III Analysis of PDEs

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Based on Lectures by Prof. Clément Mouhot

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Contents

0 Introduction

Email cm612, in E1.12. Notes are on wordpress, or by Warnick typed by Minter.

Books include Evans, Brézis, John and Lieb-Loss.

0.1 Overview

The field proceeds from works on differential calculus, and trying to turn laws of physics into equations.

We are focused on the modern approach: finding estimates, limits and the function space (using topology). We are not looking at finding explicit formulas.

The course is structured as follows.

Chapter 1. Introduction (2 lectures). This is focused on turning an ODE into a PDE.

Chapter 2. The Cauchy Kovalevskoya Theory (4-5 lectures). Here we look at a PDE with analytic function, where we want to solve for analytic solutions. This lets us construct locally a solution.

Chapter 3. Functional toolbox (4 lectures). Here we introduce Hölder and Lebesgue spaces, as well as weak derivatives, Sobolev spaces, inequalities, approximations by convolution, and extensions or traces of functions.

Chapter 4. Elliptic PDEs (6-7 lectures). Here we look at the Laplace equation and its variants $\Delta u = 0$ on U, and $u|_{\partial U} = g$. We are most interested in Lax-Milgram theory, and may look at Fredholm theory, and spectral theory.

Chapter 5. Hyperbolic PDEs (7 lectures). The main equations are the scalar transport equation (where we look at the Burgers equation), and the wave equation.

1 From ODEs to PDEs

In *differential equations*, the unknown is a function. In an ODE (ordinary differential equation), we first fix a function

$$F = F(x, y_1, \ldots, y_{k+1}).$$

Here $k \geq 1$. We solve for $u : U \subseteq \mathbb{R} \to \mathbb{R}$ the relation, for all $x \in U$,

$$F(x, u(x), u'(x), \ldots, u^{(k)}(x)) = 0. \tag{$*$}$$

Here the domain U is an open, connected, regular set in $\mathbb{R}$.

Example 1.1.

Consider

$$F = F(x, z, y) = f(x, z) - y.$$

Then the equation $(*)$ becomes

$$u'(x) = f(x, u(x)).$$

This can be solved by Picard-Lindelöf, with certain restrictions on f.

In a PDE, we no loner have x in $\mathbb{R}$, but in $\mathbb{R}^n$. Therefore the relation $(*)$ must be modified to include:

$$u(x) = u(x_1, \ldots, x_n), \qquad \frac{\partial u}{\partial x_i}(x), \qquad \frac{\partial^2 u}{\partial x_i \partial x_j}(x), \qquad \ldots$$

Definition 1.1. Give $n \geq 2$, $U \subseteq \mathbb{R}^n$ a domain, a *partial differential equation* of rank or order $k \geq 1$ is a relation of the form, for all $x \in U$,

$$F(x, u(x), Du(x), \ldots, D^k u(x)) = 0, \tag{$**$}$$

where $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \times \cdots \times \mathbb{R}^{n^k} \to \mathbb{R}$.

We solve for $u : U \subseteq \mathbb{R}^n \to \mathbb{R}$. If $u \in C^k(U)$, and satisfies $(**)$ identically as an equality between continuous functions, we say that u is a *classical solution* to the PDE.

Remark.

1. When possible (but not for elliptic PDEs) it is useful to identify one of the components of x, say x_1, as a time $x_1 = t$. We then say that the PDE takes the form of an *evolution problem*.

   Finding such a 'time variable' can be a difficulty in itself.

2. We can also consider the more general case $u(x) \in \mathbb{R}^m$, for $m \geq 1$, and F values in $\mathbb{R}^N$, for $N \geq 1$. When $m \geq 2$, we say it is a *system* of PDEs.

3. Can we consider a PDE as yet another ODE but in infinite dimensions, at least when it is in the form

$$\frac{\partial u}{\partial t} = G\left(\left(\frac{\partial u}{\partial x_i} \right)_{i=2}^{n}, \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)_{i,j=2}^{n}, \dots \right).$$

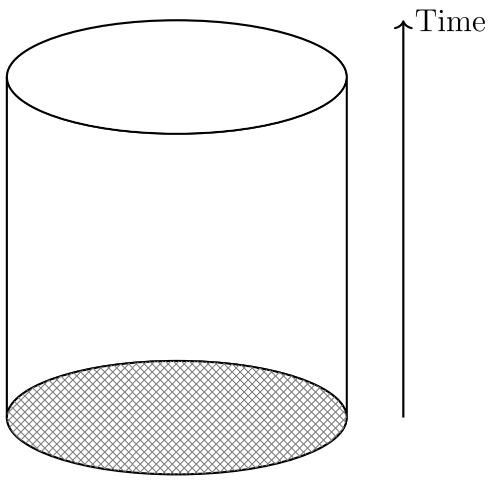
   No. First, losing the total order on the parameter x leads to some geometric phenomena. This is repsonsible for some differences (reversibility, or whether it is an evolution problem).

   Second, if we interpret this is an ODE $u'(t) = g(u)$, then u lives in functional space which is infinite-dimensional, whereas in an ODE we have a trajectory in $\mathbb{R}^N$. Even at a linear level, operators can be unbounded, and the topologies are no longer equivalent.

4. We also have boundary conditions. We know that just the condition $u'(t) = f(t, u(t))$ is not enough; we also need to specify, for example, $f(0) = u_0$.

   For PDEs in evolution form $\partial_t u = G$, then our boundary condition becomes $u(0, \cdot) = u_0(\cdot)$, where this is now a function. Moreover, we can consider boundary conditions on other variables.

5. Also PDEs come in so many different forms, that each structure must be understood.



Boundary condition at time $t = 0$

1.1 The Cauchy Problem

A basic question of mathematical analysis is to solve

$$u'(t) = F(t).$$

If F is continuous, then by FTC we get

$$u(t) = u(t_0) + \int_{t_0}^{t} F(z)\,\mathrm{d}z.$$

This is solved. We have shown there exists solutions, and there's a unique solution given $u(t_0) = u_0$, that depends continuously on boundary data u_0.

A more complicated ODE is where $F = F(t, u(t))$, so

$$u'(t) = F(t, u(t)), \qquad u(t_0) = u_0.$$

There are three main results for functions of this form:

Result 1. Cauchy-Kovalevskaya for ODEs. In the open region where F is real analytic (locally the sum of a Taylor series), there exists a unique local analytic solution: given (t_0, u_0) in this region, there is a neighbourhood around it so that a unique analytical solution u exists.

This has limited use: it is only for F analytic, it does not cover all PDEs, and it is rare to be able to continue the solution.

We can extend this to PDEs.

Result 2. Picard-Lindelöf. In the region where f' is continuous and Lipschitz in the second variable, there exist a local, unique solution C^1 solution u, which depends continuously on u_0.

This inspired the Cauchy problem and well-posedness. We can extend this to linear PDEs, known as Hille-Yosida theorem.

Result 3. Cauchy-Peano. In the region where f is merely continuous, there exists locally a C^1 solution u. In general, it is not unique.

This is done through an iterative scheme and compactness, and is the inspiration for theories of weak solutions in PDEs.

Note that in a larger space, existence is easier, but uniqueness is harder, and vice versa. Hence finding a sweet-spot is critical.

Example 1.2.

The ODE
$$u'(t) = \sqrt{u(t)}, \qquad u(0) = 0$$
has a solution which exists by Cauchy-Peano, but is not unique. Another example is
$$u'(t) = \frac{4u(t)t}{u(t)^2 + t^2}, \qquad u(0) = 0.$$

Another key question is local versus global solutions, i.e. finding a global solution to
$$u'(t) = F(t, u(t)), \qquad u(0) = u_0,$$
for all $t \geq 0$. We have a few criterion for when global solutions exist.

Criterion 1. F is uniform Lipschitz.

Here we can just apply Picard-Lindelöf to continue a solution. It is not easy to export this to PDEs.

Criterion 2. Assume the hypothesis of Picard-Lindelöf, as well as a growth condition on F:
$$|F(t, u)| \leq C(1 + |u|).$$
Then the solution can be continued globally.

The idea behind this is that, a priori, a solution C^1 has to satisfy
$$\frac{\mathrm{d}}{\mathrm{d}t}|u(t)|^2 \leq 2C(1 + |u(t)|^2),$$
$$u'(t) = F(t, u(t)).$$

This is similar to what we call an energy estimate in PDEs.

Example 1.3.

The ODE
$$u'(t) = u(t)^2, \qquad u(t_0) = u_0 > 0$$
has no global solutions. This is because when you square a big number it gets bigger. However if we swap the sign, the solution is global. This is because when you square a small number, it gets smaller.

The ODE
$$u'(t) = \sin(u(t)), \qquad u(0) = u_0$$

has global solutions, by criterion 1. Similarly,

$$u'(t) = \sin(u(t)^2), \qquad u(0) = u_0$$

has global solutions, this time by criterion 2.

1.2 Well-posedness for PDEs

Sometimes there is no explicit formula or even series for a solution to a PDE. In these cases we need to construct solutions abstractly.

Two breakthroughs happened when looking at when PDEs have solutions. The first is the definition of a Cauchy problem, and the second is looking at well-posedness.

Definition 1.2. A *Cauchy problem* is the combination of a PDE, and some boundary data; prescribing values of the unknown u, and possibly its derivatives, on parts of the domain.

Such a problem is said to be *well-posed* if:

- A solution exists (in some function space, e.g. $C^k(U), H^k(U)$, at least locally).
- The solution is unique among possible solutions in the function space.
- The solutions depends continuously on the boundary data.

