# III General Relativity

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Based on Lectures by Prof. Claude Warnick

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# Contents

# 0 Introduction

Office hours: 8:40AM MWF, in MR2. Normal room E1.14. Will follow roughly Reall's course.

General relativity is our best theory of gravitation on the largest scales. It is:

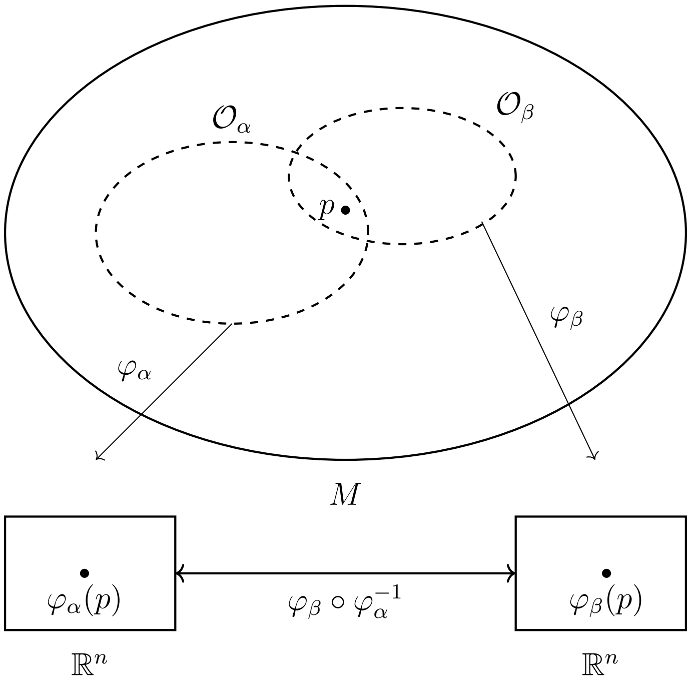
- Classical : No quantum effects.
- Geometrical: Space and time are combined in a curved spacetime.
- Dynamical: In contrast to Newton's theory of gravity, Einstein's gravitational field has its own non-trivial dynamics.

# 1 Differentiable Manifolds

The basic object of study in differential geometry is the (differentiable) manifold. This is an object which 'locally looks like $\mathbb{R}^n$', and has enough structure to let us do calculus.

**Definition 1.1.** A *differentiable manifold* of dimension $n$ is a set $M$, together with a collection of coordinate charts $(O_\alpha, \phi_\alpha)$, where:

- $O_\alpha \subseteq M$ are subsets of $M$ such that
$$\bigcup_\alpha O_\alpha = M.$$
- $\phi_\alpha$ is a bijective map from $O_\alpha$ to $U_\alpha$, an open subset of $\mathbb{R}^n$.
- If $O_\alpha \cap O_\beta \neq \emptyset$, then
$$\phi_\beta \circ \phi_\alpha^{-1}$$
is a smooth map from $\phi_\alpha(O_\alpha \cap O_\beta) \subseteq U_\alpha$ to $\phi_\beta(O_\alpha \cap O_\beta) \subseteq U_\beta$.




*Remark.*

- We could replace smooth with finite differentiability (e.g. $k$-times differentiable).

- The charts define a topology on $M$: $U \subseteq M$ is open if and only if $\phi_\alpha(U \cap O_\alpha)$ is open in $\mathbb{R}^n$ for all $\alpha$. Every open subset of $M$ is itself a manifold, by restricting the charts to $U$.

The collection $\{(O_\alpha, \phi_\alpha)\}$ is called an *atlas*. Two atlases are *compatible* if their union is an atlas.

An atlas $A$ is *maximal* if there exists no atlas $B$ which is compatible with $A$, and strictly larger than $A$. Every atlas is contained in a maximal atlas (by taking the union of all compatible atlases). Hence we can assume without loss of generality that we work with a maximal atlas.

