

# **IB Linear Algebra**

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# 1 Vector Spaces and Subspaces

Let  $F$  be an arbitrary field.

**Definition 1.1** ( $F$  vector space). A  $F$  vector space is an abelian group  $(V, +)$  equipped with a function

$$\begin{aligned} F \times V &\rightarrow V \\ (\lambda, v) &\mapsto \lambda v \end{aligned}$$

such that

- $\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$ ,
- $(\lambda_1 + \lambda_2)v = \lambda_1 v + \lambda_2 v$ ,
- $\lambda(\mu v) = (\lambda\mu)v$ ,
- $1 \cdot v = v$ .

We know how to

- Sum two vectors
- Multiply a vector  $v \in V$  by a scalar  $\lambda \in F$ .

## Example 1.1.

- (i) Take  $n \in \mathbb{N}$ , then  $F^n$  is the set of column vectors of length  $n$  with elements in  $F$ . We have

$$\begin{aligned} v \in F^n, v &= \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, x_i \in F, \\ v + w &= \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix}, \\ \lambda v &= \begin{pmatrix} \lambda v_1 \\ \vdots \\ \lambda v_n \end{pmatrix}. \end{aligned}$$

Then  $F^n$  is a  $F$  vector space.

(ii) For any set  $X$ , take

$$\mathbb{R}^X = \{f : X \rightarrow \mathbb{R}\}.$$

Then  $\mathbb{R}^X$  is an  $\mathbb{R}$  vector space.

(iii) Take  $M_{n,m}(F)$ , the set of  $n \times m$   $F$  valued matrices. Then  $M_{n,m}(F)$  is a  $F$  vector space.

*Remark.* The axiom of scalar multiplication implies that for all  $v \in V$ ,  $0 \cdot v = \mathbf{0}$ .

**Definition 1.2** (Subspace). Let  $V$  be a vector space over  $F$ . A subset  $U$  of  $V$  is a vector subspace of  $V$  (denoted  $U \leq V$ ) if

- $0 \in U$ ,
- $(u_1, u_2) \in U \times U$  implies  $u_1 + u_2 \in U$ ,
- $(\lambda, u) \in F \times U$  implies  $\lambda u \in U$ .

Note if  $V$  is an  $F$  vector space, and  $U \leq V$ , then  $U$  is an  $F$  vector space.

**Example 1.2.**

- (i) Take  $V = \mathbb{R}^{\mathbb{R}}$ , the space of functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Let  $\mathcal{C}(\mathbb{R})$  be the space of continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Then  $\mathcal{C}(\mathbb{R}) \leq \mathbb{R}^{\mathbb{R}}$ .
- (ii) Take the elements of  $\mathbb{R}^3$  which sum up to  $t$ . This is a subspace if and only if  $t = 0$ .

Note that the union of two subspaces is generally not a subspace, as it is usually not closed under addition.

**Proposition 1.1.** Let  $V$  be an  $F$  vector space, and  $U, W \leq V$ . Then  $U \cap W \leq V$ .

**Proof:** Since  $0 \in U, 0 \in W$ ,  $0 \in U \cap W$ . Now consider  $(\lambda, \mu) \in F^2$ , and  $(v_1, v_2) \in (U \cap W)^2$ . Take  $\lambda_1 v_1 + \lambda_2 v_2$ . Since  $u_1, v_1 \in U$ , this is in  $U$ . Similarly, it is in  $W$ . So it is in  $U \cap W$ , and  $U \cap W \leq V$ .

**Definition 1.3** (Sum of subspaces). Let  $V$  be an  $F$  vector space. Let  $U, W \leq V$ . Then the **sum** of  $U$  and  $W$  is the set

$$U + W = \{u + w \mid (u, w) \in U \times W\}.$$

**Proof:** Note  $0 = 0 + 0 \in U + W$ . Take  $\lambda_1 f + \lambda_2 g$ , where  $f, g \in U + W$ . Then we can write  $f = f_1 + f_2, g = g_1 + g_2$ , where  $f_1, g_1 \in U, f_2, g_2 \in W$ . Then

$$\lambda_1 f + \lambda_2 g = \lambda_1(f_1 + f_2) + \lambda_2(g_1 + g_2) = (\lambda_1 f_1 + \lambda_2 g_1) + (\lambda_1 f_2 + \lambda_2 g_2) \in U + W.$$

*Remark.*  $U + W$  is the smallest subspace of  $V$  which contains both  $U$  and  $W$ .

