III Algebraic Geometry

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Based on Lectures by Prof. Mark Gross

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Contents

0 Introduction

Introductory reading by Hassett and Reid.

More commutative algebra by Atiyah and Macdonald, and Matsumura.

Standard AG texts by Hartshorne, Görtz-Wedhorn, and Ravi Vakil.

0.1 Recap

In undergraduate, we fix an algebraically closed field K, and define affine n-space $\mathbb{A}^n = K^n$, and for an ideal $I \subseteq K[x_1, \dots, x_n]$ we define

$$Z(I) = \{(a_1, \dots, a_n) \in \mathbb{A}^n \mid f(a_1, \dots, a_n) = 0 \text{ for all } f \in I\} \subseteq \mathbb{A}^n.$$

We can define a topology on $\mathbb{A}^n$ by taking the closed sets to be the sets of the form $Z(I)$.

This turns out to be no good. We will instead introduce schemes. Natural questions are; why schemes, and why not varieties? Well,

- With varieties, we always work with algebraically closed fields, to relate the algebra with the geometry. If $K = \mathbb{R}$, and $I = (x^2 + y^2 + 1) \subseteq K[x, y]$, then $Z(I) = \emptyset$, losing information about I.
- We may be interested in Diophantine equations, where the natural space is $\mathbb{Z}$.
- Even if K is algebraically closed, we lose information passing from I to $Z(I)$. For example if $I = (x^2)$, $Z = \{0\}$, then $I(Z(I)) = (x)$.

  But it is natural to consider ideals like (x^2), for example considering $(y - x^2, y - \alpha) \subseteq \mathbb{C}[x, y]$. This produces two points for $\alpha \neq 0$, but only one point if $\alpha = 0$, but with some multiplicity.

0.2 Categorical Philosophy

Let **Set** be the category of sets. **Set** is the category with objects being all sets with morphisms between objects being maps of sets. If X, Y are sets, we write $\mathrm{Hom}(X, Y)$ for the set of maps between X and Y.

Note that there is a bijection $\mathrm{Hom}(\{*\}, X) \to X$ given by $(f : \{*\} \to X) \mapsto f(*)$.

We can use this philosophy to understand points on affine algebraic varieties. Nota $\mathbb{A}^0$ is a point. If X is an affine variety, then the points of X should be in one-to-one correspondence with $\mathrm{Hom}(\mathbb{A}^0, X)$.

Recall morphisms of affine varieties. Denote $A(X)$ by $K[x_1, \ldots, x_n]/I(X)$, where $I(X) = \{f \in K[x_1, \ldots, x_n] \mid f|_X = 0\}$. $A(X)$ is the *coordinate ring* of X, a K-algebra.

We showed that if X, Y are affine varieties, then

$$\mathrm{Hom}(X, Y) = \mathrm{Hom}(A(Y), A(X)).$$

So,

$$\mathrm{Hom}(\mathbb{A}^0, X) = \mathrm{Hom}(K[x_1, \ldots, x_n]/I(X), K).$$

Note giving a K-algebra homomorphism $K[x_1, \ldots, x_n]/I(X) \to K$ can be done by specifying the images of x_1, say $x_1 \mapsto a_1$, such that, for any $f \in I(X)$, $f(a_1, \ldots, a_n) = 0$. So there is a one-to-one correspondence between such K-algebra homomorphisms, and points of X.

If K is algebraically closed, the maximal ideals of $K[x_1, \ldots, x_n]$ are precisely ideals of the form $(x_1 - a_1, \ldots, x_n - a_n)$ by Hilbert's Nullstellensatz. Similarly, for $A(X)$, the maximal ideals are $(x_1 - a_1, \ldots, x_n - a_n)$ mod $I(X)$, with $(a_1, \ldots, a_n) \in X$.

Thus there is a bijection between points on X, and the maximal ideals of $A(X)$. This gives three objects, X, the homomorphisms and the maximal ideals, which are all bijective.

Now suppose K is not algebraically closed. Consider the K-algebra homomorphisms $A(X) \to L$, where L is an extension of K. If $x_i \mapsto a_i$, then $f(a_1, \ldots, a_n) = 0$ for all $f \in I(X)$. Thus,

$$\mathrm{Hom}_K(A(X), L) = \{(a_1, \ldots, a_n) \in \mathbb{A}^n \mid f(a_1, \ldots, a_n) = 0 \text{ for all } f \in I(X)\}.$$

These correspond to L-valued points.

We could also work over $\mathbb{Z}$. Take an ideal $I \subseteq \mathbb{Z}[x_1, \ldots, x_n]$, and $A = \mathbb{Z}[x_1, \ldots, x_n]/I$.

Then ring homomorphisms $A \to \mathbb{Z}$ are in one-to-one correspondence with points $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ such that $f(a_1, \ldots, a_n) = 0$ for all $f \in I$.

Moreover maps $A \to \mathbb{F}_p$ give solutions mod p, and $A \to \mathbb{Q}$ give rational solutions.

0.3 What we want

Given a ring A, we want to define a gadget

$$X = \mathrm{Spec}\, A,$$

and an R-valued point of X is a ring-homomorphism $A \to R$. We write the set of R-valued points as

$$X(R) = \mathrm{Hom}(A, R).$$

Morphisms $\operatorname{Spec} B \to \operatorname{Spec} A$ should be the same as ring homomorphisms $A \to B$. In category theory,

Definition 0.1. The category of affine schemes it the *opposite category* of rings.

Reminder: In this course, all of our rings are unital, are commutative, and ring homomorphism $\phi : A \to B$ satisfy $\phi(1) = 1$.

Definition 0.2. A *scheme* is a geometric object which is locally an affine scheme.

Currently this is a nonsensical definition, which we will be trying to make sense of. The motivating example is the manifold, which locally looks like an open subset of $\mathbb{R}^n$.

