III Functional Analysis

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Based on Lectures by Dr. András Zsák

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Contents

0 Introduction

Allen has good notes.

Books include Bollobás, Rudin, S.J. Taylor (measure theory), Rudin again and Murphy.

0.1 Overview

The course is structured as follows.

Chapter 1. Hahn-Banach extension theorems.

Chapter 2. Dual spaces of $L_p(\mu)$ and $C(K)$.

Chapter 3. Weak topologies.

Chapter 4. Convexity and Krein-Milman theorem.

Chapter 5. Banach algebras.

Chapter 6. Holomorphic functional calculus.

Chapter 7. C^*-algebras.

Chapter 8. Borel functional calculus and spectral theory.

1 Hahn-Banach Extension Theorems

Let X be a normed space. The *dual space* X^* of X is

$$X^* = \{f : X \to \text{scalars} \mid f \text{ linear, continuous (or bounded)}\}.$$

This is a normed space in the operator norm. For $f \in X^*$,

$$\|f\| = \sup\{|f(x)| \mid x \in B_X\},$$

where B_X is the unit ball in X, i.e. $\{x \in X \mid \|x\| \leq 1\}$. We also have $S_X = \{x \in X \mid \|x\| = 1\}$, the unit sphere.

Recall that X^* is a Banach space.

Example 1.1.

$\ell_p^* \cong \ell_q$, for $1 \leq p < \infty$, $1 < q \leq \infty$, and $1/p + 1/q = 1$.

We also have $c_0^* \cong \ell_1$.

Also if H is a Hilbert space, then $H^* \cong H$, by the Riesz representation theorem. This is conjugate linear in the complex case.

Definition 1.1. We write $X \sim Y$ if NVS's X and Y are isomorphic, so there exists a linear bijection $T : X \to Y$ where T and T^{-1} are bounded.

If X, Y are both Banach spaces, and $T : X \to Y$ is a continuous linear bijection, then T^{-1} is continuous by the open mapping theorem.

Write $X \cong Y$ if X and Y are isometrically isomorphic, i.e. there exists a surjective linear map $T : X \to Y$ such that T is isometric, i.e. $\|Tx\| = \|x\|$.

Note this automatically implies T is a linear bijection, and T^{-1} is isometric.

For a normed space X, and $x \in X$, $f \in X^*$ we write

$$\langle x, f \rangle = f(x).$$

This is bilinear, and $|\langle x, f \rangle| = |f(x)| \leq \|f\| \cdot \|x\|$. When X is a Hilbert space, X^* is identified with X, and $\langle \cdot, \cdot \rangle$ is the inner product.

Definition 1.2. Let X be a real vector space. A functional $p : X \to \mathbb{R}$ is:

(i) *positive homogeneous* if $p(tx) = tp(x)$ for all $x \in X$, $t \geq 0$.

(ii) *subadditive* if $p(x + y) \leq p(x) + p(y)$.

Theorem 1.1 (Hahn-Banach). *Let X be a real vector space, and $p : X \to \mathbb{R}$ be a positive homogeneous, subadditive functional on X. Let Y be a subspace of X, and $g : Y \to \mathbb{R}$ be linear such that $g(y) \leq p(y)$ for all $y \in Y$.*

Then there exists linear $f : X \to \mathbb{R}$ such that $f|_Y = g$, and $f(x) \leq p(x)$ for all $x \in X$.

To prove this, we need Zorn's lemma, and the theory of posets. Let $(P, \leq)$ be a poset.

For $A \subseteq P$, $x \in P$, say x is an *upper bound* for A if $a \leq x$ for all $a \in A$. For $C \subseteq P$, say C is a *chain* if $\leq$ is a linear order on C. Say $x \in P$ is a *maximal element* if, for all $y \in P$, $x \leq y$ implies $y = x$.

Theorem 1.2 (Zorn's lemma). *If P is a non-empty poset and every non-empty chain in P has an upper bound, then P has a maximal element.*

Proof: Consider the poset given by pairs (Z, h), where Z is a subspace of X containin Y, and $h : Z \to \mathbb{R}$ linear, with $h|_Y = g$, and $h(z) \leq p(z)$.

Here $(Z_1, h_1) \leq (Z_2, h_2)$ if $Z_1 \subseteq Z_2$ and $h_2|_{Z_1} = h_1$. This can be checked to be a partial order.

Now we check our conditions. First $P \neq \emptyset$ as $(Y, g) \in P$. Moreover, given a non-empty chain $C = \{(Z_i, h_I) \mid i \in I\}$ in P, we can set $Z = \bigcup_{i \in I} Z_i$, and define $h : Z \to \mathbb{R}$ by $h|_{Z_i} = h_i$. Then $(Z, h) \in P$ and is an upper bound for C.

Thus by Zorn's, P has a maximal element (W, f). Now we need to show that $W = X$, and we will be done.

