III Symmetries, Particles and Fields

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Contents

1 Introduction to Symmetries

Recall Newton's second law:

$$m\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = \mathbf{F}(\mathbf{x}).$$

This simplifies if we know F is rotationally symmetric, i.e. $\mathbf{F}(\mathbf{x}) = F(r)\hat{\mathbf{r}}$. Then $\mathbf{L} = \mathbf{x} \times \mathbf{p}$ is conserved, and trajectories lie in planes containing the origin.

Now consider Lagrangian mechanics, with Lagrangian $L(q_i, \dot{q}_i, t)$. The principle of least action says

$$S = \int_{t_1}^{t_2} \mathrm{d}t\, L(q_i(t), \dot{q}_i(t), t)$$

is minimized by classical trajectories. Hence Euler-Lagrange gives

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0.$$

Nöether's theorem says that invariance of L under some coordinate transform corresponds to an associated conserved quantity.

Example 1.1.

Consider a particle in three dimension, with a potential:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z),$$

which is independent of t, hence invariant under $t \mapsto t + \delta t$. This implies that the Hamiltonian $H = T + U$ is conserved. If we transform into canonical momenta $p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i$, then

$$H(x_i, p_i, t) = \sum \dot{x}_i p_i - L = \sum \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} - L$$

is invariant by Euler-Lagrange:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \sum \ddot{x}_i \frac{\partial L}{\partial \dot{x}_i} - \sum x_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \dot{x}_i \frac{\partial L}{\partial x_i} - \ddot{x}_i \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial t} = 0.$$

If L is invariant under $x \mapsto x + \delta x$, then

$$\frac{\partial L}{\partial x} = 0 \implies \frac{\partial L}{\partial \dot{x}} = p_x = \text{constant}.$$

If L is invariant under rotations about the z-axis, then the z-component of angular momentum, $xp_y - yp_x$, is constant. The best way to see is transform L into cylindrical coordinates:

$$L = \frac{1}{2}m(\dot{\rho}^2 - \rho^2\dot{\theta} + \dot{z}^2) - U(\rho, z).$$

So the invariance under rotations means

$$\frac{\partial L}{\partial \theta} = 0 \implies \frac{\partial L}{\partial \dot{\theta}} = 0 = m\rho^2\dot{\theta} = xp_y - yp_x$$

is constant.

1.1 Symmetries in Quantum Mechanics

Given a system whose states are element of a Hilbert space $\mathcal{H}$, a symmetry means there exists some invertible operator U acting on $\mathcal{H}$ which preserves inner products, up to an overall phase $e^{i\phi}$.

Definition 1.1. Let $|\psi\rangle, |\phi\rangle$ be any normalized vectors in $\mathcal{H}$. Denote $|U\psi\rangle = U|\psi\rangle$, and $|U\phi\rangle = U|\phi\rangle$.

U is a *symmetry transformation* if

$$|\langle U\phi|U\psi\rangle| = |\langle\phi|\psi\rangle|.$$

Proposition 1.1 (Wigner's Theorem)**.** *Symmetry transformation operators are either linear and unitary, or antilinear and antiunitary.*

Antilinear and antiunitary means

$$U(a|\psi\rangle + \beta|\phi\rangle) = a^*U|\psi\rangle + b^*U|\phi\rangle,$$

$$\langle U\phi|U\psi\rangle = \langle\phi|\psi\rangle^*.$$

Suppose we have a system with a time-independent Hamiltonian. Then we can write down

$$\langle\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle,$$

by Schrödinger's equation with $\hbar = 1$. In the first case, note

$$\begin{aligned}\langle U\phi|U\psi(t)\rangle &= \langle\phi|\psi(t)\rangle \\ &= \langle\phi|e^{-iHt}|\psi(0)\rangle.\end{aligned}$$

We should find the same result by transforming $|\psi(0)\rangle$ before time evolution:

$$\begin{aligned}\langle U\phi|U\psi(t)\rangle &= \langle U\phi|e^{-iHt}|U\psi(0)\rangle \\ &= \langle \phi|U^\dagger e^{-iHt}U|\psi(0)\rangle\,.\end{aligned}$$

Equating these, we find

$$U^\dagger e^{-iHt}U = e^{-iHt} \implies [U,H]=0.$$

Example 1.2.

If H commutes with $\mathbf{p}$, then H cannot depend on $\mathbf{x}$, as

$$[x_i,p_j] = i\delta_{ij} \neq 0$$

generally. So H is invariant under translation $\mathbf{x} \mapsto \mathbf{x}+\mathbf{a}$, and this is generated by unitary operators $\exp(i\mathbf{p}\cdot\mathbf{a})$.

If H is rotationally symmetric, then any momentum operator $\mathbf{J}$ or $\mathbf{L}$ commutes with H.

2 Lie Groups and Algebras

2.1 Lie Groups

Recall the definition of a group: a set together with a relation which has an identity, inverses and is associative.

Also recall a group is abelian if $g \cdot h = h \cdot g$ for all $g, h \in G$.

Definition 2.1. A *manifold* is a space which looks Euclidean, like $\mathbb{R}^n$, on small scales, in small neighbourhoods.

A *differentiable manifold* is one which satisfies certain smoothness conditions.

Definition 2.2. A *Lie group* consists of a differentiable manifold G along with a binary operation $\cdot$, such that the group axioms hold, and that $\cdot$ and inverse are smooth operations.

