

III Quantum Field Theory

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0 Introduction

We are following Tong's notes mostly, and Matt Schwartz book for a part of it.

Our goal is to combine quantum mechanics and special relativity. The result is that the number of particles is not preserved.

Key points of this theory is that it is robust and systematic, and governed by a few principles:

- locality,
- symmetries,
- renormalization.

There are some units and conventions we will need. In terms of the base units L, T, M (length, time and mass), then

$$\begin{aligned}[c] &= [LT^{-1}], \\ [\hbar] &= [L^2MT^{-1}], \\ [G] &= [L^3M^{-1}T^{-2}].\end{aligned}$$

We take natural units, so $c = 1 = \hbar$, meaning $L = T = M^{-1}$. We refer to the mass dimension of quantities, so $[G] = [M^{-2}] = -2$. Note M also has dimensions of energy.

We will be using the relativistic notation

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

When talking about spacetime, we let $X^\mu = (t, x, y, z)$.

1 Classical Field Theory

In classical mechanics, a natural object is the action

$$S(t_1, t_2) = \int_{t_1}^{t_2} dt \left(m \sum_{i=1}^3 \left(\frac{dx_i}{dt} \right)^2 - V(X^\mu) \right).$$

Some basic facts of the action is that:

1. Equations of motion are given by extremizing S .
2. Boundary conditions are supplied externally.
3. S is built on symmetries of the system.

We declare in field theory that the fundamental object is a *field*:

$$\phi_a(t, \mathbf{x}) : \mathbb{R}^{3,1} \rightarrow \mathbb{R} \text{ or } \mathbb{C} \text{ or } \mathbb{R}^n.$$

Here a denotes the type of the field. The first consequence is that we are dealing with an infinite number of degrees of freedom.

Example 1.1. (Electromagnetism)

In EM, the *gauge field* is

$$A^\mu(x) = (\phi(x), \mathbf{A}(x)).$$

The Maxwell equations are

$$\begin{aligned} \mathbf{E} &= -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} & \nabla \cdot \mathbf{E} &= \rho, \\ \mathbf{B} &= \nabla \times \mathbf{A}, & \nabla \times \mathbf{B} &= \mathbf{J} + \frac{\partial\mathbf{E}}{\partial t}. \end{aligned}$$

We have two identities:

$$\nabla \cdot \mathbf{B} = 0, \quad \frac{d\mathbf{B}}{dt} = \nabla \times \mathbf{E}.$$

1.1 Lagrangians

Recall the *Lagrangian* is $L = T - V$, and the action can be written

$$S = \int dt L.$$

We can write

$$L = \int d^3x \mathcal{L}(\phi_a, \partial_\mu \phi_a),$$

for \mathcal{L} the *Lagrangian density* (or just Lagrangian). Then

$$S = \int dt L = \int d^4x \mathcal{L}.$$

The equations of motion can be determined by extremizing over fields. One crucial assumption is that \mathcal{L} depends on ϕ_a and $\partial_\mu \phi_a$, and not any higher derivatives. Then,

$$\begin{aligned} \delta S &= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \delta \phi_a + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \delta (\partial_\mu \phi_a) \right] \\ &= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \delta \phi_a - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \right) \delta \phi_a + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \delta \phi_a \right) \right]. \end{aligned}$$

The last term is a total derivative, and if we assume that fields decay at infinity this evaluates to 0. Hence requiring $\delta S = 0$ gives

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0.$$

Example 1.2. (Free massive scalar field)

The ‘simplest’ Lagrangian is

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \\ &= \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} m^2 \phi^2. \end{aligned}$$

In traditional classical mechanics, the first term is the kinetic energy, and the other two terms give the potential energy.

In QFT, kinetic terms are any bilinear terms of the fields. So this Lagrangian is all kinetic terms, and no potential terms.

The equations of motion in this field are

$$\partial_\mu \partial^\mu \phi + m^2 \phi = \square \phi + m^2 \phi = 0.$$

This is the *Klein-Gordon equation*.

1.2 Hamiltonians

In this setup, one starts by defining the *canonical momentum*

$$\Pi^a(x) = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi_a)} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a}.$$

The *Hamiltonian density* is

$$\mathcal{H} = \Pi^a \partial_t \phi_a - \mathcal{L}.$$

The *Hamiltonian* is

$$H = \int d^4x \mathcal{H}.$$

Example 1.3. (Scalar field with potential)

Here our Lagrangian is

$$\mathcal{L} = \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi).$$

The canonical momentum is

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}.$$

The Hamiltonian is

$$H = \int d^4x (\Pi \partial_t \phi - \mathcal{L}) = \int d^4x \left(\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\nabla \phi)^2 + V(\phi) \right).$$

1.3 Symmetries

Symmetries will:

- Dictate the actions we write.
- Dictate the class of fields (operators) used.
- Control the observables we will compute.

1.3.1 Lorentz Invariance

The *Lorentz group* is defined by

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu,$$

which preserves the interval

$$s^2 = x^\mu x^\nu \eta_{\mu\nu} = t^2 - \mathbf{x}^2.$$

So $s^2 \rightarrow s'^2 = s^2$. This condition implies that

$$\eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma},$$

or in terms of matrices,

$$\Lambda^T \eta \Lambda = \eta.$$

Example 1.4.