Assume not. Fix $z \in X \setminus W$, and a real number $\alpha \in \mathbb{R}$. Define $f_1 : W_1 = W + \mathbb{R} \cdot z \to \mathbb{R}$ by

$$f_1(w + \lambda z) = f(w) + \lambda \alpha.$$

Then f_1 is linear, and $f_1|_W = f$. To be done, we need to choose α so that $f_1(w_1) \leq p(w_1)$ for all $w_1 \in W_1$.

Thus we need

$$\begin{aligned} & f(w) + \lambda\alpha \leq p(w + \lambda z) \\ \iff & f(w) + \alpha \leq p(w + z) \\ & f(w) - \alpha \leq p(w - z), \end{aligned}$$

for all $w \in W$. This means

$$f(x) - p(x - z) \le \alpha \le p(y + z) - f(y),$$

which is true if and only if

$$f(x) - p(x - z) \le p(y + z) - f(y),$$

for all $x, y \in W$, by taking α to be the supremum of the left hand side as x ranges over W. But this is true as

$$f(x) + f(y) = f(x + y) \le p(x + y) = p(x - z + y + z) \le p(x - z) + p(y + z),$$

for all $x, y \in W$.

Definition 1.3. A *seminorm* on a real or complex vector space X is a functional $p : X \to \mathbb{R}$ such that:

(i) $p(x) \ge 0$, for all $x \in X$.

(ii) $p(\lambda x) = |\lambda| p(x)$, for all scalars λ, and for all $x \in X$.

(iii) $p(x + y) \le p(x) + p(y)$ for all $x, y \in X$.

This is the definition of the norm, without requiring $p(x) = 0 \implies x = 0$.

Of course, any seminorm is positive heterogeneous, and subadditive.

Theorem 1.3 (Hahn-Banach)**.** *Let X be a real or complex vector space, and p a seminorm on X. Let Y be a subspace of X, and g be a linear functional on Y such that $|g(y)| \le p(y)$, for all $y \in Y$.*

Then there exists linear functional f on X, such that $f|_Y = g$, and $|f(x)| \le p(x)$ for all $x \in X$.

Proof: We split into two cases, the real and the complex case.

In the real case, we have $g(y) \le |g(y)| \le p(y)$ for all $y \in Y$, so by the first version of Hahn-Banach, there exists a linear map $f : X \to \mathbb{R}$ such that $f|_Y = g$ and $f(x) \le p(x)$.

We are almost done, except we need $|f(x)| \le p(x)$. Here we use the fact that p is a seminorm, so

$$-f(x) = f(-x) \le p(-x) = p(x).$$

Hence $|f(x)| \leq p(x)$.

Now we start with the complex case. Splitting into real and imaginary parts does not work, as f, g real linear does not imply $f + ig$ complex linear. To do this, we show the following claim:

Claim: For any real-linear $h_1 : X \to \mathbb{R}$, there is a unique complex linear $h : X \to \mathbb{C}$ such that $\Re(h) = h_1$.

We start with uniqueness. If $h_1 = \Re(h)$, then for $x \in X$,

$$\begin{aligned} h(x) &= h_1(x) + i\Im(h(x)) \\ &= -ih(ix) = -i(h_1(ix) + i\Im(h(ix))). \end{aligned}$$

So, $\Im(H(x)) = -h_1(ix)$, and thus

$$h(x) = h_1(x) - ih_1(ix).$$

For existence, we just check this h defined above works, and it does (clearly real-linear, just need to check multiplication by i is correct).

We return back to our proof. Let $g_1 = \Re(g) : Y \to \mathbb{R}$, which is real-linear. For $y \in Y$, note

$$|g_1(y)| \leq |g(y)| \leq p(y).$$

By the real case, there exists a real linear $f_1 : X \to \mathbb{R}$ such that $f_1|_Y = g_1$, and $|f_1(x)| \leq p(x)$ for all $x \in X$.

By the claim, $f_1 = \Re(f)$ for unique complex-linear functions $f : X \to \mathbb{C}$, and note

$$\Re(f|_Y) = f_1|_Y = g_1 = \Re(g).$$

Therefore by uniqueness, $f|_Y = g$. We are almost done apart form domination. Note that for $x \in X$, $|f(x)| = \lambda f(x)$, for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Then,

$$\begin{aligned} |f(x)| &= f(\lambda x) = f_1(\lambda x) + i\Im(f(\lambda x)) \\ &= f_1(\lambda x) \leq p(\lambda x) = |\lambda| p(x) = p(x). \end{aligned}$$

Remark. For a complex vector space X, let $X_{\mathbb{R}}$ be the real vector space obtained from X by restricting scalar multiplication to the reals.

If X is a complex normed space, then $f \mapsto \Re(f)$ on $(X^*)_{\mathbb{R}} \to (X_{\mathbb{R}})^*$ is an isometric isomorphism.

Corollary 1.1. *Let X be a real or complex vector space, and let p be a seminorm* *on X. Then for any $x_0 \in X$, there exists a linear functional f on X such that $f(x_0) = p(x_0)$, and $|f(x)| \leq p(x)$, for all $x \in X$.*

Proof: Let $Y = \text{span}\{x_0\}$, and define g on Y be

$$g(\lambda x_0) = \lambda p(x_0).$$

Then g is linear on Y, and

$$|g(\lambda x_0)| = |\lambda| p(x_0) = p(\lambda x_0),$$

for all scalars λ. Thus by Hahn-Banach, there exists a linear functional f on X such that $f|_Y = g$, and $|f(x)| \leq p(x)$. So $f(x_0) = g(x_0) = p(x_0)$.