1.3 Terminology and Examples

Definition 1.3. A PDE with vector field F is *linear* if F is a linear function of u and its derivative. So,

$$\sum_{|\alpha|\leq k} a_\alpha(x)\partial^\alpha u(x) = f(x).$$

Here $f(x)$ is the *source*, or the RHS.

A PDE is *semilinear* when F is linear in the highest-order derivatives of u:

$$\sum_{|\alpha|=k} a_\alpha(x)\partial^\alpha u + a_0[x, u(x), Du(x), \dots, D^{k-1}u] = 0.$$

A PDE is *quasilinear* if F is linear in highest-order deriatives of u, but can depend nonlinearly on the lower-order derivatives:

$$\sum_{|\alpha|=k} a_\alpha[x, u(x), Du(x), \dots, D^{k-1}u]\partial^\alpha u(x) + a_0[x, u(x), Du(x), \dots, D^{k-1}u] = 0.$$

A PDE is *fully nonlinear* if it is none of the types above.

Example 1.4.

- Linear PDE: Take the Laplace,

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0.$$

- Semilinear PDE:

$$\Delta u = \left(\frac{\partial u}{\partial x_1} \right)^2.$$

- Quasilinear PDE:

$$u \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial x} \text{ on } \mathbb{R}^2.$$

- Fully nonlinear:

$$\frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y} \right)^2 = 0.$$

We also have some examples from physics:

- Newtons' equations.
- Euler incompressibility equation.
- Navier-Stokes equation.
- Boltzmann equation.
- Vlasov equation.
- Schrödinger equation.
- Einstein equations.
- Dirac equation.

Moreover here are equations from math:

- Cauchy-Riemann equations.
- Ricci flow. $\partial_t g_{ij} = -2R_{ij}$.

2 The Cauchy-Kovalevskaya Theory

This is the only "general" theorem that can be salvaged from ODEs. Some concepts that arise are:

- Non-characteristic Cauchy data.
- Principal symbols.
- Basic classification of PDEs.

However the analyticity used in this theory is most often not satisfying, in the functional setting.

2.1 Real Analyticity

Definition 2.1. Given $U \subseteq \mathbb{R}^n$ open, a function $f : U \to \mathbb{R}$ is *real analytic* near $\tilde{x} \in U$ if there is $r > 0$ and real constants (f_α) so that the series

$$\sum_{\alpha \geq 0} f_\alpha (x - \tilde{x})^\alpha$$

converges for $x \in B(\tilde{x}, r)$ to $f(x)$.

If $f : U \to \mathbb{R}^n$, for $n \geq 2$, then it is real analytic if f_i is real analytic for $i = 1, \ldots, n$.

f is *real analytic* in U if it is real analytic near each point of U. This is sometimes denoted as

$$f \in C^\omega(U).$$

Example 2.1.

Simple examples of real analytic functions include polynomials, exponential functions, trigonometric functions.

The map $z \mapsto \overline{z}$, i.e. conjugation, is not $\mathbb{C}$-differentiable, but it is real analytic in $\mathbb{R}^2$.

The function

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0, \\ 0 & x = 0 \end{cases}$$

is C^∞, but not real analytic. In fact any C_c^∞ function cannot be real analytic.

Liouville's theorem does not hold, by either sin or $1/(1 + x^2)$.

Real analyticity is local, meaning if f is real analytic near $\tilde{x}$, then f is real analytic in $B(\tilde{x}, r) \subseteq U$ for some $r > 0$.

Proposition 2.1. *Given $U \subseteq \mathbb{R}^n$ open and non-empty, then $f : U \to \mathbb{R}$ is real analytic on U if and only if $f \in C^\infty$, and for any $K \subseteq U$ compact, there are $C(K)$, $r(K) > 0$ so that the following growth conditions holds: for all $x \in K$, $\alpha \in \mathbb{N}^n$,*

$$|\partial^\alpha f(x)| \leq C(K) \frac{\alpha!}{r(K)^{|\alpha|}}.$$

Remark.

- When $U \subseteq \mathbb{R}$, another equivalent definition is, f is real analytic on U if it can be locally extended to a $\mathbb{C}$-differentiable function near each point of U.
- When $U = \mathbb{R}^n$, real analyticity is also equivalent to exponential decay in the Fourier variables.

Proof: Recall that, if

$$\sum_{\alpha \geq 0} f_\alpha (x - \tilde{x})^\alpha$$

converges at x such that $|x - \tilde{x}| = r$, then the general term is bounded by

$$|f\alpha| \leq C r^{-|\alpha|}.$$

Hence for $|x - \tilde{x}| < r$, we have absolute convergence.

Recall for a function, the *radius of convergence* is the largest $r \geq 0$ so that we have a point of convergence at a distance r.

The easy implication is the forwards. Suppose that in $B(\tilde{x}, r) \subseteq U$, we have the power series

$$f(x) = \sum f_\alpha (x - \tilde{x})^\alpha,$$

with radius of convergence at least r. Then from a standard theorem, f is smooth in $B(\tilde{x}, r)$ with

$$\partial^\alpha f(\tilde{x}) = (f_\alpha)\alpha!.$$

We know that $|f_\alpha| \leq C\bar{r}^{-|\alpha|}$, for some $\tilde{r} < \bar{r} < r$. Then for all $x \in \bar{B}(\tilde{x}, \tilde{r})$, and $\beta \in \mathbb{N}^n$,

$$\begin{aligned}
|\partial^\beta f(x)| &= \left|\partial^\beta \left(\sum_{\alpha \geq 0} f_\alpha (x-\tilde{x})^\alpha\right)\right| \\
&\leq \sum_{\alpha \geq \beta} |f_\alpha| \frac{\alpha!}{(\alpha-\beta)!} |x-\tilde{x}|^{|\alpha-\beta|} \\
&\leq C \sum_{\alpha \geq \beta} \bar{r}^{-|\alpha|} \frac{\alpha!}{(\alpha-\beta)!} \tilde{r}^{|\alpha-\beta|} \\
&\leq C \bar{r}^{|\beta|} \sum_{\alpha \geq \beta} \left(\frac{\tilde{r}}{\bar{r}}\right)^{|\alpha-\beta|} \frac{\alpha!}{(\alpha-\beta)!}.
\end{aligned}$$

Let $\lambda = \tilde{r}/\bar{r} < 1$. Then by observation,

$$(1-\lambda)^{-1} = \sum_{j \geq 0} \lambda^j.$$

Taking the m'th partial derivative,

$$\frac{m!}{(1-\lambda)^{m+1}} = \sum_{j \geq m} \frac{j!}{(j-m)!} \lambda^{j-m}.$$

If we apply this, then

$$\begin{aligned}
|\partial^\beta f(x)| &\leq C r^{|\beta|} \sum_{\alpha \geq \beta} \frac{\alpha!}{(\alpha-\beta)!} \lambda^{|\alpha-\beta|} \\
&\leq C |r|^{|\beta|} \frac{\beta!}{(1-\lambda)^{|\beta|+n}} \\
&\leq \frac{C\beta!}{(1-\lambda)^n} \left(\frac{r}{1-\lambda}\right)^{|\beta|}.
\end{aligned}$$

For the other direction, consider our assumption on $K = \bar{B}(\tilde{x}, r) \subseteq U$, there exists $\tilde{C}, \tilde{r} > 0$ such that for all $x \in K$, $\alpha \in \mathbb{N}^n$,

$$|\partial^\alpha f(x)| \leq \tilde{C} \tilde{r}^{-|\alpha|} \alpha!.$$

Choose $x \in B(\tilde{x}, \tilde{r}/2)$, and Taylor expand, so

$$f(x) = \sum_{|\alpha| \leq k} \partial^\alpha f(x) \frac{(x-\tilde{x})^\alpha}{\alpha!} + \sum_{|\alpha| = k+1} R_\alpha(x)(x-\tilde{x})^\alpha.$$

If $n = 1$, we have

$$R_\alpha(x) = \frac{|\alpha|}{\alpha!} \int_0^1 (1-t)^{|\alpha|-1} \partial^\alpha f(\tilde{x} + t(x - \tilde{x}))\, \mathrm{d}t.$$

From the growth condition, the main part of the expansion is a partial sum of an absolute series, and

$$\left| \sum_{|\alpha|=k+1} R_\alpha(x)(x-\tilde{x})^\alpha \right| \leq \sum_{|\alpha|=k+1} |R_\alpha(x)| \left(\frac{\tilde{r}}{2}\right)^{k+1},$$

$$\begin{aligned} |R_\alpha(x)| &\leq \tilde{C} \frac{|\alpha|}{\alpha!} \int_0^1 (1-t)^{|\alpha|-1} \tilde{r}^{-(k+1)}\, \mathrm{d}t \\ &\leq \tilde{C} \frac{\tilde{r}^{-(k+1)}}{\alpha!}. \end{aligned}$$

So,

$$\begin{aligned} I = \left| \sum_{|\alpha|=k+1} R_\alpha(x)(x-\tilde{x})^\alpha \right| &\leq \tilde{C} \tilde{r}^{-(k+1)} \left(\frac{\tilde{r}}{2}\right)^{k+1} \cdot \binom{k+n}{n-1} \\ &\leq C'(k+n)^{n-1} 2^{-(k+1)} \to 0 \end{aligned}$$

as $k \to \infty$, which shows the convergence of the Taylor series.

Definition 2.2. Let

$$f = \sum_{\alpha \geq 0} f_\alpha x^\alpha, \qquad g = \sum_{\alpha \geq 0} g_\alpha x^\alpha$$

be two formal power series. Then g *majorizes* f, or g is a *majorant* of f, written $g \gg f$ if $g_\alpha \geq |f_\alpha|$ for all $\alpha \in \mathbb{N}^\alpha$.

