IB Geometry

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Contents

1 Surfaces

Definition 1.1. A *topological surface* is a topological space Σ such that

(a) for all $p \in \Sigma$, there is an open neighbourhood $p \in U \subset \Sigma$ such that U is homeomorphic to $\mathbb{R}^2$, or a disc $D^2 \subset \mathbb{R}^2$, with its usual Euclidean topology.

(b) Σ is Hausdorff and second countable.

Remark. $\mathbb{R}^2 \simeq D(0,1) = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$.

1. A space X is *Hausdorff* if for $p \neq q$ in X, there exist disjoint open sets U, V with $p \in U, q \in V$.

   A space is *second countable* if it has a countable base, i.e there exist open sets $\{U_i\}_{i \in \mathbb{N}}$, such that every open set is a union of some of the U_i.

   The key point of defining surfaces is point (a), point (b) is for ruling out surfaces that are too weird.

2. If X is Hausdorff or second countable, then so are subspaces of X. Moreover Euclidean space has these properties (to show it is second countable, consider open balls $B(c, r)$ with $c \in \mathbb{Q}^n \subset \mathbb{R}^n$, and $r \in \mathbb{Q}_+ \subset \mathbb{R}_+$).

Example 1.1.

(i) The plane $\mathbb{R}^2$.

(ii) Any open set in $\mathbb{R}^2$ is a surface, i.e. $R^2 \setminus Z$ where Z is closed is a surface.

(iii) Graphs of functions. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous. Then the graph of f is

$$\Gamma_f = \{(x, y, f(x, y)) \mid (x, y) \in \mathbb{R}^2\}.$$

This is a subspace of $\mathbb{R}^3$, so we can endow it with the subspace topology. We claim it is a subspace homeomorphic to $\mathbb{R}^2$.

Recall that if X, Y are topological spaces, then the product topology $X \times Y$ has a basis of open sets $U \times V$, where $U \subset X$, $V \subset Y$ are open

A feature is that if $g : Z \to X \times Y$ is continuous if and only if $\Pi_x \circ g : Z \to X$ and $\Pi_y \circ g : Z \to Y$ are continuous, where Π_x, Π_y are the canonical projectors.

We can now show that if $f : X \to Y$ is continuous, then $\Gamma_f \subset X \times Y$ is homeomorphic to X, as $s(x) = (x, f(x))$ is a continuous function from X to Γ_f, $\Pi_x|_{\Gamma_f}$ and s are inverse homeomorphisms.

In particular, for our example $\Gamma_f \simeq \mathbb{R}^2$. So any $f : \mathbb{R}^2 \to \mathbb{R}$ continuous produces a surface Γ_f.

(iv) The sphere: $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$ (with the subspace topology). To show this is a surface, we can consider the stereographic projection $\Pi_+ : S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2$:

$$(x,y,z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

Then Π_+ is continuous and has an inverse

$$(u,v) \mapsto \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right).$$

So Π_+ is a continuous bijection with continuous inverse, and hence a homeomorphism.

Similarly, taking a stereographic projection from the south pole $\Pi_- : S^2 \setminus \{(0,0,-1)\} \to \mathbb{R}^2$, by

$$(x,y,z) \mapsto \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$$

is another homeomorphism. Hence S^2 is a topological surface, as the open sets $S^2 \setminus \{(0,0,1)\}$ and $S^2 \setminus \{(0,0,-1)\}$ cover S^2, and it is Hausdorff and second countable as it is a subspace of $\mathbb{R}^3$.

(v) The *real projective plane*. The group $\mathbb{Z}_2$ acts on S^2 by homeomorphisms, via the antipodal map

$$\begin{aligned} a &: S^2 \to S^2 \\ a(x,y,z) &\mapsto (-x,-y,-z) \end{aligned}$$

Definition 1.2. The real projective plane is the quotient of S^2 by identifying every point with its antipodal image:

$$\mathbb{RP}^2 = S^2/\mathbb{Z}_2 = S^2/\sim.$$

Lemma 1.1. *As a set, $\mathbb{RP}^2$ is naturally in bijection with the set of straight lines through 0.*

This is because any straight line through $0 \in \mathbb{R}^3$ intersects S^2 in exactly a pair of antipodal points, and each such pair determines a straight line.

Lemma 1.2. $\mathbb{RP}^2$ *is a topological surface with the quotient topology.*

Recall the quotient topology: given the quotient map $q : X \to Y$, we say $V \subset Y$ is open if and only if $q^{-1}(V) \subset X$ is open in X.

Proof: First we show that $\mathbb{RP}^2$ is Hausdorff. If $[p] \neq [q] \in \mathbb{RP}^2$, then $\pm p$, $\pm q$ are distinct, antipodal pairs.

We take open discs centred on p and q and their antipodal images, such that no two discs intersect. The images of these discs give open images of $[p]$ and $[q]$ in $\mathbb{RP}^2$. Indeed, $q(B_\delta(p))$ is open since $q^{-1}(q(B_\delta(p))) = B_\delta(p) \cup (-B_\delta(p))$.

Now we show $\mathbb{RP}^2$ is second countable. Let U be a countable base of S^2, and let $\overline{U} = \{q(u) \mid u \in U\}$. Then $q(u)$ is open, as $q(u) = u \cup (-u)$, and $\overline{U}$ is clearly countable as U is.

Take $V \subset \mathbb{RP}^2$ open. By definition, $q^{-1}(V)$ is open, so let $q^{-1}(V) = \bigcup U_\alpha$, for $U_\alpha \in U$. Then

$$V = q(q^{-1}(V)) = q\left(\bigcup_\alpha U_\alpha\right) = \bigcup_\alpha q(U_\alpha).$$

Finally, let $p \in S^2$ and $[p] \in \mathbb{RP}^2$ be its image. Let $\overline{D}$ be a small closed disc neighbourhood of $p \in S^2$, so that $q|_{\overline{D}}$ is injective and continuous, and has image a Hausdorff space.

Now recall that a countinuous bijection from a compact space to a Hausdorff space is a homeomorphism.

So $q|_{\overline{D}} : \overline{D} \to q(\overline{D})$ is a homeomorphism. This induces a homeomorphism

$$q|_D : D \to q(D) \subset \mathbb{RP}^2,$$

where D is an open disc contained in $\overline{D}$. So $[p] \in q(D)$ has an open neighbourhood in $\mathbb{RP}^2$ homeomorphic to an open disc.

Example 1.2.

We continue looking at examples of surfaces.

(vi) Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Then the *torus* is $S^1 \times S^1$ with the subspace topology of $\mathbb{C}^2$ (this is the same as taking the product topology).

Lemma 1.3. *The torus is a topological surface.*

Proof: We consider the map

$$\begin{aligned} \mathbb{R}^2 &\xrightarrow{e} S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C} \\ (s,t) &\mapsto (e^{2\pi i s}, e^{2\pi i t}). \end{aligned}$$

We can view this map using the following diagram:

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "e"] \arrow[d, "q"] & S^1 \times S^1 \\ \mathbb{R}^2/\mathbb{Z}^2 \arrow[ur, dashed, "\hat{e}"] & \end{tikzcd}$$

There is an equivalence relation on $\mathbb{R}^2$ given by translating by $\mathbb{Z}^2$. Now consider the map

$$[0,1]^2 \hookrightarrow \mathbb{R}^2 \xrightarrow{q} \mathbb{R}^2/\mathbb{Z}^2$$

is onto, so $\mathbb{R}^2/\mathbb{Z}^2$ is compact. Now note that $\hat{e}$ is a continuous bijection, so since it is onto a Hausdorff space, it is a homeomorphism.

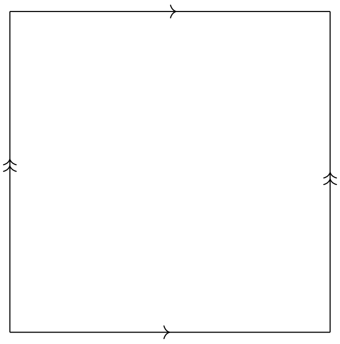
Similar to $\mathbb{RP}^2$, for $[p] \in q(p)$, take a small closed disc $\overline{D} \subset \mathbb{R}^2$ such that, for all $(m,n) \in \mathbb{Z}^2$, $\overline{D} \cap (\overline{D} + (m,n)) = \emptyset$.

Then $e|_{\overline{D}}$ and $q|_{\overline{D}}$ are injective. Now restricting to an open disc as before, we get an open disc as a neighbourhood of $[p]$, so $S^1 \times S^1$ is a topological surface.

Another viewpoint for a torus is by imposing on $[0,1]^2$ the equivalence relations

$$(x,0) \sim (x,1), \qquad (0,y) \sim (1,y).$$

Figure 1: Identification of a Torus



Example 1.3.

We look at yet another example of a surface.

(vii) Let P be a planar Euclidean polyon. Assume that the edges are oriented and paired, and for simplicity assume the Euclidean lengths of e and $\hat{e}$ are equal if $\{e, \hat{e}\}$ are paired.

Label by letters, and describe the orientation by a sign of $\pm$ relative to the clockwise orientation in $\mathbb{R}^2$.

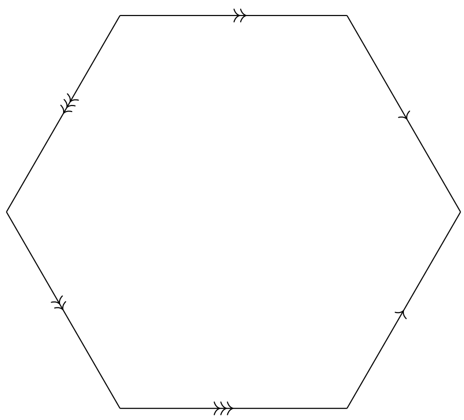
More precisely, if $\{e, \hat{e}\}$ are paired edges, there is a unique isometry from e to $\hat{e}$ respecting their orientations, say

$$f_{e\hat{e}} : e \to \hat{e}.$$

These maps generate an equivalence relation on P, where we identify $x \in \partial P$ with $f_{e\hat{e}}(x)$ whenever $x \in e$.

Lemma 1.4. *$P/\sim$ (with the quotient topology) is a topological surface.*

Figure 2: Orientation of Edges of a Hexagon



Proof: We begin by looking at a special case of the torus T^2 as $[0,1]^2/\sim$. Then if p is an interior point, we pick $\delta > 0$ small such that $\overline{B_\delta(p)}$ lies in the interior of the polygon P. Now we argue as before: the quotient map is injective on $\overline{B_\delta(p)}$ and is a homeomorphism on its interior.

Now suppose p is on an edge of P, but not a vertex. The idea is to take the two points in $q^{-1}(p)$, take half discs around them, and join them up to form a disc.

Say $p = (0, y_0) \sim (1, y_0) = p'$. Take δ small enough so the half discs of radius δ do not meet the vertices and don't intersect. Let U be the half disc around p and V the half disc around p'.

Define a map as follows:

$$\begin{aligned} U &: (x, y) \xrightarrow{f_u} (x, y - y_0), \\ V &: (x, y) \xrightarrow{f_v} (x - 1, y - y_0). \end{aligned}$$

We want to show these maps glue well together. To do this, we use the following fact:

If $X = A \cup B$, A and B are closed, and $f : A \to Y$ and $g : B \to Y$ are continuous and $f|_{A\cap B} = g|_{A\cap B}$, then they define a continuous map on X.

Now f_u and f_v are continuous on $U, V \subset [0, 1]^2$, so they induce continuous maps on $q(U)$ and $q(V)$.

In T^2, the intersection of the discs overlap on the paired edges, but our maps agree, so they are compatible with the equivalence relation. Hence f_u and f_v give a continuous map on an open image of $[p] \in T^2$ to $\mathbb{R}^2$. By the usual argument, we can show if $[p] \in T^2$ lies on an edge of P it has a neighbourhood homeomorphic to a disc.

Finally, we look at a vertex of $[0, 1]^2$. In the image, there is really only one vertex. To find a homeomorphism to the open disc, we can take four quarter circles at each corner, and glue them appropriately.

For a general polygon, it is a similar idea. Interior and edge points are done analogously to T^2. For vertices, it is a bit different. We have different equivalence classes of vertices caused by orienting the edges in different ways.

If v is a vertex of P with k vertices in its equivalence class, then we have k sectors in P. Any sector can be identified with out favourite sector in $\mathbb{R}^2$, i.e. $(r, \theta) \in \mathbb{R}^2$ with $0 \leq r < \delta$ and $\theta \in [0, 2\pi/k]$. Gluing these together, we get an open disc as a neighbourhood of v.

This works unless $k = 1$, in which case we have two paired edges coming into or out of a vertex in P. But this is homeomorphic to a cone, which is homeomorphic to a disc.

These neighbourhoods of points in $P/\sim$ show that P is locally homeomorphic to a disc, and we can easily check that $P/\sim$ is Hausdorff and second countable.

Example 1.4.

One more example now.

(viii) We now consider connecting surfaces. Given topological surfaces Σ_1 and Σ_2, we can remove an open disc from each, and glue the resulting boundary circles.

Explicitly, we take $\Sigma_1 \setminus D_1 \cup \Sigma_2 \setminus D_2$ as a disjoint union, and impose the quotient relation

$$\theta \in \partial D_1 \sim \theta \in \partial D_2,$$

where θ parametrizes $S^1 = \partial D_i$.

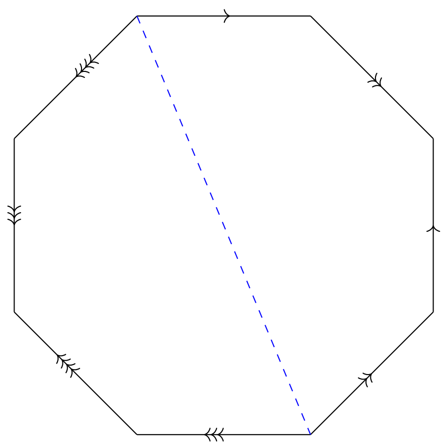
The result $\Sigma_1 \sharp \Sigma_2$ is called the *connected sum* of Σ_1 and Σ_2.

In principle, this depends on the choices of discs, and it takes some effort to prove that it is well-defined.

Lemma 1.5. *The connected sum $\Sigma_1 \sharp \Sigma_2$ is a topological surface.*

We will not prove this lemma in this course.

