III Combinatorics

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Contents

0 Introduction

We have the following list of things.

1: Set systems.

2: Isoperimetric inequalities.

3: Intersection families.

Books include ‘Combinatorics’ by Bollobás, and ‘Combinatorics of Finite Sets’, by Anderson.

1 Set Systems

Let X be a set. A *set system* on X, also called a family of subsets of X, is a family $\mathcal{A} \subseteq \mathcal{P}(X)$. For example,

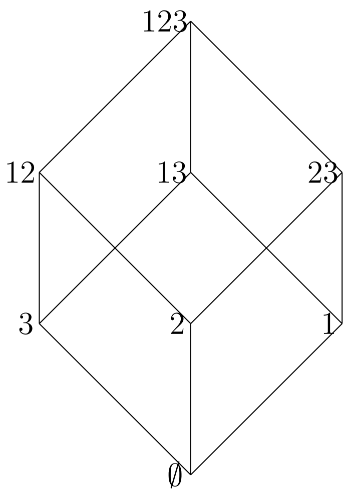
$$X^{(r)} = \{A \subseteq X \mid |A| = r\}.$$

Usually, $X = [n] = \{1, 2, \dots, n\}$, so $|X^{(r)}| = \binom{n}{r}$. Thus,

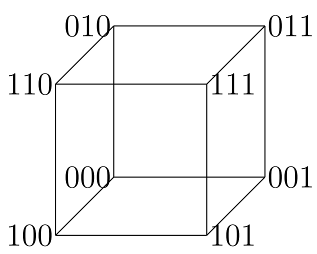
$$[4]^{(2)} = \{12, 13, 14, 23, 24, 34\}.$$

We make $\mathcal{P}(X)$ into a graph by joining A and B if $|A \triangle B| = 1$. This is the *discrete cube* Q_n.

Literally just a cube.



Alternatively, can view Q_n as an n-dimensional unit cube $\{0,1\}^n$, by identifying e.g. $\{1, 3\}$ with the binary string $101000\cdots$.



Say $\mathcal{A} \subseteq \mathcal{P}(X)$ is a *chain* if, for all $A, B \in \mathcal{A}$, $A \subseteq B$ or $B \subseteq A$. For example,

$$\mathcal{A} = \{23, 12357, 1235, 123567\}$$

is a chain.

Say $\mathcal{A}$ is an *antichain* if, for all $A, B \in \mathcal{A}$ and $A \neq B$, we have $A \not\subseteq B$. For example, $\mathcal{A} = \{23, 137\}$ is an antichain.

How large can a chain be? We can achieve $|\mathcal{A}| = n + 1$ by taking

$$\mathcal{A} = \{\emptyset, 1, 12, 123, \ldots, [n]\}$$

Cannot beat this as each $0 \leq r \leq n$, $\mathcal{A}$ can contain at most one r-set (a member of $X^{(r)}$).

How large can an antichain be? We can achieve $|\mathcal{A}| = n$, e.g. $\mathcal{A} = \{1, 2, \ldots, n\}$. More generally, we can take $\mathcal{A} = X^{(r)}$, and the best is when $r = \lfloor n/2 \rfloor$.

Theorem 1.1 (Sperner's Lemma)**.** *Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an antichain. Then,*

$$|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

The idea is follows: we know that a chain meets a layer in at most one point, since a layer is an antichain. If we decompose the cube into chains, we have at most one element of an antichain in each chain.

Proof: We will decompose $\mathcal{P}(X)$ into $\binom{n}{\lfloor n/2 \rfloor}$ chains, then we are done. To achieve this, it is sufficient to find:

(i) For each $r < n/2$, a matching from $X^{(r)}$ to $X^{(r+1)}$.

(ii) For each $r \geq n/2$, a matching from $X^{(r)}$ to $X^{(r-1)}$.

Then we put these together to form our chains; each passing through $X^{(\lfloor n/2 \rfloor)}$.

By taking complements, it is enough to prove (i).

