1, such as

$$f_n(x) = \begin{cases} n & 0 < x < \frac{1}{n}, \\ 0 & \text{otherwise.} \end{cases}$$

We consider another definition of convergence.

Definition 1.5 (Uniform Convergence)**.** Let $X \subset \mathbb{R}$, $f_n : X \to \mathbb{R}$, $f : X \to \mathbb{R}$. We say (f_n) **converges uniformly** to f, and write $f_n \to f$ uniformly, if for all $\varepsilon > 0$, there exists N, such that for all $x \in X$ and all $n \geq N$, $|f_n(x) - f(x)| < \varepsilon$.

In particular, $f_n \to f$ uniformly implies $f_n \to f$ pointwise.

Equivalently, $f_n \to f$ uniformly if for sufficiently large n, $f_n - f$ is bounded, and

$$\sup_{x \in X} |f_n(x) - f(x)| \to 0.$$

Theorem 1.5. *Let $X \subset \mathbb{R}$, $f_n : X \to \mathbb{R}$ be continuous, and let $f_n \to f : X \to \mathbb{R}$ uniformly. Then f is continuous.*

Proof: Let $x \in X$, and pick $\varepsilon > 0$. As $f_n \to f$ uniformly, we can find N such that for all $n \geq N$ and $\in X$,

$$|f_n(y) - f(y)| < \varepsilon.$$

In particular, we may take $n = N$. As f_N is continuous, we can find $\delta > 0$ such that for all $y \in X$,

$$|y - x| < \delta \implies |f_N(y) - f_N(x)| < \varepsilon.$$

Now let $y \in X$ with $|y - x| < \delta$. Then

$$|f(y) - f(x)| \leq |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.$$

But 3ε can be made arbitrarily small, so f is continuous.

Remark. This is often called a '3ε proof' (or a '$\varepsilon/3$ proof').

Theorem 1.6. *Let $f_n : [a, b] \to \mathbb{R}$ be integrable and let $f_n \to f : [a, b] \to \mathbb{R}$ uniformly. Then f is integrable and*

$$\int_a^b f_n \to \int_a^b f$$

as $n \to \infty$.

Proof: As $f_n \to f$ uniformly, we can pick n sufficiently large such that $f_n - f$ is bounded. Also f_n is bounded, so by the triangle inequality $f = (f - f_n) + f_n$ is bounded.

Let $\varepsilon > 0$. As $f_n \to f$ uniformly, there is some N such that for all $n \geq N$ and $x \in [a, b]$, we have $|f_n(x) - f(x)| < \varepsilon$. By Riemann's criterion, there is some dissection $\mathcal{D}$ of $[a, b]$ for which

$$S(f_N, \mathcal{D}) - s(f_N, \mathcal{D}) < \varepsilon.$$

Let $\mathcal{D} = \{x_0, x_1, \ldots, x_k\}$, where $a = x_0 < x_1 < \cdots < x_k = b$. Now,

$$\begin{aligned}
S(f, \mathcal{D}) &= \sum_{i=1}^{k} (x_i - x_{i-1}) \sup_{x \in [x_{i-1}, x_i]} f(x) \\
&\leq \sum_{i=1}^{k} (x_i - x_{i-1}) \sup_{x \in [x_{i-1}, x_i]} (f_N(x) + \varepsilon) \\
&= \sum_{i=1}^{k} (x_i - x_{i-1}) \sup_{x \in [x_{i-1}, x_i]} f_N(x) + \sum_{i=1}^{k} (x_i - x_{i-1}) \varepsilon \\
&= S(f_N, \mathcal{D}) + (b - a)\varepsilon.
\end{aligned}$$

Similarly, $s(f, \mathcal{D}) \geq s(f_N, \mathcal{D} - (b - a)\varepsilon$, so

$$S(f, \mathcal{D}) - s(f, \mathcal{D}) \leq S(f_N, \mathcal{D}) - s(f_N, \mathcal{D}) + 2(b - a)\varepsilon < (2(b - a) + 1)\varepsilon..$$

But this can be made arbitrarily small, so by Riemann's criterion, f is integrable over $[a, b]$.