**Example 1.1.**

1. If $U \subseteq \mathbb{R}^n$ is open, we can take $O = U$, and $\phi : U \to \mathbb{R}^n$ to be the identity on $U$. Then $\{(O, \phi)\}$ is an atlas.
2. Take $S^1$. If $p \in S^1 \setminus \{(-1,0)\} = O_1$, there is a unique $\theta_1 \in (-\pi, \pi)$ such that
$$p = (\cos\theta_1, \sin\theta_1).$$
If $p \in S^1 \setminus \{(1,0)\} = O_2$, there is a unique $\theta_2 \in (0, 2\pi)$ such that
$$p = (\cos\theta_2, \sin\theta_2).$$
These maps from $(-\pi,\pi)$ and $(0,2\pi)$ to $O_1, O_2$ give $\phi_1^{-1}, \phi_2^{-1}$ respectively. Note that $\phi_1(O_1 \cap O_2) = (-\pi, 0) \cup (0, \pi)$, and the transition function is
$$\phi_2 \circ \phi_1^{-1}(\theta) = \begin{cases} \theta & \theta \in (0,\pi), \\ \theta + 2\pi & \theta \in (-\pi, 0). \end{cases}$$
This is smooth where defined, and similarly for $\phi_2 \circ \phi_1^{-1}$. Hence $S^1$ is a manifold.
3. More generally, we can consider $S^n$, and can define charts by stereographic projections. If $\{\mathbf{E}_1, \dots, \mathbf{E}_{n+1}\}$ is the standard basis for $\mathbb{R}^{n+1}$, and $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the standard basis for $\mathbb{R}^n$, write
$$\mathbf{P} = P^1\mathbf{E}_1 + \cdots + P^{n+1}\mathbf{E}_{n+1}.$$
Set $O_1 = S^n \setminus \{\mathbf{E}_{n+1}\}$, and write
$$\phi_1(\mathbf{P}) = \frac{1}{1 - P^{n+1}}(P^1\mathbf{e}_1 + \cdots + P^n\mathbf{e}_n).$$

In a similar way we may define $O_2, \phi_2$, for $-\mathbf{E}_{n+1}$. The transition map is then

$$\phi_2 \circ \phi_1^{-1}(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|^2},$$

which is smooth on $\mathbb{R}^n \setminus \{0\} = \phi_1(O_1 \cap O_2)$.

"Nice" surfaces in $\mathbb{R}^n$ are manifolds with no cusps, cornered or self-intersections, for example $S^n \subseteq \mathbb{R}^{n+1}$.

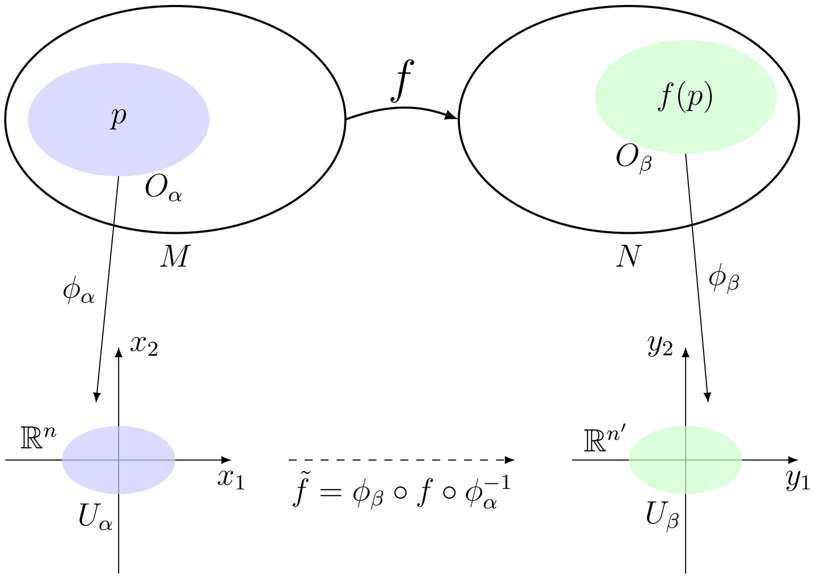
## 1.1 Smooth Functions on Manifolds

Suppose $M, N$ are manifolds of dimension $n, n'$ respectively. Let $f : M \to N$.

Then let $p \in M$, and pick charts $(O_\alpha, \phi_\alpha)$ for $M$, and $(O'_\beta, \phi'_\beta)$ for $N$, with $p \in O_\alpha$ and $f(p) \in O_\beta$. Then,

$$\phi'_\beta \circ f \circ \phi_\alpha^{-1}$$

maps a neighbourhood of $\phi_\alpha(p)$ in $U_\alpha \subseteq \mathbb{R}^n$, to $U'_\beta \in \mathbb{R}^{n'}$. If this function is smooth for all possible choices of this chart, we say that $f : M \to N$ is smooth.




*Remark.*

- A smooth map $\psi : M \to N$ which has a smooth inverse is called a *diffeomorphism*.
- If $N = \mathbb{R}$ or $\mathbb{C}$, we sometimes call $f$ a *scalar field*.
- If $M = I \subseteq \mathbb{R}$, an open interval, then $f : I \to N$ is a *smooth curve* in $N$.

- If $f$ is smooth in one atlas, then it is smooth in all compatible atlases.