1. Rotations: Say $t' = t$, and $\Lambda^i{}_j = R^i{}_j$, for $R \in O(3)$. A rotation in the x - y plane is given by

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

2. Boosts: these mix time and space. A boost in the (t, x) plane is

$$\Lambda = \begin{pmatrix} \cosh \eta & -\sinh \eta & 0 & 0 \\ -\sinh \eta & \cosh \eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here η is the rapidity, and

$$\cosh \eta = \frac{1}{\sqrt{1-v^2}}, \quad \sinh \eta = \frac{v}{\sqrt{1-v^2}}.$$

More generally, if we take the determinant of $\Lambda^T \eta \Lambda = \eta$, we find

$$\det(\Lambda)^2 = 1 \implies \det \Lambda = \pm 1.$$

If $\det \Lambda = 1$, this is called a *proper Lorentz transformation*. If $\det \Lambda = -1$, this is a *improper Lorentz transformation*.

Proper Lorentz transformations can be continuously connected to the identity, whereas improper Lorentz transformation include symmetries such as parity, and time reversal.

If we focus on $\det \Lambda = 1$, we can then write

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \varepsilon^\mu{}_\nu + \mathcal{O}(\varepsilon^2).$$

What are the properties of $\varepsilon^\mu{}_\nu$? Plugging this formula into the definition of a Lorentz transformation, we find

$$\begin{aligned} \eta_{\rho\sigma} &= \eta_{\mu\nu}(\delta^\mu{}_\rho + \varepsilon^\mu{}_\rho + \cdots)(\delta^\nu{}_\sigma + \varepsilon^\nu{}_\sigma + \cdots) \\ &= \eta_{\mu\nu}\delta^\mu{}_\rho\delta^\nu{}_\sigma + \eta_{\mu\nu}\varepsilon^\mu{}_\rho\delta^\nu{}_\sigma + \eta_{\mu\nu}\delta^\mu{}_\rho\varepsilon^\nu{}_\sigma + \mathcal{O}(\varepsilon^2) \\ &= \eta_{\rho\sigma} + \varepsilon_{\sigma\rho} + \varepsilon_{\rho\sigma} + \cdots \end{aligned}$$

Hence $\varepsilon_{\sigma\rho} = -\varepsilon_{\rho\sigma}$, which gives an anti-symmetric tensor. Therefore, in 3+1 dimensions, we have 6 components. Therefore, we have 6 generators for the Lorentz group, 3 rotations and 3 boosts.

In this context, a field is an object that depends on some coordinates, and has a definite transformation under Lorentz: if $x \rightarrow x' = \Lambda x$, then

$$\phi_a(x) \rightarrow \phi'_a(x) = D[\Lambda]_a{}^b \phi_b(\Lambda^{-1}x).$$

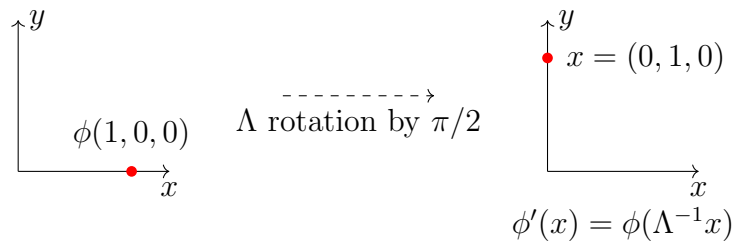
Here D forms a representation of the Lorentz group, so

- $D[\Lambda_1]D[\Lambda_2] = D[\Lambda_1\Lambda_2]$,
- $D[\Lambda^{-1}] = D[\Lambda]^{-1}$,
- $D[1] = 1$.

In the above, we have $\phi_b(\Lambda^{-1}x)$. Why are we using the active transformation? Consider a trivial representation $D[\Lambda] = 1$. Then we want

$$\phi'(x) = \phi(\Lambda^{-1}x).$$

This is the definition of the scalar field.



Example 1.5.

The trivial representation gives a scalar field.

Another example is a vector representation, so

$$D[\Lambda]^\mu{}_\nu = \Lambda^\mu{}_\nu.$$

So then

$$A^\mu(x) \rightarrow A'^\mu(x) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x),$$

and

$$\partial_\mu \phi \rightarrow \partial_\mu \phi'(x) = (\Lambda^{-1})^\mu{}_\nu \partial_\nu \phi(\Lambda^{-1}x).$$

Now we will see how symmetries constrain actions. For example consider

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial_\nu \phi \eta^{\mu\nu} - \frac{1}{2} m^2 \phi^2, \quad S = \int d^4x \mathcal{L}.$$

We can verify this is invariant under Lorentz.