Figure 3: Octagon



As another example the octagon is homeomorphic to a double torus: cutting along the blue line reveals two copies of a torus, which are joined together.

Similarly, we can find $\mathbb{RP}^2$ as the quotient of a square: this can be seem by morphing it into a circle with antipodes identified, which is then homeomorphic to $\mathbb{RP}^2$, seen by 'squishing down' $\mathbb{RP}^2$ or projecting it onto a plane.

Figure 4: Identification of $\mathbb{RP}^2$

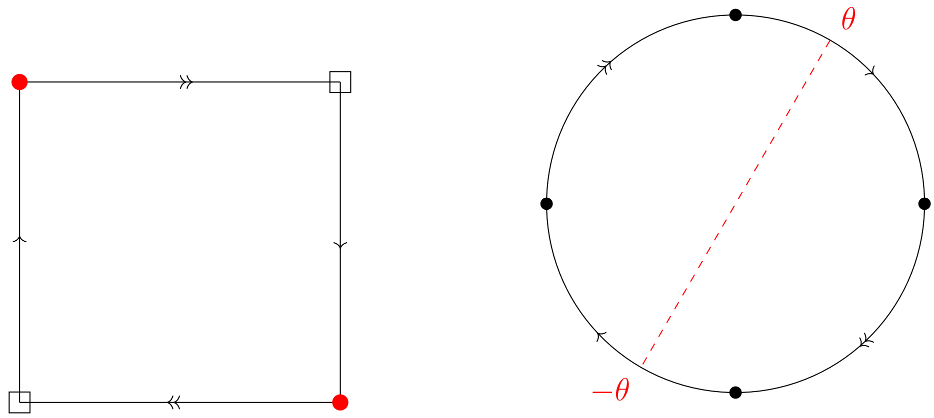
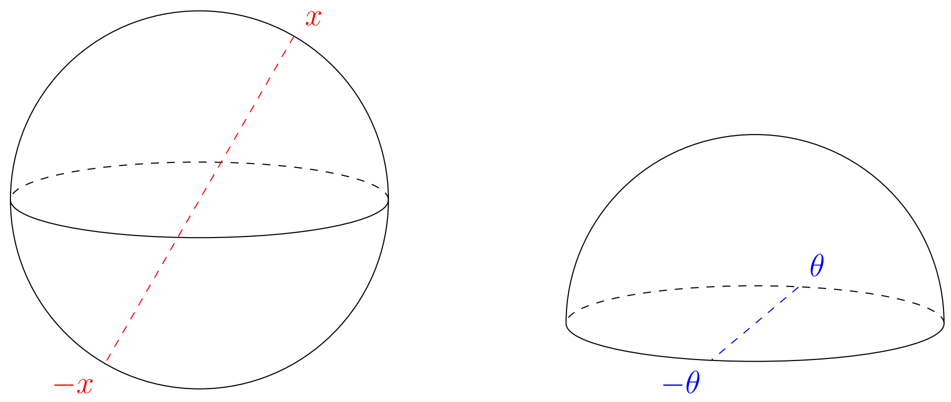


Figure 5: Squishing down $\mathbb{RP}^2$



1.1 Triangulation and Euler Characteristic

Definition 1.3. A *subdivision* of a compact topological surface Σ comprises of:

(i) a finite set V of *vertices*,

(ii) a finite collection of edges $E = \{e_i : [0,1] \to \Sigma\}$ such that

- for all i, e_i is a continuous injection on its interior and $e_i^{-1}(V) = \{0,1\}$,
- e_i and e_j have disjoint images except perhaps at their endpoints in V.

(iii) We require that each connected component of

$$\Sigma \setminus \left(\bigcup_i e_i([0,1]) \cup V \right)$$

is homeomorphic to an open disc, called a *face*.

Hence the closure of a face $\overline{F} \setminus F$ has boundary lying in

$$\bigcup_i e_i([0,1]) \cup V.$$

A subdivision is a *triangulation* if every closed face (closure of a face) contains exactly three edges, and two closed faces are disjoint, meet in exactly one edge or just one vertex.

Example 1.5.

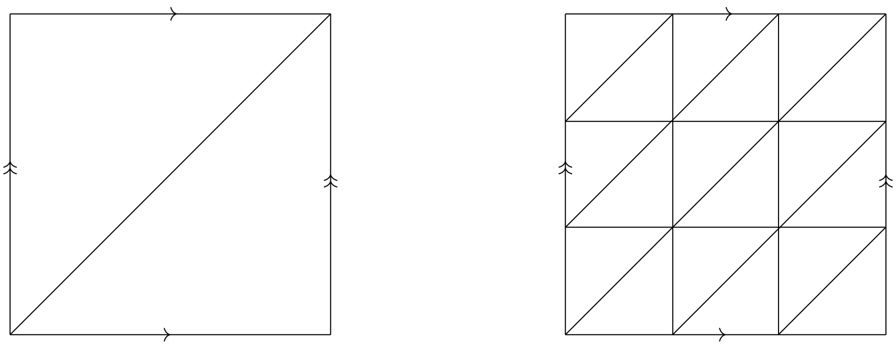
A cube displays a subdivision of S^2, and a tetrahedron displays a triangulation of S^2.

Moreover figure 1 displays a subdivision of T^2, with one vertex, two edges and one face.

In figure 6, only the right triangulation is a valid triangulation: in the left figure, the two triangles share more than one edge.

As well, figure 7 is a degenerate subdivision of the sphere, with one vertex, no edges and one face.

Figure 6: Triangulations of the Torus



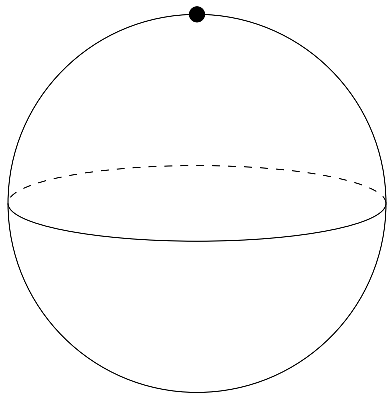
(a) Invalid (b) Valid

Definition 1.4. The *Euler characteristic* of a subdivision is

$$|V| - |E| + |F|.$$

Theorem 1.1.

Figure 7: Subdivision of S^2



(i) *Every compact topological surface admits subdivisions and triangulations.*

(ii) *The Euler characteristic, denoted* $\chi(\Sigma)$, *does not depend on the subdivision and defined a topological invariant of the surface.*

Remark. This is hard to prove, particularly (ii). There are cleaner approaches to this (seen in algebraic topology).

Example 1.6.

1. $\chi(S^2) = 2$.
2. $\chi(T^2) = 0$.
3. Let Σ_1, Σ_2 be compact topological spaces, and we form $\Sigma_1 \sharp \Sigma_2$. We remove open discs $D_i \subset \Sigma_i$ which is a face of a triangulation in each surface. Hence,

$$\chi(\Sigma_1 \sharp \Sigma_2) = \chi(\Sigma_1) + \chi(\Sigma_2) - 2.$$

In particular if Σ_g is a surface with g holes, i.e.

$$\Sigma_g = \sharp_{i=1}^g T^2,$$

then $\chi(\Sigma_g) = 2 - 2g$. g is called the *genus*.

2 Abstract Smooth Surfaces

Definition 2.1. A pair (U, φ) where $U \subset \Sigma$ is open and $\varphi : U \to V \subset \mathbb{R}^2$ is called a *chart*.

The inverse $\sigma = \varphi^{-1} : V \to U \subset \Sigma$ is called a *local parametrization* of Σ.

Definition 2.2. A collection of charts

$$\{(U_i, \varphi_i)_{i \in I}\}$$

such that

$$\bigcup_{i \in I} U_i = \Sigma$$

is called an *atlas* of Σ.

Example 2.1.

1. If $Z \subset \mathbb{R}^2$ is closed, then $\mathbb{R}^2 \setminus Z$ is a topological surface with an atlas with one chart: $(\mathbb{R}^2 \setminus Z, \varphi = \mathrm{id})$.
2. For S^2 we have an atlas with 2 charts: the two stereographic projections.

Definition 2.3. Let (U_i, φ_i) for $i = 1, 2$ be two charts containing $p \in \Sigma$. The map

$$\varphi_2 \circ \varphi_1^{-1}|_{\varphi_1(U_1 \cap U_2)}$$

is called the *transition map* between charts.

Note that

$$\varphi_1(U_1 \cap U_2) \overset{\varphi_2 \circ \varphi_1^{-1}}{\to} \phi_2(U_1 \cap U_2)$$

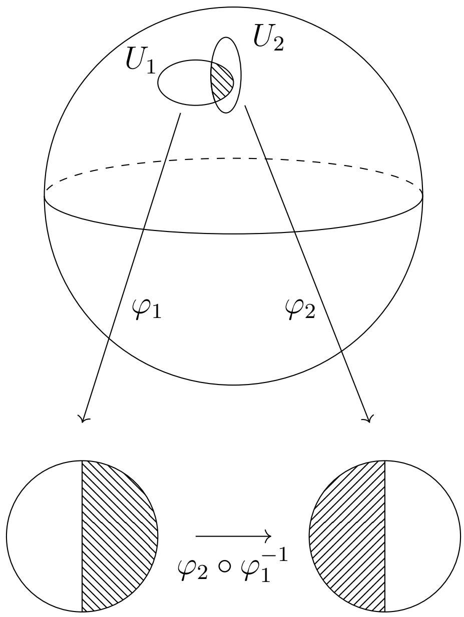
is a *homeomorphism*.

Recall if $V \subset \mathbb{R}^n$ and $V' \subset \mathbb{R}^m$ are open, a map $f : V \to V'$ is called *smooth* if it is infinitely differentiable, so it has continuous partial derivatives of all orders.

A homeomorphism $f : V \to V'$ is called a *diffeomorphism* if it is smooth and it has a smooth inverse.

Definition 2.4. An *abstract smooth surface* Σ is a topological surface with an atlas of charts $\{U_i, \varphi_i)\}$ such that all transition maps are diffeomorphisms.

Figure 8: Transition Map on S^2



Example 2.2.

1. The atlas of two charts with stereographic projections gives S^2 the structure of an abstract smooth surface.
2. The torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ is an abstract smooth surface. Recall that we obtained charts from (the inverses of) the projection restricted to small discs in $\mathbb{R}^2$. In particular, consider the atlas

$$\{(e(D_\varepsilon(x,y)), e^{-1} \text{ on its image})\},$$

where $\varepsilon < 1/3$. Here the transition maps are translations, so T^2 inherits the structure of a smooth surface.

Definition 2.5. Let Σ be an abstract smooth surface and $f : \Sigma \to \mathbb{R}^n$ a map. We say that f is *smooth* at $p \in \Sigma$ if whenever (U, φ) is a chart at p belonging to the smooth atlas of Σ, then the map

$$f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^n$$

is smooth at $\phi(p) \in \mathbb{R}^2$.

Note if this holds for one chart at p, then it holds for all charts at p, as

$$f \circ \varphi_1^{-1} = f \circ \varphi_2^{-1} \circ (\varphi_2 \circ \varphi_1^{-1}),$$

and $(\varphi_2 \circ \varphi_1^{-1})$ is a diffeomorphism.

Related, if Σ_1, Σ_2 are abstract smooth surfaces, then a map $f : \Sigma_1 \to \Sigma_2$ is *smooth* if it is smooth at the local charts: there are charts (U, φ) at p and (V, ψ) at $f(p)$ with $f(U) \subset V$ such that $\psi \circ f \circ \varphi^{-1}$ is smooth at $\varphi(p)$.

Again, if f is smooth at p, then the smoothness of the local representation of f at p will hold for all charts at p and $f(p)$ in the given atlases.

Definition 2.6. Σ_1 and Σ_2 are *diffeomorphic* if there exists $f : \Sigma_1 \to \Sigma_2$ that is smooth with smooth inverse.

Definition 2.7. If $Z \subset \mathbb{R}^n$ is an arbitrary subset, we say that $f : Z \to \mathbb{R}^m$ is smooth near $p \in Z$ if there exists open B with $p \in B \subset \mathbb{R}^n$ and smooth $F : B \to \mathbb{R}^m$ such that

$$F|_{B \cap Z} = f|_{B \cap Z}.$$

So f is locally the restriction of a smooth map defined on an open set.

Definition 2.8. If $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ are subsets, we say that X and Y are *diffeomorphic* if there exists $f : X \to Y$, smooth with smooth inverse.

Definition 2.9. A *smooth surface* in $\mathbb{R}^3$ is a subset $\Sigma \subset \mathbb{R}^3$ such that for all $p \in \Sigma$, there exists an open set $p \in U \subset \Sigma$ such that U is diffeomorphic to an open set in $\mathbb{R}^2$.

In other words, for all $p \in \Sigma$, there exists an open ball B such that $p \in B \subset \mathbb{R}^3$ and $F : B \to V \subset \mathbb{R}^2$ smooth, with

$$F|_{B \cap \Sigma} : B \cap \Sigma \to V$$

a homeomorphism with inverse $V \to B \cap \Sigma$ smooth.

Hence we have two notions of a smooth surface: one abstract, and one taking advantage of the ambient space $\mathbb{R}^3$.

Theorem 2.1. *For a subset $\Sigma \subset \mathbb{R}^3$, the following are equivalent:*

(a) *Σ is a smooth surface in $\mathbb{R}^3$.*

(b) *Σ is locally the graph of a smooth function over one of the coordinate planes, so for all $p \in \Sigma$, there exists open $p \in B \subset \mathbb{R}^3$ and open $V \subset \mathbb{R}^2$ such that*

$$\Sigma \cap B = \{(x, y, g(x, y) \mid g : V \to \mathbb{R}\},$$

with g smooth.

(c) *Σ is locally cut out by a smooth function with non-zero derivative, so for all $p \in \Sigma$, there open exists $p \in B \subset \mathbb{R}^n$ and $f : B \to \mathbb{R}$ such that*

$$\Sigma \cap B = f^{-1}(0), \qquad Df|_x \neq 0,$$

for all $x \in B$.