## 1.1 Subspaces and Quotients

**Definition 1.4** (Quotient). Let  $V$  be an  $F$  vector space. Let  $U \leq V$ . The quotient space  $V/U$  is the abelian group  $V/U$  equipped with the scalar product multiplication

$$\begin{aligned} F \times V/U &\rightarrow V/U \\ (\lambda, v + U) &\mapsto \lambda v + U \end{aligned}$$

**Proposition 1.2.**  $V/U$  is an  $F$  vector space.

## 2 Spans, Linear Independence and the Steinitz Exchange Lemma

**Definition 2.1** (Span of a family of vectors). Let  $V$  be a  $F$  vector space. Let  $S \subset V$  be a subset. We define

$$\begin{aligned}\langle S \rangle &= \{\text{finite linear combinations of elements of } S\} \\ &= \left\{ \sum_{\delta \in J} \lambda_{\delta} v_{\delta}, v_{\delta} \in S, \lambda_{\delta} \in F, J \text{ finite} \right\}.\end{aligned}$$

By convention, we let  $\langle \emptyset \rangle = \{0\}$ .

*Remark.*  $\langle S' \rangle$  is the smallest vector subspace which contains  $S$ .

**Example 2.1.** Take  $V = \mathbb{R}^3$ , and

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} \right\}.$$

Then we have

$$\langle S' \rangle = \left\{ \begin{pmatrix} a \\ b \\ 2b \end{pmatrix}, (a, b) \in \mathbb{R}^2 \right\}.$$

Take  $V = \mathbb{R}^n$ , and let  $e_i$  be the  $i$ 'th basis vector. Then  $V = \langle e_1, \dots, e_n \rangle$ .

Take  $X$  a set, and  $V = \mathbb{R}^X$ . Let  $S_x : X \rightarrow \mathbb{R}$ , such that  $y \mapsto 1$  if  $x = y$ , otherwise  $y \mapsto 0$ . Then

$$\langle (S_x)_{x \in X} \rangle = \{f \in \mathbb{R}^X \mid f \text{ has finite support}\}.$$

**Definition 2.2.** Let  $V$  be a  $F$  vector space. Let  $S'$  be a subset of  $V$ . We may say that  $S$  **spans**  $V$  if  $\langle S \rangle = V$ .

**Definition 2.3** (Finite dimension). Let  $V$  be a  $F$  vector space. We say that  $V$  is **finite dimensional** if it is spanned by a finite set.

**Example 2.2.** Consider  $P[x]$ , the polynomials over  $\mathbb{R}$ , and  $P_n[x]$ , the polynomials over  $\mathbb{R}$  with degree  $\leq n$ . Then since

$$\langle 1, x, \dots, x^n \rangle = P_n[x],$$

$P_n[x]$  is finite dimensional, however  $P[x]$  is not.

**Definition 2.4** (Independence). We say that  $(v_1, \dots, v_n)$ , elements of  $V$  are **linearly independent** if

$$\sum_{i=1}^n \lambda_i v_i = 0 \implies \lambda_i = 0 \forall i.$$

*Remark.*

1. We also say that the family  $(v_1, \dots, v_n)$  is **free**.
2. Equivalently,  $(v_1, \dots, v_n)$  are not linearly independent if one of these vectors is a linear combination of the remaining  $(n - 1)$ .
3. If  $(v_i)$  is free, then  $v_i = 0$  for all  $i$ .

**Definition 2.5** (Basis). A subset  $S$  of  $V$  is a **basis** of  $V$  if and only if

- (i)  $\langle S' \rangle = V$ ,
- (ii)  $S$  is linearly independent.

*Remark.* A subset  $S$  that generates  $V$  is a generating family, so a basis  $S$  is a free generating family.

**Example 2.3.** For  $V = \mathbb{R}^n$ , then  $(e_i)$  is a basis of  $V$ .

If  $V = \mathbb{C}$ , then for  $F = \mathbb{C}$ ,  $\{1\}$  is a basis.

If  $V = P[x]$ , then  $S = \{x^n, n \geq 0\}$  is a basis for  $V$ .

**Lemma 2.1.**  $V$  is a  $F$  vector space. Then  $(v_1, \dots, v_n)$  is a basis of  $V$  if and only if any vector  $v \in V$  has a unique decomposition

$$v = \sum_{i=1}^n \lambda_i v_i.$$

*Remark.* We call  $(\lambda_1, \dots, \lambda_n)$  the coordinates of  $v$  in the basis  $(v_1, \dots, v_n)$ .