Theorem 1.4 (Hahn-Banach)**.** *Let X be a real or complex normed space.*

(i) *Given a subspace Y of X and $g \in Y^*$, here exists $f \in X^*$ such hat $f|_Y = g$, and $\|f\| = \|g\|$.*

(ii) *For $x_0 \in X \setminus \{0\}$, here exists $f \in S_{X^*}$ such that $f(x_0) = \|x_0\|$.*

Proof:

(i) Apply previous Hahn-Banach with $p(x) = \|g\|\|x\|$. Then for $y \in Y$,

$$|g(y)| \leq \|g\| \cdot \|y\| = p(y).$$

Hence there exists a linear functional f on X such that $f|_Y = g$, and

$$|f(x)| \leq p(x) = \|g\| \cdot \|x\|.$$

Therefore, $f \in X^*$, and $\|f\| = \|g\|$. Since f extends g, $\|f\| = \|g\|$.

(ii) Let $p = \|\cdot\|$. By the previous corollary, there exists a linear functional f on X such that $f(x_0) = \|x_0\|$, and $|f(x) \leq \|x\|$.

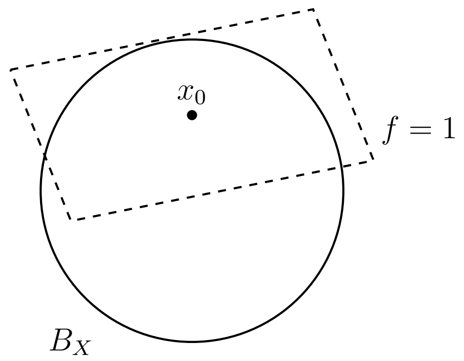
So $f \in X^*$, $\|f\| \leq 1$, but by equality at x_0, $\|f\| = 1$.

Remark.

1. We can think of this as a linear version of Tietze's extension theorem. Recall:

   If L is a closed subset of a compact Hausdorff space K and $g : L \to \mathbb{R}$ or $\mathbb{C}$ is continuous, then there exists continuous $f : K \to \mathbb{R}$ or $\mathbb{C}$ such that $f|_L = g$, and $\|f\|_\infty = \|g\|_\infty$.

2. Part (ii) implies that X^* separates points of X, i.e. if $x \neq y$ in X, then there exists $f \in X^*$ such that $f(x) \neq f(y)$, by taking $x_0 = x - y$.
3. The f in (ii) is called the *norming functional* at x_0. Therefore,
$$\|x_0\| = \max\{|g(x)| \mid g \in B_{X^*}\}.$$
Another name is the *support functional* at x_0. We can think of where $f = 1$ as the "tangent plane at x_0".



1.1 Bidual

Let X be a normed space. Then $X^{**} = (X^*)^*$ is the *bidual* or *second dual* of X.

For $x \in X$, define $\hat{x}$ on X^* by $f \mapsto f(x)$, i.e. evaluation at x.

Then $\hat{x}$ is linear, and
$$|\hat{x}(f)| = |f(x)| \leq \|f\|\|x\|,$$
for all $f \in X^*$. So $\hat{x} \in X^{**}$, and $\|\hat{x}\| \leq \|x\|$. The map $x \mapsto \hat{x}$ is the *canonical embedding* of X into X^{**}.

Theorem 1.5. *The canonical embedding is an isometric isomorphism of X into X^{**}.*

Proof: Linearity: note
$$\begin{aligned}\widehat{\lambda x + \mu y}(f) &= f(\lambda x + \mu y) = \lambda f(x) + \mu f(y) \\ &= (\lambda\hat{x} + \mu\hat{y})(f).\end{aligned}$$

Isometric: for $x \in X$,
$$\|\hat{x}\| = \sup\{|f(x)| \mid f \in B_{X^*}\} = \|x\|,$$

by Hahn-Banach.

Remark.

1. Note that
$$\langle f, \hat{x} \rangle = \langle x, f \rangle,$$
for $x \in X$, $f \in X^*$.
2. $\hat{X} = \{\hat{x} \mid x \in X\} \cong X$. Therefore,
$$\hat{X} \text{ is closed in } X^{**} \iff X \text{ is complete.}$$
3. In general, the closure in X^{**} of $\hat{X}$ is a Banach space containing an isometric copy of X as a dense subspace.

Definition 1.4. A normed space X is *reflexive* if the canonical embedding $X \to X^{**}$ is surjective.

Example 1.2.

1. Any finite-dimensional space is reflexive.
2. ℓ_p for $1 < p < \infty$ is reflexive.
3. Any Hilbert space is reflexive.
4. $L_p(\mu)$ for $1 < p < \infty$ is reflexive.
5. $c_0, \ell_1, \ell_\infty, L_1([0,1])$ are not reflexive.