If f, g are $\mathbb{R}^m$-valued, then each component $g_j \gg f_j$, for $j = 1, \dots, m$.

Proposition 2.2. *Given f, g formal power series:*

(i) *If $g \gg f$ and g converges for $\|x\| < r$, then f converges for $\|x\| < r$ as well.*

(ii) *If f converges for $\|x\| < r$, and $\tilde{r} \in (0, r/\sqrt{n})$, there is a majorant $g \gg f$ which converges in $\|x\| < \tilde{r}$.*

Proof:

(i) Let $x \in B(0,r)$, then

$$\sum_{\alpha \le k} |f_\alpha x^\alpha| \le \sum_{|\alpha| \le k} |f_\alpha||x_1|^{\alpha_1} \cdots |x_n|^{\alpha_n} \\ \le \sum_{|\alpha| \le k} g_\alpha |x_1|^{\alpha_1} \cdots |x_n|^{\alpha_n}.$$

If $y = (|x_1|, \dots, |x_n|)$, then $\|y\| = \|x\| < r$, and since g converges for $\|x\| < r$ it converges for y.

(ii) Take $\tilde{r} \in (0, r/\sqrt{n})$ and $y = (\tilde{r}, \dots, \tilde{r})$. Then $\|y\| = \sqrt{n}\tilde{r} < r$

Now f converges on y, so the general term is bounded:

$$|f_\alpha| \le C\tilde{r}^{-|\alpha|}.$$

Now consider

$$\bar{f}(x) = \frac{C}{1 - (x_1 + \cdots + x_n)/\tilde{r}},$$

for $x \in B(0, \tilde{r}/\sqrt{n})$. Then

$$\bar{f}(x) = C \sum_{k \ge 0} \tilde{r}^{-k} \left(\sum_{|\alpha| = k} \frac{|\alpha|!}{\alpha!} x^\alpha \right) \\ = C \sum_{\alpha \ge 0} \frac{\tilde{r}^{-|\alpha|}|\alpha|!}{\alpha!} x^\alpha.$$

We can check that $\alpha! \le |\alpha|!$, so $f \ll \bar{f}$.

Another majorant we can consider is

$$\bar{f}(x) = C \prod_{i=1}^{n} \left(\frac{1}{1 - (x_i/\tilde{r})} \right) = C \prod_{\alpha \ge 0} \left(\frac{x}{\tilde{r}} \right)^\alpha \\ = C \sum_{\alpha \ge 0} \tilde{r}^{-|\alpha|} x^\alpha.$$

2.2 Cauchy-Kovalevskaya for ODEs

Theorem 2.1. *Let $a, b > 0$, and $u_0 \in \mathbb{R}$. Consider $F : (u_0 - b, u_0 + b) \to \mathbb{R}$ real analytic, and $u : (-a, a) \to (u_0 - b, u_0 + b)$ a C^1 solution to*

$$u'(t) = F(u(t)).$$

Then u is real analytic on $(-a, a)$.

Proof: We prove this in many ways. The first is by Picard iteration.

Define

$$\begin{aligned} u_{l+1}(z) &= u_0 + \int_0^z F(u_l(z))\,\mathrm{d}z, \\ u_0(z) &= u_0, \end{aligned}$$

where we exited F to be homomorphic near the origin. We may prove that (u_l) is Cauchy in the $\|\cdot\|_\infty$ norm, on a small enough neighbourhood of $t = 0$. Here we use the bound on F'.

Now $u_l \to u$ converges uniform locally. By induction, we can show u_l is $\mathbb{C}$-differentiable, so by Morera's theorem, u is $\mathbb{C}$-differentiable.

The second proof is by separation of variables. If $F(0) = 0$, then $u = 0$, and there is nothing to prove.

Otherwise, $F \neq 0$ near 0, and we may write

$$G(y) = \int_0^y \frac{\mathrm{d}x}{F(x)},$$

for $y \in (-b', b')$ for some $0 < b' < b$ small enough. Then we find that

$$\frac{\mathrm{d}}{\mathrm{d}t} G(u(t)) = \frac{F(u(t))}{F(u(t))} = 1.$$

For $t \in (-a', a')$, $G(u(0)) = G(0) = 0$, and $G(u(t)) = t$, and

$$G'(0) = \frac{1}{F(0)} \neq 0.$$

So there exists a smaller $(-a'', a'') \subseteq (-a', a')$ such that G^{-1} is defined, and F is real analytic. Then since G is real analytic, G^{-1} is as well, so

$$u(t) = G^{-1}(t)$$

is real analytic on $(-a'', a'')$.

The third proof is by embedding the equation in a larger continuum of equations. For $z \in \mathbb{C}$, consider

$$\begin{aligned} u_z'(t) &= zF(u_z(t)), \\ u_1(0) &= 0, \end{aligned}$$

where the original equation is $z = 1$.

For $|z| < 2$, and $|t| < \varepsilon$ small enough, Picard-Lindelöf gives a solution uniformly in $|z| < 2$ by having Lipschitz constant on zF uniformly in $|z| < 1$.

Defining

$$\partial z = \left(\frac{\partial x - i\partial y}{2}\right), \qquad \partial \bar{z} = \left(\frac{\partial x + i\partial y}{2}\right),$$

then a function f is complex differentiable if and only if

$$\partial \bar{z}(f) = 0,$$

Taking our function to be u_z', we find

$$\partial t \partial \bar{z}[u_z(t)] = zF'(u_z(t))\partial \bar{z}[u_z(t)].$$

We can integrate this to find

$$\partial \bar{z}[u_z(t)] = \exp\left[\int_0^t zF'(u_z(s))\,\mathrm{d}s\right]\partial \bar{z}[u_z(0)],$$

where the last term is 0, hence the entire thing is 0. So, for $|t|$ small enough and $|z| < 2$, $z \mapsto u_z(t)$ is $\mathbb{C}$-differentiable. Hence, we can write

$$u_1(t) = \sum_{n=0}^{\infty} \frac{1^n}{n!} \frac{\partial^n}{\partial z^n}[u_z(t)]\bigg|_{z=0}.$$

For $|z| < 2$ real,

$$\frac{\partial^n}{\partial z^n}[u(zt)] = t^n u^{(n)}(0),$$

where the latter is real differentiable. This implies convergence and equality,

$$u(t) = u_1(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} u^{(n)}(0).$$

Another proof is by majorant. If u, F are smooth with

$$u'(t) = F(u(t)),$$

then we will induct on $u \in C^k((-a,a))$, $k \geq 1$. Then note $F \circ u \in C^k(\cdot)$, so $u' \in C^k$, hence $u \in C^{k+1}$. For example,

$$\begin{aligned}
u^{(1)}(t) &= F^{(0)}(u(t)), \\
u^{(2)}(t) &= F^{(1)}(u(t)) \times u^{(1)}(t) = F^{(1)}(u(t)) \times F^{(0)}(u(t)), \\
u^{(3)}(t) &= F^{(2)}(u(t)) \times F^{(0)}(u(t))^2 + F^{(1)}(u(t))^2 F^{(0)}(u(t)),
\end{aligned}$$

and we can go on. By induction, we can show $u^{(k)}(t)$ is a polynomial in $F^{(0)}(u(t)), F^{(1)}(u(t)), \ldots, F^{(k-1)}(u(t))$, with non-negative integer coefficients, so write

$$u^{(k)}(t) = p_k(F^{(0)}(u(t)), F^{(1)}(u(t)), \ldots, F^{(k)}(u(t))).$$

For example,

$$\begin{aligned}
p_1(x_1) &= x_1, \\
p_2(x_1, x_2) &= x_1 x_2, \\
p_3(x_1, x_2, x_3) &= x_1^2 x_3 + x_1 x_2^2.
\end{aligned}$$

These polynomials are universal; they do not depend on F. If $G \gg F$, then $|G^{(k)}(0)| > |F^{(k)}(0)|$ for all k, and so

$$\begin{aligned}
p_k(F^{(0)}(0), \ldots, F^{(k-1)}(0)) &\leq p_k(|F^{(0)}(0)|, \ldots, |F^{(k-1)}(0)|) \\
&\leq p_k(G^{(0)}(0), \ldots, G^{(k-1)}(0)).
\end{aligned}$$

Assume that we have $G \gg F$, and that v is a solution to

$$\begin{aligned}
v'(t) &= G(v(t)), \\
v(0) &= 0,
\end{aligned}$$

and v is real analytic near 0. Then,

$$v^{(k)}(0) = p_k(G^{(0)}(0), \ldots, G^{(k-1)}(0)),$$

so that $v^{(k)}(0) > |u^{(0)}(0)|$, for all $k \geq 0$. Since v is real analytic,

$$v(t) = \sum_{k \geq 0} v^{(k)}(0) \frac{t^k}{k!},$$

which is absolutely convergent near 0. Define

$$\tilde{u}(t) = \sum_{k \geq 0} p_k(F^{(0)}(0), \dots, F^{(k-1)}(0)) \frac{t^k}{k!},$$

ons the same disc of convergence. This $\tilde{u}$ is real analytic near 0, and since $\tilde{u}(t)$ and $F(\tilde{u}(t))$ are real analytic and all derivatives agree at $t = 0$, they are equal near $t = 0$.

Now all we need to do is construct G and v. This is possible since

$$|F^{(k)}(0)| \leq Ck!r^{-k},$$

for $k \geq 0$ and some $C, r > 0$. So we can define

$$G(x) = \frac{Cr}{r - x},$$

for $|x| < r$. Then the solution to

$$\begin{aligned} v'(t) &= G(v(t)), \\ v(0) &= 0 \end{aligned}$$

is

$$v(t) = r - r\sqrt{1 - \frac{2Ct}{r}}.$$

This is real analytic for $|t| < r/2C$.

