

# III Statistical Field Theory

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Based on Lectures by Prof. Harvey Reall

October 24, 2024

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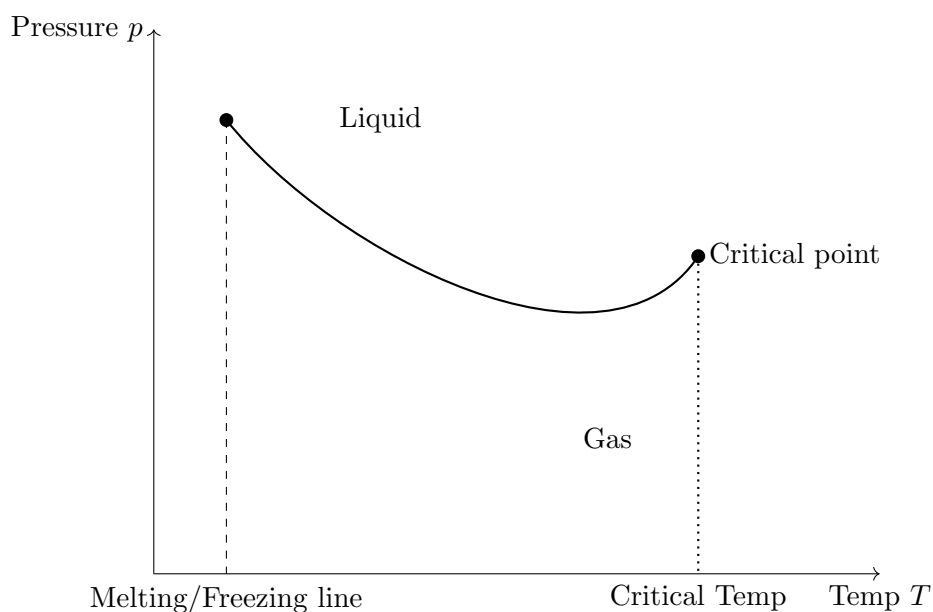
## 0 Introduction

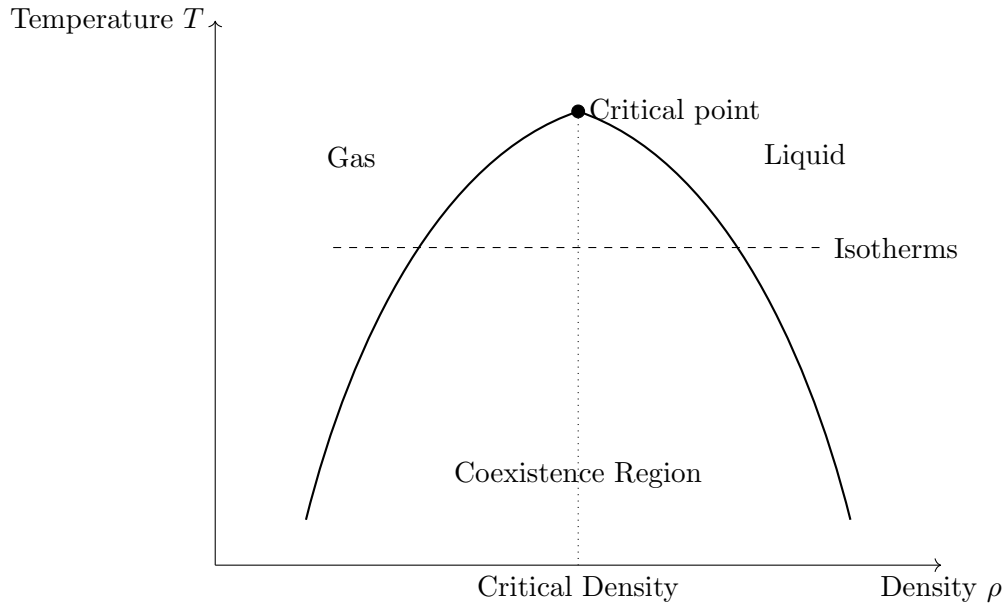
Office hours: Friday 2-3pm, in B2.09. We are following Tong's notes and example sheets.

Books include the one by Goldenfeld, and Kardar.

### 0.1 Motivation

*Universality*: sometimes very different physical systems exhibit the same behaviour, for example a liquid-gas system.





Experimentally,

$$|\rho_t - \rho_c| \propto |T - T_c|^\beta,$$

for  $\beta \simeq 0.327$ .

Now consider a ferromagnet, where  $T_c$  is the Curie temperature. For  $T > T_c$ , the magnetization is  $M = 0$ , however for  $T < T_c$ ,  $T \sim T_c$ , we find

$$M \propto (T_c - T)^\beta,$$

where the coefficient  $\beta$  is experimentally also 0.327. We want to know why this is, and also why  $\beta$  is this constant.

In this course, we are looking at the classical statistical mechanics of fields.

# 1 From Spins to Fields

## 1.1 The Ising Model

This is a simple model for a magnet. In  $d$  spatial dimensions, consider a lattice with  $N$  sites. On the  $i$ 'th site, we have a 'spin'  $s_i \in \{-1, 1\}$ .

A configuration of spins  $\{s_i\}$  has energy

$$E = -B \sum_i s_i - J \sum_{\langle i, j \rangle} s_i s_j.$$

An important question is, how does the physics depend on the parameters  $B, J$  and  $T$ ?

If  $J > 0$ , then the spins prefer to align, as  $\uparrow\uparrow$  or  $\downarrow\downarrow$ . This is a ferromagnet. If  $J < 0$ , the spins prefer to antialign, as  $\uparrow\downarrow$  or  $\downarrow\uparrow$ . This is an antiferromagnet.

Assume that  $J > 0$ . If  $B > 0$ , then the spins prefer to be  $\uparrow$ , and for  $B < 0$ , the spins prefer to be  $\downarrow$ .

