II Algebraic Topology

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0 Introduction

A basic problem in topology is the following:

(Extension problem). If X is a space and $A \subset X$ a subspace, and $f : A \to Y$ is a continuous function, is there a continuous $F : X \to Y$ such that $F|_A = f$?

In fact, the answer is no.

Theorem 0.1. *There is no continuous $F : D^n \to S^{n-1}$ such that*

$$S^{n-1} \overset{incl}{\hookrightarrow} D^n \xrightarrow{F} S^{n-1}$$

is the identity.

What does this have to do with algebra? In fact, we will reduce this problem to showing the corresponding problem in algebra.

Theorem 0.2. *There is no group homomorphism $F : \{0\} \to \mathbb{Z}$ such that*

$$\mathbb{Z} \to \{0\} \xrightarrow{F} \mathbb{Z}$$

is the identity.

Of course, this is now trivial. One more thing we can show is:

Theorem 0.3. $\mathbb{R}^n \cong \mathbb{R}^m \iff n = m$.

Once again, this will be proved by relating it to a problem in algebra.

We can do it the other way round as well.

Theorem 0.4 (Fundamental Theorem of Algebra)**.** *Any non-constant polynomial in $\mathbb{C}$ has a root.*

This can be proven by looking at $\mathbb{C}$ as a topological space.

0.1 Recollections

We say that a continuous function is called a *map*.

Here are a couple of useful lemmas.

Lemma 0.1 (Gluing Lemma)**.** *Let $f : X \to Y$ be a function between topological spaces, and let $C, K \subset X$ be closed sets such that $X = C \cup K$.*

Then f is continuous $\iff f|_C, f|_K$ is continuous.

Lemma 0.2 (Lebesgue Number Lemma)**.** *Let (X, d) be a metric space, and assume it is compact. For any open cover $\mathcal{U} = \{U_\alpha\}$, there is $\delta > 0$ such that each $B_\delta(x)$ is contained in some U_α.*

Here is a way of making reasonable spaces out of reasonable spaces.

Definition 0.1. For a space X and a map $f : S^{n-1} \to X$, the space obtained by attaching an n-cell to X is

$$X \cup_f D^n = (X \sqcup D^n)/(z \in S^{n-1} \subset D^{n-1} \sim f(z) \in X).$$

Definition 0.2. A (finite) *cell complex* is a space X obtained by the following:

(i) Start with a finite set X^0 with the discrete topology.

(ii) If X^{n-1} has been defined, form X^n by attaching a finite collection of n-cells along some maps $\{f_n : S^{n-1} \to X^{n-1}\}$. This X^n is called an *n-skeleton*.

(iii) Stop with $X = X^k$, where k is called the *dimension* of X.

1 Homotopy

In the following let $I = [0, 1]$.

Definition 1.1. Let $f, g : X \to Y$ be maps. A *homotopy* from f to g is a map $H : X \times I \to Y$ such that

$$H(x, 0) = f(x), \qquad H(x, 1) = g(x).$$

If such an H exists, we say f is *homotopic* to g, and we write $f \simeq g$.

If $A \subset X$ is a subspace, we say H is a *homotopy relative to* A if $H(a, t) = H(a, 0)$ for all $t \in I$ and $a \in A$. We write $f \simeq g$ rel A.

Proposition 1.1. *Being homotopic relative to A is an equivalence relation on the set of maps from X to Y.*

Proof: We show it is an equivalence relation.

(i) $f \simeq f$ via $H(x, t) = f(x)$.

(ii) If $f \simeq g$ via H, let $H'(x, t) = H(x, 1 - t)$. This is a homotopy from g to f.

(iii) If $f \simeq g$ via H, and $g \simeq h$ via H', then let

$$H''(x,t) = \begin{cases} H(x, 2t) & 0 \le t \le 1/2, \\ H'(x, 2t-1) & 1/2 \le t \le 1. \end{cases}$$

This is continuous on $X \times [0, 1/2]$ and $X \times [1/2, 1]$, so by the gluing lemma it is continuous on $X \times I$.

Definition 1.2. A map $f : X \to Y$ is a *homotopy equivalence* if there is a $g : Y \to X$ such that

$$f \circ g \simeq \mathrm{id}_Y, \qquad g \circ f \simeq \mathrm{id}_X .$$

Say X is *homotopy equivalent to* Y, written $X \simeq Y$, if a homotopy equivalence $f : X \to Y$ exists.

Example 1.1.

Let $X = S^1$, and $Y = \mathbb{R}^2 \setminus \{0\}$.

Let $i : X \to Y$ be the inclusion, and let $r : Y \to X$ by $x \mapsto x/|x|$ be the normalizing function. This is continuous.

Now $r \circ i = \mathrm{id}_X$. On the other hand, $i \circ r : Y \to Y$ is not the identity. But

$$H(x,t) = \frac{x}{t + |x|(1-t)}$$

is a homotopy from id_Y to $i \circ r$.

So $\mathbb{R}^2 \setminus \{0\} \simeq S^1$, but they are not homeomorphic.

Definition 1.3. X is called *contractible* if $X \simeq \{*\}$, the one-point space.

Lemma 1.1. *Let $f_0, f_1 : X \to Y$ and $g_0, g_1 : Y \to Z$ be homotopic maps.*

Then $g_0 \circ f_0 \simeq g_1 \circ f_1$.

Proof: Let's show that $g_0 \circ f_0 \simeq g_0 \circ f_1 \simeq g_1 \circ f_1$.

For the first equivalence, if H is a homotopy from f_0 to f_1, then $g_0 \circ H : X \times I \to Z$ is a homotopy of $g_0 \circ f_0$ to $g_0 \circ f_1$.

Now if G is a homotopy from g_0 to g_1, then $G \circ (f + 1 \times \mathrm{id}_I) : X \times I \to Z$ is a homotopy of $g_0 \circ f_1$ to $g_1 \circ f_1$.

Proposition 1.2.

(i) *$X \simeq X$.*

(ii) *If $X \simeq Y$, then $Y \simeq X$.*

(iii) *If $X \simeq Y$ and $Y \simeq Z$, then $X \simeq Z$.*

Proof:

(i) Take $f = g = \mathrm{id}_X$, and the stationary homotopies.

(ii) Given $f : X \to Y$ and $g : Y \to X$ with $f \circ g \simeq_H \mathrm{id}_Y$ and $g \circ f \simeq_G \mathrm{id}_X$, then this is the same data as $Y \simeq X$.

(iii) Suppose we have maps

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \arrow[r, bend left, "f'"] & Z \arrow[l, bend left, "g'"] \end{tikzcd}$$

with $f \circ g \simeq \mathrm{id}_Y$, $g \circ f \simeq \mathrm{id}_X$, $f' \circ g' \simeq \mathrm{id}_Z$, $g' \circ f' \simeq \mathrm{id}_Y$.

Consider $f' \circ f : X \to Z$ and $g \circ g' : Z \to X$. We claim these generate the required functions. Indeed,

$$(g \circ g') \circ (f' \circ f) = g \circ (g' \circ f') \circ f \simeq g \circ \mathrm{id}_Y \circ f = g \circ f \simeq \mathrm{id}_X,$$

and the other composition is similar.

Definition 1.4. If $i : A \to X$ is the inclusion of a subspace, then

(i) a *retraction* is a $r : X \to A$ such that $r \circ i = \mathrm{id}_A$.

(ii) a *deformation retraction* is a retraction such that also $i \circ r \simeq \mathrm{id}_X$.

1.1 Paths

Definition 1.5. For a space X and points $x_0, x_1 \in X$, a *path* from x_0 to x_1 is a map $\gamma : I = [0,1] \to X$ such that $\gamma(0) = x_0, \gamma(1) = x_1$. If $x_0 = x_1$, we call γ a *loop* based at x_0.

If γ is a path from x_0 to x_1, and γ' is a path from x_1 to x_2, then we can form the *concatenation* $\gamma \cdot \gamma' : I \to X$ via

$$(\gamma \cdot \gamma')(t) = \begin{cases} \gamma(2t) & 0 \le t \le 1/2, \\ \gamma'(2t) & 1/2 \le t \le 1. \end{cases}$$

This is continuous via the gluing lemma, and is a path from x_0 to x_2.

Also define the *inverse* $\gamma^{-1} : I \to X$ via

$$\gamma^{-1}(t) = \gamma(1-t).$$

Define the *constant path* $c_{x_0} : I \to X$ via $c_{x_0}(t) = x_0$, and define an equivalence relation on X via

$$x_0 \sim x_1 \iff \text{there exists a path } \gamma \text{ from } x_0 \text{ to } x_1.$$

Definition 1.6. The equivalence class of $\sim$ are called *path components* of X. We say X is *path-connected* if there is only one equivalence class. Let

$$\pi_0(X) = X/\sim .$$

Proposition 1.3. *For a map $f : X \to Y$, there is a well-defined function*

$$\pi_0(f) : \pi_0(X) \to \pi_0(Y)$$

given by $\pi_0(f)([x]) = [f(x)]$. Furthermore,

(i) *If* $f \simeq g$, *then* $\pi_0(f) = \pi_0(g)$.

(ii) *For* $A \xrightarrow{h} b \xrightarrow{k} C$, *then* $\pi_0(k \circ h) = \pi_0(k) \circ \pi_0(h)$.

(iii) $\pi_0(\mathrm{id}_X) = \mathrm{id}_{\pi_0(X)}$.

Proof: To see this is well defined, if $[x] = [x']$, then there is a path γ from x to x'. Then $(f \circ \gamma) : I \to Y$ is a path from $f(x)$ to $f(x')$, so $[f(x)] = [f(x')]$.

Now properties (ii) and (iii) are immediate. For (i), let $H : X \times I \to Y$ be a homotopy from f to g. Now

$$H|_{\{x\} \times I} : \{x\} \times I \to Y$$

is a path from $f(x)$ to $g(x)$, so $[f(x)] = [g(x)]$, hence $\pi_0(f)([x]) = \pi_0(g)([x])$.

Corollary 1.1. *If* $f : X \to Y$ *is a homotopy equivalence, then* $\pi_0(f)$ *is a bijection.*

Proof: If $g : Y \to X$ is a homotopy inverse, then

$$\pi_0(f) \circ \pi_0(g) = \pi_0(f \circ g) = \pi_0(\mathrm{id}_Y) = \mathrm{id}_{\pi_0(Y)},$$

and similarly $\pi_0(g) \circ \pi_0(f) = \mathrm{id}_{\pi_0(X)}$, so $\pi_0(f)$ is a bijection.

Example 1.2.

The space $\{-1, +1\}$ with the discrete topology is not contractible. This is because any path in this space is constant, so

$$\pi_0(\{-1, +1\}) = \{-1, +1\},$$

whereas $\pi_0(\{*\}) = \{*\}$ has cardinality 1.

Example 1.3.

The space $[-1, 1]$ does not retract onto $\{-1, +1\}$. Suppose it does, then

$$\begin{tikzcd}
\mathrm{id} : \{-1,+1\} \arrow[r, hook, "\mathrm{inj}"] & {[-1,1]} \arrow[r, "r"] \arrow[d, Rightarrow, "\pi_0"] & \{-1,+1\} \\
\mathrm{id} : \{-1,+1\} \arrow[r, "\pi_0(\mathrm{inc})"] & \pi_0[-1,1] \arrow[r, "\pi_0(r)"] & \{-1,+1\}.
\end{tikzcd}$$

Definition 1.7. Two paths $\gamma, \gamma' : I \to X$ both from x_0 to x_1 are called *homotopic as paths* if they are homotopic relative to $\{0, 1\} \subset I$ as in the previous lecture. We say $\gamma \simeq \gamma'$, relative to x_0 and x_1.

Lemma 1.2. *If $\gamma_0 \simeq \gamma_1$ are paths from x_0 to x_1, and $\gamma'_0 \simeq \gamma'_1$ as paths from x_1 to x_2, then $\gamma_0 \cdot \gamma'_0 \simeq \gamma_1 \cdot \gamma'_1$ as paths from x_0 to x_2.*

Proof: Let H be the homotopy from γ_0 to γ_1 rel. x_0 and x_1, and H' be the homotopy from γ'_0 to γ_1 rel. x_1 and x_2. Then define

$$H''(s,t) = \begin{cases} H(2s,t) & 0 \le s \le 1/2, \\ H'(2s-1,t) & 1/2 \le s \le 1. \end{cases}$$

This is a homotopy from $\gamma_0 \cdot \gamma'_0$ to $\gamma_1 \cdot \gamma'_1$ rel. x_0 and x_2.

Proposition 1.4. *Let γ_0 be a path from x_0 to x_1, γ_1 be a path from x_1 to x_2, and γ_2 be a path from x_2 to x_3. Then,*

(i) $(\gamma_0 \cdot \gamma_1) \cdot \gamma_2 \simeq \gamma_0 \cdot (\gamma_1 \cdot \gamma_2)$ *rel. x_0 and x_3.*

(ii) $\gamma_0 \cdot c_{x_1} \simeq \gamma_0 \simeq c_{x_0} \cdot \gamma_0$ *rel. x_0 and x_1.*

(iii) $\gamma_0 \cdot \gamma_0^{-1} \simeq c_{x_0}$*, rel. x_0 and x_0, and* $\gamma_0^{-1} \cdot \gamma_0 \simeq c_{x_1}$ *rel. x_1 and x_1.*

Proof: Look at a cute little diagram for all three cases. This motivates the following homotopy for the first case:

$$H(s,t) = \begin{cases} \gamma_0(\frac{4s}{t+1}) & 0 \le s \le \frac{t+1}{4}, \\ \gamma_1(4s-1-t) & \frac{t+1}{4} \le s \le \frac{t+2}{4}, \\ \gamma_2(1 - \frac{4(1-s)}{2-t}) & \frac{t+2}{4} \le s \le 1. \end{cases}$$

For the second case we can take

$$H(s,t) = \begin{cases} \gamma_0(\frac{2s}{t+1}) & 0 \le s \le \frac{t+1}{2}, \\ x_1 & \frac{t+1}{2} \le s \le 1. \end{cases}$$

For the third case, we have

$$H(s,t) = \begin{cases} \gamma_0(2s) & 0 \le s \le \frac{1-t}{2}, \\ \gamma_0(1-t) & \frac{1-t}{2} \le s \le \frac{1+t}{2}, \\ \gamma_0(2-2s) & \frac{1+t}{2} \le s \le 1. \end{cases}$$

1.2 The Fundamental Group

Theorem 1.1. *Let X be a space, and $x_0 \in X$. Let $\pi_1(X, x_0)$ be the set of homotopy classes of loops in X, starting and ending at x_0. Then the rule*

$$[\gamma] \cdot [\gamma'] = [\gamma \cdot \gamma']$$

makes $(\pi_1(X, x_0), \cdot, [c_{x_0}])$ into a group.

Proof: The lemma from last lecture shows that this is well defined, and the previous proposition shows that this group exists.

Definition 1.8. A *basic space* is a space X with a chosen point $x_0 \in X$, called the *basic point*. A *map* $f : (X, x_0) \to (Y, y_0)$ of basic spaces is a map $X \xrightarrow{f} Y$ such that $f(x_0) = y_0$. A *based homotopy* is a homotopy relative to $\{x_0\} \in X$.

Proposition 1.5. *To a based map $f : (X, x_0) \to (Y, y_0)$ there is associated a function $\pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ given by*

$$\pi_1(f)([\gamma]) = [f \circ \gamma].$$

It satisfies:

(i) *It is a group homomorphism.*

(ii) *If f is a based homotopic to f', then $\pi_1(f) = \pi_1(f')$.*

(iii) *If $(A, a) \xrightarrow{h} (B, b) \xrightarrow{k} (C, c)$ are based maps, then $\pi_1(k \circ h) = \pi_1(k) \circ \pi_1(h)$.*

(iv) $\pi_1(\mathrm{id}_X) = \mathrm{id}_{\pi_1(X, x_0)}$.

Proof: The proposed function is well-defined, as if $\gamma \simeq \gamma'$ as paths, then $f \circ \gamma \simeq f \circ \gamma'$ as paths.

(i) We verify the group axioms. Note that $f \circ c_{x_0} = c_{y_0}$, so $\pi_1(f)$ preserves the identity element, and

$$f \circ (\gamma \cdot \gamma') = (f \circ \gamma) \cdot (f \circ \gamma'),$$

so $\pi_1(f)$ is a homomorphism.

Then (ii), (iii) and (iv) are elementary.