Note under the Lorentz transformation, the fields transform as

$$\begin{aligned} \phi(x) &\rightarrow \phi(x) = \phi(\Lambda^{-1}x) = \phi(y), \\ \partial_\mu \phi &\rightarrow (\Lambda^{-1})^\nu{}_\mu \partial_\nu \phi(y). \end{aligned}$$

If we replace this in the Lagrangian,

$$\begin{aligned} \mathcal{L} &\rightarrow \frac{1}{2} (\Lambda^{-1})^\rho{}_\mu \partial_\mu \phi(y) (\Lambda^{-1})^\sigma{}_\nu \partial_\sigma \phi(y) - \frac{1}{2} m^2 \phi^2(y) \\ &= \frac{1}{2} \eta^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi - \frac{1}{2} m^2 \phi^2. \end{aligned}$$

Therefore,

$$\mathcal{L}(x) \rightarrow \mathcal{L}'(x) = \mathcal{L}(y).$$

Hence

$$S = \int d^4x \mathcal{L}(x) \rightarrow \int d^4x \mathcal{L}(y) = \int d^4y \mathcal{L}(y)$$

is conserved (note the change in variable $x \rightarrow y$ has Jacobian 1, as $\det \Lambda = 1$).

1.4 Nöether's Theorem

This has two parts:

1. Every continuous symmetry of the Lagrangian gives rise to a current j^μ , and the equations of motions imply

$$\partial_\mu j^\mu = 0 \implies \frac{\partial j^0}{\partial t} + \nabla \cdot \mathbf{j} = 0.$$

2. Provided suitable boundary conditions, a conserved current will give rise to a conserved charge Q , where

$$Q = \int d^3x j^0.$$

Definition 1.1. A transformation is *continuous* if there is an infinitesimal parameter in it. They can either be:

- internal: they do not act on the coordinates, but instead on the fields.
- local: they act on the coordinates and the fields.

In both cases, the differential of a continuous transformation is

$$\delta\phi_a = \phi'_a(x) - \phi_a(x).$$

Such a transformation is a symmetry of the system if the action is invariant, so

$$S[\phi] \rightarrow S[\phi'] = \int d^4x \mathcal{L}(x),$$

and moreover

$$\delta(S) = S[\phi'] - S[\phi] = 0.$$

This implies that for the Lagrangian,

$$\delta\mathcal{L} = \mathcal{L}'(x) - \mathcal{L}(x) = \partial_\mu F^\mu,$$

the same up to a total derivative.

Proof: Let us quantify the change in \mathcal{L} :

$$\begin{aligned} \delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi_a} \delta\phi_a + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a} \delta\partial_\mu\phi_a \\ &= \left(\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a} \right) \right) \delta\phi_a + \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a} \delta\phi_a \right) = \partial_\mu F^\mu, \end{aligned}$$

as it is a symmetry. Hence,

$$-\left(\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a}\right)\right) \delta \phi_a = \partial_\mu \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a} \delta \phi_a - F^\mu\right)}_{j^\mu}.$$

If the equations of motion are imposed, then

$$\partial_\mu j^\mu = 0,$$

where

$$j^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a} \delta \phi_a - F^\mu.$$

For the second part of the statement, we have

$$Q = \int d^3x j_0.$$

Then the total time derivative is

$$\frac{dQ}{dt} = \int_V d^3x \frac{\partial j^0}{\partial t} = - \int_V d^3x \nabla \cdot j = - \int_{\partial V} dA \cdot j = 0,$$

given suitable boundary conditions on j , i.e. fields decay.

1.5 Energy-Momentum Tensor

Consider local transformations given by translations:

$$x^\mu \rightarrow x'^\mu = x^\mu - \varepsilon^\mu.$$

Under translation, the fields transform as

$$\phi_a(x) \rightarrow \phi'_a(x) = \phi_a(x + \varepsilon) = \phi_a(x) + \varepsilon^\mu \partial_\mu \phi_a + \mathcal{O}(\varepsilon^2).$$

Hence we find

$$\delta \phi_a = \phi'_a(x) - \phi_a(x) = \varepsilon^\mu \partial_\mu \phi_a.$$

The Lagrangian changes as

$$\delta \mathcal{L} = \varepsilon^\mu \partial_\mu \mathcal{L} = \partial_\mu (\varepsilon^\mu \mathcal{L}).$$

The conserved current is then

$$\begin{aligned} j^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \varepsilon^\nu \partial_\nu \phi_a - \varepsilon^\mu \mathcal{L} \\ &= \varepsilon^\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\nu \phi_a - \delta^\mu_\nu \mathcal{L} \right) \\ &= \varepsilon^\nu T^\mu_\nu, \end{aligned}$$

where T^μ_ν is the energy-momentum tensor. If the equations of motion are 0, then varying ε^ν , we find

$$\partial_\mu j^\mu = 0 \implies \partial_\mu T^\mu_\nu = 0.$$

We can construct four conserved charges:

$$\begin{aligned} E &= \int d^3x T^{00}, \\ P^i &= \int d^3x T^{0i}. \end{aligned}$$

These are the energy, and the three momenta.

Example 1.6. (Free massive scalar field)

Here we have

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\nu \phi_a - \delta^\mu_\nu \mathcal{L},$$

so

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}.$$

Then we find

$$T^{00} = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2.$$

So

$$E = \int d^3x T^{00} = H, \quad P^i = \int d^3x T^{0i} = \int d^3x \dot{\phi} \partial^i \phi.$$

Remark. The definition of the EM tensor is

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\nu \phi_a - \delta^\mu_\nu \mathcal{L}.$$

From this equation, we do not ensure that T is symmetric. How can we make it symmetric?