2.2 Matrix Lie Groups

For example, the general linear group $\mathsf{GL}(n, \mathbb{F})$ is the group of invertible $n \times n$ matrices over a field $\mathbb{F}$. So,

$$\mathsf{GL}(n, \mathbb{F}) = \{M \in \mathrm{Mat}_n(\mathbb{F}) \mid \det M \neq 0\}.$$

The group operation is simply matrix multiplication.

The dimension of $\mathsf{GL}(n, \mathbb{R})$ is n^2, as there are n^2 free parameters. For $\mathsf{GL}(n, \mathbb{C})$, we have real dimension $2n^2$, and complex dimension n^2.

Important subgroups of $\mathsf{GL}(n, \mathbb{F})$ are:

- The special linear group

  $$\mathsf{SL}(n, \mathbb{F}) = \{M \in \mathsf{GL}(n, \mathbb{F}) \mid \det M = 1\}.$$

- $\mathsf{SL}(n, \mathbb{R})$ has dimension $n^2 - 1$.
- The orthogonal group

  $$\mathsf{O}(n) = \{M \in \mathsf{GL}(n, \mathbb{R}) \mid M^T M = I\}.$$

  This implies $\det M = \pm 1$. We can also define

  $$\mathsf{SO}(n) = \{M \in \mathsf{O}(n) \mid \det M = 1\}.$$

- Pseudo-orthogonal group. Define an $(n+m) \times (n+m)$ matrix by

$$\eta = \begin{pmatrix} I_n & 0 \\ 0 & -I_M \end{pmatrix}.$$

Then we can define

$$\mathsf{O}(n,m) = \{M \in \mathsf{GL}(n+m,\mathbb{R}) \mid M^T \eta M = \eta\}.$$

Note $M \in \mathsf{SO}(n,m) \iff \det M = 1$.

- Unitary.

$$\mathsf{U}(n) = \{M \in \mathsf{GL}(n,\mathbb{C}) \mid M^\dagger M = I\}.$$

Similarly have $\mathsf{SU}(n)$.

- Pseudounitary.

$$\mathsf{U}(n,m) = \{M \in \mathsf{GL}(n,\mathbb{C}) \mid M^\dagger \eta M = \eta\}.$$

- Symplectic group. Define a fixed, antisymmetric $2n \times 2n$ matrix, e.g.

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Then,

$$\mathsf{Sp}(2n,\mathbb{R}) = \{M \in \mathsf{GL}(2n,\mathbb{R}) \mid M^T \Omega M = \Omega\}.$$

We can show that $\det M = 1$ using the Pfaffian.

Definition 2.3. Given a $2n \times 2n$ antisymmetric matrix A, its *Pfaffian* is given by

$$\mathrm{Pf} A = \frac{1}{2^n n!} \varepsilon_{i_1 i_2 \cdots i_{2n}} A_{i_1 i_2} A_{i_3 i_4} \cdots A_{i_{2n-1} i_{2n}}.$$

2.3 Group Elements as Transformations

We can define actions of group elements $g \in G$ on a set X.

Definition 2.4. The *left action* of G on X is a map $L : G \times X \to X$ such that $L(e,x) = x$, and

$$L(g_2, L(g_1, x)) = L(g_2 g_1, x),$$

for all $x \in X$ and $g_1, g_2 \in G$. In more usual notation, for all $g \in G$, we can associate a map $g : X \to X$ as $g(x) = gx$.

Definition 2.5. The *right action* of G on X is defined by $g : X \to X$ such that $g(x) = xg^{-1}$, for all $x \in X$, $g \in G$. The inverse preserves under composition, so

$$g_2(g_1(x)) = xg_1^{-1}g_2^{-1} = (g_2g_1)(x).$$

Definition 2.6. The action of *conjugation* by G on X is the action defined by

$$g(x) = gxg^{-1},$$

for $g \in G$, $x \in X$.

Definition 2.7. Given a group G and a set X, an *orbit* of an element $x \in X$ is the set of elements of X in the image of G.

Example 2.1.

If the action is a left action, then the orbit of $x \in X$ is

$$Gx = \{gx \mid g \in G\}.$$

It can be shown that the set of orbits under G partition X.

In $\mathbb{R}^n$, orthogonal matrices $\mathsf{O}(n)$ represent rotations and reflections, and preserve the inner product; similarly for $\mathsf{U}(n)$.

We can parametrize $\mathsf{SO}(2)$ as

$$\mathsf{SO}(2) = \left\{ R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \,\middle|\, \theta \in [0, 2\pi] \right\}.$$

cos, sin are smooth. We can show that $R(\theta_2)R(\theta_1) = R(\theta_1 + \theta_2)$.

$\mathsf{SO}(3)$ gives rotations of vectors in $\mathbb{R}^3$. The axis of rotation is given by a unit vector $\mathbf{n} \in S^2$, and we also have an angle θ.

Note that rotation by $\theta \in [-\pi, 0]$ about $\mathbf{n}$ is equivalent to rotation by $-\theta$ about $-\mathbf{n}$, so we can confine $\theta \in [0, \pi]$.

Hence we can depict the manifold of $\mathsf{SO}(3)$ as a ball of radius π in $\mathbb{R}^3$, where antipodal points are identified: $\pi\mathbf{n} = -\pi\mathbf{n}$.

The *pseudo-orthogonal group* $\mathsf{SO}(n, m)$ act on vectors in $\mathbb{R}^{n+m}$, and preserve the scalar product $v_2^T \eta v_1$ for $v_1, v_2 \in \mathbb{R}^{n+m}$.