**Example 1.2.**

1. Recall $S^1$. Let $f(x,y) = x$, $f : S^1 \to \mathbb{R}$. Using our previous charts,

$$\begin{aligned} f \circ \phi_1^{-1} : (-\pi, \pi) &\to \mathbb{R} \\ \theta_1 &\mapsto \cos\theta_1. \end{aligned}$$

   Similarly,

$$\begin{aligned} f \circ \phi_2^{-1} : (0, 2\pi) &\to \mathbb{R} \\ \theta_2 &\mapsto \cos\theta_2. \end{aligned}$$

   Hence $f$ is smooth.

2. If $(O, \phi)$ is a coordinate chart on $M$, write

$$\phi(p) = (x^1(p), x^2(p), \ldots, x^n(p)),$$

   for $p \in O$. Then $x^i(p)$ defines a map from $O$ to $\mathbb{R}$. This is smooth for each $i = 1, \ldots, n$. If $(O', \phi')$ is another overlapping coordinate chart, then $x^i \circ (\phi')^{-1}$ is the $i$'th component of $\phi \circ (\phi')^{-1}$, hence smooth.

3. We can define a smooth function chart-by-chart. For simplicity, let $N = \mathbb{R}$, and let $\{(O_\alpha, \phi_\alpha)\}$ be an atlas on $M$. Define smooth functions

$$F_\alpha : U_\alpha \to \mathbb{R},$$

   and suppose that $F_\alpha \circ \phi_\alpha = F_\beta \circ \phi_\beta$ on $O_\alpha \cap O_\beta$, for all $\alpha, \beta$.

   Then for $p \in M$, we can define

$$f(p) = F_\alpha \circ \phi_\alpha(p),$$

   for any chart $(O_\alpha, \phi_\alpha)$ with $p \in O_\alpha$. Now $f$ is smooth, as

$$f \circ \phi_p^{-1} = F_\alpha \circ \phi_\alpha \circ \phi_\beta^{-1}.$$

   In practice, we often don't distinguish between $f$ and its coordinate chart representations $F_\alpha$.

## 1.2 Curves and Vectors

For a surface in $\mathbb{R}^3$, we have a notion of the 'tangent space' at a point $p$, which consists of all vectors tangent to the surface at that point.

The tangent spaces are vector spaces (copies of $\mathbb{R}^2$). Different points have different tangent spaces.

In order to define the tangent space for a manifold, we first consider tangent vectors of a curve.

Recall that $\lambda : I \to M$ a smooth map, is a smooth curve in $M$.

If $\lambda(t)$ is a smooth curve in $\mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function, the chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}t}[f(\lambda(t))] = X(t) \cdot \nabla f(\lambda(t)),$$

where $X(t) = \mathrm{d}\lambda/\,\mathrm{d}t(t)$ is the tangent vector to $\lambda$ at $t$.

**Definition 1.2.** Let $\lambda : I \to M$ be a smooth curve with $\lambda(0) = p$. The *tangent vector* to $\lambda$ at $p$ is the linear map $X_p$ from the space of smooth functions $f : M \to \mathbb{R}$, given by

$$X_p(f) = \frac{\mathrm{d}}{\mathrm{d}t} f(\lambda(t))\bigg|_{t=0}.$$

We observe that:

(i) $X_p$ is linear: $X_p(f + ag) = X_p(f) + aX_p(g)$, for $f, g$ smooth, and $a \in \mathbb{R}$.

(ii) $X_p$ satisfies the Leibniz rule:

$$X_p(fg) = X_p(f)g(p) + f(p)X_p(g).$$

If $(O, \phi)$ is a chart and $p \in O$, write

$$\phi(p) = (x^1(p), \ldots, x^n(p)).$$

Let $F = f \circ \phi^{-1}$, and $x^i(t) = x^i(\lambda(t))$. Now $F : \mathbb{R}^n \to \mathbb{R}$, and $x : \mathbb{R} \to \mathbb{R}^n$. Then,

$$f \circ \lambda(t) = f \circ \phi^{-1} \circ \phi \circ \lambda(t) = F \circ x(t),$$

and by applying the regular chain rule for functions $\mathbb{R}^m \to \mathbb{R}^n$,

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\lambda(t))\bigg|_{t=0} = \frac{\partial F}{\partial x^\mu}(x) \cdot \frac{\mathrm{d}x^\mu}{\mathrm{d}t}\bigg|_{t=0}.$$

**Proposition 1.1.** *The set of tangent vectors to curves at $p$ forms a vector space, $T_pM$, of dimension $n = \dim M$. We call $T_pM$ the* tangent space *to $M$ at $p$.*

**Proof:** Given $X_p, Y_p$ tangent vectors, we need to show that $\alpha X_p + \beta Y_p$ is a tangent vector for $\alpha, \beta \in \mathbb{R}$.

Let $\lambda, \kappa$ be smooth curves with $\lambda(0) = \kappa(0) = p$, and whose tangent vectors at $p$ are $X_p, Y_p$, respectively.

