III Combinatorics

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Contents

0 Introduction

We have the following list of things.

1: Set systems.

2: Isoperimetric inequalities.

3: Intersection families.

Books include 'Combinatorics' by Bollobás, and 'Combinatorics of Finite Sets', by Anderson.

1 Set Systems

Let X be a set. A *set system* on X, also called a family of subsets of X, is a family $\mathcal{A} \subseteq \mathcal{P}(X)$. For example,

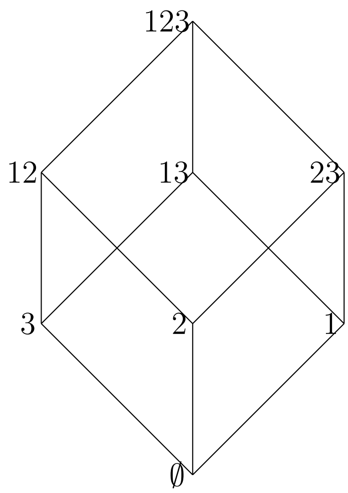
$$X^{(r)} = \{A \subseteq X \mid |A| = r\}.$$

Usually, $X = [n] = \{1, 2, \ldots, n\}$, so $|X^{(r)}| = \binom{n}{r}$. Thus,

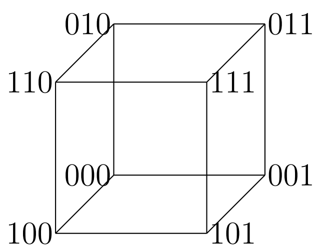
$$[4]^{(2)} = \{12, 13, 14, 23, 24, 34\}.$$

We make $\mathcal{P}(X)$ into a graph by joining A and B if $|A \triangle B| = 1$. This is the *discrete cube* Q_n.

Literally just a cube.



Alternatively, can view Q_n as an n-dimensional unit cube $\{0,1\}^n$, by identifying e.g. $\{1,3\}$ with the binary string $101000\cdots$.



Say $\mathcal{A} \subseteq \mathcal{P}(X)$ is a *chain* if, for all $A, B \in \mathcal{A}$, $A \subseteq B$ or $B \subseteq A$. For example,

$$\mathcal{A} = \{23, 12357, 1235, 123567\}$$

is a chain.

Say $\mathcal{A}$ is an *antichain* if, for all $A, B \in \mathcal{A}$ and $A \neq B$, we have $A \not\subseteq B$. For example, $\mathcal{A} = \{23, 137\}$ is an antichain.

How large can a chain be? We can achieve $|\mathcal{A}| = n + 1$ by taking

$$\mathcal{A} = \{\emptyset, 1, 12, 123, \ldots, [n]\}$$

Cannot beat this as each $0 \leq r \leq n$, $\mathcal{A}$ can contain at most one r-set (a member of $X^{(r)}$).

How large can an antichain be? We can achieve $|\mathcal{A}| = n$, e.g. $\mathcal{A} = \{1, 2, \ldots, n\}$. More generally, we can take $\mathcal{A} = X^{(r)}$, and the best is when $r = \lfloor n/2 \rfloor$.

Theorem 1.1 (Sperner's Lemma)**.** *Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an antichain. Then,*

$$|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

The idea is follows: we know that a chain meets a layer in at most one point, since a layer is an antichain. If we decompose the cube into chains, we have at most one element of an antichain in each chain.

Proof: We will decompose $\mathcal{P}(X)$ into $\binom{n}{\lfloor n/2 \rfloor}$ chains, then we are done. To achieve this, it is sufficient to find:

(i) For each $r < n/2$, a matching from $X^{(r)}$ to $X^{(r+1)}$.

(ii) For each $r \geq n/2$, a matching from $X^{(r)}$ to $X^{(r-1)}$.

Then we put these together to form our chains; each passing through $X^{(\lfloor n/2 \rfloor)}$.

By taking complements, it is enough to prove (i).

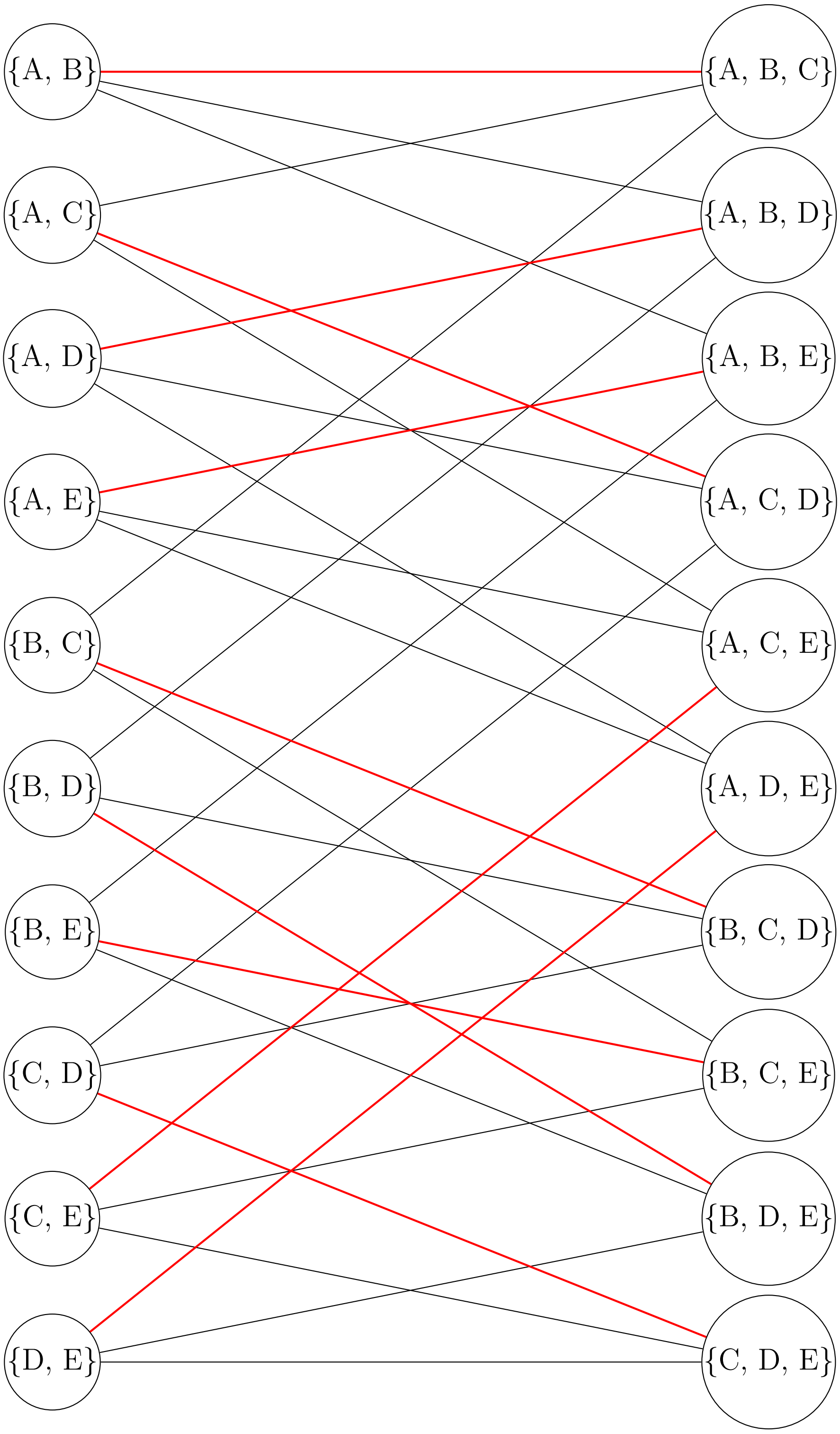
Let G be the bipartite subgraph of Q_n spanned by $X^{(r)} \cup X^{(r+1)}$: we seek a matching from $X^{(r)}$ to $X^{(r+1)}$.

For any $S \subseteq X^{(r)}$, the number of edges from S to $\Gamma(S)$ is $|S|(n - r)$, since each edge in S has $n - r$ edges.

Moreover there are at most $|\Gamma(S)|(r+1)$ edges, counting from $\Gamma(S)$. Therefore,

$$|\Gamma(S)| \geq \frac{|S|(n - r)}{r + 1} \geq |S|.$$

So we are done, by Hall's matching theorem.



When do we have equality in Sperner's? The above proof tells us nothing.

Our aim is to prove the following: if $\mathcal{A}$ is an antichain, then

$$\sum_{r=0}^{n} \frac{|\mathcal{A} \cap X^{(r)}|}{\binom{n}{r}} \leq 1.$$

In other words, the percentages of each layer occupied add up to at most 1. This trivially implies Sperner's.

1.1 Shadows

For $\mathcal{A} \subseteq X^{(r)}$, the *shadow* of $\mathcal{A}$ is $\partial\mathcal{A} = \partial^{-}\mathcal{A} \subseteq X^{(r-1)}$ defined by

$$\partial\mathcal{A} = \{B \in X^{(r-1)} \mid B \subseteq A \text{ for some } A \in \mathcal{A}\}.$$

For example, if $\mathcal{A} = \{123, 124, 134, 137\}$, then

$$\partial\mathcal{A} = \{12, 13, 23, 14, 24, 34, 17, 37\}.$$

Proposition 1.1 (Local LYM)**.** *Let* $\mathcal{A} \subseteq X^{(r)}$*. Then,*

$$\frac{|\partial\mathcal{A}|}{\binom{n}{r-1}} \geq \frac{|\mathcal{A}|}{\binom{n}{r}}.$$

So, the fraction of the local occupancy by $\partial\mathcal{A}$, is at least the occupancy by $\mathcal{A}$.

Remark. LYM = Lubell, Meshalkin, Yamamoto.

Proof: We look at the number of $\mathcal{A}$ to $\partial\mathcal{A}$ edges in the bipartite graph Q_n; counting from above, there are exactly $|\mathcal{A}|r$.

However counting from below, it is at most $|\partial\mathcal{A}|(n-r+1)$. So,

$$\frac{|\partial\mathcal{A}|}{|\mathcal{A}|} \geq \frac{r}{n-r+1} = \frac{\binom{n}{r-1}}{\binom{n}{r}}.$$

So we are done.

Remark. When do we have equality? We lose equality if an element in $\partial\mathcal{A}$ is connected to an element not in $\mathcal{A}$, so for this not to occur, we need that for all $A \in \mathcal{A}$, and $i \in A$, $j \notin \mathcal{A}$, that $A - \{i\} \cup \{j\} \in \mathcal{A}$.

But this is very strong, and in fact either $\mathcal{A} = \emptyset$ or $X^{(r)}$.

Theorem 1.2 (LYM Inequality)**.** *Let $A \subseteq \mathcal{P}(X)$ be an antichain. Then,*

$$\sum_{r=0}^{n} \frac{|\mathcal{A} \cap X^{(r)}|}{\binom{n}{r}} \leq 1.$$

As a bit of notation, we write $\mathcal{A}_r$ for $\mathcal{A} \cap X^{(r)}$.

We will look at two proofs. The first idea is to bubble down with local LYM.