Now for any n sufficiently large such that $f_n - f$ is bounded,

$$\begin{aligned}
\left| \int_a^b f_n - \int_a^b f \right| &= \left| \int_a^b (f_n - f) \right| \\
&\leq \int_a^b |f_n - f| \\
&\leq (b - a) \sup_{x \in [a, b]} |f_n(x) - f(x)| \to 0
\end{aligned}$$

as $n \to \infty$ since $f_n \to f$ uniformly.

Unfortunately, uniform convergence cannot preserve all properties.

Example 1.2. Take $f_n : [-1, 1] \to \mathbb{R}$, where each f_n is differentiable and $f_n \to f$ uniformly, but f is not differentiable. Take

$$f_n = \sqrt{\left(\frac{1}{n} + x^2\right)}.$$

Then f_n is differentiable, and also uniformly converges to $f(x) = |x|$. But f is not differentiable.

In fact, we need uniform convergence of the **derivatives**.

Theorem 1.7. *Let $f_n : (u, v) \mapsto \mathbb{R}$ with $f_n \to f : (u, v) \to \mathbb{R}$ pointwise. Suppose further that each f_n is continuously differentiable and that $f_n' \to g : (u, v) \to \mathbb{R}$ uniformly. Then f is differentiable with $f' = g$.*

Proof: Fix $a \in (u, v)$. Let $x \in (u, v)$. By FTC, we have each f_n' is integrable over $[a, x]$ and

$$\int_a^x f_n' = f_n(x) - f_n(a).$$

But $f_n' \to g$ uniformly, so by theorem 5, g is integrable over $[a, x]$ and

$$\int_a^x g = \lim_{n\to\infty} \int_a^x f_n'(x) = f(x) - f(a).$$

So we have shown that for all $x \in (u, v)$,

$$f(x) = f(a) + \int_a^x g.$$

By theorem 4, g is continuous so by FTC, f is differentiable with $f' = g$.

Remark. It would have sufficed to assume that $f_n(x) \to f(x)$ for a single value of x.

Definition 1.6. Let $X \subset \mathbb{R}$ and let $f_n : X \to \mathbb{R}$ for each $n \geq 1$. We say (f_n) is **uniformly Cauchy** if for all $\varepsilon > 0$, there exists N such that for all $m, n \geq N$ and for all $x \in X$,

$$|f_m(x) - f_n(x)| < \varepsilon.$$

It is easy to show that a uniformly convergent sequence is uniformly Cauchy.

Theorem 1.8 (General Principle of Uniform Convergence)**.** *Let (f_n) be a uniformly Cauchy sequence of functions $X \to \mathbb{R}$. Then (f_n) is uniformly convergent.*

Proof: Let $x \in X$, and $\varepsilon > 0$. Then there exists N, such that for all $m, n \geq N$ and for all $y \in X$, $|f_m(y) - f_n(y)| < \varepsilon$. In particular, $|f_m(x) - f_n(x)| < \varepsilon$, so $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$, so by GPC, it converges pointwise, say $f_n(x) \to f(x)$ as $n \to \infty$.

Let $\varepsilon > 0$. Then we can find an N such that for all $m, n \geq N$ and all $y \in X$, $|f_m(y) - f_n(y)| < \varepsilon$. Fixing y, m and letting $n \to \infty$, $|f_m(y) - f(y)| \leq \varepsilon$. But since y is arbitrary, this shows $f_n \to f$ uniformly.

We will also try to take Bolzano-Weierstrass over to the space of functions.

Definition 1.7. Let $X \subset \mathbb{R}$ and let $f_n : X \to \mathbb{R}$ for each $n \geq 1$. We say (f_n) is **pointwise bounded** if for all x, there exists M such that for all n, $|f_n(x)| \leq M$.