Let $(O, \phi)$ be a chart with $p \in O$ and $\phi(p) = 0$, and define

$$\gamma(t) = \phi^{-1}(\alpha\phi(\lambda(t)) + \beta\phi(\kappa(t))).$$

This exists, as this is just the sum of elements in $\mathbb{R}^n$.

Now $\gamma(0) = \phi^{-1}(0) = p$. If $Z_p$ is the tangent to $v$ at $p$, then

$$\begin{aligned} Z_p(f) = \frac{\mathrm{d}}{\mathrm{d}t} f(v(t))\bigg|_{t=0} &= \frac{\partial F}{\partial x^\mu}\bigg|_0 \frac{\mathrm{d}}{\mathrm{d}t}[\alpha x^\mu(\lambda(t)) + \beta x^\mu(\kappa(t))]\bigg|_{t=0} \\ &= \alpha \frac{\partial F}{\partial x^\mu}\bigg|_0 \frac{\mathrm{d}}{\mathrm{d}t} x^\mu(\lambda(t))\bigg|_{t=0} + \beta \frac{\partial F}{\partial x^\mu}\bigg|_0 \frac{\mathrm{d}}{\mathrm{d}t} x^\mu(\kappa(t))\bigg|_{t=0} \\ &= \alpha X_p(f) + \beta Y_p(f). \end{aligned}$$

Thus $T_pM$ is a vector space.

To see that $T_pM$ is $n$-dimensional, consider the curves

$$\lambda_\mu(t) = \phi^{-1} \underbrace{(0, \ldots, 0, t, 0, \ldots, 0)}_{\mu\text{'th component}}.$$

We denote the tangent vector to $\lambda_\mu$ at $p$ by $(\partial/\partial x^\mu)_p$. To see why, note that

$$\left(\frac{\partial}{\partial x^\mu}\right)_p f = \frac{\partial F}{\partial x^\mu}\bigg|_{\phi(p)=0}.$$

These vectors are linearly independent. Otherwise there exists $\alpha^\mu$ not all zero such that

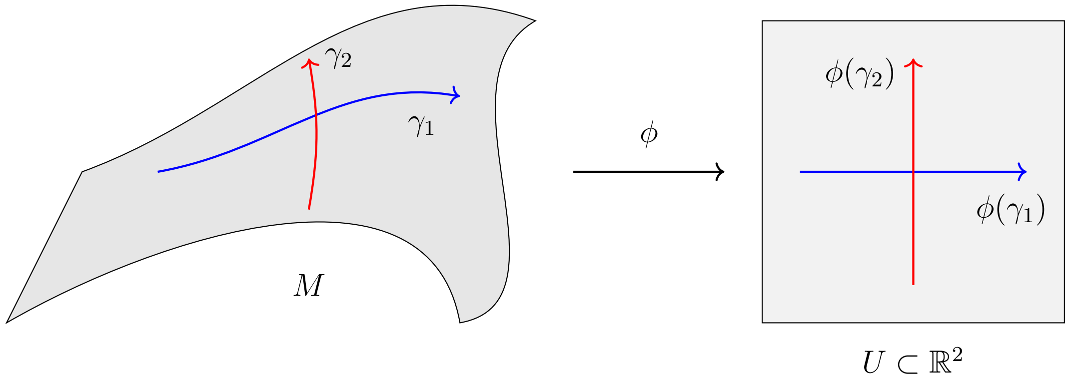
$$\alpha^\mu \left(\frac{\partial}{\partial x^\mu}\right)_p = 0 \implies \alpha^\mu \frac{\partial F}{\partial x^\mu}\bigg|_0 = 0,$$

for all $F$. Setting $F = x^\nu$ gives $\alpha^\nu = 0$.

Further, these vectors form a basis for $T_pM$, since if $\lambda$ is any curve with tangent $X_p$ at $p$, then

$$X_p(f) = \frac{\partial F}{\partial x^\mu}\bigg|_0 \frac{\mathrm{d}}{\mathrm{d}t} x^\mu(\lambda(t))\bigg|_{t=0} = X^\mu \left(\frac{\partial}{\partial x^\mu}\right) f.$$

Here, we set
$$X^\mu = \frac{\mathrm{d}}{\mathrm{d}t} x^\mu(\lambda(t))\bigg|_{t=0}.$$
These are the components of $X_p$ with respect to the basis.




Notice that the basis $\{(\partial/\partial x^\mu)_p\}$ depends on the coordinate chart $\phi$. Suppose we choose another chart $(O', \phi')$, again centred at $p$. Write $\phi' = (x'^1, \ldots, x'^n)$.