Theorem 2.2. *Let $a, b > 0$, $u_0 \in \mathbb{R}^m$ for $m \geq 1$, and $F : B(u_0, b) \to \mathbb{R}^m$ real analytic.*

Let $u : (-a, a) \to B(u_0, b)$ be a C^1 solution to $u'(t) = F(u(t))$, with $u(0) = u_0$.

Then u is real analytic in $(-a, a)$.

Proof: We can extend proofs 1 and 3 from the scalar case.

To extend the method of majorants, for $C, r > 0$ well chosen, set

$$G(x_1, \dots, x_m) = (G_1(x_1, \dots, x_m), \dots, G_m(x_1, \dots, x_m)),$$

$$G_1 = \dots = G_m = \frac{Cr}{r - x_1 - \dots - x_m}.$$

We can reduce the proof to proving that the solution v to the auxiliary problem

$$\begin{aligned} v'(t) &= G(v(t)), \\ v(0) &= 0. \end{aligned}$$

By symmetry, we solve in the form

$$v_1(t) = \cdots = v_m(t) = w'(t),$$

$$w'(t) = \frac{Cr}{r - mw(t)}.$$

This solves as

$$w(t) = \frac{r}{m} - \frac{r}{m}\sqrt{1 - \frac{2Cmt}{r}}.$$

2.3 Cauchy-Kovalevskaya for PDEs

The CK theorem only extends for k-th order quasilinear PDEs, so for $x \in U \subseteq \mathbb{R}^m$,

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \ldots, Du, u, x)\partial_x^\alpha u + a_0(D^{k-1}u, \ldots, Du, u, x) = 0.$$

Remark. For $k = 1$, we have an alternative proof by the method of characteristics.

2.4 Cauchy Problem and Statement

Definition 2.3. Give $U \subseteq \mathbb{R}^n$ open and non-empty, we say that $\Sigma \subseteq U$ is a *smooth* (resp. real analytic) hypersurface near $x \in \Sigma \subseteq U$ if there exists $\varepsilon > 0$ and $\Phi : B(x, \varepsilon) \to V \in \mathbb{R}^n$ so that

$$\phi\left(\Sigma \cap B(x, \varepsilon)\right) = \{y_n = 0\} \cap V,$$

and $\phi(x) = 0$, with:

- ϕ bijective,
- ϕ, ϕ^{-1} smooth (resp. real analytic).

$\Sigma \subseteq U$ is a smooth (resp. real analytic) hypersurface if it satisfies the previous definition around any $x \in \Sigma$.

Remark. Σ a submanifold is smooth (resp. real analytic) which is embedded with normal unit vector???

The definition implies there admits a normal unit vector $N : \Sigma \to \mathbb{R}^n$, which is perpendicular to the tangent space, and smooth (resp. real analytic).

Given a parametrization $\Psi : B_{\mathbb{R}^{n-1}}(0, \varepsilon) \times (-\varepsilon, \varepsilon) \to U_x$, define

$$\Psi(y) = \tilde{\Psi}(\tilde{y}) + y_n N(\psi(\tilde{y})),$$

where $\tilde{y} = (y_1, \dots, y_{n-1})$, and $\tilde{\Psi} : B_{\mathbb{R}^{n-1}}(0, \varepsilon) \to \Sigma \cap U_x$. Then for this parametrization,

$$\partial y_n \Psi(y) = N(\tilde{\Psi}(\tilde{\psi})).$$

The tangent space to Σ is given, for $x' \in U_x$, by

$$x' + \operatorname{span}\{\partial y_1 \tilde{\Psi}(y'), \dots, \partial_{n-1} \tilde{\Psi}(y')),$$

where $\Psi(y, 0) = \tilde{\Psi}(y', 0) = x'$. So,

- $\phi = \Psi^{-1}$ defines a chart.
- Take $\varphi = \phi_n$, the last component. Then note $\Sigma \cap U_x = \{\varphi = 0\} \cap U_x$. Moreover,

  $$\nabla_x \varphi = N$$

  on $U_x \cap \Sigma$.

Indeed, φ satisfies $\varphi(\psi(y)) = y_n$. Differentiating this condition,

$$\nabla \varphi \cdot \partial y_i \psi = 0,$$

so $\nabla \varphi$ is collinear to N. Differentiating along y_n,

$$\nabla_x \varphi \cdot N = 1,$$

so $\phi_x \varphi = N$ on Σ.

Definition 2.4. Given $\Sigma \subseteq U \subseteq \mathbb{R}^n$ a smooth or real analytic hypersurface, and $j \geq 1$, we define the j'th *normal derivative* of the function u to Σ as

$$\partial_N^j u = \sum_{|\alpha|=j} (\partial_x^\alpha u(x)) N(x)^\alpha = \sum_{\alpha_1 + \cdots + \alpha_n = j} \left(\frac{\partial^j u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right) N_1(x)^{\alpha_1} \cdots N_n(x) \alpha^n.$$

Remark. For $j = 1$,

$$\partial_N^1 u = (\nabla u \cdot N).$$

Definition 2.5. Given $\Sigma \subseteq \mathbb{R}^n$ as before, and $g_0, g_1, \ldots, g_{k-1} : \Sigma \to \mathbb{R}$ smooth (resp. real analytic), the *Cauchy problem* is finding solutions to

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \ldots, Du, u, x)\partial_x^\alpha u + a_0(D^{k-1}u, \ldots, Du, u, x) = 0,$$

with $u(x) = g_0(x), \partial_N^1 u = g_1(x), \ldots, \partial_N^{k-1} u = g_{k-1}$ on Σ.

The natural question to ask is, if we are given the above Cauchy data on Σ, does this determine all derivatives locally on Σ?

Let us start with the flat case: $U = \mathbb{R}^n$, and $\Sigma = \{x_n = 0\}$. Then $N(x) = e_n$ is constant, and

$$\partial_N^j u(x) = \partial_{x_n}^j u(x),$$

on $x = (x', 0) \in \Sigma$. The second condition gives

$$\partial_x^\alpha u(x) = \partial_{x'}^{\alpha'} \partial_{x_n}^j u(x) = \partial_{x'}^{\alpha'} g_j(x),$$

for $\alpha' \in \mathbb{N}^{n-1}$ and $j = 0, \ldots, k-1$. The first missing partial derivatives is $\partial_{x_n}^k u$ on Σ. But we can find this using the PDE: if

$$A(x) = a_{(0,\ldots,0,k)}(D^{k-1}u, \ldots, Du, u, x) \neq 0,$$

then the first condition gives

$$\partial_{x_n}^k u(x) = -\sum_{|\alpha|=k} \frac{a_\alpha(\cdots)}{A(x)} \partial_x^\alpha u - \frac{a_0(\cdots)}{A(x)}$$

on Σ. We can continue; differentiating again,

$$0 = \sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \ldots, Du, u, x)\partial_x^\alpha \partial_{x_n} u(x) + \tilde{a}_0(D^k u, D^{k-1}u, \ldots),$$

where

$$\tilde{a}_0(\cdots) = \sum_{|\alpha|=k} \partial_{x_n}(a_\alpha(\cdots))\partial_x^\alpha u + \partial_{x_u}(a_0(\cdots)).$$

If $A(x) \neq 0$, then

$$g_{k+1}(x) = \partial_{x_n}^{k+1} u = \sum_{|\alpha|=k} \frac{a_\alpha(\cdots)}{A(x)} \partial_x^\alpha \partial_{x_n} u - \frac{\tilde{a}_0(\cdots)}{A(x)}.$$

This is a function of $g_0, \ldots, g_k$, so in turn a function of only $g_0, \ldots, g_{k-1}$. So for the flat case, we ar happy if $A(x)$ is non-zero. In general, we want

$$A(x) = \sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \ldots, Du, u, x)N(x)^\alpha \neq 0$$

on Σ.

If u is a solution in the general case, then $v = v(y) = u(\psi(y))$ is a solution to

$$0 = \sum_{|\alpha|=k} b_\alpha(D^{k-1}v(y), \ldots, Dv(y), v(y), y) + b_0(D^{k-1}v(y), \ldots, Dv(y), v(y), y).$$

This is a PDE of the same type, and the coefficients b_α depends on the coefficients a_α. The boundary conditions on v are

$$\begin{aligned} \partial_{y_n}^j v(y) = \partial_{y_n}^j [u(\psi(y))] &= \sum_{|\alpha|=j} \partial_x^\alpha u(\psi(y))(\partial_{y_n}\psi)^\alpha \\ &= \partial_N^j u, \end{aligned}$$

so we get that

$$\partial_{y_n}^j v(\tilde{y}, 0) = g_j(\mathit{something})$$

The non-characteristic condition in the original variables means

$$\nabla \phi_n = N,$$

on Σ. This is

$$\sum_{|\alpha|=k} a_\alpha(\cdots)\partial_x^\alpha u.$$

Collecting all $\partial_{y_n}^k v$ terms, we find

$$\partial_x^\alpha u = \partial_x^\alpha(v \circ \phi) = (\partial_{y_n}^k v)(\nabla_x \phi) + \text{higher order terms}.$$

He yaps on idk what he is saying.

Theorem 2.3. *Let $\Sigma \subseteq U \subseteq \mathbb{R}^n$ be a real analytic hyerpsurface, with a PDE and Cauchy data as before, where a_α, a_0 and g_j are real analytic.*

Then for any $x \in \Sigma \subseteq U$, there is a neighbourhood around x in which there exists a unique real analytic solution to the PDE with prescribed Cauchy data.

To prove this, the idea is to use the method of majorants, with universal polynomials and a non-charactericity condition.