Proof: Obviously

$$\frac{|\mathcal{A}_n|}{\binom{n}{n}} \leq 1.$$

Now, ∂A_n and $\mathcal{A}_{n-1}$ are disjoint, as $\mathcal{A}$ is an antichain. So,

$$\frac{|\partial \mathcal{A}_n|}{\binom{n}{n-1}} + \frac{|\mathcal{A}_{n-1}|}{\binom{n}{n-1}} = \frac{|\partial \mathcal{A}_n \cup \mathcal{A}_{n-1}|}{\binom{n}{n-1}} \leq 1,$$

whence we get

$$\frac{|\mathcal{A}_n|}{\binom{n}{n}} + \frac{|\mathcal{A}_{n-1}|}{\binom{n}{n-1}} \leq 1,$$

by local LYM. We now continue again. Notice $\partial(\partial \mathcal{A}_n \cup \mathcal{A}_{n-1})$ is disjoint from $\mathcal{A}_{n-2}$, we find

$$\frac{|\partial(\partial \mathcal{A}_n \cup \mathcal{A}_{n-1})|}{\binom{n}{n-2}} + \frac{|\mathcal{A}_{n-2}|}{\binom{n}{n-2}} \leq 1,$$

whence

$$\frac{|\partial \mathcal{A}_n \cup \mathcal{A}_{n-1}|}{\binom{n}{n-1}} + \frac{|\mathcal{A}_{n-2}|}{\binom{n}{n-2}} \leq 1.$$

We can now continue inductively.

When do we have equality? We must have had equality in each use of local LYM. Hence equality in LYM needs that the maximum r with $\mathcal{A}_r \neq \emptyset$, then $\mathcal{A}_r = X^{(r)}$.

Hence equality in Sperner needs either $\mathcal{A} = X^{(n/2)}$, if n is even, or $\mathcal{A} = X^{(\lfloor n/2 \rfloor)}$ or $X^{(\lceil n/2 \rceil)}$, for n odd.

Now time for another proof.

Proof: Choose uniformly at random a maximal chain $\mathcal{C}$. For any r-set A, note that

$$\mathbb{P}(A \in \mathcal{C}) = \frac{1}{\binom{n}{r}}.$$

So for our antichain $\mathcal{A}$,

$$\mathbb{P}(\mathcal{C} \text{ meets } \mathcal{A}_r) = \frac{|\mathcal{A}_r|}{\binom{n}{r}},$$

as these events are disjoint. Hence, since $\mathcal{C}$ can meet $\mathcal{A}$ at one point at most,

$$\mathbb{P}(\mathcal{C} \text{ meets } \mathcal{A}) = \sum_{r=0}^{n} \frac{|\mathcal{A}_r|}{\binom{n}{r}},$$

from which we get

$$\sum_{r=0}^{n} \frac{|\mathcal{A}_r|}{\binom{n}{r}} \leq 1.$$

Equivalently, the number of maximal chains is $n!$, and the number through any fixed r-set is $r!(n-r)!$, so

$$\sum_r |\mathcal{A}_r| r!(n-r)! \leq n!.$$

We now return to shadows. For $\mathcal{A} \subseteq X^{(r)}$, we have

$$|\partial\mathcal{A}| \geq |\mathcal{A}| \frac{r}{n-r+1}.$$

We know that equality is rare: it only happens for $\mathcal{A} = \emptyset$, or $X^{(r)}$. What happens in between?

In other words, given $|\mathcal{A}|$, how should we choose $\mathcal{A} \subseteq X^{(r)}$ to minimise $|\partial\mathcal{A}|$?

It is believable that if $|\mathcal{A}| = \binom{k}{r}$, then we should take $\mathcal{A} = [k]^{(r)}$. In between adjacent binomials, it is believable that we should take $[k]^{(r)}$, plus some r-sets in $[k+1]^{(r)}$.

Example 1.1.

For $\mathcal{A} \subseteq X^{(3)}$ with

$$|\mathcal{A}| = \binom{8}{3} + \binom{4}{2},$$

we could take

$$\mathcal{A} = [8]^3 \cup \{9 \cup B \mid B \in [4]^{(2)}\}.$$

In some ways our set $\mathcal{A}$ should be of minimal 'order', under some ordering on $X^{(r)}$.

1.2 Total Orders

Let A, B be distinct r-sets, and say $A = a_1 \dots a_r$, $B = b_1 \dots b_r$, where $a_1 < \dots < a_r$, $b_1 < \dots < a_r$.

We say that $A < B$ in the *lexographic* or *lex* ordering if for some j we have $a_i = b_i$ for all $i < j$, and $a_j < b_j$. So lex cares about small elements.

Example 1.2.

Lex on $[4]^{(2)}$ orders the elements as $12, 13, 14, 23, 24, 34$.

Lex on $[6]^{(3)}$ orders the elements as

$$\begin{aligned}&123, 124, 125, 126, 134, 135, 136, 145, 146, 156,\\ &\quad 234, 235, 236, 245, 246, 256, 345, 346, 356, 456.\end{aligned}$$

We say that $A < B$ in the *colexographic* or *colex* ordering if for some j, we have $a_i = b_i$ for all $i > j$, and $a_j < b_j$. So colex cares about big elements.

Example 1.3.

Colex on $[4]^{(2)}$ orders the elements as $12, 13, 23, 14, 24, 34$.

Colex on $[6]^{(3)}$ orders the elements as

$$\begin{aligned}&123, 124, 134, 234, 125, 135, 235, 145, 245, 345,\\ &\quad 126, 136, 236, 146, 246, 346, 156, 256, 356, 456.\end{aligned}$$

Note that in colex, $[n-1]^{(r)}$ is an initial segment of $[n]^{(r)}$. This is not true in lex.

This allows us to view colex as an enumeration of $\mathbb{N}^{(r)}$.

Remark. $A < B$ in colex $\iff$ $A^c < B^c$ in lex, with ground set ordering ordering reversed.

Colex in particular may be the ordering we want to solve the above problem, minimizing $|\partial\mathcal{A}|$. Our aim will then be to show that initial segments of colex are the best for ∂, i.e. if $\mathcal{A} \subseteq X^{(r)}$ and $\mathcal{C} \subseteq X^{(r)}$ is the initial segment of colex with $|\mathcal{C}| = |\mathcal{A}|$, then

$$|\partial\mathcal{C}| \leq |\partial\mathcal{A}|.$$

In particular,

$$|\mathcal{A}| = \binom{k}{r} \implies |\partial\mathcal{A}| = \binom{k}{r-1}.$$

1.3 Compression

The idea is to try to transform $\mathcal{A} \subseteq X^{(r)}$ into some $\mathcal{A} \subseteq X^{(r)}$ such that:

(i) $|\mathcal{A}'| = |\mathcal{A}|$.

(ii) $|\partial\mathcal{A}'| \leq |\partial\mathcal{A}|$.

(iii) $\mathcal{A}'$ looks more like $\mathcal{C}$ than $\mathcal{A}$ did.

Ideally, we would like a family of such ‘compressions’

$$\mathcal{A} \to \mathcal{A}' \to \cdots \to \mathcal{B},$$

such that either $\mathcal{B} = \mathcal{C}$, or $\mathcal{B}$ is so similar to $\mathcal{C}$ that we can directly check that

$$|\partial\mathcal{B}| \geq |\partial\mathcal{C}|.$$

The fact that colex prefers 1 to 2 inspires the following: fix $1 \leq i < j \leq n$. The *ij-compression* C_{ij} is defined as follows:

For $A \in X^{(r)}$, set

$$C_{ij}(A) = \begin{cases} A \cup i - j & \text{if } j \in A,\, i \notin A\ , \\ A & \text{else.} \end{cases}$$

For $\mathcal{A} \subseteq X^{(r)}$, set

$$C_{ij}(\mathcal{A}) = \{C_{ij}(A) \mid A \in \mathcal{A}\} \cup \{A \in \mathcal{A} \mid C_{ij}(A) \in \mathcal{A}\}.$$

So $C_{ij}(\mathcal{A}) \subseteq X^{(r)}$, and $|C_{ij}(\mathcal{A})| = |\mathcal{A}|$. Say $\mathcal{A}$ is ij-compressed if $C_{ij}(\mathcal{A}) = \mathcal{A}$.

Lemma 1.1. *Let $\mathcal{A} \subseteq X^{(r)}$, and $1 \leq i < j \leq n$. Then*

$$|\partial C_{ij}(\mathcal{A})| \leq |\partial\mathcal{A}|.$$

Proof: Write $\mathcal{A}'$ for $C_{ij}(\mathcal{A})$, and let $B \in \partial\mathcal{A}' - \partial\mathcal{A}$. We will show that $i \in B, j \notin B$, and $B \cup j - i \in \partial\mathcal{A} - \partial\mathcal{A}'$, which will show that we are done.

We have that $B \cup x \in \mathcal{A}'$, for some x, with $B \cup x \notin \mathcal{A}$. So, $i \in B \cup x$, $j \notin B \cup x$, and $(B \cup x) \cup j - i \in \mathcal{A}$.

We cannot have $x = i$, otherwise $(B \cup x) \cup j - i = B \cup j$, giving $B \in \partial\mathcal{A}$. So $i \in B$, and $j \notin B$.

Also, notice $B \cup j - i \in \partial\mathcal{A}$, since $(B \cup x) \cup j - i \in \mathcal{A}$.

Suppose $B \cup j - i \in \partial\mathcal{A}'$, so $(B \cup j - i) \cup y \in \mathcal{A}'$ for some y. We cannot have $y = i$, else $B \cup j \in \mathcal{A}'$, so $B \cup j \in \mathcal{A}$, contradicting $B \notin \partial\mathcal{A}$. So $j \in (B \cup j - i) \cup y$, and $i \notin (B \cup j - i) \cup y$.

Whence both $(B \cup j - i) \cup y$ and $B \cup y$ belong to $\mathcal{A}$, by definition of $\mathcal{A}'$, contradicting $B \notin \partial\mathcal{A}$.

Remark. We have actually shown that $\partial C_{ij}(\mathcal{A}) \subseteq C_{ij}\partial\mathcal{A}$.

Say $\mathcal{A} \subseteq X^{(r)}$ is *left-compressed* if $C_{ij}(\mathcal{A}) = \mathcal{A}$ for all $i \leq j$.

Corollary 1.1. *Let $\mathcal{A} \subseteq X^{(r)}$. Then there exists a left-compressed $\mathcal{B} \subseteq X^{(r)}$ with $|\mathcal{B}| = |\mathcal{A}|$, and $|\partial\mathcal{B}| \leq |\partial\mathcal{A}|$.*

Proof: Define a sequence $\mathcal{A}_0, \mathcal{A}_1, \ldots$ as follows. Let $\mathcal{A}_0 = \mathcal{A}$.

Having defined $\mathcal{A}_0, \ldots, \mathcal{A}_k$, if $\mathcal{A}_k$ is left-compressed then we can stop the sequence with $\mathcal{A}_k$.

If not, choose $i < j$ such that $\mathcal{A}_j$ is not ij-compresssed, and set $\mathcal{A}_{k+1} = C_{ij}(\mathcal{A}_k)$.

