

III Symmetries, Particles and Fields

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Based on Lectures by Prof. Matthew Wingate

October 24, 2024

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1 Introduction to Symmetries

Recall Newton's second law:

$$m \frac{d^2 \mathbf{x}}{dt^2} = \mathbf{F}(\mathbf{x}).$$

This simplifies if we know F is rotationally symmetric, i.e. $\mathbf{F}(\mathbf{x}) = F(r)\hat{\mathbf{r}}$. Then $\mathbf{L} = \mathbf{x} \times \mathbf{p}$ is conserved, and trajectories lie in planes containing the origin.

Now consider Lagrangian mechanics, with Lagrangian $L(q_i, \dot{q}_i, t)$. The principle of least action says

$$S = \int_{t_1}^{t_2} dt L(q_i(t), \dot{q}_i(t), t)$$

is minimized by classical trajectories. Hence Euler-Lagrange gives

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0.$$

Nöether's theorem says that invariance of L under some coordinate transform corresponds to an associated conserved quantity.

Example 1.1.

Consider a particle in three dimension, with a potential:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z),$$

which is independent of t , hence invariant under $t \mapsto t + \delta t$. This implies that the Hamiltonian $H = T + U$ is conserved. If we transform into canonical momenta $p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i$, then

$$H(x_i, p_i, t) = \sum \dot{x}_i p_i - L = \sum \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} - L$$

is invariant by Euler-Lagrange:

$$\frac{dH}{dt} = \sum \ddot{x}_i \frac{\partial L}{\partial \dot{x}_i} - \sum x_i \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_i} \right) - \dot{x}_i \frac{\partial L}{\partial x_i} - \ddot{x}_i \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial t} = 0.$$

If L is invariant under $x \mapsto x + \delta x$, then

$$\frac{\partial L}{\partial x} = 0 \implies \frac{\partial L}{\partial \dot{x}} = p_x = \text{constant}.$$

If L is invariant under rotations about the z -axis, then the z -component of angular momentum, $xp_y - yp_x$, is constant. The best way to see is transform L into cylindrical coordinates:

$$L = \frac{1}{2}m(\dot{\rho}^2 - \rho^2\dot{\theta} + \dot{z}^2) - U(\rho, z).$$

So the invariance under rotations means

$$\frac{\partial L}{\partial \theta} = 0 \implies \frac{\partial L}{\partial \dot{\theta}} = 0 = m\rho^2\dot{\theta} = xp_y - yp_x$$

is constant.

1.1 Symmetries in Quantum Mechanics

Given a system whose states are element of a Hilbert space \mathcal{H} , a symmetry means there exists some invertible operator U acting on \mathcal{H} which preserves inner products, up to an overall phase $e^{i\phi}$.

Definition 1.1. Let $|\psi\rangle, |\phi\rangle$ be any normalized vectors in \mathcal{H} . Denote $|U\psi\rangle = U|\psi\rangle$, and $|U\phi\rangle = U|\phi\rangle$.

U is a *symmetry transformation* if

$$|\langle U\phi|U\psi\rangle| = |\langle\phi|\psi\rangle|.$$

Proposition 1.1 (Wigner's Theorem). *Symmetry transformation operators are either linear and unitary, or antilinear and antiunitary.*

Antilinear and antiunitary means

$$U(a|\psi\rangle + \beta|\phi\rangle) = a^*U|\psi\rangle + b^*U|\phi\rangle,$$

$$\langle U\phi|U\psi\rangle = \langle\phi|\psi\rangle^*.$$

Suppose we have a system with a time-independent Hamiltonian. Then we can write down

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle,$$

by Schrödinger's equation with $\hbar = 1$. In the first case, note

$$\begin{aligned}\langle U\phi|U\psi(t)\rangle &= \langle\phi|\psi(t)\rangle \\ &= \langle\phi|e^{-iHt}|\psi(0)\rangle.\end{aligned}$$

We should find the same result by transforming $|\psi(0)\rangle$ before time evolution:

$$\begin{aligned}\langle U\phi|U\psi(t)\rangle &= \langle U\phi|e^{-iHt}|U\psi(0)\rangle \\ &= \langle \phi|U^\dagger e^{-iHt}U|\psi(0)\rangle.\end{aligned}$$

Equating these, we find

$$U^\dagger e^{-iHt}U = e^{-iHt} \implies [U, H] = 0.$$

Example 1.2.