First we have a reduction step: without loss of generality, our base point is $x = 0$, and with ϕ and ψ we reduce to $\Sigma = \{x_n = 0\}$. Then the boundary conditions are

$$\partial_{x_n}^j u = g_j$$

on Σ. By our condition,

$$A(x) = a_{(0,\ldots,0,k)}(\cdots) \neq 0,$$

so we can divide by it, and reduce to $A = 1$. Moreover we can reduce to $g_0 = g_1 = \cdots = g_{k-1} = 0$ by subtracting to u an appropriate real analytic function, for example

$$G(y) = \sum_{j=0}^{k-1} g_j(\tilde{x}) \frac{x_n^j}{j!},$$

with

$$\partial_{x_k}^j (u - G) = 0$$

on $\{x_n = 0\}$.

Finally, we reduce to a first-order equation for a system of equations, by changing our unknown u into

$$(u, Du, D^2 u, \ldots, D^{k-1} u).$$

This produces a much nicer PDE of the form

$$\partial_{x_n} u = \sum_{j=1}^{k-1} b_j(u(x), \tilde{x}) \partial_{x_j} u + b_0(u(x), \tilde{x}),$$

where we avoid x_n dependency by adding if necessary one more component x_n to u. Our boundary conditions are $u = 0$ on $\Sigma = \{x_n = 0\}$, and also $b_j : \mathbb{R}^m \times \mathbb{R}^{n-1} \to \mathbb{R}^{m \times m}$ to something, $b_0 : \mathbb{R}^m \times \mathbb{R}^{n-1} \to \mathbb{R}^m$ are real analytic near zero.

The second step is to prove there are universal polynomials with non-negative integer coefficients $p_{\alpha,i}$ such that

$$\partial_x^\alpha u_i(0) = p_{\alpha,i}((D^\beta b_j)_{\ell_1,\ell_2}, (D^\beta b_0)_\ell)(0,0),$$

for $|\beta| \leq |\alpha| - 1$, ℓ appropriate. We prove this by induction on α_n.

Proof: This is done by induction on a_n: Differentiate $u(\tilde{x}, 0) = 0$ near $\tilde{x} = 0$ to get

$$\partial_x^\alpha u(0) = 0,$$

which gives $\alpha_n = 0$. For $\alpha_n = 1$, we look at

$$\partial_{\tilde{x}}^{\tilde{\alpha}} \partial_{x_n}^1 u.$$

We are actually now just going to look at the general case, when we differentiate the PDE with

$$\partial_{\tilde{x}}^{\tilde{\alpha}} \partial_{x_n}^{\alpha_n - 1},$$

which gives

$$\partial_{\tilde{x}}^{\tilde{\alpha}} \partial_{x_n}^{\alpha_n} u(0) = \partial_{\tilde{x}}^{\tilde{\alpha}} \partial_{x_n}^{\alpha_n - 1} \left[\sum b_j(\cdot) \partial x_j u + b_0 \right].$$

For $\alpha_n = 1$, we have no ∂_{x_n} on the RHS, so we get only $D_2^{\tilde{\alpha}} b_0(0,0)$. To prove the induction step, note

$$\partial_{\tilde{x}}^{\tilde{\alpha}} \partial_{x_n}^{\ell+1} u(0) = \partial_{\tilde{x}}^{\tilde{\alpha}} \partial_{x_n}^{\ell} \left[\sum_{j=1}^{n-1} b_j(u(x), \tilde{x}) \partial x_j u + b_0(\cdots) \right],$$

which is a polynomial with coefficients in $\mathbb{N}$, after applying the induction hypothesis for ℓ.

Our candidate solution is

$$u(x) = \sum_{\alpha \geq 0} \partial_x^\alpha u(0) \frac{x^\alpha}{\alpha!},$$

where $\partial_x^\alpha u$ is defined by $p_{\alpha,i}(Db_j(0), Db_0(0))$. If we are able to find a majorant $b_j^* \gg b_j$, and $b_0^* \gg b_0$, then

$$\begin{aligned} |\partial_x^\alpha u_i(0)| &\leq |p_{\alpha,i}(Db_j(0), Db_0(0))| \\ &\leq p_{\alpha,i}(|Db_j(0)|, |Db_0(0)|) \\ &\leq p_{\alpha,i}(Db_j^*(0), Db_0^*(0)) \\ &= \partial_x^\alpha v_i(0), \end{aligned}$$

where v is a solution to

$$\partial x_n v = \sum b_j^*(v(x), \tilde{x}) \partial_{x_j} v + b_)(v(x), \tilde{x}),$$

and $v = 0$ on Σ. Now $v \gg u$, and if v is real analytic near 0, then so is u. Then

$$\partial x_n u - \sum b_j(u(x), \tilde{x}) \partial_{x_j} u - b_0(u(x), \tilde{x}) = S(x)$$

is an entire series with non-zero radius of convergence around 0, with $\partial_x^\alpha S(0) = 0$ for all $\alpha \in \mathbb{N}$. Hence $S = 0$ near 0, and u is a solution which is unique by real analyticity.

Now all that is left is to define the majorants. Define

$$\begin{aligned} b_j^* &= gJ_{m\times m}, \\ b_0^* &= gU_m, \end{aligned}$$

where J is the all 1's matrix, and U is the all 1's vector. Then we can define

$$g(y_1, \ldots, y_m, x_1, \ldots, x_{n-1}) = \frac{Cr}{r - (x_1 + \cdots + x_{n-1}) - (y_1 + \cdots + y_m)}.$$

Then we find that

$$\partial^\alpha g(0) = C\alpha! r^{-|\alpha|},$$

which dominates the entries of b_j and b_0 for suitable $C, r > 0$. The solution to the auxiliary equation then becomes, for $n = 2$,

$$v(x) = \frac{nmCt}{r - (x_1 + \cdots + x_{n-1}) + \sqrt{(r - x_1 - \cdots - x_{n-1})^2 - 2nmCrx_n}} U_m,$$

and for $n \geq 3$ we find

$$v(x) = \frac{1}{nm}\left[(r - x_1 - \cdots - x_{n-1}) - \sqrt{(r - x_1 - \cdots - x_{n-1})^2 - 2nmCrx_n}\right] U_m.$$

These are real analytic near 0, so we are done.

The question then becomes, how did we get these formulas? By symmetry, the first $n-1$ variables must play the same role, and the components of v must be the same, so

$$v_1 = \cdots = v_m = w,$$

and

$$w = w(x_1 + \cdots + x_{n-1}, x_n) = w(\xi, x_n).$$

This gives a scalar equation

$$\begin{aligned} \partial_t \omega &= \frac{Cr}{r - \xi - mw}[(n-1)m\partial_\xi w + 1], \\ w(\xi, 0) &= 0. \end{aligned}$$

As an exercise, we can find $\xi(t), \eta(t)$ solving

$$\begin{aligned} \xi'(t) &= \frac{-(n-1)mCr}{r - \xi(t) - m\eta(t)}, \\ \eta'(t) &= -\frac{Cr}{r - \xi(t) - m\eta(t)}, \end{aligned}$$

where we initialize $\xi(0) = \xi_0$, $\eta(0) = 0$. Then $w(\xi(t), t) = \eta(t)$ defines w that solves the PDE

$$\xi'(t)(\partial_\xi w)(\xi(t), t) + (\partial_t w)(\xi(t), t) = \eta'(t),$$

at $(\xi(t), t)$. We can then solve for $\xi(t)$ and $\eta(t)$, and invert $\xi_0 \mapsto \xi(t)$ for t small enough.

2.5 Limitations

There are many limits of CK.

(i) Consider the heat equation:

$$\partial_t u = \partial_x^2 u,$$
$$u(0, x) = \frac{1}{1 + x^2},$$

where the surface is $\Sigma = \{t = 0\}$. The non-characteristic condition means that $a_{(2,0)} \neq 0$, but this is not true in the above. This is due to the fact we cannot solve the PDE for negative times.

(ii) If we have a diffusion equation of the form

$$\partial_t^k u = \sum_{|\alpha|=l} a_\alpha(\cdots)\partial_x^\alpha u,$$

then the necessary condition is $k \geq l$.

(iii) Analyticity is not good for propagation phenomena, or for shocks or singularities.

Finally it is not good for regularisation. Consider for example $\Delta u = 0$, for $u \in C^2$. Then this means $u \in C^\infty$, which we do not get from CK.

The good aspect of the Cauchy-Kovalevskaya is that it leads to classification.

2.6 Classification

Definition 2.6. Given a linear differential operator

$$p_u = \sum_{|\alpha| \leq k} a_\alpha(x)\partial_x^\alpha u,$$

the *principal symbol* at x is

$$\sigma_p(x, \xi) = \sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha,$$

for $\xi \in \mathbb{R}^n$, and the *characteristic cone* at x is

$$\mathcal{C}_x = \{\xi \in \mathbb{R} \mid \sigma_p(x, \xi) = 0\}.$$

Non-charactericity means

$$\sigma_p(x, N(x)) \neq 0$$

at $x \in \Sigma$.

We can check this definition for the:

- Laplace equation.
- Wave equation.
- Transport equation
$$\sum_{j=1}^{n} c_j(x)\partial_{x_j} u = 0.$$
- Heat equation.
- Schrödinger equation.

These all have problems for CK:

- For the Laplace equation, even though there is no characteristic surface, this means the Cauchy problem is ill-posed.
- The wave equation, although solvable by CK, wishes to be solved in a local, non-analytic way.
- The heat equation is ill-suited, as it is irreversible.
- The Schrödinger equation is a hybrid; it is a dispersive relation.

3 Functional Toolbox

Our motivation is to get access to Banach and Hilbertian geometric tools.

