

# III Algebraic Geometry

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## 0 Introduction

Introductory reading by Hassett and Reid.

More commutative algebra by Atiyah and Macdonald, and Matsumura.

Standard AG texts by Hartshorne, Görtz-Wedhorn, and Ravi Vakil.

### 0.1 Recap

In undergraduate, we fix an algebraically closed field  $K$ , and define affine  $n$ -space  $\mathbb{A}^n = K^n$ , and for an ideal  $I \subseteq K[x_1, \dots, x_n]$  we define

$$Z(I) = \{(a_1, \dots, a_n) \in \mathbb{A}^n \mid f(a_1, \dots, a_n) = 0 \text{ for all } f \in I\} \subseteq \mathbb{A}^n.$$

We can define a topology on  $\mathbb{A}^n$  by taking the closed sets to be the sets of the form  $Z(I)$ .

This turns out to be no good. We will instead introduce schemes. Natural questions are; why schemes, and why not varieties? Well,

- With varieties, we always work with algebraically closed fields, to relate the algebra with the geometry. If  $K = \mathbb{R}$ , and  $I = (x^2 + y^2 + 1) \subseteq K[x, y]$ , then  $Z(I) = \emptyset$ , losing information about  $I$ .
- We may be interested in Diophantine equations, where the natural space is  $\mathbb{Z}$ .
- Even if  $K$  is algebraically closed, we lose information passing from  $I$  to  $Z(I)$ . For example if  $I = (x^2)$ ,  $Z = \{0\}$ , then  $I(Z(I)) = (x)$ .

But it is natural to consider ideals like  $(x^2)$ , for example considering  $(y - x^2, y - \alpha) \subseteq \mathbb{C}[x, y]$ . This produces two points for  $\alpha \neq 0$ , but only one point if  $\alpha = 0$ , but with some multiplicity.

### 0.2 Categorical Philosophy

Let **Set** be the category of sets. **Set** is the category with objects being all sets with morphisms between objects being maps of sets. If  $X, Y$  are sets, we write  $\text{Hom}(X, Y)$  for the set of maps between  $X$  and  $Y$ .

Note that there is a bijection  $\text{Hom}(\{*\}, X) \rightarrow X$  given by  $(f : \{*\} \rightarrow X) \mapsto f(*)$ .

We can use this philosophy to understand points on affine algebraic varieties. Note  $\mathbb{A}^0$  is a point. If  $X$  is an affine variety, then the points of  $X$  should be in one-to-one correspondence with  $\text{Hom}(\mathbb{A}^0, X)$ .

Recall morphisms of affine varieties. Denote  $A(X)$  by  $K[x_1, \dots, x_n]/I(X)$ , where  $I(X) = \{f \in K[x_1, \dots, x_n] \mid f|_X = 0\}$ .  $A(X)$  is the *coordinate ring* of  $X$ , a  $K$ -algebra.

We showed that if  $X, Y$  are affine varieties, then

$$\mathrm{Hom}(X, Y) = \mathrm{Hom}(A(Y), A(X)).$$

So,

$$\mathrm{Hom}(\mathbb{A}^0, X) = \mathrm{Hom}(K[x_1, \dots, x_n]/I(X), K).$$

Note giving a  $K$ -algebra homomorphism  $K[x_1, \dots, x_n]/I(X) \rightarrow K$  can be done by specifying the images of  $x_i$ , say  $x_i \mapsto a_i$ , such that, for any  $f \in I(X)$ ,  $f(a_1, \dots, a_n) = 0$ . So there is a one-to-one correspondence between such  $K$ -algebra homomorphisms, and points of  $X$ .

If  $K$  is algebraically closed, the maximal ideals of  $K[x_1, \dots, x_n]$  are precisely ideals of the form  $(x_1 - a_1, \dots, x_n - a_n)$  by Hilbert's Nullstellensatz. Similarly, for  $A(X)$ , the maximal ideals are  $(x_1 - a_1, \dots, x_n - a_n) \bmod I(X)$ , with  $(a_1, \dots, a_n) \in X$ .

Thus there is a bijection between points on  $X$ , and the maximal ideals of  $A(X)$ . This gives three objects,  $X$ , the homomorphisms and the maximal ideals, which are all bijective.

Now suppose  $K$  is not algebraically closed. Consider the  $K$ -algebra homomorphisms  $A(X) \rightarrow L$ , where  $L$  is an extension of  $K$ . If  $x_i \mapsto a_i$ , then  $f(a_1, \dots, a_n) = 0$  for all  $f \in I(X)$ . Thus,

$$\mathrm{Hom}_K(A(X), L) = \{(a_1, \dots, a_n) \in \mathbb{A}^n \mid f(a_1, \dots, a_n) = 0 \text{ for all } f \in I(X)\}.$$

These correspond to  $L$ -valued points.

We could also work over  $\mathbb{Z}$ . Take an ideal  $I \subseteq \mathbb{Z}[x_1, \dots, x_n]$ , and  $A = \mathbb{Z}[x_1, \dots, x_n]/I$ .

Then ring homomorphisms  $A \rightarrow \mathbb{Z}$  are in one-to-one correspondence with points  $(a_1, \dots, a_n) \in \mathbb{Z}^n$  such that  $f(a_1, \dots, a_n) = 0$  for all  $f \in I$ .

Moreover maps  $A \rightarrow \mathbb{F}_p$  give solutions mod  $p$ , and  $A \rightarrow \mathbb{Q}$  give rational solutions.