This must terminate, as for example

$$\sum_{A \in \mathcal{A}_k} \sum_{i \in A} i$$

is strictly decreasing in k.

Then the final term $\mathcal{B} = \mathcal{A}_k$ satisfies that $|\mathcal{B}| = |\mathcal{A}|$, and $|\partial\mathcal{B}| \leq |\partial\mathcal{A}|$, by the previous lemma.

Remark.

1. Similarly we may choose all $\mathcal{B} \subseteq X^{(r)}$ with $|\mathcal{B}| = |\mathcal{A}|$, and $|\partial\mathcal{B}| \leq |\partial\mathcal{A}|$, and

then choose one with smallest sum of elements.

2. We can choose the order of the C_{ij} so that no C_{ij} is applied twice.
3. Any initial segment of colex is left-compressed. The converse is false, for example lex: $\{123, 124, 125, 126\}$.

This is not exactly what we want; we want to show that this is colex.

The fact that colex prefers 23 to 14 inspires the following. Let $U, V \subseteq X$ with $|U| = |V|$, $U \cap V = \emptyset$, and $\max V > \max U$.

Define the *UV-compression* as follows: for $A \subseteq X$,

$$C_{UV}(A) = \begin{cases} A \cup U - V & \text{if } V \subseteq A,\, U \cap A = \emptyset, \\ A & \text{otherwise.} \end{cases}$$

For $\mathcal{A} \subseteq X^{(r)}$, set

$$C_{UV}(\mathcal{A}) = \{C_{UV}(A) \mid A \in \mathcal{A}\} \cup \{A \in \mathcal{A} \mid C_{UV}(A) \in \mathcal{A}\}.$$

For example if $\mathcal{A} = \{123, 124, 147, 237, 238, 149\}$, then

$$C_{23,14}(\mathcal{A}) = \{123, 124, 147, 237, 238, 239\}.$$

So $C_{UV}(\mathcal{A}) \subseteq X^{(r)}$, and $|C_{UV}(\mathcal{A})| = |\mathcal{A}|$. Say $\mathcal{A}$ is *UV-compressed* if $C_{UV}(\mathcal{A}) = \mathcal{A}$.

Sadly, we can have $|\partial C_{UV}(\mathcal{A})| > |\partial \mathcal{A}|$. For example if $\mathcal{A} = \{147, 137\}$, then $|\partial \mathcal{A}| = 5$, but $C_{23,14}(\mathcal{A}) = \{237, 147\}$ has $|\partial C_{23,14}(\mathcal{A})| = 6$.

We can prove the following at least:

Lemma 1.2. *Let $\mathcal{A} \subseteq X^{(r)}$ be UV-compressed for all U, V with $|U| = |V|$, $U \cap V = \emptyset$ and $\max V > \max U$. Then $\mathcal{A}$ is an initial segment of colex.*

Proof: Suppose not. Then there exists $A, B \in X^{(r)}$ with $B < A$ in colex, but $A \in \mathcal{A}$, $B \notin \mathcal{A}$.

Set $V = A \setminus B$, $U = B \setminus A$. Then clearly $|V| = |U|$, and U, V are disjoint, with $\max V > \max U$ since $B < A$. So, $C_{UV}(A) = B$, contradicting $\mathcal{A}$ UV-compressed.

But we can show the following:

Lemma 1.3. *Let $U, V \subseteq X$ with $|U| = |V|$, $U \cap V = \emptyset$, and $\max U < \max V$. For $\mathcal{A} \subseteq X^{(r)}$, suppose that for all u, there exists v such that $\mathcal{A}$ is $(U - u, V - v)$-compressed. Then,*

$$|\partial C_{UV}(\mathcal{A})| \leq |\partial \mathcal{A}|.$$

Proof: Let $\mathcal{A}' = C_{UV}(\mathcal{A})$. For $B \in \partial\mathcal{A}' - \partial\mathcal{A}$, we will show that $U \subseteq B$, $V \cap B = \emptyset$, and $B \cup V - U \in \partial\mathcal{A} - \partial\mathcal{A}'$.

We have that $B \cup x \in \mathcal{A}'$, and $B \cup x \notin \mathcal{A}$. So $U \subseteq (B \cup x)$, $V \cap (B \cup x) = \emptyset$, and $(B \cup x) \cup V - U \in \mathcal{A}$, by the definition of C_{UV}.

If $x \in U$, then there exists $y \in U$ such that $\mathcal{A}$ is $(U - x, V - y)$-compressed, by assumption. So from $(B \cup x) \cup V - U \in \mathcal{A}$, we have $B \cup y \in \mathcal{A}$, contradicting $B \notin \partial\mathcal{A}$.

Thus $x \notin U$, and so $U \subseteq B$, $V \cap B = \emptyset$.

We certainly have $B \cup V - U \in \partial\mathcal{A}$, as $(B \cup x) \cup V - U \in \mathcal{A}$, so we just need to show that $B \cup V - U \notin \partial\mathcal{A}'$.

Suppose that $B \cup V - U \in \partial\mathcal{A}'$, so that $(B \cup V - U) \cup w \in \mathcal{A}'$, for some w.

If $w \in U$, then we know that $\mathcal{A}$ is $(U - w, V - z)$-compressed for some $z \in V$, so $B \cup z \in \mathcal{A}$, contradicting $B \notin \partial\mathcal{A}$.

If $w \notin U$, we have that $V \subseteq (B \cup V - U) \cup w$, and $U \cap ((B \cup V - U) \cup w) = \emptyset$, so by definition of C_{UV}, we must have that both $(B \cup V - U) \cup w$ and $B \cup w \in \mathcal{A}$, contradicting $B \notin \partial\mathcal{A}$.

Theorem 1.3 (Kruskal-Katona). *Let $\mathcal{A} \subseteq X^{(r)}$, where $1 \leq r \leq n$, and let $\mathcal{C}$ be the initial sequence of colex on $X^{(r)}$, with $|\mathcal{C}| = |\mathcal{A}|$. Then,*

$$|\partial\mathcal{C}| \leq |\partial\mathcal{A}|.$$

In particular, if $|\mathcal{A}| = \binom{k}{r}$, then

$$|\partial\mathcal{A}| \geq \binom{k}{r-1}.$$

Proof: Let

$$P = \{(U, V) \mid |U| = |V| > 0, U \cap V = \emptyset, \max U < \max V\} \cup \{(\emptyset, \emptyset)\}.$$

Define sets $\mathcal{A}_0, \mathcal{A}_1, \ldots$ of sets systems in X as follows: set $\mathcal{A}_0 = \mathcal{A}$.

Having defined $\mathcal{A}_0, \ldots, \mathcal{A}_k$, if $\mathcal{A}_k$ is (U, V)-compressed for all $(U, V) \in P$, then we are done.

Otherwise, we have $(U, V) \in P$ with $|U| = |V| > 0$ and disjoint, such that A_k is not (U, V)-compressed. Choose (U, V) minimal.

Note that for all $u \in U$, there is $v \in V$ such that $(U - u, V - v) \in P$, namely take $v = \min V$. So by the previous lemma, we get

$$|\partial C_{UV}(\mathcal{A}_k)| = |\partial \mathcal{A}_k|.$$

Set $\mathcal{A}_{k+1} = C_{UV}(\mathcal{A}_k)$, and continue. This must terminate, as

$$\sum_{A \in \mathcal{A}_k} \sum_{i \in A} 2^i$$

is strictly decreasing in k. Hence the final term $\mathcal{B}$ satisfies $|\mathcal{B}| = |\mathcal{A}|$, $|\partial \mathcal{B}| \leq |\partial \mathcal{A}|$ and is (U, V)-compressed for all $(U, V) \in P$.

So, $\mathcal{B} = \mathcal{C}$ by lemma 1.2.

Remark.

1. Equivalently, if we write

$$|\mathcal{A}| = \binom{k_r}{r} + \binom{k_{r-1}}{r-1} + \cdots + \binom{k_s}{s},$$

   where $k_r > k_{r-1} > \cdots > k_s$, and $s \geq 1$, then

$$|\partial \mathcal{A}| \geq \binom{k_r}{r-1} + \binom{k_{r-1}}{r-2} + \cdots + \binom{k_s}{s-1}.$$

2. When do we have equality in Kruskal-Katona? We can check that if $|\mathcal{A}| = \binom{k}{r}$ and $|\partial \mathcal{A}| = \binom{k}{r-1}$, then $\mathcal{A} = Y^{(r)}$ for some $Y \subseteq X$ with $|Y| = k$.
3. However, it is not true in general that if $|\partial \mathcal{A}| = |\partial \mathcal{C}|$ then $\mathcal{A}$ is isomorphic to $\mathcal{C}$ (isomorphism means the sets are equal up to a permutation of the ground set X).

For $\mathcal{A} \subseteq X^{(r)}$, $0 \leq r \leq n$, the *upper shadow* of $\mathcal{A}$ is

$$\partial^+ \mathcal{A} = \{A \cup x \mid A \in \mathcal{A}, x \notin A\} \subseteq X^{(r+1)}.$$

Corollary 1.2. *Let $\mathcal{A} \subseteq X^{(r)}$, where $0 \leq r \leq n$, and let $\mathcal{C}$ be the initial segment of lex on $X^{(r)}$ with $|\mathcal{C}| = |\mathcal{A}|$. Then,*

$$|\partial^+ \mathcal{A}| \geq |\partial^+ \mathcal{C}|.$$

Proof: From Kruskal-Katona, note $A < B$ in colex $\iff$ $A^c < B^c$ in lex, with the ground set order reversed.

From the fact that the shadow of an initial segment is an initial segment, we get the following:

Corollary 1.3. *Let $\mathcal{A} \subseteq X^{(r)}$, and $\mathcal{C}$ the initial segment of colex on $X^{(r)}$ with $|\mathcal{C}| = |\mathcal{A}|$. Then,*

$$|\partial^t \mathcal{C}| < |\partial^t \mathcal{A}|,$$

for all $1 \leq t \leq r$.

Proof: If $|\partial^t \mathcal{C}| \leq |\partial^t \mathcal{A}|$, then $|\partial^{t+1} \mathcal{C}| \leq |\partial^{t+1} \mathcal{A}|$ by Kruskal-Katona, since $\partial^t \mathcal{C}$ is an initial segment of colex.

So, if $|\mathcal{A}| = \binom{k}{r}$, then

$$|\partial^t \mathcal{A}| \geq \binom{k}{r-t}.$$

Note that our proof of Kruskal-Katona uses lemmas 1.2 and 1.3, not lemma 1.1 and its corollary.

1.4 Intersecting Families

Say $\mathcal{A} \subseteq \mathcal{P}(X)$ is *intersecting* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{A}$.

How large can an intersecting family be? We can have $|\mathcal{A}| = 2^{n-1}$, by taking

$$\mathcal{A} = \{A \in \mathcal{P}(X) \mid 1 \in A\}.$$

Proposition 1.2. *Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be intersecting. Then $|\mathcal{A}| \leq 2^{n-1}$.*

Proof: For any $A \subseteq X$, at most one of A, A^c can belong to $\mathcal{A}$.