For example,

$$\mathsf{SO}(1, 1) = \left\{ \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix} \,\middle|\, \psi \in \mathbb{R} \right\}.$$

$\mathsf{SO}(1, 1)$ is an example of a non-compact group.

2.4 Parametrization of Lie Groups

At least in small neighbourhoods, we can assign coordinates

$$x = (x^1, \dots, x^n) \in \mathbb{R}^n,$$

such that $g(x) \in G$. Closure says that $g(y)g(x) = g(z)$, and smoothness says that the components of z are continuously differentiable functions of x and y, so

$$z^n = \phi^n(x, y).$$

We can choose the coordinates at the origin such that $g(0) = e$. Then $g(0)g(x) = g(x)$, so

$$\phi^r(x, 0) = x^r, \qquad \phi^r(0, y) = y^r.$$

From inverses, for each x there exists $\bar{x}$ such that $g(\bar{x}) = g(x)^{-1}$, hence

$$\phi^r(\bar{x}, x) = 0 = \phi^r(x, \bar{x}).$$

Finally, associativity means $g(z)[g(y)g(x)] = [g(z)g(y)]g(x)$, hence

$$\phi^r(\phi(x, y), z) = \phi^r(x, \phi(y, z)).$$

2.5 Lie Algebras

Lie groups are hard to quantify. Instead, we look at lie algebras, which are a linearization of the lie group.

A lie group is homogeneous: any neighbourhood can be mapped to any other neighbourhood. We will linearize near the identity of G.

Definition 2.8. A *Lie algebra* is a vector space V, which additionally has a vector product, the *Lie bracket* $[\cdot, \cdot] : V \times V \to V$ possessing the following properties: for $X, Y, Z \in V$,

1. antisymmetry: $[X, Y] = -[Y, X]$.
2. Jacobi identity: $[X, [Y, Z]] + [Y, [X, Z]] + [Z, [X, Y]] = 0$.
3. linearity: for $\alpha, \beta \in \mathbb{F}$, $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$.

Remark. Any vector space which has a vector product $*$ can be made into a Lie algebra with Lie bracket

$$[X, Y] = X * Y - Y * X.$$

Given a Lie algebra V, choose a basis $\{T_a\}$. The basis vectors are called the *generators* of the Lie algebra.

Write their Lie brackets as

$$[T_a, T_b] = f^c{}_{ab} T_c,$$

with $f^c{}_{ab} \in \mathbb{F}$ called the *structure constants*. The properties imply:

- antisymmetry $\implies f^c{}_{ba} = -f^c{}_{ab}$.
- Jacobi $\implies f^e{}_{ad} f^d{}_{bc} + f^e{}_{cd} f^d{}_{ab} + f^e{}_{bd} f^d{}_{ca} = 0$.

General elements of Lie algebras are linear combinations of $\{T_a\}$. So $X \in V$ can be written as $X^a T_a$, where $X^a \in \mathbb{F}$, and

$$[X, Y] = X^a Y^b f^c{}_{ab} T_c.$$

2.6 Lie Groups and their Lie Algebras

We start with $\mathsf{SO}(2)$, where

$$g(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The identity is $e = I_2 = g(0)$. Near the identity, θ is small, and

$$\sin\theta = \theta - \frac{\theta^3}{3} + \cdots, \qquad \cos\theta = 1 - \frac{\theta^2}{2} + \cdots$$

Hence,

$$g(\theta) = I_2 + \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \frac{\theta^2}{2} I_2 + \mathcal{O}(\theta^3) = e + \theta \left.\frac{\mathrm{d}g}{\mathrm{d}\theta}\right|_0 + \mathcal{O}(\theta^2).$$

The linear term is the “tangent” to the manifold. We have a one-dimensional tangent space at e, and we claim that this is the Lie algebra of $\mathsf{SO}(2)$, i.e.

$$L(\mathsf{SO}(2)) = T_e(\mathsf{SO}(2)) = \left\{ \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \,\middle|\, a \in \mathbb{R} \right\}.$$

For $\mathsf{SO}(n)$, we can show the dimension is $\frac{n(n-1)}{2} = d$. Choose coordinates $x_1, \ldots, x_d$, and consider a single-parameter family of $\mathsf{SO}(n)$ elements

$$M(t) = M(x(t)) \in \mathsf{SO}(n),$$

such that $M(0) = I_n$. Then,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(M^T(t)M(t)) = \frac{\mathrm{d}M^T}{\mathrm{d}t}M + M^T\frac{\mathrm{d}M}{\mathrm{d}t}.$$

Looking at $t = 0$, we find

$$\frac{\mathrm{d}M^T}{\mathrm{d}t} = -\frac{\mathrm{d}M}{\mathrm{d}t},$$

hence matrices in the tangent space are anti-symmetric. Moreover they are also traceless.

For unitary groups, we again let $M(t)$ be a curve in $\mathsf{SU}(n)$ with $M(0) = I$. For small t, write

$$M(t) = I + tX + \mathcal{O}(t^2).$$

From unitarity, $I = M^\dagger M$, so looking at the expansion,

$$I = I + t(X + X^\dagger) + \mathcal{O}(t^2),$$

hence $X^\dagger = -X$, is anti-hermitian. We also claim X is traceless for $\mathsf{SU}(n)$. Indeed, looking at $\det M$, its expansion is

$$1 = \det M = 1 + t\,\mathrm{Tr}(X) + \mathcal{O}(t^2).$$

2.7 Lie Algebras of a Matrix Lie Group

Consider two curves through the identity e of some Lie group, $g_1(x(t))$ and $g_2(y(t))$, with $X_1 = \dot{g}_1|_0$, $X_2 = \dot{g}_2|_0$. The product is

$$g_3(z(t)) = g_2(y(t))g_1(x(t)) \in G.$$

Then,

$$\dot{g}_3|_0 = (\dot{g}_1 g_2 + g_1\dot{g}_2)|_0 = X_1 + X_2 \in T_e(G).$$

The Lie bracket arises from the group commutator.