Remark. If X is reflexive, then X is a Banach space, and $X \cong X^{**}$.

However, there exists a Banach space X such that $X \cong X^{**}$, but X is not reflexive. So even though $\ell_p^{**} \cong \ell_q^* \cong \ell_p$, this is not enough to show ℓ_p is reflexive.

1.2 Dual Operators

Let X, Y be normed spaces. Then,
$$\mathcal{B}(X, Y) = \{T : X \to Y \mid T \text{ linear, bounded}\}.$$
Recall that $\mathcal{B}(X, Y)$ is a normed space with the operator norm:
$$\|T\| = \sup\{\|Tx\| \mid x \in B_X\}.$$
If Y is complete, then $\mathcal{B}(X, Y)$ is complete.

For $T \in \mathcal{B}(X, Y)$, its *dual operator* $T^* : Y^* \to X^*$ is given by
$$T^*(g) = g \circ T.$$

This is well-defined, and in the bracket notation

$$\langle x, T^* g\rangle = \langle Tx, g\rangle.$$

It is easy to see that T^* is linear, and moreover it is bounded. Note

$$\begin{aligned} \|T^*\| &= \sup_{g \in B_{Y^*}} \|T^* g\| = \sup_{g \in B_{Y^*}} \sup_{x \in B_X} |\langle x, T^* g\rangle| = \sup_{x \in B_X} \sup_{g \in B_{Y^*}} |\langle Tx, g\rangle| \\ &\overset{HB}{=} \sup_{x \in B_x} \|Tx\| = \|T\|. \end{aligned}$$

Remark. If X, Y are Hilbert spaces, and we identify X, Y with X^*, Y^* respectively, then T^* becomes the *adjoint* of T.

Example 1.3.

If $1 \le p < \infty$, and $R : \ell_p \to \ell_p$ is the right-shift, then $R^* : \ell_q \to \ell_q$ is the left-shift.

We have the following properties:

- $(\mathrm{id}_X)^* = \mathrm{id}_{X^*}$.
- $(\lambda S + \mu T)^* = \lambda S^* + \mu T^*$.
- $(ST)^* = T^* S^*$.
- $T \mapsto T^* : \mathcal{B}(X, Y) \to \mathcal{B}(Y^*, X^*)$ is an into isometric isomorphism.
- The following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d] & Y \arrow[d] \\ X^{**} \arrow[r, "T^{**}"] & Y^{**} \end{tikzcd}$$

  In other words $\widehat{Tx} = T^{**}\hat{x}$, for all $x \in X$.

Indeed, for all $x \in X$, $g \in Y^*$,

$$\begin{aligned} \langle g, T^{**}\hat{x}\rangle &= \langle T^* g, \hat{x}\rangle = \langle x, T^* g\rangle \\ &= \langle Tx, g\rangle = \langle g, \widehat{Tx}\rangle. \end{aligned}$$

1.3 Quotient spaces

Let X be a NVS and Y be a closed subspace. Then X/Y is a normed space in the *quotient norm*:

$$\|x+Y\| = \inf\{\|x+y\| \mid y \in Y\} = d(x,Y).$$

Here closed is important, so that $\|x+Y\| = 0 \implies x \in Y$.

The quotient map $q : X \to X/Y$ is linear, surjective and bounded with $\|q\| = 1$, since for $x \in X$

$$\|q(x)\| \leq \|x\|.$$

Letting D_X be the open unit ball of X, we can show $q(D_X) = D_{X/Y}$. Indeed if $x \in D_X$, then $\|q(x)\| \leq \|x\| < 1$. If $\|x+Y\| < 1$, then there exists $y \in Y$ with $\|x+y\| < 1$. So $x+y \in D_X$ and $q(x+y) = x+Y$.

So $\|q\| = 1$, unless $Y = X$. Also, q is an open map.

Assume $T : X \to Z$ is a bounded linear map, and $Y \subseteq \ker T$. Then there exists a unique map $\tilde{T} : X/Y \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "q"'] & & Z \\ & X/Y \arrow[ur, "\tilde{T}"'] & \end{tikzcd}$$

Moreover, $\tilde{T}$ is linear and bounded, and $\|\tilde{T}\| = \|T\|$, since

$$\tilde{T}(D_{X/Y}) = \tilde{T}(q(D_X)) = T(D_X).$$

Theorem 1.6. *Let X be a normed space. If X^* is separable, then so is X.*

Remark. The converse is false in general, by taking $X = \ell_1$, then $X^* = \ell_\infty$.

Proof: Since X^* is separable, so is S_{X^*}. Let (f_n) be a dense sequence in S_{X^*}. For all $n \in \mathbb{N}$, choose $x_n \in B_X$ such that $|f_n(x_n)| > 1/2$.

Set $Y = \overline{\text{span}}\{x_n \mid n \in \mathbb{N}\}$, the closed linear span of x_n Then we claim $Y = X$.

Assume not. Then we first find $f \in S_{X^*}$ such that $f|_Y = 0$. Since $X/Y \neq \{0\}$, we have $(X/Y)^* \neq \{0\}$, by Hahn-Banach. Choose any $g \in S_{(X/Y)^*}$.