Note that there are many other extremal example, for example

$$\mathcal{A} = \{A \in \mathcal{P}(X) \mid |A| > n/2\}.$$

What if $\mathcal{A} \subseteq X^{(r)}$? If $r > n/2$, then we can just take $A = X^{(r)}$, and if $r = n/2$, then we can choose one of A, A^c.

So the interesting case is $r < n/2$. We could try again

$$\mathcal{A} = \{A \in X^{(r)} \mid 1 \in A\}.$$

Then this has size

$$\binom{n-1}{r-1} = \frac{r}{n}\binom{n}{r}.$$

We could also take, for example

$$\mathcal{B} = \{A \in X^{(r)} \mid |A \cap \{1,2,3\}| \geq 2\}.$$

But for $n = 8$, $r = 3$, we see $|\mathcal{A}| = 21$, and $|\mathcal{B}| = 16$.

Theorem 1.4 (Erdos-Ko-Rado Theorem). *Let $\mathcal{A} \subseteq X^{(r)}$ be intersecting, where $r < n/2$. Then*

$$|\mathcal{A}| \leq \binom{n-1}{r-1}.$$

Proof: We do multiple proofs. First, note that

$$A \cap B = \emptyset \iff A \not\subseteq B^c.$$

This motivates the idea, 'bubble down with Kruskal-Katona'.

Let $\tilde{\mathcal{A}} = \{A^c \mid A \in \mathcal{A}\} \subseteq X^{(n-r)}$. Then we know that $\partial^{n-2r}\tilde{\mathcal{A}}$ and $\mathcal{A}$ must be disjoint families of r-sets.

Suppose that $|\mathcal{A}| > \binom{n-1}{r-1}$. Then

$$|\tilde{\mathcal{A}}| = |\mathcal{A}| > \binom{n-1}{r-1} = \binom{n-1}{n-r}.$$

Hence, by Kruskal-Katona, we have

$$|\partial^{n-2r}\tilde{\mathcal{A}}| \geq \binom{n-1}{r}.$$

But this gives

$$|\mathcal{A}| + |\partial^{n-2r}\tilde{\mathcal{A}}| > \binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r} = |X^{(r)}|,$$

contradiction.

Note that this calculation had to give the correct answer, as the shadow calculation would all be exact if $\mathcal{A} = \{A \in X^{(r)} \mid 1 \in A\}$.

Now we consider a second proof. Pick a circle ordering of [n], i.e. a bijection $C : [n] \to \mathbb{Z}_n$. How many sets in $\mathcal{A}$ are intervals in this ordering?

At most r, since if $C_1 \dots C_r \in \mathcal{A}$, then for each $2 \le i$, at most one of the intervals $C_i C_{i+1} \dots C_{i+r-1}$ and $C_{i-r} C_{i-r+1} \dots C_{i-1}$ can belong to $\mathcal{A}$.

For each r-set A, in how many of then $n!$ cyclic orderings is it an interval? We have n choices for where it is placed, $r!$ orderings for the elements of A, and $(n-r)!$ orderings for the elements of A^c. Hence,

$$|\mathcal{A}| n r! (n-r)! \le n! r \implies |\mathcal{A}| \le \frac{n! r}{n r! (n-r)!} = \binom{n-1}{r-1}.$$

Remark.

1. Again the numbers had to work out.
2. Equivalently, we are double-counting the edges in a bipartite graph, where one class is the vertex classes, and the other class is the cyclic orderings, and an edge is present if A is an interval in C.
3. This method is called *averaging*, or *Katona's method.*
4. When do we have equality? It is actually unique; if $\mathcal{A} \subseteq X^{(r)}$ is intersecting, and $|\mathcal{A}|$ is maximal, then

$$\mathcal{A} = \{A \in X^{(r)} \mid i \in A\},$$

for some $1 \le i \le n$. This can be seen from proof 1, by analysing the equality case in KK, or by looking at proof 2 a bit more carefully.

2 Isoperimetric Inequalities

This section deals with problems of the following form: how do we minimize the boundary of a set of a given size?

For example in $\mathbb{R}^2$, given an area, the disc minimizes the perimeter. For $\mathbb{R}^3$, given a volume, the solid sphere minimizes the surface area. In S^2, given a surface area, the circular cap minimizes the perimeter.

We want to discretize this. For a set A of vertices of a graph G, the *boundary* of A is

$$b(A) = \{x \in G \mid x \notin A, xy \in E \text{ for some } y \in A\}.$$

An *isoperimetric inequality* on G is an equality of the form

$$|b(A)| \geq f(|A|),$$

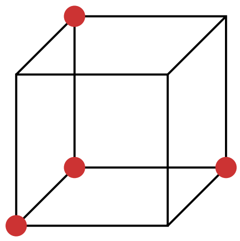
for all $A \subseteq G$, and some function f.

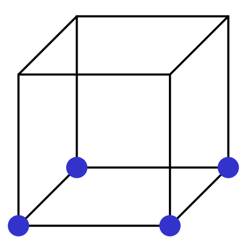
Often it is simpler to look at the neighbourhood of A, $N(A) = A \cup b(A)$, so

$$N(A) = \{x \in G \mid d(x, A) \leq 1\}.$$

A good example for A might be a ball $B(x, r) = \{y \in G \mid d(x, y) \leq r\}$. What happens for Q_n?

For $|A| = 4$ in Q_3, we may either take a ball, or Q_2. The ball has boundary 3, while Q_2 has boundary 4.





A good guess is that balls are the best, i.e. sets of the form

$$B(\emptyset, r) = X^{(\leq r)} = X^{(0)} \cup X^{(1)} \cup \cdots \cup X^{(r)}.$$

What if the size of our set is between two levels, i.e. $|X^{(\leq r)}| \leq |A| \leq |X^{(\leq r+1)}|$?

Our guess is to take A with $X^{(\leq r)} \leq A \leq X^{(\leq r+1)}$. If $A = X^{(\leq r)} \cup B$, where $B \subseteq X^{(r+1)}$, then

$$B(A) = (X^{(r+1)} - B) \cup \partial^+(B).$$

So we would take B to be an initial segment of lex, by Kruskal-Katona.

In the *simplicial ordering* of $\mathcal{P}(X)$, we set $x < y$ if either $|x| < |y|$, or $|x| = |y|$, but $x < y$ in lex.

Our aim is to show that initial segments of the simplicial ordering minimize the boundary. We do it by compression, in the spirit of KK.

Fix $A \subset \mathcal{P}(X)$. For $1 \leq i \leq n$, the *i-selection* of A are the families $A_-^{(i)}$, $A_+^{(i)} \subseteq \mathcal{P}(X - i)$ given by

$$\begin{aligned} A_-^{(i)} &= \{x \in A \mid i \notin x\}, \\ A_+^{(i)} &= \{x - i \mid x \in A, i \in x\}. \end{aligned}$$

The *i-compression* of A is the family $C_i(A) \subseteq \mathcal{P}(X)$ given by, $(C_i(A))_-^{(i)}$ is the first $|A_-^{(i)}|$ elements of the simplicial ordering of $\mathcal{P}(X - i)$, and $(C_i(A))_+^{(i)}$ be the first $|A_+^{(i)}|$ elements of the simplicial ordering on $\mathcal{P}(X - i)$.

This is essentially doing a compression on each of the two i-level sub-hypercubes simultaneously.

A subset is *i-compressed* if $C_i(A) = A$. Here a *Hamming ball* is a family with $X^{(\leq r)} \subseteq A \subseteq X^{(\leq r+1)}$ for some r.

Theorem 2.1 (Harper's Theorem)**.** *Let $A \subseteq Q_n$, and let C be the initial segment of the simplicial order with $|C| = |A|$. Then $|N(A)| \geq |N(C)|$. In particular, if*

$$|A| = \sum_{i=0}^{k} \binom{n}{i} \implies |N(A)| \geq \sum_{i=0}^{k+1} \binom{n}{i}.$$

Remark.

1. If we knew A was a Hamming ball, we would be done by KK.
2. Conversely, this theorem implies KK, as we could take $B \subseteq X^{(r)}$, and then apply theorem 1 to $A = X^{(\leq r-1)} \cup B$.

Proof: We proceed by induction on n. For $n = 1$, this is trivial.

Now suppose we are given $n > 1$, $A \subseteq Q_n$, and $1 \leq i \leq n$. Then we claim that

$$|N(C_i(A))| \leq |N(A)|.$$

Write B for $C_i(A)$. Then we have

$$N(A)_- = N(A_-) \cup A_+,$$
$$N(A)_+ = N(A_+) \cup A_-,$$

and of course

$$N(B)_- = N(B_-) \cup B_+,$$
$$N(B)_+ = N(B_+) \cup B_-.$$

Now, $|B_+| = |A_+|$, and $|N(B_-)| \leq |N(A_-)|$ by induction. But B_+ is an initial segment of the simplicial ordering, and $N(B_-)$ is as well, as a neighbourhood of an initial segment is an initial segment.

So, B_+ and $N(B_-)$ are nested. Hence $|N(B)_-| \leq |N(A)_-|$. Similarly, $|N(B)_+| \leq |N(A)_+|$, giving $|N(B)| \leq |N(A)|$. Define a sequence $A_0, A_1, \ldots \subseteq Q_n$ as follows: Set $A_0 = A$, and having chose $A_0, \ldots, A_k$, if A_k is i-compressed for all i, then stop the sequence with A_k.

If not, pick i with $C_i(A_k) \neq A_k$, and set $A_{k+1} = C_i(A_k)$, and continue. This must terminate, because the sum of the position of x in the simplicial order, over all $x \in A_k$, is strictly decreasing.

The final family $B = A_k$ satisfies $|B| = |A|$, and $|N(B)| \leq |N(A)|$, and is i-compressed for all i.

Does B being i-compressed for all i imply B is an initial segment? No; consider a copy of Q_2 in Q_3. However,

Lemma 2.1. *Let $B \subseteq Q_n$ be i-compressed for all i, but not an initial segment of the simplicial order. Then either:*

- *n is odd, say $n = 2k+1$, and $B = X^{(\leq k)} - \{k+2, k+3, \ldots, 2k+1\} \cup \{1, 2, \ldots, k+1\}$,*
- *n is even, say $n = 2k$, and $B = X^{(\leq k)} - \{1, k+2, \ldots, 2k\} \cup \{2, 3, \ldots, k+1\}$.*

Then we are done, as in each case $|N(B)| \geq |N(C)|$.

Proof: Suppose that B is not an initial segment of the simplicial ordering, so there is $x < y$ in the simplicial ordering with $x \notin B$, $y \in B$.

For each $1 \leq i \leq n$, we cannot have $i \in x$ and $i \in y$, since B is i-compressed, and we also cannot have $i \notin x$, $i \notin y$ for the same reason.

So $x = y^c$. Thus for each $y \in B$, there is at most one earlier x with $x \notin B$, namely $x = y^c$, and for each $x \notin B$, there is at most one later y with $y \in B$, namely $y = x^c$.

So $B = \{z \mid z \leq y\} - \{x\}$, with x the predecessor of y, and $x = y^c$. Hence if $n = 2k + 1$, then x must be the last k-set, and if $n = 2k$ then x is the last k-set with 1.