Then if $F \ = f \circ (\phi')^{-1}$, then

$$\begin{aligned} F(x) &= f \circ \phi^{-1}(x) = f \circ (\phi')^{-1} \circ \phi' \circ \phi^{-1}(x) \\ &= F'(x'(x)). \end{aligned}$$

Therefore,

$$\left(\frac{\partial}{\partial x^\mu}\right)_p f = \frac{\partial F}{\partial x^\mu}\bigg|_{\phi(p)} = \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{\partial F'}{\partial x'^\nu}\right)_{\phi'(p)} = \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)_{\phi(p)} \cdot \left(\frac{\partial}{\partial x'^\nu}\right)_p f.$$

We deduce that
$$\left(\frac{\partial}{\partial x^\mu}\right)_p = \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{\partial}{\partial x'^\nu}\right)_p.$$

So let $X^\mu$ be the components of $X_p$ with respect to $\{(\partial/\partial x^\mu)_p\}$, and $X'^\mu$ be the components with respect to $\{(\partial/\partial x'^\mu)_p\}$. So,

$$\begin{aligned} X_p = X^\mu \left(\frac{\partial}{\partial x^\mu}\right)_p &= X'^\mu \left(\frac{\partial}{\partial x'^\mu}\right)_p \\ &= X^\mu \left(\frac{\partial x'^\sigma}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{\partial}{\partial x'^\sigma}\right)_p. \end{aligned}$$

Therefore,

$$X'^{\mu} = \left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right)_{\phi(p)} X^{\nu}.$$

We do not have to choose a coordinate basis such as $\{(\partial/\partial x^{\mu})_p\}$. With respect to a general basis $\{e_\mu\}$ for $T_pM$, we write $X_p = X^\mu e_\mu$, for $X^\mu \in \mathbb{R}$ the components.

We always use summation conventions: we always contract one upstairs and one downstairs index.

## 1.3 Covectors

Recall that if $V$ is a vector space over $\mathbb{R}$, the *dual space* $V^*$ is the space of linear maps from $V$ to $\mathbb{R}$. If $V$ is $n$-dimensional, then so is $V^*$.

Given a basis $\{e_\mu\}$ for $V$, we define the *dual basis* $\{f^\mu\}$ for $V^*$ by requiring that

$$f^\mu(e_\nu) = \delta^\mu{}_\nu.$$

If $V$ is finite dimensional, then $V^{**} = (V^*)^*$ is isomorphic to $V$: to an element $x$ of $V$, we assign a linear map $\Lambda_x : V^* \to \mathbb{R}$ by

$$\Lambda_x(\omega) = \omega(x).$$

**Definition 1.3.** The dual space of $T_pM$ is denoted $T_p^*M$, and is called the *cotangent space* to $M$ at $p$.

An element of this space is a covector at $p$. If $\{e_\mu\}$ is a basis for $T_pM$ and $\{f^\mu\}$ the dual basis for $T_p^*M$, then we can expand a covector $\eta$ as

$$\eta = \eta_\mu f^\mu,$$

for $\eta_\mu \in \mathbb{R}$ the components of $\eta$.

Note that:

- $\eta(e_\nu) = \eta_\mu f^\mu(e_\nu) = \eta_\mu \delta^\mu{}_\nu = \eta_\nu$.
- $\eta(X) = \eta(X^\mu e_\mu) = X^\mu \eta(e_\mu) = X^\mu \eta_\mu$.

**Definition 1.4.** If $f : M \to \mathbb{R}$ is a smooth function, define $(\mathrm{d}f)_p \in T_p^*M$, the *differential* of $f$ at $p$, by

$$(\mathrm{d}f)_p(X) = X(f),$$

for any $X \in T_pM$. $(\mathrm{d}f)_p$ is sometimes also called the *gradient* of $f$ at $p$.

If $f$ is a constant, then $X(f) = 0 \implies (\mathrm{d}f)_p = 0$.

If $(O, \phi)$ is a coordinate chart with $p \in O$ and $\phi = (x^1, \ldots, x^n)$, then we can set $f = x^\mu$ to find $(\mathrm{d}x^\mu)_p$. Now,

$$(\mathrm{d}x^\mu)_p \left(\frac{\partial}{\partial x^\nu}\right)_p = \left(\frac{\partial x^\mu}{\partial x^\nu}\right)_{\phi(p)} = \delta^\mu{}_\nu.$$

Hence, $\{(\mathrm{d}x^\mu)_p\}$ is the dual basis to $\{(\partial/\partial x^\mu)_p\}$. In this basis, we can compute

$$[(\mathrm{d}f)_p]_\mu = (\mathrm{d}f)_p \left(\frac{\partial}{\partial x^\mu}\right)_p = \left(\frac{\partial}{\partial x^\mu}\right)_p f = \left(\frac{\partial F}{\partial x^\mu}\right)_{\phi(p)},$$

justifying the language of the 'gradient'.