From now on, we let $\pi_1(f) = f_*$.

Proposition 1.6. *Let u be a path from x_0 to x_1 in X. It induces a homomorphism $u_\# : \pi_1(X, x_0) \to \pi_1(X, x_1)$ satisfying*

$$[\gamma] \mapsto [u^{-1} \cdot \gamma \cdot u].$$

This satisfies:

(i) *If $u \simeq u'$ as paths, then $u_\# = u'_\#$.*

(ii) $(c_{x_0})_\# = \mathrm{id}_{\pi_1(X,x_0)}$.

(iii) *If v is a path from x_1 to x_2, then $(u \cdot v)_\# = v_\# \circ u_\#$.*

(iv) *If $f : X \to Y$ sends x_0 to y_0 and x_1 to y_1, then the following diagram commutes:*

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "f_*"] \arrow[d, "u_\#"] & \pi_1(Y, y_0) \arrow[d, "(f \circ u)_\#"] \\
\pi_1(X, x_1) \arrow[r, "f_*"] & \pi_1(Y, y_1)
\end{tikzcd}$$

(v) *If u is a path from x_0 to x_0, then $u_\#$ is conjugation by $[u] \in \pi_1(X, x_0)$.*

Proof: $u_\#$ is a group homorphism via

$$\begin{aligned}
u_\#([\gamma])u_\#([\gamma']) &= [u^{-1} \cdot \gamma \cdot u][u^{-1} \cdot \gamma' \cdot u] = [u^{-1} \cdot \gamma \cdot u \cdot u^{-1} \cdot \gamma' \cdot u] \\
&= [u^{-1} \cdot \gamma \cdot c_{x_1} \cdot \gamma' \cdot u] \\
&= [u^{-1} \cdot \gamma \cdot \gamma' \cdot u] = u_\#([\gamma] \cdot [\gamma']).
\end{aligned}$$

It is an isomorphism as $(u^{-1})_\#$ is an inverse.

For (iv), note

$$\begin{aligned}
((f \circ u)_\#)([\gamma]) &= (f \circ u)_\#([f \circ \gamma]) = [(f \circ u)^{-1} \cdot f \circ \gamma \cdot f \circ u] \\
&= [f \circ (u^{-1} \cdot \gamma \cdot u)] = f_*([u^{-1} \cdot \gamma \cdot u]) \\
&= (f_* \circ u_\#)([\gamma]).
\end{aligned}$$

Lemma 1.3. *If $H : X \times I \to Y$ is a homotopy from f to g and $x_0 \in X$ is a base point, then $u = H(x_0, t) : I \to Y$ is a path from $f(x_0)$ to $g(x_0)$.*

Then the following is a commutative diagram:

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "f_*"] \arrow[d, "g_*"] & \pi_1(Y, f(x_0)) \arrow[dl, "u_\#"] \\
\pi_1(Y, g(x_0)) &
\end{tikzcd}$$

Proof: Draw a cool looking diagram. Look at the map

$$I \times I \xrightarrow{\gamma \times \mathrm{id}} X \times I \xrightarrow{H} Y.$$

We want $g \circ \gamma \simeq u^{-1} \cdot (f \circ \gamma) \cdot u$ as loops. If we look at the square, the top edge in the square is homotopic to the concatenation of the other edges. Applying $H \circ (\gamma \times \mathrm{id}_I)$ gives us the required homotopy.

Theorem 1.2. *If $f : X \to Y$ is a homotopy equivalence, and $x_0 \in X$, then*

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$$

is an isomorphism.

Proof: Let $g : Y \to X$ be a homotopy inverse, with $f \circ g \simeq_H \mathrm{id}_Y$, $g \circ f \simeq_{H'} \mathrm{id}_X$. Let $u' : I \to X$ be $u'(t) = H'(x_0, 1-t)$ be a path from x_0 to $g \circ f(x_0)$. Then,

$$u'_{\#} = (g \circ f)_* : \pi_1(X, x_0) \overset{f_*}{\hookrightarrow} \pi_1(Y, f(x_0)) \overset{g_*}{\twoheadrightarrow} \pi_1(X, g \circ f(x_0))$$

gives an isomorphism, by the previous lemma on the maps id and $g \circ f$. We want the first map to be an isomorphism, hence it is enough to show the second map is injective. But now consider $u(t) = H(f(x_0), 1-t)$, and this gives us an isomorphism

$$u_{\#} = (f \circ g)_* : \pi_1(Y, f(x_0)) \overset{g_*}{\hookrightarrow} \pi_1(X, g \circ f(x_0)) \overset{f_*}{\twoheadrightarrow} \pi_1(Y, f \circ g \circ f(x_0)).$$

Hence the first map is injective, and so $f_* = g_*^{-1} \circ u'_{\#}$ is an isomorphism.

Definition 1.9. A space X is *simply-connected* if it is path connected and $\pi_1(X, x_0) = \{e\}$ for some (hence all) $x_0 \in X$.

Example 1.4.

A contractible space is simply connected; $X \simeq *$, so $\pi_0(X)$ and $\pi_1(X, x_0)$ are trivial.

Lemma 1.4. *X is simply connected if and only if, for all $x_0, x_1 \in X$, there is a unique homotopy class of points from x_0 to x_1.*

Proof: Let X be simply-connected, and $x_0, x_1 \in X$. As X is path-connected, there exists a path from x_0 to x_1. If γ, γ' are two such paths, then $\gamma^{-1} \cdot \gamma'$ is a loop based at x_1, so $[\gamma^{-1} \cdot \gamma] \in \pi_1(X, x_1) = \{e\}$, hence $\gamma^{-1} \cdot \gamma' \simeq c_{x_1}$, rel. x_1. Hence

$$\gamma' \simeq \gamma \cdot \gamma^{-1} \cdot \gamma' \simeq \gamma \cdot c_{x_1} \simeq \gamma,$$

relative to the endpoints.

Conversely, if X has the stated property, then

(i) It is path connected, as there exists a path between any two points.

(ii) Any loop based at x_0 is homotopic to c_{x_0} as loops, by uniqueness.

Hence $\pi_1(X, x_0) = \{e\}$.

2 Covering Spaces

Definition 2.1. A *covering map* $p : \tilde{X} \to X$ is a continuous map such that for any $x \in X$, there exists an open neighbourhood $U \ni x$ such that

$$p^{-1}(U) = \bigsqcup_{\alpha \in I} V_\alpha,$$

where

$$p|_{V_\alpha} : V_\alpha \to U$$

is a homeomorphism.

Example 2.1.

(i) A homomorphism is a covering map.

(ii) If $p : \tilde{X} \to X$ and $q : \tilde{Y} \to Y$, then $p \times q : \tilde{X} \times \tilde{Y} \to X \times Y$ are covering maps.

Example 2.2.

Let $S^1 \subset \mathbb{C}$ be the unit complex numbers, and

$$\begin{aligned} p : \mathbb{R} &\to S^1 \\ t &\mapsto e^{2\pi i t} = (\cos(2\pi t), \sin(2\pi t)). \end{aligned}$$

Let $U_{y>0} = \{x + iu \in S^1 \mid y > 0\}$. Then,

$$p^{-1}(U_{y>0}) = \bigsqcup_{j \in \mathbb{Z}} (j, j + 1/2).$$

Now we can verify that $p|_{(j,j+1/2)}$ is a homeomorphism, via

$$\begin{aligned} (j, j + 1/2) &\to U_{y>0} \\ j + \frac{\arccos(x)}{2\pi} &\leftarrow\!\shortmid x + iu. \end{aligned}$$

Similarly, $U_{y<0}$, $U_{x>0}$ and $U_{x<0}$ satisfy these properties, so p is a covering map.

Example 2.3.

Let S^1 be as before, and

$$\begin{aligned} p : S^1 &\to S^1 \\ z &\mapsto z^n, \end{aligned}$$

for $n > 0$ an integer. Let $y \in S^1$, and consider $p^{-1}(y)$, the n'th roots of y. Choosing a root ξ and letting $\zeta = e^{2\pi i/n}$, then these roots are $\{\xi, \zeta\xi, \zeta^2\xi, \dots, \zeta^{n-1}\xi\}$.

Then $S^1 - \{y\}$ is open, and $p^{-1}(S^1 - \{y\}) = S^1 - \{\xi, \zeta\xi, \dots, \zeta^{n-1}\xi\}$. Now let

$$V_0 = \left\{ z \in S^1 \mid 0 < \arg\left(\frac{\zeta}{\xi}\right) < \frac{2\pi}{n} \right\},$$

and define $V_i = \zeta^i \cdot V_0$. Then each $x \neq y \in S^1$ has a unique n'th root in each V_i, so $p|_{V_i} : V_i \to S - \{y\}$ is a bijection, in fact a homeomorphism. Hence p is a covering map.

Example 2.4.

Let $S^2 \subset \mathbb{R}^3$ be the unit vectors, and let

$$\mathbb{RP}^2 = S^2/\{x \sim -x\}.$$

We will define

$$\begin{aligned} p : S^2 &\to \mathbb{RP}^2 \\ x &\mapsto [x] \end{aligned}$$

Let $V = \{(x, y, z) \in S^2 \mid z \neq 0\}$, and $U = p(V)$. Then $p^{-1}(U) = V$ is open, so U is open in $\mathbb{RP}^2$. Now $p^{-1}(U) = V = V_{z>0} \sqcup V_{z<0}$.

Then we claim that $p|_{V_{z>0}} : V_{z>0} \to U$ and $p|_{V_{z<0}}$ are homeomorphisms.

Indeed, to construct an inverse $g : U \to V_{z>0}$, we use the definition of the product topology. Consider

$$\begin{aligned} t : V &\to V_{z>0} \\ (x, y, z) &\mapsto \begin{cases} (x, y, z) & z > 0, \\ (-x, -y, -z) & z < 0. \end{cases} \end{aligned}$$

Note that t descends to $T : U \to V_{z>0}$ as a map of sets, hence it is a continuous map by the definition of the quotient topology. It is the inverse to $p|_{V_{z>0}}$.

Doing the same with the x and y-coordinates will let us cover $\mathbb{RP}^2$.

Definition 2.2. Let $p; \tilde{X} \to X$ be a covering map and $f : Y \to X$ be a map. A *lift* of f along p is a map $\tilde{f} : Y \to \tilde{X}$ such that $p \circ \tilde{f} = f$.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, dotted, "\tilde{f}"] \arrow[r, "f"] & X \end{tikzcd}$$

Lemma 2.1 (Uniqueness of Lifts)**.** *If $\tilde{f}_0$ and $\tilde{f}_1$ are lifts of $f : Y \to X$ along a covering map $p : \tilde{X} \to X$, then the set*

$$S = \{y \in Y \mid \tilde{f}_0(y) = \tilde{f}_1(y)\}$$

is both an open and closed set. Hence if Y is connected, then $S = \emptyset$ or $S = Y$.

Proof: We first show S is open. Let $s \in S$, and let $U \ni f(s)$ be an open neighbourhood which is evenly-covered, meaning

$$p^{-1}(U) = \bigsqcup_{\alpha \in I} V_\alpha.$$

But $\tilde{f}_0(s)$ and $\tilde{f}_1(s)$ agree, so they exist in the same V_α. Then if we set $N = \tilde{f}_0^{-1}(V_\alpha) \cap \tilde{f}_1^{-1}(V_\alpha)$, we have

$$p|_{V_\alpha} \circ \tilde{f}_0|_N = f|_N = p|_{V_\alpha} \circ \tilde{f}_1|_N,$$

but $p|_{V_\alpha}$ is a homeomorphism, so $\tilde{f}_0|_N = \tilde{f}_1|_N$, hence $s \in N \subset S$, and so S is open.

Now we show that S is closed. Let $y \in \bar{S}$, and $\tilde{f}_0(y) \neq \tilde{f}_1(y)$. Let $U \ni f(y)$ be an open neighbourhood that is evenly-covered. Then $\tilde{f}_0(y) \in V_\beta$ and $\tilde{f}_1(y) \in V_\gamma$, with $\beta \neq \gamma$.

But now $\tilde{f}_0^{-1}(V_\beta) \cap \tilde{f}_1^{-1}(V_\gamma)$ is an open neighbourhood containing $y \in \bar{S}$, hence it must intersect in S. But then V_β and V_γ must intersect, which is a contradiction.

Theorem 2.1 (Homotopy Lifting Lemma). *Let $p : \tilde{X} \to X$ be a covering space, $H : Y \times I \to X$ from f_0 to f_1 be a homotopy, and $\tilde{f}_0$ be a lift of f_0. Then there exists a unique homotopy $\tilde{H} : Y \times I \to \tilde{X}$, such that*

(i) $\tilde{H}(\cdot, 0) = \tilde{f}_0(\cdot)$.

(ii) $p \circ \tilde{H} = H$.

Proof: Let $\{U_\alpha\}$ be an open cover of X by sets which are evenly-covered. So

$$p^{-1}(U_\alpha) = \bigsqcup_{\beta \in I_\alpha} V_\beta,$$

and $p|_{V_\beta}$ is a homeomorphism from V_β to U_α. Now $\{H^{-1}(U_\alpha)\}$ is an open cover of $Y \times I$, and for each $y_0 \in Y$, it gives an open cover of $\{y_0\} \times I$.

By the Lebesgue number lemma, there is a $N = N(y_0)$ such that each path

$$H|_{\{y_0\}\times[\frac{i}{N},\frac{i+1}{N}]} : \{y_0\} \times \left[\frac{i}{N}, \frac{i+1}{N}\right] \to X$$

has image inside some U_α. In fact, as $\{y_0\} \times I$ is compact, there is an open $W_{y_0} \ni y_0$ such that $H(W_{y_0} \times [\frac{i}{N}, \frac{i+1}{N}])$ lies in some U_α.

We can construct a lift $\tilde{H}|_{W_{y_0}\times I}$ as follows:

(i) Note $H|_{W_{y_0}\times[0,\frac{1}{N}]} : W_{y_0}\times[0, \frac{1}{N}] \to U_\alpha \in X$, and hence $\tilde{f}_0|_{W_{y_0}} : W_{y_0} \to \tilde{X}$ with image in some V_β lying on U_α. Define

$$\tilde{H}|_{W_{y_0}\times[0,\frac{1}{N}]} : W_{y_0} \times \left[0, \frac{1}{N}\right] \xrightarrow{H|} U_\alpha \xrightarrow{p|_{V_\beta}^{-1}} V_\beta \subseteq \tilde{X}.$$

(ii) We proceed in the same way, lifting $H|_{W_{y_0}\times[\frac{1}{N},\frac{2}{N}]}$ starting at $\tilde{H}|_{W_{y_0}\times\{\frac{1}{N}\}}$, and continue.

At the end of this process, we get a $\tilde{H}|_{W_{y_0}\times I}$ lifting $H|_{W_{y_0}\times I}$ and extending $\tilde{f}_0$ at time 0.

We can do this for each $y_0 \in Y$, so it is enough to check that on $(W_{y_0} \times I) \cap (W_{y_1} \times I) = (W_{y_0} \times W_{y_1}) \times I$, the two lifts agree.

For a $y_2 \in W_{y_0} \cap W_{y_1}$, the two choices gives lifts of $H|_{\{y_0\}\times I}$ which agree with $\tilde{f}_0(y_2)$ at time 0. By the uniqueness lemma, these lifts must agree on the whole of $\{y_2\} \times I$. So they agree.

Now uniqueness of the lifted homotopy is immediate from uniqueness of lifts.

If $Y = \{*\}$, we get the following corollary.

Corollary 2.1 (Path Lifting). *If $p : \tilde{X} \to X$ is a covering map, $\gamma : I \to X$ a path from x_0 to x_1, and $\tilde{x}_0 \in \tilde{X}$ is such that $p(\tilde{x}_0) = x_0$, then there is a unique path $\tilde{\gamma} : I \to \tilde{X}$ such that $\tilde{\gamma}(0) = \tilde{x}_0$ and $p \circ \tilde{\gamma} = \gamma$.*

Corollary 2.2. *Let $p : \tilde{X} \to X$ be a covering map, $\gamma, \gamma' : I \to X$ be paths from x_0 to x_1, and $\tilde{\gamma}, \tilde{\gamma}' : I \to \tilde{X}$ be their lifts starting at $\tilde{x}_0 \in p^{-1}(x_0)$.*

If $\gamma \simeq \gamma'$ as paths, then $\tilde{\gamma} \simeq \tilde{\gamma}'$ as paths, in particular $\tilde{\gamma}(1) = \tilde{\gamma}'(1)$.