0.4 Introductory Definitions

Definition 0.3. Let A be a ring. Then,

$$\operatorname{Spec} A = \{p \subseteq A \mid p \text{ is a prime ideal}\}.$$

In general, if we have an L-valued point of $X = Z(I) \subseteq \mathbb{A}^n$, we get a ring homomorphism $\phi : A(X) \to L$, which has image an integral subdomain of L, and so $\operatorname{Ker} \phi$ is prime.

Definition 0.4. For $I \subseteq A$ an ideal, define

$$V(I) = \{p \in \operatorname{Spec} A \mid p \supseteq I\}.$$

Again recall p is no longer a point, but a prime ideal.

Proposition 0.1. *The sets $V(I)$ form the closed sets of a topology on* $\operatorname{Spec} A$*, the* Zariski topology*.*

Proof: Need to check a handful of things.

- $V(A) = \emptyset$, so $\emptyset$ is closed.
- $V(0) = \operatorname{Spec} A$, so $\operatorname{Spec} A$ is closed.

- If $\{I_j\}_{j \in J}$ is a collection of ideals, then note

$$\bigcap_{j \in J} V(I_j) = V\left(\sum_{j \in J} I_j\right).$$

- We show that $V(I_1) \cup V(I_2) = V(I_1 \cap I_2)$. Indeed, if $p \supseteq I_1$ or $p \supseteq I_2$, then $p \supseteq I_1 \cap I_2$.

  In the other direction, then $p \supseteq I_1 \cap I_2$, then $p \supseteq I_1$ or $p \supseteq I_2$. This was proven in Part II. Or see Atiyah + Macdonald.

  This is easy: if $I_1, I_2 \not\subseteq p$, then there exists $i_1, i_2 \in I_1, I_2$ respectively that are not in p. But now $i_1 i_2 \in I_1 \cap I_2 \subseteq p$, so $i_1 i_2 \in p$.

  However p is prime, so either $i_1 \in p$ or $i_2 \in p$, contradiction.

Example 0.1.

Consider $A = K[x_1, \ldots, x_n]$ with K algebraically closed. For $I \subseteq A$, the maximal ideals of A corresponding to points of $Z(I)$ are precisely the maximal ideals containing I.

1 Sheaves

Fix a topological space X.

Definition 1.1. A *presheaf* $\mathcal{F}$ on X consists of data, such that:

- For every open set $U \subseteq X$, we have an abelian group $\mathcal{F}(U)$ (or more generally any element of a category).
- Whenever $V \subseteq U \subseteq X$ is open, there is a *restriction* homomorphism

$$\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V),$$

  such that $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$, and if $W \subseteq V \subseteq U \subseteq X$, then

$$\rho_{UW} = \rho_{VW} \circ \rho_{UV}.$$

Remark. This is precisely a contravariant functor from the category of open sets to the category of abelian groups. As mentioned, we may replace the category of abelian groups with any category.

Definition 1.2. If $\mathcal{F}, \mathcal{G}$ are presheaves on X, then a *morphism* $f : \mathcal{F} \to \mathcal{G}$ is data for each $U \subseteq X$, a group homomorphism $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that whenever $V \subseteq U$, we have a commutative diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "f_U"] \arrow[d, "\rho^{\mathcal{F}}_{UV}"] & \mathcal{G}(U) \arrow[d, "\rho^{\mathcal{G}}_{UV}"] \\ \mathcal{F}(V) \arrow[r, "f_V"] & \mathcal{G}(V) \end{tikzcd}$$

Definition 1.3. A presheaf $\mathcal{F}$ on X is a *sheaf* if it satisfies:

1. If $U \subseteq X$ is covered by $\{U_i\}$, and $s \in \mathcal{F}(U)$ such that $s|_{U_i} = \rho_{UU_i}(s) = 0$ for all i, then $s = 0$.
2. If $U, \{U_i\}$ are as in the above, and $s_i \in \mathcal{F}(U_i)$ for each i such that

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

   for all i, j, then there exists $s \in \mathcal{F}(U)$ such that

$$s|_{U_i} = s_i$$

   for all i.

Remark.

- If $\mathcal{F}$ is a sheaf, then $\mathcal{F}(\emptyset) = 0$, since the empty cover is a cover of $\emptyset$.
- The two conditions together can be stated by saying

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\alpha} \bigoplus_{i \in I} \mathcal{F}(U_i) \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} \bigoplus_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$

is *exact* for all $U \subseteq X$ open, and open covers $\{U_i\}$ of U. Here,

$$\begin{aligned} \alpha(s) &= (s|_{U_i})_{i \in I}, \\ \beta_1((s_i)_{i \in I}) &= (s_i|_{U_i \cap U_j})_{i,j \in I}, \\ \beta_2((s_i)_{i \in I}) &= (s_j|_{U_i \cap U_j})_{i,j \in I}. \end{aligned}$$

In category theory, α is the *equalizer* of β_1, β_2.

Exactness means that:

  - α is injective (property 1).
  - $\beta_1 \circ \alpha = \beta_2 \circ \alpha$.
  - For any $(s_i) \in \bigoplus \mathcal{F}(U_i)$ with $\beta_1((s_i)) = \beta_2((s_i))$, there exists an $s \in \mathcal{F}(U)$ with $\alpha(s) = (s_i)$ (property 2).

This definition works even if $\mathcal{F}(U)$ is a set, rather than an abelian group.

Example 1.1.

1. For X any topological space,

$$\mathcal{F}(U) = \{\text{continuous functions } f : U \to \mathbb{R}\}$$

is a sheaf.

2. If $X = \mathbb{C}$ with the Euclidean topology, then

$$\mathcal{F}(U) = \{f : U \to \mathbb{C} \mid f \text{ bounded and holomorphic}\}$$

is not a sheaf, as gluing fails because it does not preserve boundedness.