Moreover, some spaces are equal or close to physical quantities (e.g. energy) that is minimized or propogated in a PDE.

3.1 Hölder Spaces

Definition 3.1. Given $U \subseteq \mathbb{R}^n$ open, and $k \in \mathbb{N} \cup \{\infty\}$ then

- $C^k(U)$ is the set of functions $u : U \to \mathbb{R}$ which are k-times differentiable, with $\partial_x^\alpha u$ continuous on U, for all $|\alpha| \leq k$.
- $C_b^k(U)$ is the set of function $u : U \to K$ in $C^k(U)$, and so that $\partial_x^\alpha u$ is bounded on U for all $|\alpha| \leq k$.
- $C^k(\bar{U}) \subseteq C_b^k(U) \subseteq C^k(U)$ is the set of functions in $C_b^k(U)$, so that $\partial_x^\alpha u$ is uniformly continuous on U, for all $|\alpha| \leq k$.

Remark.

1. The ∇ notation is slighly incosistent on $C^k(\overline{\mathbb{R}^n})$ and $C^k(\mathbb{R}^n)$, even though these are the same.
2. $C^k(U)$ is a Fréchet space.

Both $C_b^k(U)$ and $C^k(\overline{U})$ are normed (Banach) vector spaces with

$$\|u\|_{C_b^k(U)} = \|u\|_{C^k(\overline{U})} = \sup_{x \in U} \sup_{|\alpha| \leq k} |\partial_x^\alpha u(x)|.$$

We can use Hölder space to interpolate between the C^k.

Definition 3.2. Let $U \subseteq \mathbb{R}^n$ be open, $\gamma \in [0, 1]$, and $u : U \to \mathbb{R}$. The u is *Hölder continuous* with respect to index γ if there exists $C > 0$ such that for all $x, y \in U$,

$$|u(x) - u(y)| \leq C|x - y|^\gamma.$$

The space of such functions which are bounded is the 0'th order Hölder space with index γ, $C^{0,\gamma}(\overline{U})$.

For $k \in \mathbb{N}$, $C^{k,\gamma}(\overline{U})$ is defined as

$$C^{k,\gamma}(\overline{U}) = \{u \in C^k(\overline{U}) \mid \partial_x^\alpha u \in C^{0,\gamma}(\overline{U}) \text{ for all } |\alpha| \leq k\}.$$

These spaces are Banach spaces, with the norm

$$\|u\|_{C^{k,\gamma}(\overline{U})} = \|u\|_{C^k(\overline{U})} + \sum_{|\alpha|\leq k} [\partial_x^\alpha u]_{C^{0,\gamma}(\overline{U})},$$

where

$$[u]_{C^{0,\gamma}(\overline{U})} = \sup_{x\neq y\in U} \frac{|u(x)-u(y)|}{|x-y|^\gamma}.$$

We have $\gamma \leq 1$, as if $\gamma > 1$, then we immediately get differentiability with derivative 0.

3.2 Lebesgue Spaces

Definition 3.3. Let $U \subseteq \mathbb{R}^n$ be open, and $p \in [1, +\infty]$. Then the global *Lebesgue space* $L^p()$ is the set of classes of equivalence (for almost-everywhere equality) of measurable functions u such that $|u|^p$ is integrable:

$$\int_U |u|^p < +\infty.$$

The *local Lebesgue space* $L^p_{\text{loc}}(U)$ is the classes of equivalence of measurable functions u such that $u \in L^p(V)$ for any $V \subseteq\subseteq U$.

Here, $V \subseteq\subseteq U$ means that V is an open set such that $\overline{V} = U$, and $\overline{V}$ is compact.

There are three key results:

1. Monotone convergence theorem: if $f_n \geq 0$, and $f_n \uparrow$, then

$$\sup \int f_n = \sup f_n.$$

2. Fatou's lemma: if $f_n \geq 0$ measurable, then

$$\int \liminf f_n \leq \liminf \int f_n.$$

3. Dominated convergence theorem: if $f_n \to f$ a.e. and $|f_n| \leq g$ integrable, then

$$\int f_n \to \int f.$$

4. $L^p(U)$ is a Banach space, where for $p < \infty$,

$$\|u\|_{L^p(U)} = \left(\int_u |u(x)|^p \, \mathrm{d}x\right)^p,$$

and for $p = \infty$,

$$\|u\|_{L^\infty(U)} = \text{esssup}_U |u|.$$

To prove this, we can show a Cauchy sequence in L^p has a subsequence converging almost-everywhere, and we can use Fatou's lemma to deduce L^p convergence.

For $p = 2$, we have a Hilbert space, with

$$\langle u, v \rangle = \int_U uv.$$

3.3 Weak (generalized) Derivatives

We say that v is an α-*weak derivative* of u, denoted $D^\alpha u = v$, if for all $\phi \in C_c^\infty(U)$,

$$\int_U u(\partial_x^\alpha \phi) = (-1)^\alpha \int_U v\phi.$$

Remark.

1. If $u \in C^k(U)$, $\partial_x^\alpha u$ for $|\alpha| \leq k$ is also an α-weak derivative.
2. The α-weak derivative of u is unique, as

$$\int_U (v_1 - v_2)\phi = 0 \text{ for all } \phi \in C_c^\infty(U) \implies v_1 = v_2 \text{ in } L^1_{\text{loc}}(U).$$

Definition 3.4. If $U \subseteq \mathbb{R}^n$ is open, $k \in \mathbb{N}$ and $p \in [1, +\infty]$, we say that $u \in W^{k,p}(U)$ (*Sobolev space*) if $u \in L^p(U)$ and for all $|\alpha| \leq k$, $D^\alpha u \in L^p(U)$.

Moreover $W^{k,p}_{\text{loc}}(U)$ are $u \in L^1_{\text{loc}}(U)$, which are in $W^{k,p}(V)$ for all $V \subseteq\subseteq U$.

This is a Banach space, with norm

$$\|u\|_{W^{k,p}(U)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|^p_{L^p(U)} \right)^{1/p} \tag{1}$$

for $p < \infty$, and

$$\|u\|_{W^{k,\infty}(U)} = \sup_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(U)}.$$

For $p = 2$, we write

$$W^{k,2}(U) = H^k(U),$$

as a Hilbert space.

We can also define $W^{k,p}_0(U)$ as the closure of $C_c^\infty(U)$ for the $W^{k,p}(U)$ norm, and analogously $H^k_0(U) = W^{k,2}_0(U)$.

Example 3.1.

Consider $u(x) = |x|^{-s}$ in $B(0,1) \subseteq \mathbb{R}^n$, for some $s < n$. Then $u \in L^1(B(0,1))$

For $s < n-1$, $u \in W^{1,1}(B(0,1))$, and we can find $D_{x_1}u$. Also for $s < \frac{n-p}{p}$, this is in $W^{2,p}(B(0,1))$.

3.4 Approximations in Sobolev Spaces

In this section we wish to approximate Sobolev functions by functions with classical derivatives.

Definition 3.5. A *standard mollifier* is a family (φ_ε), such that $\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^n)$, $\text{supp}\varphi_\varepsilon \subseteq B(0,\varepsilon)$, $\varphi_\varepsilon \geq 0$ and

$$\int_{\mathbb{R}^n} \varphi_\varepsilon(x)\,\mathrm{d}x = 1.$$

Proposition 3.1. *Let $U \subseteq \mathbb{R}^n$ be open, and $p \in [1,+\infty]$. Then,*

(i) *There exists a family of standard mollifiers.*

(ii) *For $u \in L^1_{\text{loc}}(U)$, define $u_\varepsilon = \varphi_\varepsilon * u$. This mollification satisfies $u_\varepsilon \in C_c^\infty(U_\varepsilon)$, where $U_\varepsilon = \{x \in U \mid d(x,\partial U) > \varepsilon\}$.*

Moreover $u_\varepsilon \to u$ almost-everywhere.

(iii) *If $u \in C^k(U)$, then $\partial_x^\alpha u_\varepsilon \to \partial_x^\alpha u$ uniformly on compact subsets of U, for any $|\alpha| \leq k$.*

(iv) *If $u \in W^{k,p}(U)$, then $u_\varepsilon \to u$ in $W^{k,p}_{\text{loc}}(U)$, i.e. convergence in $W^{k,p}(V)$ for any $V \subseteq\subseteq U$.*

(v) *If U is bounded and $u \in W^{k,p}(U)$, then there exists (u_j), with $u_j \in C^\infty(U) \cap W^{k,p}(U)$ so that $u_j \to u$ in $W^{k,p}(U)$ (global approximation not uniform near the boundary).*

(vi) *If U is bounded and ∂U is locally the graph of a Lipschitz function, then for $u \in W^{k,p}(U)$, there exists (u_j) where $u_j \in C^\infty(\bar{U})$ (which implies $u_j \in W^{k,p}(\Omega)$) so that $u_j \to u$ in $W^{k,p}(U)$ (global approximation in $W^{k,p}(U)$ by smooth functions uniformly regular at ∂U).*

Proof:

(i) We can use, for instance

$$\varphi(x) = \begin{cases} C\exp\left(-\frac{1}{1-|x|^2}\right) & |x| \leq 1, \\ 0 & |x| > 1. \end{cases}$$

We choose $C > 0$ such that $\int \varphi = 1$, and then scale:

$$\varphi_\varepsilon(x) = \varepsilon^{-k}\varphi\left(\frac{x}{\varepsilon}\right).$$

(ii) Importantly, translations are continuous in L^1, i.e.

$$\tau_\varepsilon u = u(\cdot + \varepsilon) \xrightarrow{L^1} u.$$

We can check this by density of simple functions. Hence $u_\varepsilon \to u$ in $L^1_{\text{loc}}(U)$, for $\varepsilon < \varepsilon_0$. We can also use Lebesgue differentiation theorem.