This completes the proof of Harper's theorem.

Remark.

1. We can also prove Harper's theorem using UV-compressions.
2. We can also prove KK using i-compressions.

For $A \subseteq Q_n$ and $t = 1, 2, 3, \ldots$, the t-neighbourhood of A is

$$A_{(t)} = N^t(A) = \{x \in Q_n \mid d(x, A) \leq t\}.$$

Corollary 2.1. *Let $A \subseteq Q_n$ with*

$$|A| \geq \sum_{i=0}^{r} \binom{n}{i}.$$

Then for all $t \leq n - r$,

$$|A_{(t)}| \geq \sum_{i=0}^{r+t} \binom{n}{i}.$$

Proof: Use Harper's theorem and induction (the neighbourhood of an initial segment of simplicial is another initial segment).

To get a feeling for the strength of the corollary, we will need some estimates on the size of things like

$$\sum_{i=0}^{r} \binom{n}{i}$$

Proposition 2.1. *Let $0 < \varepsilon < 1/4$. Then,*

$$\sum_{i=0}^{\lfloor (\frac{1}{2} - \varepsilon) n \rfloor} \binom{n}{i} \leq \frac{1}{\varepsilon} e^{-\varepsilon^2 n / t} 2^n.$$

Proof: For $i \leq \lfloor (\frac{1}{2} - \varepsilon)n \rfloor$, we have

$$\frac{\binom{n}{i-1}}{\binom{n}{i}} = \frac{i}{n-i+1} \leq \frac{(\frac{1}{2}-\varepsilon)n}{(\frac{1}{2}+\varepsilon)n} = \frac{\frac{1}{2}-\varepsilon}{\frac{1}{2}+\varepsilon} = 1 - \frac{2\varepsilon}{\frac{1}{2}+\varepsilon} \leq 1 - 2\varepsilon.$$

Hence, summing this as a GP,

$$\sum_{i=0}^{\lfloor (\frac{1}{2}-\varepsilon)n \rfloor} \binom{n}{i} \leq \frac{1}{2\varepsilon} \binom{n}{\lfloor (\frac{1}{2}-\varepsilon)n \rfloor}.$$

The same argument tells us that

$$\binom{n}{\lfloor (\frac{1}{2}-\varepsilon)n \rfloor} \leq \binom{n}{\lfloor (\frac{1}{2}-\frac{\varepsilon}{2})n \rfloor} (1-\varepsilon)^{\frac{\varepsilon n}{2}-1} \leq 2^n \cdot 2(1-\varepsilon)^{\varepsilon n/2} \leq 2^n \cdot 2e^{-\varepsilon^2 n/2}.$$

Thus we get

$$\sum_{i=0}^{\lfloor (\frac{1}{2}-\varepsilon)n \rfloor} \binom{n}{i} \leq \frac{1}{2\varepsilon} 2e^{-\varepsilon^2 n/2} 2^n.$$

Theorem 2.2. *Let* $0 < \varepsilon < 1/4$, $A \subseteq Q_n$. *Then*

$$\frac{|A|}{2^n} \geq \frac{1}{2} \implies \frac{|A_{(\varepsilon n)}|}{2^n} \geq 1 - \frac{2}{\varepsilon} e^{-\varepsilon^2 n/2}.$$

In other words, half-sized sets have exponentially large εn-neighbourhoods.

Proof: It is enough to show that if εn is an integer, then

$$\frac{|A_{(\varepsilon n)}|}{2^n} \geq 1 - \frac{1}{\varepsilon} e^{-\varepsilon^2 n/2}.$$

We have that

$$|A| \geq \sum_{i=0}^{\lceil n/2-1 \rceil} \binom{n}{i},$$

so by Harper, we have

$$|A_{(\varepsilon n)}| \geq \sum_{i=0}^{\lceil n/2-1+\varepsilon n \rceil} \binom{n}{i}.$$

So

$$|A^c_{(\varepsilon n)}| \leq \sum_{\lceil n/2+\varepsilon n\rceil}^{n} \binom{n}{i} = \sum_{i=0}^{\lfloor n/2-\varepsilon n\rfloor} \binom{n}{i} \leq \frac{1}{\varepsilon} e^{-\varepsilon^2 n/2} \cdot 2^n.$$

Remark. The same would show that, for small sets,

$$\frac{|A|}{2^n} \geq \frac{2}{\varepsilon} e^{-\varepsilon^2 n/2} \implies \frac{|A_{(2\varepsilon n)}|}{2^n} \geq 1 - \frac{2}{\varepsilon} e^{-\varepsilon^2 n/2}.$$

2.1 Concentration of Measure

Say $f : Q_n \to \mathbb{R}$ is *Lipschitz* if $|f(x) - f(y)| \leq 1$ for all x, y adjacent. For $f : Q_n \to \mathbb{R}$, say $M \in \mathbb{R}$ is a *Lévy mean* or the *median* of f if

$$|\{x \in Q_n \mid f(x) \leq M\}| \leq 2^{n-1} \qquad \text{and} \qquad |\{x \in Q_n \mid f(x) \geq M\}| \geq 2^{n-1}.$$

We are now ready to show that every well-behaved function on the cube Q_n is roughly constant nearly everywhere.

Theorem 2.3. *Let $f : Q_n \to \mathbb{R}$ be Lipschitz with median M. Then,*

$$\frac{|\{x \mid |f(x) - M| \leq \varepsilon n\}|}{2^n} \geq 1 - \frac{4}{\varepsilon} e^{-\varepsilon^2 n/2},$$

for any $0 < \varepsilon < 1/4$.

Note that this is the concentration of measure phenomenon.

Proof: Let $A = \{x \mid f(x) \leq M\}$. Then

$$\frac{|A|}{2^n} \geq \frac{1}{2} \implies \frac{|A_{(\varepsilon n)}|}{2^n} \geq 1 - \frac{2}{\varepsilon} e^{-\varepsilon^2 n/2}.$$

But f is Lipschitz, so if $x \in A_{(\varepsilon n)}$, then $f(x) \leq M + \varepsilon n$. Then,

$$\frac{|\{x \mid f(x) \leq M + \varepsilon n\}|}{2^n} \geq 1 - \frac{2}{\varepsilon} e^{-\varepsilon^2 n/2}.$$

Similarly,

$$\frac{|\{x \mid f(x) \geq M - \varepsilon n\}|}{2^n} \geq 1 - \frac{2}{\varepsilon} e^{-\varepsilon^2 n/2}.$$

Putting this together,

$$\frac{|\{x \mid |f(x) - M| \leq \varepsilon n\}|}{2^n} \geq 1 - \frac{4}{\varepsilon} e^{-\varepsilon^2 n/2},$$

Let G be a graph of diameter D. Write

$$\alpha(G,\varepsilon) = \max\left\{1 - \frac{|A_{(\varepsilon D)}|}{|G|} \mid A \subseteq G, \frac{|A|}{|G|} \geq \frac{1}{2}\right\}.$$

So $\alpha(G,\varepsilon)$ says that half-sized sets have larger εD-neighbourhoods.

We say that a sequence of graphs is a *Lévy family* if $\alpha(G_n,\varepsilon) \to 0$ as $n \to \infty$, for each $\varepsilon > 0$.

This theorem tells us that the sequence (Q_n) is a Lévy family, and it is even a *normal Lévy family*, meaning that $\alpha(G_n,\varepsilon)$ is exponentially small in n, for each $\varepsilon > 0$.

For any Lévy family we have concentration of measure. Most naturally occurring families of graphs are Lévy families, for example (S_n) where S_n is made into a graph by joining permutations joined by a transposition.

We can also define $\alpha(X,\varepsilon)$ similarly for an metric measure space X, of finite measure and finite diameter.

Example 2.1.

(S^n) is a Lévy family. This requires two ingredients.

1. An isoperimetric inequality on S_n: for $A \subseteq S_n$ and C a circular cap with $|C| = |A|$, we have $|A_{(\varepsilon)}| \geq |C_{(\varepsilon)}|$.

   This is proven by compression; consider laying the sphere out on some way, and then vertically projecting each point if possible. This is known as two-point symmetrisation.

2. Then we estimate the size. A circular cap of measure $1/2$ is the cap of angle $\pi/2$. Then $C_{(\varepsilon)}$ is the circular cap of angle $\pi/2 + \varepsilon$. This has measure about
$$\int_{\varepsilon}^{\pi/2} \cos^{n-1} t \,\mathrm{d}t \to 0.$$

Moreover this is a normal Lévy family.

We have deduced concentration of measure from an isoperimetric family. Conversely,

Proposition 2.2. *Let G be a graph such that for any Lipschitz function $f : G \to \mathbb{R}$ with median M, we have*

$$\frac{|\{x \in G \mid |f(x) - M| > t\}}{|G|} \leq \alpha$$

for some given t, α. Then for all $A \in G$, if $\frac{|A|}{|G|} \geq \frac{1}{2}$, we have

$$\frac{|A_{(t)}|}{|G|} \geq 1 - \alpha.$$

Proof: The function $f(x) = d(x, A)$ is Lipschitz, and has 0 as its median since at least half of the values take 0. So

$$\frac{|\{x \in G \mid x \notin A_{(t)}\}|}{|G|} \leq \alpha.$$

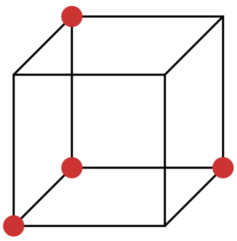
2.2 Edge-isoperimetric Inequalities

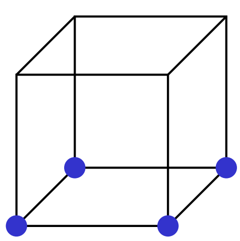
For a subset A of vertices of a graph G, the *edge-boundary* of A is

$$\partial_e A = \partial A = \{xy \in G \mid x \in A, y \notin A\}.$$

An inequality of the form $|\partial A| \geq f(|A|)$ for all $A \subseteq G$ is an *edge-isoperimetric inequality* on G.

What happens in Q_n? Given $|A|$, which $A \subseteq Q_n$ could we take to maximize $|\partial A|$? For example, if $|A| = 4$, then the





It follows that maybe subcubes are the best. What if we have $A \subseteq Q_n$ with $2^k < |A| < 2^{k+1}$? Then we should take $A = \mathcal{P}([k])$ along with some sets in $\mathcal{P}([k+1])$.

So we define the following ordering: for $x, y \in Q_n$ and $x \neq y$, we say that $x < y$ in the *binary ordering* on Q_n if $\max x \triangle y \leq y$. Equivalently,

$$x < y \iff \sum_{i \in x} 2^i < \sum_{i \in y} 2^i.$$

For example in Q_3, the sets are ordered

$$\emptyset, 1, 2, 12, 3, 13, 23, 123.$$

For $A \subseteq Q_n$ and $1 \leq i \leq n$, we define the *i-binary-compression* $B_i(A) \subseteq Q_n$ by giving it i-sections:

$$\begin{aligned}(B_i(A))_-^{(i)} &= \text{initial segment of binary on } \mathcal{P}(X-i) \text{ of size } |A_-^{(i)}|,\\ (B_i(A))_+^{(i)} &= \text{initial segment of binary on } \mathcal{P}(X-i) \text{ of size } |A_+^{(i)}|.\end{aligned}$$

So $|B_i(A)||A|$. We say that A is *i-binary-compressed* if $B_i(A) = A$.