3. Let G be a group, and set $\mathcal{F}(U) = G$ for all $U \subseteq X$. Then $\rho_{UV} = \mathrm{id}$. This is a presheaf known as the *constant presheaf.*

If we give G the discrete topology, set

$$\mathcal{F}(U) = \{f : U \to G \text{ continuous}\}.$$

These are all locally constant functions, and is obviously a sheaf, called the *constant sheaf.*

4. If X is a variety, denote by $\mathcal{O}_X(U)$ the set of regular functions $f : U \to K$. Then $\mathcal{O}_X(U)$ is a sheaf, called the *structure sheaf* of X.

Definition 1.4. Let $\mathcal{F}$ be a presheaf in X, $p \in X$. Then the *stalk* of $\mathcal{F}$ at p is

$$\mathcal{F}_p = \{(U, s) \mid U \text{ an open neighbourhood of } p, s \in F(U)\}/\cong,$$

where $(U, s) \cong (V, t)$ if there exists $W \subseteq U \cap V$, a neighbourhood of p, such that

$$s|_W = t|_W.$$

The equivalence class of $(U, s) \in \mathcal{F}_p$ is written as s_p, and is the *germ* of s at p.

So the stalk is the set of germs. The stalks should be thought of as the local information of the sheaf around p. Note that given a morphism $f : \mathcal{F} \to \mathcal{G}$, we obtain $f_p : \mathcal{F}_p \to \mathcal{G}_p$ by

$$f_p(U, s) = (U, f_U(s)).$$

Proposition 1.1. *Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves (i.e. a morphism of presheaves). Then f is an isomorphism if and only if f_p is an isomorphism, for all $p \in X$.*

Proof: The forward direction is obvious.

For the other direction, assume f_p is an isomorphism for all p. We will show that $f(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism for all U, and then we can define the inverse to f by $(f^{-1})_U = (f_U)^{-1}$.

First we show f_U is injective. Suppose $s \in \mathcal{F}(U)$ is such that $f_U(s) = 0$. Then for all $p \in U$,

$$f_p((U, s)) = (U, f_U(s)) = (U, 0) = 0 \in \mathcal{G}_p.$$

Thus $s_p = 0$ since f_p is injective. So there is an open neighbourhood $V_p \subseteq U$ of p such that $s|_{V_p} = 0$.

But $\{V_p\}$ covers U, so by property 1, $s = 0$.

Now we show f_U is surjective. Let $t \in \mathcal{G}(U)$. Then for all $p \in U$, there exists $s_p \in \mathcal{F}_p$ such that $f_p(s_p) = t_p$, i.e. there exists an open neighbourhood V_p at $p \in U$ and a germ $(V_p, \tilde{s}_p)$ representing s_p such that

$$(V_p, f_{V_p}(\tilde{s}_p)) \cong (U, t) = t_p.$$

Shrinking V_p if necessary, we can assume that $f_{V_p}(\tilde{s}_p) = t|_{V_p}$. Now on $V_p \cap V_q$,

$$f_{V_p\cap V_q}(\tilde{s}_p|_{V_p\cap V_q} - \tilde{s}_q|_{V_p\cap V_q}) = t|_{V_p\cap V_q} - t|_{V_p\cap V_q} = 0.$$

Since we have shown that $f_{V_p\cap V_q}$ is injective, we get

$$\tilde{s}_p|_{V_p\cap V_q} = \tilde{s}_q|_{V_p\cap V_q},$$

and so by property 2, there exists $s \in \mathcal{F}(U)$ such that

$$s|_{V_p} = \tilde{s}_p,$$

for all p. Now,

$$f_U(s)|_{V_p} = f_{V_p}(s|_{V_p}) = f_{V_p}(\tilde{s}_p) = t|_{V_p}.$$

Therefore, $f_U(s) - t = 0$, so by property 1, $f_U(s) = t$. Hence f_U is surjective.

Remark. If $f_p : \mathcal{F}_p \to \mathcal{G}_p$ is injective for all p, then $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is still injective.

But instead if $f_p : \mathcal{F}_p \to \mathcal{G}_p$ is surjective for all p, it need not be the case that $f_U : \mathcal{F}(U) \to \mathcal{G}(\mathcal{U})$ is surjective.

1.1 Sheafification

Given a presheaf $\mathcal{F}$, there exists a sheaf $\mathcal{F}^+$ and a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$, satisfying the following universal property:

For any sheaf $\mathcal{G}$ and morphism $\phi : \mathcal{F} \to \mathcal{G}$, there exists a unique morphism $\phi^+ : \mathcal{F}^+ \to \mathcal{G}$ such that

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\theta"] \arrow[dr, "\phi"'] & \mathcal{F}^+ \arrow[d, "\phi^+"] \\ & \mathcal{G} \end{tikzcd}$$

commutes.

The pair $(\mathcal{F}^+, \theta)$ is unique up to isomorphism. Also $\mathcal{F}_p \cong \mathcal{F}_p^+$ via θ_p, for all $p \in X$.

The sheafification is defined as follows: define $\mathcal{F}^+(U)$ to be the functions

$$s : U \to \bigsqcup_{p\in U} \mathcal{F}_p$$

such that:

(i) $s(p) \in \mathcal{F}_p$ for all p,

(ii) for each $p \in U$, there exists an open neighbourhood $p \in V \subseteq U$ and an element $t \in \mathcal{F}(V)$ such that

$$(V, t) = s(q),$$

for all $q \in V$.

We define $\theta : \mathcal{F} \to \mathcal{F}^+$ given by

$$\mathcal{F}(U) \ni s \mapsto (p \mapsto (U, s) \in \mathcal{F}_p).$$

We can check that this satisfies the universal property stated previously.

Definition 1.5. Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. We define:

1. The *presheaf kernel* of f as

$$(\ker f)(U) = \ker(f_U : \mathcal{F}(U) \to \mathcal{G}(U)).$$

2. The *presheaf cokernel* of f as

$$(\operatorname{coker} f)(U) = \operatorname{coker} f_U.$$

3. The *presheaf image* as

$$(\operatorname{im} f)(U) = \operatorname{im} f_U.$$

Remark. If $f : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then $\ker f$ is a sheaf. First note that any sub-presheaf of $\mathcal{F}$ satisfies property 1.