Definition 2.9. The *group commutator* of $g_1, g_2 \in G$ is

$$[g_1, g_2] = g_1^{-1}g_2^{-1}g_1g_2 \in G.$$

Let $g_1(t), g_2(t)$ be two curves through the identity, and

$$g_i(t) = c + tX_i + t^2W_i + \mathcal{O}(t^3).$$

Then,

$$g_1(t)g_2(t) = e + t(X_1 + X_2) + t^2(X_1X_2 + W_1 + W_2) + \mathcal{O}(t^3),$$
$$g_2(t)g_1(t) = e + t(X_1 + X_2) + t^2(X_2X_1 + W_1 + W_2) + \mathcal{O}(t^3).$$

Therefore,

$$h(t) = [g_2(t)g_1(t)]^{-1}g_1(t)g_2(t) = e + t^2[X_1, X_2] + \cdots .$$

So if $h(t) \in G$, then the tangent to $h(t)$ at e is $[X_1, X_2] \in L(G)$.

Now we can think of tangent spaces to $G \subseteq \mathsf{GL}(n, \mathbb{F})$ at a general element p, $T_p(G)$.

Let $g(t)$ be a curve in the manifold through p with $g(t_0) = p$, so

$$g(t_0 + \varepsilon) = g(t_0) + \varepsilon\dot{g}(t_0) + \mathcal{O}(\varepsilon^2).$$

Both $g(t_0)$ and $g(t_0 + \varepsilon)$ are in G, so there exists $h_p(\varepsilon) \in G$ such that

$$g(t_0 + \varepsilon) = g(t_0)h_p(\varepsilon),$$

where $h_p(0) = e$. For small ε,

$$h_p(\varepsilon) = e + \varepsilon X_p + \mathcal{O}(\varepsilon^2)$$

for some $X_p \in L(G) = T_e(G)$. Neglecting higher order terms,

$$\begin{aligned} e + \varepsilon X_p = h_p(\varepsilon) &= g(h_0)^{-1}g(t_0 + \varepsilon) \\ &= g(t_0)^{-1}[g(t_0) + \varepsilon\dot{g}(t_0)] \\ &= e + \varepsilon g(t_0)^{-1}\dot{g}(t_0), \end{aligned}$$

where $g(t)^{-1}\dot{g}(t) = X_p \in L(G)$.

Conversely, for any $X \in L(G)$, there exists a unique curve $g(t)$ with $g^{-1}(t)\dot{g}(t) = X$, and $g(0) = g_0$. This is a consequence of the existence and uniqueness of solutions to ODEs. The solution of this ODE is

$$g(t) = g_0 \exp tX,$$

where

$$\exp tX = \sum_{k=0}^{\infty} \frac{1}{k!}(tX)^k.$$

Given an $X \in L(G)$, the curve

$$g_X(t) = \exp tX,$$

which forms an abelian subgroup of G generated by X. Note $g_x(t)$ is isomorphic to $(\mathbb{R}, +)$ if only $g_x(0) = e$, and S^1 if $g_x(t_0) = e$ for some $t_0 \neq 0$.

2.8 Lie Groups from Lie Algebras

Given a Lie algebra $L(G)$ of a Lie group G, we can define the *exponential map*

$$\exp : L(G) \to G.$$

For matrix Lie groups,

$$X \mapsto \exp X = \sum_{k=0}^{\infty} \frac{X^k}{k!}.$$

Locally, the map is bijective, but globally it may not be.

Example 2.2.

Recall the group

$$\mathsf{U}(1) = \{e^{i\theta} \mid \theta \in [0, 2\pi)\}.$$

The Lie algebra is

$$L(\mathsf{U}(1)) = \{ix \mid x \in \mathbb{R}\}.$$

The exponential is not one-to-one, since $e^{2\pi n i} = 1$.

Another example is $G = \mathsf{O}(n)$. Let $X \in L(O(n)) \in \mathrm{Skew}_n(\mathbb{R})$.

Let $M = \exp tX$, then

$$M^T = [\exp tX]^T = \exp(-tX),$$

so $MM^T = I = M^T M$, hence $M \in O(n)$.

Suppose that $\operatorname{Tr} X = 0$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues. Then,

$$\det M = \det(\exp tX) = \exp(t(\lambda_1 + \cdots + \lambda_n)) = \exp(t \operatorname{Tr} X) = 1.$$

So $M \in \mathsf{SO}(n)$. Thus elements of $\mathsf{O}(n)$ with determinant -1 are not in the image of the exp map. We can think of $\mathsf{O}(n)$ as a disconnected manifold.

We can show that if $A \in \mathrm{Skew}_n(\mathbb{R})$, then $A \in L(\mathsf{SO}(n))$ or $L(\mathsf{O}(n))$.