(iii) This is a standard calculation, by integration by parts:

$$\partial_x^\alpha u_\varepsilon = \int \partial_x^\alpha u(y)\varphi_\varepsilon(x-y)\,\mathrm{d}y = \cdots.$$

(iv) We repeat (ii) on each weak derivative.

(v) We use a covering argument. We decompose U into $U_l = \{x \in U \mid d(x, \partial U) > 1/l\}$, and define

$$V_l = U_{l+3} \setminus \overline{U_{l+1}}.$$

Then,

$$\bigcup_{l\geq 1} V_l \cup (V_0) = U,$$

where $V_0 = U \setminus \bigcup_{l\geq 1} V_c$.

We now take partitions of unity (ξ_l), subordinate to (V_l) i.e. $\xi_l \in C_c^\infty(V_l)$, and

$$\sum_{l\geq 0} \xi_l = 1.$$

Then for each $l > 0$, we pick $\varepsilon_l 0$ so that

$$u^l = \varphi_{\varepsilon_l} * (\xi_l u)$$

is supported in

$$W_l = U_{l+4} \setminus \overline{U_l},$$

and is $2^{-l-1} \cdot \delta$ close to $\xi_l u$ in $W^{k,p}(\cdots)$. Then we define

$$\tilde{u} = \sum_{l=0} u^l,$$

which satisfies

$$\begin{aligned}\|u - \tilde{u}\|_{W^{k,p}(U)} &= \left\| \sum_{l \geq 0} \xi_l u - \sum_{l \geq 0} \varphi_{\varepsilon_l} * (\xi_l u) \right\|_{W^{k,p}(U)} \\ &\leq \sum_{l \geq 0} 2^{-l-1} \delta \leq \delta.\end{aligned}$$

(vi) For each $x \in \partial U$, there is $r_x > 0$ such that $\partial U \cap B(x, r_0)$ is the graph of a Lipschitz function. By compactness of ∂U (since U is bounded), we get a finite cover

$$\partial U \subseteq \bigcup_{l=1}^{L} (\partial U \cap B(x^l, r_l)).$$

Hence let us write

$$U = V_0 \cup \bigcup_{l=1}^{L} V_l,$$

where $V_l = U \cap B(x^l, r_l)$. Then on each V_l in $B(x^l, r_l)$ we write the boundary as a graph of a Lipschitz function. Without loss of generality, say $x_n = \Gamma(x_1, \ldots, x_{n-1})$ defines $\partial U \cap B(x^l, r_l)$, where Γ is Lipschitz.

Then, there exists $\lambda \in \mathbb{R}_+$ such that for all $x \in \partial U \cap B(x^l, r_l)$, and for all $\varepsilon > 0$ small enough,

$$B(x + \lambda \varepsilon e_n, \varepsilon) \subseteq U.$$

Then,

$$\varphi_{\hat{\varepsilon}} * (\tau_{\lambda \varepsilon e_n} u)$$

is well-defined for all $\hat{\varepsilon} < \varepsilon$ small, and for $\varepsilon \ll 1$, $\hat{\varepsilon} \ll \varepsilon$, this constructs u_l such that

$$\|u^l - u\|_{W^{k,p}(U \cap B(x^l, r_l))} \leq \frac{\delta}{k+1}.$$

We can then recontruct, by taking (ξ_l) a partition of unity subordinate to (V_l), and letting

$$v = \sum_{l=0}^{L} \xi_l u^l.$$

This satisfies $\|v - u\|_{W^{k,p}(U)} \leq \delta$, again by triangle inequality.

3.5 Extensions and Traces

Theorem 3.1 (Extensions of $W^{1,p}$ functions)**.** *Given $U \subseteq \mathbb{R}^n$ open and bounded with C^1 boundary, and $V \subseteq \mathbb{R}^n$ open and bounded such that $U \subseteq\subseteq V$, if $p \in [+1, \infty]$ then there exists an extension operator, which is linear and bounded*

$$E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$$

such that $\operatorname{supp} E(u) \subseteq V$ *for $u \in W^{1,p}(U)$, and $E(u)|_U = u$ almost everywhere.*

Remark. Bounded means there is $C = C(U, V, p) > 0$ such that

$$\|E(u)\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}.$$

Proof: Since ∂U is C^1, it is locally the graph of a C^1 function. So for all $x \in \partial U$, there is $B(x, r)$ so that $\partial U \cap B(x, r)$ can be written as

$$x_n = \Gamma(x_1, \dots, x_{n-1}),$$

for some $\Gamma \in C^1$. Then we can write

$$U \cap B(x, r) = \{x' \in B(x, r) \mid x'_n > \Gamma(x'_1, \dots, x'_{n-1})\}.$$

Now we wish to flatten out the boundary. Take $\Phi : B(x, r) \to B(y, r')$, where $y = (\tilde{x}, 0)$, and $\tilde{x} = (x_1, \dots, x_{n-1})$. We can define $\Phi(x') = y'$ where $y'_n = x'_n - \Gamma(\tilde{x}')$, and $\tilde{y}' = \tilde{x}'$.

Then Φ maps $\partial U \cap B(x, r)$ into $\{y'_n = 0\}$, and $U \cap B(x, r)$ into $\{y'_n > 0\}$. Also Φ is C^1 and has $\det D\Phi = 1$.

So we may take $\Psi = \Phi^{-1}$ on $B(y, r'')$, so $\Psi(y') = x'$.

Take $u \in C^\infty(\overline{U}) \cap W^{1,p}(U)$. Then locally around $x \in \partial U$, $v = u \circ \Psi$ is defined on

$$B_+ = B(y, r'') \cap \{y'_n > 0\}.$$

Define (higher-order) symmetrization $\bar{v}(y')$ on $B(y, r'')$ by

$$\bar{v} = v,$$

on B_+, and

$$\bar{v}(y') = -3v(\tilde{y}', -y_n') + 4v(\tilde{y}', -y_n'/2).$$

This is continuous at $y_n' = 0$, and $\partial\bar{v}$ is continuous at $y_n' = 0$. This can be defined on $B_- = B(y, r'') \cap \{y_n' < 0\}$.

We can check that $\bar{v} \in C^1$ on $B(y, r'')$, and

$$\|\bar{v}\|_{W^{1,p}(B(y,r''))} \leq C\|v\|_{W^{1,p}(B_+)}.$$

This is done by relating $\partial\bar{v}$ to ∂v. Define $\bar{u} = \bar{v} \circ \Phi$ on $B(x, r''')$, for $r''' > 0$.

Since $\Phi \in C^1$, $\bar{u}$ is C^1 on $B(x, r''')$ and

$$\|\bar{u}\|_{W^{1,p}(B(x,r'''))} \leq C'\|u\|_{W^{1,p}(U \cap B(x,r'''))}.$$

Now we want to reconstruct the entire function. Take a cover $V_l = B(x^l, r_l)$ for $x^l \in \partial U$, for $l = 1, \ldots, L$, and say

$$U \setminus \left(\bigcup_{l=1}^{L} V_l\right) \subseteq V_0 \subseteq\subseteq U.$$

Take (ξ_l) a partition of unity subordinate to (V_l). Let $u^0 = u$ on V_0 and u^l as the symmetrisation on $l = 1, \ldots, L$. Then let

$$Eu = \sum_{l=0}^{L} \xi_l u^l.$$

Then $\operatorname{supp} Eu \subseteq V = \bigcup V_l \subseteq U$, and

$$\|Eu\|_{W^{1,p}} \leq \sum_{l=0}^{L} \|u^l\|_{W^{1,p}(V_l)} = C''\|u\|_{W^{1,p}(U)}.$$

We can now get rid of $u \in C^\infty(\overline{U})$, by using a density argument.

Let $u_j \in C^\infty(\overline{U})$ such that $u_j \to u$ in $W^{1,p}(U)$. E is linear and bounded, so (Eu_j) is Cauchy. So $Eu_j \to Eu$ in $W^{1,p}(V)$. Moreover the limit does not depend on the approximation as if $u_j^1, u_j^2 \to u$, then

$$\|E(u_j^1 - u_j^2)\|_{W^{1,p}(V)} \to 0.$$

Remark. If we want to extend in $W^{k,p}(U)$, in our proof structure we need $\partial U \in C^k$, and a higher order symmetrisation,

$$\bar{v}(y') = \sum_{m=1}^{k} c_m v(\tilde{y}', -y'_n/m),$$

and find suitable coefficients c_m.

Theorem 3.2. *Let $U \subseteq \mathbb{R}^n$ be open and bounded and with ∂U C^1. Let $p \in [1, +\infty]$. Then there exists a linear bounded trace operator $T : W^{1,p}(U) \to L^p(\partial U)$ so that $T(u) = u|_{\partial U}$ for $u \in W^{1,p}(U) \in C^\infty(\overline{U})$.*

Remark.

1. As $u \in C^\infty(\overline{U})$, we can extend u on ∂U by uniform continuity.
2. T is bounded means that there is $C = C(U, p)$ so that

$$\|Tu\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)}.$$

3. If $u \in W^{k,p}(U)$, we can define the trace of $Du, D^2u, \ldots, D^{k-1}u$.
4. If $u \in W_0^{1,p}(U)$, i.e. within

$$\overline{C_c^\infty(U)}^{W^{1,p}(U)},$$

then $Tu = 0$ on ∂U almost everywhere, since if $u_j \to u$ for $u_j \in C_c^\infty(U)$ then $Tu_j = 0$, and T is bounded.

In fact, the converse is true: if $Tu = 0$ for $u \in W^{1,p}(U)$, then $u \in W_0^{1,p}(U)$.