(d) *Σ is locally the image of an* allowable parametrization, *so if $p \in \Sigma$, there exists open $p \in U \subset \Sigma$ and smooth $\sigma : V \to U$, such that σ is a homeomorphism and $D\sigma|_x$ has rank 2 for all $x \in V$.*

Remark. (b) says that if Σ is a smooth surface in $\mathbb{R}^3$, then each $p \in \Sigma$ belongs to a chart (U, φ) where φ is the restriction of $\pi_{xy}, \pi_{yz}, \pi_{xz}$ from $\mathbb{R}^3$ to $\mathbb{R}^2$.

2.1 Inverse Function Theorem

Theorem 2.2. *Let $U \subset \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}^n$ be a continuously differentiable map. Let $p \in U$, $f(p) = q$, and suppose $Df|_p$ is invertible. Then there is an open neighbourhood V of q and a differentiable map $g : V \to \mathbb{R}^n$ with $g(q) = p$, with image an open neighbourhood $U' \subset U$ of p, such that*

$$f \circ g = \mathrm{id}_V, \qquad g \circ f = \mathrm{id}_{u'}\,.$$

If f is smooth, then so is g.

Remark. $(Dg|_q) = (Df|_p)^{-1}$ by the chain rule.

If we have a map $f : \mathbb{R}^n \to \mathbb{R}^m$ where $n > m$, then

$$Df|_p = \left(\frac{\partial f_i}{\partial x_j}\right)_{m \times n}$$

having full rank means that, permuting coordinates if necessary, we can assume that the first m columns are linearly independent.

Theorem 2.3 (Implicit Function theorem)**.** *Let $p = (x_0, y_0) \in U$, where $U \subset \mathbb{R}^k \times \mathbb{R}^\ell$ is open, and $f : U \to \mathbb{R}^\ell$ be a continuously differentiable map with $f(p) = 0$, and*

$$\left(\frac{\partial f_i}{\partial y_j}\right)_{\ell \times \ell} \text{ is an isomorphism at } p.$$

Then there exists an open neighbourhood $x_0 \in V \subset \mathbb{R}^k$ and a continuously differentiable map $g : V \to \mathbb{R}^\ell$ taking x_0 to y_0, such that if $(x, y) \in U \cap (V \times \mathbb{R}^\ell)$, then

$$f(x, y) = 0 \iff y = g(x).$$

Proof: Introduce $F : U \to \mathbb{R}^k \times \mathbb{R}^\ell$, where $(x, y) \mapsto (x, f(x, y))$. Then

$$DF = \begin{pmatrix} I & * \\ 0 & \frac{\partial f_i}{\partial y_j} \end{pmatrix}.$$

So $DF|_{(x_0,y_0)}$ is an isomorphism. The inverse function says that F is locally invertible near $F(x_0, y_0) = (x_0, f(x_0, y_0)) = (x_0, 0)$.

Take a product of open neighbourhoods $(x_0, 0) \in V \times V'$, where $V \subset \mathbb{R}^k$, $V' \subset \mathbb{R}^\ell$ are open. Then there is some continuously differentiable inverse $G : V \times V' \to U' \subset U$ such that $F \circ G = \mathrm{id}_{V \times V'}$

Write $G(x, y) = (\varphi(x, y), \psi(x, y))$. Then,

$$F \circ G(x, y) = (\varphi(x, y), f(\varphi(x, y), \psi(x, y))) = (x, y).$$

Hence $\varphi(x, y) = x$, $f(x, \psi(x, y)) = y$. Thus, $f(x, y) = 0 \iff y = \psi(x, 0)$.

Define $g : V \to \mathbb{R}^\ell$ by $g(x) = \psi(x, 0)$. Then $g(x_0) = y_0$, and this is the required function g.

Example 2.3.

1. Take $f : \mathbb{R}^2 \to \mathbb{R}$ smooth, and $f(x_0, y_0) = 0$. Assume

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \neq 0.$$

   Then there exists smooth $g : (x_0 - \varepsilon, x_0 + \varepsilon) \to \mathbb{R}$. Such that $g(x_0) = y_0$ and $f(x, y) = 0 \iff y = g(x)$.

   Since $f(x, g(x)) = 0$ by chain rule

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} g'(x) = 0 \implies g'(x) = -\frac{\partial f/\partial x}{\partial f/\partial y}.$$

2. Let $f : \mathbb{R}^3 \to \mathbb{R}$ smooth and $f(x_0, y_0, z_0) = 0$, and assume

$$Df|_{(x_0,y_0,z_0)} \neq 0.$$

   Permuting coordinates if necessary, we may assume that

$$\frac{\partial f}{\partial z}|_{(x_0,y_0,z_0)} \neq 0.$$

Then there exists an open neighbourhood $(x_0, y_0) \in V \subset \mathbb{R}^2$ and a smooth $g : V \to \mathbb{R}$, with $g(x_0, y_0) = z_0$, such that for an open set $(x_0, y_0, z_0) \in U$,

$$f^{-1}(0) \cap U = \{(x, y, g(x, y)) \mid (x, y) \in V\}.$$

We return to theorem 2.1, which we can now prove.

Proof: Note (b) implies all other statements. If Σ is locally $\{(x, y, g(x, y)) \mid (x, y) \in V\}$, then we get a chart from the projection Π_{xy}, which is smooth and defined on an open neighbourhood of Σ, hence (b) implies (a).

Also, it is cut out by

$$f(x, y, z) = z - g(x, y).$$

Clearly $\frac{\partial f}{\partial z} = 1 \neq 0$, so (b) implies (c).

Also, $\sigma(x, y) = (x, y, g(x, y))$ is allowable and smooth, with

$$\sigma_x = (1, 0, g_x), \qquad \sigma_y = (0, 1, g_y)$$

linearly independent. So (b) implies (d).

Now (a) implies (d), as this is part of the definition of being a smooth surface in $\mathbb{R}^3$.

Moreover, (c) implies (b) from the above example of the implicit function theorem.

We finally show that (d) implies (b). Let $p \in \Sigma$, and $\sigma : V \to U \subset \Sigma$ with $\sigma(0) = p \in U$, and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$. Then

$$D\sigma = \begin{pmatrix} \partial\sigma_1/\partial u & \partial\sigma_1/\partial v \\ \partial\sigma_2/\partial u & \partial\sigma_2/\partial v \\ \partial\sigma_3/\partial u & \partial\sigma_3/\partial v \end{pmatrix}.$$

So there exists two rows defining an invertible matrix, as $D\sigma$ has rank two. Suppose the first two rows are. Then $\Pi_{xy} \circ \sigma : V \to \mathbb{R}^2$ satisfies $D(\Pi_{xy} \circ \sigma)|_0$ is an isomorphism.

By the inverse function theorem, this is locally invertible, so if we let $\phi = \Pi_{xy} \circ \sigma$, then Σ is the graph of $(x, y, \sigma_3(\phi^{-1}(x, y)))$.

Using this, we can find many examples of smooth surfaces in $\mathbb{R}^3$.

Example 2.4.

1. The *ellipsoid* $E \subset \mathbb{R}^3$ is $f^{-1}(0)$ for $f : \mathbb{R}^3 \to \mathbb{R}$ satisfying
$$f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1.$$
For all $p \in E = f^{-1}(0)$, $Df|_p \neq 0$, so E is a smooth surface in $\mathbb{R}^3$.
2. Let $\gamma : [a,b] \to \mathbb{R}^3$ be a smooth map with image in the xz-plane, so
$$\gamma(t) = (f(t), 0, g(t)).$$
Assume γ is injective, with $\gamma'(t) \neq 0$. Rotating this around the z-axis, we get a surface of revolution with allowable parametrization
$$\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u)).$$
For $(u,v) \in [a,b] \times [\theta, \theta+2\pi)$ for $\theta \in [0,2\pi]$ fixed, σ is a homeomorphism onto its image. Indeed,
$$\sigma_u = (f'\cos v, f'\sin v, g'),$$
$$\sigma_v = (-f\sin v, f\cos v, 0).$$
Moreover,
$$\|\sigma_u \times \sigma_v\|^2 = f^2(f'^2 + g'^2) \neq 0,$$
proving σ is allowable.

2.2 Orientability

Consider $V, V' \subset \mathbb{R}^2$ open, with $f : V \to V'$ a diffeomorphism. Then at any $x \in V$,
$$Df|_x \in \mathsf{GL}(2,\mathbb{R}).$$

Let $\mathsf{GL}^+(2,\mathbb{R}) \subset \mathsf{GL}(2,\mathbb{R})$ be the subgroup of matrices of positive determinant.

Definition 2.10. We say that f is *orientation preserving* if $Df|_x \in \mathsf{GL}^+(2,\mathbb{R})$ for all $x \in V$.

Definition 2.11. An abstract smooth surface Σ is *orientable* if it admits an atlas such that the transition maps are orientation preserving diffeomorphisms of open sets of $\mathbb{R}^2$.

A choice of such atlas is an *orientation* of Σ, and we say that Σ is *oriented.*

Lemma 2.1. *If Σ_1, Σ_2 are abstract smooth surfaces, and they are diffeomorphic, then Σ_1 is orientable if and only if Σ_2 is orientable.*

Proof: Suppose $f : \Sigma_1 \to \Sigma_2$ is a diffeomorphism and Σ_2 is orientable and equipped with an oriented atlas.

Consider the atlas on Σ_1 given by

$$(f^{-1}U, \psi \circ f|_{f^{-1}U}),$$

where (U, ψ) is a chart of Σ_2. The transition function between two such charts is exactly the transition function in the Σ_2 atlas.

The transition function between two such charts is exactly the transition function in the Σ_2-atlas.

Remark.

1. There is no sensible classification of all smooth or topological surfaces, for example $\mathbb{R}^2 \setminus Z$ for Z closed.

   By contrast, compact smooth surfaces up to diffeomorphism are classified by Euler characteristic and orientability.

2. There is a definition of orientation preserving homeomorphism that needs some algebraic topology.

   The Möbius band is the surface in figure 9. It turns out that abstract smooth surfaces are orientable if and only if it contains no subsurface homeomorphic to the Möbius band.

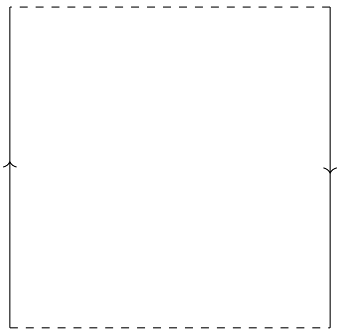
   So we can say that a topological surface is orientable if and only if it contains no subsurface homeomorphic to a Möbius band.

3. We get other structures by demanding the transition maps to be such that

   $$D(\varphi_1 \varphi_2^{-1})|_x \in G \subset \mathsf{GL}(2, \mathbb{R}).$$

   For example, we can take $G = \mathsf{GL}(1, \mathbb{C}) \subset \mathsf{GL}(2, \mathbb{R})$, which give *Riemann surfaces*.

Figure 9: Möbius band



Example 2.5.

1. If we take S^2 with the atlas of the two stereographic projection, we can compute the transition map as
$$(u, v) \mapsto \left(\frac{u}{u^2 + v^2}, \frac{v}{u^2 + v^2} \right),$$
on $\mathbb{R}^2 \setminus \{0\}$. This is orientation preserving.
2. In T^2, the transition maps are translations of $\mathbb{R}^2$, so T^2 is oriented.

For surfaces in $\mathbb{R}^3$, we would like to have orientability dictated by some ambient feature.

Definition 2.12. Let $\Sigma \subset \mathbb{R}^3$ be a smooth surface, and $p \in \Sigma$. Fix an allowable parametrization $\sigma : V \to U \subset \Sigma$, $\sigma(0) = p$.

Then, the *tangent plane* $T_p\Sigma$ of Σ at p is the image of $D\sigma|_0$, as a subset of $\mathbb{R}^3$. This is a 2D vector subspace of $\mathbb{R}^3$.

The *affine tangent plane* of Σ at p is $p + T_p\Sigma \subset \mathbb{R}^3$.

Lemma 2.2. *$T_p\Sigma$ is well-defined, so it is independent of the choice of allowable parametrization near p.*

Proof: Let $\sigma : V \to U \subset \Sigma$, $\tilde{\sigma} : \tilde{V} \to \tilde{U} \subset E$, with $\sigma(0) = \tilde{\sigma}(0) = p$ be two parametrizations near p.

Since $\sigma^{-1} \circ \tilde{\sigma}$ is a transition map,
$$\tilde{\sigma} = \sigma \circ (\sigma^{-1} \circ \tilde{\sigma}).$$

As $D(\sigma^{-1} \circ \tilde{\sigma})|_0$ is an isomorphism, we get $\operatorname{im}(D\tilde{\sigma}|_0) = \operatorname{im}(D\sigma|_0)$.

Definition 2.13. Let $\Sigma \subset \mathbb{R}^3$ be a smooth surface. The *normal direction* at p is $(T_p\Sigma)^\perp$.

At each $p \in \Sigma$, we have two unit normal vectors.

Definition 2.14. A smooth surface in $\mathbb{R}^3$ is *two-sided* if it admits a continuous global choice of unit normal vectors.

Lemma 2.3. *A smooth surface in $\mathbb{R}^3$ is orientable with its abstract smooth surface structure if and only if it is two-sided.*

Proof: Let $\sigma : V \to U \subset \Sigma$ be allowable, with $\sigma(0) = p$. Define the *positive* unit normal with respect to σ at p as the unique $n_\sigma(p)$ such that

$$\{\sigma_u, \sigma_v, n_\sigma(p)\}, \qquad \{e_1, e_2, e_3\}$$

induce the same orientation in $\mathbb{R}^3$ (i.e. they are related by a change of basis matrix with positive determinant). Explicitly,

$$n_\sigma(p) = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}.$$

Let $\tilde{\sigma}$ be another allowable parametrization at p, and suppose Σ is orientable as an abstract smooth surface with σ and $\tilde{\sigma}$ belonging to the same oriented atlas.

Again, we can write $\sigma = \tilde{\sigma} \circ \varphi$, where $\varphi = \tilde{\sigma}^{-1} \circ \sigma$. Then

$$D\varphi|_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

By the chain rule, we have

$$\begin{aligned} \sigma_u &= \alpha\tilde{\sigma}_{\tilde{u}} + \gamma\tilde{\sigma}_{\tilde{v}}, \\ \sigma_v &= \beta\tilde{\sigma}_{\tilde{u}} + \delta\tilde{\sigma}_{\tilde{v}}. \end{aligned}$$

Hence we get

$$\sigma_u \times \sigma_v = \det(D\phi|_0)\tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}}.$$

So the positive unit normal at p does not depend on the parametrization. Since $n_\sigma(p)$ is continuous, Σ is two-sided.