Proof: Let $H : I \times I \to X$ be a homotopy from γ to γ' relative to the endpoints.

The homotopy lifting lemma gives a $\tilde{H} : I \times I \to \tilde{X}$. At $t = 0$, it must be $\tilde{\gamma}$, as it is forced. The edges at x_0 is a lift of the constant function, hence must be $c_{\tilde{x}_0}$. Therefore the edge at $t = 1$ is a lift of γ' starting at $\tilde{x}_0$, so is $\tilde{\gamma}'$. Finally the rightmost edge is a lift of the constant c_{x_1}, so must be $c_{\tilde{x}_1}$.

Hence this $\tilde{H}$ is a homotopy of paths, as required.

Corollary 2.3. *Let $p : \tilde{X} \to X$ be a covering map, and X be path-connected. Then the sets $p^{-1}(x)$ are all in bijection with each other.*

Proof: Let $\gamma : I \to X$ be a path from x_0 to x_1. Define

$$\begin{aligned} \gamma_* : p^{-1}(x_0) &\to p^{-1}(x_1) \\ y_0 &\mapsto \tilde{\gamma}(1), \end{aligned}$$

where $\tilde{\gamma}$ is the lift of γ starting at y_0. $(\gamma^{-1})_* : p^{-1}(x_1) \to p^{-1}(x_0)$ is defined similarly.

Then $(\gamma^{-1})_* \circ \gamma_*(y_0)$ is the end point of the lift of $\gamma \cdot \gamma^{-1}$ which starts at y_0, which is the end point of c_{y_0} as these are homotopic, which is itself y_0.

Lemma 2.2. *Let $p : \tilde{X} \to X$ be a covering map, $x_0 \in X$ be a base point, and $\tilde{x}_0 \in p^{-1}(x_0) \in \tilde{X}$. Then $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is an injective homomorphism.*

Proof: Let $\gamma : I \to \tilde{X}$ be a loop based at $\tilde{x}_0$, and suppose that $p_*[\gamma] = [c_{x_0}]$, so $p \circ \gamma \simeq_H c_{x_0}$ as loops. Now lift H to a homotopy $\tilde{H}$ starting at γ: then $\tilde{H}$ is a homotopy of paths from γ to a lift of c_{x_0}, which must be $c_{\tilde{x}_0}$. Thus $[\gamma] = [c_{\tilde{x}_0}] = e \in \pi_1(\tilde{X}, \tilde{x}_0)$.

In the proof of the previous corollary, we constructed, for a path $\gamma : I \to X$ from x_0 to x_1 a bijection $\gamma_* : p^{-1}(x_0) \to p^{-1}(x_1)$. It only depended on the homotopy class of the path γ.

This defines a (right) action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$, via $y_0 \cdot [\gamma] = \tilde{\gamma}(1)$, for $\tilde{\gamma}(1)$ the lift of γ starting at y_0.

Lemma 2.3. *Let $p : \tilde{X} \to X$ be a covering map, with X path-connected, and $x_0 \in X$. Then:*

(i) *$\pi_1(X, x_0)$ acts transitively on $p^{-1}(x_0) \iff \tilde{X}$ is path-connected.*

(ii) *The stabiliser of $y_0 \in p^{-1}(x_0)$ is $\mathrm{Im}(\pi_1(\tilde{X}, y_0) \xrightarrow{p_*} \pi_1(X, x_0)) \le \pi_1(X, x_0)$.*

(iii) *If $\tilde{X}$ is path-connected, then there is a bijection*

$$\frac{\pi_1(X, x_0)}{p_* \pi_1(\tilde{X}, y_0)} \leftrightarrow p^{-1}(x_0),$$

induced by acting on $y_0 \in p^{-1}(x_0)$.

Proof: Let $\tilde{X}$ be path-connected, and $y_0, z_0 \in p^{-1}(x_0)$. Let $\tilde{\gamma} : I \to \tilde{X}$ be a path from y_0 to z_0, so $\gamma = p \circ \tilde{\gamma} : I \to X$ is a loop based at x_0.

Then the lift of γ starting at y_0 is $\tilde{\gamma}$, which ends at z_0, so $y_0 \cdot [\gamma] = z_0$. Hence the action is transitive.

Conversely, suppose that the action is transitive. Take $z_0 \in p^{-1}(x_0)$, and y_0 arbitrary. We show there is a path from y_0 to z_0. Choose a path from $p(y_0)$ to $p(z_0) = x_0$, then take a lift of it starting at y_0. It ends at $z_1 \in p^{-1}(x_0)$. Then as the action on $\pi_1(X, x_0)$ on $p^{-1}(x_0)$ is transitive, there exists a loop which starts at z_1 and ends at z_0.

Taking the concatenation of these paths gives a path from y_0 to z_0. Then for any other y_1, taking the concatenation of the path from z_0 to y_1 gives a path from y_0 to $_1$.

For (ii), suppose that $y_0 \cdot [\gamma] = y_0$, i.e. the lift of γ starting at y_0 ends at y_0. Then $[\gamma] = p_*[\tilde{\gamma}]$, hence $\mathrm{Stab}_{\pi_1(X,x_0)}(p^{-1}(x_0)) \le \mathrm{Im}(p_*)$. If $[\gamma] = p_*[\gamma']$, then $[\gamma']$ is the lift of γ starting at y_0, but also it ends at y_0, so $y_0 \cdot [\gamma] = y_0$.

Then (iii) is orbit-stabiliser.

Definition 2.3. If $p : \tilde{X} \to X$ is a covering map, we say:

- It is *n-sheeted* if $p^{-1}(x_0)$ has cardinality $n \in \mathbb{N} \cup \{\infty\}$.

- It is a *universal cover* if $\tilde{X}$ is simply-connected.

Corollary 2.4. *If $p : \tilde{X} \to X$ is a universal cover, then each $\tilde{x}_0 \in p^{-1}(x_0)$ determines a bijection*

$$\begin{aligned} \ell : \pi_1(X, x_0) &\to p^{-1}(x_0) \\ [\gamma] &\mapsto \tilde{\gamma}(1), \end{aligned}$$

for $\tilde{\gamma}$ the lift of γ starting at $\tilde{x}$.

This induces a group-law on $p^{-1}(x_0)$ via

$$y_0 * z_0 = \ell(\ell^{-1}(y_0) \cdot \ell^{-1}(z_0)).$$

Spelling this out, this is given by:

- Choose a path $\tilde{\gamma} : I \to \tilde{X}$ from $\tilde{x}_0$ to z_0.
- Let γ be the lift of $p \circ \tilde{\gamma}$ starting at y_0.
- Then $y_0 * z_0 = \gamma(1)$.

2.1 Fundamental Group of S^1

Theorem 2.2. *Let $u : I \to S^1$ be $u(s) = e^{2\pi is}$, which is based at $1 \in S^1 \subset \mathbb{C}$. Then there is an isomorphism $\pi_1(S^1, 1) \cong (\mathbb{Z}, +, 0)$ which sends u to $1 \in \mathbb{Z}$.*

Proof: We have [: $\mathbb{R} \to S^1$ by $t \mapsto e^{2\pi it}$ is a covering map. Note $\mathbb{R}$ is contractible so it is simply-connected. Hence this is a universal cover, so

$$\pi_1(S^1, 1) \leftrightarrow p^{-1}(1) = \mathbb{Z} \subset \mathbb{R}$$

is a bijection. To compute $\ell^{-1}(m)$, we can take $\tilde{u}_m : I \to \mathbb{R}$ by $t \mapsto mt$, so $u_m = p \circ \tilde{u}_m$ is a loop in S^1. Take $\tilde{x}_0 = 0 \in \mathbb{Z}$. So $n * m$ is the end point of the lift of u_m starting at n, which is $t \mapsto n + mt$. This ends at $n + m$, hence our group operation is addition.

Theorem 2.3. *The disc D^2 does not retract to its boundary S^1.*

Proof: Suppose $r : D^2 \to S^1$ is a retraction, with $i : S^1 \hookrightarrow D^2$ the inclusion. Then $r \circ i = \mathrm{id}_{S^1}$. Now,

$$\mathrm{id} : \pi_1(S^1, 1) \cong \mathbb{Z} \xrightarrow{i_*} \pi_1(D^2, 1) \cong \{0\} \xrightarrow{r_*} \pi_1(S^1, 1).$$

This is not possible, so r does not exist.

Corollary 2.5 (Brouwer Fixed Point Theorem). *Any map $f : D^2 \to D^2$ has a fixed point.*

Proof: Suppose not. Define $r : D^2 \to S^1$, by intersecting the ray starting at $f(x)$ and going through x with S^1. Then this fixes the boundary, and would be continuous, but this would give a retraction X.

Let's look at another application of the fundamental group of a circle.

Corollary 2.6 (Fundamental Theorem of Algebra). *Any non-constant polynomial over $\mathbb{C}$ has a root in $\mathbb{C}$.*

Proof: Let $p(z) = z^n + a_1 z^{n-1} + \cdots + z_n$, a non-constant polynomial. Choose $r > |a_1| + \cdots + |a_n| + 1$. On the circle $|z| = r$, we have

$$|z^n| = |z^{n-1}|r > |z^{n-1}|(|a_1| + \cdots + |a_n|) > |a_1 z^{n-1} + \cdots + a_n|,$$

which means for any $t \in [0,1]$, the polynomial $p_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_n)$ does not have a root in the circle $|z| = r$. Consider the homotopy of loops in $S^1 \subset \mathbb{C}$

$$F(s,t) = \frac{p_t(re^{2\pi i s})/p_t(r)}{|p_t(re^{2\pi i s})/p_t(r)|}.$$

Note that $F(0,t) = 1 = F(1,t)$, so this is indeed a loop homotopy.

At $t = 0$, this is the loop $s \mapsto e^{2\pi i s n}$, which is $n \in \mathbb{Z} \sim \pi_1(S^1, 1)$.

At $t = 1$, it is the loop $s \overset{f_r}{\mapsto} [p(re^{2\pi i s})/p(r)]/|p(re^{2\pi i s})/p(r)|$, which also represents $n \in \mathbb{Z}$.

Now suppose p has no roots. Then f_r is a continuous loops for all $r \in [0, \infty)$, and varying r gives a homotopy from f_r to f_0. But

$$s \mapsto f_0(s) = \frac{p(0)/p(0)}{|p(0)/p(0)|} = 1,$$

hence we must have $n = 0 \in \mathbb{Z}$, so p was constant.

2.2 Universal Covers

Universal covers are really helpful in determining the fundamental group of an object. We now look to methodically construct universal covers.

Our first observation is as follows: if $p : \tilde{X} \to X$ is a universal cover, with $x \in X$

and $U \ni x$ an evenly-covered neighbourhood, let $p^{-1}(U) = \bigsqcup V_\tau$. Let $\gamma : I \to U$ be a loop based at x_0 this lifts to

$$\begin{tikzcd} \tilde{\gamma} : I \arrow[rr] \arrow[dr, "\gamma"] & & V_\alpha \subseteq \tilde{X} \\ & U \arrow[ur, "p|_{V_\alpha}^{-1}"'] & \end{tikzcd}$$

This is homotopic to a constant loop in $\tilde{X}$, as $\tilde{X}$ is simply connected. Applying p, this shows $\gamma \simeq c_{x_0}$ in X. Hence every $x \in X$ has a neighbourhood $U \ni x$ such that $\pi_1(U, x_1) \to \pi_1(X, x)$ is trivial.

Note not every space has this property, for example the Hawaiian earring, obtained by gluing circles of shrinking radius at the same point as arbitrarily small neighbourhoods with non-trivial fundamental group.

Our second observation is as follows: suppose $p : \tilde{X} \to X$ is a universal cover, and $x_0 = p(\tilde{x}_0)$ is a basepoint. Then there is a unique path α from $\tilde{x}_0$ to y, up to homotopy. Hence y is the end point of the lift $p \circ \alpha$, starting at $\tilde{x}_0$.

Therefore, there is a bijection

$$\begin{aligned} \tilde{x} &\leftrightarrow \{\text{homotopy classes of paths in } X \text{ starting at } x_0\}, \\ y &\mapsto p \circ \alpha, \\ \tilde{\gamma}(1) &\leftarrow\!\shortmid [\gamma]. \end{aligned}$$

So we aim to build a universal cover by creating a topology on the homotopy classes.

Theorem 2.4. *Let X be path connected, locally path connected, and semi-locally simply connected. Then it has a universal cover.*

Proof: This is non-examinable as it is super annoying.

As a set, let $\tilde{X}$ be the set of homotopy classes of paths in X starting at x_0. Then define

$$\begin{aligned} p : \tilde{X} &\to X, \\ [\gamma] &\mapsto \gamma(1). \end{aligned}$$

We have a couple of aims.

(i) We want to make a topology on $\tilde{X}$.

(ii) We need to show p is continuous.

(iii) Moreover, we need p to be a covering map.

(iv) Finally we show $\tilde{X}$ is indeed simply-connected.

Consider the sets

$$\begin{aligned}\mathcal{U} = \{U \subseteq X \mid\ & U \text{ open, path-connected and} \\ & \pi_1(U, x) \to \pi_1(X, x) \text{ is trivial for all } x \in U\}.\end{aligned}$$

We will show this is a basis for the topology on X. Indeed, let $V \ni x$ be an open neighbourhood.

(i) Since X is semi-locally simply-connected, $U' \ni X$ such that $\pi_1(U', x) \to \pi_1(X, x)$ is trivial.

(ii) As X is locally path-connected, we can find $V \cap U' = U \ni x$ which is path-connected.

(iii) The map

$$\begin{tikzcd} \pi_1(U,x) \arrow[rr] \arrow[dr] & & \pi_1(X,x) \\ & \pi_1(U',x) \arrow[ur, "\text{trivial}"'] & \end{tikzcd}$$

is trivial.

(iv) Let $y \in U$ be another point, and $u : I \to U$ be a path from $x \to y$. Then

$$\begin{tikzcd} \pi_1(U,y) \arrow[r] \arrow[d, "u_\#"] & \pi_1(X,y) \\ \pi_1(U,x) \arrow[r, "\text{trivial}"] & \pi_1(X,x) \arrow[u, "u_\#"'] \end{tikzcd}$$

shows that the top map is trivial.

Hence we have a basis. For $[\alpha] \in \tilde{X}$ and a $U \in \mathcal{U}$ such that $\alpha(1) \in U$, define

$$([\alpha], U) = (\alpha, U) = \{[\beta] \in \tilde{X} \mid [\beta] = [\alpha \cdot \alpha'] \text{ for some path } \alpha' \in U\}.$$

We claim these sets form a basis for a topology on $\tilde{X}$. Indeed, let $[\beta] \in (\alpha_0, U_0) \cap (\alpha_1, U_1)$. Hence there are α_0', α_1' with $[\alpha_0 \cdot \alpha_0'] = [\beta] = [\alpha_1 \cdot \alpha_1']$.

Let $\beta(1) \in W \subseteq U_0 \cap U_1$ with $W \in \mathcal{U}$. We want to show that $(\beta, W) \subseteq (U_0, \alpha_0) \cap (U_1, \alpha_1)$. Indeed, if $[\gamma] \in (\beta, W)$, then there is a path δ in W with

$[\gamma] = [\beta \cdot \delta] = [\alpha_0 \cdot \alpha_1' \cdot \delta]$, and notice $\alpha_1' \cdot \delta \in U_0$, so $[\gamma] \in (U_0, \alpha_0)$. Similarly, $[\gamma] \in (U_1, \alpha_1)$.

To see that p is continuous, it is enough to check that $p^{-1}(U)$ is open for $U \in \mathcal{U}$. But indeed, if $[\alpha] \in p^{-1}(U)$, then $[\alpha] \in (\alpha, U) \subseteq p^{-1}(U)$, so $p^{-1}(U)$ is open.

To see that p is a covering map, we first need to show that each

$$p|_{(\alpha,U)} : (\alpha, U) \to U$$

is a homeomorphism. As U is path-connected, for any $y \in U$ there is a path γ in U from $\alpha(1)$ to y, and $p([\alpha \cdot \gamma]) = y$, so the map is surjective. Also, if $[\beta], [\beta'] \in (\alpha, U)$ map to the same thing under $p|_{(\alpha,U)}$, then β and β' end at the same point. Therefore there exists paths γ, γ' in U such that $[\beta] = [\alpha \cdot \gamma]$ and $[\beta'] = [\alpha \cdot \gamma']$. So then

$$[\beta'] = [\alpha \cdot \gamma \cdot \gamma^{-1} \cdot \gamma'] = [\alpha \cdot \gamma] = [\beta],$$

as $[\gamma^{-1} \cdot \gamma']$ is a loop in U, hence is homotopic to a constant loop in X. Hence $p|_{(\alpha,U)}$ is a bijection, and continuous. It is also open, as $p((\beta, V)) = V$, so $p|_{(\alpha,U)}$ is a homeomorphism.