### 0.3 What we want

Given a ring  $A$ , we want to define a gadget

$$X = \mathrm{Spec} A,$$

and an  $R$ -valued point of  $X$  is a ring-homomorphism  $A \rightarrow R$ . We write the set of  $R$ -valued points as

$$X(R) = \text{Hom}(A, R).$$

Morphisms  $\text{Spec } B \rightarrow \text{Spec } A$  should be the same as ring homomorphisms  $A \rightarrow B$ . In category theory,

**Definition 0.1.** The category of affine schemes is the *opposite category* of rings.

**Reminder:** In this course, all of our rings are unital, are commutative, and ring homomorphisms  $\phi : A \rightarrow B$  satisfy  $\phi(1) = 1$ .

**Definition 0.2.** A *scheme* is a geometric object which is locally an affine scheme.

Currently this is a nonsensical definition, which we will be trying to make sense of. The motivating example is the manifold, which locally looks like an open subset of  $\mathbb{R}^n$ .

## 0.4 Introductory Definitions

**Definition 0.3.** Let  $A$  be a ring. Then,

$$\text{Spec } A = \{p \subseteq A \mid p \text{ is a prime ideal}\}.$$

In general, if we have an  $L$ -valued point of  $X = Z(I) \subseteq \mathbb{A}^n$ , we get a ring homomorphism  $\phi : A(X) \rightarrow L$ , which has image an integral subdomain of  $L$ , and so  $\text{Ker } \phi$  is prime.

**Definition 0.4.** For  $I \subseteq A$  an ideal, define

$$V(I) = \{p \in \text{Spec } A \mid p \supseteq I\}.$$

Again recall  $p$  is no longer a point, but a prime ideal.

**Proposition 0.1.** The sets  $V(I)$  form the closed sets of a topology on  $\text{Spec } A$ , the Zariski topology.

**Proof:** Need to check a handful of things.

- $V(A) = \emptyset$ , so  $\emptyset$  is closed.
- $V(0) = \text{Spec } A$ , so  $\text{Spec } A$  is closed.

- If  $\{I_j\}_{j \in J}$  is a collection of ideals, then note

$$\bigcap_{j \in J} V(I_j) = V\left(\sum_{j \in J} I_j\right).$$

- We show that  $V(I_1) \cup V(I_2) = V(I_1 \cap I_2)$ . Indeed, if  $p \supseteq I_1$  or  $p \supseteq I_2$ , then  $p \supseteq I_1 \cap I_2$ .

In the other direction, then  $p \supseteq I_1 \cap I_2$ , then  $p \supseteq I_1$  or  $p \supseteq I_2$ . This was proven in Part II. Or see Atiyah + Macdonald.

This is easy: if  $I_1, I_2 \not\subseteq p$ , then there exists  $i_1, i_2 \in I_1, I_2$  respectively that are not in  $p$ . But now  $i_1 i_2 \in I_1 \cap I_2 \subseteq p$ , so  $i_1 i_2 \in p$ .

However  $p$  is prime, so either  $i_1 \in p$  or  $i_2 \in p$ , contradiction.

### Example 0.1.

Consider  $A = K[x_1, \dots, x_n]$  with  $K$  algebraically closed. For  $I \subseteq A$ , the maximal ideals of  $A$  corresponding to points of  $Z(I)$  are precisely the maximal ideals containing  $I$ .

# 1 Sheaves

Fix a topological space  $X$ .

**Definition 1.1.** A *presheaf*  $\mathcal{F}$  on  $X$  consists of data, such that:

- For every open set  $U \subseteq X$ , we have an abelian group  $\mathcal{F}(U)$  (or more generally any element of a category).
- Whenever  $V \subseteq U \subseteq X$  is open, there is a *restriction* homomorphism

$$\rho_{UV} : \mathcal{F}(U) \rightarrow \mathcal{F}(V),$$

such that  $\rho_{UU} = \text{id}_{\mathcal{F}(U)}$ , and if  $W \subseteq V \subseteq U \subseteq X$ , then

$$\rho_{UW} = \rho_{VW} \circ \rho_{UV}.$$

*Remark.* This is precisely a contravariant functor from the category of open sets to the category of abelian groups. As mentioned, we may replace the category of abelian groups with any category.

**Definition 1.2.** If  $\mathcal{F}, \mathcal{G}$  are presheaves on  $X$ , then a *morphism*  $f : \mathcal{F} \rightarrow \mathcal{G}$  is data for each  $U \subseteq X$ , a group homomorphism  $f_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$  such that whenever  $V \subseteq U$ , we have a commutative diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f_U} & \mathcal{G}(U) \\ \downarrow \rho_{UV}^{\mathcal{F}} & & \downarrow \rho_{UV}^{\mathcal{G}} \\ \mathcal{F}(V) & \xrightarrow{f_V} & \mathcal{G}(V) \end{array}$$

**Definition 1.3.** A presheaf  $\mathcal{F}$  on  $X$  is a *sheaf* if it satisfies:

1. If  $U \subseteq X$  is covered by  $\{U_i\}$ , and  $s \in \mathcal{F}(U)$  such that  $s|_{U_i} = \rho_{UU_i}(s) = 0$  for all  $i$ , then  $s = 0$ .
2. If  $U, \{U_i\}$  are as in the above, and  $s_i \in \mathcal{F}(U_i)$  for each  $i$  such that

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

for all  $i, j$ , then there exists  $s \in \mathcal{F}(U)$  such that

$$s|_{U_i} = s_i$$

for all  $i$ .

*Remark.*

- If  $\mathcal{F}$  is a sheaf, then  $\mathcal{F}(\emptyset) = 0$ , since the empty cover is a cover of  $\emptyset$ .
- The two conditions together can be stated by saying

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\alpha} \bigoplus_{i \in I} \mathcal{F}(U_i) \xrightarrow[\beta_2]{\beta_1} \bigoplus_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$

is *exact* for all  $U \subseteq X$  open, and open covers  $\{U_i\}$  of  $U$ . Here,

$$\begin{aligned} \alpha(s) &= (s|_{U_i})_{i \in I}, \\ \beta_1((s_i)_{i \in I}) &= (s_i|_{U_i \cap U_j})_{i,j \in I}, \\ \beta_2((s_i)_{i \in I}) &= (s_j|_{U_i \cap U_j})_{i,j \in I}. \end{aligned}$$

In category theory,  $\alpha$  is the *equalizer* of  $\beta_1, \beta_2$ .