It can be shown that if $(O', \phi')$ is another chart with $p \in O'$, then

$$(\mathrm{d}x^\mu)_p = \left(\frac{\partial x^\mu}{\partial x'^\nu}\right)_{\phi'(p)} (\mathrm{d}x'^\nu)_p,$$

where $x(x') = \phi \circ \phi'^{-1}$, and hence if $\eta_\mu$, $\eta'_\mu$ are components with respect to these bases, then

$$\eta'_\mu = \left(\frac{\partial x^\nu}{\partial x'^\mu}\right)_{\phi'(p)} \eta_\nu.$$

## 1.4 The (Co)tangent bundle

We can glue together the tangent spaces $T_pM$ as $p$ varies to get a new $2n$ dimensional manifold, $TM$, the *tangent bundle*. We let

$$TM = \bigcup_{p \in M} \{p\} \times T_pM,$$

the set of ordered pairs $(p, X)$, with $p \in M$, $X \in T_pM$. If $\{(O_\alpha, \phi_\alpha)\}$ is an atlas on $M$, we obtain an atlas for $TM$ by setting

$$\tilde{O}_\alpha = \bigcup_{p \in O_\alpha} \{p\} \times T_pM,$$

and

$$\tilde{\phi}_\alpha((p, X)) = (\phi_\alpha(p), X^\mu) \in U_\alpha \times \mathbb{R}^n = \tilde{U}_\alpha,$$

where $X^\mu$ are the components of $X$ with respect to the coordinate basis of $\phi_\alpha$.

It can be shown that if $(O, \phi)$ and $(O', \phi')$ are two charts on $M$, on $\tilde{U} \cap \tilde{U}'$, if we write $\phi' \circ \phi^{-1}(x) = x'(x)$, then

$$\tilde{\phi}' \circ \tilde{\phi}^{-1}(x, X^\mu) = \left( x'(x), \frac{\partial x'^\mu}{\partial x^\nu} X^\nu \right).$$

This lets us deduce that $TM$ is a manifold.

A similar construction permits us to define the *cotangent bundle*

$$T^*M = \bigcup_{p \in M} \{p\} \times T_p^* M.$$

The map $\pi : TM \to M$ which takes $(p, X) \mapsto p$ is smooth (show this!)

## 1.5 Abstract Index Notation

We have used Greek letters $\mu, \nu$ to label components of vectors (or covectors) with respect to the bases $\{e_\mu\}$. Equations involving these quantities refer to the specific basis, for example if we write $X^\mu = \delta^\mu$, this says $X$ only has one non-zero component *in the current basis*, which will not be true in other bases.

However, we know that some equations hold in all bases, e.g.

$$\eta(X) = X^\mu \eta_\mu.$$

To capture this, we can use *abstract index notation*. We denote a vector by $X^a$, where the latin index $a$ does not denote a component, rather it tells us that $X^a$ is a vector. Similarly we denote a covector $\eta$ by $\eta_a$.

If an equation is true in all bases, then we can replace Greek indices by latin indices:

$$\eta(X) = X^a \eta_a = \eta_a X^a,$$

or

$$X(f) = X^a (\mathrm{d}f)_a.$$

An equation in abstract index notation can always be turned into an equation for components, by picking a basis and changing $a \to \mu$, $b \to \nu$.

## 1.6 Tensors

In Newtonian physics, we know that some quantities are described by higher rank orders, e.g. the inertia tensor or the metric.

**Definition 1.5.** A *tensor* of type $(r, s)$ is a multilinear map

$$T : (T_p^* M)^r \times (T_p M)^s \to \mathbb{R}.$$

**Example 1.3.**

1. A tensor of type $(0,1)$ is a linear map $T_pM \to \mathbb{R}$, i.e. just a covector.
2. A tensor of type $(1,0)$ is a linear map $T^*pM \to \mathbb{R}$, i.e. an element of $(T^*pM)^* \cong T_pM$, a vector.
3. We can define a $(1,1)$ tensor $\delta$ by

$$\delta(\omega, X) = \omega(X),$$

where $\omega \in T_p^*M$, $X \in T_pM$.

If $\{e_\mu\}$ is a basis for $T_pM$ and $\{f^\mu\}$ its dual basis, then the components of an $(r,s)$ tensor $T$ are

$$T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s} = T(f_1^\mu, \ldots, f^{\mu_r}, e_{\nu_1}, \ldots, e_{\nu_s}).$$

In abstract index notation, we denote $T$ by $T^{a_1 \cdots a_r}{}_{b_1 \cdots b_s}$. Tensors at $p$ form a vector space over $\mathbb{R}$ of dimension $n^{r+s}$.