To check property 2, given $s_i \in (\ker f)(U_i) \subseteq \mathcal{F}(U_i)$ for $\{U_i\}$ an open cover of U, suppose $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. Then the s_i's glue to given an $s \in \mathcal{F}(U)$. Now,

$$f_U(s)|_{U_i} = f_{U_i}(s_i) = 0,$$

so by property 1, $f_U(s) = 0$. Hence $s \in (\ker f)(U)$.

Example 1.2.

Take $X = \mathbb{P}^1$, or the Riemann sphere, and let $P, Q \in X$ be distinct points.

Let $\mathcal{G}$ be the sheaf of regular functions on X (or the holomorphic functions on X), and let $\mathcal{F}$ be the sheaf of regular functions on X vanishing at P and Q (or the holomorphic functions vanishing at P, Q).

Note $\mathcal{F}(U) = \mathcal{G}(U)$ if $U \cap \{P, Q\} = \emptyset$. By Liouville's theorem, $\mathcal{G}(X) = K$, and $\mathcal{F}(X) = 0$.

Let $U = X \setminus \{P\}$, $V = X \setminus \{Q\}$, and $f : \mathcal{F} \to \mathcal{G}$ the obvious inclusion. Note $\mathcal{G}(U) = K[x]$ by affine geometry, $\mathcal{F}(U) = (x)$. So,

$$\begin{aligned}(\operatorname{coker} f)(X) &= \mathcal{G}(X)/\mathcal{F}(X) = K,\\ (\operatorname{coker} f)(U \cap V) &= \mathcal{G}(U \cap V)/\mathcal{F}(U \cap V) = 0,\\ (\operatorname{coker} f)(U) &= \mathcal{G}/\mathcal{F}(U) = K[x]/(x) = K,\\ (\operatorname{coker} f)(V) &= K.\end{aligned}$$

Choose $s_U \in (\operatorname{coker} f)(U)$, $s_V \in (\operatorname{coker} f)(V)$. But now $s_U|_{U\cap V} = s_V|_{U\cap V} = 0$, and this would imply that if $\operatorname{coker} f$ were a sheaf, that

$$K \oplus K \subseteq (\operatorname{coker} f)(X).$$

Remark. This shows that $\operatorname{coker} f$ need not be a sheaf. The same is true of $\operatorname{im} f$.

Definition 1.6. Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. The *sheaf kernel* of f is $\ker f$, the presheaf kernel.

The *sheaf cokernel* of f is the sheafification of the presheaf cokernel. We still write this as $\operatorname{coker} f$.

Note that to show $\operatorname{coker} f$ is a presheaf, we need to use the third isomorphism theorem.

We can also show that the sheaf image of f is a subsheaf of $\mathcal{G}$ (prove this).

We say that f is *injective* if $\ker f = 0$, and f is *surjective* if $\operatorname{im} f = \mathcal{G}$. We say that a sequence

$$\cdots \longrightarrow \mathcal{F}^{i-1} \xrightarrow{f^i} \mathcal{F}^i \xrightarrow{f^{i+1}} \mathcal{F}^{i+1} \longrightarrow \cdots$$

is *exact* if $\ker f^{i+1} = \operatorname{im} f^i$.

If $\mathcal{F}' \subseteq \mathcal{F}$ is a subsheaf, we write $\mathcal{F}/\mathcal{F}'$ for the sheaf associated to the presheaf

$$U \mapsto \mathcal{F}(U)/\mathcal{F}'(U).$$

This is $\operatorname{coker}(\iota : \mathcal{F}' \hookrightarrow \mathcal{F})$, where ι is the inclusion.

Lemma 1.1. *Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then,*

$$\begin{aligned}(\ker f)_p &= \ker(f_p : \mathcal{F}_p \to \mathcal{G}_p),\\ (\operatorname{im} f)_p &= \operatorname{im} f_p.\end{aligned}$$

Proof: We have a map $(\ker f)_p \to \ker f_p$, defined as follows: if $(U, s) \in (\ker f)_p$, then $s \in (\ker f)(U)$, and $(U, s) \in \mathcal{F}_p$. So,

$$f_p(U, s) = (U, f(s)) = (U, 0) = 0 \in \mathcal{G}_p,$$

thus $(U, s) \in \ker f_p$.

It is easy to see this map is injective: if $(U, s) = 0$ in $\mathcal{F}_p$, then in some neighbourhood $s|_V = 0$. So $(U, s) \sim (V, s|_V) = 0$ in $(\ker f)_p$.

We now tackle surjectivity. If $(U, s) \in \ker f_p$, then $0 = f_p(U, s) = (U, f_U(s))$ in $\mathcal{G}_p$, so in some neighbourhood V, $f_U(s)|_V = 0$.

Thus $f_V(s|_V) = 0$, so $(U, s) \sim (V, s|_V)$, and $s|_V \in (\ker f)(V)$. Thus $(V, s|_V) \in (\ker f)_p$, which maps to $(U, s) \in \ker f_p$.

We now prove the appropriate theorem for images. Let $\operatorname{im}' f$ denote the presheaf image. Recall that if $\mathcal{F}$ is a presheaf, then $\mathcal{F}_p \cong \mathcal{F}_p^+$, via θ_p. So it is enough to show there is an isomorphism

$$(\operatorname{im}' f)_p \cong \operatorname{im} f_p.$$

First, define

$$\begin{aligned} (\operatorname{im}' f)_p &\to \operatorname{im} f_p \\ (U, s) &\mapsto (U, s), \end{aligned}$$

with $s \in f_U(t)$ for some $t \in \mathcal{F}(U)$, which lives in $\operatorname{im} f_p$ since

$$f_p(U, t) = (U, f_U(t)) = (U, s)$$

First we show injectivity. If $(U, s) = 0$ in $\mathcal{G}_p$, then there exists $p \in V \subseteq U$ such that $s|_V = 0$. But then,

$$(U, s) \sim (V, s|_V) = (V, 0) = 0$$

in $(\operatorname{im}' f)_p$.