Now let's consider changing temperature. Intuitively, for low temperature  $T$ , the system prefers to minimize  $E$ , which gives an ordered state. For high  $T$ , we want to maximize  $S$ , the entropy, which gives a disordered state.

In the canonical ensemble, we have

$$p[s_i] = \frac{e^{-\beta E[s_i]}}{Z},$$

where we recall  $\beta = 1/T$ . Moreover we always assume  $k_B = 1$ . Here  $Z$  is the partition function

$$Z(T, B) = \sum_{\{s_i\}} e^{-\beta E[s_i]}.$$

The *thermodynamic free energy* is

$$F_{\text{thermo}}(T, B) = \langle E \rangle - TS = -T \log Z.$$

Another observable is the *magnetization*

$$m = \frac{1}{N} \left\langle \sum_{i=1}^N s_i \right\rangle \in [-1, 1].$$

This distinguishes ordered phases, where  $m \neq 0$ , and disordered phases, where  $m = 0$ . Using the partition function,

$$m = \sum_{\{s_i\}} \frac{e^{-\beta E[s_i]}}{Z} \cdot \frac{1}{N} \sum_i s_i = \frac{1}{N\beta} \frac{\partial}{\partial B} \log Z.$$

Therefore it suffices to find the partition function. For  $d = 1$ , this is easy. For  $d = 2$  there is no analytic solution except for the square lattice with  $B = 0$ .

For the other cases there is no exact solution. Our aim is to approximate in a way that correctly captures long-distance behaviour. We define  $m$  for any  $\{s_i\}$  by

$$m = \frac{1}{N} \sum s_i.$$

Then write

$$Z = \sum_m \sum_{\{s_i\}|m} e^{-\beta E[s_i]} := \sum_m e^{-\beta F(m)}.$$

Notice changing  $s_i$  changes  $m$  by  $2/N$  so  $m$  is quantized into distances of  $2/N$ . For large  $N$ , we can approximate this as continuous, so

$$Z \approx \frac{N}{2} \int_{-1}^1 dm e^{-\beta F(m)}.$$

$F(m)$  is the *effective free energy*. This depends on  $T, B$  and  $m$ . It contains more information than  $F_{\text{thermo}}$ . If  $f(m) = F(m)/N$ , then

$$Z \propto \int_{-1}^1 dm e^{-\beta N f(m)}.$$

For  $N$  large,  $\beta f(m) = \mathcal{O}(1)$ , as it is intensive, so this integral will be dominated by the minimum of  $f$ , where

$$\left. \frac{\partial f}{\partial m} \right|_{m=m_{\min}} = 0.$$

Here  $m_{\min}$  is the equilibrium value of the magnetization. By the saddle-point approximation,

$$Z \propto e^{-\beta N f(m_{\min})} = e^{-\beta F(m_{\min})}.$$

Thus,

$$F_{\text{thermo}}(T, B) = F(m_{\min}(T, B), T, B).$$

However computing  $F(m)$  is hard. For a first attempt, we use the “mean field approximation”. Here we replace  $s_i$  with  $m$ , so

$$E = -B \sum_i m - J \sum_{\langle i,j \rangle} m^2 = -BNm - \frac{1}{2} N J q m^2.$$

In this,  $q$  is the number of nearest neighbours. For  $d = 1$ ,  $q = 2$ . In general it is  $2d$ . In this approximation,

$$Z \approx \sum_m \Omega(m) e^{-\beta E[m]},$$

where  $\Omega(m)$  is the number of configurations with average value  $m$ .

Let  $N_\uparrow$  be the number of up spins, and  $N_\downarrow = N - N_\uparrow$  the number of down spins. Then,

$$m = \frac{N_\uparrow - N_\downarrow}{N} = \frac{2N_\uparrow - N}{N},$$

so

$$\Omega(m) = \frac{N!}{N_\uparrow!(N - N_\uparrow)!} = \frac{N!}{N_\uparrow!N_\downarrow!}.$$

By Stirling's approximation,

$$\log n! = n \log n - n$$

for large  $n$ , so

$$\log \Omega \approx N \log N - N_\uparrow \log N_\uparrow - N_\downarrow \log N_\downarrow.$$

Dividing by  $N$  and substituting in  $m$  for  $N_\uparrow, N_\downarrow$ ,

$$\frac{\log \Omega}{N} \approx \log 2 - \frac{1}{2}(1+m) \log(1+m) - \frac{1}{2}(1-m) \log(1-m).$$

Since

$$e^{-\beta N f(m)} = \Omega(m) e^{-\beta E(m)}$$

in the mean field approximation, taking the logarithm we find

$$f(m) = -Bm - \frac{1}{2}Jqm^2 - T \left[ \log 2 - \frac{1}{2}(1+m) \log(1+m) - \frac{1}{2}(1-m) \log(1-m) \right].$$

We minimize

$$\begin{aligned} \frac{\partial f}{\partial m} = 0 &\implies \beta(B + Jqm) = \frac{1}{2} \log \left( \frac{1+m}{1-m} \right) \\ &\implies m = \tanh[\beta(B + Jqm)], \end{aligned}$$

where  $B_{\text{eff}} = B + Jqm$ . The intuition is that each spin feels an effective magnetic field, given by the actual magnetic field, and the overall spin.