Conversely, if Σ is two-sided, we have a global choice of n, so we can consider the subatlas of the smooth atlas such that we have charts (U, φ), where $\varphi^{-1} = \sigma$ and $\{\sigma_u, \sigma_v, n\}$ is an oriented basis of $\mathbb{R}^3$.

Then from the above formula, the transition maps between such charts are orientation preserving, so Σ is orientable.

Remark. Give a curve $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^3$, smooth with image in Σ and with $\gamma(0) = p$, we can write

$$\gamma(t) = \sigma(u(t), v(t)).$$

Differentiating,

$$\sigma'(0) = D\sigma|_0(u'(0), v'(0)).$$

This is on the tangent plane. Hence

$$T_p\Sigma = \{\gamma'(0) \mid \gamma \text{ smooth on } \Sigma\}.$$

Example 2.6.

1. Take $S^2 \subset \mathbb{R}^3$. Then the obvious unit normal at p is p.

   To prove this, take any $\gamma : (-\varepsilon, \varepsilon) \to S^2$, with $\gamma(0) = p$. Since $|\gamma(t)|^2 = 1$, differentiating at $t = 0$,

   $$2\langle \gamma'(0), p\rangle = 0 \implies (T_pS^2)^\perp = \mathbb{R}p.$$

   Therefore we can take $n(p) = p$, and this is continuous so S^2 is two-sided.

2. The Möbius band is constructed as follows: start with the unit circle in the xy-plane, and take an open interval of length 1. Rotate this line in the cz-plane as we move around the circle, such that it has rotated by $\theta/2$ after moving an angle θ.

   After a full turn, the segment returns to its original position but with the endpoints inverted. We can describe the surface with

   $$\sigma(t, \theta) = ((1 - t\sin\theta/2)\cos\theta, (1 - t\sin\theta/2)\sin\theta, t\cos\theta/2),$$

   where (t, θ) belongs to

   $$V_1 = \{t \in (-1/2, 1/2), \theta \in (0, 2\pi)],$$

   or

   $$V)2 = \{t \in (-1/2, 1/2), \theta \in (-\pi, \pi)\}.$$

We can check that if σ_i is σ on V_i, then σ_i is allowable. A computation shows that

$$\sigma_t \times \sigma_\theta(0, \theta) = (-\cos\theta\cos\theta/2, -\sin\theta\cos\theta/2, -\sin\theta/2).$$

As $\theta \to 0^+$, $n_\theta = (-1, 0, 0)$, but as $\theta \to 2\pi^-$, $n_\theta = (1, 0, 0)$. Hence the Möbius band is not two-sided.

3 Surfaces in 3-space

Let $\gamma : (a, b) \to \mathbb{R}^3$ be smooth. The length of γ is

$$L(\gamma) = \int_a^b \|\gamma'(t)\| \, \mathrm{d}t.$$

If $s : (A, B) \to (a, b)$ is monotone increasing, and we let $\tau(t) = \gamma(s(t))$, then

$$L(\tau) = \int_A^B \|\tau'(t)\| \, \mathrm{d}t = \int_A^B \|\gamma'(s(t))\| s'(t) \, \mathrm{d}t = \int_a^b \|\gamma'(s)\| \, \mathrm{d}s = L(\gamma).$$

Lemma 3.1. *If $\gamma : (a, b) \to \mathbb{R}^3$ and $\gamma'(t) \neq 0$ for all t, then γ can be parametrized by arc-length, i.e. by a parameter s such that $\|\gamma'(s)\| = 1$ for all s.*

Let $\Sigma \subset \mathbb{R}^3$ be a smooth surface, and $\sigma : V \to U \subset \Sigma$ be allowable. If $\gamma : (a, b) \to U$ is smooth, write

$$\gamma(t) = \sigma(u(t), v(t)).$$

Then we have

$$\gamma'(t) = \sigma_u u'(t) + \sigma_v v'(t).$$

Therefore,

$$\|\gamma'(t)\|^2 = E(u'(t))^2 + 2F u'(t) v'(t) + G(v'(t))^2.$$

Where

$$\begin{aligned} E &= \langle \sigma_u, \sigma_u \rangle = \|\sigma_u\|^2, \\ F &= \langle \sigma_u, \sigma_v \rangle = \langle \sigma_v, \sigma_u \rangle, \\ G &= \langle \sigma_v, \sigma_v \rangle = \|\sigma_v\|^2. \end{aligned}$$

They are defined only on σ, and not γ.

Definition 3.1. The *first fundamental form* (FFF) in the parametrization σ is the expression

$$E \, \mathrm{d}u^2 + 2F \, \mathrm{d}u \, \mathrm{d}v + G \, \mathrm{d}v^2.$$

This satisfies

$$L(\gamma) = \int_a^b \sqrt{Eu'^2 + 2Fu'v' + Gv'^2} \, \mathrm{d}t,$$

where $\gamma(t) = \sigma(u(t), v(t))$.

Remark. The first fundamental form is sometimes defined as the quadratic form in $T_p\Sigma$, given by the restriction of the standard inner product in $\mathbb{R}^3$:

$$\mathrm{I}_p(w) = |w|^2 = \langle w, w\rangle_{\mathbb{R}^3}.$$

After picking σ with $\sigma(0) = p$ and after writing $w = D\sigma|_0(u', v')$, we have

$$\mathrm{I}_p(w) = Eu'^2 + 2Fu'v' + Gv'^2.$$

This is an example of a *Riemannian metric*.

Example 3.1.

The xy-plane $\mathbb{R}^3$ is parametrized as $\sigma(u, v) = (u, v, 0)$, so $\sigma_u = (1, 0, 0)$, $\sigma_v = (0, 1, 0)$, and the first fundamental form is $\mathrm{d}u^2 + \mathrm{d}v^2$. Here $E = G = 1$, $F = 0$.

In polar coordinates, $\sigma(r, \theta) = (r\cos\theta, r\sin\theta, 0)$. Then we have

$$\sigma_r = (\cos\theta, \sin\theta, 0),$$
$$\sigma_\theta = (-r\sin\theta, r\cos\theta, 0).$$

The first fundamental form is $\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$. Here $E = 1$, $F = 0$ and $G = r^2$.

Definition 3.2. Let $\Sigma, \Sigma' \subset \mathbb{R}^3$ be smooth surfaces. We say that Σ and Σ' are *isometric* if there exists $f : \Sigma \to \Sigma'$ a diffeomorphism such that for every smooth curve $\gamma : (a, b) \to \Sigma$,

$$L_\Sigma(\gamma) = L_{\Sigma'}(f \circ \gamma).$$

Example 3.2.

If $\Sigma' = f(\Sigma)$, where $f : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear motion, i.e.

$$f(x) = Ax + b$$

where $A \in \mathsf{O}(3)$ and $b \in \mathbb{R}^3$, then $f : \Sigma \to \Sigma'$ is an isometry because

$$|(f \circ \gamma)'(t)| = |A\gamma'(t)| = |\gamma'(t)|.$$

Hence the lengths of curves are preserved.

Often we are interested in local statements.

Definition 3.3. Σ, Σ' are *locally isometric* near points $p \in \Sigma$ and $p' \in \Sigma'$, if there exist open neighbourhoods $p \in U \subset \Sigma$ and $p' \in U' \subset \Sigma'$, which are isometric.

Lemma 3.2. $\Sigma, \Sigma' \subset \mathbb{R}^3$ *are locally isometric near* $p \in \Sigma$ *and* $p' \in \Sigma'$ *if and only if there exist allowable parametrizations*

$$\sigma : V \to U \subset \Sigma,$$
$$\sigma' : V \to U' \subset \Sigma',$$

for which the first fundamental forms are equal in V*.*

Proof: We know (by definition) that the first fundamental form of σ determines the length of all curves on $\sigma(V) = U$.

If we have σ and σ' as in the lemma, then $\sigma' \circ \sigma^{-1} : U \to U'$ is an isometry, since

$$\begin{aligned}
\left|\frac{\mathrm{d}}{\mathrm{d}t}\sigma' \circ \sigma^{-1} \circ \gamma\right|^2 &= \left|\frac{\mathrm{d}}{\mathrm{d}t}\sigma'(u(t), v(t))\right|^2 \\
&= E'\dot{u}^2 + 2F'\dot{u}\dot{v} + G'\dot{v}^2 \\
&= E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2 \\
&= \left|\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\right|^2.
\end{aligned}$$

Hence $L(\sigma' \circ \sigma^{-1} \circ \gamma) = L(\gamma)$.

For the converse, we will show first that the lengths of curves in U determine the first fundamental form of σ. Indeed, suppose

$$\sigma : B(0, \delta) \to U,$$

with $\sigma(0) = p$. Then take a curve $\gamma_\varepsilon : [0, \varepsilon] \to U$ with $t \mapsto \sigma(t, 0)$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}L(\gamma_\varepsilon) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_0^\varepsilon \sqrt{E(t,0)}\,\mathrm{d}t = \sqrt{E(\varepsilon, 0)}.$$

So the length of the curve determines $E(0,0)$. Similarly, $\chi_\varepsilon : t \mapsto \sigma(0,t)$ determines $G(0,0)$ and $\lambda_\varepsilon : t \mapsto \sigma(t,t)$ gives $\sqrt{(E + 2F + G)(0,0)}$, so knowing E and G, we get F.

So if $f : U \to U'$ is a local isometry, take any allowable parametrization $\sigma' : V \to U'$. Then $\sigma = f^{-1} \circ \sigma'$ is such that the first fundamental forms of σ and σ' agree.

Example 3.3.

Take a cone: with $u > 0$ and $v \in (0, 2\pi)$, we get

$$\sigma(u, v) = (au\cos v, au\sin v, u).$$

This parametrizes the complement of one line of the cone. The first fundamental form is $(1 + a^2)\,\mathrm{d}u^2 + a^2u^2\,\mathrm{d}v^2$.

Cutting open the cone and unfolding it, we get a plane sector of angle $\theta_0 = \frac{2\pi a}{\sqrt{1+a^2}}$. Parametrize the plane sector by

$$\sigma(r, \theta) = \left(\sqrt{1+a^2}r\cos\left(\frac{a\theta}{\sqrt{1+a^2}}\right), \sqrt{1+a^2}r\sin\left(\frac{a\theta}{\sqrt{1+a^2}}\right), 0\right).$$

This holds for $r > 0$ and $\theta \in (0, 2\pi)$. We can check that the first fundamental form is $(1 + a^2)\,\mathrm{d}r^2 + r^2a^2\,\mathrm{d}\theta^2$. So the cone is locally isometric to the plane.

Lemma 3.3. *Let $\sigma, \tilde{\sigma}$ be allowable parametrizations with transition map $f : \tilde{\sigma}^{-1} \circ \sigma$, with first fundamental forms*

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}, \qquad \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix}.$$

Then these are related by

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = (Df)^T \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} Df.$$

Proof: We have

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \sigma_u \cdot \sigma_u & \sigma_u \cdot \sigma_v \\ \sigma_v \cdot \sigma_u & \sigma_v \cdot \sigma_v \end{pmatrix} = (D\sigma)^T D\sigma.$$

Since $\sigma = \tilde{\sigma} \circ f$, we have $D\sigma = D\tilde{\sigma} Df$, so this equals

$$(D\tilde{\sigma}Df)^T(D\tilde{\sigma}Df) = (Df)^T(D\tilde{\sigma})^T D\tilde{\sigma}Df.$$

3.1 Angles

For two vectors $v, w \in \mathbb{R}^3$, we can define the angle between them as $\theta \in [0, \pi]$ satisfying $v \cdot w = |v||w|\cos\theta$.

Similarly, if $v, w \in T_p\Sigma$, we can define

$$\cos\theta = \frac{v \cdot w}{|v||w|}.$$

Here $w = D\sigma|_0(w_0)$, $v = D\sigma|_0(v_0)$. We can therefore easily calculate

$$v \cdot w = v_0^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} w_0.$$

So therefore, we can compute the angles using the first fundamental form of σ.

Lemma 3.4. *σ is* conformal *(preserves angles) exactly when $E = G$, and $F = 0$.*

Proof: Consider curves $\alpha(t) = (u(t), v(t))$ in V, and $\tilde{\alpha}(t) = (\tilde{u}(t), \tilde{v}(t))$, satisfying $\alpha(0) = \tilde{\alpha}(0) \in V$.

The curves $\sigma \circ \alpha$ and $\sigma \circ \tilde{\alpha}$ meet at p with angle θ, given by

$$\cos\theta = \frac{E\dot{u}\dot{\tilde{u}} + F(\dot{u}\dot{\tilde{v}} + \dot{\tilde{u}}\dot{v}) + G\dot{v}\dot{\tilde{v}}}{(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)^{1/2}(E\dot{\tilde{u}}^2 + 2F\dot{\tilde{u}}\dot{\tilde{v}} + G\dot{\tilde{v}}^2)^{1/2}}.$$

If σ is conformal at $\alpha(t) = (t, 0)$, $\tilde{\alpha}(t) = (0, t)$ meeting at angle $\pi/2$ in V we get that

$$0 = F.$$

Similarly, using $\alpha(t) = (t, t)$ and $\tilde{\alpha}(t) = (t, -t)$, we get $E = G$.

Conversely, if σ is such that $E = G$ and $F = 0$, then with respect to σ, the first fundamental form is just

$$\rho(\mathrm{d}u^2 + \mathrm{d}v^2).$$

Hence, we get

$$\cos\theta = \frac{\dot{u}\dot{\tilde{u}} + \dot{v}\dot{\tilde{v}}}{(\dot{u}^2 + \dot{v}^2)^{1/2}(\dot{\tilde{u}}^2 + \dot{\tilde{v}}^2)^{1/2}}.$$

This shows that angles do not change.

3.2 Areas

Recall that the area of a parallelogram spanned by vectors v and w is

$$|v \times w| = (|v|^2|w|^2 - (v \cdot w)^2)^{1/2}.$$

Suppose we have $\sigma : V \to U$, with $\sigma(0) = p$, and consider $\sigma_u, \sigma_v \in T_p\Sigma$.