Theorem 2.4 (Edge-isoperimetric Inequality in Q_n)**.** *Let $A \subseteq Q_n$ and C be the initial segment of binary on Q_n with $|C| = |A|$. Then $|\partial C| \leq |\partial A|$. In particular, if $|A| = 2^k$, then $|\partial A| \geq 2^k(n-k)$.*

Remark. This is sometimes called the theorem of Harper, Lindsey, Bernstein and Hart.

Proof: We proceed by induction on n. $n = 1$ is trivial.

For $n > 1$, $A \subseteq Q_n$, $1 \leq i \leq n$, we claim that $|\partial B_i(A)| \leq |\partial A|$.

Indeed, write B for $B_i(A)$. Then

$$\begin{aligned}|\partial A| &= |\partial(A_-)| + |\partial(A_+)| + |A_+ \triangle A_-|,\\ |\partial B| &= |\partial(B_-)| + |\partial(B_+)| + |B_+ \triangle B_-|.\end{aligned}$$

By induction, $|\partial(B_-)| \leq |\partial(A_-)|$ and $|\partial(B_+)| \leq |\partial(A_+)|$. Also, the set B_+ and B_- are nested, as each is an initial segment of binary on $\mathcal{P}(X-i)$.

Therefore, certainly we have $|B_+ \triangle B_-| \leq |A_+ \triangle A_-|$. So $|\partial B| \leq |\partial A|$.

Define a sequence $A_0, A_1, \ldots \subseteq Q_n$ as follows. Set $A_0 = A$. Having defined $A_0, \ldots, A_k$, if A_k is i-binary-compressed for all i, then stop the sequence. If not, choose i with $B_i(A_k) \neq A_k$, and put $A_{k+1} = B_i(A_k)$.

This must terminate as the function

$$k \mapsto \sum_{x \in A_k} (\text{position of } x \text{ in binary})$$

is strictly decreasing. Now the final family $B = A_k$ satisfies $|B| = |A|$ and $|\partial B| \leq |\partial A|$.

Note that B need not be an initial segment of binary, for example take $\{\emptyset, 1, 2, 3\} \subseteq Q_3$. However these are the only counterexamples.

Lemma 2.2. *Let $B \subseteq Q_n$ be binary compressed for all i, that is not an initial* *segment of binary. Then*

$$B = \mathcal{P}(n-1) - \{12\dots n-1\} \cup \{n\}.$$

Then we are done, as clearly $|\partial B| \geq |\partial C|$, since $C = \mathcal{P}(n-1)$.

Proof: As B is not an initial segment there exists $x < y$ with $x \notin B$, $y \in B$.

Thus for all i, we cannot have $i \in x, y$ or $i \notin x, y$, as B is i-binary-compressed. So $y = x^c$.

Thus for each $y \in B$, there is at most one earlier $x \in B$, and for each $x \notin B$ there is at most one later $y \in B$. So $B = \{z \mid z \leq y\} - \{x\}$, where y is the predecessor of y and $y = x^c$.

So we must have $y = \{n\}$.

This concludes the proof of theorem 2.4.

Remark. It is vital in this proof, and the proof of Harper's that the extremal sets were nested.

The *isoperimetric number* of a graph G is

$$i(G) = \min\left\{\frac{|\partial A|}{|A|} \mid A \subseteq G, \frac{|A|}{|G|} \leq \frac{1}{2}\right\}.$$

Corollary 2.2. $i(Q_n) = 1$.

Proof: Taking $A = \mathcal{P}(n-1)$, we get that $i(Q_n) \leq 1$ (every edge is between x and $x+n$).

To show that $i(Q_n) \geq 1$, we need to show that if C is an initial segment of binary with $|C| \leq 2^{n-1}$, then $|\partial C| \geq |C|$. But $C \subseteq \mathcal{P}(n-1)$, so certainly $|\partial C| \geq |C|$.

2.3 Inequalities in the Grid

For any $k = 2, 3, \dots$, the *grid* is the graph on $[k]^n$ in which x is joined to y if for some i, we have $x_j = y_j$ for $j \neq i$, and $|x_i - y_i| = 1$.

For example, we can draw the 4 grid.

Note that for $k = 2$, this is exactly Q_n. Do we have analogues of Sperner's and KK for the grid? We start with the vertex-isoperimetric problem; which sets $A \subseteq [k]^n$ (of a given size) minimise $|N(A)|$?

In $[k]^2$, we can either consider a diagonal or a square. The diagonal cut seems to be better.

This suggests that we go up in levels according to

$$|x| = \sum_{i=1}^{n} |x_i|,$$

i.e. we take $\{x \in [k]^n \mid |x| \leq r\}$. What if

$$|\{x \in [k]^n \mid |x| \leq r\}| < |A| < |\{x \in [k]^n \mid |x| \leq r+1\}|?$$

In this case, we take $A = \{x \in [k]^n \mid |x| \leq r\}$, and some point with $|x| = r + 1$. The points we pick should be those which keep x_1 large.

This suggests in the *simplicial ordering* on $[k]^n$, we set $x < y$ if either $|x| < |y|$ or $|x| = |y|$ and $x_i > y_i$ where i is the minimal element where they differ.

Example 2.2.

On $[3]^2$, the ordering is

$$(1,1),(2,1),(1,2),(3,1),(2,2),(1,3),(3,2),(2,3),(3,3).$$

On $[4]^3$, the ordering is

$$\begin{aligned}(1,1,1),&(2,1,1),(1,2,1),(1,1,2),(3,1,1),(2,2,1),(2,1,2),(1,3,1),\\ &(1,2,2),(1,1,3),(4,1,1),(3,2,1),\ldots\end{aligned}$$

For $A \subseteq [k]^n$ for some $k \geq 2$ and $1 \leq i \leq n$, the *i-sections* of A are the sets $A_1, \ldots, A_k \subseteq [k]^{n-1}$ defined by

$$A_t = A_t^{(i)} = \{x \in [k]^{n-1} \mid (x_1, x_2, \ldots, x_{i-1}, t, x_i, x_{i+1}, \ldots, x_{n-1}) \in A\}.$$

The *i-compression* of A is $C_i(A) \subseteq [k]^n$ defined by giving its i-sections:

$$C_i(A) = \text{initial segment of } [k]^{n-1} \text{ of size } |A_t|.$$

Essentially, we are going layer by layer in the i'th direction, and shoving everything into an initial segment of simplicial.

Theorem 2.5 (Vertex-Isoperimetric Inequality in the Grid)**.** *Let $A \subseteq [k]^n$, and let C be the initial segment of simplicial of $[k]^n$ with $|C| = |A|$.*

Then $|N(C)| \leq |N(A)|$. In particular, if $|A| \geq |\{x \mid |x| \leq r\}$, then $|N(A)| \geq |\{x \mid |x| \leq r+1\}$.

Proof: We proceed by induction on n. $n = 1$ is trivial, if $A \subseteq [k]^1 \neq \emptyset$, then $|N(A)| \geq |A| + 1 = |N(C)|$.

Given $n > 1$ and $A = [k]^n$, fix $1 \leq i \leq n$. We claim that $|N(C_i(A))| \leq |N(A)|$.

Indeed, write B for $C_i(A)$. For any $1 \leq t \leq k$, we have

$$N(A)_t = N(A_t) \cup A_{t-1} \cup A_{t+1},$$

and similarly $N(B)_t = N(B_t) \cup B_{t-1} \cup B_{t+1}$. Now $|B_{t-1}| = |A_{t-1}|$, and $|B_{t+1}| = |A_{t+1}|$, and $|N(B_t)| \leq |N(A_t)|$, by the induction hypothesis. But the sets B_{t-1}, B_{t+1} and $N(B_t)$ are nested as each is an initial segment of simplicial on $[k]^{n-1}$.

Hence $|N(B)_t| \leq |N(A)_t|$, so $|N(B)| \leq |N(A)|$.

Among all $B \subseteq [k]^n$ with $|B| = |A|$ and $|N(B)| \leq |N(A)|$, pick one with sum of simplicial positionings the smallest. Then B is i-compressed for all i.

We are very far from being done. To continue, we split into two cases.

Case 1: $n = 2$. What we know is precisely that B is a down-set. Let $r = \min\{|x| \mid x \notin B\}$, and $s = \max\{|x| \mid x \in B\}$. We may assume that $r \leq s$, since if $r = s+1$, then $B = \{x \mid |x| \leq s\}$, hence $B = C$.

If $r = s$, then
$$\{x \mid |x| \leq r-1\} \subseteq B \subseteq \{x \mid |x| \leq r\},$$
so clearly $|N(B)| \geq |N(C)|$.

If $r < s$, then we cannot have $\{x \mid |x| = s\} \subseteq B$, because then also $\{x \mid |x| = r\} \subseteq B$ as B is a down set.

So there is y, y' with $|y| = |y'| = s$, $y \in B$ and $y' \notin B$, and $y' = y \pm (e_1 - e_2)$, by DIVT.

Similarly, we cannot have $\{x \mid |x| = r\} \cap B = \emptyset$, otherwise $\{x \mid |x| = s\} \cap B = \emptyset$, again as B is a down set. So there is x, x' with $|x| = |x'| = r$, $x \notin B$, $x' \in B$ and $x' = x \pm (e_1 - e_2)$.

Now let $B' = B \cup \{x\} - \{y\}$. From B we lost at least one point in the neighbourhood, namely $y + e_1$ or $y + e_2$, and gained at most one point, so $|N(B')| \leq |N(B)|$. This contradicts the minimality of B.

Case 2: $n \geq 3$. For any $1 \leq i \leq n-1$ and any $x \in B$ with $x_n > 1$ and $x_i < k$, we must have that

$$x - e_n + e_i \in B,$$

as B is j-compressed, for any $j \neq 1, n$.

So, considering the n-sections of B, we have that $N(B_t) \subseteq B_{t-1}$ for all $t = 2, \ldots, k$. Recall that $N(B)_t = N(B_t) \cup B_{t+1} \cup B_{t-1}$, so in fact $N(B)_t = B_{t-1}$. Thus,

$$|N(B)| = |B_{k-1}| + |B_{k-2}| + \cdots + |B_1| + |N(B_1)| = |B| - |B_k| + |N(B_1)|.$$

Similarly, $|N(C)| = |C| - |C_k| + |N(C_1)|$. So to show $|N(C)| \leq |N(B)|$, it is enough to show that $|B_k| \leq |C_k|$, and $|B_1| \geq |C_1|$, since $|N(C_1)|$ is minimal for its size.

To show $|B_k| \leq |C_k|$, define a set $D \subseteq [k]^n$ as follows: put $D_k = B_k$, and for $t = k-1, k-2, \ldots, 1$ set $D_t = N(D_{t-1})$.

