IB Complex Analysis

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Contents

1 Complex Differentiation

Our goal in this course is to study the theory of complex-valued differentiable functions in one complex variable. Example include:

- Polynomials $p(z) = a_d z^d + \cdots + a_1 z + a_0$, with coefficients in $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$ or $\mathbb{C}$.
- The infinite series
$$\sum_{n=1}^{\infty} \frac{1}{n^z},$$
which we showed convergence for z having real part greater than 1.
- Harmonic functions $u(x, y) : \mathbb{R}^2 \to \mathbb{R}$, $u_{xx} + u_{yy} = 0$.

In this course, we make the convention that $\theta = \arg(z) \in [0, 2\pi)$.

1.1 Basic Notions

- $U \subset \mathbb{C}$ is *open* if for all $u \in U$, there exists $\varepsilon > 0$ such that
$$\Delta(x, \varepsilon) = \{z \in \mathbb{C} \mid |z - u| < \varepsilon\} \subset U.$$
- A *path* in $U \subset \mathbb{C}$ is a continuous map $\gamma : [a, b] \to U$. We say the path is C^1 if γ' exists and is continuous (we take one-sided derivatives at the endpoints).

  γ is *simple* if it is injective.
- $U \subset \mathbb{C}$ is *path-connected* if for all $z, w \in U$, there exists a path in U with endpoints at z, w.

Remark. If U is open, and $z, w \in U$ are connected by a path γ in U, then there exists a path γ in U connected z, w consisting of finitely many horizontal and vertical segments.

Definition 1.1. A *domain* is a non-empty, open, path-connected subset of $\mathbb{C}$.

Definition 1.2.

(i) $f : U \to \mathbb{C}$ is *differentiable* at $u \in U$ if
$$f'(u) = \lim_{z \to u} \frac{f(z) - f(u)}{z - u}$$
exists.

(ii) $f : U \to \mathbb{C}$ is *holomorphic* at $u \in U$ if there exists $\varepsilon > 0$ such that f is differentiable at z, for all $z \in \Delta(u, \varepsilon)$. We may also call such a function *analytic*.

(iii) $f : \mathbb{C} \to \mathbb{C}$ is *entire*if it is holomorphic everywhere.

Remark. All differentiation rules (sum, products, ...) in $\mathbb{R}$ hold, by the same proofs.

Identifying $\mathbb{C}$ with $\mathbb{R}^2$, we may write $f : U \to \mathbb{C}$ as $f(x+iy) = u(x,y) + iv(x,y)$, where u, v are the real and imaginary parts of f.

From analysis and topology, recall that $u : U \to \mathbb{R}$ as a function of two real variables if $(\mathbb{R}^2)$ differentiable at $(c,d) \in \mathbb{R}^2$with $Du|_{(c,d)} = (\lambda, \mu)$ if

$$\frac{u(x,y) - u(c,d) - [\lambda(x-c) + \mu(y-d)]}{\sqrt{(x-c)^2 + (y-d)^2}} \to 0,$$

as $(x,y) \to (c,d)$. However, **this is a weaker condition** than differentiability over $\mathbb{C}$.

Proposition 1.1 (Cauchy-Riemann equations)**.** *Let $f : U \to \mathbb{C}$ on an open set $U \subset \mathbb{C}$. Then f is differentiable at $w = c+id \in U$ if and only if, writing $f = u+iv$, we have u, v are $\mathbb{R}^2$-differentiable at (c,d), and*

$$u_x = v_y, \qquad\qquad u_y = -v_x.$$

Proof: f is differentiable at w if and only if $f'(w) = p + iq$ exists, so

$$\lim_{z \to w} \frac{f(z) - f(w) - (z-w)(p+iq)}{|z-w| = 0}.$$

Writing $f = u + iv$ and considering the real and imaginary parts in the quotient above, this holds if and only if

$$\lim_{(x,y)\to(c,d)} \frac{u(x,y) - u(c,d) - [p(x-c) - q(y-d)]}{\sqrt{(x-c)^2 + (y-d)^2}} = 0,$$

and

$$\lim_{(x,y)\to(c,d)} \frac{v(x,y) - v(c,d) - [q(x-c) + p(y-d)]}{\sqrt{(x-c)^2 + (y-d)^2}} = 0.$$

This holds if and only if u, v are $\mathbb{R}^2$-differentiable at (c,d), and $u_x = v_y$, $u_y = -v_x$.