They span a parallelogram in $T_p\Sigma$ of area

$$(|\sigma_u|^2|\sigma_v|^2 - (\sigma_u \cdot \sigma_v)^2)^{1/2} = \sqrt{EG - F^2}.$$

Definition 3.4. We define the *area* of U as

$$\int_V \sqrt{EG - F^2}\, \mathrm{d}u\, \mathrm{d}v.$$

To show this is independent of parametrization, suppose $\sigma : V \to U$ and $\tilde{\sigma} : \tilde{V} \to U$ are allowable parametrizations. Then if $\varphi = \sigma^{-1} \circ \tilde{\sigma}$ is a transition map, by our previous lemma,

$$\begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} = (D\varphi)^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} D\varphi.$$

Taking the determinants, we get

$$\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2} = |\det(D\varphi)|\sqrt{EG - F^2}.$$

Now the change of variables between V and $\tilde{V}$ gives

$$\int_{\tilde{V}} \sqrt{\tilde{E}\tilde{G} - \tilde{F}^2}\, \mathrm{d}\tilde{u}\, \mathrm{d}\tilde{v} = \int_V \sqrt{EG - F^2}\, \mathrm{d}u\, \mathrm{d}v.$$

This shows that the area of U is intrinsic, and well-defined.

Example 3.4.

Consider the graph

$$\Sigma = \{(u, v, f(u, v)) \mid (u, v) \subset \mathbb{R}^2\}$$

with $f : \mathbb{R}^2 \to \mathbb{R}$ smooth, and with obvious parametrization $\sigma(u, v) = (u, v, f(u, v))$. Then $\sigma_u = (1, 0, f_u)$, $\sigma_v = (0, 1, f_v)$. This gives

$$EG - F^2 = 1 + f_u^2 + f_v^2.$$

Therefore, if $U_R \subset \Sigma$ is $\sigma(B(0, R))$, then

$$\operatorname{area}(U_R) = \int_{B(0,R)} \sqrt{1 + f_u^2 + f_v^2}\, \mathrm{d}u\, \mathrm{d}v \geq \pi R^2,$$

with equality only if $f_u = f_v = 0$, i.e. f is constant.

Hence the projection from Σ to $\mathbb{R}^2_{xy}$ is not area preserving, unless Σ is a plane parallel to $\mathbb{R}^2_{xy}$.

Contrast this to the theorem from Archimedes, which says the horizontal radius projection (with centre the z-axis) from S^2 to the cylinder is area-preserving.

3.3 Second Fundamental Form

Let's try to measure how much $\Sigma \subset \mathbb{R}^3$ deviates from its own tangent planes. Taking $\sigma : V \to U \subset \Sigma$ and using Taylor's theorem

$$\begin{aligned}\sigma(u+h, v+l) &= \sigma(u,v) + h\sigma_u(u,v) + l\sigma_v(u,v) \\ &\quad + \frac{1}{2}[h^2\sigma_{uu}(u,v) + 2hl\sigma_{uv}(u,v) + l^2\sigma_{vv}(u,v)] + \mathcal{O}(h^3, l^3),\end{aligned}$$

where (h,l) are small enough so that (u,v) and $(u+h, v+l) \in V$.

Taking a projection in the normal direction,

$$\langle n, \sigma(u+h, v+l) - \sigma(u,v)\rangle = \frac{1}{2}[\langle n, \sigma_{uu}\rangle h^2 + 2\langle n, \sigma_{uv}\rangle hl + \langle n, \sigma_{vv}\rangle l^2] + \mathcal{O}(h^3, l^3).$$

Definition 3.5. The *second fundamental form* of $\Sigma \subset \mathbb{R}^3$ in the parametrization σ is the quadratic form:

$$L\,\mathrm{d}u^2 + 2M\,\mathrm{d}u\,\mathrm{d}v + N\,\mathrm{d}v^2,$$

where

$$\begin{aligned} L &= \langle n, \sigma_{uu}\rangle, \\ M &= \langle n, \sigma_{uv}\rangle, \\ N &= \langle n, \sigma_{vv}\rangle, \end{aligned}$$

and

$$n = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}.$$

Lemma 3.5. *Let V be connected and $\sigma : V \to U \subset \Sigma$ such that the second fundamental form vanishes identically. Then U lies in an affine plane in $\mathbb{R}^3$.*

Proof: Recall that $\langle n, \sigma_u\rangle = \langle n, \sigma_v\rangle = 0$. Hence, differentiating,

$$\begin{aligned} \langle n_u, \sigma_u\rangle + \langle n, \sigma_{uu}\rangle &= 0, \\ \langle n_v, \sigma_v\rangle + \langle n, \sigma_{vv}\rangle &= 0, \\ \langle n_v, \sigma_u\rangle + \langle n, \sigma_{uv}\rangle &= 0. \end{aligned}$$

Hence we can alternatively write

$$\begin{aligned} L &= \langle n, \sigma_{uu} \rangle = -\langle n_u, \sigma_u \rangle, \\ M &= \langle n, \sigma_{uv} \rangle = -\langle n_v, \sigma_u \rangle = -\langle n_u, \sigma_v \rangle, \\ N &= \langle n, \sigma_{vv} \rangle = -\langle n_v, \sigma_v \rangle. \end{aligned}$$

So if the second fundamental form vanishes, then n_u is orthogonal to σ_u and σ_v. But we also know that $\langle n, n \rangle = 1$, so n_u is orthogonal to $\{n, \sigma_u, \sigma_v\}$. However, these form a basis, so $n_u = 0$. Similarly, $n_v = 0$.

So n is constant (as V is connected, using the mean value inequality). This implies that $\langle \sigma, n \rangle$ is constant, and U is contained in a plane as desired.

Recall that the first fundamental form in the parametrization σ was

$$(D\sigma)^T D\sigma = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \sigma_u \cdot \sigma_u & \sigma_u \cdot \sigma_v \\ \sigma_v \cdot \sigma_u & \sigma_v \cdot \sigma_v \end{pmatrix}.$$

Analogously, the second fundamental form can be written as

$$-(Dn)^T D\sigma = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} n_u \cdot \sigma_u & n_u \cdot \sigma_v \\ n_v \cdot \sigma_u & n_v \cdot \sigma_v \end{pmatrix},$$

using the alternative expressions for L, M and N in the previous proof.

So if $\sigma : V \to U$, $\tilde{\sigma} : \tilde{V} \to U$ are two parametrizations with transition map $\varphi : \tilde{V} \to V$ with $\varphi = \sigma^{-1} \circ \tilde{\sigma}$, then

$$n_{\tilde{\sigma}}(\tilde{u}, \tilde{v}) = \pm n_\sigma(\varphi(\tilde{u}, \tilde{v})),$$

where the sign depends on $\det(D\varphi)$, due to out discussion on orientability. Hence

$$\begin{pmatrix} \tilde{L} & \tilde{M} \\ \tilde{M} & \tilde{N} \end{pmatrix} = -(Dn_{\tilde{\sigma}})^T D\tilde{\sigma} = \pm (D\varphi)^T \begin{pmatrix} L & M \\ M & N \end{pmatrix} D\varphi.$$

Example 3.5.

The cylinder has $\sigma(u, v) = (a \cos u, a \sin u, v)$. Note that $\sigma_{uv} = \sigma_{vv} = 0$, so $M = N$.

We can check that $\sigma_{uu} = (-a \cos u, -a \sin u, 0)$ and $n = (\cos u, \sin u, v)$, so

$L = -a$ and the second fundamental form is

$$\begin{pmatrix} -a & 0 \\ 0 & 0 \end{pmatrix}.$$

Definition 3.6. Let $\Sigma \subset \mathbb{R}^3$ be a smooth oriented surface. The *Gauss map* is

$$n : \Sigma \to S^2$$

is the map $p \to n(p)$, where $n(p)$ is the unit normal vector at p, defined by the orientation of Σ.

Lemma 3.6. *The Gauss map $n : \Sigma \to S^2$ is smooth.*

Proof: Smoothness can be checked locally. If $\sigma : V \to U$ is allowable and compatible with the orientation, then at $\sigma(u, v) = p \in \Sigma$ satisfies

$$n(\sigma(u, v)) = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|},$$

where $n \circ \sigma : V \to S^2$ is smooth, since σ is.

Note that $T_p\Sigma = T_{n(p)}S^2$. Thus, we can view

$$Dn|_p : T_p\Sigma \to T_{n(p)}S^2 = T_p\Sigma,$$

as the differential of the Gauss map. We can also view $Dn|_p$ acting on tangent vectors in terms of curves: if $\gamma : (-\varepsilon, \varepsilon) \to \Sigma$ has $\gamma(0) = p$ and $\gamma'(0) = v$, then

$$Dn|_p(v) = Dn|_p(\gamma'(0)) = (n \circ \gamma)'(0).$$

Recall the first fundamental form $\mathrm{I}_p : T_p\Sigma \times T_p\Sigma \to \mathbb{R}$, by

$$\mathrm{I}_p(v, w) = \langle v, w \rangle_{\mathbb{R}^3}.$$

Lemma 3.7. *$D_n|_p : T_p\Sigma \to T_p\Sigma$ is self-adjoint with respect to I_p, i.e.*

$$\mathrm{I}_p(Dn|_p(v), w) = \mathrm{I}_p(v, Dn|_p(w)).$$

Proof: Take σ a parametrization with $\sigma(0) = p$. Then $\{\sigma_u, \sigma_v\}$ is a basis of $T_p\Sigma$.

To prove the map is self-adjoint, it suffices to check that

$$\langle Dn|_p(\sigma_u), \sigma_v\rangle = \langle n_u, \sigma_v\rangle \overset{*}{=} \langle \sigma_u, n_v\rangle = \langle \sigma_u, Dn|_p(\sigma_v)\rangle.$$

This is true, as if we parametrize n as $n(\sigma(u,v))$, then

$$n_u = Dn|_p(\sigma_u), \qquad n_v = Dn|_p(\sigma_v).$$

Now since

$$\langle n, \sigma_u\rangle = \langle n, \sigma_v\rangle = 0,$$

differentiating the first with respect to v gives

$$\langle n_v, \sigma_u\rangle + \langle n, \sigma_{uv}\rangle = 0,$$

and differentiating the second with respect to u gives

$$\langle n_u, \sigma_v\rangle + \langle u, \sigma_{vu}\rangle = 0.$$

Let's try to find the matrix $Dn|_p$ in the basis $\{\sigma_u, \sigma_v\}$:

$$\begin{aligned} n_u &= Dn|_p(\sigma_u) = a_{11}\sigma_u + a_{21}\sigma_v, \\ n_v &= Dn|_p(\sigma_v) = a_{12}\sigma_u + a_{22}\sigma_v. \end{aligned}$$

Taking the product of these equations with σ_u and σ_v, we find

$$-\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

or if Q is the second fundamental form matrix, P is the first fundamental form matrix, then our matrix satisfies

$$Q = -PA = -A^T P.$$

If $v = D\sigma|_0(\hat{v})$ and $w = D\sigma|_0(\hat{w})$, then

$$\begin{aligned} \hat{v}^T \begin{pmatrix} L & M \\ M & N \end{pmatrix} \hat{w} &= -v^T \begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} w \\ &= \mathrm{I}_p(v, -Dn|_p(w)) = \mathrm{I}_p(-Dn|_p(v), w). \end{aligned}$$

Then the second fundamental form has an intrinsic form given by the symmetric bilinear form

$$\mathrm{I\!I}_p : T_p\Sigma \times T_p\Sigma \to \mathbb{R},$$

given by

$$\mathrm{I\!I}_p(v, w) = \mathrm{I}_p(-Dn|_p(v), w).$$

Definition 3.7. Let $\Sigma \subset \mathbb{R}^3$ be a smooth surface. The *Gauss curvature* $K : \Sigma \to \mathbb{R}$ of Σ is the function

$$p \mapsto \det(Dn|_p).$$

Remark. This is always well-defined, even if Σ is not oriented; we can always choose a local expression of n. If we replace it by $-n$, the determinant will not change (since $Dn|_p$ is two-dimensional).

If we pick σ, then using the formula

$$-\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

and taking the determinant, we get

$$LN - M^2 = (EG - F^2)K, \implies K = \det(A) = \frac{LN - M^2}{EG - F^2}.$$

Example 3.6.

We look at a cylinder again. We have already calculated the second fundamental form in $\sigma(u, v) = (a \cos u, a \sin u, v)$ to be

$$\begin{pmatrix} -a & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence since the determinant is 0, we get $K(p) = 0$ for all p.

Indeed, the Gauss map is from Σ to the equator of S^2, so if $\gamma : (-\varepsilon, \varepsilon) \to \Sigma$ is a vertical curve, then $Dn|_p(\gamma'(0)) = (n \circ \gamma)'(0) = 0$, hence $\det(Dn|_p) = 0$.

Definition 3.8. Σ is said to be *flat* if $K = 0$ on Σ.

Example 3.7.

If Σ is a graph of a smooth function f, then it is easy to check that

$$E = 1 + f_u^2,\ G = 1 + f_v^2,\ F = f_u f_v,\ EG - F^2 = 1 + f_u^2 + f_v^2,$$

$$L = \frac{f_{uu}}{\sqrt{EG - F^2}},\ M = \frac{f_{uv}}{\sqrt{EG - F^2}},\ N = \frac{f_{uv}}{\sqrt{EG - F^2}},$$

$$K = \frac{f_{uu} f_{vv} - f_{uv}^2}{(1 + f_u^2 + f_v^2)^2}.$$

So the curvature depends on the Hessian of f.

Definition 3.9. Let $\Sigma \subset \mathbb{R}^3$, $p \in \Sigma$. We say that p is:

- *elliptic* if $K(p) > 0$,
- *hyperbolic* if $K(p) < 0$,
- *parabolic* if $K(p) = 0$.

Example 3.8.

1. Looking at the graph $f(u,v) = \frac{u^2+v^2}{2}$, then at $(0,0)$,
$$K(0,0,0) = 1.$$
Hence the point is elliptic.
2. Looking at the graph $f(u,v) = \frac{u^2-v^2}{2}$, at $(0,0)$,
$$K(0,0,0) = -1.$$
Hence the point is hyperbolic.

Lemma 3.8.

