IB Geometry

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Contents

1 Surfaces

Definition 1.1. A *topological surface* is a topological space Σ such that

(a) for all $p \in \Sigma$, there is an open neighbourhood $p \in U \subset \Sigma$ such that U is homeomorphic to $\mathbb{R}^2$, or a disc $D^2 \subset \mathbb{R}^2$, with its usual Euclidean topology.

(b) Σ is Hausdorff and second countable.

Remark. $\mathbb{R}^2 \simeq D(0,1) = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$.

1. A space X is *Hausdorff* if for $p \neq q$ in X, there exist disjoint open sets U, V with $p \in U, q \in V$.

   A space is *second countable* if it has a countable base, i.e there exist open sets $\{U_i\}_{i\in\mathbb{N}}$, such that every open set is a union of some of the U_i.

   The key point of defining surfaces is point (a), point (b) is for ruling out surfaces that are too weird.

2. If X is Hausdorff or second countable, then so are subspaces of X. Moreover Euclidean space has these properties (to show it is second countable, consider open balls $B(c, r)$ with $c \in \mathbb{Q}^n \subset \mathbb{R}^n$, and $r \in \mathbb{Q}_+ \subset \mathbb{R}_+$).

Example 1.1.

(i) The plane $\mathbb{R}^2$.

(ii) Any open set in $\mathbb{R}^2$ is a surface, i.e. $R^2 \setminus Z$ where Z is closed is a surface.

(iii) Graphs of functions. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous. Then the graph of f is

$$\Gamma_f = \{(x, y, f(x, y)) \mid (x, y) \in \mathbb{R}^2\}.$$

This is a subspace of $\mathbb{R}^3$, so we can endow it with the subspace topology. We claim it is a subspace homeomorphic to $\mathbb{R}^2$.

Recall that if X, Y are topological spaces, then the product topology $X \times Y$ has a basis of open sets $U \times V$, where $U \subset X$, $V \subset Y$ are open

A feature is that if $g : Z \to X \times Y$ is continuous if and only if $\Pi_x \circ g : Z \to X$ and $\Pi_y \circ g : Z \to Y$ are continuous, where Π_x, Π_y are the canonical projectors.

We can now show that if $f : X \to Y$ is continuous, then $\Gamma_f \subset X \times Y$ is homeomorphic to X, as $s(x) = (x, f(x))$ is a continuous function from X to Γ_f, $\Pi_x|_{\Gamma_f}$ and s are inverse homeomorphisms.

In particular, for our example $\Gamma_f \simeq \mathbb{R}^2$. So any $f : \mathbb{R}^2 \to \mathbb{R}$ continuous produces a surface Γ_f.

(iv) The sphere: $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ (with the subspace topology). To show this is a surface, we can consider the stereographic projection $\Pi_+ : S^2 \setminus \{(0, 0, 1)\} \to \mathbb{R}^2$:

$$(x, y, z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

Then Π_+ is continuous and has an inverse

$$(u, v) \mapsto \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right).$$

So Π_+ is a continuous bijection with continuous inverse, and hence a homeomorphism.

Similarly, taking a stereographic projection from the south pole $\Pi_- : S^2 \setminus \{(0, 0, -1)\} \to \mathbb{R}^2$, by

$$(x, y, z) \mapsto \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$$

is another homeomorphism. Hence S^2 is a topological surface, as the open sets $S^2 \setminus \{(0, 0, 1)\}$ and $S^2 \setminus \{(0, 0, -1)\}$ cover S^2, and it is Hausdorff and second countable as it is a subspace of $\mathbb{R}^3$.

(v) The *real projective plane*. The group $\mathbb{Z}_2$ acts on S^2 by homeomorphisms, via the antipodal map

$$\begin{gathered} a : S^2 \to S^2 \\ a(x, y, z) \mapsto (-x, -y, -z) \end{gathered}$$

Definition 1.2. The real projective plane is the quotient of S^2 by identifying every point with its antipodal image:

$$\mathbb{RP}^2 = S^2/\mathbb{Z}_2 = S^2/\sim .$$

Lemma 1.1. *As a set, $\mathbb{RP}^2$ is naturally in bijection with the set of straight lines through* 0.

This is because any straight line through $0 \in \mathbb{R}^3$ intersects S^2 in exactly a pair of antipodal points, and each such pair determines a straight line.

Lemma 1.2. $\mathbb{RP}^2$ *is a topological surface with the quotient topology.*

Recall the quotient topology: given the quotient map $q : X \to Y$, we say $V \subset Y$ is open if and only if $q^{-1}(V) \subset X$ is open in X.

Proof: First we show that $\mathbb{RP}^2$ is Hausdorff. If $[p] \neq [q] \in \mathbb{RP}^2$, then $\pm p$, $\pm q$ are distinct, antipodal pairs.

We take open discs centred on p and q and their antipodal images, such that no two discs intersect. The images of these discs give open images of $[p]$ and $[q]$ in $\mathbb{RP}^2$. Indeed, $q(B_\delta(p))$ is open since $q^{-1}(q(B_\delta(p))) = B_\delta(p) \cup (-B_\delta(p))$.