Let $f = g \circ q$. Then $\|f\| = \|g\| = 1$, so $f \in S_{X^*}$, and $f|_Y = 0$.

Choose $n \in \mathbb{N}$ such that $\|f - f_n\| < 1/10$. Now,

$$\frac{1}{2} < |f_n(x_n)| = |(f_n - f)(x_n)| \leq \|f_n - f\| \cdot \|x_n\| < \frac{1}{10},$$

a contradiction.

Theorem 1.7. *Let X be a separable normed space. Then X is isometrically isomorphic to a subspace of ℓ_∞.*

Consider a map $T : X \to \ell_\infty$. The n'th coordinate is then a linear function of x, that is bounded, hence is a functional. So we can think of

$$Tx = (f_n(x)).$$

We also want $\|Tx\|_\infty = \|x\|$, which we can do by choosing a norming functional (or an appropriate approximate).

Proof: Let (x_n) be a dense sequence in X. For each $n \in \mathbb{N}$, choose $f_n \in S_{X^*}$ such that $f_n(x_n) = \|x_n\|$.

Define $T : X \to \ell_\infty$ by

$$T(x) = (f_1(x), f_2(x), \ldots).$$

Note that $|f_n(x)| \leq \|x\|$, so T is well-defined, linear and bounded with norm at most 1.

But for each n,

$$\|Tx_n\|_\infty \geq |f_n(x_n)| = \|x_n\|,$$

so $\|Tx_n\|_\infty = \|x_n\|$. Since (x_n) is dense, and continuity of T, we have $\|Tx\| = \|x\|$ for all $x \in X$.

Remark. We say that ℓ_∞ is *isometrically universal* for the class $\mathcal{SB}$ of all separable Banach spaces.

Theorem 1.8 (Vector-valued Liouville's Theorem)**.** *Let X be a complex Banach space, and $f : \mathbb{C} \to X$ bounded and holomorphic. Then f is constant.*

Proof: Since f is bounded, there is $M \in \mathbb{R}$ such that for all $z \in \mathbb{C}$, $\|f(z)\| \leq M$.

f is holomorphic means that

$$\lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

exists, and is denoted by $f'(z)$, for all $z \in \mathbb{C}$.

Fix $\phi \in X^*$. Since ϕ is linear and continuous,

$$\lim_{w \to z} \frac{\phi(f(w)) - \phi(f(z))}{w - z} = \phi \left(\lim_{w \to z} \frac{f(w) - f(z)}{w - z} \right).$$

So $\phi \circ f : \mathbb{C} \to \mathbb{C}$ is entire.

Also, for all $z \in \mathbb{C}$, $|\phi(f(z))| \leq \|\phi\| \cdot \|f(z)\| \leq M\|\phi\|$. So by Liouville, $\phi \circ f$ is constant, hence $\phi(f(z)) = \phi(f(0))$ for all $z \in \mathbb{C}$.

Fix $z \in \mathbb{C}$. Since X^* separates the points of X, $f(z) = f(0)$.

1.4 Locally Convex Spaces

Definition 1.5. A *locally convex space* (LCS) is a pair $(X, \mathcal{P})$ where X is a real or complex vector space, and $\mathcal{P}$ is a family of seminorms on X such that $\mathcal{P}$ separates the points of X, i.e. for all $x \in X \setminus \{0\}$, there exists $p \in \mathcal{P}$ with $p(x) \neq 0$.

The family $\mathcal{P}$ defines a topology on X as follows: $U \subseteq X$ is open if and only if, for all $x \in U$, there are seminorms $p_1, \dots, p_n \in \mathcal{P}$ and $\varepsilon > 0$ such that

$$\{y \in X \mid p_k(y - x) < \varepsilon \text{ for } k = 1, \dots, n\} \subseteq U.$$

So the open balls form a base of the topology.

Remark.

1. Addition and scalar multiplication are continuous.
2. This is Hausdorff, as $\mathcal{P}$ separates the points.
3. $x_n \to x$ in X if and only if $p(x_n - x) \to 0$ for all $p \in \mathcal{P}$.
4. Let Y be a subspace of X. Let $\mathcal{P}_Y = \{p|_Y \mid p \in \mathcal{P}\}$. Then $(Y, \mathcal{P}_Y)$ is a LCS, and the topology of $(Y, \mathcal{P}_Y)$ is the subspace topology induced by the topology of the LCS $(X, \mathcal{P})$.
5. Let $\mathcal{P}, \mathcal{Q}$ be two families ofseminorms on X, both separating points of X. Say $\mathcal{P}, \mathcal{Q}$ are *equivalent*, and we write $P \sim Q$, if they generate the same topology on X.

The topology of a LCS $(X, \mathcal{P})$ is metrizable if and only if there is a countable $Q \sim P$.

Definition 1.6. A *Fréchet space* is a complete metrizable LCS.

Example 1.4.