Define $\gamma(t) = \exp tA$, a curve of matrices on some manifold. From the above,

$$(\gamma(t))^T(\gamma(t)) = I, \qquad \det \gamma(t) = 1,$$

so $\gamma(t) \in \mathsf{SO}(n)$. By construction, $A = \gamma(t)|_0$, the tangent to the curve at the identity. So $A \in L(\mathsf{SO}(n))$, hence

$$\dim \mathrm{SO}(n) = \dim L(\mathrm{SO}(n)) = \dim \mathrm{Skew}_n(\mathbb{R}) = \frac{n(n-1)}{2}.$$

We can also determine the group product from the Lie bracket.

Lemma 2.1 (Baker-Campbell-Hausdorff Formula). *For $X, Y \in L(G)$, we have* $\exp tX \exp tY = \exp tZ$, *where*

$$Z = X + Y + \frac{t}{2}[X, Y] + \frac{t^2}{12}([X, [X, Y]] + [Y, [X, Y]]) + \mathcal{O}(t^3).$$

We can show this order-by-order in t. Since $L(G)$ is closed under the Lie bracket, $Z \in L(G)$, so $\exp tZ \in G$.

3 Representations

Groups are transformations under which some things are invariant.

Representations are how group actions transform vectors in a vector space.

An example is $\mathsf{GL}(n, \mathbb{F})$, the group of invertible matrices. These form linear maps (automorphisms) on the vector space $\mathbb{F}^n$.

3.1 Lie Group Representations

Definition 3.1. A *representation* D of a group G is a smooth group homomorphism

$$D : G \to \mathsf{GL}(V),$$

from G to the group of automorphisms on some vector space V, called the *representation space* associated with D.

That is, for all $g \in G$, $D(g) : V \to V$ is an invertible, linear map such that for a vector $v \in V$,

$$v \mapsto D(g)v.$$

- Linearity: $D(g)(\alpha v_1 + \beta v_2) = \alpha D(g)v_1 + \beta D(g)v_2$.
- Group structure: $D(g_2 g_1) = D(g_2)D(g_1)$.
- Identity: $D(0) = \mathrm{id}_V$.
- Inverses: $D(g)^{-1} = D(g^{-1})$.

Definition 3.2. The *dimension* of a representation D is the dimension of its vector space. If V is finite-dimensional, say $\dim V = N$, then $\mathsf{GL}(V)$ is isomorphic to $\mathsf{GL}(N, \mathbb{F})$.

Recall that the kernel of a map $D : G \to \mathsf{GL}(V)$ consists of the elements of G which map to the identity id_V.

Definition 3.3. A representation D is *faithful* if $D(g) = \mathrm{id}_V$, only for $g = e$, i.e. $\ker D = \{e\}$.

Faithfulness implies that D is injective.

Example 3.1.

We look at some examples with $G = (\mathbb{R}, +)$.

(i) For some fixed $k \in \mathbb{R}$, $D(x) = e^{k\alpha}$ is a one-dimensional representation,

as
$$D(\alpha)D(\beta) = e^{k\alpha}e^{k\beta} = e^{k(\alpha+\beta)} = D(\alpha+\beta).$$
For $k \neq 0$, this is faithful, as $D(\alpha) = 1 \implies \alpha = 0$. For $k = 0$, $D(\alpha) = 1$ for all α, so $\ker D = G$. This is the trivial representation.

(ii) Another one-dimensional representation is $D(\alpha) = e^{ik\alpha}$. This is again not faithful.

(iii) A two-dimensional representation is
$$D(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

(iv) Let V be the space of all functions, and define
$$D(\alpha)f(x) = f(x-\alpha).$$
This is an infinite-dimensional representation, which is faithful, as $f(x) = f(x-\alpha)$ for all f implies $\alpha = 0$.

Definition 3.4. The *trivial representation* D_0 is where
$$D_0(g) = 1,$$
for all $g \in G$.

Quantities which are invariant under group transformation transform in the trivial representation.

We can also form a trivial representation for any dimension, by $D(g) = I_m$ for all g. This is reducible; it is M copies of the one-dimensional trivial representation.

Definition 3.5. If G is a matrix Lie group, then the *fundamental* or *defining representation* D_f is given by
$$D_f(g) = g,$$
for all $g \in G$.

This is clearly faithful, and if $G \subseteq \mathsf{GL}(n, \mathbb{F})$, then $\dim D_f = n$.

Let G be a matrix Lie group and consider its Lie algebra as a vector space $V = L(G)$.

Definition 3.6. The *adjoint representation* $D^{\text{adj}} = \text{Ad}$ is the map $\text{Ad} : G \to \mathsf{GL}(L(G))$ such that for all $g \in G$, $\text{Ad}_g : L(G) \to L(G)$ by
$$\text{Ad}_g X = gXg^{-1}$$

for all $X \in L(G)$. This is just action by conjugation.

We can check that this satisfies the group operations, and the Lie bracket satisfies

$$\mathrm{Ad}_g([X,Y]) = g[X,Y]g^{-1} = [gXg^{-1}, gYg^{-1}] = [\mathrm{Ad}_g X, \mathrm{Ad}_g Y].$$

3.2 Lie Algebra Representation

Definition 3.7. A *representation* d of a Lie algebra $L(G)$ is a linear map from $L(G)$ to $\mathfrak{gl}(V)$, which is the Lie algebra of $\mathsf{GL}(V)$, that preserves the Lie bracket.

That is, for each $X \in L(G)$, $d(X) : V \to V$ is a linear map such that

$$v \mapsto d(X)v,$$

for $v \in V$, such that

$$d([X,Y]) = [d(X), d(Y)].$$

The dimension of this d is then $\dim V$.