Now we claim that $p^{-1}(U)$ can be partitioned into (α, U)'s. We have seen that they cover, so it suffices to show that if two intersect, then they are equal. Let $[\gamma] \in (\alpha, U) \cap (\beta, U)$, i.e. there are paths $\alpha', \beta' \in U$ such that $[\gamma] = [\alpha \cdot \alpha'] = [\beta \cdot \beta']$. Let $[\delta] \in (\alpha, U)$. Let $[\delta] \in (\alpha, U)$, then

$$[\delta] = [\alpha \cdot \alpha''] = [\alpha \cdot \alpha' \cdot (\alpha')^{-1} \cdot \alpha''] = [\beta \cdot \beta' \cdot (\alpha')^{-1} \cdot \alpha''],$$

hence $[\delta] \in (\beta, U)$. Thus $(\alpha, U) \subseteq (\beta, U)$, and the reverse inclusion obviously holds.

Finally, we need to show that $\tilde{X}$ is simply-connected. Note that if $\gamma : I \to X$ is a path, then its lift $\tilde{\gamma} : I \to \tilde{X}$ starting at $[c_{x_0}]$ ends at $[\gamma]$, because

$$s \mapsto [t \mapsto \gamma(st)] : I \to \tilde{X}$$

is the lift. So if a loop γ in X lifts to a loop in $\tilde{X}$ based at $[c_{x_0}]$, then $[\gamma] = [c_{x_0}]$, i.e. $p_* \pi_1(\tilde{X}, [c_{x_0}]) = \{e\} \in \pi_1(X, x_0)$. But p_* is injective, so $\pi_1(\tilde{X}, [c_{x_0}]) = \{e\}$.

2.3 The Galois Correspondence

If $p : \tilde{X} \to X$ is a covering map, where $\tilde{X}$ is path connected, then for $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}(x_0)$, then

$$p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$$

is injective, giving a subgroup of $\pi_1(X, x_0)$. If $\tilde{x}_0' \in p^{-1}(x_0)$ is another basepoint, let γ be a path in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_0'$, then $p \circ \gamma$ is a loop based at x_0, and we have

$$[p \circ \gamma]^{-1} p_* \pi_1(\tilde{X}, \tilde{x}_0)[p \circ \gamma] = p_* \pi_1(\tilde{X}, \tilde{x}_0') \leq \pi_1(X, x_0).$$

So fixing a based space (X, x_0), we get a map

$$\{\text{based path-connected covering maps } p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)\} \overset{p \mapsto \mathrm{Im}(p_*)}{\to} \{\text{subgroup of } \pi_1(X, x_0)\}.$$

Moreover if we loosen our restriction on being based, we get a map

$$\{\text{path-connected covering maps } p : \tilde{X} \to X\} \to \{\text{conjugacy classes of subgroups of } \pi_1(X, x_0)\}.$$

Proposition 2.1 (Surjectivity)**.** *Let X be path-connected, locally path-connected, and semi-locally simply-connected. Then for any $H \in \pi_1(X, x_0)$, there is a $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ with $p_* \pi_1(\tilde{X}, \tilde{x}_0) = H$.*

Proof: Let $\bar{X} \overset{q}{\to} X$ be the universal cover we have constructed. Define $\sim_H$ on $\bar{X}$ by

$$[\gamma] \sim_H [\gamma'] \iff \gamma(1) = \gamma'(1) \text{ and } [\gamma \cdot (\gamma')^{-1}] \in H \leq \pi_1(X, x_0).$$

First, note that $\sim_H$ is an equivalence relation:

(i) $[\gamma] \sim_H [\gamma]$.

(ii) If $[\gamma] \sim_H [\gamma']$, then $[\gamma \cdot (\gamma')^{-1}] \in H$, so $[\gamma' \cdot (\gamma)^{-1}] \in H$.

(iii) If $[\gamma] \sim_H [\gamma']$ and $[\gamma'] \sim_H [\gamma'']$ with $\gamma(1) = \gamma'(1) = \gamma''(1)$, then

$$[\gamma \cdot (\gamma'')^{-1}] = [\gamma \cdot (\gamma')^{-1} \cdot \gamma' \cdot (\gamma'')^{-1}] = [\gamma \cdot (\gamma')^{-1}][\gamma' \cdot (\gamma'')^{-1}] \in H.$$

So $\sim_H$ is an equivalence relation.

Define $\bar{X}_h = \bar{X} / \sim_H$, the quotient space, and $p_H : \tilde{X}_H \to X$ be the induced map. If $[\gamma] \in (\alpha, U)$, $[\gamma'] \in (\beta, U)$ satisfies $[\gamma] \sim_H [\gamma']$, then (α, U) and (β, U) are identified by $\sim_H$, as $[\gamma \cdot q] \sim_H [\gamma' \cdot q]$ for any path $q \in U$.

It remains to show that $(p_H)_* \pi_1(\bar{X}_h, [[c_{x_0}]]) = H \leq \pi_1(X, x_0)$. If $[\gamma] \in H$, then the lift of γ to $\bar{X}$ starting at $[c_{x_0}]$ ends at $[\gamma]$, so the lift to $\bar{X}_H$ ends at $[[\gamma]] = [[c_{x_0}]]$, hence is a loop. So $H \leq (p_H)_* \pi_1(\bar{X}_H, [[c_{x_0}]])$.

On the other hand, if $[\gamma] \in (p_H)_a st\pi_1(\bar{X}_h, [[c_{x_0}]])$, then the lift $\bar{\gamma}$ of γ to $\bar{X}$ starting at $[c_{x_0}]$ ends at $[\gamma]$, so $[\gamma] \sim_H [c_{x_0}]$, as it becomes a loop in $\bar{X}_H$ by assumption. So $[\gamma] \in H$.

Now we just need to show injectivity.

Proposition 2.2 (Based Uniqueness). *Let (X, x_0) satisfy the usual conditions for the existence of a universal cover.*

If $p_1 : (\tilde{X}_1, \tilde{x}_1) \to (X, x_0)$ and $p_2 : (\tilde{X}_2, \tilde{x}_2) \to (X, x_0)$ are path-connected covering spaces, then there exists a based homeomorphism $\tilde{X}_1, \tilde{x}_1) \to (\tilde{X}_2, \tilde{x}_2)$ such that $p_2 \circ h = h_1$, if and only if

$$(p_1)_* \pi_1(\tilde{X}_1, \tilde{x}_1) = (p_2)_* \pi_1(\tilde{X}_2, \tilde{x}_2).$$

Proof: If h exists, then $\mathrm{Im}((p_1)_*) = \mathrm{Im}((p_2)_* \circ h_*) = \mathrm{Im}((p_2)_*)$, as h_* is an isomorphism.

On the other hand, suppose the images are both $H \leq \pi_1(X, x_0)$. We will show that $\tilde{X}_1$ and $\tilde{X}_2$ are both homeomorphic to $\bar{X}_H$.

Consider $r : \bar{X} \to \tilde{X}_1$, by $[\gamma] \mapsto \tilde{\gamma}(1)$, the end point of the lift $\tilde{\gamma}$ of γ to $\tilde{X}_1$ starting at $\tilde{x}_1$.

$$\begin{aligned} r([\gamma]) = r([\gamma']) &\iff \tilde{\gamma} \text{ and } \tilde{\gamma}' \text{ both end at the same point of } \tilde{X}_1 \\ &\iff [\gamma' \cdot \gamma^{-1}] \in (p_1)_* \pi_1(\tilde{X}_1, \tilde{x}_1) = H \\ &\iff [\gamma] \sim_H [\gamma']. \end{aligned}$$

So r descends to a map $q : (\bar{X}_H, [[c_{x_0}]]) \to (\tilde{X}_1, \tilde{x}_1)$, a bijection. It is also an open map, as $\bar{X}_H$ and $\tilde{X}_1$ are both locally homeomorphic to X. So q is a homeomorphism.

Corollary 2.7 (Unbased Uniqueness). *Suppose the usual hypothesis on X.*

If $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ are path connected covering spaces, then there exists a homeomorphism $h : \tilde{X}_1 \to \tilde{X}_2$ such that $p_2 \circ h = p_1$ if and only if

$$(p_1)_* \pi_1(\tilde{X}_1, \tilde{x}_1) \text{ and } (p_2)_* \pi_1(\tilde{X}_2, \tilde{x}_2) \text{ are conjugate in } \pi_1(X, x_0),$$

for any $\tilde{x}_1 \in p_1^{-1}(x_0)$, $\tilde{x}_2 \in p_2^{-1}(x_0)$.

Proof: If h exists, choose $\tilde{x}_1 \in p_1^{-1}(x_0)$ and $\tilde{x}_2 = h(\tilde{x}_1)$. Then the previous proposition applies, and the groups obtained are equal.

Conversely, suppose $[\gamma] \in \pi_1(X, x_0)$ is such that

$$[\gamma]^{-1}(p_1)_*\pi_1(\tilde{X}_1, \tilde{x}_1)[\gamma] = (p_2)_*\pi_1(\tilde{X}_2, \tilde{x}_2).$$

Then lifting γ to $\tilde{X}_1$ starting at $\tilde{x}_1$, it ends at $\tilde{x}_1' \in p_1^{-1}(x_0)$. Then choosing $\tilde{x}_1'$ as our basepoint, the left hand side is exactly $(p_1)_*\pi_1(\tilde{X}_1, \tilde{x}_1')$, using the change of basepoint isomorphism. So our previous proposition gives a based homeomorphism $h : (\tilde{X}_1, \tilde{x}_1') \to (\tilde{X}_2, \tilde{x}_2)$, which is also an unbased homeomorphism.

3 Free Groups and Presentations

Let $S = \{s_\alpha\}_{\alpha \in I}$ be a set, called the *alphabet*. Then $S^{-1} = \{s_\alpha^{-1}\}_{\alpha \in I}$. We suppose that $S \cap S^{-1} = \emptyset$.

A *word* in the alphabet S is a (possibly empty(finite sequence $(x_1, x_2, \dots, x_n)$ of elements in $S \cup S^{-1}$. A word is called *reduced* if it does not contain $(s_\alpha, s_\alpha^{-1})$ or $(s_\alpha^{-1}, s_\alpha)$ as a subword.

An elementary reduction of the word $(x_1, x_2, \dots, x_i, s_\alpha, s_\alpha^{-1}, x_{i+1}, \dots, x_n)$ (or s_α^{-1}, s_α) is $(x_1, x_2, \dots, x_i, x_{i+1}, \dots, x_n)$.

Definition 3.1. The *free group* on the alphabet S, denoted $F(S)$, is the set of reduced (possibly empty) words in this alphabet.

The group operation is given by concatenation, and performing elementary reductions until the word is reduced (note this is not obviously a group, as we have a choice on how to do our reductions—we can alternatively define it as a subgroup of a permutation group).

We have that $()(x_1, \dots, x_n) = (x_1, \dots, x_n) = (x_1, \dots, x_n)()$, and

$$(x_n^{-1}, \dots, x_1^{-1})(x_1, \dots, x_n) = () = (x_1, \dots, x_n)(x_n^{-1}, \dots, x_1^{-1}).$$

Moreover there is a function $\iota : S \to F(S)$, given by sending s to (s).

Lemma 3.1 (Universal Property of Free Groups)**.** *For any group H, the function*

$$\{\textit{homomorphisms } \varphi : F(S) \to H\} \xrightarrow{\circ\iota} \{\textit{functions } \phi : S \to H\}$$

is a bijection.

Proof: Given $\phi : S \to H$, we want $\varphi : F(S) \to H$ such that $\varphi((s)) = \phi(s)$. Let, on a not necessarily reduced word $(s_{\alpha_1}^{\varepsilon_1}, \dots, s_{\alpha_n}^{\varepsilon_n})$,

$$\varphi((s_{\alpha_1}^{\varepsilon_1}, \dots, s_{\alpha_n}^{\varepsilon_n})) = \phi(s_{\alpha_1})^{\varepsilon_1} \cdots \phi(s_{\alpha_n})^{\varepsilon_n} \in H.$$

If the word contained $(s_\alpha, s_\alpha^{-1})$, then the result contains $\phi(s_\alpha)\phi(s_\alpha)^{-1} = e \in H$. So φ is well-defined. As the group operation on $F(S)$ is given by concatenation, we see that φ is a homomorphism.

Definition 3.2. Let S be a set, and $R \subseteq F(S)$. Then

$$\langle S \mid R \rangle = F(S)/\langle\langle R \rangle\rangle,$$

with

$$\langle\langle R\rangle\rangle = \{(r_1^{\varepsilon_1})^{g_1}\cdots(r_n^{\varepsilon_n})^{g_n} \mid r_i \in R, \varepsilon_i \in \{\pm 1\}, g \in F(S)\} \lhd F(S),$$

where $h^g = g^{-1}hg$. Call this a presentation of the group $\langle S \mid R\rangle$. If S and R are finite, call it a *finite presentation*.

Lemma 3.2 (Universal Property of Group Presentations)**.** *For any group H, the function*

$$\begin{aligned}&\{\textit{group homomorphisms } \psi : \langle S \mid R\rangle \to H\}\\ &\quad\to \{\textit{functions } \phi : S \to H \textit{ such that } R \subseteq \operatorname{Ker}(\varphi : F(S) \to H)\}\end{aligned}$$

is a bijection, by

$$\psi \mapsto (\phi : S \xrightarrow{\iota} F(S) \twoheadrightarrow \langle S \mid R\rangle \xrightarrow{\psi} H).$$

Proof: Suppose ψ, ψ' determine functions $\phi = \phi' : S \to H$. Then

$$\begin{tikzcd} F(S) \arrow[r, "\text{quot}"] & \langle S \mid R\rangle \arrow[r, bend left, "\psi"] \arrow[r, bend right, "\psi'"'] & H \end{tikzcd}$$

are equal by the universal property of free groups. As quotient is onto, $\psi = \psi'$.

Conversely, given $\phi : S \to H$ such that $\varphi(r) = e$ for all $r \in R$, consider

$$\varphi : F(S) \to H.$$

Now $R \subseteq \operatorname{Ker}\varphi$, so as $\operatorname{Ker}\varphi$ is normal, $\langle\langle R\rangle\rangle \leq \operatorname{Ker}\varphi$. Thus φ descends to a homomorphism

$$\psi : \langle S \mid R\rangle = \frac{F(S)}{\langle\langle R\rangle\rangle} \to H,$$

as is required.

Example 3.1.

If G is a group, the function $\mathrm{id} : G \to G$ gives a homomorphism $f : F(G) \to G$, which is onto. Let $R = \operatorname{Ker}(\phi)$. Then

$$\langle G \mid R\rangle = \frac{F(G)}{\langle\langle R\rangle\rangle} \cong G.$$

Essentially, this is saying that a group is what you get when you take the elements, related by their relations.

Example 3.2.

Let $G = \langle a, b \mid a \rangle$ and $H = \langle t \mid \rangle$. Now consider $\phi : \{a, b\} \to H$ by $a \mapsto e$ and $b \mapsto t$. Then $\varphi(a) = e$, so we get a homomorphism

$$\psi : \langle a, b, \mid a \rangle \to \langle t \mid \rangle.$$

Moreover, we can take $\phi : \{t\} \to G$ by $t \mapsto b$. Then $\psi' : \langle t \mid \rangle \to \langle a, b \mid a \rangle$. Since $\psi' \circ \psi([a]) = [e] = [a]$, and $\psi \circ \psi([b]) = [b]$, as $[a]$ and $[b]$ generate we have $\psi' \circ \psi = \mathrm{id}$. Moreover, $\psi \circ \psi' = \mathrm{id}$, so indeed

$$\langle a, b \mid a \rangle = \langle t \mid \rangle.$$

Example 3.3.

Let $G = \langle a, b \mid ab^{-3}, ba^{-2} \rangle$. Then $[a][b]^{-3} = e$, so $[a] = [b]^3$. Moreover $[b][a]^{-2} = e$, so $[b] = [a]^2$.