To show surjectivity, we know that for $(U, s) \in \operatorname{im} f_p$, that there is $(V, t) \in \mathcal{F}_p$ with $f_p(V, t) \sim (U, s)$. We can replace U with the smaller open set V, so can assume that $U = V$, and then

$$f_p(U, t) = (U, f_U(t)) \sim (U, s)$$

in $\mathcal{G}_p$. Shrinking U further, we can assume $f_U(t) = s$, and hence

$$(U, s) = (U, f_U(t)) \in (\operatorname{im}' f)_p.$$

Proposition 1.2. *$f : \mathcal{F} \to \mathcal{G}$ is injective if and only if $f_p : \mathcal{F}_p \to \mathcal{G}_p$ is injective for all p.*

$f : \mathcal{F} \to \mathcal{G}$ is surjective if and only if $f_p : \mathcal{F}_p \to \mathcal{G}_p$ is surjective for all p.

Proof: We have

$$\begin{aligned} f_p \text{ is injective} &\iff \ker f_p = 0 && \forall p \\ &\iff (\ker f)_p = 0 && \forall p \text{ by the lemma} \\ &\iff \ker f = 0. \end{aligned}$$

Note this uses the easy fact that if $\mathcal{F}$ is a sheaf, and $\mathcal{F}_p = 0$ for all p, then $\mathcal{F} = 0$. Also,

$$\begin{aligned} f_p \text{ is surjective} &\iff \operatorname{im} f_p = \mathcal{G}_p && \forall p \\ &\iff (\operatorname{im} f)_p = \mathcal{G}_p && \forall p \text{ by the lemma} \\ &\iff \operatorname{im} f = \mathcal{G}. \end{aligned}$$

Hence if $\mathcal{F} \subseteq \mathcal{G}$ are sheaves with $\mathcal{F}_p = \mathcal{G}_p$, we can check that $\mathcal{F} = \mathcal{G}$.

1.2 Passing between Spaces

Let $f : X \to Y$ be a continuous map of topological spaces.

Let $\mathcal{F}$ be a sheaf in X. Define a sheaf $f_*\mathcal{F}$ on Y by

$$(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U)).$$

This is the *push-forward* of $\mathcal{F}$. This can be checked to be a sheaf.

If $\mathcal{G}$ is a sheaf on Y, define the *pull-back* sheaf $f^{-1}\mathcal{G}$ to be the sheaf associated to the presheaf

$$U \mapsto \{(V, s) \mid V \subseteq f(U), V \text{ open}, s \in \mathcal{G}(V)\}/\sim,$$

where $(V, s) \sim (V', s')$ if there exists W with $f(U) \subseteq W \subseteq V \cap V'$ with $s|_W = s'|_W$.

Example 1.3.

If $f : \{p\} \hookrightarrow Y$, then $f^{-1}\mathcal{G} = \mathcal{G}_p$, by identifying a sheaf $\mathcal{F}$ on a topological space X with the group $\mathcal{F}(X)$.

More generally, if $\iota : Z \hookrightarrow X$ is an inclusion, we often write $\mathcal{F}|_Z$ for the sheaf $\iota^{-1}\mathcal{F}$.

If $\iota : U \hookrightarrow X$ is an open subset, then in fact $i^{-1}\mathcal{F} = \mathcal{F}|_Z$ is the sheaf $V \mapsto \mathcal{F}(V)$, for $V \subseteq U$ open.

If $s \in \mathcal{F}(U)$, we call s a *section* of $\mathcal{F}$ over U. We often also write

$$\mathcal{F}(U) = \Gamma(U, \mathcal{F}),$$

thinking of $\Gamma(U, \cdot)$ as a covariant functor

$$\Gamma(U, \cdot) : \mathbf{Presheaves}_X \to \mathbf{Ab}.$$

2 Affine Schemes

Let A be a ring. $\operatorname{Spec} A$ is a topological space analogous to the sheaf of regular functions.

Let $S \subseteq A$ be a multiplicatively closed subset, i.e. $1 \in S$ and whenever $a, b \in S$, we have $a \cdot b \in S$. We define

$$S^{-1}A = \{(a, s) \mid a \in A, s \in S\}/\sim,$$

where $(a, s) \sim (a', s')$ if there exists $s'' \in S$ such that

$$s''(as' - a's) = 0.$$

This is called the *localization* of A at S.

Example 2.1.

1. Say $S = \{1, f, f^2, \dots\}$ for some $f \in A$. Then
$$A_f = S^{-1}A = \left\{\frac{a}{f^n} \mid a \in A, n \geq 0\right\}/\sim.$$
2. Take $P \subseteq A$ a prime ideal, and $S = A \setminus P$. Then we write $A_P = S^{-1}A$.

Our goal is to now construct the sheaf

$$\mathcal{O} = \mathcal{O}_{\operatorname{Spec} A}.$$

For $U \subseteq \operatorname{Spec} A$ open, we write

$$\mathcal{O}(U) = \Big\{ s : U \to \coprod_{p \in U} A_p \mid s(p) \in A_p \wedge \text{for each } p \in U, \exists\, q \in V \subseteq U \\ \text{and } a, f \in A \text{ such that } \forall q \in V, f \notin q \wedge s(q) = \frac{a}{f} \in A_q \Big\}$$

This is a sheaf, as it is only defined locally.

The scheme has the following properties. First, for any $p \in \operatorname{Spec} A$, $\mathcal{O}_p = A_p$.

Proof: We have a map $\mathcal{O}_p \to A_p$ given by $(U, s) \mapsto s(p)$.