**Proof:** Since  $\langle v_1, \dots, v_n \rangle = V$ , we must have

$$v = \sum_{i=1}^n \lambda_i v_i$$

for some  $\lambda_i$ . Now assume

$$\begin{aligned} v &= \sum_{i=1}^n \lambda_i v_i = \sum_{i=1}^n \lambda'_i v_i, \\ \implies \sum_{i=1}^n (\lambda_i - \lambda'_i) v_i &= 0. \end{aligned}$$

Since  $v_i$  are free,  $\lambda_i = \lambda'_i$ .

**Lemma 2.2.** *If  $(v_1, \dots, v_n)$  spans  $V$ , then some subset of this family is a basis of  $V$ .*

**Proof:** If  $(v_1, \dots, v_n)$  are linearly independent, we are done. Otherwise assume they are not independent, then by possibly reordering the vectors, we have

$$v_n \in \langle v_1, \dots, v_{n-1} \rangle.$$

Then we have  $V = \langle v_1, \dots, v_n \rangle = \langle v_1, \dots, v_{n-1} \rangle$ . By iterating, we must eventually get to an independent set.

**Theorem 2.1** (Steinitz Exchange Lemma). *Let  $V$  be a finite dimensional vector space over  $F$ . Take*

- (i)  $(v_1, \dots, v_m)$  free,
- (ii)  $(w_1, \dots, w_n)$  generating.

*Then  $m \leq n$ , and up to reordering,  $(v_1, \dots, v_m, w_{m+1}, \dots, w_n)$  spans  $V$ .*

**Proof:** Induction. Suppose that we have replaced  $l$  of the  $w_i$ , reordering if necessary, so

$$\langle v_1, \dots, v_l, w_{l+1}, \dots, w_n \rangle = V.$$

If  $m = l$ , we are done. Otherwise,  $l < m$ . Then since these vectors span  $V$ , we have



$$v_{l+1} = \sum_{i \leq l} a_i v_i + \sum_{i > l} \beta_i w_i.$$

Since  $(v_1, \dots, v_{l+1})$  is free, some of the  $\beta_i$  are non-zero. Upon reordering, we may let  $\beta_{l+1} \neq 0$ . Then,

$$w_{l+1} = \frac{1}{\beta_{l+1}} \left[ v_{l+1} - \sum_{i \leq l} \alpha_i v_i - \sum_{i > l+1} \beta_i w_i \right].$$

Hence,  $V = \langle v_1, \dots, v_l, w_{l+1}, \dots, w_n \rangle = \langle v_1, \dots, v_l, v_{l+1}, w_{l+1}, \dots, w_n \rangle = \langle v_1, \dots, v_{l+1}, w_{l+2}, \dots, w_n \rangle$ . Iterating this process, we eventually get  $l = m$ , which then proves  $m \leq n$ .

### 3 Basis, Dimension and Direct Sums

**Corollary 3.1.** *Let  $V$  be a finite dimensional vector space over  $F$ . Then any two bases of  $V$  have the same number of vectors, called the **dimension** of  $V$ .*

**Proof:** take  $(v_1, \dots, v_n), (w_1, \dots, w_m)$  bases of  $V$ .

(i) As  $(v_i)$  is free and  $(w_i)$  is generating,  $n \leq m$ .

(ii) As  $(w_i)$  is free and  $(v_i)$  is generating,  $m \leq n$ .

So  $m = n$ .

**Corollary 3.2.** *Let  $V$  be a vector space over  $F$  with dimension  $n \in \mathbb{N}$ .*

- (i) *Any set of independent vectors has at most  $n$  elements, with equality if and only if it is a basis.*
- (ii) *Any spanning set of vectors has at least  $n$  elements, with equality if and only if it is a basis.*

**Proof:** Exercise (fill this in).

**Proposition 3.1.** *Let  $U, W$  be finite dimensional subspaces of  $V$ . If  $U$  and  $W$  are finite dimensional, then so is  $U + W$ , and*

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

**Proof:** Pick  $(v_1, \dots, v_l)$  a basis of  $U \cap W$ . Extend this to a basis  $(v_1, \dots, v_l, u_1, \dots, u_m)$  of  $U$ , and a basis  $(v_1, \dots, v_l, w_1, \dots, w_n)$  of  $W$ . Then we show  $(v_1, \dots, v_l, u_1, \dots, u_m, w_1, \dots, w_n)$  is a basis of  $U + W$ .