## 1.2 Landau Theory of Phase Transitions

At a phase transition, some quantity (an *order parameter*) is not smooth. For us, this is  $m$ . For small  $m$ ,

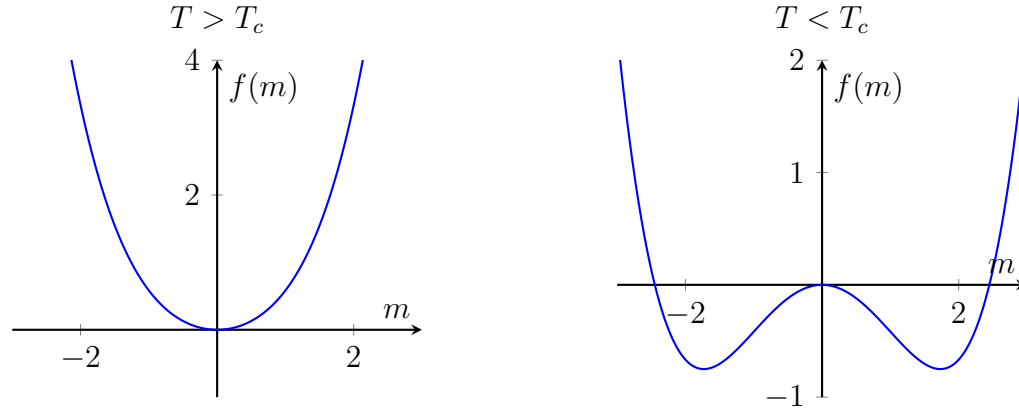
$$f(m) \approx -T \log 2 - Bm + \frac{1}{2}(T - Jq)m^2 + \frac{1}{12}Tm^4 + \dots$$

In equilibrium,  $m = m_{\min}$ : how does this behave as we vary  $T$  and  $B$ ?

First we look at  $B = 0$ . Note the first part does not change the minimization, so

$$f(m) \approx \frac{1}{2}(T - T_c)m^2 + \frac{1}{12}Tm^4 + \dots,$$

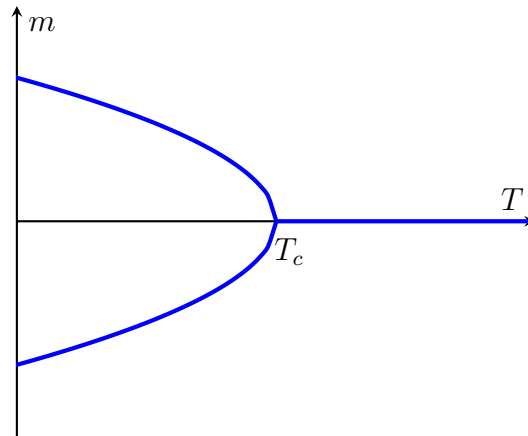
where  $T_c = Jq$ .



For  $T < T_c$ ,  $m_{\min} = 0$ . For  $T > T_c$ ,  $m_{\min} = \pm m_0$ , where

$$m_0 = \sqrt{\frac{3(T_c - T)}{T}}.$$

Hence here is a phase transition at  $T = T_c$ . For  $T > T_c$ , we have  $m = 0$ , which is a disordered phase, and for  $T < T_c$ ,  $m \neq 0$ , giving an ordered phase.



Here,  $m$  is continuous at  $T = T_c$ , giving a continuous phase transition, or second order phase transition.



Note that  $F$  is invariant under  $\mathbb{Z}_2$  symmetry: if we swap  $m \rightarrow -m$ , and  $B \rightarrow -B$ . For  $T < T_c$ , either  $m = +m_0$  or  $m = -m_0$ , the  $\mathbb{Z}_2$  symmetry does not preserve the ground state, known as ‘spontaneous symmetry breaking’ (SSB).

At finite  $N$ ,  $Z$  is *analytic* in  $T, B$ . Therefore the phase transition only occurs for  $N \rightarrow \infty$ , and SSB also only occurs for  $N \rightarrow \infty$ : in this case

$$m = \lim_{B \rightarrow 0} \lim_{N \rightarrow \infty} \frac{1}{N} \sum \langle s_i \rangle.$$

Note the order of the limits are important, otherwise we get 0.

For finite  $N$ , the  $\mathbb{Z}_2$  symmetry tells us  $F_{\text{thermo}}$  is even in  $B$ , so

$$\langle m \rangle = -\frac{1}{N} \frac{\partial F_{\text{thermo}}}{\partial B} \Big|_{B=0} = 0.$$

The heat capacity is

$$C = \frac{\partial \langle E \rangle}{\partial T}, \quad \langle E \rangle = -\frac{\partial \log Z}{\partial \beta}.$$

Thus we find

$$C = \beta^2 \frac{\partial^2 \log Z}{\partial \beta^2},$$

and notice that

$$\log Z = -\beta N f(m_{\min}) = \begin{cases} \text{const} & T > T_c, \\ \frac{3N}{4} \frac{(T_c - T)^2}{T^2} + \text{const} & T < T_c, \end{cases}$$

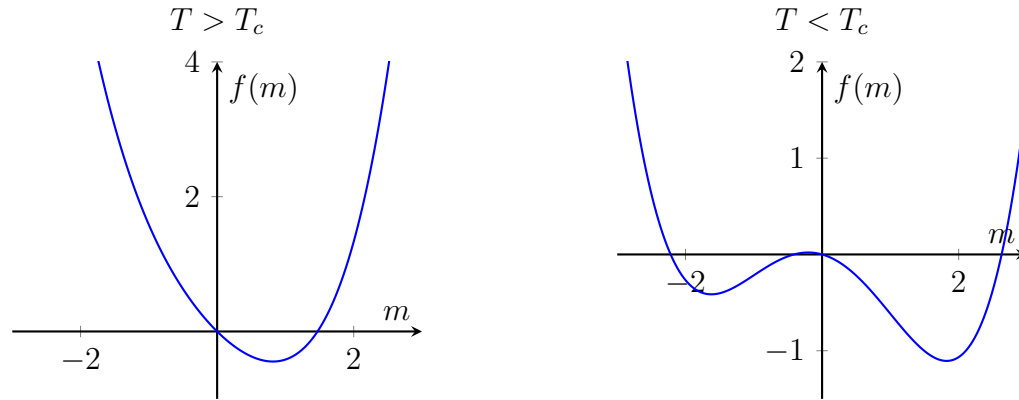
so

$$c = \frac{C}{N} \rightarrow \begin{cases} 0 & T \rightarrow T_c^+, \\ 3/2 & T \rightarrow T_c^-. \end{cases}$$

Therefore  $c$  is discontinuous at  $T = T_c$ .