Definition 3.8. The trivial representation is

$$d_0(X) = 0.$$

Definition 3.9. The fundamental representation is

$$d_f(X) = X.$$

Definition 3.10. The adjoint representation is

$$\mathrm{ad} : L(G) \to \mathfrak{gl}(L(G)),$$

such that for all $X \in L(G)$, $\mathrm{ad}_X : L(G) \to L(G)$ such that

$$\mathrm{ad}_X Y = [X,Y].$$

We can get representations of $L(G)$ from representations of G.

As before, consider the tangent cuves in G:

$$g(t) = e + tX + \cdots .$$

Expand corresponding elements of our representation D of G as

$$D(g(t)) = \mathrm{id}_V + td(X) + \cdots .$$

Then we can use this expansion to define d from D. We can check the Lie bracket works. We know

$$D(g_1^{-1}g_2^{-1}g_1g_2) = D(g_1)^{-1}D(g_2)^{-1}D(g_1)D(g_2).$$

Expanding $g_1(t) = e + TX_1 + \cdots$, $g_2(t) = e + tX_2 + \cdots$, we find the left-hand side gives

$$D(e + t^2[X_1, X_2] + \cdots),$$

and the right-hand side is

$$\mathrm{id}_V + t^2[d(X_1), d(X_2)] + \cdots .$$

Hence we see that

$$d([X_1, X_2]) = [d(X_1), d(X_2)].$$

For example, in the adjoint representation ad_X from Ad_g, for $Y \in L(G)$ note

$$\begin{aligned}\mathrm{Ad}_g Y = gYg^{-1} &= (I + tX)Y(I - tX) + \cdots \\ &= Y + t[X, Y] + \cdots \\ &= (I + t\mathrm{ad}_X + \mathcal{O}(t^2))Y.\end{aligned}$$

So $\mathrm{ad}_X Y = [X, Y]$.

3.3 Useful Concepts

Definition 3.11. Representations D_1 and D_2 of G, or d_1 and d_2 of $L(G)$, are *equivalent* if there exists invertible linear maps R and S such that

$$\begin{aligned}D_2(g) &= RD_1(g)R^{-1}, \\ d_2(X) &= Sd_1(X)S^{-1},\end{aligned}$$

for all $g \in G$ or $X \in L(G)$.

Definition 3.12. A representation d of $L(G)$ with representation space V has an *invariant subspace* $W \subseteq V$ if $d(X)w \in W$, for all $X \in L(G)$ and $w \in W$.

Example 3.2.

If all $d(X)$ are upper-triangular matrices, then $W = \{(z, 0)^T\}$ are invariant.

Definition 3.13. An *irreducible representation* (irrep) is a representation with no nontrivial invariant subspaces. Otherwise the representation is *reducible*.

Definition 3.14. A *direct sum* of vector spaces U and W, is

$$U \oplus W = \{(u, w) \text{ or } u \oplus w \mid u \in U, w \in W\},$$

where $(u_1, w_1) + (u_2, w_2) = (u_1 + u_2, w_1 + w_2)$ and $\alpha(u, w) = (\alpha u, \alpha w)$. Note that

$$\dim U \oplus W = \dim U + \dim W.$$

Definition 3.15. A *totally reducible representation* d of $L(G)$ can be decomposed into irreducible pieces, i.e. can be written as a direct sum with representation space

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k,$$

such that $d(X)w_i \in W_i$ for all $X \in L(G)$. Then there exists some basis where

$$d(X) = \begin{pmatrix} d_1(X) & 0 & \cdots & 0 \\ 0 & d_2(X) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n(X) \end{pmatrix}$$

is a block diagonal.

Definition 3.16. An N-dimensional representation D is *unitary* if $D(g) \in \mathsf{U}(N)$, and $d(X) \in L(\mathsf{U}(N))$.

If all $D(g)$ are real, then $D(g) \in \mathsf{O}(N)$, and D is said to be *orthogonal*.

Theorem 3.1 (Maschke). *A finite-dimensional unitary representation is either irreducible, or totally reducible.*

Proof: Sketch.

We show that for each invariant subspace W, the orthogonal component $W_\perp$ is also invariant, i.e. $V = \oplus W + W_\perp$.

Then we can similarly decompose W and $W_\perp$. As V is finite dimensional, we must terminate.

Maschke's theorem can be extended to:

- All finite representations of discrete groups (by Weyl's trick).
- All finite representations of compact Lie groups.

Example 3.3.

Consider V to be all 2π-periodic functions, and a representation by

$$D(\alpha)f(x) = f(x - \alpha).$$

The following subspaces are invariant:

$$W_n = \{a_n \cos nx + b_n \sin nx \mid a_n, b_n \in \mathbb{R}\},$$

for each $n \in \mathbb{Z}_{\geq 0}$. This is invariant as

$$a_n \cos n(x - \alpha) + b_n \sin n(x - \alpha) = a'_n \cos nx + b'_n \sin nx$$

for some $a'_n, b'_n \in \mathbb{R}$.

Recall the Fourier decomposition of any 2π-periodic function: it can be written as

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

Hence we see

$$V = W_0 \oplus W_1 + W_2 \oplus \cdots = \oplus_{n=0}^{\infty} W_n.$$

This is a direct sum of covariant derivatives, each occurring once.

Definition 3.17. Let V, W be vector spaces. The *tensor product space* $V \otimes W$ is spanned by elements, which are product vectors $v \otimes w$ with $v \in V$, $w \in W$. Here,

$$v \otimes (\lambda_1 w_1 + \lambda_2 w_2) = \lambda_v \otimes w_1 + \lambda_2 v \otimes w_2.$$

Hence, $\dim V \otimes W = (\dim V)(\dim W)$.