If H commutes with \mathbf{p} , then H cannot depend on \mathbf{x} , as

$$[x_i, p_j] = i\delta_{ij} \neq 0$$

generally. So H is invariant under translation $\mathbf{x} \mapsto \mathbf{x} + \mathbf{a}$, and this is generated by unitary operators $\exp(i\mathbf{p} \cdot \mathbf{a})$.

If H is rotationally symmetric, then any momentum operator \mathbf{J} or \mathbf{L} commutes with H .

2 Lie Groups and Algebras

2.1 Lie Groups

Recall the definition of a group: a set together with a relation which has an identity, inverses and is associative.

Also recall a group is abelian if $g \cdot h = h \cdot g$ for all $g, h \in G$.

Definition 2.1. A *manifold* is a space which looks Euclidean, like \mathbb{R}^n , on small scales, in small neighbourhoods.

A *differentiable manifold* is one which satisfies certain smoothness conditions.

Definition 2.2. A *Lie group* consists of a differentiable manifold G along with a binary operation \cdot , such that the group axioms hold, and that \cdot and inverse are smooth operations.

2.2 Matrix Lie Groups

For example, the general linear group $\mathrm{GL}(n, \mathbb{F})$ is the group of invertible $n \times n$ matrices over a field \mathbb{F} . So,

$$\mathrm{GL}(n, \mathbb{F}) = \{M \in \mathrm{Mat}_n(\mathbb{F}) \mid \det M \neq 0\}.$$

The group operation is simply matrix multiplication.

The dimension of $\mathrm{GL}(n, \mathbb{R})$ is n^2 , as there are n^2 free parameters. For $\mathrm{GL}(n, \mathbb{C})$, we have real dimension $2n^2$, and complex dimension n^2 .

Important subgroups of $\mathrm{GL}(n, \mathbb{F})$ are:

- The special linear group

$$\mathrm{SL}(n, \mathbb{F}) = \{M \in \mathrm{GL}(n, \mathbb{F}) \mid \det M = 1\}.$$

- $\mathrm{SL}(n, \mathbb{R})$ has dimension $n^2 - 1$.
- The orthogonal group

$$\mathrm{O}(n) = \{M \in \mathrm{GL}(n, \mathbb{R}) \mid M^T M = I\}.$$

This implies $\det M = \pm 1$. We can also define

$$\mathrm{SO}(n) = \{M \in \mathrm{O}(n) \mid \det M = 1\}.$$

- Pseudo-orthogonal group. Define an $(n + m) \times (n + m)$ matrix by

$$\eta = \begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix}.$$

Then we can define

$$\mathrm{O}(n, m) = \{M \in \mathrm{GL}(n + m, \mathbb{R}) \mid M^T \eta M = \eta\}.$$

Note $M \in \mathrm{SO}(n, m) \iff \det M = 1$.

- Unitary.

$$\mathrm{U}(n) = \{M \in \mathrm{GL}(n, \mathbb{C}) \mid M^\dagger M = I\}.$$

Similarly have $\mathrm{SU}(n)$.

- Pseudounitary.

$$\mathrm{U}(n, m) = \{M \in \mathrm{GL}(n, \mathbb{C}) \mid M^\dagger \eta M = \eta\}.$$

- Symplectic group. Define a fixed, antisymmetric $2n \times 2n$ matrix, e.g.