We say (f_n) is **uniformly bounded** if there exists M, such that for all x and n, $|f_n(x)| \leq M$.

We would like a uniform Bolzano-Weierstrass, saying if (f_n) is a uniformly bounded sequence of functions, then it has a uniformly convergent subsequence. But this is not true.

Example 1.3. Take $f_n : \mathbb{R} \to \mathbb{R}$,

$$f_n(x) = \begin{cases} 1 & x = n, \\ 0 & x \neq n \end{cases}.$$

Then (f_n) is uniformly bounded, but if $m \neq n$, then $f_m(m) = 1$ and $f_n(m) = 0$, so $|f_m(m) - f_n(m)| = 1$, hence (f_n) are not uniformly Cauchy, so cannot be uniformly convergent.

1.4 Application to Power Series

Recall that if $\sum a_n x^n$ is a real power of series with radius of convergence $R > 0$, then we can differentiate and integrate it term-by-term within $(-R, R)$.

Definition 1.8. Let $f_n : X \to \mathbb{R}$ for each $n \geq 0$. We say that the series

$$\sum_{n=0}^{\infty} f_n$$

converges uniformly if the sequence of partial sums (F_n) does, where $F_n = f_0 + f_1 + \cdots + f_n$.

If we can prove that $\sum a_n x^n$ is uniformly convergent, then we can apply earlier theorems to show differentiability. However this is not quite true, for example take

$$\sum_{n=0}^{\infty} x^n.$$

However, we do have another approach. We can show that if $0 < r < R$, then we do have uniform convergence on $(-r, r)$, and then given $x \in (-R, R)$, we can choose $|x| < r < R$ and use the above to show all the properties we want. This is known as the **local uniform convergence of power series**.

Lemma 1.2. *Let $\sum a_n x^n$ be a real power series with radius of convergence $R > 0$. Let $0 < r < R$. Then $\sum a_n x^n$ converges uniformly on $(-r, r)$.*

Proof: Define $f, f_m : (-r, r) \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad f_m(x) = \sum_{n=0}^{m} a_n x^n.$$

Recall that $\sum a_n x^n$ converges absolutely for all x with $|x| < R$. Let $x \in (-r, r)$. Then

$$\begin{aligned} |f(x) - f_m(x)| &= \left| \sum_{n=m+1}^{\infty} a_n x^n \right| \\ &\le \sum_{n=m+1}^{\infty} |a_n||x|^n \le \sum_{n=m+1}^{\infty} |a_n| r^n, \end{aligned}$$

which converges by absolute convergence at r. hence if m is sufficiently large, $f - f_m$ is bounded and

$$\sup_{x \in (-r,r)} |f(x) - f_m(x)| \le \sum_{n=m+1}^{\infty} |a_n| r^n \to 0$$

as $m \to \infty$.

Theorem 1.9. *Let $\sum a_n x^n$ be a real power series with radius of convergence $R > 0$.*

Define $f : (-R, R) \to \mathbb{R}$ *by*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then,

(i) f *is continuous;*

(ii) *For any* $x \in (-R, R)$, f *is integrable over* $[0, x]$ *with*

$$\int_0^x = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

Proof: Let $x \in (-R, R)$. Pick r such that $|x| < r < R$. By the above lemma, $\sum a_n y^n$ converges uniformly on $(-r, r)$. But the partial sum functions are all continuous on $(-r, r)$, hence $f|_{(-r,r)}$ is continuous. Thus f is a continuous function on $(-R, R)$.