An element is in a *product state* if it can be written $\phi = v \otimes w$. In general we can write $\phi_A = \phi_{a\alpha} = v_a w_\alpha$.

Not all elements of $V \otimes W$ are product states.

3.4 Tensor Products Representations

Let $D^{(1)}$ and $D^{(2)}$ be representations of a group G with vector spaces V and W such that

$$D^{(1)}(g) : v_\alpha \mapsto D^{(1)}(g)_{\alpha\beta} v_\beta,$$
$$D^{(2)}(g) : w_\alpha \mapsto D^{(2)}(g)_{\alpha\beta} w_\beta.$$

Then we get a tensor product representation $D^{(1)j \otimes D^{(2)}}$, by

$$(D^{(1)} \otimes D^{(2)})(g)(v \otimes w) = (D^{(1)}(g)v) \otimes (D^{(2)}(g)w).$$

Let $g_t \in G$ be a curve in the Lie group G with $g_0 = e$, and $\dot{g}_0 = X \in L(G)$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(D^{(1)} \otimes D^{(2)})(g_t)v \otimes w\right] = \left[\frac{\mathrm{d}}{\mathrm{d}t}D^{(1)}(g_t)v\right] \otimes w + v \otimes \left[\frac{\mathrm{d}}{\mathrm{d}t}D(g_t)w\right].$$

Let $d^{(1)}$ and $d^{(2)}$ be Lie algebra representations corresponding to $D^{(1)}$ and $D^{(2)}$. Their tensor product is

$$(d^{(1)} \otimes d^{(2)})(X) = d^{(1)}(X) \otimes \mathrm{id}_W + \mathrm{id}_V \otimes d^{(2)}(X).$$

Here is an important corollary to Maschke's theorem: representations of $d^{(1)} \otimes d^{(2)}$ can be, if finite, written as the direct sum of irreps of $L(G)$:

$$d^{(1)} \otimes d^{(2)} = \tilde{d}_1 \oplus \cdots \oplus \tilde{d}_k = \bigoplus_{i=1}^{k} \tilde{d}_i.$$

This is decomposition into irreps. Note that the dimensions must be equal.

4 Angular Momentum

$\mathsf{SO}(3)$ describes rotations in three-dimensions. This implies that angular momentum is quantized in QM.

Sometimes we have half integer quantum numbers, which gives $\mathsf{SU}(2)$ representations.

4.1 Relationships between $\mathsf{SO}(3)$ and $\mathsf{SU}(2)$

Consider the Lie algebra of $\mathsf{SU}(2)$, the 2×2 traceless, anti-Hermitian matrices. We can choose a basis

$$T_a = -\frac{i}{2}\sigma_a,$$

for $a = 1, 2, 3$. These are the *Pauli matrices*. Recall that

$$\sigma_a \sigma_b = I\delta_{ab} + i\varepsilon_{abc}\sigma_c \implies [T_a, T_b] = \varepsilon_{abc}T_c.$$

Hence the structure constants are

$$f^c{}_{ab} = \varepsilon_{abc}.$$

Similarly, $\mathfrak{so}(3) = L(\mathsf{SO}(3))$ are the 3×3 skew matrices, with basis

$$\tilde{T}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \tilde{T}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad \tilde{T}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

i.e. $(\tilde{T}_a)_{bc} = -\varepsilon_{abc}$. Then again

$$[\tilde{T}_a, \tilde{T}_b] = \varepsilon_{abc}\tilde{T}_c,$$

which gives the same structure constants as $\mathfrak{su}(2)$.

To show an isomorphism between two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, we need a linear isomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ such that

$$\phi([X, Y]) = [\phi(X), \phi(Y)],$$

for all $X, Y \in \mathfrak{g}$. However, we find that $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ are not isomorphic.

Let's look at the group manifolds of the Lie groups.

- $\mathsf{SO}(3)$ was discussed earlier. This can be thought of as a 3-ball of radius π, with antipodes identified.

- $\mathsf{SU}(2)$ can be written as
$$U = a_0 \cdot I + i\mathbf{a} \cdot \boldsymbol{\sigma},$$
with $(a_0, \mathbf{a})$ real and $a_0^2 + |\mathbf{a}|^2 = 1$. This manifold is a unit sphere in $\mathbb{R}^4$, i.e. S^3.

Definition 4.1. Let H be a subgroup of G. Then for any $g \in G$, we can form a *left coset* of H as
$$gH = \{gh \mid h \in H\},$$
and the *right coset* as
$$Hg = \{hg \mid h \in H\}.$$

Definition 4.2. $H \lhd G$ is a *normal subgroup* of G, if $gH = Hg$ for all $g \in G$. Define a set G/H to be
$$G/H = \{gH \mid g \in G\}.$$
Define coset multiplication as
$$(g_2H)(g_1H) = (g_2g_1)H.$$

Theorem 4.1. *For $H \lhd G$, G/H is a group under coset multiplication, with $H = eH$ as the identity of G/H. Such a group is called the* quotient group*.*

Definition 4.3. The *centre* of a group is the set of all $x \in G$ which satisfy $xg = gx$ for all $g \in X$.