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Then,

$$\mathrm{Sp}(2n, \mathbb{R}) = \{M \in \mathrm{GL}(2n, \mathbb{R}) \mid M^T \Omega M = \Omega\}.$$

We can show that $\det M = 1$ using the Pfaffian.

Definition 2.3. Given a $2n \times 2n$ antisymmetric matrix A , its *Pfaffian* is given by

$$\mathrm{Pf} A = \frac{1}{2^n n!} \varepsilon_{i_1 i_2 \dots i_{2n}} A_{i_1 i_2} A_{i_3 i_4} \dots A_{i_{2n-1} i_{2n}}.$$

2.3 Group Elements as Transformations

We can define actions of group elements $g \in G$ on a set X .

Definition 2.4. The *left action* of G on X is a map $L : G \times X \rightarrow X$ such that $L(e, x) = x$, and

$$L(g_2, L(g_1, x)) = L(g_2 g_1, x),$$

for all $x \in X$ and $g_1, g_2 \in G$. In more usual notation, for all $g \in G$, we can associate a map $g : X \rightarrow X$ as $g(x) = gx$.

Definition 2.5. The *right action* of G on X is defined by $g : X \rightarrow X$ such that $g(x) = xg^{-1}$, for all $x \in X$, $g \in G$. The inverse preserves under composition, so

$$g_2(g_1(x)) = xg_1^{-1}g_2^{-1} = (g_2g_1)(x).$$

Definition 2.6. The action of *conjugation* by G on X is the action defined by

$$g(x) = gxg^{-1},$$

for $g \in G$, $x \in X$.

Definition 2.7. Given a group G and a set X , an *orbit* of an element $x \in X$ is the set of elements of X in the image of G .

Example 2.1.

If the action is a left action, then the orbit of $x \in X$ is

$$Gx = \{gx \mid g \in G\}.$$

It can be shown that the set of orbits under G partition X .

In \mathbb{R}^n , orthogonal matrices $O(n)$ represent rotations and reflections, and preserve the inner product; similarly for $U(n)$.

We can parametrize $SO(2)$ as

$$SO(2) = \left\{ R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \mid \theta \in [0, 2\pi] \right\}.$$

\cos, \sin are smooth. We can show that $R(\theta_2)R(\theta_1) = R(\theta_1 + \theta_2)$.

$SO(3)$ gives rotations of vectors in \mathbb{R}^3 . The axis of rotation is given by a unit vector $\mathbf{n} \in S^2$, and we also have an angle θ .

Note that rotation by $\theta \in [-\pi, 0]$ about \mathbf{n} is equivalent to rotation by $-\theta$ about $-\mathbf{n}$, so we can confine $\theta \in [0, \pi]$.

Hence we can depict the manifold of $SO(3)$ as a ball of radius π in \mathbb{R}^3 , where antipodal points are identified: $\pi\mathbf{n} = -\pi\mathbf{n}$.

The *pseudo-orthogonal group* $SO(n, m)$ act on vectors in \mathbb{R}^{n+m} , and preserve the scalar product $v_2^T \eta v_1$ for $v_1, v_2 \in \mathbb{R}^{n+m}$.

For example,

$$SO(1, 1) = \left\{ \begin{pmatrix} \cosh \psi & \sinh \psi \\ \sinh \psi & \cosh \psi \end{pmatrix} \mid \psi \in \mathbb{R} \right\}.$$

$SO(1, 1)$ is an example of a non-compact group.

2.4 Parametrization of Lie Groups

At least in small neighbourhoods, we can assign coordinates

$$x = (x^1, \dots, x^n) \in \mathbb{R}^n,$$

such that $g(x) \in G$. Closure says that $g(y)g(x) = g(z)$, and smoothness says that the components of z are continuously differentiable functions of x and y , so

$$z^n = \phi^n(x, y).$$

We can choose the coordinates at the origin such that $g(0) = e$. Then $g(0)g(x) = g(x)$, so

$$\phi^r(x, 0) = x^r, \quad \phi^r(0, y) = y^r.$$

From inverses, for each x there exists \bar{x} such that $g(\bar{x}) = g(x)^{-1}$, hence

$$\phi^r(\bar{x}, x) = 0 = \phi^r(x, \bar{x}).$$

Finally, associativity means $g(z)[g(y)g(x)] = [g(z)g(y)]g(x)$, hence

$$\phi^r(\phi(x, y), z) = \phi^r(x, \phi(y, z)).$$

2.5 Lie Algebras

Lie groups are hard to quantify. Instead, we look at lie algebras, which are a linearization of the lie group.