First we show surjectivity. Any element of A_p can be written as a/f, for

$a \in A$ and $f \notin p$. We define

$$D(f) = (\operatorname{Spec} A) \setminus V((f)) = \{q \in \operatorname{Spec} A \mid f \notin q\}.$$

This is an open neighbourhood of $p \in \operatorname{Spec} A$. Then a/f defines a section s of $\mathcal{O}$ over $D(f)$, via

$$s(q) = \frac{a}{f} = A_q.$$

In particular, $s(p) = a/f \in A_p$. So, $(D(f), s) \mapsto a/f$ under this map.

For injectivity, let $p \in U \subseteq \operatorname{Spec} A$, and $s \in \mathcal{O}(U)$, defining a germ $(U, s) \in \mathcal{O}_p$.

Suppose $s(p) = 0$. We want to show $(U, s) = 0$ in $\mathcal{O}_p$. We may shrink U and assume there are $a, f \in A$ such that $s(q) = a/f$ for all $q \in U$. In particular, $f \notin q$.

Since $s(p) = 0$, $a/f = 0$ in A_p. Thus there exists $h \in A \setminus p$ such that

$$h(a \cdot 1 - 0 \cdot f) = 0,$$

i.e. $h \cdot a = 0$. Let $V = D(f) \cap D(h)$. Then for $q \in V$, $a/f = 0$ in A_q.

Since $h \notin q$ and $h \cdot a = 0$, $s|_V = 0$, so $(U, s) = 0$. Moreover, V is non-empty as it contains p.

Another property is as follows: for any $f \in A$, $\mathcal{O}(D(f)) = A_f$. In particular,

$$\mathcal{O}(\operatorname{Spec} A) = \mathcal{O}(D(1)) = A_1 = A.$$

Proof: Let $\psi : A_f \to \mathcal{O}(D(f))$ be given by

$$\frac{a}{f^n} \mapsto \left(p \mapsto \frac{a}{f^n} \in A_p\right).$$

We need to show this map is injective, and surjective.

Injectivity: To show ψ is injective, if $\psi(a/f^n) = 0$, then for all $p \in D(f)$, $a/f^n = 0$ in A_p. So there exists $h \notin p$ such that $h \cdot a = 0$, where h may depend on p. Let

$$I = \{g \in A \mid g \cdot a = 0\}.$$

So $h \in I$, for all p. But also $h \notin p$, so $I \not\subseteq p$. This is true for all $p \in D(f)$, so

$$V(I) \cap D(f) = \emptyset.$$

Thus,

$$f \in \bigcap_{q \in V(I)} q = \sqrt{I},$$

the radical of I. Thus $f^m \in I$ for some $n > 0$. Thus $f^m \cdot a = 0$, so in particular $a/f^n = 0$ in A_f, by choosing $h = f^m$.

Surjectivity: is a lot harder. Let $s \in \mathcal{O}(D(f))$. Our goal is to show that $s = \psi(a/f^n)$ for some a and n. Let $\{V_i\}$ be an open cover of $D(f)$ such that $s|_{V_i}$ is represented by a_i/g_i with $g_i \notin p$, for all $p \in V_i$.

Thus $V_i \subseteq D(g_i)$. By an example sheet question, the open sets of te form $D(h)$ form a basis for the Zariski topology. We can thus assume $V_i = D(h_i)$ for some $h_i \in A$.

Since $D(h_i) \subseteq D(g_i)$, $V((h_i)) \subseteq V((g_i))$, so $\sqrt{(h_i)} \subseteq \sqrt{(g_i)}$. Therefore $h_i^n \in (g_i)$, for some n, so $h_i^n = c_i g_i$ for some $c_i \in A$. Thus,

$$\frac{a_i}{g_i} = \frac{c_i a_i}{h_i^n}.$$

We can thus replace h_i with h_i^n, since a prime ideal containing one contains the other, and can now assume that $s|_{V_i}$ is represented by an element of the for a_i/h_i, where $V_i = D(h_i)$.

So far, we have proven that $D(f)$ is covered with open sets $V_i = D(h_i)$, and $s|_{V_i}$ represented by a_i/h_i.

Claim: $D(f)$ can be covered with a finite number of the $D(h_i)$'s, in other words $D(f)$ is *quasicompact*, meaning compact without the Hausdorff condition.

The proof is as follows:

$$\begin{aligned}
& D(f) \subseteq \bigcup_i D(h_i) \\
\iff \quad & V((f)) \supseteq \bigcap V((h_i)) = V\left(\sum_i (h_i)\right) \\
\iff \quad & \sqrt{(f)} \subseteq \sqrt{\sum_i (h_i)} \\
\iff \quad & f \in \sqrt{\sum_i (h_i)} \\
\iff \quad & f^n \in \sum_I (h_i)
\end{aligned}$$

for some n. Thus,

$$f^n = \sum b_i h_i$$

for some finite set of h_i's. Thus, by running the above proof backwards, we find that the finite number of these $D(h_i)$ cover $D(f)$.

Now we have a finite cover of $D(f)$. On $D(h_i) \cap D(h_j)$, a_i/h_i and a_j/h_j both represent $s|_{D(h_i)\cap D(h_j)}$. But,

$$\psi : A_{h_i h_j} \to \mathcal{O}(D(h_i, h_j)) = \mathcal{O}(D(h_i) \cap D(h_j))$$

is injective, by proof of injectivity. So, $a_i/h_i = a_j/h_j$ in $A_{h_i h_j}$. Thus for some n,

$$(h_i h_j)^n (h_j a_i - h_i a_j) = 0.$$

We can choose n to work for all i, j. Rewriting this as

$$h_j^{n+1}(h_i^n a_i) - h_i^{n+1}(h_j^n a_j) = 0,$$

and replacing each h_i with h_i^{n+1} and a_i with $h_i^n a_i$, the ratio is the same, and so s is still represented by a_i/h_i, but also $h_j a_i = h_i a_j$ for all i, j. Let

$$a = \sum b_i a_i,$$

where we recall that

$$f^n = \sum b_i h_i.$$

Then for any j,

$$h_j a = \sum h_j b_i a_i = \sum_i h_i b_i a_j = f^n a_j,$$

so $a_j/h_j = a/f^n$, for all j. So $\psi(a/f^n) = s$.