(a) *In a sufficiently small neighbourhood of an elliptic point p, Σ lies entirely on one side of the affine tangent plane $p + T_p\Sigma$.*

(b) *In a sufficiently small neighbourhood of a hyperbolic, Σ meets both sides of its affine tangent plane.*

Proof: Take a parametrization σ. Near p,
$$K = \frac{LN - M^2}{EG - F^2},$$
and $EG - F^2 > 0$. Recall also that if
$$w = n\sigma_n + l\sigma_v \in T_p\Sigma,$$
then $\frac{1}{2}\mathrm{II}_p(w,w)$ measured the distance from $\sigma(h,l)$ to $p + T_p\Sigma$ measured in the inner product space with positive normal:
$$\frac{1}{2}(Lh^2 + 2Mhl + Nl^2) = \mathcal{O}(h^3, l^3).$$
If p is elliptic, then
$$\begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

has eigenvalues of the same sign, so it positive or negative definite at p. Hence in a neighbourhood of p, this signed distance only has one sign locally.

If p is hyperbolic, then $\mathbb{I}_p$ is indefinite, so Σ meets both sides of $p + T_p\Sigma$.

Remark. If p is parabolic, we cannot conclude either.

Take a cylinder. Then all points lie on the side of the tangent plane.

But if we take parametrization $\sigma(u, v) = (u, v, u^3 - 3v^2u)$, then at $p = \sigma(0, 0)$, $K(p) = 0$, but locally Σ meets both sides of its tangent plane (this is known as the *Monkey saddle*).

Proposition 3.1. *Let Σ be a compact surface in $\mathbb{R}^3$. Then Σ has an elliptic point.*

Proof: Σ compact means $\Sigma \in \overline{B(0, R}$, for some R large enough. Decrease R to the minimal such value, so that some point p is on the boundary of $B(0, R)$.

Up to applying a rotation and translation, we may assume that the point of contact is on the z-axis.

Locally near p, we can view Σ as the graph of a smooth function f, such that

$$f - \sqrt{R^2 - u^2 - v^2} \leq 0.$$

We have $f : V \to \mathbb{R}$, with V open in $\mathbb{R}^2$. Since f has a local maximum at $(0, 0)$, $f_v = f_u = 0$ at $(0, 0)$. Now let

$$F(u, v) = f(u, v) - \sqrt{R^2 - u^2 - v^2} \leq 0.$$

An easy computation shows that $F_u = F_v = 0$ at $(0, 0)$. Moreover,

$$\begin{aligned} F_{uu} &= f_{uu} + 1/R, \\ F_{uv} &= f_{uv}, \\ F_{vv} &= f_{vv} + 1/R, \end{aligned}$$

hence by a Taylor expansion and using the fact that 0 is a local maximum

$$(f_{uu} + 1/R)h^2 + 2f_{uv}hl + (f_{vv} + 1/R)l^2 \leq 0,$$

for small enough h, l. Rearranging,

$$f_{uu}h^2 + 2f_{uv}hl + f_{vv}l^2 \leq -\frac{1}{R}(h^2 + l^2),$$

hence we get
$$\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} f_{uu} & f_{uv} \\ f_{uv} & f_{vv} \end{pmatrix}$$
is negative definite at $(0,0)$. Since at $(0,0)$, $E = G = 1$ and $F = 0$, $K(p) > 0$.

Theorem 3.1. *Let $\Sigma \subset \mathbb{R}^3$, $p \in \Sigma$ with $K(p) \neq 0$. Let $U \subset \Sigma$ be a small open neighbourhood of p.*

Consider a sequence $p \in A_i \subset U \subset \Sigma$, such that the A_i "shrink" to p, in the sense that for all $\varepsilon > 0$, $A_i \subset B(p, \varepsilon)$, for all i large enough.

Then,
$$|K(p)| = \lim_{i \to \infty} \frac{\mathrm{area}_{S^2}(n(A_i))}{\mathrm{area}_{\Sigma}(A_i)},$$
i.e. the Gauss curvature is an infinitesimal measure of how much the Gauss map n distorts areas.

Proof: This is all local, so take $\sigma : V \to U$ with $\sigma(0) = p$, and let $V_i = \sigma^{-1}(A_i) \subset V$ be open. Since A_i shrink to p,
$$\bigcap_{i \geq 1} V_i = \{(0,0)\}.$$
Moreover we have
$$\begin{aligned} \mathrm{area}_{\Sigma}(A_i) &= \int_{V_i} \sqrt{EG - F^2} \, \mathrm{d}u \, \mathrm{d}v \\ &= \int_{V_i} \|\sigma_u \times \sigma_v\| \, \mathrm{d}u \, \mathrm{d}v. \end{aligned}$$
Now $n \circ \sigma : V \to S^2 \subset \mathbb{R}^2$ has differential $Dn|_p \circ D\sigma|_0$, which has rank 2 since $K(p) \neq 0$.

Thus $n \circ \sigma$ defines an allowable parametrization in an open neighbourhood of $n(p) \subset S^2$ by the inverse function theorem. Thus,
$$\mathrm{area}_{S^2}(n(A_i)) = \int_{V_i} \|n_u \times n_v\| \, \mathrm{d}u \, \mathrm{d}v,$$
as $\|n_u \times n_v\| = \|Dn(\sigma_u) \times Dn(\sigma_v)\|$. Recall from the last lecture that
$$\begin{aligned} Dn(\sigma_u) &= a_{11}\sigma_u + a_{21}\sigma_v \\ Dn(\sigma_v) &= a_{12}\sigma_u + a_{22}\sigma_v \end{aligned}$$

hence

$$\begin{aligned} Dn(\sigma_u) \times Dn(\sigma_v) &= (a_{11}\sigma_u + a_{21}\sigma_v) \times (a_{12}\sigma_u + a_{22}\sigma_v) \\ &= (a_{11}a_{22} - a_{12}a_{21})(\sigma_u \times \sigma_v) = K(p)(\sigma_u \times \sigma_v), \end{aligned}$$

so

$$\begin{aligned} \mathrm{area}_{S^2}(n(A_i)) &= \int_{V_i} \|n_u \times n_v\| \,\mathrm{d}u\,\mathrm{d}v = \int_{V_i} |\det Du| \|\sigma_u \times \sigma_v\| \,\mathrm{d}u\,\mathrm{d}v \\ &= \int_{V_i} |K(u,v)| \|\sigma_u \times \sigma_v\| \,\mathrm{d}u\,\mathrm{d}v. \end{aligned}$$

Since K is continuous, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|K(u,v) - K(0,0)| < \varepsilon$ for all $(u,v) \in B((0,0),\delta) \subset V$. So if $i \geq i_0$, we have

$$|K(p)| - \varepsilon \leq |K(u,v)| \leq |K(p)| + \varepsilon.$$

Hence,

$$\begin{aligned} (|K(p)| - \varepsilon) \int_{V_i} \|\sigma_u \times \sigma_v\| \,\mathrm{d}u\,\mathrm{d}v &\leq \int_{V_i} |K(u,v)| \|\sigma_u \times \sigma_v\| \,\mathrm{d}u\,\mathrm{d}v \\ &\leq (|K(p)| + \varepsilon) \int_{V_i} \|\sigma_u \times \sigma_v\| \,\mathrm{d}u\,\mathrm{d}v, \end{aligned}$$

which when dividing gives

$$|K(p)| - \varepsilon \leq \frac{\mathrm{area}_{S^2}(n(A_i))}{\mathrm{area}_{\Sigma}(A_i)} \leq |K(p)| + \varepsilon,$$

for all $i \geq i_0$.

The Gauss curvature is constraint by two theorems. The first is called the "theorem egrigium" (meaning "remarkable theorem").

Theorem 3.2. *The Gauss curvature of a smooth surface in $\mathbb{R}^3$ is isometry invariant, i.e. if $f : \Sigma_1 \to \Sigma_2$ is an isometry, then*

$$K_1(p) = K_2(f(p)),$$

for all $p \in \Sigma$.

In fact K can be computed exclusively in terms of I_p even though it was defined using I_p and II_p.

Another important theorem is the following:

Theorem 3.3 (Gauss-Bonnet theorem)**.** *If Σ is a compact smooth surface in $\mathbb{R}^3$, then*

$$\int_\Sigma K \, \mathrm{d}A_\Sigma = 2\pi\chi(\Sigma),$$

where χ is the Euler-characteristic.

This provides a link between the (local) geometry of the surface, and the (global) topology of the surface.

Proofs for these are found in part II differential geometry.

4 Geodesics

Recall that if $\gamma : [a, b] \to \mathbb{R}^3$ is smooth, then

$$\text{length}(\gamma) = L(\gamma) = \int_a^b |\gamma'(t)| \,\mathrm{d}t.$$

Definition 4.1. The *energy* of γ is

$$E(\gamma) = \int_a^b |\gamma'(t)|^2 \,\mathrm{d}t.$$

If we let Ω_{pq} be the set of all curves $\gamma : [a, b] \to \mathbb{R}^3$ with $\gamma(a) = p$, $\gamma(b) = q$, then the energy is a map

$$E : \Omega_{pq} \to \mathbb{R}.$$

In fact we really want to generalize this to surfaces $\Sigma \subset \mathbb{R}^3$ and curves $\gamma : [a, b] \to \Sigma$.

Definition 4.2. Let $\gamma : [a, b] \to \Sigma \subset \mathbb{R}^3$ be smooth. A *one-parameter variation* (with fixed endpoints) of γ is a smooth map $\Gamma : (-\varepsilon, \varepsilon) \times [a, b] \to \Sigma$ such that, if $\gamma_s = \Gamma(s, \cdot)$, then

(a) $\gamma_0(t) = \gamma(t)$ for all t.

(b) $\gamma_s(a)$, $\gamma_s(b)$ are independent of s.

Definition 4.3. A smooth curve $\gamma : [a, b] \to \Sigma$ is a *geodesic* if for any variation γ_s of γ with fixed endpoints, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}s} E(\gamma_s)\right|_{s=0} = 0,$$

i.e. γ is a critical point of the energy function on curves from $\gamma(a)$ to $\gamma(b)$.

Suppose γ has image contained in the image of a parametrization σ. Then we write

$$\gamma_s(t) = \sigma(u(s, t), v(s, t)).$$

Suppose the first fundamental form is $E \,\mathrm{d}u^2 + 2F \,\mathrm{d}u \,\mathrm{d}v + G \,\mathrm{d}v^2$. Then we define

$$R = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2,$$

and the energy is

$$E(\gamma_s) = \int_a^b R \,\mathrm{d}t.$$

We also have

$$\begin{aligned}\frac{\partial R}{\partial s} =&(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2)\frac{\partial u}{\partial s} + (E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2)\frac{\partial v}{\partial s}\\ &+ 2(E\dot{u} + F\dot{v})\frac{\partial \dot{u}}{\partial s} + 2(F\dot{u} + G\dot{v})\frac{\partial \dot{v}}{\partial s},\end{aligned}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}s}E(\gamma_s) = \int_a^b \frac{\partial R}{\partial s}\,\mathrm{d}t.$$

Note that

$$\frac{\partial \dot{u}}{\partial s} = \frac{\partial^2 u}{\partial s \partial t}, \qquad \frac{\partial \dot{v}}{\partial s} = \frac{\partial^2 v}{\partial s \partial t},$$

and so integrating by parts and noticing that u_s, v_s vanish at the endpoints a and b, we get

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}E(\gamma_s)\right|_{s=0} = \int_a^b \left[A\frac{\partial u}{\partial s} + B\frac{\partial v}{\partial s}\right]\mathrm{d}t,$$

where

$$\begin{aligned}A &= E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2 - 2\frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v}),\\ B &= E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2 - 2\frac{\mathrm{d}}{\mathrm{d}t}(F\dot{u} + G\dot{v}).\end{aligned}$$

Note that we have absolute freedom in choosing the variational vector field

$$w(t) = \left(\frac{\partial u}{\partial s}(0,t), \frac{\partial v}{\partial s}(0,t)\right).$$

Hence we see that γ is geodesic if and only if $A = B = 0$. Thus γ is a geodesic if and only $\gamma(t) = \sigma(u(t), v(t))$ satisfies the *geodesic equations*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v}) &= \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2),\\ \frac{\mathrm{d}}{\mathrm{d}t}(F\dot{u} + G\dot{v}) &= \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2).\end{aligned}$$

Remark.

1. If $w(t)$ has $w(a) = w(b) = 0$, then

$$\gamma_s(t) = \gamma(u(t), v(t) + sw(t))$$

for s small enough is a variation of γ with fixed end points and variational vector field w.

2. Recall from IA Analysis example sheets that if
$$\int_a^b f(x)g(x)\,\mathrm{d}x = 0,$$
for all $g : [a,b] \to \mathbb{R}$ with $g(a) = g(b) = 0$, then $f = 0$. We are using exactly this to obtain the geodesic equations.

Probably the best way to think about these is via the *Euler-Lagrange* equations of the *Lagrangian*:
$$L(u,v,\dot{u},\dot{v}) = \frac{1}{2}(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2).$$
This is the purely kinetic energy. Recall from Variational Principles that the Euler-Lagrange equations are:
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i},$$
and here $q_1 = u$, $q_2 = v$.