Theorem 4.2. *The centre $Z(G) \lhd G$ is a normal subgroup of G.*

$\mathsf{SU}(2)$ has centre $Z(\mathsf{SU}(2)) = \{\pm I\}$. The cosets are
$$UZ(\mathsf{SU}(2)) = \{\pm U\}.$$
The set of all such cosets forms the quotient group
$$\mathsf{SU}(2)/\mathbb{Z}_2,$$
whose manifold is S^3 with antipodes identified. The manifold of $\mathsf{SU}(2)/\mathbb{Z}_2$ is just the upper half of S^3, with opposite points on the equator identified. However this can be projected onto $\mathsf{SO}(3)$, showing
$$\mathsf{SO}(3) \cong \mathsf{SU}(2)/\mathbb{Z}_2.$$
We are able to show an explicit map. Define $p : \mathsf{SU}(2) \to \mathsf{SO}(3)$ by, for $A \in \mathsf{SU}(2)$, $p(A) = R$ where
$$R_{ij} = \frac{1}{2}\operatorname{tr}(\sigma_i A \sigma_j A^\dagger).$$
This is a two-to-one map, with $p(-A) = p(A)$. This is a *double cover* of $\mathsf{SO}(3)$. Hence $\mathsf{SU}(2)$ is the double cover of $\mathsf{SO}(3)$.

Proposition 4.1. *Every Lie algebra is the algebra of exactly one simply-connected Lie group.*

For example, $\mathsf{U}(2)$ and $\mathsf{SU}(2)$ have the same Lie algebra, but $\mathsf{U}(2)$ is not connected.

Definition 4.4. A manifold is *simply connected* if any closed loop can be smoothly shrunk to a point, i.e. $\pi_1(X)$ is trivial.

4.2 Representations of $\mathfrak{su}(2)$

It is convenient to enlarge our real vector space to the field $\mathbb{C}$. Given a vector space V, say

$$V = \{\lambda^a T_a \mid \lambda^a \in \mathbb{R}\} = \mathrm{span}_{\mathbb{R}}\{T_a\}.$$

The *complexification* of V is

$$V_{\mathbb{C}} = \mathrm{span}_{\mathbb{C}}\{T_a\}.$$

Example 4.1.

Recall

$$\mathfrak{su}(n) = \{X \in \mathrm{Mat}_n(\mathbb{C}) \mid X^\dagger = -X, \mathrm{Tr}\, X = 0\}.$$

Then the complexification is

$$\mathfrak{su}(n)_{\mathbb{C}} = \{X \in \mathrm{Mat}_n(\mathbb{C}) \mid \mathrm{Tr}\, X = 0\} \cong \mathfrak{sl}(n, \mathbb{C}) = L(\mathsf{SL}(n, \mathbb{C})).$$

Let $\mathfrak{g} = L(G)$ be a real Lie algebra, and denote its complexification by $\mathfrak{g}_{\mathbb{C}}$. Representations of $L(G)$ can be extended to $L(G)_{\mathbb{C}}$ by imposing

$$d(X + iY) = d(X) + id(Y).$$

Conversely, if we have a representation $d_{\mathbb{C}}$ of $L(G)_{\mathbb{C}}$, we can restrict to a representation of $L(G)$ by writing

$$d(X) = d_{\mathbb{C}}(X)$$

for $X \in L(G) \subseteq L(G)_{\mathbb{C}}$.

Definition 4.5. A *real form* of a complex Lie algebra $\mathfrak{h}$ is a real Lie algebra $\mathfrak{g}$ whose complexification is $\mathfrak{h}$.

In general, a complex Lie algebra can have multiple nonisomorphic real forms. For $L(\mathsf{SU}(2))$,

$$L(\mathsf{SU}(2))_{\mathbb{C}} = \mathrm{span}_{\mathbb{C}}\{\sigma_a \mid a = 1, 2, 3\}.$$

A more conveniant basis is

$$\begin{aligned} H &= \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ E_+ &= \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ E_- &= \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Then the commutation relations are

$$[H, E_\pm] = \pm 2E\pm, \qquad [E_+, E_-] = H.$$

This is the Canton-Weyl basis.

Recall that $\mathrm{ad}_X Y = [X, Y]$, so $[H, E_\pm] = \mathrm{ad}_H E_\pm = \pm 2E_\pm$. We also have that $\mathrm{ad}_H H = [H, H] = 0$.

We see that E_-, H, E_+ are eigenvectors of ad_H with eigenvalues $-2, 0, 2$. These eigenvalues are the *roots* of $L(\mathsf{SU}(2))$.

Let d be a finite dimensional irreducible representation (an irrep) of $\mathsf{SU}(2)$ with representation space V. Write an eigenvector of $d(H)$ as v_λ, where

$$d(H)v_\lambda = \lambda v_\lambda.$$

Definition 4.6. The eigenvectors of $d(H)$ are the *weights* of d.

The operators $d(E_\pm)$ are *ladder* or *step* or *raising* or *lowering* operators, as

$$d(H)(d(E_\pm)v_\lambda) = (\lambda \pm 2)(d(E_\pm)v_\lambda).$$

Hence $d(E_\pm)v_\lambda$ is an eigenvector of $d(H)$ with eigenvalue $\lambda \pm 2$, or $d(E_\pm)v_\lambda = 0$.

If d is a finite dimensional representation, then there is a finite number of eigenvalues. Hence there must be some Λ such that

$$d(H)v_\Lambda = \Lambda v_\Lambda \qquad \text{and} \qquad d(E_+)v_\Lambda = 0.$$

Such a Λ is called a *highest weight*. Applying $d(E_-)$ n times,

$$v_{\Lambda - 2n} = (d(E_-))^n v_\Lambda.$$

This process must terminate for some N, so the eigenbasis for this representation is

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