1. Define

$$\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\rho \Gamma^{\rho\mu\nu},$$

where

$$\Gamma^{\rho\mu\nu} = -\Gamma^{\mu\rho\nu},$$

such that $\partial_\mu \Theta^{\mu\nu} = 0$.

2. Couple fields to $g_{\mu\nu}$. Then

$$\Theta^{\mu\nu} = \left(-\frac{2}{\sqrt{-g}} \frac{\partial}{\partial g_{\mu\nu}} (\sqrt{-g} \mathcal{L}) \right) \Big|_{g=\eta}.$$

Example 1.7. (Complex scalar field)

A complex scalar field is

$$\psi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x)),$$

where ϕ_i are real scalar fields. A Lagrangian for this field is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \psi \partial^\mu \psi^* - V(|\psi|^2).$$

The equations of motion turn out to be

$$\partial_\mu \partial^\mu \psi + \frac{\partial V}{\partial \psi^*} = 0, \quad \partial_\mu \partial^\mu \psi^* + \frac{\partial V}{\partial \psi} = 0.$$

In this system, there is an internal symmetry given by

$$\psi(x) \rightarrow \psi'(x) = e^{i\alpha} \psi(x), \quad \psi^*(x) \rightarrow \psi'^*(x) = e^{-i\alpha} \psi^*(x).$$

Here $\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L}$, so $\mathcal{S} \rightarrow \mathcal{S}' = \mathcal{S}$.

α is a continuous parameter of the transformation, so $\delta\psi = \psi'(\alpha) - \psi(\alpha) = i\alpha\psi$, and $\delta\psi^* = -i\alpha\psi^*$.

We can construct the current as

$$\begin{aligned} j^\mu &= \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} \delta\psi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi^*} \delta\psi^* \\ &= \partial^\mu \psi^* \delta\psi + \partial^\mu \psi \delta\psi^* \\ &= i\alpha(\psi \partial^\mu \psi^* - \psi^* \partial^\mu \psi). \end{aligned}$$

It is also possible to parametrize our transformation as

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rightarrow \begin{pmatrix} \phi'_1 \\ \phi'_2 \end{pmatrix} = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

2 Quantum Fields

2.1 Free Theory

We will begin by taking a Hamiltonian approach, and will follow the rules of QM. In this framework,

$$[X^i, P^j] = i\hbar\delta^{ij}.$$

In QFT, we now have $\phi_a(x)$ and $\Pi^a(x)$, where $\Pi^a = \partial\mathcal{L}/\partial\dot{\phi}_a$. The new rule is

$$[\phi_a(\mathbf{x}, t), \Pi^b(\mathbf{y}, t)] = i\delta^3(\mathbf{x} - \mathbf{y})\delta_a^b.$$

This looks like it breaks relativity; we will see it does not. Our goal is to implement such a quantum field theory for the free massive real scalar field. The plan will be to go through:

- Canonical quantization.
- Hamiltonian.
- Fock space.
- Causality.
- Propagators.

2.2 Canonical Quantization

Our theory is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2.$$

The equations of motion are

$$\partial_\mu\partial^\mu\phi + m^2\phi = 0.$$

Solutions to this include

$$\phi \sim \exp(i\mathbf{k} \cdot \mathbf{x} + i\omega t),$$

where ω, \mathbf{k}, m are related by

$$-\omega^2 + \mathbf{k}^2 + m^2 = 0 \implies \omega = \pm\sqrt{\mathbf{k}^2 + m^2}.$$

We adopt the notation

$$\omega = \sqrt{\mathbf{k}^2 + m^2}.$$

Hence, we can write

$$\phi(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3} [a(\mathbf{k})e^{i\mathbf{k} \cdot \mathbf{x} - i\omega t} + b(\mathbf{k})e^{i\mathbf{k} \cdot \mathbf{x} + i\omega t}].$$

Note that ϕ is real, so $\phi^* = \phi$ implies

$$a^*(-\mathbf{k}) = b(\mathbf{k}), \quad b^*(-\mathbf{k}) = a(\mathbf{k}).$$

Hence we can write

$$\begin{aligned} \phi(x) &= \int \frac{d^3k}{(2\pi)^3} [a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} + a^*(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega t}] \\ &= \int \frac{d^3k}{(2\pi)^3} [a(\mathbf{k})e^{-ikx} + a^*(\mathbf{k})e^{ikx}], \end{aligned}$$

where

$$kx = k^\mu x_\mu = \omega t - \mathbf{k} \cdot \mathbf{x}, \quad k^2 = \omega^2 - \mathbf{k}^2 = m^2.$$

We choose to normalize $a(\mathbf{k})$ and $a^*(\mathbf{k})$ such that

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega}} [a(\mathbf{k})e^{-ikx} + a^*(\mathbf{k})e^{ikx}].$$