It is clearly a generating family, so we will show it is free. Suppose

$$\sum_{i=1}^l \alpha_i v_i + \sum_{i=1}^m \beta_i u_i + \sum_{i=1}^n \gamma_i w_i = 0.$$

Then we get

$$\sum_{i=1}^n \gamma_i w_i \in U \cap W,$$

implying that

$$\sum_{i=1}^l s_i v_i = \sum_{i=1}^n \gamma_i w_i.$$

But since  $(v_1, \dots, w_n)$  is a basis of  $W$ , we get  $\gamma_i = 0$ . Similarly,  $\beta_i = 0$ . Thus,

$$\sum_{i=1}^l \alpha_i v_i = 0.$$

Since  $(v_i)$  is a basis of  $U \cap W$ ,  $\alpha_i = 0$ .

**Proposition 3.2.** *Let  $V$  be a finite dimensional vector space over  $F$ . Let  $U \leq V$ . Then  $U$  and  $V/U$  are both finite dimensional and*

$$\dim V = \dim U + \dim(V/U).$$

**Proof:** Let  $(u_1, u_2, \dots, u_l)$  be a basis of  $U$ . Extend this to a basis  $(u_1, \dots, u_l, w_{l+1}, \dots, w_n)$  of  $V$ . Then we show that  $(w_{l+1} + U, \dots, w_n + U)$  is a basis of  $V/U$ . (Fill this in).

*Remark.* If  $U \leq V$ , then we say  $U$  is proper if  $U \neq V$ . Then for finite dimensions,  $U$  proper implies  $\dim U < \dim V$ , as  $\dim(V/U) > 0$ .

**Definition 3.1** (Direct sum). Let  $V$  be a vector space over  $F$ , and  $U, W \leq V$ . We say  $V = U \oplus W$  if and only if any element of  $v \in V$  can be uniquely decomposed as  $v = u + w$  for  $u \in U, w \in W$ .

*Remark.* If  $V = U \oplus W$ , we say that  $W$  is a complement of  $U$  in  $V$ . There is no uniqueness of such a complement.

In the sequel, we use the following notation. Let  $\mathcal{B}_1 = \{u_1, \dots, u_l\}$  and  $\mathcal{B}_2 = \{w_1, \dots, w_m\}$  be collections of vectors. Then

$$\mathcal{B}_1 \cup \mathcal{B}_2 = \{u_1, \dots, u_l, w_1, \dots, w_m\}$$

with the convention that  $\{v\} \cup \{v\} = \{v, v\}$ .

**Lemma 3.1.** *Let  $U, W \leq V$ . Then the following are equivalent:*

- (i)  $V = U \oplus W$ ;
- (ii)  $V = U + W$  and  $U \cap W = \{0\}$ ;
- (iii) For any basis  $\mathcal{B}_1$  of  $U$ ,  $\mathcal{B}_2$  of  $W$ , the union  $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$  is a basis of  $V$ .

**Proof:** We show (ii) implies (i). Let  $V = U + W$ , then clearly  $U, W$  generate  $V$ . We only need to show uniqueness. Suppose  $u_1 + w_1 = u_2 + w_2$ . Then

$$u_1 - u_2 = w_2 - w_1 \in U \cap W = \{0\}.$$

Hence  $u_1 = u_2$  and  $w_1 = w_2$ , as required.

Now we show (i) implies (iii). Let  $\mathcal{B}_1$  be a basis of  $U$ , and  $\mathcal{B}_2$  a basis of  $W$ . Then  $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$  generates  $U + W = V$ , and  $\mathcal{B}$  is free, as if  $\sum \lambda_i v_i = u + w = 0$ , then  $0 = 0 + 0$  uniquely, so  $u = 0, w = 0$ , giving  $\lambda_i = 0$  for all  $i$ .

Finally, we show (iii) implies (ii). Let  $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ . Then since  $\mathcal{B}$  is a basis of  $V$ ,

$$v = \sum_{u_i \in \mathcal{B}_1} \lambda_i u_i + \sum_{w_i \in \mathcal{B}_2} \lambda_i w_i = u + w.$$

Now if  $v \in U \cap W$ ,

$$v = \sum_{u \in \mathcal{B}_1} \lambda_u u = \sum_{w \in \mathcal{B}_2} \lambda_w w.$$

This gives

$$\sum_{u \in \mathcal{B}_1} \lambda_u u - \sum_{w \in \mathcal{B}_2} \lambda_w w = 0.$$

Since  $\mathcal{B}_1 \cup \mathcal{B}_2$  is free, we get  $\lambda_u = \lambda_w = 0$ , so  $U \cap W = \{0\}$ .