Exactness means that:

- $\alpha$  is injective (property 1).
- $\beta_1 \circ \alpha = \beta_2 \circ \alpha$ .
- For any  $(s_i) \in \bigoplus \mathcal{F}(U_i)$  with  $\beta_1((s_i)) = \beta_2((s_i))$ , there exists an  $s \in \mathcal{F}(U)$  with  $\alpha(s) = (s_i)$  (property 2).

This definition works even if  $\mathcal{F}(U)$  is a set, rather than an abelian group.

### Example 1.1.

1. For  $X$  any topological space,

$$\mathcal{F}(U) = \{\text{continuous functions } f : U \rightarrow \mathbb{R}\}$$

is a sheaf.

2. If  $X = \mathbb{C}$  with the Euclidean topology, then

$$\mathcal{F}(U) = \{f : U \rightarrow \mathbb{C} \mid f \text{ bounded and holomorphic}\}$$

is not a sheaf, as gluing fails because it does not preserve boundedness.

3. Let  $G$  be a group, and set  $\mathcal{F}(U) = G$  for all  $U \subseteq X$ . Then  $\rho_{UV} = \text{id}$ . This is a presheaf known as the *constant presheaf*.

If we give  $G$  the discrete topology, set

$$\mathcal{F}(U) = \{f : U \rightarrow G \text{ continuous}\}.$$



These are all locally constant functions, and is obviously a sheaf, called the *constant sheaf*.

4. If  $X$  is a variety, denote by  $\mathcal{O}_X(U)$  the set of regular functions  $f : U \rightarrow K$ . Then  $\mathcal{O}_X(U)$  is a sheaf, called the *structure sheaf* of  $X$ .

**Definition 1.4.** Let  $\mathcal{F}$  be a presheaf in  $X$ ,  $p \in X$ . Then the *stalk* of  $\mathcal{F}$  at  $p$  is

$$\mathcal{F}_p = \{(U, s) \mid U \text{ an open neighbourhood of } p, s \in \mathcal{F}(U)\} / \cong,$$

where  $(U, s) \cong (V, t)$  if there exists  $W \subseteq U \cap V$ , a neighbourhood of  $p$ , such that

$$s|_W = t|_W.$$

The equivalence class of  $(U, s) \in \mathcal{F}_p$  is written as  $s_p$ , and is the *germ* of  $s$  at  $p$ .

So the stalk is the set of germs. The stalks should be thought of as the local information of the sheaf around  $p$ . Note that given a morphism  $f : \mathcal{F} \rightarrow \mathcal{G}$ , we obtain  $f_p : \mathcal{F}_p \rightarrow \mathcal{G}_p$  by

$$f_p(U, s) = (U, f_U(s)).$$

**Proposition 1.1.** Let  $f : \mathcal{F} \rightarrow \mathcal{G}$  be a morphism of sheaves (i.e. a morphism of presheaves). Then  $f$  is an isomorphism if and only if  $f_p$  is an isomorphism, for all  $p \in X$ .

**Proof:** The forward direction is obvious.

For the other direction, assume  $f_p$  is an isomorphism for all  $p$ . We will show that  $f(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$  is an isomorphism for all  $U$ , and then we can define the inverse to  $f$  by  $(f^{-1})_U = (f_U)^{-1}$ .

First we show  $f_U$  is injective. Suppose  $s \in \mathcal{F}(U)$  is such that  $f_U(s) = 0$ . Then for all  $p \in U$ ,

$$f_p((U, s)) = (U, f_U(s)) = (U, 0) = 0 \in \mathcal{G}_p.$$

Thus  $s_p = 0$  since  $f_p$  is injective. So there is an open neighbourhood  $V_p \subseteq U$  of  $p$  such that  $s|_{V_p} = 0$ .

But  $\{V_p\}$  covers  $U$ , so by property 1,  $s = 0$ .

Now we show  $f_U$  is surjective. Let  $t \in \mathcal{G}(U)$ . Then for all  $p \in U$ , there exists  $s_p \in \mathcal{F}_p$  such that  $f_p(s_p) = t_p$ , i.e. there exists an open neighbourhood  $V_p$  at  $p \in U$  and a germ  $(V_p, \tilde{s}_p)$  representing  $s_p$  such that

$$(V_p, f_{V_p}(\tilde{s}_p)) \cong (U, t) = t_p.$$

Shrinking  $V_p$  if necessary, we can assume that  $f_{V_p}(\tilde{s}_p) = t|_{V_p}$ . Now on  $V_p \cap V_q$ ,

$$f_{V_p \cap V_q}(\tilde{s}_p|_{V_p \cap V_q} - \tilde{s}_q|_{V_p \cap V_q}) = t|_{V_p \cap V_q} - t|_{V_p \cap V_q} = 0.$$

Since we have shown that  $f_{V_p \cap V_q}$  is injective, we get

$$\tilde{s}_p|_{V_p \cap V_q} = \tilde{s}_q|_{V_p \cap V_q},$$

and so by property 2, there exists  $s \in \mathcal{F}(U)$  such that

$$s|_{V_p} = \tilde{s}_p,$$

for all  $p$ . Now,

$$f_U(s)|_{V_p} = f_{V_p}(s|_{V_p}) = f_{V_p}(\tilde{s}_p) = t|_{V_p}.$$

Therefore,  $f_U(s) - t = 0$ , so by property 1,  $f_U(s) = t$ . Hence  $f_U$  is surjective.