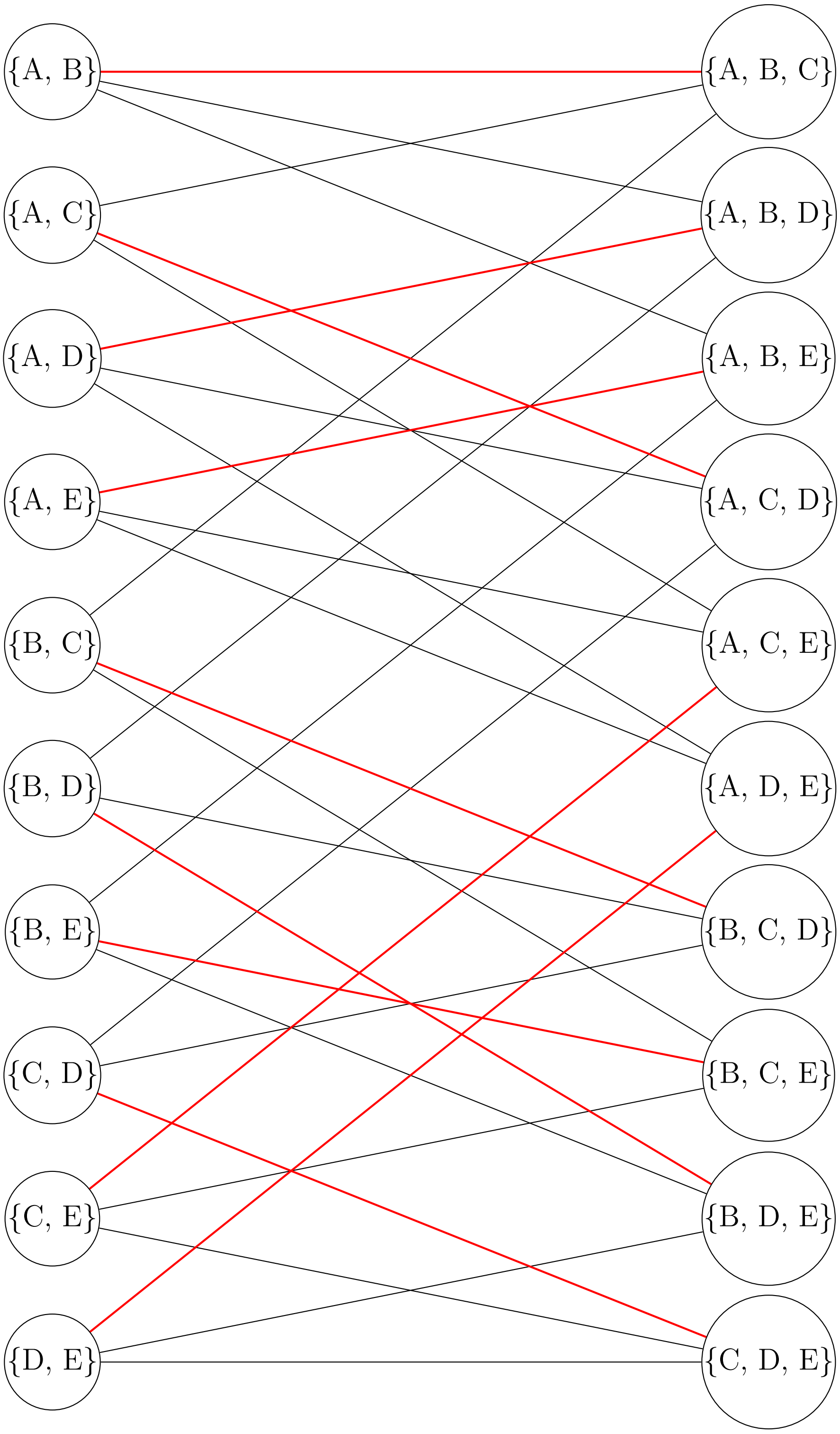
Let G be the bipartite subgraph of Q_n spanned by $X^{(r)} \cup X^{(r+1)}$: we seek a matching from $X^{(r)}$ to $X^{(r+1)}$.

For any $S \subseteq X^{(r)}$, the number of edges from S to $\Gamma(S)$ is $|S|(n - r)$, since each edge in S has $n - r$ edges.

Moreover there are at most $|\Gamma(S)|(r+1)$ edges, counting from $\Gamma(S)$. Therefore,

$$|\Gamma(S)| \geq \frac{|S|(n - r)}{r + 1} \geq |S|.$$

So we are done, by Hall's matching theorem.



When do we have equality in Sperner's? The above proof tells us nothing.

Our aim is to prove the following: if $\mathcal{A}$ is an antichain, then

$$\sum_{r=0}^{n} \frac{|\mathcal{A} \cap X^{(r)}|}{\binom{n}{r}} \leq 1.$$

In other words, the percentages of each layer occupied add up to at most 1. This trivially implies Sperner's.