**Example 1.4.**

1. Consider $\delta$ above. Then,

$$\delta^\mu{}_\nu = \delta(f^\mu, e_\nu) = f^\mu(e_\nu) = \begin{cases} 1 & \mu = \nu, \\ 0 & \mu \neq \nu. \end{cases}$$

We can write $\delta$ as $\delta^a{}_b$ in AIN.

2. Consider a $(2,1)$ tensor $T$, and let $\omega, \eta \in T_p^*M$, $X \in T_pM$. Then,

$$\begin{aligned} T(\omega, \eta, X) &= T(\omega_\mu f^\mu, \eta_\nu f^\nu, X^\sigma e_\sigma) \\ &= \omega_\mu \eta_\nu X^\sigma T(f^\mu, f^\nu, e_\sigma) \\ &= \omega_\mu \eta_\nu X^\sigma T^{\mu\nu}{}_\sigma. \end{aligned}$$

In AIN,

$$T(\omega, \nu, X) = \omega_a \eta_b X^c T^{ab}{}_c.$$

This can be generalized to higher ranks.

## 1.7 Change of Bases

We have already seen how the components of $X$ or $\eta$ with respect to a coordinate basis ($X^\mu, \eta_\nu$ respectively) change under a change of coordinates.

But we do not only have to consider coordinate bases.

Suppose $\{e_\mu\}$ and $\{e'_\mu\}$ are two bases for $T_pM$ with dual bases $\{f^\mu\}$ and $\{f'^\mu\}$.

As these are bases, we can expand

$$f'^\mu = A^\mu{}_\nu f^\nu, \qquad e'_\mu = B^\nu{}_\mu e_\nu,$$

for some $A^\mu{}_\nu, B^\mu{}_\nu \in \mathbb{R}$. But,

$$\begin{aligned}\delta^\mu{}_\nu &= f'^\mu(e'_\nu) = A^\mu{}_\tau f^\tau(B^\sigma{}_\nu e_\sigma)\\ &= A^\mu{}_\tau B^\sigma{}_\nu f^\tau(e_\sigma) = A^\mu{}_\tau B^\sigma{}_\nu \delta^\tau{}_\sigma\\ &= A^\mu{}_\sigma B^\sigma{}_\nu.\end{aligned}$$

Therefore, looking at these as matrices,

$$B^\mu{}_\nu = (A^{-1})^\mu{}_\nu.$$

If

$$e_\mu = \left(\frac{\partial}{\partial x^\mu}\right)_p, \qquad e'_\mu = \left(\frac{\partial}{\partial x'^\mu}\right)_p,$$

then we have already seen

$$A^\mu{}_\nu = \left(\frac{\partial x'^\mu}{\partial x^\nu}\right)_{\phi(p)}, \qquad B^\mu{}_\nu = \left(\frac{\partial x^\mu}{\partial x'^\nu}\right)_{\phi(p)},$$

which indeed satisfies $A^\mu{}_\sigma B^\sigma{}_\nu = \delta^\mu{}_\nu$, by the chain rule.

A chance of bases induces a transformation of tensor components, for example if $T$ is a $(1,1)$-tensor, then

$$\begin{aligned}T^\mu{}_\nu &= T(f^\mu, e_\nu)\\ T'^\mu{}_\nu &= T(f'^\mu, e'_\nu) = T(A^\mu{}_\sigma f^\sigma, (A^{-1})^\tau{}_\nu e_\tau)\\ &= A^\mu{}_\sigma (A^{-1})^\tau{}_\nu T(f^\sigma, e_\tau)\\ &= A^\mu{}_\sigma (A^{-1})^\tau{}_\nu T^\sigma{}_\tau.\end{aligned}$$

## 1.8 Tensor Operations

Given an $(r,s)$-tensor, we can form a $(r-1,s-1)$-tensor by *contraction*.

For simplicity, assume $T$ is a $(2,2)$-tensor. Define a $(1,1)$-tensor $S$ by

$$S(\omega, X) = T(\omega, f^\mu, X, e_\mu).$$

To see that this is independent of the choice of basis, note

$$\begin{aligned} T(\omega, f'^{\mu}, X, e'_{\mu}) &= T(\omega, A^{\mu}{}_{\sigma} f^{\sigma}, X, (A^{-1})^{\tau}{}_{\mu} e_{\tau}) = A^{\mu}{}_{\sigma} (A^{-1})^{\tau}{}_{\mu} T(\omega, f^{\sigma}, X, e_{\tau}) \\ &= \delta^{\tau}{}_{\sigma} T(\omega, f^{\sigma}, X, e_{\tau}) = T(\omega, f^{\tau}, X, e_{\tau}) = S(\omega, X). \end{aligned}$$

So this does not depend on the choice of basis. $S$ and $T$ have components related by

$$S^{\mu}{}_{\nu} = T^{\mu\sigma}{}_{\nu\sigma}.$$

In any basis in AIN we write

$$S^{a}{}_{b} = T^{ac}{}_{bc}.$$

We can generalise to contract any upstairs index with any downstairs index in a general $(r,s)$-tensor.