Moreover, $[0, x] \subset (-r, r)$ so we also have $\sum a_n y^n$ converges uniformly on $[0, x]$. Each partial sum on $[0, x]$ is a polynomial, so can be integrated with

$$\int_0^x \sum_{n=0}^{m} a_n y^n \,\mathrm{d}y = \sum_{n=0}^{m} \int_0^x a_n y^n \,\mathrm{d}y = \sum_{n=0}^{m} \frac{a_n}{n+1} x^{n+1}.$$

Thus, f is integrable over $[0, x]$ with

$$\int_0^x f = \lim_{m\to\infty} \int_0^x \sum_{n=0}^{m} a_n y^n \,\mathrm{d}y = \lim_{m\to\infty} \sum_{n=0}^{m} \frac{a_n}{n+1} x^{n+1} = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

For differentiation, we need the following lemma:

Lemma 1.3. *Let* $\sum a_n x^n$ *be a real power series with radius of convergence* $R > 0$. *Then the power series* $\sum n a_n x^{n-1}$ *has radius of convergence at least* R.

Proof: Let $x \in \mathbb{R}$, $0 < |x| < R$. Pick w with $|x| < w < R$. Then $\sum a_n w^n$ is absolutely convergent, so $a_n w^n \to 0$. Therefore, there exists M such that $|a_n w^n| \leq M$ for all n. For each n,

$$|n a_n x^{n-1}| = |a_n w^n| \left|\frac{x}{w}\right|^n \frac{1}{|x|} n.$$

Fix n, let $\alpha = |x/w| < 1$, and let $c = M/|x|$, a constant. Then $|na_n x^{n-1}| \leq cn\alpha^n$. By comparison test, it suffices to show $\sum n\alpha^n$ converges. Note

$$\left|\frac{(n+1)\alpha^{n+1}}{n\alpha^n}\right| = \left(1+\frac{1}{n}\right)\alpha \to \alpha < 1$$

as $n \to \infty$, so this converges by the ratio test.

Theorem 1.10. *Let $\sum a_n x^n$ be a real power series with radius of convergence $R > 0$. Let $f : (-R, R) \to \mathbb{R}$ be defined by*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then f is differentiable and for all $x \in (-R, R)$,

$$f'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}.$$

Proof: Let $x \in (-R, R)$. Pick r with $|x| < r < R$. Then $\sum a_n y^n$ converges uniformly on $(-r, r)$. Moreover, the power series $\sum na_n y^{n-1}$ had radius of convergence at least R, and so also converges uniformly on $(-r, r)$.

The partial sum functions $f_m(y)$ are polynomials, so are differentiable with

$$f_m'(y) = \sum_{n=1}^{m} na_n y^{n-1}.$$

we now have f_m' converging uniformly on $(-r, r)$ to the function

$$g(y) = \sum_{n=1}^{\infty} na_n y^{n-1}.$$

Hence, $f|_{(-r,r)}$ is differentiable and for all $y \in (-r, r)$, $f'(y) = g(y)$. In particular, f is differentiable at x with $f'(x) = g(x)$. This gives f is a differentiable function on $(-R, R)$ with derivative g as desired.

1.5 Uniform Continuity

Let $X \subset \mathbb{R}$. Let $f : X \mapsto \mathbb{R}$. Recall that f is **continuous** if for all $\varepsilon > 0$ and for all $x \in X$, there exists $\delta > 0$, such that for all $y \in X$ with $|x - y| < \delta$, we have

$|f(x) - f(y)| < \varepsilon$.

Definition 1.9. We say f is **uniformly continuous** if for all $\varepsilon > 0$, there exists $\delta > 0$, such that for all $x, y \in X$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$.

Remark. Clearly if f is uniformly continuous, then f is continuous. The converse is not true.

Example 1.4. Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Then f is continuous as it is a polynomial. Suppose $\delta > 0$. Then,

$$f(x + \delta) - f(x) = (x + \delta)^2 - x^2 = 2\delta x + \delta^2 \to \infty$$

as $x \to \infty$. So the condition fails for $\varepsilon = 1$.

Even on the bounded interval $(0, 1)$, take $f(x) = 1/x$. This is clearly continuous, but cannot be uniformly continuous as it approaches infinity as x approaches 0.

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