1.1 Shadows

For $\mathcal{A} \subseteq X^{(r)}$, the *shadow* of $\mathcal{A}$ is $\partial\mathcal{A} = \partial^{-}\mathcal{A} \subseteq X^{(r-1)}$ defined by

$$\partial\mathcal{A} = \{B \in X^{(r-1)} \mid B \subseteq A \text{ for some } A \in \mathcal{A}\}.$$

For example, if $\mathcal{A} = \{123, 124, 134, 137\}$, then

$$\partial\mathcal{A} = \{12, 13, 23, 14, 24, 34, 17, 37\}.$$

Proposition 1.1 (Local LYM)**.** *Let $\mathcal{A} \subseteq X^{(r)}$. Then,*

$$\frac{|\partial\mathcal{A}|}{\binom{n}{r-1}} \geq \frac{|\mathcal{A}|}{\binom{n}{r}}.$$

So, the fraction of the local occupancy by $\partial\mathcal{A}$, is at least the occupancy by $\mathcal{A}$.

Remark. LYM = Lubell, Meshalkin, Yamamoto.

Proof: We look at the number of $\mathcal{A}$ to $\partial\mathcal{A}$ edges in the bipartite graph Q_n; counting from above, there are exactly $|\mathcal{A}|r$.

However counting from below, it is at most $|\partial\mathcal{A}|(n-r+1)$. So,

$$\frac{|\partial\mathcal{A}|}{|\mathcal{A}|} \geq \frac{r}{n-r+1} = \frac{\binom{n}{r-1}}{\binom{n}{r}}.$$

So we are done.

Remark. When do we have equality? We lose equality if an element in $\partial\mathcal{A}$ is connected to an element not in $\mathcal{A}$, so for this not to occur, we need that for all $A \in \mathcal{A}$, and $i \in A$, $j \notin \mathcal{A}$, that $A - \{i\} \cup \{j\} \in \mathcal{A}$.

But this is very strong, and in fact either $\mathcal{A} = \emptyset$ or $X^{(r)}$.

Theorem 1.2 (LYM Inequality)**.** *Let $A \subseteq \mathcal{P}(X)$ be an antichain. Then,*

$$\sum_{r=0}^{n} \frac{|\mathcal{A} \cap X^{(r)}|}{\binom{n}{r}} \leq 1.$$

As a bit of notation, we write $\mathcal{A}_r$ for $\mathcal{A} \cap X^{(r)}$.

We will look at two proofs. The first idea is to bubble down with local LYM.

Proof: Obviously

$$\frac{|\mathcal{A}_n|}{\binom{n}{n}} \leq 1.$$

Now, ∂A_n and $\mathcal{A}_{n-1}$ are disjoint, as $\mathcal{A}$ is an antichain. So,

$$\frac{|\partial \mathcal{A}_n|}{\binom{n}{n-1}} + \frac{|\mathcal{A}_{n-1}|}{\binom{n}{n-1}} = \frac{|\partial \mathcal{A}_n \cup \mathcal{A}_{n-1}|}{\binom{n}{n-1}} \leq 1,$$

whence we get

$$\frac{|\mathcal{A}_n|}{\binom{n}{n}} + \frac{|\mathcal{A}_{n-1}|}{\binom{n}{n-1}} \leq 1,$$

by local LYM. We now continue again. Notice $\partial(\partial \mathcal{A}_n \cup \mathcal{A}_{n-1})$ is disjoint from $\mathcal{A}_{n-2}$, we find

$$\frac{|\partial(\partial \mathcal{A}_n \cup \mathcal{A}_{n-1})|}{\binom{n}{n-2}} + \frac{|\mathcal{A}_{n-2}|}{\binom{n}{n-2}} \leq 1,$$

whence

$$\frac{|\partial \mathcal{A}_n \cup \mathcal{A}_{n-1}|}{\binom{n}{n-1}} + \frac{|\mathcal{A}_{n-2}|}{\binom{n}{n-2}} \leq 1.$$

We can now continue inductively.

When do we have equality? We must have had equality in each use of local LYM. Hence equality in LYM needs that the maximum r with $\mathcal{A}_r \neq \emptyset$, then $\mathcal{A}_r = X^{(r)}$.

Hence equality in Sperner needs either $\mathcal{A} = X^{(n/2)}$, if n is even, or $\mathcal{A} = X^{(\lfloor n/2 \rfloor)}$ or $X^{(\lceil n/2 \rceil)}$, for n odd.

Now time for another proof.