Now we show $\mathbb{RP}^2$ is second countable. Let U be a countable base of S^2, and let $\overline{U} = \{q(u) \mid u \in U\}$. Then $q(u)$ is open, as $q(u) = u \cup (-u)$, and $\overline{U}$ is clearly countable as U is.

Take $V \subset \mathbb{RP}^2$ open. By definition, $q^{-1}(V)$ is open, so let $q^{-1}(V) = \bigcup U_\alpha$, for $U_\alpha \in U$. Then

$$V = q(q^{-1}(V)) = q\left(\bigcup_\alpha U_\alpha\right) = \bigcup_\alpha q(U_\alpha).$$

Finally, let $p \in S^2$ and $[p] \in \mathbb{RP}^2$ be its image. Let $\overline{D}$ be a small closed disc neighbourhood of $p \in S^2$, so that $q|_{\overline{D}}$ is injective and continuous, and has image a Hausdorff space.

Now recall that a countinuous bijection from a compact space to a Hausdorff space is a homeomorphism.

So $q|_{\overline{D}} : \overline{D} \to q(\overline{D})$ is a homeomorphism. This induces a homeomorphism

$$q|_D : D \to q(D) \subset \mathbb{RP}^2,$$

where D is an open disc contained in $\overline{D}$. So $[p] \in q(D)$ has an open neighbourhood in $\mathbb{RP}^2$ homeomorphic to an open disc.

Example 1.2.

We continue looking at examples of surfaces.

(vi) Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Then the *torus* is $S^1 \times S^1$ with the subspace topology of $\mathbb{C}^2$ (this is the same as taking the product topology).

Lemma 1.3. *The torus is a topological surface.*

Proof: We consider the map

$$\begin{aligned}\mathbb{R}^2 &\xrightarrow{e} S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C} \\ (s,t) &\mapsto (e^{2\pi i s}, e^{2\pi i t}).\end{aligned}$$

We can view this map using the following diagram:

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "e"] \arrow[d, "q"] & S^1 \times S^1 \\ \mathbb{R}^2/\mathbb{Z}^2 \arrow[ur, dashed, "\hat{e}"] & \end{tikzcd}$$

There is an equivalence relation on $\mathbb{R}^2$ given by translating by $\mathbb{Z}^2$. Now consider the map

$$[0,1]^2 \hookrightarrow \mathbb{R}^2 \xrightarrow{q} \mathbb{R}^2/\mathbb{Z}^2$$

is onto, so $\mathbb{R}^2/\mathbb{Z}^2$ is compact. Now note that $\hat{e}$ is a continuous bijection, so since it is onto a Hausdorff space, it is a homeomorphism.

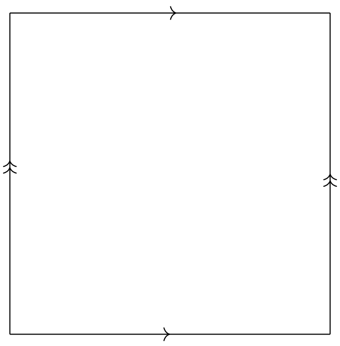
Similar to $\mathbb{RP}^2$, for $[p] \in q(p)$, take a small closed disc $\overline{D} \subset \mathbb{R}^2$ such that, for all $(m,n) \in \mathbb{Z}^2$, $\overline{D} \cap (\overline{D} + (m,n)) = \emptyset$.

Then $e|_{\overline{D}}$ and $q|_{\overline{D}}$ are injective. Now restricting to an open disc as before, we get an open disc as a neighbourhood of $[p]$, so $S^1 \times S^1$ is a topological surface.

Another viewpoint for a torus is by imposing on $[0,1]^2$ the equivalence relations

$$(x,0) \sim (x,1), \qquad (0,y) \sim (1,y).$$

Figure 1: Identification of a Torus



Example 1.3.

We look at yet another example of a surface.

(vii) Let P be a planar Euclidean polyon. Assume that the edges are oriented and paired, and for simplicity assume the Euclidean lengths of e and $\hat{e}$ are equal if $\{e, \hat{e}\}$ are paired.

Label by letters, and describe the orientation by a sign of $\pm$ relative to the clockwise orientation in $\mathbb{R}^2$.

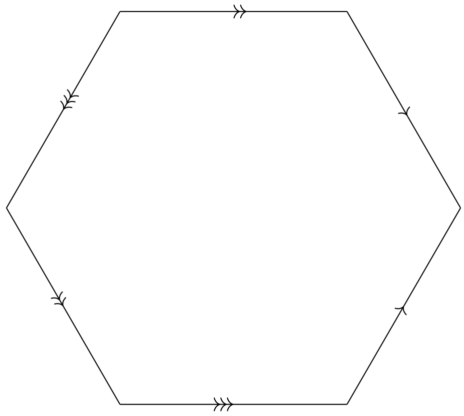
More precisely, if $\{e, \hat{e}\}$ are paired edges, there is a unique isometry from e to $\hat{e}$ respecting their orientations, say

$$f_{e\hat{e}} : e \to \hat{e}.$$

These maps generate an equivalence relation on P, where we identify $x \in \partial P$ with $f_{e\hat{e}}(x)$ whenever $x \in e$.