Definition 2.1. A *ringed space* is a pair $(X, \mathcal{O}_X)$ where:

- X is a topological space,
- $\mathcal{O}_X$ is a sheaf of rings.

A *morphism* of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is data such that:

- $f : X \to Y$ is continuous.
- $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves of rings, i.e.

  $$f_U^\# : \mathcal{O}_Y(U) \to (f_*\mathcal{O}_X)(U) = \mathcal{O}_X(f^{-1}(U))$$

  is a ring homomorphism.

Example 2.2.

1. Let X be a variety, and $\mathcal{O}_X$ the sheaf of regular functions on X, so

   $$\mathcal{O}_X(U) = \{f : U \to K \text{ such that } f \text{ is regular}\}.$$

   A morphism $f : X \to Y$ of varieties is a continuous map $f : X \to Y$ inducing a map, for all $U \subseteq Y$, by $\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$, where $\varphi \mapsto \varphi \circ f$.
2. Let X be a topological space, then we can consider

   $$\mathcal{O}_X(U) = \{f : U \to \mathbb{R} \mid f \text{ continuous}\}.$$

3. Let X be a C^∞-manifold, and

   $$\mathcal{O}_X(U) = \{f : U \to \mathbb{R} \mid fC^\infty\}.$$

   Then $f : X \to Y$ a continuous map between C^∞ manifolds is C^∞ if and only if, for any C^∞ function $\varphi : U \to \mathbb{R}$, and $U \subseteq Y$, $f \circ \varphi : f^{-1}(U) \to \mathbb{R}$ is C^∞.

Remark. All of these example have the feature that $\mathcal{O}_{X,p}$, for $p \in X$, is a local ring, i.e. one with a unique maximal ideal.

Indeed, take $m_p = \{(U, f) \mid f(p) = 0\}$. Note $(U, f) \sim (V, f|_V)$ for some V with $f|_V$ non-vanishing if $f(p) \neq 0$.

So every element in $\mathcal{O}_{X,p} \setminus m_p$ is invertible, hence m_p is the unique maximal ideal.

Note that given a morphism of ringed space $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ in the previous three examples, we get a map

$$f_p^\# : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$$

for $p \in X$ defined by

$$f_p^\#(U, \varphi) = (f^{-1}(U), \varphi \circ f).$$

Note that $(U, \varphi) \in m_{f(p)}$ if and only if $(f^{-1}(U), \varphi \circ f) \in m_p$.

In particular, $(f_p^\#)^{-1}(m_p) = m_{f(p)}$.

Definition 2.2. A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that $\mathcal{O}_{X,p}$ is a local ring for all $p \in X$.

A *morphism* $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of locally ringed spaces is a morphism of ringed spaces such that the induced morphism

$$f_p^\# : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$$

is a local homomorphism.

Here $f_p^\#$ is defined by

$$\mathcal{O}_{Y,f(p)} \ni (U, \varphi) \mapsto (f^{-1}(U), f_U^\#(\varphi)) \in \mathcal{O}_{X,p}.$$

A homomorphism $g : A \to B$ of local rings is a ring homomorphism such that $g^{-1}(m_B) = m_A$. Note that $g(A \setminus m_a) = g(A^*) \subseteq B^* = B \setminus m_B$, so we always have $g^{-1}(m_B) \subseteq m_A$.

The motivating example is $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, which we will show is a locally ringed space. Recall that $\mathcal{O}_{\operatorname{Spec} A,p} = A_p$, which has a unique maximal ideal

$$m = \left\{ \frac{a}{s} \;\middle|\; a \in p, s \notin p \right\}.$$

There is a natural map $A \to A_p$ by $a \mapsto a/1$, and $m \subseteq A_p$ is the ideal generated by the image of p under this map. This is often written as pA_p.

We call $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ an *affine scheme*.

Theorem 2.1. *The category of affine schemes with morphisms being morphisms of locally ringed spaces is equivalent to the opposite category of rings.*

Proof: We need to show that:

- If $\varphi : A \to B$ is a homomorphism of rings, we get a morphism of locally ringed spaces
$$(f, f^{\#}) : (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}).$$
- Any morphism of locally ringed spaces is induced by a homomorphism $\varphi : A \to B$ of rings.

For the first part, we define $f : \operatorname{Spec} B \to \operatorname{Spec} A$ by
$$f(p) = \varphi^{-1}(p).$$
Note that $\varphi^{-1}(p)$ is prime, as if $a \cdot b \in \varphi^{-1}(p)$, then $\varphi(a)\varphi(b) = \varphi(ab) \in p$, so say $\varphi(a) \in p$. Then $a \in \varphi^{-1}(p)$.

Now we show f is continuous. Note for any $I \subseteq A$,
$$\begin{aligned} f^{-1}(V(I)) &= \{p \in \operatorname{Spec} B \mid f(p) \supseteq I\} = \{p \in \operatorname{Spec} B \mid \varphi^{-1}(p) \supseteq I\} \\ &= \{p \in \operatorname{Spec} B \mid p \supseteq \varphi(I)\} \\ &= V(\varphi(I)). \end{aligned}$$

Now we need to define $f^{\#} : \mathcal{O}_{\operatorname{Spec} A} \to f_* \mathcal{O}_{\operatorname{Spec} B}$, where $f^{\#}_V : \mathcal{O}_{\operatorname{Spec} A}(V) \to \mathcal{O}_{\operatorname{Spec} B}(f^{-1}(V))$.