For  $B > 0$ , we find

$$f(m) = -Bm + \frac{1}{2}(T - T_c)m^2 + \frac{1}{12}Tm^4 + \dots$$

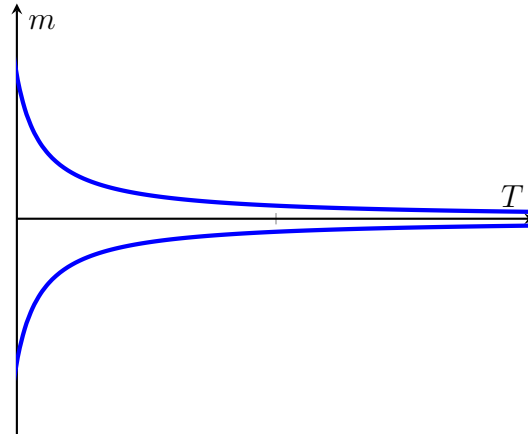


In this case,

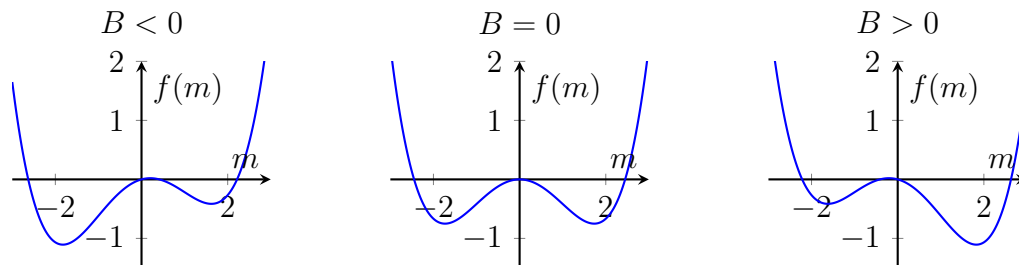
$$m_{\min} = \frac{B}{T},$$

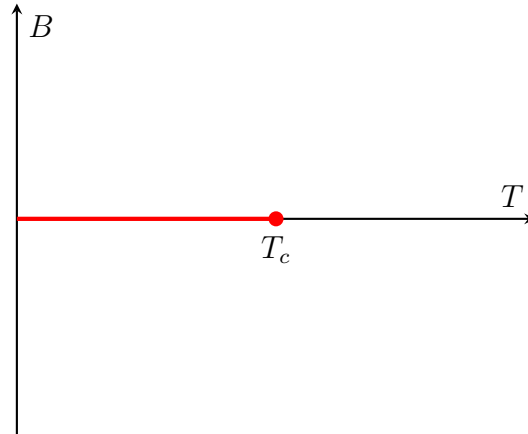
for  $T \rightarrow \infty$ .

In this case  $m_{\min}$  depends smoothly on  $T$ , so there is no phase transition if  $T$  is varied at fixed  $B \neq 0$ .



But, if we vary  $B$  at fixed  $T < T_c$ , then  $m$  jumps discontinuously from  $m_0$  to  $-m_0$  as  $B$  decreases from positive to negative, which is an example of a first order phase transition.





We can draw the phase transition points as above.

Consider the behaviour near the critical point. If we fix  $T = T_c$ , then

$$f \approx -Bm + \frac{1}{12}Tm^4 + \dots,$$

so minimizing, we see that  $m^3 \sim B$ , and  $m \sim B^{1/3}$ .

We can also define the *magnetic susceptibility* as

$$\chi = \left( \frac{\partial m}{\partial B} \right)_T.$$

For  $T > T_c$ ,

$$f(m) = -Bm + \frac{1}{2}(T - T_c)m^2 + \dots,$$

so

$$m \approx \frac{B}{T - T_c} \implies \chi = \frac{1}{T - T_c}.$$

For  $T < T_c$ , we write  $m = m_0 + \delta m$ , and solving for  $\delta m$  to leading order, we find

$$m = m_0 + \frac{B}{2(T_c - T)} \implies \chi = \frac{1}{2(T_c - T)}.$$

Hence,

$$\chi \sim \frac{1}{|T - T_c|}.$$

We have been using the MFT approximation. Does this give the correct results?

- $d = 1$ . No, there is no phase transition.

- $d = 2, 3$ . The phase diagram is qualitatively correct, but the qualitative predictions at the critical point are incorrect.
- $d \geq 4$ . Yes.

Similarly for other systems, MFT gets the phase structure wrong for  $d \leq d_l$ , the ‘lower critical dimension’, and correct for  $d \geq d_c$ , the ‘upper critical dimension’. In the Ising model,  $d_l = 1$  and  $d_c = 4$ .

For  $d_l < d < d_c$ , the theory is interesting.