Lemma 1.4. *$P/\sim$ (with the quotient topology) is a topological surface.*

Figure 2: Orientation of Edges of a Hexagon



Proof: We begin by looking at a special case of the torus T^2 as $[0,1]^2/\sim$. Then if p is an interior point, we pick $\delta > 0$ small such that $\overline{B_\delta(p)}$ lies in the interior of the polygon P. Now we argue as before: the quotient map is injective on $\overline{B_\delta(p)}$ and is a homeomorphism on its interior.

Now suppose p is on an edge of P, but not a vertex. The idea is to take the two points in $q^{-1}(p)$, take half discs around them, and join them up to form a disc.

Say $p = (0, y_0) \sim (1, y_0) = p'$. Take δ small enough so the half discs of radius δ do not meet the vertices and don't intersect. Let U be the half disc around p and V the half disc around p'.

Define a map as follows:

$$\begin{aligned} U &: (x, y) \xrightarrow{f_u} (x, y - y_0), \\ V &: (x, y) \xrightarrow{f_v} (x - 1, y - y_0). \end{aligned}$$

We want to show these maps glue well together. To do this, we use the following fact:

If $X = A \cup B$, A and B are closed, and $f : A \to Y$ and $g : B \to Y$ are continuous and $f|_{A\cap B} = g|_{A\cap B}$, then they define a continuous map on X.

Now f_u and f_v are continuous on $U, V \subset [0, 1]^2$, so they induce continuous maps on $q(U)$ and $q(V)$.

In T^2, the intersection of the discs overlap on the paired edges, but our maps agree, so they are compatible with the equivalence relation. Hence f_u and f_v give a continuous map on an open image of $[p] \in T^2$ to $\mathbb{R}^2$. By the usual argument, we can show if $[p] \in T^2$ lies on an edge of P it has a neighbourhood homeomorphic to a disc.

Finally, we look at a vertex of $[0, 1]^2$. In the image, there is really only one vertex. To find a homeomorphism to the open disc, we can take four quarter circles at each corner, and glue them appropriately.

For a general polygon, it is a similar idea. Interior and edge points are done analogously to T^2. For vertices, it is a bit different. We have different equivalence classes of vertices caused by orienting the edges in different ways.

If v is a vertex of P with k vertices in its equivalence class, then we have k sectors in P. Any sector can be identified with out favourite sector in $\mathbb{R}^2$, i.e. $(r, \theta) \in \mathbb{R}^2$ with $0 \leq r < \delta$ and $\theta \in [0, 2\pi/k]$. Gluing these together, we get an open disc as a neighbourhood of v.

This works unless $k = 1$, in which case we have two paired edges coming into or out of a vertex in P. But this is homeomorphic to a cone, which is homeomorphic to a disc.

These neighbourhoods of points in $P/\sim$ show that P is locally homeomorphic to a disc, and we can easily check that $P/\sim$ is Hausdorff and second countable.

Example 1.4.

One more example now.

(viii) We now consider connecting surfaces. Given topological surfaces Σ_1 and Σ_2, we can remove an open disc from each, and glue the resulting boundary circles.

Explicitly, we take $\Sigma_1 \setminus D_1 \cup \Sigma_2 \setminus D_2$ as a disjoint union, and impose the quotient relation

$$\theta \in \partial D_1 \sim \theta \in \partial D_2,$$

where θ parametrizes $S^1 = \partial D_i$.

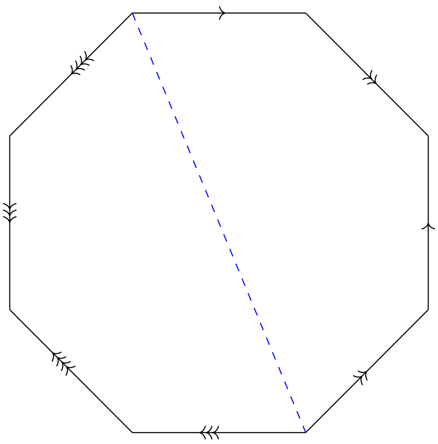
The result $\Sigma_1 \sharp \Sigma_2$ is called the *connected sum* of Σ_1 and Σ_2.

In principle, this depends on the choices of discs, and it takes some effort to prove that it is well-defined.

Lemma 1.5. *The connected sum $\Sigma_1 \sharp \Sigma_2$ is a topological surface.*

We will not prove this lemma in this course.

Figure 3: Octagon



As another example the octagon is homeomorphic to a double torus: cutting along the blue line reveals two copies of a torus, which are joined together.

Similarly, we can find $\mathbb{RP}^2$ as the quotient of a square: this can be seem by morphing it into a circle with antipodes identified, which is then homeomorphic to $\mathbb{RP}^2$, seen by 'squishing down' $\mathbb{RP}^2$ or projecting it onto a plane.