*Remark.* If  $f_p : \mathcal{F}_p \rightarrow \mathcal{G}_p$  is injective for all  $p$ , then  $f_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$  is still injective.

But instead if  $f_p : \mathcal{F}_p \rightarrow \mathcal{G}_p$  is surjective for all  $p$ , it need not be the case that  $f_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$  is surjective.

## 1.1 Sheafification

Given a presheaf  $\mathcal{F}$ , there exists a sheaf  $\mathcal{F}^+$  and a morphism  $\theta : \mathcal{F} \rightarrow \mathcal{F}^+$ , satisfying the following universal property:

For any sheaf  $\mathcal{G}$  and morphism  $\phi : \mathcal{F} \rightarrow \mathcal{G}$ , there exists a unique morphism  $\phi^+ : \mathcal{F}^+ \rightarrow \mathcal{G}$  such that

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\theta} & \mathcal{F}^+ \\ & \searrow \phi & \downarrow \phi^+ \\ & & \mathcal{G} \end{array}$$

commutes.

The pair  $(\mathcal{F}^+, \theta)$  is unique up to isomorphism. Also  $\mathcal{F}_p \cong \mathcal{F}_p^+$  via  $\theta_p$ , for all  $p \in X$ .

The sheafification is defined as follows: define  $\mathcal{F}^+(U)$  to be the functions

$$s : U \rightarrow \bigsqcup_{p \in U} \mathcal{F}_p$$

such that:

- (i)  $s(p) \in \mathcal{F}_p$  for all  $p$ ,
- (ii) for each  $p \in U$ , there exists an open neighbourhood  $p \in V \subseteq U$  and an element  $t \in \mathcal{F}(V)$  such that

$$(V, t) = s(q),$$

for all  $q \in V$ .

We define  $\theta : \mathcal{F} \rightarrow \mathcal{F}^+$  given by

$$\mathcal{F}(U) \ni s \mapsto (p \mapsto (U, s) \in \mathcal{F}_p).$$

We can check that this satisfies the universal property stated previously.

**Definition 1.5.** Let  $f : \mathcal{F} \rightarrow \mathcal{G}$  be a morphism of presheaves. We define:

1. The *presheaf kernel* of  $f$  as

$$(\ker f)(U) = \ker(f_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)).$$

2. The *presheaf cokernel* of  $f$  as

$$(\operatorname{coker} f)(U) = \operatorname{coker} f_U.$$

3. The *presheaf image* as

$$(\operatorname{im} f)(U) = \operatorname{im} f_U.$$

*Remark.* If  $f : \mathcal{F} \rightarrow \mathcal{G}$  is a morphism of sheaves, then  $\ker f$  is a sheaf. First note that any sub-presheaf of  $\mathcal{F}$  satisfies property 1.

To check property 2, given  $s_i \in (\ker f)(U_i) \subseteq \mathcal{F}(U_i)$  for  $\{U_i\}$  an open cover of  $U$ , suppose  $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ . Then the  $s_i$ 's glue to give an  $s \in \mathcal{F}(U)$ . Now,

$$f_U(s)|_{U_i} = f_{U_i}(s_i) = 0,$$

so by property 1,  $f_U(s) = 0$ . Hence  $s \in (\ker f)(U)$ .

### Example 1.2.

Take  $X = \mathbb{P}^1$ , or the Riemann sphere, and let  $P, Q \in X$  be distinct points.

Let  $\mathcal{G}$  be the sheaf of regular functions on  $X$  (or the holomorphic functions on  $X$ ), and let  $\mathcal{F}$  be the sheaf of regular functions on  $X$  vanishing at  $P$  and  $Q$  (or the holomorphic functions vanishing at  $P, Q$ ).

Note  $\mathcal{F}(U) = \mathcal{G}(U)$  if  $U \cap \{P, Q\} = \emptyset$ . By Liouville's theorem,  $\mathcal{G}(X) = K$ , and  $\mathcal{F}(X) = 0$ .

Let  $U = X \setminus \{P\}$ ,  $V = X \setminus \{Q\}$ , and  $f : \mathcal{F} \rightarrow \mathcal{G}$  the obvious inclusion. Note  $\mathcal{G}(U) = K[x]$  by affine geometry,  $\mathcal{F}(U) = (x)$ . So,

$$\begin{aligned} (\operatorname{coker} f)(X) &= \mathcal{G}(X)/\mathcal{F}(X) = K, \\ (\operatorname{coker} f)(U \cap V) &= \mathcal{G}(U \cap V)/\mathcal{F}(U \cap V) = 0, \\ (\operatorname{coker} f)(U) &= \mathcal{G}/\mathcal{F}(U) = K[x]/(x) = K, \\ (\operatorname{coker} f)(V) &= K. \end{aligned}$$

Choose  $s_U \in (\operatorname{coker} f)(U)$ ,  $s_V \in (\operatorname{coker} f)(V)$ . But now  $s_U|_{U \cap V} = s_V|_{U \cap V} = 0$ , and this would imply that if  $\operatorname{coker} f$  were a sheaf, that

$$K \oplus K \subseteq (\operatorname{coker} f)(X).$$

*Remark.* This shows that  $\operatorname{coker} f$  need not be a sheaf. The same is true of  $\operatorname{im} f$ .

**Definition 1.6.** Let  $f : \mathcal{F} \rightarrow \mathcal{G}$  be a morphism of sheaves. The *sheaf kernel* of  $f$  is  $\ker f$ , the presheaf kernel.

The *sheaf cokernel* of  $f$  is the sheafification of the presheaf cokernel. We still write this as  $\operatorname{coker} f$ .