**Definition 3.2.** Let  $V$  be a vector space over  $F$ , and  $V_1, \dots, V_l \leq V$ . Then

(i) The sum of the subspaces is

$$\sum_{i=1}^l V_i = \{v_1 + \cdots + v_l \mid v_j \in V_j, 1 \leq j \leq l\}.$$

(ii) The sum is direct:

$$\sum_{i=1}^l V_i = \bigoplus_{i=1}^l V_i$$

if and only if

$$v_1 + \cdots + v_l = v'_1 + \cdots + v'_l \implies v_1 = v'_1, \dots, v_l = v'_l.$$

**Proof:** Exercise.

**Proposition 3.3.** *The following are equivalent:*

(i)

$$\sum_{i=1}^l V_i = \bigoplus_{i=1}^l V_i,$$

(ii)

$$\forall i, V_i \cap \left( \sum_{j < i} V_j \right) = \{0\},$$

(iii) *For any basis  $\mathcal{B}_i$  of  $V_i$ ,*

$$\mathcal{B} = \bigcup_{i=1}^l \mathcal{B}_i \text{ is a basis of } \sum_{i=1}^l V_i.$$

## 4 Linear maps, Isomorphisms and the Rank-Nullity Theorem

**Definition 4.1** (Linear map). Let  $V, W$  be vector spaces over  $F$ . A map  $\alpha : V \rightarrow W$  is **linear** if and only if for all  $\lambda_1, \lambda_2 \in F$  and  $v_1, v_2 \in V$ , we have

$$\alpha(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \alpha(v_1) + \lambda_2 \alpha(v_2).$$

### Example 4.1.

- (i) Take an  $m \times n$  matrix  $M$ , Then we can take the linear map  $\alpha : \mathbb{R}^m \rightarrow \mathbb{R}^n$  defined by  $X \mapsto MX$ .
- (ii) Take the linear map  $\alpha : \mathcal{C}[0, 1] \rightarrow \mathcal{C}[0, 1]$  by

$$f \mapsto \alpha(f)(x) = \int_0^x f(t) dt.$$

- (iii) Fix  $x \in [a, b]$ . Then we can take a linear map  $\mathcal{C}[a, b] \rightarrow \mathbb{R}$  by  $f \mapsto f(x)$ .

*Remark.* Let  $U, V, W$  be  $F$ -vector spaces.

- (i) The identity map  $\text{id}_V : V \rightarrow V$  by  $x \mapsto x$  is a linear map.
- (ii) If  $U \rightarrow V$  is  $\beta$  linear, and  $V \rightarrow W$  is  $\alpha$  linear, then  $U \rightarrow W$  is linear by  $\alpha \circ \beta$ .

**Lemma 4.1.** Let  $V, W$  be  $F$ -vector spaces, and  $\mathcal{B}$  a basis of  $V$ . Let  $\alpha_0 : \mathcal{B} \rightarrow W$  be any map, then there is a unique linear map  $\alpha : V \rightarrow W$  extending  $\alpha_0$ .

**Proof:** For  $v \in V$ , we can write

$$v = \sum_{i=1}^n \lambda_i v_i,$$

where  $\mathcal{B} = (v_1, \dots, v_n)$ . Then by linearity, we must have

$$\alpha(v) = \alpha\left(\sum_{i=1}^n \lambda_i v_i\right) = \sum_{i=1}^n \lambda_i \alpha_0(v_i).$$

This is unique as  $\mathcal{B}$  is a basis.

*Remark.* This is true in infinite dimensions as well.

Often, to define a linear map, we define its value on a basis and extend by linearity. As a corollary, if  $\alpha_1, \alpha_2 : V \rightarrow W$  are linear and agree on a basis of  $V$ , they are equal.

**Definition 4.2** (Isomorphism). Let  $V, W$  be vector spaces over  $F$ . A map  $\alpha : V \rightarrow W$  is called an **isomorphism** if and only if  $\alpha$  is linear and bijective. If such an  $\alpha$  exists, we say  $V \cong W$ .