Figure 4: Identification of $\mathbb{RP}^2$

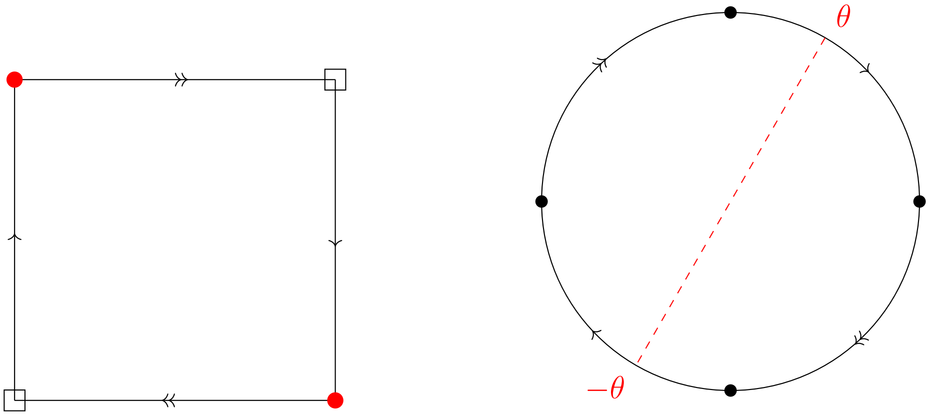
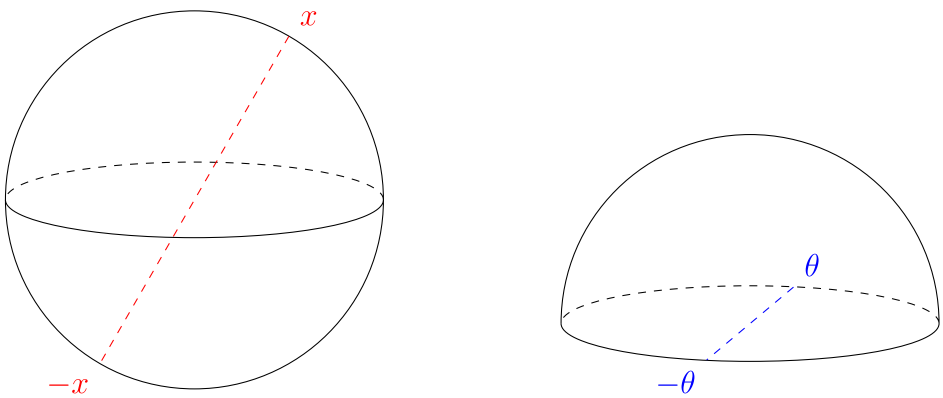


Figure 5: Squishing down $\mathbb{RP}^2$



1.1 Triangulation and Euler Characteristic

Definition 1.3. A *subdivision* of a compact topological surface Σ comprises of:

(i) a finite set V of *vertices*,

(ii) a finite collection of edges $E + \{e_i : [0,1] \to \Sigma\}$ such that

- for all i, e_i is a continuous injection on its interior and $e_i^{-1}(V) = \{0,1\}$,
- e_i and e_j have disjoint images except perhaps at their endpoints in V.

(iii) We require that each connected component of

$$\Sigma \setminus \left(\bigcup_i e_i([0,1]) \cup V \right)$$

is homeomorphic to an open disc, called a *face*.

Hence the closure of a face $\overline{F} \setminus F$ has boundary lying in

$$\bigcup_i e_i([0,1]) \cup V.$$

A subdivision is a *triangulation* if every closed face (closure of a face) contains exactly three edges, and two closed faces are disjoint, meet in exactly one edge or just one vertex.

Example 1.5.

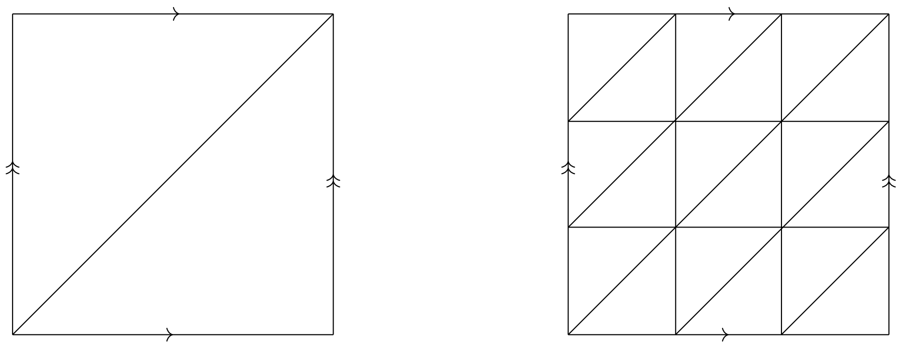
A cube displays a subdivision of S^2, and a tetrahedron displays a triangulation of S^2.

Moreover figure 1 displays a subdivision of T^2, with one vertex, two edges and one face.

In figure 6, only the right triangulation is a valid triangulation: in the left figure, the two triangles share more than one edge.

As well, figure 7 is a degenerate subdivision of the sphere, with one vertex, no edges and one face.

Figure 6: Triangulations of the Torus



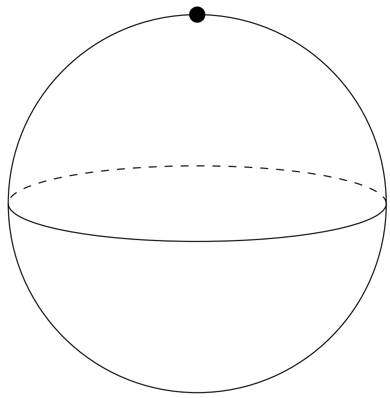
(a) Invalid (b) Valid

Definition 1.4. The *Euler characteristic* of a subdivision is

$$|V| - |E| + |F|.$$

Theorem 1.1.