A lie group is homogeneous: any neighbourhood can be mapped to any other neighbourhood. We will linearize near the identity of G .

Definition 2.8. A *Lie algebra* is a vector space V , which additionally has a vector product, the *Lie bracket* $[\cdot, \cdot] : V \times V \rightarrow V$ possessing the following properties: for $X, Y, Z \in V$,

1. antisymmetry: $[X, Y] = -[Y, X]$.
2. Jacobi identity: $[X, [Y, Z]] + [Y, [X, Z]] + [Z, [X, Y]] = 0$.
3. linearity: for $\alpha, \beta \in \mathbb{F}$, $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$.

Remark. Any vector space which has a vector product $*$ can be made into a Lie algebra with Lie bracket

$$[X, Y] = X * Y - Y * X.$$

Given a Lie algebra V , choose a basis $\{T_a\}$. The basis vectors are called the *generators* of the Lie algebra.

Write their Lie brackets as

$$[T_a, T_b] = f_{ab}^c T_c,$$

with $f_{ab}^c \in \mathbb{F}$ called the *structure constants*. The properties imply:

- antisymmetry $\implies f_{ba}^c = -f_{ab}^c$.
- Jacobi $\implies f_{ad}^e f_{bc}^d + f_{cd}^e f_{ab}^d + f_{bd}^e f_{ca}^d = 0$.

General elements of Lie algebras are linear combinations of $\{T_a\}$. So $X \in V$ can be written as $X^a T_a$, where $X^a \in \mathbb{F}$, and

$$[X, Y] = X^a Y^b f_{ab}^c T_c.$$

2.6 Lie Groups and their Lie Algebras

We start with $\mathrm{SO}(2)$, where

$$g(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

The identity is $e = I_2 = g(0)$. Near the identity, θ is small, and

$$\sin \theta = \theta - \frac{\theta^3}{3} + \dots, \quad \cos \theta = 1 - \frac{\theta^2}{2} + \dots$$

Hence,

$$g(\theta) = I_2 + \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \frac{\theta^2}{2} I_2 + \mathcal{O}(\theta^3) = e + \theta \left. \frac{dg}{d\theta} \right|_0 + \mathcal{O}(\theta^2).$$

The linear term is the “tangent” to the manifold. We have a one-dimensional tangent space at e , and we claim that this is the Lie algebra of $\mathrm{SO}(2)$, i.e.

$$L(\mathrm{SO}(2)) = T_e(\mathrm{SO}(2)) = \left\{ \begin{pmatrix} 0 & -a & a & 0 \end{pmatrix} \mid a \in \mathbb{R} \right\}.$$

For $\mathrm{SO}(n)$, we can show the dimension is $\frac{n(n-1)}{2} = d$. Choose coordinates x_1, \dots, x_d , and consider a single-parameter family of $\mathrm{SO}(n)$ elements

$$M(t) = M(x(t)) \in \mathrm{SO}(n),$$

such that $M(0) = I_n$. Then,

$$0 = \frac{d}{dt}(M^T(t)M(t)) = \frac{dM^T}{dt}M + M^T\frac{dM}{dt}.$$

Looking at $t = 0$, we find

$$\frac{dM^T}{dt} = -\frac{dM}{dt},$$

hence matrices in the tangent space are anti-symmetric. Moreover they are also traceless.