Then $D \subseteq B$ so $|D| \leq |B|$, and in fact D is an initial segment of simplicial so $D \subseteq C$, showing $|B_k| = |D_k| \leq |C_k|$.

To show $|B_1| \geq |C_1|$, define a set $E \subseteq [k]^n$ as follows: put $E_1 = B_1$, and for $t = 2, 3, \ldots, k$ set $E_t = \{x \in [k]^{n-1} \mid N(\{x\}) \subseteq E_{t-1}\}$.

Then $E \supseteq B$ so $|E| \supseteq |B|$ and E is an initial segment of simplicial, so $E \supseteq C$, whence $|B_1| = |E_1| \geq |C_1|$.

This shows the theorem.

Corollary 2.3. *Let $A \subseteq [k]^n$ with $|A| \geq |\{x \mid |x| \leq r\}|$. Then $|A_{(t)}| \geq |\{x \mid |x| \leq r + t\}|$.*

Remark. We can check from the corollary that for k fixed, the sequence $([k]^n)$ is a Lévy family.

We now look at the edge-isoperimetric inequality in the grid. Which set $A \subseteq [k]^n$ should we take to minimize $|\partial A|$? For example, in $[k]^2$, we can look at squares and triangles, which suggest that squares are the best.

However, there are nasty things that happen as we grow the size of of $|A|$. When $|A|$ is $1/4$ of the total size of $[k]^2$, the square does equally as well as the column, and for greater sizes columns do better.

A similar thing occurs at 3/4 of the total size, when the column equals a cosquare, and at large sizes the cosquare does the best. So we have phase transitions at $|A| = k^2/4$ and $3k^2/4$, when the extremal sets are not nested. This seems to rule out compression methods (insert image).

And in $[k]^3$, we have even more phase transitions:

$$[a]^3 \to [a]^2 \times [k] \to [a] \times [k]^2 \to ([a]^2 \times [k])^c \to ([a]^3]^c.$$

Hence in $[k]^n$, up to halfway we get $n-1$ of these phase transitions.

Note that if $A = [a]^d \times [k]^{n-d}$, then $|\partial A| = da^{d-1}k^{n-d} = d|A|^{1-1/d}k^{n/d-1}$.

Theorem 2.6. *Assuming that $A \subseteq [k]^n$ and $|A| \le k^n/2$, then*

$$|\partial A| \ge \min\{d|A|^{1-1/d}k^{n/d-1} \mid 1 \le d \le n\}.$$

This just says that some set of the form $[a]^d \times [k]^{n-d}$ is the best. The following is non-examinable.

Proof: We give a sketch of the proof. The idea is induction on n: $n = 1$ is trivial.

Take $A \subseteq [k]^n$, with $|A| \le k^n/2$ and $n > 1$. Without loss of generality, assume that A is a down-set, by compressing in each direction $1 \le i \le n$. For any $1 \le i \le n$, define $C_i(A)$ by defining its sections:

$$C_i(A)_t = \text{extremal set of size } |A_t| \text{ in } [k]^{n-1}.$$

This will be a set of the form $[a]^d \times [k]^{n-1-d}$, or a complement. Say $B = C_i(A)$. We want to say something like $|\partial B| \le |\partial A|$. Since A is a down set, we can write

$$|\partial A| = |\partial A_1| + \cdots + |\partial A_k| + |A_1| - |A_k|,$$

since the sections of A are nested. But for B, we do not have the same expression, since it is not a down-set: extremal sets may not be nested. Indeed, we may have $|\partial B| > |\partial A|$, by choosing the sections of A on opposite sides of a phase transition.

The idea we have to solve this is to introduce a "fake" boundary ∂', where $\partial' A \le \partial A$ and $\partial' = \partial$ on extremal sets. Then we can try to show that C_i

decreases ∂'. Consider defining

$$\partial' A = \sum_t |\partial A_t| + |A_1| - |A_k|,$$

where A_t is the t'th section in the i'th direction. Then $\partial' A \leq |\partial A|$ for all A, we have equality for extremal sets (as equality holds for down-sets), and $\partial' C_i(A) \leq \partial' A$. However, for any $j \neq i$, $C_j(A)$ does not decrease the edge perimeter. One potential fix is trying

$$\partial'' A = \sum_i (|A_1^{(i)}| - |A_k^{(i)}|).$$

But this also fails, if we take e.g. A to be the outer-shell of $[k]^n$. Collecting our results,

$$\begin{aligned} |\partial A| \geq \partial' A \geq \partial' B &= \sum_t |\partial B_t| + |B_1| - |B_k| \\ &= \sum_t f(|B_t|) + |B_1| - |B_k|, \end{aligned}$$

where f is the extremal function in $[k]^{n-1}$. Note that f is the pointwise minimum of functions of the form $cx^{1-1/d}$ and $c(k^{n-1} - x)^{1-1/d}$, which are concave. So f is concave.

Consider varying $|B_2|, \ldots, |B_{k-1}|$ while keeping $|B_2| + \cdots + |B_{k-1}|$ constant, and ensuring $|B_1| \geq |B_2| \geq \cdots \geq |B_{k-1}| \geq |B_k|$. To minimize this concave function, we should take

$$C_t = \begin{cases} B_1 & 1 \leq t \leq r, \\ B_k & r+1 \leq t \leq k, \end{cases}$$

for some r. Then we have

$$|\partial A| = \partial' A \geq \partial' B \geq \partial C = rf(|B_1|) + (k-r)f(|B_k|) + |B_1| - |B_k|.$$

But now C is still not a down set. Now to improve this further, vary $|B_1|$ while keeping $r|B_1| + (k-r)|B_k|$ fixed (for fixed r), and $|B_1| \geq |B_k|$. This is concave in $|B_1|$, as it is a sum of concave functions. Hence we either make $|B_1|$ as big or as small as possible.

If we let D be this set, then D must have one of the following forms:

- $|B_1| = |B_k|$, i.e. $D_t = D_1$ for all t.
- $|B_k| = 0$, so $D_t = D_1$ for all $t \leq r$, and $D_t = \emptyset$ for $t > r$.
- B_1 is maximal, so $D_t = [k]^{n-1}$ for $t \leq r$, and $D_t = D_k$ for $t > r$.

But finally, this D has become a down-set. Hence

$$|\partial A| = \partial' A \geq \partial' B \geq \partial' C \geq \partial' D = |\partial D|.$$

So we can compress by going directly $A \to D$. This allows us to finish as before.

Remark. We were a bit sloppy as we assumed the sections B_i were exactly of the form $[a]^d \times [k]^{n-d}$, however this may not be the case if $|B_i|$ is not of this form. To fix this, work instead in $[0,1]^n$, and then take a discrete approximation.

This concludes the non-examinable discussion.

Remark. Very few isoperimetric inequalities are known (even approximately). For example, take isoperimetric in a layer: in $X^{(r)}$, join two vertices x, y if $|x \cap y| = r - 1$. This is open; the nicest special case if $r = n/2$, where it is conjectured that balls are the best, i.e. sets of the form

$$\{x \in [2r]^{(r)} \mid |x \cap [r]| \geq t\}.$$

3 Intersecting Families

3.1 t-intersecting Families

$A \subseteq \mathcal{P}(X)$ is called *t-intersecting* if $|x \cap y| \geq t$, for all $x, y \in A$. How large can a t-intersecting family be?

Example 3.1.

Take $t = 2$. We could take $\{x \mid 1, 2 \in x\}$, which has size $1/4 \cdot 2^n$.

Or we could take $\{x \mid |x| \geq n/2 + 1\}$. This has size about $1/2 \cdot 2^n$.

Theorem 3.1 (Katona's t-intersecting Theorem)**.** *Let $A \subseteq \mathcal{P}(X)$ be t-intersecting, where $n + t$ is event. Then*

$$|A| \leq \left| X^{(\geq \frac{n+t}{2})} \right|.$$

Proof: For any $x, y \in A$, we have $|x \cap y| \geq t$, so $d(x, y^c) \geq t$. So writing $\bar{A}$ for $\{y^c \mid y \in A\}$, we have $d(A, \bar{A}) \geq t$, i.e. $A_{(t-1)}$ is disjoint from $\bar{A}$.

Suppose that $|A| \geq |X^{(\geq \frac{n+t}{2})}|$. Then by Harper,

$$|A_{(t-1)}| \geq |X^{(\geq \frac{n+t}{2} - (t-1))}| = |X^{(\geq \frac{n-t}{2} + 1)}|.$$

But $A_{(t-1)}$ is disjoint from $\bar{A}$, which has size greater than $|X^{(\leq \frac{n-t}{2})}|$, contradicting $|A_{(t-1)}| + |\bar{A}| \leq 2^n$.

What about t-intersecting families of $A \subseteq X^{(r)}$? We might guess that the best is

$$A_0 = \{x \in X^{(r)} \mid [t] \subseteq x\}.$$

We could also try

$$A_\alpha = \{x \in X^{(r)} \mid |x \cap [t + 2\alpha]| \geq t + \alpha\},$$

for $\alpha = 1, 2, \ldots, r - t$.

Example 3.2.

Take 2-intersecting subsets of $[n]^{(4)}$.

- If $n = 7$, then $|A_0| = \binom{5}{2} = 10$, $|A_1| = 1 + \binom{4}{3}\binom{3}{1} = 13$, $|A_2| = \binom{6}{4} = 15$.

- If $n = 8$, then $|A_0| = \binom{6}{2} = 15$, $|A_1| = 1 + \binom{4}{3}\binom{4}{1} = 17$, $|A_2| = \binom{6}{4} = 15$.
- If $n = 9$, then $|A_0| = \binom{7}{2} = 21$, $|A_1| = 1 + \binom{4}{3}\binom{5}{1} = 21$, $|A_2| = \binom{6}{4} = 15$.

Note that A_0 grows quadratically, A_1 grows linearly, and A_2 is constant, so A_0 is the largest for n large.

The fact that A_0 is the largest for n large inspires the following:

Theorem 3.2. *Let $A \subseteq X^{(r)}$ be t-intersecting. Then, for n sufficiently large, we have*

$$|A| \leq |A_0| = \binom{n-t}{r-t}.$$

Remark.

1. The bound we get on n is either $(16r)^r$ if we are crude, or $2tr^3$, if we are careful.
2. This is often called the second Erdős-Ko-Rado theorem.

The idea of this proof is that A_0 has $r - t$ degrees of freedom.

Proof: Extend A to a maximal t-intersecting family, so we must have some $x, y \in A$ with $|x \cap y| = t$. If not, then by maximality, we have that for all $x \in A$, $i \in x$ and $j \notin x$ then $x + j - i \in A$, whence $A = X^{(r)}$.

We may assume that there exists $z \in A$ with $x \cap y \not\subseteq z$, otherwise all $z \in A$ have $x \cap y \subseteq z$, and $x \wedge y$ is a t set, whence

$$|A| \leq \binom{n-t}{r-t} = |A_0|.$$

So each $w \in A$ must meet $x \cup y \cup z$ in at least $t + 1$ points, so

$$|A| \leq 2^{3r}\left(\binom{n}{r-t-1} + \binom{n}{r-t-2} + \cdots + \binom{n}{0}\right),$$

which is a polynomial of degree $r - t - 1$, hence smaller than $|A_0|$.