Hence we get $[a] = [a]^6$, or $e = [a]^5$. Moreover $[b] = [a]^2$, so these relation show that every element is equal to one of $e, [a], [a]^2, [a]^3$ or $[a]^4$. We will show that the group has exactly five elements.

Consider $\phi : \{a, b\} \to \mathbb{Z}/5$, such that $\phi(a) = 1$, $\phi(b) = 2$. Then $\varphi(ab^{-3}) = e - \varphi(ba^{-2})$, so we get a homomorphism $\psi : \langle a, b \mid ab^{-3}, ba^{-2} \rangle \to \mathbb{Z}/5$, which is isomorphic.

3.1 Free Products with Amalgamation

Consider group homomorphisms

$$G_1 \xleftarrow{i_1} H \xrightarrow{i_2} G_2,$$

and suppose $G_i = \langle S_i \mid R_i \rangle$ The *free product* of G_1 and G_2 is

$$G_1 * G_2 = \langle S_1 \sqcup S_2 \mid R_1 \sqcup R_2 \rangle.$$

The functions

$$S_i \to S_1 \sqcup S_2 \to F(S_1 \sqcup S_2) \to G_1 * G_2$$

induce homomorphisms

$$G_1 \xrightarrow{j_1} G_1 * G_2 \xleftarrow{j_2} G_2.$$

The *free product with amalgamation* over H is the quotient

$$G_1 *_H G_2 = G_1 * G_2 / \langle\langle j_1 i_1(h)(j_2 i_2(h))^{-1} \mid h \in H \rangle\rangle$$

Hence the square

$$\begin{tikzcd} H \arrow[r, "i_1"] \arrow[d, "i_2"] & G_1 \arrow[d, "j_1"] \\ G \arrow[r, "j_2"] & G_1 *_H G_2 \end{tikzcd}$$

commutes.

Lemma 3.3 (Universal Property of Free Products with Amalgamation)**.** *For any group K, the following is a bijection:*

$$\{\textit{group homomorphisms } \phi : G_1 *_H G_2 \to K\}$$
$$\to \{\textit{group homomorphisms } \phi_1 : G_1 \to K, \phi_2 : G_2 \to K \textit{ such that } \phi_1 \circ i_1 = \phi_2 \circ i_2\}.$$

This is by

$$\phi \mapsto [G_i \xrightarrow{j_i} G_1 *_H G_2 \xrightarrow{\phi} H].$$

In essence the following diagram commutes.

$$\begin{tikzcd} H \arrow[r, "i_1"] \arrow[d, "i_2"] & G_1 \arrow[d, "j_1"] \arrow[ddr, bend left, "\phi_1"] & \\ G \arrow[r, "j_2"] \arrow[drr, bend right, "\phi_2"] & G_1 *_H G_2 \arrow[dr, "\phi"] & \\ & & K \end{tikzcd}$$

4 The Seifert-Van Kampen Theorem

Let X be a space, $A, B \subseteq X$ be subspaces. Moreover let $x_0 \in A \cap B$. Then we get a commutative diagram

$$\begin{tikzcd} \pi_1(A \cap B, x_0) \arrow[r] \arrow[d] & \pi_1(A, x_0) \arrow[d] \\ \pi_1(B, x_0) \arrow[r] & \pi_1(X, x_0) \end{tikzcd}$$

The universal property says that this gives a homomorphism

$$\pi_1(A, x_0) *_{\pi_1(A\cap B, x_0)} \pi_1(B, x_0) \to \pi_1(X, x_0).$$

Theorem 4.1 (Seifert-Van Kampen). *Let X be a space, $A, B \subseteq X$ be open subsets which cover X and such that $A \cap B$ is path-connected.*

Then for any $x_0 \in A \cap B$, the induced map

$$\pi_1(A, x_0) *_{\pi_1(A\cap B, x_0)} \pi_1(B, x_0) \to \pi_1(X, x_0)$$

is an isomorphism.

Example 4.1.

Consider S^n for $n \geq 2$. Let $A = S^n - \{(1, 0, \dots, 0)\}$ and $B = S^n - \{(-1, 0, \dots, 0)\}$. Then $A, B \cong \mathbb{R}^n$ by a projection, and these are contractible, so $A, B \simeq \{*\}$.

Moreover, $A \cap B \cong S^{n-1} \times (-1, 1) \simeq S^{n-1}$, which is path-connected for $n \geq 2$. So

$$\pi_1(S^n, *) \cong \{e\} *_{\pi_1(S^{n-1}, *)} \{e\} = \{e\}.$$

So S^n is simply-connected.

Example 4.2.

There exists a 2-sheeted covering map $p : S^n \to \mathbb{RP}^n$. For $n \geq 2$, S^n is simply-connected, so this is a universal cover. So,

$$\pi_1(\mathbb{RP}^n, *) \overset{\text{bij}}{\to} p^{-1}(*),$$

which has two elements. So $\pi_1(\mathbb{RP}^n, *) \cong \mathbb{Z}/2$.

For the next example, we define, for (X, x_0) and (Y, y_0) based spaces,

$$X \vee Y = (X \sqcup Y)/(x_0 \sim y_0).$$

Example 4.3.

Let $S^1 \in \mathbb{C}$ have basepoint $1 \in S^1 \subseteq \mathbb{C}$ Then consider $S^1 \vee S^1$, which is covered by $(S^1 - \{-1\}) \vee S^1 = U$, and $S_1 \vee (S^1 - \{-1\}) = V$. Then $U, V \simeq S^1$, and $U \cap V \simeq \{*\}$.

So by Seifert-Van Kampen,

$$\pi_1(S^1 \vee S^1) \simeq \langle a \mid \rangle *_{\{e\}} \langle b \mid \rangle = \langle a, b \mid \rangle.$$

Example 4.4.

The function $\{a, b\} \to \mathbb{Z}/3$ given by $a, b \mapsto 1$ determines a homomorphism

$$\varphi : \langle a, b \mid \rangle = \pi_1(S^1 \vee S^1, x_0) \to \mathbb{Z}/3,$$

which is surjective. So

$$K = \mathrm{Ker}(\varphi) \leq \pi_1(S^1 \vee S^1, x_0)$$

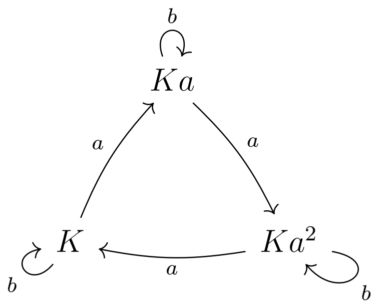
is a subgroup of index 3. This corresponds to a covering space. What is it? It is a $p : \tilde{X} \to X$ with

$$p^{-1}(x_0) \cong \frac{\pi_1(S^1 \vee S^1, x_0)}{K} \cong \mathbb{Z}/3.$$

Our thing looks like this:

$$\begin{tikzcd}
& Ka \arrow[dr, bend left, "a"] \arrow[dr, bend right, "b"'] & \\
K \arrow[ur, bend left, "a"] \arrow[ur, bend right, "b"'] & & Ka^2 \arrow[ll, bend right, "b"'] \arrow[ll, bend left, "a"]
\end{tikzcd}$$

Now if we consider $a \mapsto 1$, $b \mapsto 0$, this gives $\varphi : \pi_1(S^1 \vee S^1, x_0) \to \mathbb{Z}/3$, with kernel K. We want the covering space corresponding to K.



The universal cover of $S^1 \vee S^1$ is a tree thing.

We now defer to the proof of Seifert-Van Kampen.

Proof: We can assume that A, B are path-connected.

First, we show that ϕ is surjective. Let $\gamma : I \to X$ be a loop. Then $\{\gamma^{-1}(A), \gamma^{-1}(B)\}$ is an open cover of I, so by the Lebesgue number lemma there is an $n \gg 0$ such that each $[i/n, (i+1)/n]$ is sent into A or B (or both).

By concatenating interval which lie entirely in A or B, we can write $\gamma = \gamma_1 \cdot \gamma_2 \cdot \ldots \cdot \gamma_k$, with each γ_i having endpoints in $A \cap B$. This is done by concatenating adjacent paths which lie in both A or B.

Now choose paths u_i from $\gamma_i(1)$ to x_0 in $A \cap B$. Then

$$\gamma \simeq (\gamma_1 \cdot u_1) \cdot (u_1^{-1} \cdot \gamma_2 \cdot u_2) \cdot \ldots \cdot (u_{n-1}^{-1} \cdot \gamma_n).$$

Now each of these is a loop based at x_0, lying in A or in B. So $[\gamma] \in \mathrm{Im}(\phi)$, as required.

Now we show that ϕ is injective. Note that $\pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0)$ has the following description: it is generated by,

(i) for $\gamma : I \to A$ a loop in A, $[\gamma]_A$,

(ii) for $\gamma : I \to B$ a loop in B, $[\gamma]_B$,

with relations:

(i) If $\gamma \simeq \gamma'$ in A, then $[\gamma]_A = [\gamma']_A$, and similarly for B.

(ii) If $\gamma : I \to A \cap B$, then $[\gamma]_A = [\gamma]_B$.

(iii) $[\gamma]_A[\gamma']_A = [\gamma \cdot \gamma']_A$, and similarly for B.

Suppose that

$$\phi([\gamma_1]_{A_{i1}}[\gamma_2]_{A_{i_2}} \cdots [\gamma_n]_{A_{i_n}}) = [c_{x_0}],$$

so $\gamma_1 \cdots \gamma_n \simeq_H c_{x_0}$ in X. Take this $H : I \times I \to X$. We can subdivide $I \times I$ into squares of size $1/N$ with $n \mid N$ such that each square is sent into A or B, or both.

Choose paths u_{ij} from $H(i/N, j/N)$ to x_0 such that, if it is a vertex of a box labelled A, the path is in A, and if it is a vertex of a box labelled B, the path is in B.

Now there is an insane diagram to form a homotopy G, which has the property that it decomposes into rectangles with vertices sent to x_0, and each rectangle is in A or in B. This shows that $[\gamma_1]_{A_{i1}}[\gamma_2]_{A_{i_2}} \cdots [\gamma_n]_{A_{i_n}}$ can be transformed into the trivial word using the three kinds of relations described.

4.1 Attaching a Cell

Let $f : (S^{n-1}, *) \to (X, x_0)$. Then let

$$Y = X \cup_f D^n = \frac{X \sqcup D^n}{f(x) \sim x \in \partial D^n = S^{n-1}}.$$

Theorem 4.2. *If $n \geq 3$, then*

$$\text{inc}_*(X, x_0) \to \pi_1(Y, [x_0])$$

is an isomorphism. If $n = 2$, then

$$\text{inc}_*(X, x_0) \to \pi_1(Y, [x_0])$$

is the quotient by the normal subgroup generated by $[f] \in \pi_1(X, x_0)$.

Proof: Let $U = \text{int}(D^n)$, and $V = X \cup_f (D^n \setminus \{0\})$. Then these give an open cover of Y. Choose a path u in U, V from y_0 to some $y_1 \in \text{int}(D^n) = U$. If $n \geq 3$, then we know that $U \simeq *$, and $U \cap V \simeq S^{n-1} \times (0, 1)$ is simply-connected.

By Seifert-Van Kampen, we have that

$$\pi_1(V, y_1) = \pi_1(U, y_1) *_{\pi_1(U \cap V, y_1)} \pi_1(V, y_1) \cong \pi_1(Y, y_1)$$

Now by a change of basepoint isomorphism, we get $\pi_1(Y, y_0) \cong \pi_1(V, y_0)$. But V strongly derogation retracts to X, so this is $\pi_1(X, y_0)$.

If instead $n - 2$, then $U \simeq *$, and $U \cap V \simeq S^1 \times (0,1)$, so Seifer-Van Kampen says

$$\{e\} *_{\mathbb{Z}} \pi_1(V, y_1) \cong \pi_1(Y, y_1) *_{\pi_1(U \cap V, y_1)} \pi_1(V, y_1) \cong \pi_1(Y, y_1),$$

or looking at where 1 goes,

$$\frac{\pi_1(V, y_1)}{\langle\langle u_\#^{-1}[f]\rangle\rangle} \cong \pi_1(Y, y_1).$$

Using a change of basepoint,

$$\frac{\pi_1(V, y_0)}{\langle\langle [f]\rangle\rangle} \cong \pi_1(Y, y_0).$$

Example 4.5.

Let's look at the torus T (insert picture from geometry). This is the standard definition, however it has a cell structure with:

- 0-cell x_0.
- 1-cells a, b.
- A two-cell.

The 1-skeleton is $T^1 = S^1 \vee S^1$, so $\pi_1(T^1, x_0) = \langle a, b \mid \rangle$, and so

$$\pi_1(T, x_0) = \langle a, b \mid aba^{-1}b^{-1}\rangle.$$

Quite cool! We can check that this is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, since this is saying that $ab = ba$, i.e. the group is abelian.

Corollary 4.1. *For $G = \langle S \mid R\rangle$ with S, R finite, there is a two-dimensional based cell-complex (X, x_0) with*

$$\pi_1(X, x_0) = G.$$

Proof: Let Y be the wedge of $|S|$-many circles. Sending $s \in S$ to the s-th circle gives an isomorphism $\langle s \mid \rangle \cong \pi_1(Y, y_0)$.

Each word $r \in R$ is an element of $\langle S \mid \rangle$, so gives a loop $[\gamma_r] \in \pi_1(Y, y_0)$. Attaching 2-cells to Y along $\{\gamma_r\}_{r \in R}$ gives (X, x_0) with

$$\pi_1(X, x_0) = \frac{\langle S \mid \rangle}{\langle\langle r \in R \rangle\rangle} = \langle S \mid R \rangle.$$

4.2 Refining Seifert-Van Kampen

Definition 4.1. A subset $A \subseteq X$ is called a *neighbourhood deformation retract* (NDR) if there is an open neighbourhood $A \subseteq U \subseteq X$, and U strongly deformation retracts to A.

Theorem 4.3 (Seifert-Van Kampen)**.** *Let X be a space, $A, B \subseteq X$ closed subsets which cover X and such that $A \cap B$ is path-connected, and is a NDR in both A and B. Then*

$$\pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0) \cong \pi_1(X, x_0).$$

Proof: Let $A \cap B \subseteq U \subseteq A$, and $A \cap B \subseteq V \subseteq B$ for U, V open, and strongly deformation retractable to $A \cap B$. Observe that $(A \cup V)^c = B \setminus V$, and $(B \cup U)^c = A \setminus U$ are closed. Hence $A \cup V$, $B \cup U$ give an open cover of X.

The deformations of U and V to $A \cap B$ glue to give a deformation of $(A \cup V) \cap (B \cup U) = U \cup V$ to $A \cap B$, $A \cup V$ to A and $B \cup U$ to B.

We can now use Seifert-Van-Kampen on the open cover, and observe that

$$\begin{tikzcd}
\pi_1(B) \arrow[d, "\sim"] & \pi_1(A \cap B) \arrow[l] \arrow[r] \arrow[d, "\sim"] & \pi_1(A) \arrow[d, "\sim"] \\
\pi_1(B \cup U) & \pi_1((A \cup V) \cap (B \cup U)) \arrow[l] \arrow[r] & \pi_1(A \cup V),
\end{tikzcd}$$

so the amalgamated free products are the same, and hence we get what we want.

4.3 Surfaces

Example 4.6.

We are going to look at a torus with a ball cut out of it (the ball is actually a triangle).

Then since it has genus one and has boundary a triangle, it is in some way homeomorphic to a triangle with a little handle.

But the torus cut out strongly deformation retract to $S^1 \vee S^1$, so $\pi_1(X, x_0) = \langle a, b \mid \rangle$ with $[r] = aba^{-1}b^{-1}$.

Now consider a whole lot of bumpy pizzas glued together. We will do this g many times to get F_g. Applying Seifert-Van Kampen (the closed version) g times gives

$$\pi_1(F_g, x_0) = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g \mid \rangle.$$

The boundary is $r_1 r_2 \cdots r_g$, so attached a two cell along it we get Σ_g, to get that the fundamental group is

$$\pi_1(\Sigma_g, x_0) = \langle a_1, b_1, \ldots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \rangle.$$

Nice!

Example 4.7.

Recall we can find $\mathbb{RP}^2$ as a skeleton: it has one-skeleton a, and the two-cell is attached along $a \cdot a$, so

$$\pi_1(\mathbb{RP}^2, *) \simeq \langle a \mid a^2 \rangle = \mathbb{Z}/2.$$

Now take Y, which will be $\mathbb{RP}^2$ with a disk cut out (again a triangle). This is homeomorphic to something with a triangle r as a boundary and a non-orientable thing in the middle.