Figure 7: Subdivision of S^2



(i) *Every compact topological surface admits subdivisions and triangulations.*

(ii) *The Euler characteristic, denoted $\chi(\Sigma)$, does not depend on the subdivision and defined a topological invariant of the surface.*

Remark. This is hard to prove, particularly (ii). There are cleaner approaches to this (seen in algebraic topology).

Example 1.6.

1. $\chi(S^2) = 2$.
2. $\chi(T^2) = 0$.
3. Let Σ_1, Σ_2 be compact topological spaces, and we form $\Sigma_1 \sharp \Sigma_2$. We remove open discs $D_i \subset \Sigma_i$ which is a face of a triangulation in each surface. Hence,

$$\chi(\Sigma_1 \sharp \Sigma_2) = \chi(\Sigma_1) + \chi(\Sigma_2) - 2.$$

In particular if Σ_g is a surface with g holes, i.e.

$$\Sigma_g = \sharp_{i=1}^{g} T^2,$$

then $\chi(\Sigma_g) = 2 - 2g$. g is called the *genus*.

2 Abstract Smooth Surfaces

Definition 2.1. A pair (U, φ) where $U \subset \Sigma$ is open and $\varphi : U \to V \subset \mathbb{R}^2$ is called a *chart*.

The inverse $\sigma = \varphi^{-1} : V \to U \subset \Sigma$ is called a *local parametrization* of Σ.

Definition 2.2. A collection of charts

$$\{(U_i, \varphi_i)_{i \in I}\}$$

such that

$$\bigcup_{i \in I} U_i = \Sigma$$

is called an *atlas* of Σ.

Example 2.1.

1. If $Z \subset \mathbb{R}^2$ is closed, then $\mathbb{R}^2 \setminus Z$ is a topological surface with an atlas with one chart: $(\mathbb{R}^2 \setminus Z, \varphi = \mathrm{id})$.
2. For S^2 we have an atlas with 2 charts: the two stereographic projections.

Definition 2.3. Let (U_i, φ_i) for $i = 1, 2$ be two charts containing $p \in \Sigma$. The map

$$\varphi_2 \circ \varphi_1^{-1}|_{\varphi_1(U_1 \cap U_2)}$$

is called the *transition map* between charts.

Note that

$$\varphi_1(U_1 \cap U_2) \overset{\varphi_2 \circ \varphi_1^{-1}}{\to} \phi_2(U_1 \cap U_2)$$

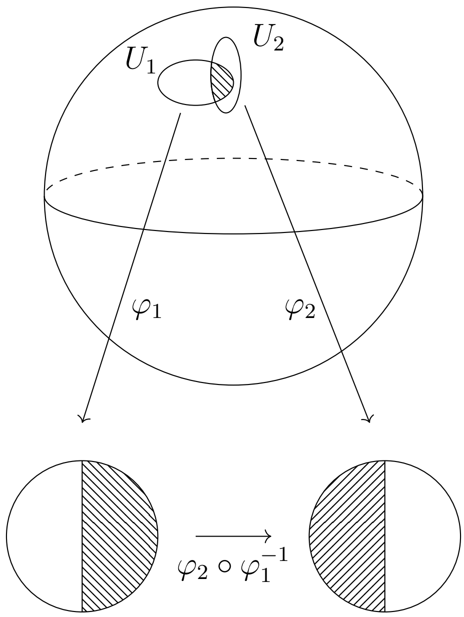
is a *homeomorphism*.

Recall if $V \subset \mathbb{R}^n$ and $V' \subset \mathbb{R}^m$ are open, a map $f : V \to V'$ is called *smooth* if it is infinitely differentiable, so it has continuous partial derivatives of all orders.

A homeomorphism $f : V \to V'$ is called a *diffeomorphism* if it is smooth and it has a smooth inverse.

Definition 2.4. An *abstract smooth surface* Σ is a topological surface with an atlas of charts $\{U_i, \varphi_i)\}$ such that all transition maps are diffeomorphisms.

Figure 8: Transition Map on S^2



Example 2.2.

1. The atlas of two charts with stereographic projections gives S^2 the structure of an abstract smooth surface.
2. The torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ is an abstract smooth surface. Recall that we obtained charts from (the inverses of) the projection restricted to small discs in $\mathbb{R}^2$. In particular, consider the atlas

$$\{(e(D_\varepsilon(x,y)), e^{-1} \text{ on its image})\},$$

where $\varepsilon < 1/3$. Here the transition maps are translations, so T^2 inherits the structure of a smooth surface.

Definition 2.5. Let Σ be an abstract smooth surface and $f : \Sigma \to \mathbb{R}^n$ a map. We say that f is *smooth* at $p \in \Sigma$ if whenever (U, φ) is a chart at p belonging to the smooth atlas of Σ, then the map

$$f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^n$$

is smooth at $\phi(p) \in \mathbb{R}^2$.

Note if this holds for one chart at p, then it holds for all charts at p, as

$$f \circ \varphi^{-1} = f \circ \varphi_2^1 \circ (\varphi_2 \circ \varphi_1^{-1}),$$

and $(\varphi_2 \circ \varphi_1^{-1})$ is a diffeomorphism.