For unitary groups, we again let $M(t)$ be a curve in $\mathrm{SU}(n)$ with $M(0) = I$. For small t , write

$$M(t) = I + tX + \mathcal{O}(t^2).$$

From unitarity, $I = M^\dagger M$, so looking at the expansion,

$$I = I + t(X + X^\dagger) + \mathcal{O}(t^2),$$

hence $X^\dagger = -X$, is anti-hermitian. We also claim X is traceless for $\mathrm{SU}(n)$. Indeed, looking at $\det M$, its expansion is

$$1 = \det M = 1 + t \operatorname{Tr}(X) + \mathcal{O}(t^2).$$

2.7 Lie Algebras of a Matrix Lie Group

Consider two curves through the identity e of some Lie group, $g_1(x(t))$ and $g_2(y(t))$, with $X_1 = \dot{g}_1|_0$, $X_2 = \dot{g}_2|_0$. The product is

$$g_3(z(t)) = g_2(y(t))g_1(x(t)) \in G.$$

Then,

$$\dot{g}_3|_0 = (\dot{g}_1g_2 + g_1\dot{g}_2)|_0 = X_1 + X_2 \in T_e(G).$$

The Lie bracket arises from the group commutator.

Definition 2.9. The *group commutator* of $g_1, g_2 \in G$ is

$$[g_1, g_2] = g_1^{-1}g_2^{-1}g_1g_2 \in G.$$

Let $g_1(t), g_2(t)$ be two curves through the identity, and

$$g_i(t) = c + tX_i + t^2W_i + \mathcal{O}(t^3).$$

Then,

$$\begin{aligned} g_1(t)g_2(t) &= e + t(X_1 + X_2) + t^2(X_1X_2 + W_1 + W_2) + \mathcal{O}(t^3), \\ g_2(t)g_1(t) &= e + t(X_1 + X_2) + t^2(X_2X_1 + W_1 + W_2) + \mathcal{O}(t^3). \end{aligned}$$

Therefore,

$$h(t) = [g_2(t)g_1(t)]^{-1}g_1(t)g_2(t) = e + t^2[X_1, X_2] + \dots$$

So if $h(t) \in G$, then the tangent to $h(t)$ at e is $[X_1, X_2] \in L(G)$.

Now we can think of tangent spaces to $G \subseteq \mathbf{GL}(n, \mathbb{F})$ at a general element p , $T_p(G)$.

Let $g(t)$ be a curve in the manifold through p with $g(t_0) = p$, so

$$g(t_0 + \varepsilon) = g(t_0) + \varepsilon \dot{g}(t_0) + \mathcal{O}(\varepsilon^2).$$

Both $g(t_0)$ and $g(t_0 + \varepsilon)$ are in G , so there exists $h_p(\varepsilon) \in G$ such that

$$g(t_0 + \varepsilon) = g(t_0)h_p(\varepsilon),$$

where $h_p(0) = e$. For small ε ,

$$h_p(\varepsilon) = e + \varepsilon X_p + \mathcal{O}(\varepsilon^2)$$

for some $X_p \in L(G) = T_e(G)$. Neglecting higher order terms,

$$\begin{aligned} e + \varepsilon X_p &= h_p(\varepsilon) = g(t_0)^{-1}g(t_0 + \varepsilon) \\ &= g(t_0)^{-1}[g(t_0) + \varepsilon \dot{g}(t_0)] \\ &= e + \varepsilon g(t_0)^{-1}\dot{g}(t_0), \end{aligned}$$

where $g(t)^{-1}\dot{g}(t) = X_p \in L(G)$.

Conversely, for any $X \in L(G)$, there exists a unique curve $g(t)$ with $g^{-1}(t)\dot{g}(t) = X$, and $g(0) = g_0$. This is a consequence of the existence and uniqueness of solutions to ODEs. The solution of this ODE is

$$g(t) = g_0 \exp tX,$$

where

$$\exp tX = \sum_{k=0}^{\infty} \frac{1}{k!} (tX)^k.$$

Given an $X \in L(G)$, the curve

$$g_X(t) = \exp tX,$$

which forms an abelian subgroup of G generated by X . Note $g_x(t)$ is isomorphic to $(\mathbb{R}, +)$ if only $g_x(0) = e$, and S^1 if $g_x(t_0) = e$ for some $t_0 \neq 0$.