Proposition 4.1. *Let $\Sigma \subset \mathbb{R}^3$ be a smooth surface. A smooth curve $\gamma : [a,b] \to \Sigma$ is a geodesic if and only if $\ddot{\gamma}(t)$ is everywhere normal to Σ.*

Proof: As the statement is purely local, we can work in a parametrization $\sigma : V \to U \subset \Sigma$, and as usual, $\gamma(t) = \sigma(u(t), v(t))$. Then,
$$\dot{\gamma}(t) = \sigma_u \dot{u} + \sigma_v \dot{v},$$
so $\ddot{\gamma}(t)$ is normal to Σ exactly when it is orthogonal to $T_{\gamma(t)}\Sigma$, which is spanned by $\{\sigma_u, \sigma_v\}$. In other words,
$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}(\sigma_u\dot{u} + \sigma_v\dot{v}), \sigma_u \right\rangle = 0,$$
$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}(\sigma_u\dot{u} + \sigma_v\dot{v}), \sigma_v \right\rangle = 0.$$
The first statement is equivalent to
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \sigma_u\dot{u} + \sigma_v\dot{v}, \sigma_u\rangle - \langle \sigma_u\dot{u} + \sigma_v\dot{v}, \frac{\mathrm{d}}{\mathrm{d}t}\sigma_u\rangle = 0.$$
Noting that $E = \langle \sigma_u, \sigma_v\rangle$ and $F = \langle \sigma_u, \sigma_v\rangle$, this is exactly
$$\frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v}) - \langle \sigma_u\dot{u} + \sigma_v\dot{v}, \sigma_{uu}\dot{u} + \sigma_{uv}\dot{v}\rangle = 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v}) - \dot{u}^2\langle \sigma_u, \sigma_{uu}\rangle + \dot{u}\dot{v}(\langle \sigma_u, \sigma_{uv}\rangle + \langle \sigma_v, \sigma_{uv}\rangle) + \dot{v}^2\langle \sigma_v, \sigma_{uv}\rangle.$$

But,

$$\begin{aligned} E = \langle \sigma_u, \sigma_v \rangle &\implies E_u = 2\langle \sigma_u, \sigma_{uu} \rangle, \\ F = \langle \sigma_u, \sigma_v \rangle &\implies F_u = \langle \sigma_u, \sigma_{uv} \rangle + \langle \sigma_v, \sigma_{uu} \rangle, \\ G = \langle \sigma_v, \sigma_v \rangle &\implies G_u = 2\langle \sigma_v, \sigma_{uv} \rangle, \end{aligned}$$

Thus the first equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v}) = \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2),$$

which is the first geodesic equation. Similarly, the second equation becomes the second geodesic equation.

Corollary 4.1. *If $\gamma : [a, b] \to \Sigma$ is a geodesic, then $|\dot{\gamma}(t)|$ is constant.*

Proof: We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \dot{\gamma}, \dot{\gamma} \rangle = 2\langle \ddot{\gamma}, \dot{\gamma} \rangle = 0,$$

since $\ddot{\gamma}$ is perpendicular to $T_{\gamma(t)}\Sigma$, and $\dot{\gamma}(t) \in T_{\gamma(t)}\Sigma$.

Thus geodesics are parametrized with *constant speed* (i.e proportional to arc-length).

4.1 Length versus Energy

Energy is sensitive to parametrization. Given $\gamma : [a, b] \to \mathbb{R}^3$ smooth, we always have:

$$(L(\gamma))^2 \leq (b - a)E(\gamma),$$

with equality if and only if $|\dot{\gamma}|$ is constant.

Indeed, using Cauchy-Schwarz,

$$\left(\int_a^b |\dot{\gamma}(t)| \,\mathrm{d}t\right)^2 \leq \left(\int_a^b |\dot{\gamma}(t)|^2 \,\mathrm{d}t\right)\left(\int_a^b 1 \,\mathrm{d}t\right),$$

with equality if and only if $|\dot{\gamma}|$ is constant.

Corollary 4.2. *A smooth curve $\gamma : [a, b] \to \Sigma \subset \mathbb{R}^3$ that minimizes length and has constant speed is a geodesic.*

Proof: We need to prove that γ is a critical point of E. Let $\tau : [a,b] \to \Sigma$ be any other curve connected $\gamma(a)$ and $\gamma(b)$. Then

$$E(\gamma) = \frac{(L(\gamma))^2}{b-a} \leq \frac{(L(\tau))^2}{b-a} \leq E(\tau),$$

hence γ is critical for E and hence a geodesic.

A geodesic may not be a global minimizer, but they are always local minimizers.

Example 4.1.

1. The plane $\mathbb{R}^2$. Let $\sigma(u,v) = (u,v,0)$. Then the first fundamental form is $\mathrm{d}u^2 + \mathrm{d}v^2$. The geodesic equations are

   $$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{u}) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}(\dot{v}) = 0.$$

   So $u(t) = \alpha t + \beta$, $v(t) = \gamma(t) + \delta$, which is a straight line parametrized by a constant speed.

2. Take the unit sphere with σ given by spherical coordinates. Then

   $$\begin{aligned} \sigma(\phi,\theta) &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \\ \sigma_\phi &= (-\sin\theta, \sin\phi, \sin\theta\cos\phi, 0), \\ \sigma_\theta &= (\cos\theta\cos\phi, -\cos\theta\sin\phi, -\sin\theta). \end{aligned}$$

   Then $E = \sin^2\theta$, $F = 0$ and $G = 1$, so

   $$L = \frac{1}{2}(\sin^2\theta\dot{\phi}^2 + \dot{\theta}^2).$$

   This is the Lagrangian $L(\theta, \phi, \dot{\theta}, \dot{\phi})$. From the Euler-Lagrange equations, we get

   $$\frac{\mathrm{d}}{\mathrm{d}t}(\sin^2\dot{\phi}) = 0, \qquad \ddot{\theta} = \sin\theta\cos\theta\dot{\phi}^2.$$

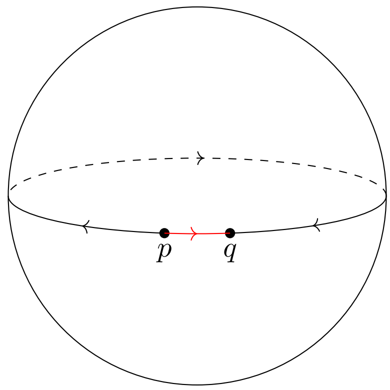
   These equations give right away that the equator $t \to (t, \pi/2)$ is a geodesic. In fact, all great circles parametrized with constant speed are geodesics.

   We can prove this by integrating the Euler-Lagrange equations, but we can see this geometrically by noticing that such curves have $\ddot{\gamma}$ perpendicular to $T_{\gamma(t)}S^2$.

Since geodesics solve a second order ODE, prescribing $v \in T_p\Sigma$ determines the geodesic completely. Thus great circles are all possible geodesics.

Note that the larger great circle γ between p and q does not minimize the length.

Figure 10: Geodesic not minimizing length



We now look at an important example for geodesics: a surface of revolution. We take

$$\eta(u) = (f(u), 0, g(u))$$

in the xz-plane, and rotate about the z-axis. Here $\eta : [a, b] \to \mathbb{R}^3$ is smooth, injective with $\eta' \neq 0$ and $f > 0$. Taking the usual parametrization

$$\sigma(u, v) = (f(u)\cos v, f(u)\sin v, g(u)),$$

with $a < u < b$ and $v \in (0, 2\pi)$, the FFF is

$$(f'^2 + g'^2)\,\mathrm{d}u^2 + f^2\,\mathrm{d}v^2.$$

We assume η is parametrized by arc-length. So the FFF becomes

$$\mathrm{d}u^2 + f^2\,\mathrm{d}v^2.$$

The Lagrangian for geodesics is

$$L = \frac{1}{2}(\dot{u}^2 + f^2\dot{v}^2).$$

The Euler-Lagrange equations for u give

$$\frac{\partial L}{\partial u} = ff'\dot{v}^2, \qquad \frac{\partial L}{\partial \dot{u}} = \dot{u} \implies \ddot{u} = ff'\dot{v}^2.$$

Doing the same for v gives
$$\frac{\mathrm{d}}{\mathrm{d}t}(f^2\dot{v}) = 0.$$
We also know that geodesics move with constant speed, so this gives
$$\dot{u}^2 + f^2\dot{v}^2 = \text{constant}, \qquad f^2\dot{v} = c.$$
As an example, consider meridians: we have $v = v_0$, and if $u(t) = t + u_0$, then $t \mapsto (t + u_0, v_0)$ is a geodesic with speed 1 through (u_0, v_0).

We also have parallels, with $u = u_0$ and $\dot{v} = a/f(u_0)$. Then we see that we need $f'(u_0) = 0$ for these to be geodesics.

Let's look at the conserved quantity $f^2\dot{v}$ in more detail. Suppose γ makes an angle θ with a parallel of radius $\rho = f$. Write as usual: $\gamma = \sigma(u(t), v(t))$. Then,
$$\dot{\gamma} = \sigma_u\dot{u} + \sigma_v\dot{v},$$
and note that σ_v is tangent to the parallel, since
$$\sigma_v = (-f\sin v, f\cos v, 0).$$
Thus,
$$\cos\theta = \frac{\langle \sigma_v, \sigma_u\dot{u} + \sigma_v\dot{v}\rangle}{|\sigma_v||\dot{\gamma}|}.$$
Assume γ is parametrized by arc-length, so $|\dot{\gamma}| = 1$. Using $F = 0$ and $G = f^2$, we get that
$$\cos\theta = \frac{f^2\dot{v}}{f} = f\dot{v},$$
and therefore if γ is a geodesic, then
$$\rho\cos\theta = \text{constant}.$$
This is known as *Clairaut's relation*. This is just another way to write the conservation law arising from $\partial L/\partial v = 0$.

Example 4.2.

Consider an ellipsoid of revolution, with $\rho\cos\theta = c$. Then as $c = \rho_0\cos\theta_0 > 0$, we have $c = \rho\cos\theta \leq \rho$.

This means that γ must move between the region bounded by the parallels of radius c.

Recall Picard's theorem for ODE's: if $I = [t_0 - a, t_0 + a] \subset \mathbb{R}$, $B = \{x \mid \|x - x_0\| \leq b\} \subset \mathbb{R}^n$, and

$$f : I \times B \to \mathbb{R}^n$$

is Lipschitz in the second variable, so

$$\|f(t, x_1) - f(t, x_2)\| \leq K\|x_1 - x_2\|,$$

then

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(t, x(t)), \qquad x(t_0) = x_0$$

has a unique solution in some interval $|t - t_0| < h$. Moreover, if f is smooth, then the solution is smooth and depends smoothly on the initial condition. In our setting, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v}) &= \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2),\\
\frac{\mathrm{d}}{\mathrm{d}t}(F\dot{u} + G\dot{v}) &= \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2),
\end{aligned}$$

and so

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \ddot{u} \\ \ddot{v} \end{pmatrix} = \mathcal{A}(u, v, \dot{u}, \dot{v}).$$

Since the FFF is invertible, we can write the geodesic equations as

$$\begin{aligned}
\ddot{u} &= A(u, v, \dot{u}, \dot{v}),\\
\ddot{v} &= B(u, v, \dot{u}, \dot{v}),
\end{aligned}$$

for some smooth A and B. We can turn this into a first order system by the usual trick: let $\dot{u} = x$, $\dot{v} = y$. Then,

$$\begin{aligned}
\dot{u} &= x, & \dot{v} &= y,\\
\dot{x} &= A(u, v, x, y), & \dot{y} &= B(u, v, x, y).
\end{aligned}$$

So Picard's theorem applies, noting that since A and B are smooth, and a local bound on $\|DA\|$ and $\|DB\|$ gives the Lipschitz conditions. We thus have:

Corollary 4.3. *Let Σ be a smooth surface in $\mathbb{R}^3$. For $p \in \Sigma$ and $v \in T_p\Sigma$, there is $\varepsilon > 0$ and a unique geodesic $\gamma : (-\varepsilon, \varepsilon) \to \Sigma$ with initial conditions $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Moreover, γ depends smoothly on (p, v).*

The local existence of geodesics gives rise to parametrizations with very nice properties.

Fix $p \in \Sigma$, and consider a geodesic arc γ going through p and parametrized by arc-length. For t small enough, let γ_t be the unique geodesic such that:

- $\gamma_t(0) = \gamma(t)$,
- $\gamma_t'(0)$ is orthogonal to $\gamma'(t)$, and has unit length.

Define $\sigma(u, v) = \gamma_v(u)$ for $u \in (-\varepsilon, \varepsilon), v \in (-\delta, \delta)$.

Lemma 4.1. *For ϵ and δ sufficiently small, σ defines an allowable parametrization of an open set in Σ.*

Proof: Smoothness follows from smoothness of geodesics with initial conditions. At $(0, 0)$, σ_u and σ_v are orthogonal and have norm 1 by construction. Thus,

$$D\sigma|_0 : \mathbb{R}^2 \to T_p\Sigma$$

is a linear isomorphism. Now applying the inverse function theorem, we deduce that σ is a local diffeomorphism at $(0, 0)$, and hence for ϵ, δ small enough it is an allowable parametrization.

Proposition 4.2. *Any smooth surface Σ in $\mathbb{R}^3$ admits a local parametrization for which the first fundamental form is of the form*

$$\mathrm{d}u^2 + G\,\mathrm{d}v^2,$$

i.e. $E = 1$, $F = 0$.

Proof: Consider $\sigma(u, v) = \gamma_v(u)$, as above. If we fix v_0, then the curve $u \to \gamma_{v_0}(u)$ is a geodesic parametrized by arc-length. So $E = \langle \sigma_u, \sigma_u \rangle = 1$.

Also one of the geodesic equations is

$$\frac{\mathrm{d}}{\mathrm{d}t}(F\dot{u} + G\dot{v}) = \frac{1}{2}(E_v\dot{u}^2 + F_v\dot{u}\dot{v} + G_v\dot{v}^2),$$

and for $v = v_0$, $u(t) = t$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(F) = 0,$$

giving $F_u = 0$. Hence F is independent of u. But when $u = 0$, then by construction of γ_v as orthogonal to γ at $\gamma(0)$, we have that $F = 0$ everywhere.

Remark.

1. These coordinates are sometimes referred to as the *Fermi coordinates*.
2. $\gamma_t(u)$ for fixed u is typically not a geodesic.
3. In these coordinates, we also have $G(0, v) = 1$, and $G_u(0, v) = 0$. The first holds as σ_v has length 1 at $u = 0$. To show the second, we use that at $u = 0$, $v = t$ is a geodesic and
$$\frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v}) = \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2),$$
which becomes $0 = \frac{1}{2}G_u(0, t)$.
4. One can show that if $E = 1$ and $F = 0$, then the Gauss curvature is given by
$$K = -\frac{(\sqrt{G})_{uu}}{\sqrt{G}}.$$
This is theorem 8.1 in Wilson's book. The computation is not hard, but beyond the scope of the course. However, we will use this result.

4.2 Constant Gaussian Curvature

First a general result. If $\Sigma \in \mathbb{R}^3$ and $f : \mathbb{R}^3 \to \mathbb{R}^3$ is a dilation by $\lambda \neq 0$, i.e.
$$f(x, y, z) = \lambda(x, y, z),$$
then
$$K_{f(\Sigma)} = \frac{1}{\lambda^2}K_\Sigma.$$
To check this, note that the coefficients E, F, G rescale by λ^2, and L, M, N scale by λ.