Related, if Σ_1, Σ_2 are abstract smooth surfaces, then a map $f : \Sigma_1 \to \Sigma_2$ is *smooth* if it is smooth at the local charts: there are charts (U, φ) at p and (V, ψ) at $f(p)$ with $f(U) \subset V$ such that $\psi \circ f \circ \varphi^{-1}$ is smooth at $\varphi(p)$.

Again, if f is smooth at p, then the smoothness of the local representation of f at p will hold for all charts at p and $f(p)$ in the given atlases.

Definition 2.6. Σ_1 and Σ_2 are *diffeomorphic* if there exists $f : \Sigma_1 \to \Sigma_2$ that is smooth with smooth inverse.

Definition 2.7. If $Z \subset \mathbb{R}^n$ is an arbitrary subset, we say that $f : Z \to \mathbb{R}^m$ is smooth near $p \in Z$ if there exists open B with $p\, inB \subset \mathbb{R}^n$ and smooth $F : B \to \mathbb{R}^m$ such that

$$F|_{B\cap Z} = f|_{B\cap Z}.$$

So f is locally the restriction of a smooth map defined on an open set.

Definition 2.8. If $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ are subsets, we say that X and Y are *diffeomorphic* if there exists $f : X \to Y$, smooth with smooth inverse.

Definition 2.9. A *smooth surface* in $\mathbb{R}^3$ is a subset $\Sigma \subset \mathbb{R}^3$ such that for all $p \in \Sigma$, there exists an open set $p \in U \subset \Sigma$ such that U is diffeomorphic to an open set in $\mathbb{R}^2$.

In other words, for all $p \in \Sigma$, there exists an open ball B such that $p \in B \subset \mathbb{R}^3$ and $F : B \to V \subset \mathbb{R}^2$ smooth, with

$$F|_{B\cap\Sigma} : B \cap \Sigma \to V$$

a homeomorphism with inverse $V \to B \cap \Sigma$ smooth.

Hence we have two notions of a smooth surface: one abstract, and one taking advantage of the ambient space $\mathbb{R}^3$.

Theorem 2.1. *For a subset $\Sigma \subset \mathbb{R}^3$, the following are equivalent:*

(a) *Σ is a smooth surface in $\mathbb{R}^3$.*

(b) *Σ is locally the graph of a smooth function over one of the coordinate planes, so for all $p \in \Sigma$, there exists open $p \in B \subset \mathbb{R}^3$ and open $V \subset \mathbb{R}^2$ such that*

$$\Sigma \cap B = \{(x, y, g(x, y) \mid g : V \to \mathbb{R}\},$$

with g smooth.

(c) Σ *is locally cut out by a smooth function with non-zero derivative, so for all* $p \in \Sigma$*, there open exists* $p \in B \subset \mathbb{R}^n$ *and* $f : B \to \mathbb{R}$ *such that*

$$\Sigma \cap B = f^{-1}(0), \qquad Df|_x \neq 0,$$

for all $x \in B$*.*

(d) Σ *is locally the image of an* allowable parametrization*, so if* $p \in \Sigma$*, there exists open* $p \in U \subset \Sigma$ *and smooth* $\sigma : V \to U$*, such that* σ *is a homeomorphism and* $D\sigma|_x$ *has rank* 2 *for all* $x \in V$*.*

Remark. (b) says that if Σ is a smooth surface in $\mathbb{R}^3$, then each $p \in \Sigma$ belongs to a chart (U, φ) where φ is the restriction of $\pi_{xy}, \pi_{yz}, \pi_{xz}$ from $\mathbb{R}^3$ to $\mathbb{R}^2$.

2.1 Inverse Function Theorem

Theorem 2.2. *Let* $U \subset \mathbb{R}^n$ *be open and let* $f : U \to \mathbb{R}^n$ *be a continuously differentiable map. Let* $p \in U$*,* $f(p) = q$*, and suppose* $Df|_p$ *is invertible. Then there is an open neighbourhood* V *of* q *and a differentiable map* $g : V \to \mathbb{R}^n$ *with* $g(q) = p$*, with image an open neighbourhood* $U' \subset U$ *of* p*, such that*

$$f \circ g = \mathrm{id}_V, \qquad g \circ f = \mathrm{id}_{u'}.$$

If f *is smooth, then so is* g*.*

Remark. $(Dg|_q) = (Df|_p)^{-1}$ by the chain rule.

If we have a map $f : \mathbb{R}^n \to \mathbb{R}^m$ where $n > m$, then

$$Df|_p = \left(\frac{\partial f_i}{\partial x_j}\right)_{m \times n}$$

having full rank means that, permuting coordinates if necessary, we can assume that the first m columns are linearly independent.