2.8 Lie Groups from Lie Algebras

Given a Lie algebra $L(G)$ of a Lie group G , we can define the *exponential map*

$$\exp : L(G) \rightarrow G.$$

For matrix Lie groups,

$$X \mapsto \exp X = \sum_{k=0}^{\infty} \frac{X^k}{k!}.$$

Locally, the map is bijective, but globally it may not be.

Example 2.2.

Recall the group

$$\mathbf{U}(1) = \{e^{i\theta} \mid \theta \in [0, 2\pi)\}.$$

The Lie algebra is

$$L(\mathbf{U}(1)) = \{ix \mid x \in \mathbb{R}\}.$$

The exponential is not one-to-one, since $e^{2\pi ni} = 1$.

Another example is $G = \mathbf{O}(n)$. Let $X \in L(\mathbf{O}(n)) \in \text{Skew}_n(\mathbb{R})$.

Let $M = \exp tX$, then

$$M^T = [\exp tX]^T = \exp(-tX),$$

so $MM^T = I = M^T M$, hence $M \in \mathbf{O}(n)$.

Suppose that $\text{Tr } X = 0$. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues. Then,

$$\det M = \det(\exp tX) = \exp(t(\lambda_1 + \dots + \lambda_n)) = \exp(t \text{Tr } X) = 1.$$

So $M \in \mathbf{SO}(n)$. Thus elements of $\mathbf{O}(n)$ with determinant -1 are not in the image of the \exp map. We can think of $\mathbf{O}(n)$ as a disconnected manifold.

We can show that if $A \in \text{Skew}_n(\mathbb{R})$, then $A \in L(\mathbf{SO}(n))$ or $L(\mathbf{O}(n))$.

Define $\gamma(t) = \exp tA$, a curve of matrices on some manifold. From the above,

$$(\gamma(t))^T(\gamma(t)) = I, \quad \det \gamma(t) = 1,$$

so $\gamma(t) \in \mathbf{SO}(n)$. By construction, $A = \gamma'(0)$, the tangent to the curve at the identity. So $A \in L(\mathbf{SO}(n))$, hence

$$\dim \mathbf{SO}(n) = \dim L(\mathbf{SO}(n)) = \dim \text{Skew}_n(\mathbb{R}) = \frac{n(n-1)}{2}.$$

We can also determine the group product from the Lie bracket.

Lemma 2.1 (Baker-Campbell-Hausdorff Formula). *For $X, Y \in L(G)$, we have $\exp tX \exp tY = \exp tZ$, where*

$$Z = X + Y + \frac{t}{2}[X, Y] + \frac{t^2}{12}([X, [X, Y]] + [Y, [X, Y]]) + \mathcal{O}(t^3).$$

We can show this order-by-order in t . Since $L(G)$ is closed under the Lie bracket, $Z \in L(G)$, so $\exp tZ \in G$.

3 Representations

Groups are transformations under which some things are invariant.

Representations are how group actions transform vectors in a vector space.

An example is $\mathrm{GL}(n, \mathbb{F})$, the group of invertible matrices. These form linear maps (automorphisms) on the vector space \mathbb{F}^n .

3.1 Lie Group Representations

Definition 3.1. A *representation* D of a group G is a smooth group homomorphism

$$D : G \rightarrow \mathrm{GL}(V),$$

from G to the group of automorphisms on some vector space V , called the *representation space* associated with D .