Next, we quantize. To do this, we calculate the conjugate momenta:

$$\Pi(x) = \dot{\phi} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{i} \sqrt{\frac{\omega}{2}} [a(\mathbf{k})e^{-ikx} - a^*(\mathbf{k})e^{ikx}].$$

We declare

$$\begin{aligned} [\phi(\mathbf{x}, t), \phi(\mathbf{x}', t)] &= 0, \\ [\Pi(\mathbf{x}, t), \Pi(\mathbf{x}', t)] &= 0, \\ [\phi(\mathbf{x}, t), \Pi(\mathbf{x}', t)] &= i\delta^3(\mathbf{x} - \mathbf{x}'). \end{aligned}$$

The claim is that these commutation relations promote a to an operator, and a^* becomes a^\dagger , also an operator. These themselves have commutation relations:

$$\begin{aligned} [a(\mathbf{k}), a(\mathbf{k}')] &= 0, \\ [a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}')] &= 0, \\ [a(\mathbf{k}), a^\dagger(\mathbf{k}')] &= (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}'). \end{aligned}$$

Proof: We show that the second set of commutation relations imply the first set. We only prove the last one as it is the only non-trivial one:

$$\begin{aligned}
[\phi(\mathbf{x}, t), \Pi(\mathbf{y}, t)] &= \int \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{2i} \sqrt{\frac{\omega_p}{\omega_q}} ([a(p)e^{i\mathbf{p}\cdot\mathbf{x}-i\omega t} + a^\dagger(p)e^{-i\mathbf{p}\cdot\mathbf{x}+i\omega t}, \\
&\quad a(q)e^{i\mathbf{q}\cdot\mathbf{y}-i\omega t} - a^\dagger(q)e^{-i\mathbf{q}\cdot\mathbf{y}+i\omega t}]) \\
&= C \int d^3p d^3q (-[a(p), a^\dagger(q)]e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i\mathbf{q}\cdot\mathbf{y}}e^{it(\omega_q-\omega_p)} \\
&\quad + [a^\dagger(p), a(q)]e^{-i\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{q}\cdot\mathbf{y}}e^{it(\omega_p-\omega_q)}) \\
&= i \int \frac{d^3p}{(2\pi)^3} e^{ip(\mathbf{x}-\mathbf{y})} = i\delta^3(\mathbf{x}-\mathbf{y}).
\end{aligned}$$

2.3 Hamiltonian

The Hamiltonian of the free theory is

$$H = \int d^3x \mathcal{H} = \frac{1}{2} \int d^3x (\Pi^2 + (\nabla\phi)^2 + m^2\phi^2),$$

which we want in terms of a, a^\dagger . Writing this out,

$$\begin{aligned}
H &= \frac{1}{2} \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} \left(-\frac{\sqrt{\omega_p\omega_q}}{2} (ae^{-ipx} - a^\dagger e^{ipx})(ae^{-iqx} - a^\dagger e^{iqx}) \right. \\
&\quad - \frac{1}{2} \frac{1}{\sqrt{\omega_p\omega_q}} (ae^{-ipx} - a^\dagger e^{ipx})(ae^{-iqx} - a^\dagger e^{iqx}) \mathbf{p} \cdot \mathbf{q} \\
&\quad \left. + \frac{m^2}{2} \frac{1}{\sqrt{\omega_p\omega_q}} (ae^{ipx} + a^\dagger e^{-ipx})(ae^{iqx} + a^\dagger e^{-iqx}) \right) \\
&= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left[\underbrace{(-\omega_p^2 + \mathbf{p}^2 + m^2)}_0 (a_p a_p e^{-2i\omega t} + a_p^\dagger a_p^\dagger e^{2i\omega t}) \right. \\
&\quad \left. + (\omega_p^2 + \mathbf{p}^2 + m^2)(a_p^\dagger a_p + a_p a_p^\dagger) \right] \\
&= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \omega_p (a_p^\dagger a_p + a_p a_p^\dagger) \\
&= \int \frac{d^3p}{(2\pi)^3} \omega_p a_p^\dagger a_p + \int \frac{d^3p}{2} \omega_p \delta(0).
\end{aligned}$$

We should be scared by the last term. If we take a vacuum state $|0\rangle$ such that

$$a_p |0\rangle = 0,$$

then in fact

$$H |0\rangle = \int \frac{d^3p}{2} \omega (2\pi)^3 \delta(0) |0\rangle = \infty.$$

To understand the nature of this, we need to see the origin of this divergence. We actually have two infinities:

- (i) Infrared divergence: $(2\pi)^3 \delta(0)$. This arises as follows:

$$(2\pi)^3 \delta(0) = \lim_{L \rightarrow \infty} \int_{-L}^L d^3x e^{i\mathbf{x} \cdot \mathbf{p}} \Big|_{\mathbf{p}=0} = \lim_{L \rightarrow \infty} \int_{-L}^L d^3x = V.$$

For an infinite size system, this blows up. The solution is to discuss energy densities, so

$$\mathcal{E}_0 = \frac{E_0}{V} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \omega_p \sim \int d^3p p^2 \rightarrow \infty.$$

This also diverges.

- (ii) Ultraviolet divergence:

$$\int_0^{p_{\max}} d^3p \sqrt{p^2 + m^2} \rightarrow \infty,$$

which is high-frequency divergence.