5. This theorem is optimal for $p = 1$, but for $p \in (1, +\infty)$ the loss of derivatives is only $1/p$. In codimension m, the loss is m/p.

Proof: We use the same structure again. To construct T, we use localization, flattening, deconstruction, and then relax $C^\infty(\overline{U})$.

In the flat boundary with the whole space, to show it is L^p we have

$$\begin{aligned}
\int_{\mathbb{R}^{n-1}} |v(\tilde{y}, 0)|^p \, \mathrm{d}\tilde{y} &= \int_{\mathbb{R}^{n-1}} \int_0^\infty \partial y_n |v(\tilde{y}, y_n)|^p \, \mathrm{d}\tilde{y} \, \mathrm{d}y_n \\
&= \int_{\mathbb{R}^n_+} p|v(y)|^{p-1} (\partial y_n v)(y) \operatorname{sgn}(v) \, \mathrm{d}y.
\end{aligned}$$

3.6 Sobolev Inequalities

Our goal is to integrate weak integrated regularity with classical regularity.

We start with $W^{1,p}$, which $W^{k,p}$ is found inside. For $p > n$ we find Hölder functions.

Lemma 3.1. *Let $n \geq 2$, and $f_1, \ldots, f_n \in L^{n-1}(\mathbb{R}^{n-1})$. Then let*

$$f(x) = \prod_{i=1}^{n} f_i(\tilde{x}_i),$$

where $\tilde{x}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. This is integrable with

$$\|f\|_{L^1(\mathbb{R}^n)} \leq \prod_{i=1}^{n} \|f_i\|_{L^{n-1}(\mathbb{R}^{n-1})}.$$

Proof: Our base case is $n = 2$. We use Fubini's; if $f(x) = f_1(x_2)f_2(x_1)$, then

$$\|f\|_{L^1(\mathbb{R}^2)} = \|f_1\|_{L^1(\mathbb{R})}\|f_2\|_{L^1(\mathbb{R})}.$$

Now we propagate from n to $n+1$. Let $f(x) = f_{n+1}(\tilde{x}_{n_1})F(x)$, where $F(x) = f_1(\tilde{x}_1) \cdots f_n(\tilde{x}_n)$.

We apply step n with x_{n+1} frozen, so

$$\int_{\mathbb{R}^n} |f(\cdot, x_{n+1})| \, \mathrm{d}x_1 \cdots \mathrm{d}x_n \leq \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \|F(\cdot, x_{n+1})\|_{L^{n/(n-1)}}$$

by Hölder. Then note

$$F(\cdot, x_{n+1})^{n/(n-1)} = \prod_{i=1}^{n} \tilde{f}_i,$$

where $\tilde{f}_i(\cdot) = f_i(\cdot, x_{n+1})^{n/(n-1)}$. Then this has norm at most

$$\begin{aligned}
&\|f_{n+1}\|_{L^n(\mathbb{R}^n)} \left(\prod_{i=1}^{n} \|\tilde{f}_i\|_{L^{n-1}(\mathbb{R}^{n-1})} \right)^{(n-1)/n} \\
&\leq \|f_{n_1}\|_{L^n} \prod_{i=1}^{n} \|f_i(\cdot, x_{n+1})\|_{L^n(\mathbb{R}^n)}.
\end{aligned}$$

Now we integrate this with respect to x_{n+1}, so

$$\int_{\mathbb{R}} LHS \,\mathrm{d}x_{n+1} \leq \|f_{n+1}\|_{L^n} \left(\prod_{i=1}^{n} \int_{x_{n+1}} \|f_n(\cdot, x_{n+1})\|_{L^n}^n\right)^{1/n}$$
$$\leq \prod_{i=1}^{n+1} \|f_i\|_{L^n(\mathbb{R}^n)},$$

again by Hölder.

Theorem 3.3 (Gagliardo-Norenberg-Sobolev). *Globally in $\mathbb{R}^n$, given $p \in [1, n)$, there is $C = C(n, p) > 0$ so that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)},$$

for all $u \in W^{1,p}(\mathbb{R}^n)$, where $\frac{1}{p} = \frac{1}{n} + \frac{1}{p^*}$.

Locally away from a boundary, given $U \subseteq \mathbb{R}^n$ open and bounded, then there is $C = C(U, n, p) > 0$ such that

$$\|u\|_{L^{p^*}(U)} \leq C\|Du\|_{L^p(U)},$$

for all $u \in W_0^{1,p}(U)$.

We also have a local version up to the boundary: let $U \subseteq \mathbb{R}^n$ be open and bounded with C^1 boundary. Then there is $C = C(U, n, p) > 0$ such that

$$\|u\|_{L^{p^*}(U)} \leq C\|u\|_{W^{1,p}(U)},$$

for all $u \in W^{1,p}(U)$.

Remark.

1. Note $p^* > p$, so this shows higher integrability.
2. Du is not changed by adding a constant to u, so $u \in L^p$ is needed in $\mathbb{R}^n$ to get $u \to 0$.

Proof: The idea is to write u as an integral of partial derivatives, then use the previous lemma.

First we consider $p = 1$, and $u \in C_c^\infty(\mathbb{R}^n)$, in the global version. For all i,

$$u(\bar{x}) = \int_{-\infty}^{\bar{x}_i} \partial_{x_i} u(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) \, \mathrm{d}y_i.$$

Then $|u(\bar{x})| \le g_i(\tilde{x}_i)$, where

$$g_i(\tilde{x}_i) = \int_{-\infty}^{+\infty} |\partial_{x_i} u(\tilde{x}_i, y_i)| \, \mathrm{d}y_i.$$

Let $f(x) = |u(x)|^{n/(n-1)} \le \prod f_i(\tilde{x}_i)$, where

$$f_i(\tilde{x}_i) = g_i(\tilde{x}_i)^{1/(n-1)}.$$

Then we find

$$\begin{aligned} \|f\|_{L^1} = \|u\|_{L^{n/(n-1)}}^{n/(n-1)} &\le \prod_{i=1}^n \|f_i\|_{L^{n-1}} \\ &= \prod_{i=1}^n \|g_i\|_{L^1}^{1/(n-1)} = \prod_{i=1}^n \|\partial x_i u\|_{L^1}^{1/(n-1)} \\ &\le \|\nabla_U\|_{L^1}^{n/(n-1)}. \end{aligned}$$

Then note that $p^* = n/(n-1)$.

Now we wish to relax to $p \in (1, n)$. Apply the $p = 1$ step to $v = |u|^\gamma$, where

$$\gamma = \frac{p(n-1)}{n-p}.$$

Then $\partial_{x_i} v = \gamma \operatorname{sgn}(u) |u|^{1-\gamma} \partial_{x_i} u$, and

$$\begin{aligned} \|v\|_{L^{n/(n-1)}} \le \|u\|_{L^{pn/(n-p)}}^\gamma &= \prod_{i=1}^n \|\gamma |u|^{\gamma-1} \partial_{x_i} u\|_{L^1}^{1/n} \\ &\le \gamma \prod_{i=1}^n \left(\|u\|_{L^{pn/(n-p)}}^{n(p-1)/(n-p)} \|\nabla u\|_{L^p} \right)^{1/n}. \end{aligned}$$

Collecting everything,

$$\|u\|_{L^{pn/(n-p)}}^{\gamma - n(p-1)/(n-p)} \le \gamma \|\nabla u\|_{L^p}.$$

The top becomes 1, and the bottom becomes p^*. Now $u \in C_c^\infty(\mathbb{R}^n)$ is reduced by using density of $C_c^\infty(\mathbb{R}^n)$ in $W^{1,p}(\mathbb{R}^n)$, by localizing to a small enough ball and then mollifying.

Now consider the local version in $W_0^{1,p}(U)$. Apply the fact that $W_0^{1,p}(U) = \overline{C_c^\infty(U)}^{W^{1,p}}$, and the previous calculation for $u \in C_c^\infty(U)$.

For the local version, we know there exists an extension operator $E : W^{1,p}(U) \to W^{1,p}(V)$. Then $Eu \in W^{1,p}(\mathbb{R}^n)$. Applying the global inequality to Eu,

$$\begin{aligned}\|u\|_{L^{p^*}(U)} = \|Eu\|_{L^{p^*}(U)} &\leq \|Eu\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \\ &= C\|Eu\|_{W^{1,p}(V)} \leq CC'\|u\|_{W^{1,p}(U)}.\end{aligned}$$

Theorem 3.4 (Morrey's Inequality)**.** *Let $o \in (n, +\infty)$. Then there is $C = C(n,p) > 0$ so that*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for $u \in C_c^\infty(\mathbb{R}^n)$, $\gamma \in (0,1)$.

In other words, any $u \in W^{1,p}(\mathbb{R}^n)$ has a $C^{0,\gamma}$ representative.

Locally, if $U \subseteq \mathbb{R}^n$ is open and bounded, and ∂U is C^1, then there is $C = C(U,n,p) > 0$ such that

$$\|u\|_{C^{0,\gamma}(U)} \leq C\|u\|_{W^{1,p}(U)}.$$

Proof: The main idea is to show that u is locally pointwise close to its averages.

If $\bar{x} \in \mathbb{R}^n$, and

$$Q_r(\bar{x}) = \{|x_i - \bar{x}_i| \leq r/2 \text{ for all } i\},$$

then we can define

$$\bar{u}_{\bar{x},r} = \frac{1}{|Q_r(\bar{x})|}\int_{Q_r(\bar{x})} u.$$

We will show that $|u(\bar{x}) - \bar{u}_{\bar{x},r}| \leq r^{\cdot}\|u\|_{W^{1,p}}$.

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