Note that to show  $\operatorname{coker} f$  is a presheaf, we need to use the third isomorphism theorem.

We can also show that the sheaf image of  $f$  is a subsheaf of  $\mathcal{G}$  (prove this).

We say that  $f$  is *injective* if  $\ker f = 0$ , and  $f$  is *surjective* if  $\operatorname{im} f = \mathcal{G}$ . We say that a sequence

$$\dots \longrightarrow \mathcal{F}^{i-1} \xrightarrow{f^i} \mathcal{F}^i \xrightarrow{f^{i+1}} \mathcal{F}^{i+1} \longrightarrow \dots$$

is *exact* if  $\ker f^{i+1} = \operatorname{im} f^i$ .

If  $\mathcal{F}' \subseteq \mathcal{F}$  is a subsheaf, we write  $\mathcal{F}/\mathcal{F}'$  for the sheaf associated to the presheaf

$$U \mapsto \mathcal{F}(U)/\mathcal{F}'(U).$$

This is  $\operatorname{coker}(\iota : \mathcal{F}' \hookrightarrow \mathcal{F})$ , where  $\iota$  is the inclusion.

**Lemma 1.1.** Let  $f : \mathcal{F} \rightarrow \mathcal{G}$  be a morphism of sheaves. Then,

$$\begin{aligned} (\ker f)_p &= \ker(f_p : \mathcal{F}_p \rightarrow \mathcal{G}_p), \\ (\operatorname{im} f)_p &= \operatorname{im} f_p. \end{aligned}$$

**Proof:** We have a map  $(\ker f)_p \rightarrow \ker f_p$ , defined as follows: if  $(U, s) \in (\ker f)_p$ , then  $s \in (\ker f)(U)$ , and  $(U, s) \in \mathcal{F}_p$ . So,

$$f_p(U, s) = (U, f(s)) = (U, 0) = 0 \in \mathcal{G}_p,$$

thus  $(U, s) \in \ker f_p$ .

It is easy to see this map is injective: if  $(U, s) = 0$  in  $\mathcal{F}_p$ , then in some neighbourhood  $s|_V = 0$ . So  $(U, s) \sim (V, s|_V) = 0$  in  $(\ker f)_p$ .

We now tackle surjectivity. If  $(U, s) \in \ker f_p$ , then  $0 = f_p(U, s) = (U, f_U(s))$  in  $\mathcal{G}_p$ , so in some neighbourhood  $V$ ,  $f_U(s)|_V = 0$ .

Thus  $f_V(s|_V) = 0$ , so  $(U, s) \sim (V, s|_V)$ , and  $s|_V \in (\ker f)(V)$ . Thus  $(V, s|_V) \in (\ker f)_p$ , which maps to  $(U, s) \in \ker f_p$ .

We now prove the appropriate theorem for images. Let  $\text{im}' f$  denote the presheaf image. Recall that if  $\mathcal{F}$  is a presheaf, then  $\mathcal{F}_p \cong \mathcal{F}_p^+$ , via  $\theta_p$ . So it is enough to show there is an isomorphism

$$(\text{im}' f)_p \cong \text{im } f_p.$$

First, define

$$\begin{aligned} (\text{im}' f)_p &\rightarrow \text{im } f_p \\ (U, s) &\mapsto (U, s), \end{aligned}$$

with  $s \in f_U(t)$  for some  $t \in \mathcal{F}(U)$ , which lives in  $\text{im } f_p$  since

$$f_p(U, t) = (U, f_U(t)) = (U, s)$$

First we show injectivity. If  $(U, s) = 0$  in  $\mathcal{G}_p$ , then there exists  $p \in V \subseteq U$  such that  $s|_V = 0$ . But then,

$$(U, s) \sim (V, s|_V) = (V, 0) = 0$$

in  $(\text{im}' f)_p$ .

To show surjectivity, we know that for  $(U, s) \in \text{im } f_p$ , that there is  $(V, t) \in \mathcal{F}_p$  with  $f_p(V, t) \sim (U, s)$ . We can replace  $U$  with the smaller open set  $V$ , so can assume that  $U = V$ , and then

$$f_p(U, t) = (U, f_U(t)) \sim (U, s)$$

in  $\mathcal{G}_p$ . Shrinking  $U$  further, we can assume  $f_U(t) = s$ , and hence

$$(U, s) = (U, f_U(t)) \in (\text{im}' f)_p.$$

**Proposition 1.2.**  $f : \mathcal{F} \rightarrow \mathcal{G}$  is injective if and only if  $f_p : \mathcal{F}_p \rightarrow \mathcal{G}_p$  is injective for all  $p$ .

$f : \mathcal{F} \rightarrow \mathcal{G}$  is surjective if and only if  $f_p : \mathcal{F}_p \rightarrow \mathcal{G}_p$  is surjective for all  $p$ .

**Proof:** We have

$$\begin{aligned} f_p \text{ is injective} &\iff \ker f_p = 0 && \forall p \\ &\iff (\ker f)_p = 0 && \forall p \text{ by the lemma} \\ &\iff \ker f = 0. \end{aligned}$$

Note this uses the easy fact that if  $\mathcal{F}$  is a sheaf, and  $\mathcal{F}_p = 0$  for all  $p$ , then  $\mathcal{F} = 0$ . Also,

$$\begin{aligned} f_p \text{ is surjective} &\iff \operatorname{im} f_p = \mathcal{G}_p && \forall p \\ &\iff (\operatorname{im} f)_p = \mathcal{G}_p && \forall p \text{ by the lemma} \\ &\iff \operatorname{im} f = \mathcal{G}. \end{aligned}$$

Hence if  $\mathcal{F} \subseteq \mathcal{G}$  are sheaves with  $\mathcal{F}_p = \mathcal{G}_p$ , we can check that  $\mathcal{F} = \mathcal{G}$ .