*Remark.* If  $\alpha : V \rightarrow W$  is an isomorphism, then  $\alpha^{-1} : W \rightarrow V$  is linear. Indeed, for  $w_1, w_2 \in W \times W$ , let  $w_1 = \alpha(v_1), w_2 = \alpha(v_2)$ . Then,

$$\begin{aligned} \alpha^{-1}(\lambda_1 w_1 + \lambda_2 w_2) &= \alpha^{-1}(\lambda_1 \alpha(v_1) + \lambda_2 \alpha(v_2)) \\ &= \alpha^{-1}(\alpha(\lambda_1 v_1 + \lambda_2 v_2)) \\ &= \lambda_1 v_1 + \lambda_2 v_2 \\ &= \lambda_1 \alpha^{-1}(w_1) + \lambda_2 \alpha^{-1}(w_2). \end{aligned}$$

**Lemma 4.2.** *Congruence is an equivalence relation on the class of all vector spaces of  $F$ :*

- (i)  $\text{id}_V : V \rightarrow V$  is an isomorphism.
- (ii)  $\alpha : V \rightarrow W$  is an isomorphism implies  $\alpha^{-1} : W \rightarrow V$  is an isomorphism.
- (iii) If  $\alpha : U \rightarrow V$  is an isomorphism,  $\beta : V \rightarrow W$  is an isomorphism, then  $\beta \circ \alpha : U \rightarrow W$  is an isomorphism.

**Proof:** Exercise.

**Theorem 4.1.** *If  $V$  is a vector space over  $F$  of dimension  $n$ , then  $V \cong F^n$ .*

**Proof:** Let  $\mathcal{B} = (v_1, \dots, v_n)$  be a basis of  $V$ . Then take

$$\alpha : V \rightarrow F^n$$

$$v = \sum_{i=1}^n \lambda_i v_i \mapsto \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

as an isomorphism.

*Remark.* In this way, choosing a basis of  $V$  is like choosing an isomorphism from  $V$  to  $F^n$ .

**Theorem 4.2.** *Let  $V, W$  be vector spaces over  $F$  with finite dimension. Then  $V \cong W$  if and only if  $\dim V = \dim W$ .*

**Proof:** If  $\dim V = \dim W$ , then  $V \cong F^n \cong W$ , so  $V \cong W$ .

Otherwise, let  $\alpha : V \rightarrow W$  be an isomorphism, and  $\mathcal{B}$  a basis of  $V$ . Then we show  $\alpha(\mathcal{B})$  is a basis of  $W$ .

- $\alpha(\mathcal{B})$  spans  $W$  from the surjectivity of  $\alpha$ .
- $\alpha(\mathcal{B})$  is free from the injectivity of  $\alpha$ .

Hence  $\dim V = \dim W$ .

**Definition 4.3** (Kernal and Image). Let  $V, W$  be vector spaces over  $F$ . Let  $\alpha : V \rightarrow W$  be a linear map. We define

- (i)  $\text{Ker } \alpha = \{v \in V \mid \alpha(v) = 0\}$ , the kernel of  $\alpha$ .
- (ii)  $\text{Im } \alpha = \{w \in W \mid \exists v \in V, \alpha(v) = w\}$ , the image of  $\alpha$ .

**Lemma 4.3.**  *$\text{Ker } \alpha$  is a subspace of  $V$ , and  $\text{Im } \alpha$  is a subspace of  $W$ .*

**Proof:** Let  $\lambda_1, \lambda_2 \in F$ , and  $v_1, v_2 \in \text{Ker } \alpha$ . Then

$$\alpha(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \alpha(v_1) + \lambda_2 \alpha(v_2) = 0.$$

So  $\lambda_1 v_1 + \lambda_2 v_2 \in \text{Ker } \alpha$ .

Now if  $w_1 = \alpha(v_1), w_2 = \alpha(v_2)$ , then

$$\lambda_1 w_1 + \lambda_2 w_2 = \lambda_1 \alpha(v_1) + \lambda_2 \alpha(v_2) = \alpha(\lambda_1 v_1 + \lambda_2 v_2).$$

Hence  $\lambda_1 w_1 + \lambda_2 w_2 \in \text{Im } \alpha$ .

**Example 4.2.** Consider  $\alpha : \mathcal{C}^\infty(\mathbb{R}) \rightarrow \mathcal{C}^\infty(\mathbb{R})$ , given by

$$f \mapsto \alpha(f) = f'' + f.$$

Then  $\alpha$  is linear, and

$$\text{Ker } \alpha = \{f \in \mathcal{C}^\infty(\mathbb{R}) \mid f'' + f = 0\} = \langle \sin t, \cos t \rangle.$$



*Remark.* If  $\alpha : V \rightarrow W$  is linear, then  $\alpha$  is injective if and only if  $\text{Ker } \alpha = \{0\}$ , as

$$\alpha(v_1) = \alpha(v_2) \iff \alpha(v_1 - v_2) = 0.$$

**Theorem 4.3.** *Let  $V, W$  be vector spaces over  $F$ , and  $\alpha : V \rightarrow W$  linear. Then*

$$\begin{aligned} V / \text{Ker } \alpha &\rightarrow \text{Im } \alpha \\ v + \text{Ker } \alpha &\mapsto \alpha(v) \end{aligned}$$