### 1.3 Critical Exponents

Near the critical point, MFT predicts the following:

- If  $B = 0$ , then as  $T \rightarrow T_c^-$ ,

$$\begin{aligned} m &\sim (T_c - T)^\beta && \text{with } \beta = \frac{1}{2}, \\ c &\sim c_\pm |T - T_c|^{-\alpha} && \text{with } \alpha = 0, \\ \chi &\sim \frac{1}{|T - T_c|^\gamma} && \text{with } \gamma = 1. \end{aligned}$$

- As  $B \rightarrow 0$ ,

$$m \sim B^{1/\delta} \quad \text{with } \delta = 3.$$

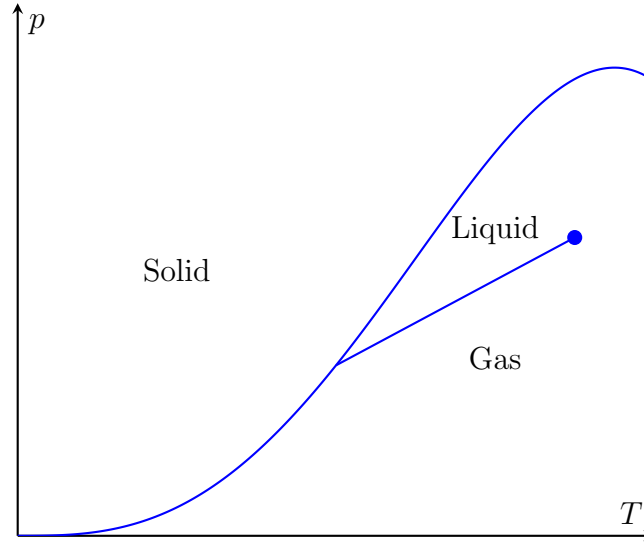
Here,  $\alpha, \beta, \gamma, \delta$  are the *critical exponents*.

These values are not correct for small  $d$ .

	MFT	$d = 2$	$d = 3$
$\alpha$	0 (disc)	0 (log)	0.1101..
$\beta$	$\frac{1}{2}$	$\frac{1}{8}$	0.3264..
$\gamma$	1	$\frac{7}{4}$	1.2371..
$\delta$	3	15	4.7898..

Table 1: Predictions and Theoretical Critical Exponents

## 1.4 Universality



In a normal material, we have liquid-gas phase transition similar to the Ising model; a line of first order phase transitions, ending at a critical point.

If we replace  $B$  with  $p$  the pressure, and  $m$  with  $v = V/N$  as our order parameter, then using an equation of state (e.g. the van der Waals) to calculate the behaviour near the critical point:

- As  $T \rightarrow T_c$ ,

$$v_{\text{gas}} - v_{\text{liquid}} \sim (T_c - T)^\beta \quad \text{where } \beta = \frac{1}{2}.$$

- If  $T = T_c$  is fixed, and  $p \rightarrow p_c$ ,

$$v_{\text{gas}} - v_{\text{liquid}} \sim (p - p_c)^{1/\delta} \quad \text{where } \delta = 3.$$

- The isothermal compressibility is

$$\kappa = -\frac{1}{v} \left( \frac{\partial v}{\partial p} \right)_T \sim \frac{1}{|T - T_c|^\gamma} \quad \text{where } \gamma = 1.$$

- The heat capacity is

$$c_v \sim c_\pm |T - T_c|^{-\alpha} \quad \text{where } \alpha = 0,$$

i.e. the heat capacity is discontinuous.

These are the same predictions as for MFT for the Ising model. As is probably expected, these are incorrect; but the correct values are the same as the correct values for  $d = 3$  Ising model!

This is an example of *universality*: different physical systems can exhibit the same behaviour at the critical points.

This suggests that the microscopic physics is unimportant at a critical point. Systems governed by the same critical point belong to the same *universality class*.

## 1.5 Landau-Ginzberg Theory

Our aim is to find a model that correctly describes long-distance physics near the critical point, which can be used to calculate critical exponents for all theories in the same universality class.

LG theory generalizes MFT, to allow for spatial variation in  $m$ .

Here,  $m(\mathbf{x})$  is a field, produced from a microscopic model by coarse-graining. In the Ising model, we divide the lattice into boxes, each with  $N' \ll N$  sites, with size  $a$ . Then we can define

$$m(\mathbf{x}) = \text{average of spins in box with centre } \mathbf{x}.$$

Take  $N' \gg 1$  the discreteness of  $m(\mathbf{x})$  can be ignored, and  $m \in [-1, 1]$ .

Assume that  $a \ll \xi$ , the length scale over which the physics varies. Treat  $m(\mathbf{x})$  as a smooth function, which does not vary on scales less than  $a$ . Then,

$$Z = \sum_{m(\mathbf{x})} \sum_{\{s_i\}|m(\mathbf{x})} e^{-\beta E[s_i]} = \sum_{m(\mathbf{x})} e^{-\beta F[m(\mathbf{x})]}.$$

Here,  $F[m(\mathbf{x})]$  is a *functional*, which depends on a functional. This is the *Landau-Ginzberg free energy*.

We write

$$Z = \int \mathcal{D}m(\mathbf{x}) e^{-\beta F[m(\mathbf{x})]},$$

which is a *functional integral*, a sum over all  $m(\mathbf{x})$  that do not vary on scales less than  $a$ .

We can interpret this as a probability of a field configuration  $m(\mathbf{x})$  as

$$p[m(\mathbf{x})] = \frac{e^{-\beta F[m(\mathbf{x})]}}{Z}.$$

The form of  $F[m(\mathbf{x})]$  is constrained by the following:

- Locality: spins only influence nearby spins, so we may write

$$F[m(\mathbf{x})] = \int d^d x f(m(\mathbf{x}), \nabla_i m(\mathbf{x}), \nabla_i \nabla_j m(\mathbf{x}), \dots),$$

not for example a function in  $x$  and  $y$ .

- Translational symmetry is inherited from discrete translational symmetry of the lattice.
- Can also inherit rotational symmetry.
- We also have a  $\mathbb{Z}_2$  symmetry, by  $s_i \rightarrow -s_i$ , and  $B \rightarrow -B$ . So we can assume that  $F$  is invariant under  $m(\mathbf{x}) \rightarrow -m(\mathbf{x})$ , and  $B \rightarrow -B$ .
- Analyticity. We assume  $F[m(\mathbf{x})]$  is defined by coarse graining over a finite number of spins. This lets us suppose  $f$  is *analytic*, i.e. we can Taylor expand near  $m = 0$ .