## 1.2 Passing between Spaces

Let  $f : X \rightarrow Y$  be a continuous map of topological spaces.

Let  $\mathcal{F}$  be a sheaf in  $X$ . Define a sheaf  $f_*\mathcal{F}$  on  $Y$  by

$$(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U)).$$

This is the *push-forward* of  $\mathcal{F}$ . This can be checked to be a sheaf.

If  $\mathcal{G}$  is a sheaf on  $Y$ , define the *pull-back* sheaf  $f^{-1}\mathcal{G}$  to be the sheaf associated to the presheaf

$$U \mapsto \{(V, s) \mid V \subseteq f(U), V \text{ open}, s \in \mathcal{G}(V)\} / \sim,$$

where  $(V, s) \sim (V', s')$  if there exists  $W$  with  $f(U) \subseteq W \subseteq V \cap V'$  with  $s|_W = s'|_W$ .

### Example 1.3.

If  $f : \{p\} \hookrightarrow Y$ , then  $f^{-1}\mathcal{G} = \mathcal{G}_p$ , by identifying a sheaf  $\mathcal{F}$  on a topological space  $X$  with the group  $\mathcal{F}(X)$ .

More generally, if  $\iota : Z \hookrightarrow X$  is an inclusion, we often write  $\mathcal{F}|_Z$  for the sheaf  $\iota^{-1}\mathcal{F}$ .

If  $\iota : U \hookrightarrow X$  is an open subset, then in fact  $i^{-1}\mathcal{F} = \mathcal{F}|_U$  is the sheaf  $V \mapsto \mathcal{F}(V)$ , for  $V \subseteq U$  open.

If  $s \in \mathcal{F}(U)$ , we call  $s$  a *section* of  $\mathcal{F}$  over  $U$ . We often also write

$$\mathcal{F}(U) = \Gamma(U, \mathcal{F}),$$

thinking of  $\Gamma(U, \cdot)$  as a covariant functor

$$\Gamma(U, \cdot) : \mathbf{Presheaves}_X \rightarrow \mathbf{Ab}.$$

## 2 Affine Schemes

Let  $A$  be a ring.  $\text{Spec } A$  is a topological space analogous to the sheaf of regular functions.

Let  $S \subseteq A$  be a multiplicatively closed subset, i.e.  $1 \in S$  and whenever  $a, b \in S$ , we have  $a \cdot b \in S$ . We define

$$S^{-1}A = \{(a, s) \mid a \in A, s \in S\} / \sim,$$

where  $(a, s) \sim (a', s')$  if there exists  $s'' \in S$  such that

$$s''(as' - a's) = 0.$$

This is called the *localization* of  $A$  at  $S$ .

### Example 2.1.

1. Say  $S = \{1, f, f^2, \dots\}$  for some  $f \in A$ . Then

$$A_f = S^{-1}A = \left\{ \frac{a}{f^n} \mid a \in A, n \geq 0 \right\} / \sim.$$

2. Take  $P \subseteq A$  a prime ideal, and  $S = A \setminus P$ . Then we write  $A_P = S^{-1}A$ .

Our goal is to now construct the sheaf

$$\mathcal{O} = \mathcal{O}_{\text{Spec } A}.$$

For  $U \subseteq \text{Spec } A$  open, we write

$$\mathcal{O}(U) = \left\{ s : U \rightarrow \bigsqcup_{p \in U} A_p \mid s(p) \in A_p \wedge \text{for each } p \in U, \exists q \in V \subseteq U \right. \\ \left. \text{and } a, f \in A \text{ such that } \forall q \in V, f \notin q \wedge s(q) = \frac{a}{f} \in A_q \right\}$$

This is a sheaf, as it is only defined locally.

The scheme has the following properties. First, for any  $p \in \text{Spec } A$ ,  $\mathcal{O}_p = A_p$ .

**Proof:** We have a map  $\mathcal{O}_p \rightarrow A_p$  given by  $(U, s) \mapsto s(p)$ .

First we show surjectivity. Any element of  $A_p$  can be written as  $a/f$ , for



$a \in A$  and  $f \notin p$ . We define

$$D(f) = (\operatorname{Spec} A) \setminus V((f)) = \{q \in \operatorname{Spec} A \mid f \notin q\}.$$

This is an open neighbourhood of  $p \in \operatorname{Spec} A$ . Then  $a/f$  defines a section  $s$  of  $\mathcal{O}$  over  $D(f)$ , via

$$s(q) = \frac{a}{f} = A_q.$$

In particular,  $s(p) = a/f \in A_p$ . So,  $(D(f), s) \mapsto a/f$  under this map.

For injectivity, let  $p \in U \subseteq \operatorname{Spec} A$ , and  $s \in \mathcal{O}(U)$ , defining a germ  $(U, s) \in \mathcal{O}_p$ .

Suppose  $s(p) = 0$ . We want to show  $(U, s) = 0$  in  $\mathcal{O}_p$ . We may shrink  $U$  and assume there are  $a, f \in A$  such that  $s(q) = a/f$  for all  $q \in U$ . In particular,  $f \notin q$ .

Since  $s(p) = 0$ ,  $a/f = 0$  in  $A_p$ . Thus there exists  $h \in A \setminus p$  such that

$$h(a \cdot 1 - 0 \cdot f) = 0,$$

i.e.  $h \cdot a = 0$ . Let  $V = D(f) \cap D(h)$ . Then for  $q \in V$ ,  $a/f = 0$  in  $A_q$ .

Since  $h \notin q$  and  $h \cdot a = 0$ ,  $s|_V = 0$ , so  $(U, s) = 0$ . Moreover,  $V$  is non-empty as it contains  $p$ .