It is absurd to think that the theory is valid for arbitrarily high energies.

The solution, which is mostly practical, is to declare

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_p a_p^\dagger a_p,$$

and with this $H |0\rangle = 0$.

The origin is due to an ambiguity in multiplying fields. The cure is *normal ordering*.

Definition 2.1. If we are given a list of fields, we define the *normal ordering* as

$$:\phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n):$$

where this is the usual product with all $a(p)$ operators placed to the right of an $a^\dagger(p)$.

2.4 Fock Space

Given $|0\rangle$, we want to construct excited states. We know

$$\begin{aligned}[H, a_p^\dagger] &= \omega_p a_p^\dagger, \\ [H, a_p] &= -\omega_p a_p.\end{aligned}$$

We can construct excited states by

$$\begin{aligned}|p\rangle &= a^\dagger(p) |0\rangle, \\ H |p\rangle &= \omega_p |p\rangle,\end{aligned}$$

where $\omega_p^2 = p^2 + m^2$. We can consider

$$\mathbf{P} = - \int d^3x \pi \nabla \phi = \int \frac{d^3k}{(2\pi)^3} k a_k^\dagger a_k,$$

where

$$\mathbf{P} |\mathbf{p}\rangle = \mathbf{p} |\mathbf{p}\rangle.$$

Here $|\mathbf{p}\rangle$ is a momentum and energy eigenstate, with eigenvalues \mathbf{p} and energy $E = \omega_p^2 = \mathbf{p}^2 + m^2$.

Also, $\mathbf{p} = 0$ is an angular momentum J^i eigenstate, i.e.

$$J^i |\mathbf{p} = 0\rangle = 0.$$

With this, we can create more states:

$$|\mathbf{p}_1 \cdots \mathbf{p}_n\rangle = a^\dagger(\mathbf{p}_1) \cdots a^\dagger(\mathbf{p}_n) |0\rangle.$$

Because a^\dagger commute, these are configurations which are symmetric under interchange, i.e.

$$|\mathbf{p}_1 \mathbf{p}_2\rangle = |\mathbf{p}_2 \mathbf{p}_1\rangle.$$

The *Fock space* is the collection of all possible combinations of a^\dagger acting on $|0\rangle$. Introducing

$$N = \int \frac{d^3p}{(2\pi)^3} a_p^\dagger a_p,$$

which is the *number operator*, we find

$$N |\mathbf{p}_1 \cdots \mathbf{p}_n\rangle = n |\mathbf{p}_1 \cdots \mathbf{p}_n\rangle.$$

For a free theory,

$$[N, H] = 0.$$

Fock space is then

$$\bigoplus_i \mathcal{H}_n$$

2.5 Relativistic Normalization

How do we normalize these states? First thing, pick

$$\langle 0|0\rangle = 1.$$

For 1-particle states, note

$$|\mathbf{p}\rangle = a_p^\dagger |0\rangle \implies \langle \mathbf{p}|\mathbf{q}\rangle = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}).$$

This is not Lorentz invariant. Our dream is that under a Lorentz transformation,

$$|\mathbf{p}\rangle \rightarrow |\mathbf{p}'\rangle = U(\Lambda) |\mathbf{p}\rangle.$$

To figure out a proper definition of $|\mathbf{p}\rangle$, we use the identity

$$|\mathbf{q}\rangle = \int \frac{d^3p}{(2\pi)^3} |\mathbf{p}\rangle \langle \mathbf{p}|\mathbf{q}\rangle,$$

hence

$$1 = \int \frac{d^3p}{(2\pi)^3} |\mathbf{p}\rangle \langle \mathbf{p}|.$$

This integral is manifestly not Lorentz invariant, due to the measure we are taking the integral over. Instead, we transform

$$\begin{aligned} \int \frac{d^3p}{(2\pi)^3} &\rightarrow \int d^4p \delta^4(\mathbf{p}^2 - m^2) \Theta(p^0) = \int d^3p \frac{1}{2\sqrt{\mathbf{p}^2 + m^2}} \\ &= \int d^3p \frac{1}{2\omega_p}. \end{aligned}$$

So instead, we should define

$$1 = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} |\mathbf{p}\rangle \langle \mathbf{p}|,$$

where

$$|\mathbf{p}\rangle = \sqrt{2\omega_p} a_p^\dagger |0\rangle.$$

This is relativistic normalization.

2.6 Causality

Here we are interested in whether measurements influence each other, i.e. whether commutators vanish. This is associated with why equal-time commutators are compatible with relativity. Define

$$\Delta(x - y) = [\phi(x), \phi(y)].$$

We can evaluate this for a free theory:

$$\begin{aligned} \Delta &= [\phi(x), \phi(y)] \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} \frac{1}{2\sqrt{\omega_k\omega_p}} ([a_k, a_p^\dagger] e^{-ikx} e^{ipy} + [a_k^\dagger, a_p] e^{ikx} e^{-ipy}) \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} (e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)}). \end{aligned}$$

This commutator satisfies a few properties:

- It is Lorentz invariant due to the appearance of our measure, and also a c-number operator.
- For time-like separation, $(x - y)_T = (t, 0, 0, 0)$, we find

$$\Delta(x - y)_T = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} (e^{-i\omega_p t} - e^{i\omega_p t}) \sim e^{-imt} - e^{imt} \neq 0.$$

- For spacelike separation, say $(x - y)_S = (0, \mathbf{x} - \mathbf{y})$, then

$$\Delta(x - y)_S = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} (e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} - e^{-i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})}) = 0,$$

so any two spacelike events have zero commutator.

