II Large Ordinals

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1 The First Interesting Ordinal

The first interesting ordinal is ϵ_0: the first fixed point of ω^α.

The interesting property has to do with PA: Peano Arithmetic. Famously Gödel showed PA is incomplete: $\mathsf{PA} \nvdash CON(\mathsf{PA})$.

ϵ_0 is the first ordinal that PA can't prove is a well-ordering.

ϵ_0 is the proof-theoretic ordinal of PA.

What does this mean? If we take $\omega + \omega$, there exists a relation on $\mathbb{N}$, which gives a well-ordering of order-type $\omega + \omega$, and is computable.

PA proves, for any p and well-ordering,

$$\mathsf{PA} \vdash (\forall x)((\forall y)(yRx \Rightarrow p(y)) \Rightarrow p(x)).$$

A digression: consider the fast-growing hierarchy: $f_1(x) = 2x, f_2(x) = 2^x = f_1^{(x)}(1), f_3(x) = 2^{2^{\cdots}} = f_2^{(x)}(1)$.

Then $f_\omega(x) = f_x(x), f_{\omega+1}(x) = f_\omega^{(x)}(1)$, and we can keep going till f_{ϵ_0}.

Now consider the statement $(\forall x)(\exists y)p(x, y)$. If the least such y (as a function of x) grows as fast as f_{ϵ_0}, then PA cannot prove it.

In 1930's, Gentzen proved that, given a proof of t from PA, one can associate an ordinal $< \epsilon_0$, and associate a rooted tree of decreasing ordinals starting from t.

Gentzen also showed if $PA \vdash \perp$, then there would be an infinitely decreasing branch, which cannot happen as ordinals.

Moreover, if PA proves ϵ_0, running this process would produce an infinite tree.

2 The second interesting ordinals

Let's look at Γ_0. First, define $0 * \alpha = w^\alpha$. Then let $1 * \alpha$ be the α'th fixed point of $\beta \to 0 * \beta$, namely $1 * \alpha = \epsilon_\alpha$.

Similarly, there are fixed points of $1 * \alpha$, so let $2 * \alpha$ be the α'th fixed point of $\beta \mapsto 1 * \beta = \epsilon_\beta$. We have $2 * 0 = \epsilon_{\epsilon_{\cdots}} = \zeta_0$.

We can continue to get $n * \alpha$, and then define $\omega * \alpha = \sup\{n * \alpha \mid n \in \mathbb{N}\}$. We can define $\omega + 1 * \alpha$ similarly.

The question is whether there exists α such that $\alpha * 0 = \alpha$. The answer is yes, as let $\alpha_0 = 0$, $\alpha_{n+1} = \alpha_n * 0$, then taking the supremum.

The fixed point is then Γ_0, which is said to be inpredicative.

We can keep going. Let $1 * 0 * 0 = \Gamma_0$, and $1 * 1 * \alpha$ be the α'th fixed point of $\beta \mapsto 1 * 0 * \beta$. Keeping on going, we get $\alpha * 0 * 0$, and the fixed point is the Ackermann ordinal.

Going further, $\sup\{1 * 0, 1 * 0 * 0, 1 * 0 * 0 * 0, \ldots\}$ is the small Veblen ordinal. We can also keep going, define $1 * 0 * 0 * \cdots$ infinitely many times, and get the fixed point to get the large Veblen ordinal.

3 The third interesting ordinal

Let's look at B, the Bachmann-Howard ordinal.

This is constructed by a function ϕ, using $0, 1, +, \cdot, \hat{}, \omega$ and ω_1, meaning an ordinal that is bigger than any other ordinal.

Then we define $\phi(\alpha)$ to be the least ordinal that you cannot construct from the above constructions and ϕ at times previously.

Computing, $\phi(0) = \epsilon_0$, and $\phi(1) = \epsilon_1$. In general, $\phi(n) = \epsilon_n$.

Then $\phi(\omega) = \epsilon_\omega$, and we can conjecture $\phi(\alpha) = \epsilon_\alpha$. However, something weird happens at ζ_0.

As predicted, $\phi(\zeta_0) = \epsilon_{\zeta_0} = \zeta_0$. But $\phi(\zeta_0 + 1) = \epsilon_{\zeta_0} = \zeta_0$.

So then we get $\phi(\alpha) = \zeta_0$ for all α with $\zeta_0 \leq \alpha < \omega_1$. Moreover $\phi(\omega_1) = \zeta_0$.

But magically, $\phi(\omega_1 + 1) = \epsilon_{\zeta_0+1}$, and so on: we get $\phi(\omega_1 + \alpha) = \epsilon_{\zeta_0+\alpha}$ until $\alpha = \zeta_1$. Then $\phi(\omega_1 \cdot 2) = \zeta_1$, $\phi(\omega_1 \cdots 3) = \zeta_2$, until $\phi(\omega_1 \cdot \alpha) = \zeta_\alpha$ until $\alpha = 3 * 0$.

Then $\phi(\omega_1^3) = 4 * 0$, and $\phi(\omega_1^\omega) = \omega * 0$, and then taking the limit, $\phi(\omega_1^{\omega_1}) = \Gamma_0$. Moreover $\phi(\omega_1^{\omega_1^2})$ is the Ackerman ordinal, $\phi(\omega_1^{\omega_1^\omega})$ is the small Veblen ordinal, and also $\phi(\omega_1^{\omega_1^{\omega_1}})$ is the large Veblen ordinal.

But notice our notation stops at $\omega_1^{\omega_1^{\cdot^{\cdot^{\cdot}}}}$, and we let $\phi(\omega_1^{\omega_1^{\cdot^{\cdot^{\cdot}}}}) = B$.

4 The fourth interesting ordinal

This is ω_1^{CK}. For each of the previous ordinals, we can write a computer program to describe an ordinal.

Call α computable if as before: if there is a well-ordering of the naturals that a computer program can check. However, there are only countably many programs, so let ω_1^{CK} be the first non-computable ordinal.