Another property is as follows: for any  $f \in A$ ,  $\mathcal{O}(D(f)) = A_f$ . In particular,

$$\mathcal{O}(\operatorname{Spec} A) = \mathcal{O}(D(1)) = A_1 = A.$$

**Proof:** Let  $\psi : A_f \rightarrow \mathcal{O}(D(f))$  be given by

$$\frac{a}{f^n} \mapsto \left( p \mapsto \frac{a}{f^n} \in A_p \right).$$

We need to show this map is injective, and surjective.

**Injectivity:** To show  $\psi$  is injective, if  $\psi(a/f^n) = 0$ , then for all  $p \in D(f)$ ,  $a/f^n = 0$  in  $A_p$ . So there exists  $h \notin p$  such that  $h \cdot a = 0$ , where  $h$  may depend on  $p$ . Let

$$I = \{g \in A \mid g \cdot a = 0\}.$$

So  $h \in I$ , for all  $p$ . But also  $h \notin p$ , so  $I \not\subseteq p$ . This is true for all  $p \in D(f)$ , so

$$V(I) \cap D(f) = \emptyset.$$

Thus,

$$f \in \bigcap_{q \in V(I)} q = \sqrt{I},$$

the radical of  $I$ . Thus  $f^n \in I$  for some  $n > 0$ . Thus  $f^n \cdot a = 0$ , so in particular  $a/f^n = 0$  in  $A_f$ , by choosing  $h = f^n$ .

**Surjectivity:** is a lot harder. Let  $s \in \mathcal{O}(D(f))$ . Our goal is to show that  $s = \psi(a/f^n)$  for some  $a$  and  $n$ . Let  $\{V_i\}$  be an open cover of  $D(f)$  such that  $s|_{V_i}$  is represented by  $a_i/g_i$  with  $g_i \notin p$ , for all  $p \in V_i$ .

Thus  $V_i \subseteq D(g_i)$ . By an example sheet question, the open sets of the form  $D(h)$  form a basis for the Zariski topology. We can thus assume  $V_i = D(h_i)$  for some  $h_i \in A$ .

Since  $D(h_i) \subseteq D(g_i)$ ,  $V((h_i)) \subseteq V((g_i))$ , so  $\sqrt{(h_i)} \subseteq \sqrt{(g_i)}$ . Therefore  $h_i^n \in (g_i)$ , for some  $n$ , so  $h_i^n = c_i g_i$  for some  $c_i \in A$ . Thus,

$$\frac{a_i}{g_i} = \frac{c_i a_i}{h_i^n}.$$

We can thus replace  $h_i$  with  $h_i^n$ , since a prime ideal containing one contains the other, and can now assume that  $s|_{V_i}$  is represented by an element of the form  $a_i/h_i$ , where  $V_i = D(h_i)$ .

So far, we have proven that  $D(f)$  is covered with open sets  $V_i = D(h_i)$ , and  $s|_{V_i}$  represented by  $a_i/h_i$ .

**Claim:**  $D(f)$  can be covered with a finite number of the  $D(h_i)$ 's, in other words  $D(f)$  is *quasicompact*, meaning compact without the Hausdorff condition.

The proof is as follows:

$$\begin{aligned}
D(f) &\subseteq \bigcup_i D(h_i) \\
\iff V((f)) &\supseteq \bigcap V((h_i)) = V\left(\sum_i (h_i)\right) \\
\iff \sqrt{(f)} &\subseteq \sqrt{\sum_i (h_i)} \\
\iff f &\in \sqrt{\sum_i (h_i)} \\
\iff f^n &\in \sum_i (h_i)
\end{aligned}$$

for some  $n$ . Thus,

$$f^n = \sum b_i h_i$$

for some finite set of  $h_i$ 's. Thus, by running the above proof backwards, we find that the finite number of these  $D(h_i)$  cover  $D(f)$ .

Now we have a finite cover of  $D(f)$ . On  $D(h_i) \cap D(h_j)$ ,  $a_i/h_i$  and  $a_j/h_j$  both represent  $s|_{D(h_i) \cap D(h_j)}$ . But,

$$\psi : A_{h_i h_j} \rightarrow \mathcal{O}(D(h_i, h_j)) = \mathcal{O}(D(h_i) \cap D(h_j))$$

is injective, by proof of injectivity. So,  $a_i/h_i = a_j/h_j$  in  $A_{h_i h_j}$ . Thus for some  $n$ ,

$$(h_i h_j)^n (h_j a_i - h_i a_j) = 0.$$

We can choose  $n$  to work for all  $i, j$ . Rewriting this as

$$h_j^{n+1} (h_i^n a_i) - h_i^{n+1} (h_j^n a_j) = 0,$$

and replacing each  $h_i$  with  $h_i^{n+1}$  and  $a_i$  with  $h_i^n a_i$ , the ratio is the same, and so  $s$  is still represented by  $a_i/h_i$ , but also  $h_j a_i = h_i a_j$  for all  $i, j$ . Let

$$a = \sum b_i a_i,$$

where we recall that

$$f^n = \sum b_i h_i.$$

Then for any  $j$ ,

$$h_j a = \sum h_j b_i a_i = \sum_i h_i b_i a_j = f^n a_j,$$

so  $a_j/h_j = a/f^n$ , for all  $j$ . So  $\psi(a/f^n) = s$ .

**Definition 2.1.** A *ringed space* is a pair  $(X, \mathcal{O}_X)$  where:

- $X$  is a topological space,
- $\mathcal{O}_X$  is a sheaf of rings.