Now $Y \simeq S^1$, so $\pi_1(Y, y_0) = \langle a \mid \rangle$, and $[r] = a^2$. Glue n of these together to form E_n. Then Seifert-Van Kampen says

$$\pi_1(E_n, y_n) = \langle a_1, \ldots, a_n \mid \rangle.$$

The boundary is then $r_1 r_2 \cdots r_n$, so attaching a 2-cell along it we get a closed surface

$$\pi_1(S_n, y_0) = \langle a_1, a_2, \ldots, a_n \mid a_1^2 a_2^2 \cdots a_n^2 \rangle.$$

5 Homology

5.1 Simplicial Complexes

Definition 5.1. A finite set of points $a_0, a_1, \dots, a_n \in \mathbb{R}^m$ is *affinely independent* if:

- $\sum_{i=1}^n t_i a_i = 0$ and
- $\sum_{i=1}^n t_i = 0$

if and only if $(t_1, \dots, t_n) = 0$.

Lemma 5.1. *$a_0, \dots, a_n \in \mathbb{R}^m$ are affinely independent if and only if $a_1 - a_0, \dots, a_n - a_0$ are linearly independent.*

Proof: Let $a_0, \dots, a_n$ be affinely independent. Suppose that

$$\sum_{i=0}^n s_i(a_i - a_0) = 0.$$

This means that

$$\left(-\sum_{i=1}^n s_i\right) a_0 + s_1 a_1 + \cdots + s_n a_n = 0,$$

and note that the sum of coefficients is 0, so we get $(s_1, \dots, s_n) = 0$ by affine independence.

We can do a similar thing for the converse.

In particular, since we have at most n linearly independent points in $\mathbb{R}^n$, we can have at most $n+1$ affinely independent points.

Definition 5.2. If $a_0, \dots, a_n \in \mathbb{R}^m$ are affinely independent (AI), then they define an *n-simplex*

$$\sigma = \langle a_0, \dots, a_n \rangle = \left\{ \sum_{i=1}^n t_i a_i \mid \sum_{i=0}^n t_i = 1, t_i \geq 0 \right\},$$

given by the convex hull of the points $a_0, \dots, a_n$. These a_i are called the *vertices* of σ, and we say they span σ.

If $x \in \langle a_0, \dots, a_n \rangle$, then x can be written uniquely as

$$x = \sum_{i=0}^n t_i a_i,$$

for real numbers $t_0, \ldots, t_n$. Call the t_i's the *barycentric coordinates* of x.

A *face* of a simplex $\sigma = \langle a_0, \ldots, a_n \rangle$ is a simplex τ spanned by a subset of $\{a_0, \ldots, a_n\}$. We write $\tau \leq \sigma$. Write $\tau < \sigma$ if τ is a proper face.

The *boundary* of a simplex σ, written $\partial\sigma$, is the union of all of its proper faces. The *interior* of σ, $\mathring{\sigma}$ is $\sigma \setminus \partial\sigma$.

Lemma 5.2. *Let σ be a p-simplex in $\mathbb{R}^m$ and τ be a p-simplex in $\mathbb{R}^n$. Then σ and τ are homeomorphic.*

Proof: Let $\sigma = \langle a_0, \ldots, a_p \rangle$ and $\tau = \langle b_0, \ldots, b_p \rangle$. Define $h : \sigma \to \tau$ by

$$\sum_{i=0}^{p} t_i a_i \mapsto \sum_{i=0}^{p} t_i b_i.$$

This is well-defined and a bijection, by uniqueness of barycentric coordinates. As the $a_i - a_0$ are linearly independent, h extends to an affine map $\hat{h} : \mathbb{R}^m \to \mathbb{R}^n$, so h is continuous. Similarly so is its inverse.

Definition 5.3. A *geometric* (or Euclidean) *simplicial complex* in $\mathbb{R}^n$ is a finite set K of simplices in $\mathbb{R}^n$ such that:

(i) If $\sigma \in K$ and $\tau \leq \sigma$, then $\tau \in K$.

(ii) If $\sigma, \tau \in K$, then either $\sigma \cap \tau = \emptyset$ or $\sigma \cap \tau$ is a face of both σ and τ.

I.e. a simplicial complex is what we get when we are allowed to glue a bunch of simplices along their faces.

The *dimension* of a simplicial complex K is the largest p such that K contains a p-simplex. The *polyhedron* of K is the space

$$|K| = \bigcap_{\sigma \in K} \sigma \subseteq \mathbb{R}^m.$$

The *d-skeleton* $K_{(d)}$ of K is the sub-simplicial complex containing all simplices of K of dimension at most d.

Observe that as we asserted K was finite, and simplices are compact, we get that $|K|$ is compact and Hausdorff.

Definition 5.4. A *triangulation* of a space X is a geometric simplicial complex K and a homeomorphism $h : |K| \to X$.

Example 5.1.

The *standard n-simplex* is $\Delta^n = \langle e_1, \ldots, e_{n+1} \rangle \subseteq \mathbb{R}^{n+1}$. It, along with its faces, defines a simplicial complex.

The *simplicial* $n-1$-*sphere* is the simplicial complex given by the proper faces of Δ^n. Its polyhedron is $\partial\Delta^n$.

In $\mathbb{R}^{n+1}$ consider the 2^{n+1} simplices given by $\langle \pm e_1, \pm e_2, \ldots, \pm e_{n+1} \rangle$. Let K be given by these and all their faces. Define $h : |K| \subset \mathbb{R}^{n+1} \to S^n$ by $x \mapsto x/|x|$. This is continuous, and a bijection. As $|K|$ and S^n are compact Hausdorff, this is a homeomorphism.

Definition 5.5. Write V_K for the set of vertices of K. A *simplicial map* f from K to L is a function $f : V_K \to V_L$ such that if $\sigma = \langle a_0, \ldots, a_n \rangle \in K$, then $\{f(a_0), \ldots, f(a_n)\}$ spans a simplex of L, called $f(\sigma)$. We write $f : K \to L$.

We write $\{f(a_0), \ldots, f(a_n)\}$ instead of saying $f(a_0), \ldots, f(a_n)$ is a simplex to account for the possibility that $f(a_i)$ are not affinely independent, otherwise we could not map a larger dimensional simplex to a smaller one.

Example 5.2.

The map $f : \Delta^1 \to \Delta^2$ by $(1, 0) \mapsto (1, 0, 0)$ and $(0, 1) \mapsto (0, 1, 0)$ is a simplicial map.

Geometrically we are including an edge into the 2-simplex.

The map $g : \Delta^2 \to \Delta^1$ given by $(1, 0, 0) \mapsto (1, 0)$, $(0, 1, 0) \mapsto (1, 0)$ and $(0, 0, 1) \mapsto (0, 1)$ is a simplicial map.

Geometrically we are reducing a 2-simplex into an edge.

Lemma 5.3. *A map $f : K \to L$ of simplicial complexes induces a continuous map $|f| : |K| \to |L|$, and $|f \circ g| = |f| \circ |g|$.*

Proof: For $\sigma \in K$, where $\sigma = \langle a_0, a_1, \ldots, a_p \rangle$, define $f_\sigma : \sigma \to |L|$ by

$$\sum_{i=0}^{p} t_i a_i \mapsto \sum_{i=0}^{p} t_i f(a_i),$$

which is linear in the t_i, so is continuous. If $\tau \leq \sigma$, then $f_\tau = f_\sigma|_\tau$, so $f_\sigma|_{\sigma\cap\sigma'} = f_{\sigma\cap\sigma'} = f_{\sigma'}|_{\sigma\cap\sigma'}$ for any other $\sigma' \in K$.

Hence the maps f_σ give rise to a continuous

$$|f| : |K| = \bigcup_{\sigma \in K} \sigma \to |L|,$$

by the gluing lemma. The formula for $|f|$ shows that it behaves as claimed under composition.

We can recover f from $|f|$ and the distinct sets and the discrete sets $V_K \subseteq |K|$, $V_L \subseteq |L|$ i.e. a simplicial map is the same as a continuous map $|K| \to |L|$ which sends vertices to vertices, and is affine on each simplex.

Definition 5.6. For a $x \in |K|$,

(i) The (open) *star* of x is the union of the interiors of the simplicies which contain x:
$$\mathrm{St}_K(x) = \bigcup_{\substack{\sigma \in K \\ x \in \sigma}} \mathring{\sigma} \subseteq \mathbb{R}^n.$$
The complement of $\mathrm{St}_K(x)$ is the union of all simplices which do not contain x, a polyhedron, so it is closed. Thus $\mathrm{St}_K 9x)$ is open.

(ii) The *link* of (x), $\mathrm{Lk}_K(x)$, is the union of those simplices which do not contain x, but are faces of a simplex which does contain x.

We can draw the stars and links for points in 2-simplices:

5.2 Simplicial Approximation

Definition 5.7. Let $f : |K| \to |L|$ be a continuous map. A *simplicial approximation* to f is a function $g : V_K \to V_L$ such that

$$f(\mathrm{St}_K(v)) \subseteq \mathrm{St}_L(g(v)),$$

for all $v \in V_K$.

Lemma 5.4. *If g is a simplicial approximation to a continuous map f, then g is a simplicial map, and f is homotopic to $|g|$. Furthermore, this homotopy may be taken relative to $\{x \in |K| \mid f(x) = |g|(x)\}$.*

Proof: To show that g defines a simplicial map, for $\sigma \in K$ we must show that the images of g of the vertices of σ span a simplex in L.

For $x \in \mathring{\sigma}$, then

$$x \in \bigcap_{v \in V_\sigma} \mathrm{St}_K(v),$$

so we have

$$f(x) \in \bigcap_{v \in V_\sigma} f(\mathrm{St}_K(v)) \subseteq \bigcap_{v \in V_\sigma} \mathrm{St}_L(g(v)).$$

If τ is the unique simplex of L with $f(x) \in \mathring{\tau}$, then each $g(v)$ is a vertex of τ. So $\{g(v)\}$ span a face of τ, which is a simplex of L.

Now we want to show that $f \simeq |g|$. If $|L| \subseteq \mathbb{R}^m$, then let $H : |K| \times I \to |L|$ be given by

$$(x, t) \mapsto tf(x) + (1-t)|g|(x).$$

This is continuous, and so we need to show that it lies in $|L|$. Let $x \in \mathring{\sigma} \subseteq |K|$, and suppose $f(x) \in \mathring{\tau} \subseteq |L|$. If $\sigma = \langle a_0, \dots, a_p \rangle$, then by the above, each $g(a_i)$ is a vertex of τ. Then

$$|g|(x) = \sum_{i=0}^{p} t_i g(a_i) \in \tau,$$

as it is a convex linear combination of vertices of τ. As $f(x) \in \tau$, each of $tf(x) + (1-t)|g|(x)$ lies in τ as well.

Definition 5.8. The *barycentre* of a simplex $\sigma = \langle a_0, a_1, \dots, a_p \rangle$ is the point

$$\hat{\sigma} = \frac{1}{p+1}(a_0 + a_1 + \cdots + a_p).$$

The *barycentric subdivision* of a simplicial complex K is

$$K' = \{\langle \hat{\sigma_0}, \dots, \hat{\sigma_p} \rangle \mid \sigma_i \in K \text{ and } \sigma_0 < \sigma_1 < \cdots < \sigma_p\}.$$

With this definition, we define $K^{(r)} = (K^{(r-1)})'$.

It is not obvious that this is a simplicial complex.

Proposition 5.1. *K' is a simplicial complex, and $|K'| = |K|$.*

Proof: There are many things to show in this proof. First we need, if $\sigma_0 < \sigma_1 < \cdots < \sigma_p$, then the $\hat{\sigma_i}$ are affinely independent. Indeed, suppose

that
$$\sum_{i=0}^{p} t_i \hat{\sigma}_i = 0 \text{ and } \sum_{i=0}^{p} t_i = 0.$$
Let $j = \max\{i \mid t_j \neq 0\}$. Then
$$\hat{\sigma}_j = -\sum_{i=0}^{j-1} \frac{t_i}{t_j} \hat{\sigma}_i \in \sigma_{j-1},$$
so $\hat{\sigma_j}$ lies in a proper face of σ_j, which is not possible. Thus all t_i must be 0.

Next we show that K' is a simplicial complex, which again has many parts. Let $\langle \hat{\sigma_0}, \ldots, \hat{\sigma_p} \rangle \in K'$. A face is given by omitting some of $\hat{\sigma_j}$. But omitting some σ_j's from $\sigma_0 < \sigma_1 < \cdots < \sigma_p$ still gives a strictly increasing chain of simplices of K.

Now let $\sigma' = \langle \hat{\sigma_0}, \ldots, \hat{\sigma_p} \rangle$ and $\tau' = \langle \hat{\tau}_0, \ldots, \hat{\tau_q} \rangle$, and consider $\sigma' \cap \tau'$. This is inside $\sigma_p \cap \tau_q$, which is a simplex δ of K. So there are simplices
$$\sigma'' = \langle \hat{\sigma_0} \cap \delta, \ldots, \hat{\sigma_p} \cap \delta \rangle, \qquad \tau'' = \langle \hat{\tau_0} \cap \delta, \ldots, \hat{\tau_q} \cap \delta \rangle$$
of K. Now $\sigma' \cap \tau' = \sigma'' \cap \tau''$. This reduces us to the case that σ'' and τ'' are contained in a simplex δ of K.

If σ'' and τ'' contain $\hat{\delta}$, let $\bar{\sigma}'', \bar{\tau}''$ be the faces of σ'', τ'' opposite to $\hat{\delta}$. Then $\sigma'' \cap \tau''$ is spanned by $\hat{\delta}$, and $\bar{\sigma}'' \cap \bar{\tau}''$. But $\bar{\sigma}'' \cap \bar{\tau}'' \subseteq \partial\delta$, which has smaller dimension than δ, so we can suppose it is a simplex of $\partial\delta$ by induction on dimension.

If neither σ'' nor τ'' contain $\hat{\sigma}$, then again $\sigma'' \cap \tau'' \subseteq \partial\delta$, so we can finish by induction on dimension.

Now we show that $|K'| = |K|$. Note that $\langle \hat{\sigma}_0, \ldots, \hat{\sigma}_p \rangle \leq \sigma_p \subseteq |K|$, so we have $|K'| \subseteq |K|$. Conversely, if $x \in \sigma = \langle a_0, \ldots, a_p \rangle \subseteq |K|$, then we can write it as
$$x = \sum_{i=0}^{p} t_i a_i.$$
We can reorder the a_i so that $t_0 \geq t_1 \geq \cdots \geq t_p$, so
$$\begin{aligned} x &= (t_0 - t_1)a_0 + 2(t_1 - t_2)\left(\frac{a_0 + a_1}{2}\right) + 3(t_2 - t_1)\left(\frac{a_1 + a_2 + a_3}{3}\right) + \cdots \\ &= (t_0 - t_1)\widehat{\langle a_0 \rangle} + 2(t_1 - t - 2)\widehat{\langle a_0, a_1 \rangle} + 3(t_2 - t_3)\widehat{\langle a_0, a_1, a_2 \rangle} + \cdots \\ &\in \langle \widehat{\langle a_0 \rangle}, \widehat{\langle a_0, a_1 \rangle}, \ldots, \widehat{\langle a_0, \ldots, a_p \rangle} \rangle \subseteq |K'|. \end{aligned}$$

The vertices of K are in bijection with the simplices of K. Choose a function $K \to V_K$ which assigns to σ some vertex of σ. Then we can take $g : V_{K'} \cong K \to V_K$.

If $\langle \hat{\sigma}_0, \dots, \hat{\sigma}_p \rangle$ is a simplex of K', then $\sigma_0 \le \dots \le \sigma_p$, and $g(\hat{\sigma}_i)$ is some vertex of σ_i, so is a vertex of σ_p. Thus $\{g(\hat{\sigma}_i)\}$ spans a face of σ, so is a simplex of K. So g is a simplicial map.