3.2 Modular Intersections

For intersecting families, we banned $|x \cap y| = 0$. What if we banned $|x \cap y| = 0 \pmod{k}$ for some k?

Example 3.3.

Take $A \subseteq X^{(r)}$ with $|x \cap y|$ odd, for all distinct $x, y \in A$. Then for odd r, we can achieve

$$|A| = \binom{\lfloor \frac{n-1}{2} \rfloor}{\frac{r-1}{2}},$$

by taking the sets to contain 1, and then unions of $\{2,3\}$, $\{4,5\}$, ...

If we wanted $|x \cap y|$ even, we can achieve $n - r + 1$ by fixing an $r - 1$ set and then another element. This is only linear in n.

Similarly, if r is even, then for $|x \cap y|$ even we can achieve

$$|A| = \binom{\lfloor \frac{n}{2} \rfloor}{\frac{r}{2}},$$

by again taking unions of $\{1,2\}$, $\{3,4\}$, ...

But for $|x \cap y|$ odd for all $x, y \in A$ we can again only achieve $n - r + 1$.

Is it possible to improve our linear bounds? It seems that $|x \cap y| = r \pmod 2$ forces the size of our set to be very small.

Remarkably, we cannot beat linear.

Proposition 3.1. *Let r be odd, and let $A \subseteq X^{(r)}$ have $|x \cap y|$ even for all distinct x, y. Then $|A| \le n$.*

The idea is to find $|A|$ linearly independent vectors in a vector space of dimension n, namely Q_n.

Proof: We view $\mathcal{P}(X)$ as $\mathbb{Z}_2^n$, the n-dimensional vector space over $\mathbb{Z}_2$, by identifying each $x \in \mathcal{P}(X)$ with $\bar{x}$, its characteristic sequence.

Then $(\bar{x}, \bar{x}) \neq 0$ for all $x \in A$, as x has odd size. Also, $(\bar{x}, \bar{y}) = 0$ for all distinct $x, y \in A$, as they have even intersection.

Hence the $\bar{x}$ for $x \in A$ are linearly independent: if $\sum \lambda_i \bar{x}_i = 0$, then dotting with $\bar{x}_j$ gives $\lambda_j = 0$ for all j, so $|A| \le n$.

Remark. Hence also if $A \subseteq X^{(r)}$, with r even and $|x \cap y|$ odd for all distinct $x, y \in A$, then $|A| \le n + 1$: just add $n + 1$ to each $x \in A$ and apply the previous proposition with $X = [n + 1]$.

Does this mod 2 behaviour generalise? We will now show that if we have s allowed values for $|x \cap y| \pmod p$, then $|A|$ is bounded by a polynomial of size s.

Theorem 3.3 (Frankl-Wilson Theorem)**.** *Let p be prime, and let $A \subseteq X^{(r)}$ be such that for all distinct $x, y \in A$, we have that $|x \cap y| \equiv \lambda_i \pmod p$ for some i, where $\lambda_1, \dots, \lambda_s$ ($s \le r$) are integers with no $\lambda_i \equiv r \pmod p$. Then*

$$|A| \le \binom{n}{s}.$$

Remark.

1. This bound is a polynomial in s that is independent of r.
2. This bound is essentially the best: if we let all $x \in A$ contain $[r-s]$, and s other elements, then we have
$$|A| = \binom{n-r+s}{s} \sim \binom{n}{s}.$$
3. We do need that no $\lambda_i \equiv r \pmod p$. Otherwise, say $n = a + \lambda p$, then we can have $A \subseteq X^{(a+kp)}$ with $|A| = \binom{\lambda}{k}$ and all $|x \cap y| = a = r \pmod p$.

The idea is to try and find $|A|$ linearly independent points in a vector space of dimension $\binom{n}{s}$, by somehow applying the polynomial $(t-\lambda_1)\cdots(t-\lambda_s)$ to $|x \cap y|$.

Proof: For each $i \le j$, let $M(i,j)$ be the $\binom{n}{i} \times \binom{n}{j}$ matrix, where rows are indexed by $X^{(i)}$ and columns are indexed by $X^{(j)}$, with

$$M(i,j)_{xy} = \begin{cases} 1 & \text{if } x \subseteq y, \\ 0 & \text{if not.} \end{cases}$$

Let V be the vector space over $\mathbb{R}$ spanned by the rows of $M(s,r)$. Then $\dim V \le \binom{n}{s}$.

For $i \le s$, consider $M(i,s)M(s,r)$. Then each row belong to V, as we have premultiplied $M(s,r)$ by a matrix. For $x \in X^{(i)}$, $y \in X^{(r)}$,

$$\begin{aligned} M(i,s)M(s,r) &= \text{number of } s\text{-sets } z \text{ with } x \subseteq z \text{ and } z \subseteq y \\ &= \begin{cases} 0 & \text{if } x \not\subseteq y, \\ \binom{r-i}{s-i} & \text{if } x \subseteq y. \end{cases} \end{aligned}$$

Hence we see

$$M(i,s)M(s,r) = \binom{r-i}{s-i} M(i,r),$$

hence all rows of $M(i,r)$ belongs to V. Now let

$$M(i) = M(i,r)^T M(i,r),$$

whose rows again belong to V as we have premultiplied. For $x, y \in X^{(r)}$, we have

$$\begin{aligned} M(i)_{xy} &= \text{number of } i\text{-sets } z \text{ with } z \subseteq x,\ z \subseteq y \\ &= \binom{|x \cap y|}{i}. \end{aligned}$$

Consider the integer polynomial $(t-\lambda_1)\cdots(t-\lambda_s)$. Then we may write it as

$$(t-\lambda_1)\cdots(t-\lambda_s) = \sum_{i=0}^{s} a_i \binom{t}{i},$$

where all $a_i \in \mathbb{Z}$. Moreover each a_i must be positive, as

$$t(t-1)\cdots(t-i+1) = n!\binom{t}{n}.$$

Now define

$$M = \sum_{i=0}^{s} a_i M(i).$$

Then for all $x, y \in X^{(r)}$,

$$M_{xy} = \sum_{i=0}^{s} a_i \binom{|x \cap y|}{i} = (|x \cap y| - \lambda_1)\cdots(|x \cap y| - \lambda_s).$$

So the submatrix of M by spanned by the rows and columns corresopnding to the elements of A is non-zero mod p on the diagonal, and zero elsewhere. Hence the rows of M corresponding to A are linearly independent over $\mathbb{Z}_p$, so also over $\mathbb{Z}$, hence also over $\mathbb{Q}$ and $\mathbb{R}$.

But these rows are a subset of V, so

$$|A| \leq \dim V \leq \binom{n}{s}.$$

Remark. We do need p to be prime. Grolmusz constructed, for each n, a value of $r \neq 0 \pmod d$ and a family $A \subseteq [n]^{(r)}$ such that, for all distinct $x, y \in A$, we have

$|x \cap y| \neq 0 \pmod{d}$, with

$$|A| > n^{c \log n / \log \log n}.$$

Corollary 3.1. *Let $A \subseteq [n]^{(r)}$ with $|x \cap y| \neq r \pmod{p}$ for each distinct $x, y \in A$, where $p < r$ is prime. Then*

$$|A| \leq \binom{n}{p-1}.$$

Proof: We are allowed $p-1$ values of $|x \cap y|$ mod p, hence we are done by Frankl-Wilson.

Note that two $n/2$ sets in $[n]$ typically meet in about $n/4$ points. But $|x \cap y| = n/4$ is very unlikely. Remarkably, we have the following result:

Corollary 3.2. *Let p be prime, and $A \subseteq [4p]^{(2p)}$ have $|x \cap y| \neq p$ for all distinct $x, y \in A$. Then*

$$|A| \leq 2\binom{4p}{p-1}.$$

Note that $\binom{4p}{p-1}$ is a tiny fraction of $\binom{4p}{2p}$. Indeed,

$$\binom{n}{n/2} \sim c\frac{2^n}{\sqrt{n}}, \qquad \binom{n}{n/4} \leq 2e^{-n/32}2^n.$$

Proof: Halving $|A|$ if necessary, we may assume that no $x, x^c \in A$. Then for $x, y \in A$ distinct,

$$|x \cap y| \neq 0, p \implies |x \cap y| \neq 0 \pmod{p}.$$

Hence $|A| \leq \binom{4p}{p-1}$ by corollary 3.1.

3.3 Borsuk's Conjecture

Let S be a bounded subset of $\mathbb{R}^n$. How few pieces can we break S into, such that each has a smaller diameter than that of S?

The example of a regular simplex in $\mathbb{R}^n$, which are $n+1$ all equidistant, shows that we may need at least $n+1$ pieces.

Borsuk's Conjecture: $n+1$ pieces are always sufficient.

This is known for $n = 1, 2$ and 3, and also for S a smooth convex body or a symmetric convex body in $\mathbb{R}^n$.

However, Borsuk is massively false.

Theorem 3.4 (Kahn, Kalai)**.** *For all n, there exists bounded $S \in \mathbb{R}^n$ such that to break S into pieces of smaller diameter, we need at least $c^{\sqrt{n}}$ pieces, for some constant $c > 1$.*

Remark.

1. Our proof will show that Borsuk is false for $n \geq 2000$.
2. We will prove this for n of the form $\binom{4p}{2}$, where p is prime.

Proof: We find $S \subseteq Q_n \subseteq \mathbb{R}^n$. In fact, we can take $S \subseteq [n]^{(r)}$, for some r.

Since $S \subseteq [n]^{(r)}$, for all $x, y \in S$,

$$\|x - y\|^n = 2(r - |x \cap y|).$$

So we seek S with $\min |x \cap y| = k$, but every subset of S with $\min |x \cap y| > k$ is very small.

Identify $[n]$ with the edge set of K_{4p}, the complete graph on $4p$ pieces. For each $x \in [4p]^{(2p)}$, let G_x be the complexe bipartite graph with vertex classes x and x^c. Let $S = \{G_x \mid x \in [4p]^{(2p)}\}$, so $S \subseteq [n]^{(4p^2)}$, and $|S| = \frac{1}{2}\binom{4p}{2p}$.

Now,

$$|G_x \cap G_y| = |x \cap y|^2 + |x^c \cap y|^2 = d^2 + (2p - d)^2,$$

where $d = |x \cap y|$, which is minimized when $d = p$, i.e. when $|x \cap y| = p$.

Now let $S' \subseteq S$ have a smaller diameter than that of S, and say $S' = \{G_x \mid x \in A\}$. Then for all $x, y \in A$ distinct, $|x \cap y| \neq p$. Thus,

$$|A| \leq 2\binom{4p}{p-1}.$$

The conclusion is that the number of pieces needed is at least

$$\frac{|S|}{\max |A|} = \frac{1}{2}\binom{4p}{2p} \cdot \frac{1}{2}\binom{4p}{p-1}^{-1} \geq \frac{c \cdot 2^{4p}\sqrt{p}}{e^{-p/8}2^{4p}}$$

for some c. But this is at least $(c')^p = (c'')^{\sqrt{n}}$ for some $c', c'' > 1$.

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