*is an isomorphism.*

**Proof:** We proceed in steps.

- $\bar{\alpha}$  is well defined: Note if  $v + \text{Ker } \alpha = v' + \text{Ker } \alpha$ , then  $v - v' \in \text{Ker } \alpha$ , so  $\alpha(v - v') = 0$ . Hence  $\alpha(v) = \alpha(v')$ .
- $\bar{\alpha}$  is linear: This follows from linearity of  $\alpha$ .
- $\bar{\alpha}$  is a bijection: First, if  $\bar{\alpha}(v + \text{Ker } \alpha) = 0$ , then  $\alpha(v) = 0$ , so  $v \in \text{Ker } \alpha$ , hence  $v + \text{Ker } \alpha = 0 + \text{Ker } \alpha$ , so  $\alpha$  is injective. Then  $\bar{\alpha}$  is surjective from the definition of the image.

**Definition 4.4** (Rank and Nullity). We define the rank  $r(\alpha) = \text{rank}(\alpha) = \dim \text{Im } \alpha$ , and the nullity  $n(\alpha) = \text{null}(\alpha) = \dim \text{Ker } \alpha$ .

**Theorem 4.4** (Rank-nullity theorem). *Let  $U, V$  be vector spaces over  $F$ , with  $\dim U < \infty$ , and let  $\alpha : U \rightarrow V$  be a linear map. Then,*

$$\dim U = r(\alpha) + n(\alpha).$$

**Proof:** We have proven that  $U / \text{Ker } \alpha \cong \text{Im } \alpha$ , but we have already proven  $\dim U / \text{Ker } \alpha = \dim U - r(\alpha)$ , which proves the theorem.

**Lemma 4.4.** *Let  $V, W$  be vector spaces over  $F$  of equal finite dimension. Let  $\alpha : V \rightarrow W$  be a linear map. Then the following are equivalent:*

- $\alpha$  is injective,
- $\alpha$  is surjective,
- $\alpha$  is an isomorphism.

This follows immediately from the rank-nullity theorem.

## 5 Linear maps and Matrices

**Definition 5.1.** If  $V, W$  are vector spaces over  $F$ , then

$$L(V, W) = \{\alpha : V \rightarrow W \text{ linear}\}.$$

**Proposition 5.1.**  $L(V, W)$  is a vector space over  $F$  with

$$(\alpha_1 + \alpha_2)(v) = \alpha_1(v) + \alpha_2(v),$$

$$(\lambda\alpha)(v) = \lambda\alpha(v).$$

Moreover, if  $V$  and  $W$  are finite dimensional, then so is  $L(V, W)$ , and

$$\dim L(V, W) = \dim V \dim W.$$

**Definition 5.2.** An  $m \times n$  matrix over  $F$  is an array with  $m$  rows and  $n$  columns with entries in  $F$ ,  $A = (a_{ij})$ . Define

$$M_{m,n}(F) = \{\text{set of } m \times n \text{ matrices over } F\}.$$

**Proposition 5.2.**  $M_{m,n}(F)$  is a vector space over  $F$ , and  $\dim M_{m,n}(F) = mn$

**Proof:** Let  $E_{ij}$  be the matrix with  $a_{xy} = \delta_{xi}\delta_{yj}$ . Then  $(E_{ij})$  is a basis of  $M_{m,n}(F)$ , as

$$N = (a_{ij}) = \sum_{i,j} a_{ij} E_{ij},$$

and  $(E_{ij})$  is free.

If  $V, W$  are vector spaces over  $F$ , and  $\alpha : V \rightarrow W$  is a linear map, we take a basis  $\mathcal{B} = (v_1, \dots, v_n)$  of  $V$ , and  $\mathcal{C} = (w_1, \dots, w_m)$  of  $W$ . Let  $v \in V$ , then

$$v = \sum_{i=1}^n \lambda_i v_i \sim \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \in F^n.$$

We let this isomorphism from  $V$  to  $F^n$  be  $[v]_{\mathcal{B}}$ . Similarly, we can obtain  $[w]_{\mathcal{C}}$  for  $w \in W$ .