A *morphism* of ringed spaces  $f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$  is data such that:

- $f : X \rightarrow Y$  is continuous.
- $f^\# : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$  is a morphism of sheaves of rings, i.e.

$$f_U^\# : \mathcal{O}_Y(U) \rightarrow (f_*\mathcal{O}_X)(U) = \mathcal{O}_X(f^{-1}(U))$$

is a ring homomorphism.

**Example 2.2.**

1. Let  $X$  be a variety, and  $\mathcal{O}_X$  the sheaf of regular functions on  $X$ , so

$$\mathcal{O}_X(U) = \{f : U \rightarrow K \text{ such that } f \text{ is regular}\}.$$

A morphism  $f : X \rightarrow Y$  of varieties is a continuous map  $f : X \rightarrow Y$  inducing a map, for all  $U \subseteq Y$ , by  $\mathcal{O}_Y(U) \rightarrow \mathcal{O}_X(f^{-1}(U))$ , where  $\varphi \mapsto \varphi \circ f$ .

2. Let  $X$  be a topological space, then we can consider

$$\mathcal{O}_X(U) = \{f : U \rightarrow \mathbb{R} \mid f \text{ continuous}\}.$$

3. Let  $X$  be a  $C^\infty$ -manifold, and

$$\mathcal{O}_X(U) = \{f : U \rightarrow \mathbb{R} \mid f \in C^\infty\}.$$

Then  $f : X \rightarrow Y$  a continuous map between  $C^\infty$  manifolds is  $C^\infty$  if and only if, for any  $C^\infty$  function  $\varphi : U \rightarrow \mathbb{R}$ , and  $U \subseteq Y$ ,  $f \circ \varphi : f^{-1}(U) \rightarrow \mathbb{R}$  is  $C^\infty$ .

*Remark.* All of these example have the feature that  $\mathcal{O}_{X,p}$ , for  $p \in X$ , is a local ring, i.e. one with a unique maximal ideal.

Indeed, take  $m_p = \{(U, f) \mid f(p) = 0\}$ . Note  $(U, f) \sim (V, f|_V)$  for some  $V$  with  $f|_V$  non-vanishing if  $f(p) \neq 0$ .

So every element in  $\mathcal{O}_{X,p} \setminus m_p$  is invertible, hence  $m_p$  is the unique maximal ideal.

Note that given a morphism of ringed space  $f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$  in the previous three examples, we get a map

$$f_p^\# : \mathcal{O}_{Y,f(p)} \rightarrow \mathcal{O}_{X,p}$$

for  $p \in X$  defined by

$$f_p^\#(U, \varphi) = (f^{-1}(U), \varphi \circ f).$$

Note that  $(U, \varphi) \in m_{f(p)}$  if and only if  $(f^{-1}(U), \varphi \circ f) \in m_p$ .

In particular,  $(f_p^\#)^{-1}(m_p) = m_{f(p)}$ .

**Definition 2.2.** A *locally ringed space* is a ringed space  $(X, \mathcal{O}_X)$  such that  $\mathcal{O}_{X,p}$  is a local ring for all  $p \in X$ .

A *morphism*  $f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$  of locally ringed spaces is a morphism of ringed spaces such that the induced morphism

$$f_p^\# : \mathcal{O}_{Y,f(p)} \rightarrow \mathcal{O}_{X,p}$$

is a local homomorphism.

Here  $f_p^\#$  is defined by

$$\mathcal{O}_{Y,f(p)} \ni (U, \varphi) \mapsto (f^{-1}(U), f_U^\#(\varphi)) \in \mathcal{O}_{X,p}.$$

A homomorphism  $g : A \rightarrow B$  of local rings is a ring homomorphism such that  $g^{-1}(m_B) = m_A$ . Note that  $g(A \setminus m_A) = g(A^*) \subseteq B^* = B \setminus m_B$ , so we always have  $g^{-1}(m_B) \subseteq m_A$ .

The motivating example is  $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$ , which we will show is a locally ringed space. Recall that  $\mathcal{O}_{\text{Spec } A, p} = A_p$ , which has a unique maximal ideal

$$m = \left\{ \frac{a}{s} \mid a \in p, s \notin p \right\}.$$

There is a natural map  $A \rightarrow A_p$  by  $a \mapsto a/1$ , and  $m \subseteq A_p$  is the ideal generated by the image of  $p$  under this map. This is often written as  $pA_p$ .

We call  $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$  an *affine scheme*.

**Theorem 2.1.** *The category of affine schemes with morphisms being morphisms of locally ringed spaces is equivalent to the opposite category of rings.*

**Proof:** We need to show that:

- If  $\varphi : A \rightarrow B$  is a homomorphism of rings, we get a morphism of locally ringed spaces

$$(f, f^\#) : (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \rightarrow (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}).$$

- Any morphism of locally ringed spaces is induced by a homomorphism  $\varphi : A \rightarrow B$  of rings.

For the first part, we define  $f : \operatorname{Spec} B \rightarrow \operatorname{Spec} A$  by

$$f(p) = \varphi^{-1}(p).$$

Note that  $\varphi^{-1}(p)$  is prime, as if  $a \cdot b \in \varphi^{-1}(p)$ , then  $\varphi(a)\varphi(b) = \varphi(ab) \in p$ , so say  $\varphi(a) \in p$ . Then  $a \in \varphi^{-1}(p)$ .

Now we show  $f$  is continuous. Note for any  $I \subseteq A$ ,

$$\begin{aligned} f^{-1}(V(I)) &= \{p \in \operatorname{Spec} B \mid f(p) \supseteq I\} = \{p \in \operatorname{Spec} B \mid \varphi^{-1}(p) \supseteq I\} \\ &= \{p \in \operatorname{Spec} B \mid p \supseteq \varphi(I)\} \\ &= V(\varphi(I)). \end{aligned}$$

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