1. A normed space $(X, \|\cdot\|)$ is a LCS with $\mathcal{P} = \{\|\cdot\|\}$.
2. Let $U \subseteq \mathbb{C}$ be a non-empty open set, and

$$\mathcal{O}(U) = \{f : U \to \mathbb{C} \mid f \text{ holomorphic}\}.$$

For $K \subseteq U$, K compact, let

$$p_K(f) = \sup_{z \in K} |f(z)|,$$

for $f \in \mathcal{O}(U)$. Let $\mathcal{P} = \{p_K \mid K \subseteq U, K \text{ compact}\}$. Then $(\mathcal{O}(U), \mathcal{P})$ is a LCS. The topology is the topology of local uniform convergence.

Note that there exists (K_n) of compact subsets of U such that $K_n \subseteq \mathrm{int} K_{n+1}$ for all n, and $\bigcup K_n = U$, and

$$\{p_{K_n} \mid n \in \mathbb{N}\} \sim \mathcal{P}.$$

So $(\mathcal{O}(U), \mathcal{P})$ is metrizable, and in fact a Fréchet space. This topology is not normable, i.e. there is no norm on $\mathcal{O}(U)$ inducing the same topology (can use Montel's theorem).

3. Take $d \in \mathbb{N}$, and $\Omega \subseteq \mathbb{R}^d$ non-empty and open. Take

$$C^\infty(\Omega) = \{f : \Omega \to \mathbb{R} \mid f \text{ infinitely differentiable}\}.$$

For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$, we have a differential operator D^α given by

$$D^\alpha f = \left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x^n}\right)^{\alpha_n}.$$

For $\alpha \in (\mathbb{Z}_{\geq 0})^d$, $K \subseteq \Omega$ compact, define

$$p_{K,\alpha}(f) = \sup\{|(D^\alpha) f(x)| \mid x \in K\}.$$

Let $\mathcal{P} = \{p_{K,\alpha} \mid \alpha \text{ multiindex}, K \text{ compact}\}$. Then $(C^\infty(\Omega), \mathcal{P})$ is a LCS, which is a Fréchet space that is not normable.

Lemma 1.1. *Let $(X, \mathcal{P})$ and $(Y, \mathcal{Q})$ be LCS, and $T : X \to Y$ a linear map. Then the following are equivalent:*

(i) *T is continuous.*

(ii) *T is continuous at 0.*

(iii) *For all $q \in \mathcal{Q}$, there are seminorms $p_1, \ldots, p_n \in \mathcal{P}$ and $C \geq 0$ such that for all x,*
$$q(Tx) \leq C \max_{1 \leq k \leq n} p_k(x).$$

Proof: It is easy to see (i) $\iff$ (ii), since translations are a homeomorphism.

We show (ii) $\implies$ (iii). Let $q \in \mathcal{Q}$, and $V = \{y \in Y \mid q(y) < 1\}$ a neighbourhood of 0 in Y. As T is continuous at 0, there exists a neighbourhood of 0 in X such that $T(U) \subseteq V$. Without loss of generality,
$$U = \{x \in X \mid p_k(X) \leq \varepsilon, k = 1, \ldots, n\}$$
for some $n \in \mathbb{N}$, and $p_1, \ldots, p_n \in \mathcal{P}$, $\varepsilon > 0$.

Let $p(x) = \max_{1 \leq k \leq n} p_k(x)$. We show that $q(Tx) \leq \frac{1}{\varepsilon} p(x)$ for all $x \in X$. Let $x \in X$. If $p(x) \neq 0$, then
$$p\left(\frac{\varepsilon x}{p(x)}\right) = \varepsilon,$$
so
$$\frac{\varepsilon x}{p(x)} \in U \implies T\left(\frac{\varepsilon x}{p(x)}\right) \in V.$$
Therefore,
$$q\left(T\left(\frac{\varepsilon x}{p(x)}\right)\right) < 1 \implies q(Tx) \leq \frac{1}{\varepsilon} p(x).$$
If $p(x) = 0$, then $\lambda x \in U$ for all scalars λ, hence $q(T(\lambda x)) < 1$ for all λ. So $q(Tx) = 0$.

Now we show (iii) $\implies$ (ii). Let V be an open neighbourhood of 0 in Y. We seek a neighbourhood U of 0 in X such that $T(U) \subseteq V$. Without loss of generality,
$$V = \{y \in Y \mid q_k(y) < \varepsilon, k = 1, \ldots, m\}.$$
For each $k = 1, \ldots, m$, there exist seminorms $p_{k,1}, \ldots, p_{k,n_k} \in \mathcal{P}$ and $C_k > 0$ such that for all $x \in X$,
$$q_k(Tx) \leq C_k \max_{1 \leq j \leq n_k} p_{k,j}(x).$$

Then,

$$U = \{x \in X \mid p_{k,j}(x) \leq \frac{\varepsilon}{C_k}, k = 1, \ldots, m, j = 1, \ldots, n_k\}$$

is a neighbourhood of 0 in X, and for each $x \in U$,

$$q_k(Tx) \leq C_k \max_{1 \leq j \leq n_k} p_{k,j}(x) < \varepsilon$$

for each $k = 1, \ldots, m$, so $Tx \in V$,

Definition 1.7. The *dual space* of a LCS $(X, \mathcal{P})$ is the space X^* of all linear functional of X which are continuous with respect to the topology of X.