Remark. If the partial u_x, u_y, v_x, v_y exist and are continuous on U, then u, v are differentiable on U. So it suffices to check the partials exist and are continuous, and the Cauchy-Riemann equations hold to deduce complex differentiability.

Example 1.1.

1. Take $f(z) = \overline{z}$. Then f has $u(x, y) = x$ and $v(x, y) = -y$, so $u_x = 1$, $v_y = -1$. So $f(z) = \overline{z}$ is not holomorphic or differentiable anywhere.
2. Any polynomial $p(z) = a_d z^d + \cdots + a_1 z + a_0$, with $a_i \in \mathbb{C}$ is entire.
3. Rational function, which are quotients of polynomials $\frac{p(z)}{q(z)}$ are holomorphic on the open set $\mathbb{C} \setminus \{\text{zeroes of } q\}$.

Note that $f = u + iv$ satisfying the Cauchy-Riemann equations at a point does not mean it is differentiable at that point.

Some proofs in regular analysis have natural extensions to complex analysis. For example, if $f : U \to \mathbb{C}$ on a domain U with $f'(z) = 0$ on U, then f is constant on U.

Now we ask: why are we interested in complex analysis?

- Unlike $\mathbb{R}^2$ differentiable functions, holomorphics functions are very constrained. For example, if f is entire and bounded (so $|f(z)| < M$ for all $z \in \mathbb{C}$), then f is constant. Contrast with sin, for example.
- We will see that f holomorphic on a domain U has holomorphic derivative on U. This implies that f is infinitely differentiable, as are u and v.

In particular, we can differentiate the Cauchy-Riemann equations to get

$$u_{xx} = v_{yx} = v_{xy} = -u_{yy},$$

so $u_{xx} + u_{yy} = 0$, and similarly $v_{xx} + v_{yy} = 0$. Hence the real and imaginary parts of a holomorphic function are harmonic.

Let $f : U \to \mathbb{C}$ be a holomorphic function on an open set U_1 and $w \in U$ with $f'f(w) \neq 0$. We want to look at the geometric behaviour of f at w.

In fact, we claim f is *conformal* at w. Let γ_1, γ_2 be C^1-paths through w, say $\gamma_1, \gamma_2 : [-1, 1] \to U_1$, such that $\gamma_1(0) = \gamma_2(0) = w$, and $\gamma_i'(0) \neq 0$. If we write $\gamma_j(t) = w + r_j(t) = e^{i\theta_j(t)}$, then we have

$$\arg(\gamma_j'(z)) = \theta_j(0),$$

and the argument of the image line is

$$\arg((f \circ \gamma_j)'(0)) = \arg(\gamma_j'(0) f'(\gamma_j(0))) = \arg(\gamma_j'(0)) + \arg(f'(w)) + 2\pi n,$$

where crucially we use $\gamma_j'(0)f'(\gamma_j(0)) \neq 0$, so the direction of γ_j at w under the application of f is rotated by $\arg(f'(w))$. This is independent of γ_j. Since the angle between γ_1 and γ_2 is the difference of the arguments f preserves the angle. This is what it means to be conformal.

Definition 1.3. Let U, V be domains in $\mathbb{C}$. A map $f : U \to V$ is a *conformal equivalence* of U and V if f is a bijective holomorphic map with $f'(z) \neq 0$, for all $z \in U$.

Remark.