Theorem 2.3 (Implicit Function theorem)**.** *Let* $p = (x_0, y_0) \in U$*, where* $U \subset \mathbb{R}^k \times \mathbb{R}^\ell$ *is open, and* $f : U \to \mathbb{R}^\ell$ *be a continuously differentiable map with* $f(p) = 0$*, and*

$$\left(\frac{\partial f_i}{\partial y_j}\right)_{\ell \times \ell} \text{ is an isomorphism at } p.$$

Then there exists an open neighbourhood $x_0 \in V \subset \mathbb{R}^k$ *and a continuously differentiable map* $g : V \to \mathbb{R}^\ell$ *taking* x_0 *to* y_0*, such that if* $(x, y) \in U \cap (V \times \mathbb{R}^\ell)$*, then*

$$f(x, y) = 0 \iff y = g(x).$$

Proof: Introduce $F : U \to \mathbb{R}^k \times \mathbb{R}^\ell$, where $(x, y) \mapsto (x, f(x, y))$. Then

$$DF = \begin{pmatrix} I & * \\ 0 & \frac{\partial f_i}{\partial y_j} \end{pmatrix}.$$

So $DF|_{(x_0,y_0)}$ is an isomorphism. The inverse function says that F is locally invertible near $F(x_0, y_0) = (x_0, f(x_0, y_0)) = (x_0, 0)$.

Take a product of open neighbourhoods $(x_0, 0) \in V \times V'$, where $V \subset \mathbb{R}^k$, $V' \subset \mathbb{R}^\ell$ are open. Then there is some continuously differentiable inverse $G : V \times V' \to U' \subset U$ such that $F \circ G = \mathrm{id}_{V \times V'}$

Write $G(x, y) = (\varphi(x, y), \psi(x, y))$. Then,

$$F \circ G(x, y) = (\varphi(x, y) f(\varphi(x, y), \psi(x, y))) = (x, y).$$

Hence $\varphi(x, y) = x$, $f(x, \psi(x, y)) = y$. Thus, $f(x, y) = 0 \iff y = \psi(x, 0)$.

Define $g : V \to \mathbb{R}^\ell$ by $g(x) = \psi(x, 0)$. Then $g(x_0) = y_0$, and this is the required function g.

Example 2.3.

1. Take $f : \mathbb{R}^2 \to \mathbb{R}$ smooth, and $f(x_0, y_0) = 0$. Assume

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \neq 0.$$

Then there exists smooth $g : (x_0 - \varepsilon, x_0 + \varepsilon) \to \mathbb{R}$. Such that $g(x_0) = y_0$ and $f(x, y) = 0 \iff y = g(x)$.

Since $f(x, g(x)) = 0$ by chain rule

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} g'(x) = 0 \implies g'(x) = -\frac{\partial f/\partial x}{\partial f/\partial y}.$$

2. Let $f : \mathbb{R}^3 \to \mathbb{R}$ smooth and $f(x_0, y_0, z_0) = 0$, and assume

$$Df|_{(x_0,y_0,z_0)} \neq 0.$$

Permuting coordinates if necessary, we may assume that

$$\frac{\partial f}{\partial z}|_{(x_0,y_0,z_0)} \neq 0.$$

Then there exists an open neighbourhood $(x_0, y_0) \in V \subset \mathbb{R}^2$ and a smooth $g : V \to \mathbb{R}$, with $g(x_0, y_0) = z_0$, such that for an open set $(x_0, y_0, z_0) \in U$,

$$f^{-1}(0) \cap U = \{(x, y, g(x, y)) \mid (x, y) \in V\}.$$

We return to theorem 2.1, which we can now prove.

Proof: Note (b) implies all other statements. If Σ is locally $\{(x, y, g(x, y)) \mid (x, y) \in V\}$, then we get a chart from the projection Π_{xy}, which is smooth and defined on an open neighbourhood of Σ, hence (b) implies (a).

Also, it is cut out by

$$f(x, y, z) = z - g(x, y).$$

Clearly $\frac{\partial f}{\partial z} = 1 \neq 0$, so (b) implies (c).

Also, $\sigma(x, y) = (x, y, g(x, y))$ is allowable and smooth, with

$$\sigma_x = (1, 0, g_x), \qquad \sigma_y = (0, 1, g_y)$$

linearly independent. So (b) implies (d).

Now (a) implies (d), as this is part of the definition of being a smooth surface in $\mathbb{R}^3$.

Moreover, (c) implies (b) from the above example of the implicit function theorem.

We finally show that (d) implies (b). Let $p \in \Sigma$, and $\sigma : V \to U \subset \Sigma$ with $\sigma(0) = p \in U$, and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$. Then

$$D\sigma = \begin{pmatrix} \partial\sigma_1/\partial u & \partial\sigma_1/\partial v \\ \partial\sigma_2/\partial u & \partial\sigma_2/\partial v \\ \partial\sigma_3/\partial u & \partial\sigma_3/\partial v \end{pmatrix}.$$

So there exists two rows defining an invertible matrix, as $D\sigma$ has rank two. Suppose the first two rows are. Then $\Pi_{xy} \circ \sigma : V \to \mathbb{R}^2$ satisfies $D(\Pi_{xy} \circ \sigma)|_0$ is an isomorphism.

By the inverse function theorem, this is locally invertible, so if we let $\phi = \Pi_{xy} \circ \sigma$, then Σ is the graph of $(x, y, \sigma_3(\phi^{-1}(x, y)))$.

Using this, we can find many examples of smooth surfaces in $\mathbb{R}^3$.

Example 2.4.

1. The *ellipsoid* $E \subset \mathbb{R}^3$ is $f^{-1}(0)$ for $f : \mathbb{R}^3 \to \mathbb{R}$ satisfying

$$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1.$$

For all $p \in E = f^{-1}(0)$, $Df|_p \neq 0$, so E is a smooth surface in $\mathbb{R}^3$.