That is, for all $g \in G$, $D(g) : V \rightarrow V$ is an invertible, linear map such that for a vector $v \in V$,

$$v \mapsto D(g)v.$$

- Linearity: $D(g)(\alpha v_1 + \beta v_2) = \alpha D(g)v_1 + \beta D(g)v_2$.
- Group structure: $D(g_2 g_1) = D(g_2)D(g_1)$.
- Identity: $D(0) = \mathrm{id}_V$.
- Inverses: $D(g)^{-1} = D(g^{-1})$.

Definition 3.2. The *dimension* of a representation D is the dimension of its vector space. If V is finite-dimensional, say $\dim V = N$, then $\mathrm{GL}(V)$ is isomorphic to $\mathrm{GL}(N, \mathbb{F})$.

Recall that the kernel of a map $D : G \rightarrow \mathrm{GL}(V)$ consists of the elements of G which map to the identity id_V .

Definition 3.3. A representation D is *faithful* if $D(g) = \mathrm{id}_V$, only for $g = e$, i.e. $\ker D = \{e\}$.

Faithfulness implies that D is injective.

Example 3.1.

We look at some examples with $G = (\mathbb{R}, +)$.

- (i) For some fixed $k \in \mathbb{R}$, $D(x) = e^{kx}$ is a one-dimensional representation,

as

$$D(\alpha)D(\beta) = e^{k\alpha}e^{k\beta} = e^{k(\alpha+\beta)} = D(\alpha + \beta).$$

For $k \neq 0$, this is faithful, as $D(\alpha) = 1 \implies \alpha = 0$. For $k = 0$, $D(\alpha) = 1$ for all α , so $\ker D = G$. This is the trivial representation.

(ii) Another one-dimensional representation is $D(\alpha) = e^{ik\alpha}$. This is again not faithful.

(iii) A two-dimensional representation is

$$D(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}.$$

(iv) Let V be the space of all functions, and define

$$D(\alpha)f(x) = f(x - \alpha).$$

This is an infinite-dimensional representation, which is faithful, as $f(x) = f(x - \alpha)$ for all f implies $\alpha = 0$.

Definition 3.4. The *trivial representation* D_0 is where

$$D_0(g) = 1,$$

for all $g \in G$.

Quantities which are invariant under group transformation transform in the trivial representation.

We can also form a trivial representation for any dimension, by $D(g) = I_m$ for all g . This is reducible; it is M copies of the one-dimensional trivial representation.

Definition 3.5. If G is a matrix Lie group, then the *fundamental* or *defining representation* D_f is given by

$$D_f(g) = g,$$

for all $g \in G$.

This is clearly faithful, and if $G \subseteq \mathbf{GL}(n, \mathbb{F})$, then $\dim D_f = n$.

Let G be a matrix Lie group and consider its Lie algebra as a vector space $V = L(G)$.

Definition 3.6. The *adjoint representation* $D^{\text{adj}} = \text{Ad}$ is the map $\text{Ad} : G \rightarrow \mathbf{GL}(L(G))$ such that for all $g \in G$, $\text{Ad}_g : L(G) \rightarrow L(G)$ by

$$\text{Ad}_g X = gXg^{-1}$$

for all $X \in L(G)$. This is just action by conjugation.

We can check that this satisfies the group operations, and the Lie bracket satisfies

$$\mathrm{Ad}_g([X, Y]) = g[X, Y]g^{-1} = [gXg^{-1}, gYg^{-1}] = [\mathrm{Ad}_g X, \mathrm{Ad}_g Y].$$

3.2 Lie Algebra Representation

Definition 3.7. A *representation* d of a Lie algebra $L(G)$ is a linear map from $L(G)$ to $\mathfrak{gl}(V)$, which is the Lie algebra of $\mathrm{GL}(V)$, that preserves the Lie bracket.

That is, for each $X \in L(G)$, $d(X) : V \rightarrow V$ is a linear map such that

$$v \mapsto d(X)v,$$

for $v \in V$, such that

$$d([X, Y]) = [d(X), d(Y)].$$

The dimension of this d is then $\dim V$.

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