Another way to make new tensors from old tensors is to form the *tensor product*. If $S$ is a $(p,q)$-tensor and $T$ is a $(r,s)$-tensor, then $S \otimes T$ is an $(p+r, q+s)$-tensor:

$$\begin{aligned} &S \otimes T(\omega^1, \ldots, \omega^p, \eta^1, \ldots, \eta^r, X_1, \ldots, X_q, Y_1, \ldots, Y_s) \\ &\quad = S(\omega^1, \ldots, \omega^p, X_1, \ldots, X_q) T(\eta^1, \ldots, \eta^r, Y_1, \ldots, Y_s). \end{aligned}$$

This is independent of basis. In AIN,

$$(S \otimes T)^{a_1 \ldots a_p b_1 \ldots b_r}{}_{c_1 \ldots c_q d_1 \ldots, d_s} = S^{a_1 \ldots a_r}{}_{c_1 \ldots c_q} T^{b_1 \ldots b_r}{}_{d_1 \ldots b_r}.$$

We can show that for any $(1,1)$-tensor $T$, in any basis we have

$$T = T^{\mu}{}_{\nu}\, e_{\mu} \otimes f^{\nu}.$$

The final tensor operations we require are *(anti)symmetrization*. If $T$ is a $(0,2)$-tensor, we can define two new tensors:

$$\begin{aligned} S(X,Y) &= \frac{1}{2}(T(X,Y) + T(Y,X)), \\ A(X,Y) &= \frac{1}{2}(T(X,Y) - T(Y,X)). \end{aligned}$$

In AIN,

$$\begin{aligned} S_{ab} &= \frac{1}{2}(T_{ab} + T_{ba}) = T_{(ab)}, \\ A_{ab} &= \frac{1}{2}(T_{ab} - T_{ba}) = T_{[ab]}. \end{aligned}$$

These operations can be applied to any pair of matching symmetries in a more general tensor, for example:

$$T^{a(bc)}{}_{de} = \frac{1}{2}(T^{abc}{}_{de} + T^{acb}{}_{de}).$$

We can also (anti)symmetrize over more than two indices.

- To symmetrize over $n$ indices, we sum over all permutations of the indices and divide by $n!$.
- To anti-symmetrize over $n$ indices, we sum over all permutation weighted by their sign, and then divide by $n!$.

For example,

$$T^{(abc)} = \frac{1}{3!}(T^{abc} + T^{bca} + T^{cab} + T^{acb} + T^{cba} + T^{bac}),$$
$$T^{[abc]} = \frac{1}{3!}(T^{abc} + T^{bca} + T^{cab} - T^{acb} - T^{cba} - T^{bac}.$$

To exclude indices from (anti)symmetrization, we use vertical lines:

$$T^{(a|b|c)} = \frac{1}{2}(T^{abc} + T^{cba}).$$

## 1.9 Tensor Bundles

The space of $(r, s)$-tensors at a point $p$ is the vector space $(T^r{}_s)_p M$. These can be glued together to form the bundle of $(r, s)$-tensors

$$T^r{}_s M = \bigcup_{p \in M} \{p\} \times (T^r{}_s)_p M.$$

If $(O, \phi)$ is a coordinate chart on $M$, set

$$\tilde{O} = \bigcup_{p \in O} \{p\} \times (T^{r_s})_p M \subseteq T^r{}_s M,$$

and

$$\tilde{\phi}(p, S_p) = (\phi(p), S^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}).$$

$T^r{}_s M$ is a manifold, with a natural smooth map $\Pi : T^r{}_s M \to M$ such that $\pi(p, S_p) = p$.

A *tensor field* is a smooth map $T : M \to T^r{}_s M$ such that $\pi \circ T = \text{id}$.

If $(O, \phi)$ is a coordinate chart on $M$, then

$$\tilde{\phi} \circ T \circ \phi(x) = (x, T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}(x)),$$

which is smooth provided the components $T^{\mu_1, \dots, \mu_r}{}_{\nu_1 \dots \nu_s}(x)$ are smooth functions of $x$.

**Example 1.5.**

If ${T^r}_s M = {T^1}_0 M \cong TM$, the tensor field is called a *vector field*. In a local coordinate patch, if $X$ is a vector field, we write $X(p) = (p, X_p)$, with

$$X_p = X^\mu(x) \left(\frac{\partial}{\partial x^\mu}\right)_p .$$

In particular, $\partial/\partial x^\mu$ are always smooth, but only defined locally.

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