**Definition 5.3.** We define a matrix of  $\alpha$  with respect to a basis  $\mathcal{B}, \mathcal{C}$  as

$$[\alpha]_{\mathcal{B}, \mathcal{C}} = ([\alpha(v_1)]_{\mathcal{C}}, [\alpha(v_2)]_{\mathcal{C}}, \dots, [\alpha(v_n)]_{\mathcal{C}}).$$

By definition, if  $[\alpha]_{\mathcal{B},\mathcal{C}} = (a_{ij})$ , then

$$\alpha(v_j) = \sum_{i=1}^m a_{ij} w_i.$$

**Lemma 5.1.** *If  $v \in V$ , then*

$$[\alpha(v)]_{\mathcal{C}} = [\alpha]_{\mathcal{B},\mathcal{C}} \cdot [v]_{\mathcal{B}},$$

*or equivalently,*

$$(\alpha(v))_i = \sum_{j=1}^n a_{ij} \lambda_j.$$

**Proof:** Let  $v \in V$ , then

$$v = \sum_{j=1}^n \lambda_j v_j.$$

Then

$$\begin{aligned} \alpha(v) &= \alpha\left(\sum_{j=1}^n \lambda_j v_j\right) = \sum_{j=1}^n \lambda_j \alpha(v_j) \\ &= \sum_{j=1}^n \lambda_j \sum_{i=1}^m a_{ij} w_i = \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} \lambda_j\right) w_i. \end{aligned}$$

**Lemma 5.2.** *If  $U \rightarrow V$  is linear under  $\beta$ ,  $V \rightarrow W$  linear under  $\alpha$ , then  $U \rightarrow W$  is linear under  $\alpha \circ \beta$ . Let  $\mathcal{A}$  be a basis of  $U$ ,  $\mathcal{B}$  a basis of  $V$ , and  $\mathcal{C}$  a basis of  $W$ . Then*

$$[\alpha \circ \beta]_{\mathcal{A},\mathcal{C}} = [\alpha]_{\mathcal{B},\mathcal{C}} \cdot [\beta]_{\mathcal{A},\mathcal{B}}.$$

**Proof:** Let  $A = [\alpha]_{\mathcal{B},\mathcal{C}}$ ,  $B = [\beta]_{\mathcal{A},\mathcal{B}}$ . Pick  $u_l \in A$ . Then

$$\begin{aligned} (\alpha \circ \beta)(u_l) &= \alpha(\beta(u_l)) = \alpha\left(\sum_j b_{jl} v_j\right) \\ &= \sum_j b_{jl} \alpha(v_j) = \sum_j b_{jl} \sum_i a_{ij} w_i \\ &= \sum_i \left(\sum_j a_{ij} b_{jl}\right) w_i. \end{aligned}$$

**Proposition 5.3.** *If  $V$  and  $W$  are vector spaces over  $F$ , and  $\dim V = n$ ,  $\dim W = m$ , then  $L(V, W) \cong M_{m,n}(F)$ , so  $\dim L(V, W) = m \times n$ .*

**Proof:** Fix  $\mathcal{B}, \mathcal{C}$  bases of  $V$  and  $W$ . We show

$$\begin{aligned}\theta : L(V, W) &\rightarrow M_{m,n}(F) \\ \alpha &\mapsto [\alpha]_{\mathcal{B}, \mathcal{C}}\end{aligned}$$

is an isomorphism.

- $\theta$  is linear:  $[\lambda_1 \alpha_1 + \lambda_2 \alpha_2]_{\mathcal{B}, \mathcal{C}} = \lambda_1 [\alpha_1]_{\mathcal{B}, \mathcal{C}} + \lambda_2 [\alpha_2]_{\mathcal{B}, \mathcal{C}}$ .
- $\theta$  is surjective: Consider  $A = (a_{ij})$ . Consider the map

$$\alpha : v_j \mapsto \sum_{i=1}^m a_{ij} w_i.$$

This can be extended by linearity, and  $[\alpha]_{\mathcal{B}, \mathcal{C}} = A$ .

- $\theta$  is injective: If  $[\alpha]_{\mathcal{B}, \mathcal{C}} = 0$ , then  $\alpha = 0$  for all  $v$ .

*Remark.* If  $\mathcal{B}, \mathcal{C}$  are bases of  $V, W$  and  $\varepsilon_{\mathcal{B}} : v \mapsto [v]_{\mathcal{B}}$ ,  $\varepsilon_{\mathcal{C}} : w \mapsto [w]_{\mathcal{C}}$ , then the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & W \\ \downarrow \varepsilon_{\mathcal{B}} & & \downarrow \varepsilon_{\mathcal{C}} \\ F^n & \xrightarrow{[\alpha]_{\mathcal{B}, \mathcal{C}}} & F^m \end{array}$$

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