One natural question to ask is what the constant curvature surfaces look like. By dilations, it suffices to understand surfaces of constant curvature $1, -1$ and 0.

Proposition 4.3. *Let $\Sigma \subset \mathbb{R}^3$ be a smooth surface.*

(a) *If $K_\Sigma = 0$, then Σ is locally isometric to*
$$(\mathbb{R}^2, \mathrm{d}u^2 + \mathrm{d}v^2).$$

(b) *If $K_\Sigma = 1$, then Σ is locally isometric to*
$$(S^2, \mathrm{d}u^2 + \cos^2 u\,\mathrm{d}v^2).$$

Proof: We know that Σ admits a parametrization with $E = 1$, $F = 0$ and $G(0, v) = 1$ and $G_u(0, v) = 0$. Also,

$$K = -\frac{(\sqrt{G})_{uu}}{\sqrt{G}}.$$

If $K = 0$, we get $(\sqrt{G})_{uu} = 0$, so

$$\sqrt{G} = A(v)u + B(v).$$

Our conditions on G gives $B = 1$ and $A = 0$. Then the FFF is $\mathrm{d}u^2 + \mathrm{d}v^2$.

If $K = 1$, then $(\sqrt{G})_{uu} + \sqrt{G} = 0$, so

$$\sqrt{G} = A(v)\sin u + B(v)\cos u.$$

The conditions on G gives $A = 0$ and $B = 1$, so the FFF is $\mathrm{d}u^2 + \cos^2 u \,\mathrm{d}v^2$. In the parametrization,

$$\sigma(u, v) = (\cos u \cos v, \cos u \sin v, \sin u)$$

of S^2, this is the FFF.

Remark. We can certainly do the same for $K = -1$, and we get the FFF is

$$\mathrm{d}u^2 + \cosh^2 u \,\mathrm{d}v^2.$$

A surface of revolution with FFF $\mathrm{d}u^2 + \cosh^2 u \,\mathrm{d}v^2$ is given by rotating

$$\eta(u) = \left(\cosh u, 0, \int_0^u \sqrt{1 - \sinh^2 x} \,\mathrm{d}x\right)$$

This has $f'^2 + g'^2 = 1$ and hence

$$K = -\frac{f''}{f} = -1.$$

Or, we can forget about $\mathbb{R}^3$ and think in more abstract terms. The change of variables

$$V = e^v \tanh u, \qquad W = e^v \mathrm{sech} u$$

turns

$$\mathrm{d}u^2 + \cosh^2 u \,\mathrm{d}v^2 \text{ into } \frac{\mathrm{d}V^2 + \mathrm{d}W^2}{W^2}.$$

This is the *standard presentation* of the hyperbolic plane.

5 Hyperbolic Surfaces

We start by discussing *abstract Riemannian metrics*.

Definition 5.1. Let $V \subset \mathbb{R}^2$ be an open set. An abstract Riemannian metric on V is a smooth map:

$$\begin{aligned} V &\to \{\text{positive definite symmetric forms}\} \subset \mathbb{R}^4, \\ p &\mapsto \begin{pmatrix} E(p) & F(p) \\ F(p) & G(p) \end{pmatrix} = g(p), \end{aligned}$$

where $E > 0, G > 0$ and $EG - F^2 > 0$. If V is a vector at $p \in V$, then its norm is:

$$\|v\|_g^2 = v^T \begin{pmatrix} E(p) & F(p) \\ F(p) & G(p) \end{pmatrix} v,$$

and if $\gamma : [a,b] \to V$ is smooth, then its length is

$$L(\gamma) = \int_a^b \|\dot{\gamma}(t)\|_g \, \mathrm{d}t = \int_a^b (E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)^{1/2} \, \mathrm{d}t,$$

where $\gamma(t) = (u(t), v(t))$.

Definition 5.2. Given (V, g), $(\tilde{V}, \tilde{g})$, we say that they are *isometric* if there exists a diffeomorphism $f : V \to \tilde{V}$ such that

$$\|Df|_p(v)\|_{\tilde{g}} = \|v\|_g,$$

for all $v \in T_pV = \mathbb{R}^2$, and $p \in V$. This is equivalent to saying that f preserves the length curves.

Note that $Df_p : T_pV \to T_{f(p)}\tilde{V}$ is a map from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Writing out the condition for two spaces to be isometric,

$$\|Df|_p(v)\|_{\tilde{g}}^2 = (Df|_p v)^T \tilde{g}_{f(p)} Df|_p v = v^T (Df|_p)^T \tilde{g}_{f(p)} Df|_p v = \|v\|_g^2 = v^T g v.$$

This holds for all v if and only if

$$(Df|_p)^T \tilde{g}_{f(p)} Df|_p = g.$$

Recall this is exactly the transformation law.

Definition 5.3. Let Σ be an abstract smooth surface, so

$$\Sigma = \bigcup_{i \in I} U_i,$$

with $U_i \subset \Sigma$ open and $\phi_i : U_i \to V_i \subset \mathbb{R}^3$ homeomorphisms such that

$$\phi_i \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

is smooth for all i, j.

A *Riemannian metric* on Σ, usually denoted by g, is a choice of Riemannian metrics g_i on each V_i which are compatible in the following sense: for all i, j, $\phi_i \phi_j^{-1}$ is an isometry between $\phi_j(U_i \cap U_j)$ and $\phi_i(U_j \cap U_j)$, i.e. if we let $f = \phi_i \phi_j^{-1}$, then

$$(Df|_p)^T \begin{pmatrix} E_i & F_i \\ F_i & G_i \end{pmatrix}_{f(p)} Df|_p = \begin{pmatrix} E_j & F_j \\ F_j & G_j \end{pmatrix}_p$$

for all $p \in \phi_j(U_i \cap U_j)$.

Example 5.1.

1. Recall the torus: $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. We exhibited charts where transition functions were restrictions of translations. Equip each $V_i \subset \mathbb{R}^2$ (the image of such a chart(with the Euclidean metric $\mathrm{d}u^2 + \mathrm{d}v^2$, i.e.

   $$V_i \mapsto \mathrm{id}_{2\times 2}\,.$$

   If f is a translation, $Df = \mathrm{id}$, so

   $$(Df)^T I Df = I.$$

   Hence the torus inherits a global Riemannian metric, everywhere locally isometric to $\mathbb{R}^2$, hence flat.

   Since geodesics are well-defined for abstract Riemannian metrics, they are also well-defined on T^2, and they are just projections of straight lines in $\mathbb{R}^2$.

   Note that this flat metric on T^2 in not induced by any embedding of T^2 in $\mathbb{R}^3$.

2. The real projective plane admits a Riemannian metric with constant curvature $+1$. Indeed, we exhibited an atlas of $\mathbb{RP}^2$ with charts of the form (U, ϕ) where $U = q(\hat{U})$, where $q : S^2 \to \mathbb{RP}^2$ is the quotient map, and $\hat{U} \subset S^2$ are open and small enough so that the quotient map is a homeomorphism.

   The transition map for this atlas are either the identity or induced by the antipodal map. But both are isometries of the round metric in S^2.

3. The Klein bottle also has a flat Riemannian metric, induced by its presentation as a quotient of $\mathbb{R}^2$.

Proposition 5.1. *Given a Riemannian metric g on a connected open set $V \subset \mathbb{R}^2$, we can define the* length metric

$$d_g(p,q) = \inf_{\gamma} L(\gamma),$$

where γ varies over all piecewise smooth paths in V from p to q, and $L(\gamma)$ is computed using g.

Then d_g is a metric in V (in the sense of a metric space).

Remark.

1. Given $p, q \in V$, there is always a piecewise smooth path connecting p and q.
2. This implies $d_g(p,q) \geq 0$. Also it is easy to check $d_g(p,q) = d_g(q,p)$, by reversing paths.

   It is also easy to see $d_g(p,r) \leq d_g(p,q) + d_g(q,r)$, from connecting paths.

   The only non-trivial claim is:

$$d_g(p,q) = 0 \iff p = q.$$

3. All this works on any abstract smooth connected surface (Σ, g) equipped with a Riemannian metric g.

Proof: We only show that $d_g(p,q) > 0$ for $p \neq q$. Since

$$g = \begin{pmatrix} E(p) & F(p) \\ F(p) & G(p) \end{pmatrix}$$

is positive definite, there is ε sufficiently small such that

$$\begin{pmatrix} E(p) - \epsilon^2 & F(p) \\ F(p) & G(p) - \epsilon^2 \end{pmatrix}$$

is also positive definite. Moreover, the matrix

$$\begin{pmatrix} E(p') - \epsilon^2 & F(p') \\ F(p') & G(p') - \epsilon^2 \end{pmatrix}$$

remains positive definite for all $p' \in B(p,\delta) \subset V$. Thus for any $p' \in B(p,\delta)$ and $v = (v_1, v_2) \in \mathbb{R}^2$, we have

$$\|v\|_{p'}^2 = E(p')v_1^2 + 2F(p')v_1v_2 + G(p')v_2^2 \geq \epsilon^2(v_1^2 + v_2^2).$$

Hence if γ is a curve in $B(p, \delta)$, then

$$L_g(\delta) \geq \varepsilon L(\gamma),$$

with regular Euclidean distance. Hence given $p \neq q$, and $\gamma : [a, b] \to V$ any curve connecting p to q, then if γ is not contained in $B(p, \delta)$, there exists some $t_0 \in [a, b]$ such that $\gamma|_{[a,t_0]}$ is in $B(p, \delta)$, but $\gamma(t_0)$ is on the boundary of the ball. Thus,

$$L_g(\gamma) \geq L_g(\gamma|_{[a,t_0]}) \geq \epsilon\delta.$$

If however γ is contained in the ball, then

$$L_g(\gamma) \geq \epsilon d(p, q),$$

with regular Euclidean distance. Taking the infimum over all such γ, we get

$$d_g(p, q) \geq \epsilon \min\{\delta, d(p, q)\} > 0.$$

Remark. It follows that d_g gives the same topology that $V \subset \mathbb{R}^2$ inherits from $\mathbb{R}^2$.

5.1 Hyperbolic Geometry

Definition 5.4. We define an abstract Riemannian metric on the disc

$$D = B(0, 1) = \{z \in \mathbb{C} \mid |z| < 1\}$$

by

$$g_{\text{hyp}} = \frac{4(\mathrm{d}u^2 + \mathrm{d}v^2)}{(1 - u^2 - v^2)^2} = \frac{4|\,\mathrm{d}z|^2}{(1 - |z|^2)^2}.$$

In other words,

$$E = G = \frac{4}{(1 - u^2 - v^2)^2}, \qquad F = 0.$$

Recall that the Möbius group:

$$\text{Möb} = \left\{ z \mapsto \frac{az + b}{cz + d} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{GL}(2, \mathbb{C}) \right\}$$

acts on $\mathbb{C} \cup \{\infty\}$.‘

Lemma 5.1.

$$\text{Möb}(D) = \{T \in \text{Möb} \mid T(D) = D\} = \left\{ z \mapsto e^{i\theta} \frac{z - a}{1 - \bar{a}z} \mid |a| < 1 \right\}.$$

Proof: First we note that

$$\left|\frac{z-a}{1-\overline{a}z}\right| = 1 \iff (z-a)(\overline{z}-\overline{a}) = (1-\overline{a}z)(1-a\overline{z})$$
$$\iff |z|^2(1-|a|^2) = 1-|a|^2 \iff |z| = 1.$$

So this map preserves $|z|=1$ and maps a to 0, hence it belongs to $\text{Möb}(D)$. To show they are all of this form, pick $T \in \text{Möb}(D)$. Then if $a = f^{-1}(0)j$, and

$$Q(z) = \frac{z-a}{1-\overline{a}z} \in \text{Möb}(D),$$

then $TQ^{-1}(0) = 0$, and preserves $|z| = 1$. Hence, it must be of the form $z \mapsto e^{i\theta} z$.

Lemma 5.2. *The Riemannian metric g_{hyp} is invariant under* $\text{Möb}(D)$*, i.e. it acts by hyperbolic isometries.*

Proof: $\text{Möb}(D)$ is generated by

$$z \mapsto e^{i\theta} z \text{ and } z \mapsto \frac{z-a}{1-\overline{a}z},$$

so we look at these individually. The first (rotation) clearly preserves

$$g_{\text{hyp}} = \frac{4|\mathrm{d}z|^2}{(1-|z|^2)^2}.$$

For the second type, let

$$\omega = \frac{z-a}{1-\overline{a}z},$$

so

$$\mathrm{d}\omega = \frac{\mathrm{d}z}{1-\overline{a}z} + \frac{(z-a)}{(1-\overline{a}z)^2}\overline{a}\,\mathrm{d}z = \frac{\mathrm{d}z(1-|a|^2)}{(1-\overline{a}z)^2},$$

giving

$$\frac{|\mathrm{d}\omega|}{1-|\omega|^2} = \frac{|\mathrm{d}z|(1-|a|^2)}{|1-\overline{a}z|^2(1-|\frac{z-a}{1-\overline{a}z}|^2)} = \frac{|az|(1-|a|^2)}{|1-\overline{a}z|^2-|z-a|^2} = \frac{|\mathrm{d}z|}{1-|z|^2}.$$

For another view, we have

$$g_{\text{hyp}} = \lambda\,\mathrm{id}\,.$$

Hence

$$\lambda(z) = \frac{4}{(1-|z|^2)^2}, \qquad f(z) = \frac{z-a}{1-\overline{a}z}.$$

To check isometry, we have

$$(Df|_z)^T(g_{\text{hyp}})_{f(z)}Df|_z = \lambda(z)\,\text{id},$$

i.e.

$$\lambda(f(z))(Df|_z)^T Df|_z = \lambda(z)\,\text{id}\,.$$

But the latter term is Cauchy-Riemann, so this simplifies to

$$\lambda(f(z))|f'(z)|^2 = \lambda(z),$$

and this is checked as previously.

Lemma 5.3.

(i) *Every pair of points in (D, g_{hyp}) is joined by a unique geodesic (up to reparametrization).*

(ii) *The geodesics are diameters of the discs and circular arcs orthogonal to ∂D.*

The whole geodesics are called *hyperbolic lines*.

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