2.7 Propagators

Here we are interested in the quantity

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} e^{-ip(x-y)} = D(x - y).$$

This can be thought of as preparing a particle at position y , and measuring its amplitude at position x . For spacelike events,

$$D(x - y) \sim e^{-m(\mathbf{x} - \mathbf{y})} \neq 0.$$

But,

$$[\phi(x), \phi(y)] = D(x - y) - D(y - x) = 0.$$

Define

$$\Delta_F(x - y) = \langle 0 | T \phi(x) \phi(y) | 0 \rangle = \begin{cases} D(x - y) & x^0 > y^0, \\ D(y - x) & y^0 > x^0. \end{cases}$$

Here T is the *time operator*. We claim that

$$\Delta_F(x - y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{-ip(x-y)}.$$

Proof: A lot of calculation:

$$\begin{aligned} \langle 0 | T \phi(x) \phi(y) | 0 \rangle &= \langle 0 | \phi(x) \phi(y) | 0 \rangle \Theta(x^0 - y^0) + \langle 0 | \phi(y) \phi(x) | 0 \rangle \Theta(y^0 - x^0) \\ &= \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_k} e^{-i\omega_k(x^0 - y^0)} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} \Theta(x^0 - y^0) \\ &\quad + \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_k} e^{-i\omega_k(y^0 - x^0)} e^{i\mathbf{k} \cdot (\mathbf{y} - \mathbf{x})} \Theta(y^0 - x^0) \\ &= \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_k} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} (e^{-i\omega_k z} \Theta(z) + e^{i\omega_k z} \Theta(-z)). \end{aligned}$$

We focus on the time-dependent part, and show that

$$e^{-i\omega_k z} \Theta(z) + e^{i\omega_k z} \Theta(-z) = \lim_{\varepsilon \rightarrow 0} \frac{(-2\omega_k)}{2\pi i} \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega z}}{\omega^2 - \omega_k^2 + \varepsilon}.$$

We start from the right hand side of the equation:

$$\begin{aligned} \frac{1}{\omega^2 - \omega_k^2 + i\varepsilon} &= \frac{1}{(\omega - (\omega_k - i\varepsilon))(\omega - (-\omega_k + i\varepsilon))} \\ &= \frac{1}{2\omega_k} \left[\frac{1}{\omega - (\omega_k - i\varepsilon)} - \frac{1}{\omega - (-\omega_k + i\varepsilon)} \right] + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Consider the integral

$$I_1 = \int_{-\infty}^{\infty} \frac{e^{-i\omega z}}{\omega - (\omega_k - i\varepsilon)}.$$

We want to use Schwartz' lemma to evaluate this in the complex plane. The function has a pole at $\omega = \omega_k - i\varepsilon$.

If $z < 0$, we can complete is in the upper-half plane, and get $I_1 = 0$. If $z > 0$, we need to close it in the lower-half plane, which encompasses the pole, and results in an integral of

$$I_1 = -2\pi i e^{-i\omega_k z} \theta(z) + \mathcal{O}(\varepsilon).$$

The negative sign is as we are integrated in a clockwise direction.

The other term in the integral is

$$I_2 = \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega z}}{\omega - (-\omega_k + i\varepsilon)}.$$

We can do a similar thing to find I_2 , and get

$$I_2 = 2\pi i e^{i\omega_k z} \Theta(-z) + \mathcal{O}(\varepsilon).$$

Collecting these terms,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega z}}{\omega^2 - \omega_k^2 + \varepsilon} &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2\omega_k} (I_1 - I_2) \\ &= \frac{1}{2\omega_k} (-2\pi i e^{-i\omega_k z} \Theta(z) - 2\pi i e^{i\omega_k z} \Theta(-z)). \end{aligned}$$

Pulling this into the expression,

$$\begin{aligned} \langle 0 | T \phi(x) \phi(y) | 0 \rangle &= \int \frac{d^3 k}{(2\pi)^3} \frac{i}{2\pi} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega z}}{\omega^2 - \omega_k^2 + i\varepsilon} \\ &= \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} e^{-ik(x-y)}. \end{aligned}$$

By convention we drop the limit.

Remark.

1. Due to the time ordering, our contour is prescribed.
2. $\Delta_F(x - y)$ is Lorentz invariant.
3. $\Delta_F(x - y)$ is a Green's function, as

$$(\partial_\mu \partial^\mu + m^2) \Delta_F(x - y) = -i \delta^4(x - y).$$

Δ_F is the Green's function associated to the Klein-Gordon equation.