2. Let $\gamma : [a, b] \to \mathbb{R}^3$ be a smooth map with image in the xz-plane, so

$$\gamma(t) = (f(t), 0, g(t)).$$

Assume γ is injective, with $\gamma'(t) \neq 0$. Rotating this around the z-axis, we get a surface of revolution with allowable parametrization

$$\sigma(u, v) = (f(u) \cos v, f(u) \sin v, g(u)).$$

For $(u, v) \in [a, b] \times [\theta, \theta + 2\pi)$ for $\theta \in [0, 2\pi]$ fixed, σ is a homeomorphism onto its image. Indeed,

$$\begin{aligned} \sigma_u &= (f' \cos v, f' \sin v, g'), \\ \sigma_v &= (-f \sin v, f \cos v, 0). \end{aligned}$$

Moreover,

$$\|\sigma_u \times \sigma_v\|^2 = f^2(f'^2 + g'^2) \neq 0,$$

proving σ is allowable.

2.2 Orientability

Consider $V, V' \subset \mathbb{R}^2$ open, with $f : V \to V'$ a diffeomorphism. Then at any $x \in V$,

$$Df|_x \in \mathsf{GL}(2, \mathbb{R}).$$

Let $\mathsf{GL}^+(2, \mathbb{R}) \subset \mathsf{GL}(2, \mathbb{R})$ be the subgroup of matrices of positive determinant.

Definition 2.10. We say that f is *orientation preserving* if $Df|_x \in \mathsf{GL}^+(2, \mathbb{R})$ for all $x \in V$.

Definition 2.11. An abstract smooth surface Σ is *orientable* if it admits an atlas such that the transition maps are orientation preserving diffeomorphisms of open sets of $\mathbb{R}^2$.

A choice of such atlas is an *orientation*of Σ, and we say that Σ is *oriented.*

Lemma 2.1. *If Σ_1, Σ_2 are abstract smooth surfaces, and they are diffeomorphic, then Σ_1 is orientable if and only if Σ_2 is orientable.*

Proof: Suppose $f : \Sigma_1 \to \Sigma_2$ is a diffeomorphism and Σ_2 is orientable and equipped with an oriented atlas.

Consider the atlas on Σ_1 given by

$$(f^{-1}U, \psi \circ f|_{f^{-1}U}),$$

where (U, ψ) is a chart of Σ_2. The transition function between two such charts is exactly the transition function in the Σ_2 atlas.

The transition function between two such charts is exactly the transition function in the Σ_2-atlas.

Remark.

1. There is no sensible classification of all smooth or topological surfaces, for example $\mathbb{R}^2 \setminus Z$ for Z closed.

   By contrast, compact smooth surfaces up to diffeomorphism are classified by Euler characteristic and orientability.

2. There is a definition of orientation preserving homeomorphism that needs some algebraic topology.

   The Möbius band is the surface in figure 9. It turns out that abstract smooth surfaces are orientable if and only if it contains no subsurface homeomorphic to the Möbius band.

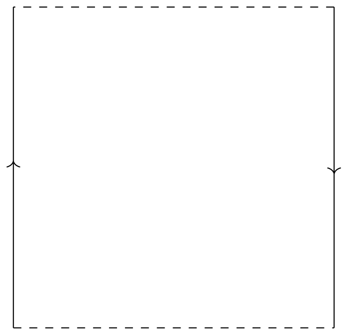
   So we can say that a topological surface is orientable if and only if it contains no subsurface homeomorphic to a Möbius band.

3. We get other structures by demanding the transition maps to be such that

   $$D(\varphi_1\varphi_2^{-1})|_x \in G \subset \mathsf{GL}(2, \mathbb{R}).$$

   For example, we can take $G = \mathsf{GL}(1, \mathbb{C}) \subset \mathsf{GL}(2, \mathbb{R})$, which give *Riemann surfaces*.

Figure 9: Möbius band



Example 2.5.

1. If we take S^2 with the atlas of the two stereographic projection, we can compute the transition map as

$$(u, v) \mapsto \left(\frac{u}{u^2 + v^2}, \frac{v}{u^2 + v^2} \right),$$

on $\mathbb{R}^2 \setminus \{0\}$. This is orientation preserving.

2. In T^2, the transition maps are translations of $\mathbb{R}^2$, so T^2 is oriented.

For surfaces in $\mathbb{R}^3$, we would like ot have orientability dictated by some ambient feature.

Definition 2.12. Let $\Sigma \subset \mathbb{R}^3$ be a smooth surface, and $p \in \Sigma$. Fix an allowable parametrization $\sigma : V \to U \subset \Sigma$, $\sigma(0) = p$.

Then, the *tangent plane* $T_p\Sigma$ of Σ at p is the image of $D\sigma|_0$, as a subset of $\mathbb{R}^3$. This is a 2d vector subspace of $\mathbb{R}^3$.

The *affine tangent plane* of Σ at p is $p + T_p\Sigma \subset \mathbb{R}^3$.

Lemma 2.2. *$T_p\Sigma$ is well-defined, so it is independent of the choice of allowable parametrization near p.*

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