1. Using the real inverse function theorem, one can show if $f : U \to V$ is a holomorphic bijection of open sets with $f'(z) \neq 0$ for all $z \in U$, then the inverse of f is also holomorphic, so also conformal by the chain rule. So conformally equivalent domains are equal from the perspective of the functions f.
2. We will later see than being injective and holomorphic on a domain implies $f'(z) \neq 0$ for all $z \in U$, so this requirement is redundant.

Example 1.2.

1. Any change of coordinates: on $\mathbb{C}$, take $f(z) = az + b$, for $a \neq 0$ and b, which is a conformal equivalence $\mathbb{C} \to \mathbb{C}$. More generally, a Möbius map
$$f(z) = \frac{az+b}{cz+d},$$
for $ad - bc \neq 0$, is a conformal equivalence from the Riemann sphere to itself. This can eb seen as adding a point at infinity to make a sphere $\mathbb{C}_\infty$ (or gluing two copies of the unit disc with coordinates z and $\frac{1}{z}$).

   If $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is continuous, then

   - if $f(\infty) = \infty$, then f is holomorphic at ∞ if and only if $g(z) = \frac{1}{f(\frac{1}{z})}$ is holomorphic at 0.
   - If $f(\infty) \neq \infty$, then f is homolorphic at ∞ if and only if $f(\frac{1}{z})$ is holomorphic at 0.
   - If $f(a) = \infty$ for $a \in \mathbb{C}$, then f is holomorphic at a if and only if $\frac{1}{f(z)}$ is holomorphic at a.

   We can then think of Möbius maps as change of coordinates for the sphere.

Choosing $z_1 \to 0$, $z_2 \to \infty$, $z_3 \to 1$ defined a Möbius map

$$f(z) = \frac{z - z_1}{z - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_1},$$

for distinct $z_1, z_2, z_3 \in \mathbb{C}$.

2. For $n \in \mathbb{N}$, $f(z) = z^n$ is a conformal equivalence from the sector $\{z \in \mathbb{C}^\times \mid 0 < \arg z < \frac{\pi}{n}\}$ to the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \Im z > 0\}$.
3. The Möbius map $f(z) = \frac{z-i}{z+i}$ is a conformal equivalence between $\mathbb{H}$ and $D(0,1)$. We can compute $f'(z) \neq 0$ on $\mathbb{H}$, and

   $$z \in \mathbb{H} \iff |z - i| < |z + i| \iff |f(z)) < 1.$$

   Note that $f^{-1}(w) = -i\frac{w+1}{w-1}$.
4. We can use these examples to write down conformal equivalences. Let U_1 be the upper half semicircle, and U_2 the lower half plane. Considering $g(z) = \frac{z+1}{z-1}$, we know that sends $D(0,1)$ to the left half-plane, so it sends U_1 to the upper left quadrant.

   Then, the upper left quadrant if mapped by the squaring map to U_2. So $f(z) = (\frac{z+1}{z-1})^2$ is a conformal equivalence from $U_1 \to U_2$.

These are all examples of the deep *Riemann mapping theorem*:

Theorem 1.1 (Riemann mapping theorem)**.** *Let $U \subset \mathbb{C}$ be a proper domain which is simply connected. Then there exists a conformal equivalence between U and $D(0,1)$.*

Here, *simply connected* means a subset $U \subset \mathbb{C}$ which is path-connected, and contractible: any loop in U can be contracted to a point. So any continuous path $\gamma : S^1 \to U$ extends to a continuous map $\hat{\gamma} : D(0,1) \to U_1$ with $\hat{\gamma}|_{S_1} = \gamma$.

In fact any domain bounded by a simple closed curve is simply connected, so all of these are conformally equivalent to $D(0,1)$.

Example 1.3.

We look at a domains in the Riemann sphere, with bounded and connected complement. This is simply connected as a subset of $\mathbb{C}_\infty$.

Now, the Mandelbrot set is bounded and connected, so the complement of the Mandelbrot set is simply connected in $\mathbb{C}_\infty$.

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