Also, if $\hat{\sigma} \in \tau' = \langle \hat{\tau}_0, \dots, \hat{\tau}_p \rangle \in K'$, then $\hat{\sigma} \in \tau_p$, so σ is a face of τ_p. Thus $v_\sigma \in \sigma \subseteq \tau_p$. So

$$\mathring{\tau}' \subseteq \mathring{\tau}_p \subseteq \mathrm{St}_K(v_\sigma).$$

Thus $\mathrm{St}_{K'}(\hat{\sigma}) \subseteq \mathrm{St}_K(v_\sigma) = \mathrm{St}_K(g(\hat{\sigma}))$, so g is a simplicial approximation to $\mathrm{id} : |K'| \to |K|$, so $|g| \simeq \mathrm{id}$.

Definition 5.9. The *mesh* of K is

$$\mu(K) = \max\{|v_0 - v_1| \mid \langle v_0, v_1 \rangle \in K\}.$$

Lemma 5.5. *Suppose K has dimension at most n. Then $\mu(K^{(r)}) \le (\frac{n}{n+1})^r \mu(K)$, so $\mu(K^{(r)}) \to 0$ as $r \to \infty$.*

Proof: It is enough to prove the case $r = 1$. Let $\langle \hat{\tau}, \hat{\sigma} \rangle \in K'$, so $\tau \le \sigma$ in K. Then

$$|\hat{\tau} - \hat{\sigma}| \le \max\{|v - \hat{\sigma}| \mid v \text{ a vertex of } \sigma\}.$$

Let $\sigma = \langle v_0, \dots, v_m \rangle$ with $m \le n$, and reorder so that the maximum is attained at $v = v_0$. Then

$$\begin{aligned} |v_0 - \hat{\sigma}| &= \left| v_0 - \frac{1}{m+1} \sum_{i=0}^m v_i \right| = \left| \frac{m+1}{m+1} v_0 - \frac{1}{m+1} \sum_{i=0}^m v_i \right| \\ &= \frac{1}{m+1} \left| \sum_{i=0}^m v_0 - v_i \right| \le \frac{1}{m+1} \sum_{i=1}^m |v_0 - v_i| \le \frac{m}{m+1} \mu(K) \\ &\le \frac{n}{n+1} \mu(K). \end{aligned}$$

Theorem 5.1 (Simplicial Approximation Theorem)**.** *Let $f : |K| \to |L|$ be a continuous map. Then there is a $r \gg 0$ and a simplicial map $g : K^{(r)} \to L$ such that g is a simplicial approximation to f.*

If f is simplicial on some $|N| \subset |K|$, we can take $g|_{V_N} = f|_{V_N}$.

Proof: The $\text{St}_L(w)$, for $w \in V_L$, is an open cover of $|L|$, so $\{f^{-1}\text{St}_K(w)\}$ is an open cover of $|K|$; let $\delta > 0$ be the Lebesgue number for this cover. Choose $r \gg 0$ such that $\mu(K^{(r)}) < \delta$. For each $v \in V_{K^{(r)}}$, we have

$$\text{St}_{K^{(r)}}(v) \subseteq B_{\mu(K^{(r)})}(v) \subseteq f^{-1}\text{St}_L(w),$$

for some $w \in V_L$. Define $g : V_{K^{(r)}} \to V_L$ by $g(v) = w$. Then

$$f(\text{St}_{K^{(r)}}(v)) \subseteq \text{St}_L(g(v)),$$

so g is a simplicial approximation to f. So g is a simplicial map. The final step is by choosing w carefully when $v \in V_N$.

Corollary 5.1. *If $n < m$, then any map $f : S^n \to S^m$ is homotopic to a constant map.*

Proof: Spheres are polyhedra. Take $S^n = |K|$, and $S^m = |L|$, then f is homotopic to g for some $g : K^{(r)} \to L$. This cannot hit any m-simplex of L, as K has dimension at most n. So $|g|$ must miss every point on the interior of some m-simplex it is not onto.

Thus, it factors through $(S^m - \{*\}) \simeq *$, so is homotopic to a constant map.

5.3 Simplicial Homology

Definition 5.10. Let K be a simplicial complex, and $\mathcal{O}_n(K)$ be the free abelian group with basis

$$\{[v_0, v_1, \dots, v_n] \mid v_i \text{ vertices of } K \text{ which span a simplex}\}.$$

The v_i are considered to be ordered, and could span a simplex of dim $< n$, i.e. we could have repeats.

Let $T_n(K) \leq \mathcal{O}_n(K)$ be the subgroup spanned by:

(i) $[v_0, \dots, v_n]$ containing a repeat.

(ii) $[v_0, \dots, v_n] - \text{sgn}(\sigma)[v_{\sigma(0)}, \dots, v_{\sigma(n)}]$ for a permutation σ of $\{0, 1, \dots, n\}$.

Then we can define

$$C_n(K) = \frac{\mathcal{O}_n(K)}{T_n(K)},$$

the quotient group.

Lemma 5.6. *There is a non-canonical isomorphism* $C_n(K) \cong \mathbb{Z}\{n\text{-simplices of } K\}$.

Proof: Choose a total order $\prec$ of V_K. Then each n-simplex of K defines a canonical ordered simplex $[\sigma] \in \mathcal{O}_n(K)$ by ordering its vertices such that $a_0 \prec a_1 \prec \cdots \prec a_n$. This gives a map $\phi : Z\{n\text{-simplices of } K\} \to \mathcal{O}_n(K)$, by $\sigma \mapsto [\sigma]$, and hence factors through $C_n(K)$.

For each $[a_0, \dots, a_n] \in \mathcal{O}_n(K)$, there is a unique permutation τ of $\{0, 1, \dots, n\}$ such that $a_{\tau(0)} \prec \cdots \prec a_{\tau(n)}$, and let $\mathrm{sgn}[a_0, \dots, a_n] = \mathrm{sgn}(\tau)$. Define $\rho : \mathcal{O}_n(K) \to \mathbb{Z}\{n\text{-simplices of } K\}$ by

$$[a_0, \dots, a_n] \mapsto \begin{cases} \mathrm{sgn}[a_0, \dots, a_n]\langle a_0, \dots, a_n\rangle & \text{no repeats,} \\ 0 & \text{repeats.} \end{cases}$$

For this to descend to $C_n(K)$, we need $T_n(K)$ to be in the kernel. Clearly $[a_0, \dots, a_n]$ with repeats are in the kernel, and

$$\begin{aligned} \rho([v_0, \dots, v_n] - \mathrm{sgn}(\sigma)[v_{\sigma(0)}, \dots, v_{\sigma(n)}]) &= \mathrm{sgn}[v_0, \dots, v_n]\langle v_0, \dots, v_n\rangle \\ &\quad - \mathrm{sgn}(\sigma)\,\mathrm{sgn}[v_{\sigma(0)}, \dots, v_{\sigma(n)}]\langle v_{\sigma(0)}, \dots, v_{\sigma(n)}\rangle \\ &= 0. \end{aligned}$$

So we get $\rho' : C_n(K) \to \mathbb{Z}\{n\text{-simplices of } K\}$. Now $\rho' = \phi'(\sigma) = \sigma$. If $[a_0, \dots, a_n]$ has no repeats, then

$$\begin{aligned} \phi' \circ \rho'([a_0, \dots, a_n]) &= \phi'(\mathrm{sgn}[a_0, \dots, a_n]\langle a_0, \dots, a_n\rangle) \\ &= \mathrm{sgn}[a_0, \dots, a_n]\,\mathrm{sgn}[a_{\tau(0)}, \dots, a_{\tau(1)}] \\ &= [a_0, \dots, a_n], \end{aligned}$$

modulo $T_n(K)$, so ϕ' and ρ' are the same.

Define a homomorphism

$$\begin{aligned} d_n : \mathcal{O}_n(K) &\to \mathcal{O}_{n-1}(K) \\ [v_0, \dots, v_n] &\mapsto \sum_{i=0}^{n} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n]. \end{aligned}$$

Lemma 5.7. d_n *sends* $T_n(K)$ *to* $T_{n-1}(K)$.

Proof: Note

$$\begin{aligned} d_n([v_0, \ldots, v_n] - \operatorname{sgn}(\sigma)[v_{\sigma(0)}, \ldots, v_{\sigma(n)}]) \\ &= \sum_{i=0}^{n} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n] \\ &- \sum_{i=0}^{n} (-1)^i \operatorname{sgn}(\sigma)[v_{\sigma(0)}, \ldots, \hat{v}_{\sigma(i)}, \ldots, v_{\sigma(n)}]. \end{aligned}$$

We need to show that this is trivial?? in $\mathcal{O}_{n-1}(K)/T_{n-1}(K)$. Suppose first $\sigma = (j, j+1)$, so $\operatorname{sgn}(\sigma) = -1$. Then

$$\begin{aligned} \sum_{i=0}^{n} (-1)^i \operatorname{sgn}(\sigma)[v_{\sigma(0)}, \ldots, \hat{v}_{\sigma(i)}, \ldots, v_{\sigma(n)}] \\ &= \sum_{i=0}^{n} (-1)^{j-1} [v_0, \ldots, \hat{v}_i, \ldots, v_{j+1}, v_j, v_{j+2}, \ldots, v_n] \\ &\quad + (-1)^{j+1} [v_0, \ldots, v_{j-1}, v_j, v_{j+1}, \ldots, v_n] \\ &\quad + (-1)^{j+1} [v_0, \ldots, v_{j-1}, v_{j+1}, v_{j+2}, v_j, \ldots, v_n] \\ &+ \sum_{i=j+2}^{n} (-1)^{j+1} [v_0, \ldots, v_{j+1}, v_j, v_{j+2}, \ldots, \hat{v}_i, \ldots, v_n]. \end{aligned}$$

In the first sum

$$[v_0, \ldots, \hat{v}_i, \ldots, v_{j-1}, v_{j+1}, v_j, \ldots, v_n] = -[v_0, \ldots, \hat{v}_i, \ldots, v_n] \mod T_{n-1}(K).$$

In the second sum, we get a similar thing. So the right hand side is equal to

$$\sum_{i=0}^{n} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n] \mod T_{n-1}(K),$$

as required. Since any σ is a product of $(j, j+1)$, we get the same for any σ.

Now suppose $[v_0, \ldots, v_n]$ is such that $v_j = v_{j+1}$, with permutations. Hence writing out $d_n[v_0, \ldots, v_n]$, notice the summations are in $T_{n-1}(K)$ as they have a repeat, and the other two terms cancel.

So d_n induces a homomorphism

$$d_n : C_n(K) \to C_{n-1}(K),$$

given by the same function.

Lemma 5.8. *The composition $d_{n-1} \circ d_n : C_n(K) \to C_{n-2}(K)$ is zero.*

Proof: At the level of $\mathcal{O}_n(K)$, compute

$$\begin{aligned} d_{n-1} \circ d_n[v_0, \dots, v_n] &= d_{n-1}\left(\sum_{i=0}^{n}(-1)^i[v_0, \dots, \hat{v}_i, \dots, v_n]\right) \\ &= \sum_{i=0}^{n}(-1)^i\left[\sum_{k=0}^{i-1}(-1)^k[v_0, \dots, \hat{v}_k, \dots, \hat{v}_i, \dots, v_n]\right. \\ &\left.+\sum_{k=i}^{n-1}(-1)^k[v_0, \dots, \hat{v}_i, \dots, \hat{v}_{k+1}, \dots, v_n]\right]. \end{aligned}$$

Then the coefficient of $[v_0, \dots, \hat{v}_a, \dots, \hat{v}_b, \dots, v_n]$ is $(-1)^a(-1)^b+(-1)^a(-1)^{b+1} = 0$. As $[v_0, \dots, v_n]$ generates, we get $d_{n-1} \circ d_n = 0$.

This means that

$$\mathrm{Im}(d_n : C_n(K) \to C_{n-1}(K)) \subseteq \mathrm{Ker}(d_{n-1} : C_{n-1}(K) \to C_{n-2}(K)).$$

Definition 5.11. The *n'th simplicial homology group* of K is

$$H_n(K) = \frac{\mathrm{Ker}(d_n : C_n(K) \to C_{n-1}(K))}{\mathrm{Im}(d_{n+1} : C_{n+1}(K) \to C_n(K))}.$$

Example 5.3.

Let K be the union of all the proper faces of the standard 2-simplex $\Delta^2 \subseteq \mathbb{R}^3$, i.e.

$$K = \{\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1, e_2\rangle, \langle e_2, e_3\rangle, \langle e_3, e_1\rangle\}.$$

Order the vertices as $e_1 \prec e_2 \prec e_3$. Then

$$\begin{aligned} C_0(K) &= \mathbb{Z}\{[e_1], [e_2], [e_3]\}, \\ C_1(K) &= \mathbb{Z}\{[e_1, e_2], [e_2, e_3], [e_1, e_3]\}, \end{aligned}$$

and $C_n(K) = 0$ for $n \geq 2$. Now note that $d_1 : C_1(K) \to C_0(K)$ is given by

$d_1[e_i, e_j] = [e_j] - [e_i]$, so in matrix form it is

$$\begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Note that $\operatorname{Im} d_1 = \langle [e_i] - [e_j] \rangle_{\mathbb{Z}}$, so

$$H_0(K) = \frac{\mathbb{Z}\{[e_1], [e_2], [e_3]\}}{\langle [e_i] - [e_j] \rangle} \cong \mathbb{Z}.$$

We have that $\operatorname{Ker} d_1 = \mathbb{Z}\{[e_1, e_2] - [e_1, e_3] + [e_2, e_3]\} \cong \mathbb{Z}$, and $\operatorname{Im} d_2 = 0$, so again $H_1(K) \cong \mathbb{Z}$.

Example 5.4.

Now let L be the standard 2-simplex $\Delta^2 \subseteq \mathbb{R}^3$, i.e.

$$L = \{\langle e_1 \rangle, \langle e_2 \rangle, \langle e_3 \rangle, \langle e_1, e_2 \rangle, \langle e_2, e_3 \rangle, \langle e_3, e_1 \rangle, \langle e_1, e_2, e_3 \rangle\}.$$

Then,

$$\begin{aligned} &C_0(L) = \mathbb{Z}\{[e_1], [e_2], [e_3]\}, \\ &C_1(L) = \mathbb{Z}\{[e_1, e_2], [e_2, e_3], [e_1, e_3]\}, C_2(L) &&= \mathbb{Z}\{[e_1, e_2, e_3]\}. \end{aligned}$$

So,

$$0 \xrightarrow{d_4} 0 \xrightarrow{d_3} C_2(K) \cong \mathbb{Z} \xrightarrow{d_2} C_1(K) \cong \mathbb{Z}^3 \xrightarrow{d_1} C_0(K) \cong \mathbb{Z}^3 \longrightarrow 0.$$

Now $d_2[e_1, e_2, e_3] = [e_1, e_2] - [e_1, e_3] + [e_1, e_2] \neq 0$, so d_2 is injective. Hence $H_2(L) = 0$. But also,

$$\begin{aligned} H_1(L) &= \frac{\operatorname{Ker} d_1}{\operatorname{Im} d_2} = 0, \\ H_0(L) &= \mathbb{Z}. \end{aligned}$$

5.4 Some Homological Algebra

Definition 5.12. A *chain complex* is a sequence $C_0, C_1, C_2, \dots$ of abelian groups, and homomorphisms $d_n : C_n \to C_{n-1}$ such that

$$d_{n-1} \circ d_n = 0,$$

for all n. We write this data as $C_\bullet$, and call d_n the *differentials* of $C_\bullet$. Then define

$$H_n(C_\bullet) = \frac{\operatorname{Ker} d_n}{\operatorname{Im} d_{n+1}}.$$

Write $Z_n(C_\bullet) = \operatorname{Ker} d_n$, and $B_n(C_\bullet) = \operatorname{Im} d_{n+1}$. We can think of Z_n as the n-cycles of $C_\bullet$, and B_n as the n-boundaries of $C_\bullet$.

A *chain map* $f_\bullet : C_\bullet \to D_\bullet$ is a sequence of homomorphisms $f_n : C_n \to D_n$ such that

$$\begin{tikzcd} C_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "d_{n+1}"] & D_{n+1} \arrow[d, "d_{n+1}"] \\ C_n \arrow[r, "f_n"] & D_n \end{tikzcd}$$

commutes, i.e. $f_n \circ d_{n+1} = d_{n+1} \circ f_{n+1}$.

A *chain homotopy* between $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ is a sequence of homomorphisms $h_n : C_n \to D_{n+1}$ such that

$$g_n - f_n = d_{n+1} \circ h_n + h_{n-1} \circ d_n.$$

Just accept this.