Lemma 1.2. *Let f be a linear functional on a LCS X. Then,*

$$f \in X^* \iff \ker f \text{ is closed.}$$

Proof: One way is obvious: if f is continuous, then $\ker f = f^{-1}(\{0\})$ must be closed.

Now consider the other direction. We can assume without loss of generality that $f \neq 0$. Fix $x_0 \in X \setminus \ker f$. Since $\ker f$ is closed, there is a neighbourhood U of 0 in X, such that $x_0 + U$ is disjoint from $\ker f$.

Without loss of generality,

$$U = \{x \in X \mid p_k(x) < \varepsilon, k = 1, \ldots, n\}$$

for seminorms $p_1, \ldots, p_n \in \mathcal{P}$.

Note that U is convex and *balanced* (if $x \in U$, $|\lambda| = 1$ a scalar then $\lambda x \in U$) since p_i are seminorms.

As f is linear, $f(U)$ is also convex and balanced. Hence it is an interval or a disc.

But since $-f(x_0) \notin f(U)$, otherwise $0 \in f(x_0 + U)$, $f(U)$ is bounded. Hence

$$f(U) \subseteq \{\lambda \text{ a scalar} \mid |\lambda| < M\}.$$

Hence for any $\delta > 0$,

$$f\left(\frac{\delta}{M} U\right) \subseteq \{\lambda \text{ a scalar} \mid |\lambda| < \delta\},$$

and $\frac{\delta}{M} U$ is a neighbourhood of 0. Thus f is continuous at 0.

Theorem 1.9 (Hahn-Banach)**.** *Let $(X, \mathcal{P})$ be a LCS.*

(i) *If Y is a subspace of X and $g \in Y^*$, then there exists $f \in X^*$ such that $f|_Y = g$.*

(ii) *If Y is a closed subspace of X and $x_0 \in X \setminus Y$, then there exists $f \in X^*$ such that $f|_Y = 0$, and $f(x_0) \neq 0$.*

Proof:

(i) By lemma 1.1, there exists $p_1, \dots, p_n \in \mathcal{P}$, and $C \geq 0$ such that for all $y \in Y$,

$$|g(y)| \leq C \max_{1 \leq k \leq n} p_k(y).$$

Define $p : X \to \mathbb{R}$ by

$$p(x) = C \max_{1 \leq k \leq n} p_k(x).$$

Then p is a seminorm on X, and on Y $|g(y)| \leq p(y)$ for all $y \in Y$.

By Hahn-Banach on seminorms, there exists a linear functional f on X such that $f|_Y = g$ and for all $x \in X$, $|f(x)| \leq p(x)$. Lemma 1.1 gives us that f is continuous.

(ii) Let $Z = \operatorname{span}(Y \cup \{x_0\})$. Define a linear functional g on Z by

$$g(y + \lambda x_0) = \lambda$$

for $y \in Y$, λ a scalar. Notice that $\ker g = Y$ is closed by supposition, so g is continuous, i.e. $g \in Z^*$. Then applying (i), we find $f \in X^*$ satisfying $f|_Z = g$, so in particular $f|_Y = 0$ and $f(x_0) = g(x_0) = 1$.

Remark. X^* separates the points of X: given $x \neq y$, apply (ii) to $Y = \{0\}$, and $x_0 = x - y$.

2 Dual Spaces of $L_p(\mu)$ and $C(K)$

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $1 \le p < \infty$. Recall

$$L_p(\mu) = \left\{ f : \Omega \to \text{scalars} \;\middle|\; f \text{ measurable}, \int_\Omega |f|^p \, \mathrm{d}\mu < \infty \right\}.$$

This is a normed space in the L_p-norm,

$$\|f\|_p = \left(\int_\Omega |f|^p \, \mathrm{d}\mu \right)^{1/p}.$$

We identify functions f, g if $f = g$ almost everywhere. If $p = \infty$, then

$$L_\infty(\mu) = \{ f : \Omega \to \text{scalars} \mid f \text{ measurable, essentially bounded} \}.$$

Essentially bounded means f is bounded, up to a null set. This is a normed space in the L_∞ norm:

$$\|f\|_\infty = \operatorname{ess\,sup} |f| = \inf \{ \sup_{\Omega \setminus N} |f| \mid N \in \mathcal{F}, \mu(N) = 0 \}.$$

The infimum can be attained by taking N_i that limit to the infimum, and then taking their union.

Remark. If $\|\cdot\|$ is a seminorm on a vector space X, then

$$N = \{ x \in X \mid \|x\| = 0 \}$$

is a subspace of X, and $\|x + N\| = \|x\|$ defines a norm on the quotient.

We will not think like this for L_p.

Theorem 2.1. *$L_p(\mu)$ is a Banach space for $1 \le p \le \infty$.*

Our aim is to describe $L_p(\mu)^*$.