From dimensional analysis,

$$a \nabla m \sim \frac{am}{\xi}, \quad a^2 \nabla \nabla m \sim \frac{a^2 m}{\xi^2}.$$

The fact  $\xi \gg a$  suggests that  $\nabla m$  is more important than  $\nabla \nabla m$ , etc.

From the above assumptions, for  $B = 0$  we can write

$$F[m(\mathbf{x})] = \int d^d x \left( \frac{1}{2} \alpha_2(T) m^2 + \frac{1}{4} \alpha_4(T) m^4 + \frac{1}{2} \gamma(T) (\nabla m)^2 + \dots \right).$$

If  $B \neq 0$ , we could include  $Bm, Bm^3$  terms.

The coefficients  $\alpha_2(T), \alpha_4(T)$  are hard to compute from first principles. From MFT, we expect that  $\alpha_2(T) \sim T - T_c$ , and  $\alpha_4(T) \sim T/3$ . But all we will assume is that the coefficients are analytic in  $T$ , with  $\alpha_4(T) > 0$ ,  $\gamma(T) > 0$ , and  $\alpha_2(T) > 0$  for  $T > T_c$ , with a simple zero at  $T = T_c$ .

Still, this field integral is hard to evaluate. Hence we will use a saddle point approximation.

Assume that the integral is dominated by the saddle point; the  $m(\mathbf{x})$  that minimizes  $F[m(\mathbf{x})]$ . We can vary  $m(\mathbf{x}) \rightarrow m(\mathbf{x}) + \delta m(\mathbf{x})$ , with

$$\begin{aligned} \delta F &= \int d^d x (\alpha_2 m \delta m + \alpha_4 m^3 \delta m + \gamma \nabla m \cdot \nabla \delta m + \dots) \\ &= \int d^d x (\alpha_2 m + \alpha_4 m^3 - \gamma \nabla^2 m + \dots) \delta m, \end{aligned}$$

by using integration by parts. The part in the brackets is often written as

$$\frac{\delta F}{\delta m(\mathbf{x})},$$

the *functional derivative*. If  $m(\mathbf{x})$  minimizes  $F$ , then  $\delta F = 0$  for all  $\delta m(\mathbf{x})$ , so

$$\gamma \nabla^2 m = \alpha_1 m + \alpha_4 m^3 + \dots$$

This is just the Euler-Lagrange equations, I think? The simplest solution is when  $m$  is constant, which corresponds to MFT/Landau theory.

For  $T > T_c$ ,  $\alpha_2 > 0$ , so  $m = 0$ , and for  $T < T_c$ ,  $\alpha_2 < 0$ , so

$$m = \pm m_0 \sqrt{\frac{-\alpha_2}{\alpha_4}}.$$

## 1.6 Domain Walls

If  $T < T_c$ , then there are two ground states  $m = \pm m_0$ . We could have  $m \rightarrow \pm m_0$  as  $x \rightarrow \pm\infty$ .

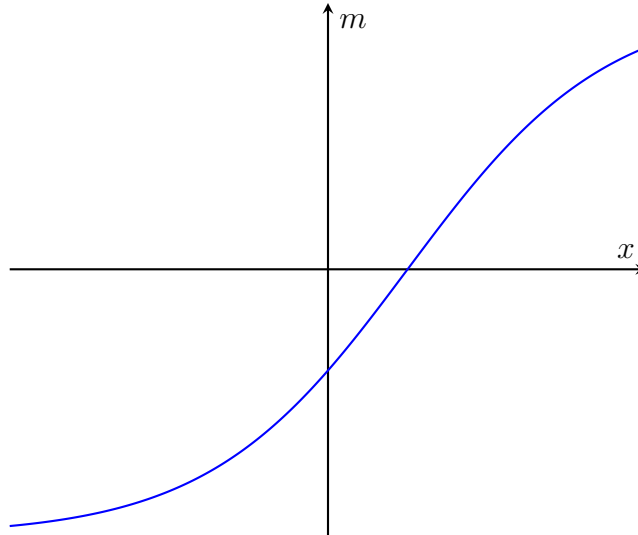
Consider a field that varies only in  $x$ , so  $m(\mathbf{x}) = m(x)$ . Then we need to solve

$$\gamma \frac{d^2 m}{dx^2} = \alpha_2 m + \alpha_4 m^3 + \dots$$

This is solved by

$$m = m_0 \tanh\left(\frac{x - X}{W}\right),$$

where  $X$  is some constant, and  $W = \sqrt{-2\gamma/\alpha_2}$ .





This is a *domain wall* with position  $X$  and width  $W$ .

If the system has size  $L$ , then the free energy is  $F[m_0] \propto L^d$ . The cost of the domain wall is

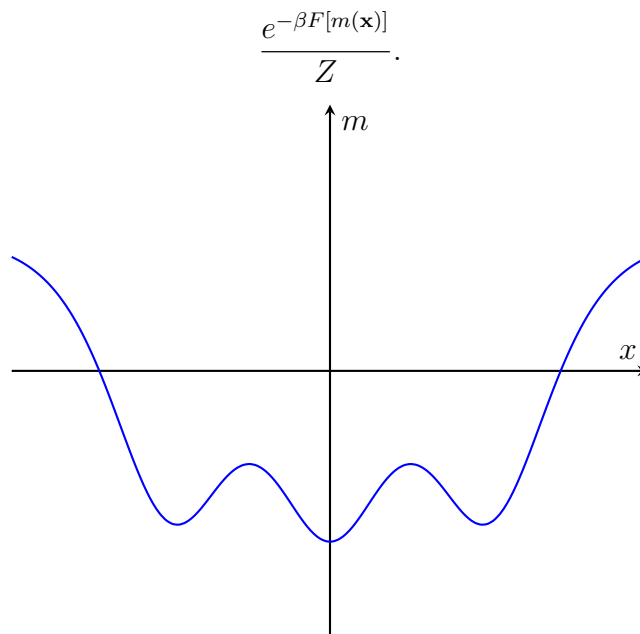
$$\Delta F = F[m(\mathbf{x})] - F[m_0] \sim L^{d-1} \sqrt{\frac{-\gamma \alpha_2^3}{\alpha_4^2}},$$

which is proportional to the area of the domain wall.