Lemma 5.9. *A chain map $f_\bullet : C_n \to D_n$ induces a homomorphism*

$$f_* : H_n(C_\bullet) \to H_n(D_\bullet)$$

by $[x] \mapsto [f_n(x)]$. Furthermore, if $g_\bullet$ is chain homotopic to $f_\bullet$, then $g_* = f_*$.

Proof: We need to show that f_* is well-defined.

(i) Let $[x] \in H_n(C_\bullet)$, i.e. $x \in C_n$ and $d_n(x) = 0$. Then

$$d_n f_n(x) = f_{n-1} d_n(x) = 0,$$

so $f_n(x) \in Z_n(D_*)$.

(ii) If $[x] = [y] \in H_n(C_\bullet)$, then $x - y \in B_n(C_*)$, so say $x - y = d_{n+1}(z)$.

Then

$$f_n(x) - f_n(y) = f_n d_{n+1}(z) = d_{n+1} f_{n+1}(z) \in B_n(D_\bullet),$$

so $[f_n(x)] = [f_n(y)]$, and so f_* is well-defined, and so a homomorphism.

Now let $g_\bullet$ be chain homotopic to $f_\bullet$, i.e.

$$g_n - f_n = d_{n+1} \circ h_n + h_{n-1} \circ d_n.$$

Let $x \in Z_n(C_\bullet)$, so

$$g_n(x) - f_n(x) = d_{n+1} h_n(x) + h_{n-1} d_n(x) \in B_n(D_\bullet),$$

so $[g_n(x)] = [f_n(x)]$.

Just as we did for homotopies of maps between spaces, one can show:

(i) Being chain homotopic defines an equivalence relation on the set of chain maps from $C_\bullet$ to $D_\bullet$. We write $f_\bullet \simeq g_\bullet$.

(ii) If $a_\bullet : A_\bullet \to C_\bullet$ is a chain map and $f_\bullet \simeq g_\bullet : C_\bullet \to D_\bullet$, then $f_\bullet \circ a_\bullet \simeq g_\bullet \circ a_\bullet$, and similarly with postcomposing.

Lemma 5.10. *If $f_\bullet : C_\bullet \to D_\bullet$ is a chain homotopy equivalence, then $f_* : H_n(C_\bullet) \to H_n(D_\bullet)$ is an isomorphism.*

Proof: Using a homotopy inverse $g_\bullet$, we have

$$f_* \circ g_* = (f_\bullet \circ g_\bullet)_* = (\mathrm{id}_{D_\bullet})_* = \mathrm{id}_{H_n(D_\bullet)},$$

and similarly $g_* \circ f_* = \mathrm{id}_{H_n(C_\bullet)}$.

Bonus exercise: let

$$\mathbb{Z}[n] = (\to 0 \overset{d_{n+1}}{\to} \mathbb{Z} \overset{d_n}{\to} 0 \overset{d_{n-1}}{\to} 0 \cdots).$$

Then describe the set of chain maps $\{\mathbb{Z}[n] \to C_\bullet\}$, up to chain homotopy.

5.5 Elementary Calculations

Return to the chain complexes $C_\bullet(K)$ associated to a simplicial complex K.

Lemma 5.11. *Let $f : K \to L$ be a simplicial map. Then the formula*

$$\begin{aligned} f_n : C_n(K) &\to C_n(L) \\ [a_0, \dots, a_n] &\mapsto [f(a_0), \dots, f(a_n)] \end{aligned}$$

is a well-defined homomorphism, and defines a chain map $f_\bullet : C_\bullet(K) \to C_\bullet(L)$, and hence gives us $f_* : H_n(K) \to H_n(L)$.

Proof: To be well-defined, we need the given formula to send $T_n(K)$ to $T_n(L)$. It does. To be a chain map, we check

$$\begin{aligned} f_{n-1} d_n [a_0, \dots, a_n] &= f_{n-1} \left(\sum_{i=0}^{n} (-1)^i [a_0, \dots, \hat{a}_i, \dots, a_n] \right) \\ &= \sum_{i=0}^{n} (-1)^i [f(a_0), \dots, f(\hat{a_i}), \dots, f(a_n)] \\ &= d_n f_n [a_0, \dots, a_n]. \end{aligned}$$

Definition 5.13. Say a simplicial complex K is a *cone* with a *cone point* $v_0 \in V_K$, if every simplex of K is a face of a simplex which has v_0 as a vertex.

Proposition 5.2. *If K is a cone with cone point v_0, then the inclusion $i : \{v_0\} \hookrightarrow K$ induces a chain homotopy equivalence $i_\bullet : C_\bullet(\{v_0\}) \to C_\bullet(K)$, and so*

$$H_n(K) \cong \begin{cases} \mathbb{Z}\{[v_0]\} & n = 0, \\ 0 & \textit{else.} \end{cases}$$

Proof: The only map $r : V_K \to \{v_0\}$ is the simplicial map $r : K \to \{v_0\}$, and $r \circ i = \mathrm{id}_{\{v_0\}}$. We show that $i \circ r \simeq \mathrm{id}_{C_\bullet(K)}$. Define $h_n : \mathcal{O}_n(K) \to \mathcal{O}_{n+1}(K)$ by

$$[a_0, \dots, a_n] \mapsto [v_0, a_0, \dots, a_n].$$

This is valid as every simplex is part of a face which contains v_0. Moreover, this map sends $T_n(K)$ into $T_{n+1}(K)$, so it descends to $h_n : C_n(K) \to C_{n+1}(K)$.

For $n > 0$, then

$$\begin{aligned}(h_{n-1} \circ d_n + d_{n+1} \circ h_n)[a_0, \dots, a_n] &= \left(\sum_{i=0}^{n} (-1)^i [v_0, a_0, \dots, \hat{a}_i, \dots, a_n]\right) \\ &\quad + \left([a_0, \dots, a_n] + \sum_{i=0}^{n} (-1)^{i+1} [v_0, a_0, \dots, \hat{a}_i, \dots, a_n]\right) \\ &= [a_0, \dots, a_n] = (\mathrm{id} - i_n \circ r_n)[a_0, \dots, a_n],\end{aligned}$$

as the last term is $[v_0, \dots, v_0] = 0$ as $n > 0$. For $n = 0$,

$$\begin{aligned}(h_{-1} \circ d_0 + d_1 \circ h_0)[a_0] &= d_1[v_0, a_0] = [a_0] - [v_0] \\ &= (\mathrm{id} - i_0 \circ r_0)[a_0].\end{aligned}$$

So h provides a chain homotopy from $\mathrm{id}_{C_\bullet(K)}$ to $i_\bullet \circ r_\bullet$.

Corollary 5.2. *The standard n-simplex $\Delta^n \subseteq \mathbb{R}^{n+1}$ and all its faces is a cone, with any vertex as a cone point. So*

$$H_i(K) \cong \begin{cases} \mathbb{Z} & i = 0, \\ 0 & \textit{else.} \end{cases}$$

Corollary 5.3. *Let K be the union of all the proper faces of $\Delta^n \subseteq \mathbb{R}^n$. Then for $n \geq 2$, we have*

$$H_i(K) = \begin{cases} \mathbb{Z} & i = 0, \\ \mathbb{Z} & i = n-1, \\ 0 & \textit{else.} \end{cases}$$

Proof: Note that K is the $(n-1)$-skeleton of L, so $i : K \hookrightarrow L$ gives us an isomorphism $C_i(K) \to C_i(L)$ for $i \leq n-1$. These chain complexes are

$$\begin{tikzcd} C_0(L) & C_1(L) \arrow[l, "d_0^L"'] & \cdots \arrow[l, "d_1^L"'] & C_{n-1}(L) \arrow[l, "d_{n-1}^L"'] & C_n(L) \arrow[l, "d_n^L"'] & 0 \arrow[l] \\ C_0(K) \arrow[u, equal] & C_1(K) \arrow[l, "d_0^L"'] \arrow[u, equal] & \cdots \arrow[l, "d_1^K"'] & C_{n-1}(K) \arrow[l, "d_{n-1}^K"'] \arrow[u, equal] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \end{tikzcd}$$

For $i \le n-2$, $H_i(K) = H_i(K)$. For $i > n-1$, $H_i(K) = 0$ as there are no i-simplices. Hence

$$H_{n-1}(K) = \frac{\operatorname{Ker}(d_{n-1}^K)}{\operatorname{Im}(d_n^K)} = \operatorname{Ker}(d_{n-1}^K) = \operatorname{Ker}(d_{n-1}^L).$$

As $H_{n-1}(L) = 0$, $\operatorname{Ker}(d_{n-1}^L)/\operatorname{Im}(d_n^L) = 0$, so they are equal. As $H_n(L) = 0$, we see that $d_n^L : \mathbb{Z} = C_n(L) \to C_{n-1}(L)$ is injective, so $H_{n-1}(K) = \operatorname{Ker}(d_{n-1}^L) = \operatorname{Im}(d_n^L) \cong \mathbb{Z}$, as it is generated by $d_n^L[e_0, \dots, e_n]$.

Lemma 5.12. *There is an isomorphism $H_0(K) \cong \mathbb{Z}\{\pi_0(|K|)\}$.*

Proof: Note we have a homomorphism $\phi : C_0(K) \to \mathbb{Z}\{\pi_0|K|\}$ by $[v] \mapsto$ the path component of $v \in |K|$.

This is onto, as any path-component of $|K|$ contains a vertex.

If $[v, w]$ is an ordered 1-simplex, then $d_1[v, w] = [w] - [v]$. But $[v]$ and $[w]$ live in the same path-component, as the 1-simplex $\langle v, w\rangle$ goes between them. So $\operatorname{Im}(d_1) \subseteq \operatorname{Ker}\phi$, hence we get an induced $\phi : H_0(K) \to \mathbb{Z}\{\pi_0|K|\}$.

If $\phi([v]) = \phi([w])$, then choose a path $\gamma : I \to |K|$ from v to w. By simplicial approximation, $I = \Delta^1$ can be subdivided so that there is is $g : (\Delta^1)^{(r)} \to K$, with $|g| \simeq \gamma$, i.e. there are 1-simplices $[v, v_1], [v_1, v_2], \dots, [v_n, w]$. Then

$$[w] - [v] = d_1([v, v_1] + [v_1, v_2] + \cdots + [v_n, w]),$$

hence $[v] - [w] \in H_0(K)$.

5.6 Mayer-Vietoris Theorem

Definition 5.14. Say that a pair of homomorphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is *exact* if $\operatorname{Im} f = \operatorname{Ker} g$. More generally, a collection of homomorphisms

$$\cdot \longrightarrow A_i \longrightarrow A_{i+1} \longrightarrow A_{i+2} \longrightarrow A_{i+3} \longrightarrow \cdots$$

is *exact*if it is exact at each A_j, where A_j has a homomorphism in and out.

A *short exact sequence* is an exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0,$$

so f is injective, $\operatorname{Im} f = \operatorname{Ker} g$ and g is surjective. The chain maps $i_\bullet : A_\bullet \to B_\bullet$ and $j_\bullet : B_\bullet \to C_\bullet$ form a *short exact sequence* of chain complexes if each

$$0 \longrightarrow A_n \xrightarrow{i_n} B_n \xrightarrow{j_n} C_n \longrightarrow 0$$

is a short exact sequence.

Theorem 5.2 (Zig-zag Lemma)**.** *If*

$$0 \longrightarrow A_\bullet \xrightarrow{i_\bullet} B_\bullet \xrightarrow{j_\bullet} C_\bullet \longrightarrow 0$$

is a short exact sequence of chain complexes, then there are natural homomorphisms $\partial_* : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ *such that*

$$\begin{tikzcd}
 & & \vdots \arrow[dll] \\
H_n(A_\bullet) \arrow[r, "i_*"'] & H_n(B_\bullet) \arrow[r, "j_*"] & H_n(C_\bullet) \arrow[dll, "\partial_*"] \\
H_{n-1}(A_\bullet) \arrow[r, "i_*"'] & H_{n-1}(B_\bullet) \arrow[r, "j_*"] & H_{n-1}(C_\bullet) \arrow[dl] \\
 & \vdots &
\end{tikzcd}$$

is an exact sequence.

Proof: We begin by constructing ∂_*, using the snake lemma. Look at this diagram:

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "d_n"] & B_n \arrow[r, "j_n"] \arrow[d, "d_n"] & C_n \arrow[r] \arrow[d, "d_n"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "i_{n-1}"] & B_{n-1} \arrow[r, "j_{n-1}"] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

Pretty cool right. We are now going to chase this diagram. Let $[x] \in H_n(C_\bullet)$. Then $d_n x = 0$. By exactness of the top row, we can choose $y \mapsto x$.

Applying $d_n y$ gives us an element of B_{n-1}. But by commutativity, $j_{n-1}(d_n y) = 0$. But by exactness, this means there is $z \in A_{n-1}$ that maps to $d_n y$.

What can we say about z? Well $i_{n-2}d_{n-1}(z) = d_{n-1}i_{n-1}(z) = d_{n-1}d_n(y) = 0$. But as i_{n-1} is injective, this shows $d_{n-1}(z) = 0$. Hence z is a cycle, and so we can let $\partial_*[x] = [z]$.

Now we check ∂_* is well-defined. Suppose $[x] = [x'] \in H_n(C_\bullet)$. Then $x - x' = d_{n+1}(a)$ for some $a \in C_{n+1}$. The same process for x' gives a $y' \in B_n$, and as j_{n+1} is surjective, we can write $a = j_{n+1}(b)$. Then

$$j_n(y - y') = x - x' = j_n d_{n+1}(b),$$

so by exactness at B_n,

$$y - y' = d_{n+1}(b) + i_n(c),$$

for some $c \in A_n$. Now z' is such that $i_{n-1}(z') = d_n(y')$. Then

$$\begin{aligned} i_{n-1}(z - z') &= d_n(y - y') = d_n(d_{n+1}(b) + i_n(c)) \\ &= d_n i_n(c) = i_{n-1} d_n(c). \end{aligned}$$

The injectivity of i_{n-1} shows that $z - z' \in d_n(c)$, so $[z] = [z'] \in H_{n-1}(A_\bullet)$.

Now we show that ∂_* is a homomorphism. Given $[x_1], [x_2] \in H_n(C_\bullet)$ with components y_1, y_2, z_1, z_2, we choose $y_1 + y_2$ to be the lift of $x_1 + x + 2$. This gives $z_1 + z_2$ as the result, so

$$\partial_*[x_1 + x_2] = [z_1 + z_2] = [z_1] + [z_2].$$

This shows ∂_* exists. Now we show the sequence we defined is exact. First we show it at $H_n(C_\bullet)$. Let $[x] \in \mathrm{Im}(j_*)$, so there exists $y \in B_n$ such that $j_n(y) = x$ and y is a cycle.

We can use this y to calculate $\partial_*[x]$. As y is a cycle, $d_n(y) = 0$, so z is 0, hence $\partial_*[x] = 0$.

Suppose now that $\partial_*[x] = 0$. We calculate this by choosing $y \in B_n$ and taking the corresponding z. Then $z = d_n(t)$, as $[z] = 0$. Thus $j_n(y - i_n(t)) = x$, and

$$d_n(y - i_n(t)) = d_n y - d_n i_n(t) = i_n(z - z) = 0.$$

So $j_*[y - i_n(t)] = [x]$, so $[x] \in \mathrm{Im}(j_*)$.

We also need exactness at $H_n(B_\bullet)$. As $j_n \circ i_n = 0$, we have $\mathrm{Im}(i_*) \subseteq \mathrm{Ker}(j_*)$. Now suppose $j_*[y] = 0$. Then $j_n(y) = d_{n+1}(a)$ for some $a \in C_{n+1}$. Let $b \in B_{n+1}$ be such that $j_{n+1}(b) = a$. Then

$$j_n(y - d_{n+1}(b)) = d_{n+1}(a) - j_n d_{n+1}(b) = d_{n+1}(a) - d_{n+1} j_{n+1}(b) = 0,$$

hence $y - d_{n+1}(b) = i_n(t)$. Since $d_n(y - d_{n+1}(b)) = 0$, we get

$$[y] = [y - d_{n+1}(b)] = [i_n(t)] = i_*[t],$$

as required. Finally we want exactness at $H_n(A_\bullet)$. Let $[z] = \partial_*[x]$. Then $i_n(z) = d_{n+1}(g)$, so $i_*[z] = 0$. Conversely let $[z]$ be such that $i_*[z] = 0$. Then $i_n(z) = d_{n+1}(g)$ for some g, so $[z] = \partial_*[j_{n+1}(y)]$.

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