Proof: Choose uniformly at random a maximal chain $\mathcal{C}$. For any r-set A, note that
$$\mathbb{P}(A \in \mathcal{C}) = \frac{1}{\binom{n}{r}}.$$
So for our antichain $\mathcal{A}$,
$$\mathbb{P}(\mathcal{C} \text{ meets } \mathcal{A}_r) = \frac{|\mathcal{A}_r|}{\binom{n}{r}},$$
as these events are disjoint. Hence, since $\mathcal{C}$ can meet $\mathcal{A}$ at one point at most,
$$\mathbb{P}(\mathcal{C} \text{ meets } \mathcal{A}) = \sum_{r=0}^{n} \frac{|\mathcal{A}_r|}{\binom{n}{r}},$$
from which we get
$$\sum_{r=0}^{n} \frac{|\mathcal{A}_r|}{\binom{n}{r}} \leq 1.$$

Equivalently, the number of maximal chains is $n!$, and the number through any fixed r-set is $r!(n-r)!$, so
$$\sum_r |\mathcal{A}_r| r!(n-r)! \leq n!.$$

We now return to shadows. For $\mathcal{A} \subseteq X^{(r)}$, we have
$$|\partial \mathcal{A}| \geq |\mathcal{A}| \frac{r}{n-r+1}.$$

We know that equality is rare: it only happens for $\mathcal{A} = \emptyset$, or $X^{(r)}$. What happens in between?

In other words, given $|\mathcal{A}|$, how should we choose $\mathcal{A} \subseteq X^{(r)}$ to minimise $|\partial \mathcal{A}|$?

It is believable that if $|\mathcal{A}| = \binom{k}{r}$, then we should take $\mathcal{A} = [k]^{(r)}$. In between adjacent binomials, it is believable that we should take $[k]^{(r)}$, plus some r-sets in $[k+1]^{(r)}$.

Example 1.1.

For $\mathcal{A} \subseteq X^{(3)}$ with
$$|\mathcal{A}| = \binom{8}{3} + \binom{4}{2},$$
we could take
$$\mathcal{A} = [8]^3 \cup \{9 \cup B \mid B \in [4]^{(2)}\}.$$

In some ways our set $\mathcal{A}$ should be of minimal 'order', under some ordering on $X^{(r)}$.

1.2 Total Orders

Let A, B be distinct r-sets, and say $A = a_1 \dots a_r$, $B = b_1 \dots b_r$, where $a_1 < \dots < a_r$, $b_1 < \dots < a_r$.

We say that $A < B$ in the *lexographic* or *lex* ordering if for some j we have $a_i = b_i$ for all $i < j$, and $a_j < b_j$. So lex cares about small elements.

Example 1.2.

Lex on $[4]^{(2)}$ orders the elements as $12, 13, 14, 23, 24, 34$.

Lex on $[6]^{(3)}$ orders the elements as
$$123,124,125,126,134,135,136,145,146,156,$$
$$234,235,236,245,246,256,345,346,356,456.$$

We say that $A < B$ in the *colexographic* or *colex* ordering if for some j, we have $a_i = b_i$ for all $i > j$, and $a_j < b_j$. So colex cares about big elements.

Example 1.3.

Colex on $[4]^{(2)}$ orders the elements as $12, 13, 23, 14, 24, 34$.

Colex on $[6]^{(3)}$ orders the elements as
$$123,124,134,234,125,135,235,145,245,345,$$
$$126,136,236,146,246,346,156,256,356,456.$$

Note that in colex, $[n-1]^{(r)}$ is an initial segment of $[n]^{(r)}$. This is not true in lex.

This allows us to view colex as an enumeration of $\mathbb{N}^{(r)}$.

Remark. $A < B$ in colex $\iff$ $A^c < B^c$ in lex, with ground set ordering ordering reversed.

Colex in particular may be the ordering we want to solve the above problem, minimizing $|\partial\mathcal{A}|$. Our aim will then be to show that initial segments of colex are the best for ∂, i.e. if $\mathcal{A} \subseteq X^{(r)}$ and $\mathcal{C} \subseteq X^{(r)}$ is the initial segment of colex with $|\mathcal{C}| = |\mathcal{A}|$, then

$$|\partial\mathcal{C}| \leq |\partial\mathcal{A}|.$$

In particular,

$$|\mathcal{A}| = \binom{k}{r} \implies |\partial\mathcal{A}| = \binom{k}{r-1}.$$

1.3 Compression

The idea is to try to transform $\mathcal{A} \subseteq X^{(r)}$ into some $\mathcal{A}\ \subseteq X^{(r)}$ such that:

(i) $|\mathcal{A}'| = |\mathcal{A}|$.

(ii) $|\partial\mathcal{A}'| \leq |\partial\mathcal{A}|$.

(iii) $\mathcal{A}'$ looks more like $\mathcal{C}$ than $\mathcal{A}$ did.

Ideally, we would like a family of such 'compressions'

$$\mathcal{A} \to \mathcal{A}' \to \cdots \to \mathcal{B},$$

such that either $\mathcal{B} = \mathcal{C}$, or $\mathcal{B}$ is so similar to $\mathcal{C}$ that we can directly check that

$$|\partial\mathcal{B}| \geq |\partial\mathcal{C}|.$$

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