Near the critical point  $\alpha_2 \rightarrow 0$  and  $W \rightarrow \infty$ , so  $\Delta F \rightarrow 0$ .

Domain walls explain why  $d_l = 1$  for the Ising model. Let  $d = 1$ , and  $-L/2 < x < L/2$ , and assume  $\alpha_2(T) < 0$ .

The boundary conditions are  $m(\pm L/2) = m_0$ , and the probability of the configuration is



Now consider a field with two domain walls. Assume  $L \gg W$ . For well separated wells,  $\Delta F_{2\text{wells}} = 2\Delta F$ . Then,

$$\frac{\mathbb{P}(2 \text{ domain walls at given } X, Y)}{p(m = m_0)} = e^{-2\beta \Delta F}.$$

Summing over all  $X, Y$ ,

$$\frac{\mathbb{P}(2 \text{ domain walls})}{\mathbb{P}(m = m_0)} \sim \int_{-L/2}^{L/2} \frac{dX}{W} \int_X^{L/2} \frac{dY}{W} e^{-2\beta \Delta(F)} \sim \left(\frac{L}{W}\right)^2 e^{-2\beta \Delta(F)}.$$

For  $d = 1$ , the RHS goes to 0 as  $L \rightarrow \infty$ , showing that energy beats entropy. But for  $d = 1$ , the RHS tends to  $\infty$ , as entropy beats energy.

Hence for  $d = 1$ , it is much more probably to see two domain walls than a constant field, so any region with constant  $m = \pm m_0$  is unstable to the formation of domain walls, so the ordered phase does not exist.

## 2 My First Path Integral

We want to go beyond the saddle point approximation to calculate

$$Z = \int \mathcal{D}m(\mathbf{x}) e^{-\beta F[m(\mathbf{x})]}.$$

We make a couple of changes: first we change from  $m(\mathbf{x})$  to  $\phi(\mathbf{x})$ , and also we assume  $B = 0$ , so

$$F[\phi(\mathbf{x})] = \int d^d x \left( \frac{1}{2} \alpha_2(T) \phi^2 + \frac{1}{4} \alpha_4 \phi^4 + \frac{1}{2} \gamma(T) (\nabla \phi)^2 + \dots \right).$$

Evaluating the path integral is:

- Easy if  $F$  is quadratic in  $\phi$ .
- Possibly if the higher order terms are small.
- Very hard otherwise.

For  $T > T_c$ , let  $\mu^2 = \alpha_2(T) > 0$ . Consider the quadratic approximation to  $F$ :

$$F[\phi(\mathbf{x})] = \frac{1}{2} \int d^d x (\mu^2 \phi^2 + \gamma (\nabla \phi)^2).$$

For  $T < T_c$ ,  $\alpha_2(T) < 0$ , so  $\langle \phi \rangle = \pm m_0$ . Let  $\tilde{\phi} = \phi - \langle \phi \rangle$ . Then,

$$F = F[m_0] + \frac{1}{2} \int d^d x (\alpha_2(T) \tilde{\phi}^2 + \gamma(T) (\nabla \tilde{\phi})^2 + \dots).$$

Here

$$\alpha'_2(T) = \alpha_2(T) + 3m_0^2 \alpha_4(T) = -2\alpha_2(T) > 0.$$

This quadratic approximation gives the same equation with  $\phi$  replaced with  $\tilde{\phi}$ , and  $\mu^2 = \alpha'_2(T) = 2|\alpha_2(T)| > 0$ .

### 2.1 Thermodynamic Free Energy

We aim to compute the corrections to  $F_{\text{thermo}}$  from the fluctuation in  $\phi(\mathbf{x})$ . It is useful to take the Fourier transform

$$\phi_{\mathbf{k}} = \int d^d x e^{-i\mathbf{k} \cdot \mathbf{x}} \phi(\mathbf{x}),$$

where since  $\phi$  is real,  $\phi_{\mathbf{k}}^* = \phi_{-\mathbf{k}}$ . Here the wavevector  $\mathbf{k}$  is often called the *momentum*. Since  $\phi$  doesn't vary on scales less than  $a$ , we get  $\phi_{\mathbf{k}} = 0$  for all  $|\mathbf{k}| = \Lambda$ , where  $\Lambda = \pi/a$  is called the ultraviolet cutoff.

The inverse Fourier transform is given by

$$\phi(\mathbf{x}) = \int \frac{d^d x}{(2\pi)^d} e^{i\mathbf{k} \cdot \mathbf{x}} \phi_{\mathbf{k}}.$$

In finite volume, consider a system occupying a cubic region with  $V = L^d$ . Then  $\mathbf{k}$  takes discrete values,  $\mathbf{k} = \frac{2\pi}{L}\mathbf{n}$ , for some  $\mathbf{n} \in \mathbb{Z}^d$ . Thus, the integral becomes

$$\int \frac{d^d k}{(2\pi)^d} \cdots = \left(\frac{1}{L}\right)^d \sum_{\mathbf{n}} \cdots = V^{-1} \sum_{\mathbf{n}} \cdots,$$

hence the inverse Fourier transform can be written as

$$\phi(\mathbf{x}) = \frac{1}{V} \sum_{\mathbf{k}} e^{-i\mathbf{k} \cdot \mathbf{x}} \phi_{\mathbf{k}}.$$