Note for $p \in \operatorname{Spec} B$, we obtain a map
$$\begin{aligned} \varphi_p : A_{\varphi^{-1}(p)} &\to B_p \\ a/s &\mapsto \varphi(a)/\varphi(s). \end{aligned}$$
This makes sense since $s \notin \varphi^{-1}(p)$ means that $\varphi(s) \notin p$. Now we claim φ_p is a local homomorphism. Consider the following commutative diagram:
$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ A_{\varphi^{-1}(p)} \arrow[r, "\varphi_p"] & B_p \end{tikzcd}$$
The maximal ideal of B_p is generated by the image of p, and the maximal ideal of $A_{\varphi^{-1}(p)}$ is generated by the image of $\varphi^{-1}(p)$.

Thus $\varphi_p^{-1}(m_{B_p})$ contains the image of the ideal generated by $\varphi^{-1}(p)$, i.e. $m_{A_{\varphi^{-1}(p)}}$. Thus,

$$\varphi_p^{-1}(m_{B_p}) = m_{A_{\varphi^{-1}(p)}}.$$

Now given $V \subseteq \operatorname{Spec} A$, we may define $f_V^\# : \mathcal{O}_{\operatorname{Spec} A}(V) \to \mathcal{O}_{\operatorname{Spec} B}(f^{-1}(V))$, given by

$$(p \mapsto s(p)) \mapsto (q \in f^{-1}(V) \mapsto \varphi_q(s(f(q)))),$$

giving $f^\# : \mathcal{O}_{\operatorname{Spec} A} \to f_* \mathcal{O}_{\operatorname{Spec} B}$.

Note we need to check the local coherence, i.e. if s is locally given by a/b, then $f_V^\#(a/b)$ is locally given by $\varphi(a)/\varphi(b)$.

By construction, $f_p^\# : \mathcal{O}_{\operatorname{Spec} A, f(p)} \to \mathcal{O}_{\operatorname{Spec} B, p}$ is $\varphi_p : A_{\varphi^{-1}(p)} \to B_p$, a local homomorphism, and hence $(f, f^\#)$ defines a morphism of locally ringed spaces.

The other part of this is to show that $(f, f^\#) : \operatorname{Spec} B \to \operatorname{Spec} A$, a morphism of locally ringed spaces, is given by a morphism of rings. First,

$$f_{\operatorname{Spec} A}^\# = \varphi : \mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) \to \mathcal{O}_{\operatorname{Spec} B}(\operatorname{Spec} B),$$

i.e. $\varphi : A \to B$. We also have $f_p^\# : \mathcal{O}_{\operatorname{Spec} A, f(p)} \to \mathcal{O}_{\operatorname{Spec} B, p}$ giving a local homomorphism. This map is compatible with $\varphi = f_{\operatorname{Spec} A}^\#$, i.e. we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ A_{f(p)} \arrow[r, "f_p^\#"] & B_p \end{tikzcd}$$

The vertical maps correspond to $a \mapsto a/1$. Thus, $(f_p^\#)^{-1}(pB_p) = f(p)A_{f(p)}$, so $\varphi^{-1}(p) = f(p0$ as the pullback of pB_p to B is p, and the pullback of $f(p)A_{f(p)}$ to A is $f(p)$. Thus $f(p) = \varphi^{-1}(p)$.

So f is induced by φ, and $f_p^\# = \varphi_p$. This implies $(f, f^\#)$ is induced by φ.

Definition 2.3. An *affine scheme* is a locally ringed space isomorphic, in the category of locally ringed spaces, to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some A.

A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ with an open cover $\{U_i\}$ with each $(U, \mathcal{O}_X|_{U_i})$ an affine scheme.

Example 2.3.

1. Let K be a field, then

$$\operatorname{Spec} K = (\{*\}, \mathcal{O} = K\}$$

Suppose X is a scheme, and $f : \operatorname{Spec} K \to X$ a morphism. Then its map f is determined by $f(*) = x_0$.

$f^\#$ gives $f_X^\# : \mathcal{O}_{X,x_0} \to \mathcal{O}_{\operatorname{Spec} K,*} = K$, a local homomorphism, so

$$(f_{x_0}^\#)^{-1}(0) = m_{x_0} \subseteq \mathcal{O}_{X,x_0}$$

is the maximal ideal. Thus we obtain an inclusion

$$K(x-0) = \mathcal{O}_{X,x_0}/m_{x_0} \hookrightarrow K,$$

where $K(x_0)$ is the *residue field* of x_0. Note $f^\#$ is determined by this map:

$$f_V^\# : \mathcal{O}_X(V) \to \mathcal{O}_{\operatorname{Spec} K}(f^{-1}(V)) = \begin{cases} 0 & \text{if } x_0 \notin V, \\ K & \text{if } x_0 \in V. \end{cases}$$

In the second case, $f_V^\#$ must factor as

$$\mathcal{O}_X(V) \to \mathcal{O}_{X,x_0} \to \mathcal{O}_{X,x_0}/m_{x_0} = K(x_0) \to K.$$

Conversely, given $x_0 \in X$ and an inclusion $K(x_0) \hookrightarrow K$, we get a morphism $f : \operatorname{Spec} K \to X$.

Note: if $X = \operatorname{Spec} A$, giving $\operatorname{Spec} K \to X$ is the same thing as giving $A \to K$, which we saw as defining a K-valued point.

What is a morphism $f : X \to \operatorname{Spec} K$? The map is constant. We need $f^\# : \mathcal{O}_{\operatorname{Spec} K} \to f_*\mathcal{O}_X$, i.e. a map $f_{\operatorname{Spec} K}^\# : K \to \mathcal{O}_X(X)$, which gives $\mathcal{O}_X(X)$ a K-algebra structure.

Note that this makes $\mathcal{O}_X(V)$ a K-algebra for all $V \subseteq X$ open via

$$K \to \mathcal{O}_X(X) \to \mathcal{O}_X(V),$$

and also all local rings $\mathcal{O}_{X,p}$ K-algebras.

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