2.1 Complex Measures

Let Ω be a set, and $\mathcal{F}$ be a σ-algebra on Ω. A *complex measure* on $\mathcal{F}$ is a countably additive set function $\nu : \mathcal{F} \to \mathbb{C}$.

The *total variation measure* of ν, denoted by $|\nu|$, is defined as follows: for $A \in \mathcal{F}$,

$$|\nu|(A) = \sup \left\{ \sum_{k=1}^n |\nu(A_k)| \;\middle|\; A = \bigcup_{k=1}^n A_k \text{ is a measurable partition of } A \right\}.$$

Then $|\nu| : \mathcal{F} \to [0, \infty]$ is a positive measure, and is the smallest measure such that for all $A \in \mathcal{F}$,

$$|\nu(A)| \leq |\nu|(A).$$

In other words, if μ is a positive measure on $\mathcal{F}$ and for all $A \in \mathcal{F}$, $|\nu(A)| \leq \mu(A)$, then $|\nu|(A) \leq \mu(A)$.

The *total variation* of ν is

$$\|\nu\|_1 = |\nu|(\Omega).$$

As currently defined this could be infinite, but we will see that this is always finite.

ν satisfies the two continuity conditions:

- If $A_n \subseteq A_{n+1}$, then
$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \nu(A_n).$$
- If $A_n \supseteq A_{n+1}$, then
$$\nu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \nu(A_n).$$

Signed measures are complex measures that take real values, i.e. countably additive set functions $\mathcal{F} \to \mathbb{R}$.

Theorem 2.2. *Let $(\Omega, \mathcal{F})$ be as before, and ν a signed measure on $\mathcal{F}$.*

Then there exists a measurable partition $\Omega = P \cup N$ of Ω such that for all $A \in \mathcal{F}$ and $A \subseteq P$, then $\nu(A) \geq 0$, and if $A \subseteq N$ then $\nu(A) \leq 0$.

Remark.

1. $\Omega = P \cup N$ is the *Hahn decomposition* of Ω (or of ν).
2. Let $\nu^+(A) = \nu(A \cap P)$ and $\nu^-(A) = -\nu(A \cap N)$ for $A \in \mathcal{F}$.

   Then ν^+, ν^- are finite positive measures such that $\nu = \nu^+ - \nu^-$, and $|\nu| = \nu^+ + \nu^-$.

   These properties determine ν^+ and ν^- uniquely. This decomposition $\nu = \nu^+ - \nu^-$ is the *Jordan decomposition* of ν.
3. Let ν be a complex measure. Then $\Re(\nu)$ and $\Im(\nu)$ are signed measures with Jordan decompositions $\nu_1 - \nu_2$ and $\nu_3 - \nu_4$. Then,

$$\nu = \nu_1 - \nu_2 + i(\nu_3 - \nu_4).$$

This is the *Jordan decomposition* of ν. Note that $\nu_k \leq |\nu|$, and

$$|\nu| \leq \nu_1 + \nu_2 + \nu_3 + \nu_4.$$

So $|\nu|$ is a finite measure since $\nu_1, \nu_2, \nu_3, \nu_4$ are all finite, so $\|\nu\|_1 < \infty$.

4. Suppose the signed measure ν has Hahn decomposition $\Omega = P \cup N$ and Jordan decomposition $\nu^+ - \nu^-$. For $A, B \in \mathcal{F}$ with $B \subseteq A$,

$$\nu^+(A) \geq \nu^+(B) \geq \nu(B),$$

and $\nu^+(A) = \nu(B)$ if $B = P \cap A$. So,

$$\nu^+(A) = \sup\{\nu(B) \mid B \in \mathcal{F}, B \subseteq A\}.$$

Proof: This is a non-examinable sketch.

Define

$$\nu^+(A) = \sup\{\nu(B) \mid B \in \mathcal{F}, B \subseteq A\} \geq 0,$$

since we may always take $B = \emptyset$. It is clear that $\nu^+(\emptyset) = 0$, and ν^+ is finitely additive.

The main claim is that $\nu^+(\Omega) < \infty$. Assume not. Inductively construct $(A_n), (B_n)$ in $\mathcal{F}$ such that $A_0 = \Omega$, and if $\nu^+(A_{n-1}) = \infty$, pick $B_n \subseteq A_{n-1}$, with $\nu(B_n) > n$.

Then pick either $A_n = B_n$ or $A_{n-1} \setminus B_n$ such that $\nu^+(A_n) = \infty$.

We can then use continuity of ν to get a contradiction, by condition on whether $A_n = B_n$ eventually, or $A_n = A_{n-1} \setminus B_n$ infinitely often.

The next claim is that the supremum is achieved, so there exists $P \in \mathcal{F}$ such that

$$\nu^+(\Omega) = \nu(P).$$

Choose $A_n \in \mathcal{F}$, with $\nu(A_n) > \nu^+(\Omega) - 2^{-n}$, and we can check

$$P = \bigcup_m \bigcap_{n \geq m} A_n$$

works. Then letting $N = \Omega \setminus P$, we can check this works as a partition.

Index