Substituting this into the form for  $F$ ,

$$F[\phi_{\mathbf{k}}] = \frac{1}{2} \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \int d^d x (\mu^2 - \gamma \mathbf{k}_1 \cdot \mathbf{k}_2) \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} e^{i(\mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{x}}.$$

Using the fact the latter integral is a  $\delta$  function, this simplifies to

$$\begin{aligned} F &= \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} (\mu^2 + \gamma \mathbf{k}^2) \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \\ &= \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} (\mu^2 + \gamma \mathbf{k}^2) |\phi_{\mathbf{k}}|^2 \\ &= \int^+ \frac{d^d k}{(2\pi)^d} (\mu^2 + \gamma \mathbf{k}^2) |\phi_{\mathbf{k}}|^2 \quad \text{integral over } k_x > 0 \\ &= \frac{1}{V} \sum_k^+ (\mu^2 + \gamma \mathbf{k}^2) |\phi_{\mathbf{k}}|^2. \end{aligned}$$

The measure is

$$\int \mathcal{D}\phi(\mathbf{x}) = \mathcal{N} \prod_{\mathbf{k}}^+ \int d(\Re \phi_{\mathbf{k}}) d(\Im \phi_{\mathbf{k}}),$$

so the partition function becomes

$$\begin{aligned} Z &= \mathcal{N} \left( \prod_{\mathbf{k}} \int d(\Re \phi_{\mathbf{k}}) d(\Im \phi_{\mathbf{k}}) \right) \exp \left[ -\frac{\beta}{V} \sum_{\mathbf{k}}^+ (\mu^2 + \gamma \mathbf{k}^2) |\phi_{\mathbf{k}}|^2 \right] \\ &= \mathcal{N} \prod_{\mathbf{k}}^+ \int d(\Re \phi_{\mathbf{k}}) d(\Im \phi_{\mathbf{k}}) \exp \left[ -\frac{\beta}{V} (\mu^2 + \gamma \mathbf{k}^2) ((\Re \phi_{\mathbf{k}})^2 + (\Im \phi_{\mathbf{k}})^2) \right]. \end{aligned}$$

Recall that

$$\int_{-\infty}^{\infty} e^{-x^2/a} = \sqrt{\pi a},$$

so we find that

$$e^{-\beta F_{\text{thermo}}} = Z = \mathcal{N} \prod_{\mathbf{k}}^+ \left[ \sqrt{\frac{\pi VT}{\mu^2 + \gamma \mathbf{k}^2}} \right]^2 = \mathcal{N} \prod_{\mathbf{k}}^+ \frac{\pi VT}{\mu^2 + \gamma \mathbf{k}^2},$$

hence

$$\frac{F_{\text{thermo}}}{V} = -\frac{T}{V} \log Z = -\frac{T}{V} \sum_{\mathbf{k}}^+ \log \left( \frac{\pi VT}{\mu^2 + \gamma \mathbf{k}^2} \right) - \frac{T}{V} \log N.$$

Now we compute the contribution of fluctuations to the heat capacity. Recall that

$$\langle E \rangle = -\frac{\partial \log Z}{\partial \beta} = \frac{\partial(\beta F_{\text{thermo}})}{\partial \beta},$$

and the heat capacity is

$$C = \frac{\partial \langle E \rangle}{T} = -\beta^2 \frac{\partial \langle E \rangle}{\partial \beta} = -\beta^2 \frac{\partial^2(\beta F_{\text{thermo}})}{\partial \beta^2},$$

from which we get

$$\frac{C}{V} = -\beta^2 \frac{\partial^2}{\partial \beta^2} \left[ -\frac{1}{V} \sum_{\mathbf{k}} \log \left( \frac{\pi VT}{\mu^2 + \gamma \mathbf{k}^2} \right) \right].$$

Take  $\mu^2 = T - T_c$ , and  $V$  a constant for simplicity. Taylor expanding,

$$\frac{C}{V} = \frac{1}{V} \sum_{\mathbf{k}}^+ \left[ 1 + \frac{2T}{\mu^2 + \gamma \mathbf{k}^2} + \frac{T^2}{(\mu^2 + \gamma \mathbf{k}^2)^2} \right] = \frac{1}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} [\dots].$$

The first term gives  $\frac{1}{2}k_0$  per degree of freedom; this is equipartition.

The other terms may diverge as  $T \rightarrow T_c$ , and the integral may not converge at  $\mathbf{k} = 0$ , an example of infrared divergence. The final term is proportional to

$$\int_0^\Lambda \frac{dk k^{d-1}}{(\mu^2 + \gamma k^2)^2} = \frac{\mu^{d-4}}{\gamma^{d/2}} \int_0^{\Lambda \sqrt{x/\mu^2}} \frac{dx x^{d-1}}{(1+x^2)^2} \sim \begin{cases} \Lambda^{d-4} & d > 4, \\ \mu^{d-4} & d < 4, \end{cases}$$

as  $T \rightarrow T_c$ . If  $d = 4$ , then this tends to  $\log \Lambda$ . Similarly, the second term is proportional to

$$\int_0^\Lambda \frac{dk k^{d-1}}{\mu^2 + \gamma k^2} \sim \begin{cases} \Lambda^{d-2} & d > 2, \\ \mu^{-1} & d = 1. \end{cases}$$

as  $T \rightarrow T_c$ . Hence for  $d \geq 4$ , the contributions of fluctuations is finite as  $T \rightarrow T_c$ , and for  $d < 4$  the contributions diverges. In general,

$$C \sim \mu^{d-4} \sim |T - T_c|^{-\alpha},$$

where  $\alpha = 2 - d/2$ . These fluctuations explain why the MFT value  $\alpha = 0$  is incorrect. But this value is also wrong.

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