4. Normal orderings are invariant up to reordering, as each a^\dagger and a commute.
5. Fun little fact:

$$T\phi(x)\phi(y) = :\phi(x)\phi(y): + \Delta_F(x-y) = :\phi(x)\phi(y): + \overbrace{\phi(x)\phi(y)}$$

To prove this, write $\phi = \phi^+ + \phi^-$, where

$$\begin{aligned}\phi^+ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} a_p e^{-ipx}, \\ \phi^- &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} a_p^\dagger e^{ipx}.\end{aligned}$$

Choose $x^0 > y^0$. Then,

$$\begin{aligned}T(\phi(x)\phi(y)) &= \phi(x)\phi(y) \\ &= \phi^+(x)\phi^+(y) + \phi^-(x)\phi^+(y) + \phi^-(y)\phi^+(x) \\ &\quad + [\phi^+(x), \phi^-(y)] + \phi^-(x)\phi^-(y) \\ &= :\phi(x)\phi(y): + D(x-y).\end{aligned}$$

If $x^0 < y^0$, we get

$$T\phi(x)\phi(y) = :\phi(x)\phi(y): + D(y-x).$$

Together this gives *Wick's theorem*, a generalisation of this:

$$\begin{aligned}T(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)) &= :\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4): \\ &\quad + \overbrace{\phi_1\phi_2} : \phi_3\phi_4 : + \overbrace{\phi_1\phi_3} : \phi_2\phi_4 : + \overbrace{\phi_1\phi_4} : \phi_2\phi_3 : \\ &\quad + \overbrace{\phi_2\phi_3} : \phi_1\phi_4 : + \overbrace{\phi_2\phi_4} : \phi_1\phi_3 : + \overbrace{\phi_3\phi_4} : \phi_1\phi_2 : \\ &\quad + \overbrace{\phi_1\phi_2} \overbrace{\phi_3\phi_4} + \overbrace{\phi_1\phi_3} \overbrace{\phi_2\phi_4} + \overbrace{\phi_1\phi_4} \overbrace{\phi_2\phi_3}.\end{aligned}$$

3 Interactions

3.1 Couplings

The free theory we have been studying so far is “simple”: we can construct explicitly the Fock space.

We want to consider more general Lagrangians. An obstruction is that we cannot generally solve the equations of motion, hence do not have access to the Hilbert space.

In this way, we take a perturbative approach to QFT, where

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}},$$

where say

$$\mathcal{L}_0 = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2,$$

and

$$\mathcal{L}_{\text{int}} = \sum_{n \geq 0} \frac{\lambda_n}{n!} \phi^n.$$

A naive approach is to believe that $\lambda_n \ll 1$. This will in general not be true.

In some way we want to quantify smallness. Recall our units

$$c = 1 = \hbar,$$

and so

$$[L] = [T] = [M^{-1}],$$

where $[M] = 1$. Applying this to the action,

$$S = \int d^4x \mathcal{L},$$

where $[S] = [\hbar] = 0$, which is dimensionless. Since we are taking an integral over 4-space, this means that $[\mathcal{L}] = 4$.

Applying to \mathcal{L}_0 , we see that $[m] = 1$, the partial have unit 1, so hence $[\phi] = 1$. Therefore,

$$[\mathcal{L}_{\text{int}}] = [\lambda_n \phi^n] \implies [\lambda_n] = 4 - n.$$

In different cases, we want to consider our λ_n in relation to given scales.

1. Take $n = 3$, so $[\lambda_n] = 1$. Then a dimensionless quantity we could consider is λ_n/E , where E is some energy scale.

If $\lambda_n \ll E$, then the perturbation is small. If $\lambda_n \gg E$, then the perturbation is large.

If for example $E > m$, then $\lambda_n \ll m$ is a small perturbation. This is an example of a *relevant coupling*.

2. If $n = 4$, then $[\lambda_n] = 0$, so it is meaningful to write $\lambda_n \gg 1$ or $\lambda_n \ll 1$. This is a *marginal coupling*.

3. If $n > 4$, then $[\lambda_n] < 0$. So the dimensional combination is $\lambda_n(E)^{n-1}$, which are not important at low energies, and only become important at higher energies. In this case λ_n is an *irrelevant coupling*.

QFT is simple because at the start, we only care about relevant and marginal couplings.

In our case, the classification or identification of the couplings is classical. In advanced QFT, we will see that quantum effects may change the couplings.

3.2 LSZ Reduction Formula

The basic quantity to study in QFT is a scattering matrix (S-matrix). That is, given an ingoing particle which interacts in some way, we want to calculate the outgoing particle.

To fully quantify the theory, we need to:

1. Define our states (asymptotic states).
2. Understand how to relate the in and out states (S-matrices).
3. Understand how to evaluate S .

To understand our states, consider our Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}},$$

which is analogously given by a Hamiltonian

$$H = H_0 + H_{\text{int}}.$$

We will assume there is a vacuum state $|\Omega\rangle$, for the entire system H . $|0\rangle$ will still be our vacuum state for H_0 .

We will also assume that the Hamiltonian is responsible for time evolution, so

$$i\partial_t\phi = [\phi, H],$$

and in general

$$i\partial_t\Theta = [\Theta, H]$$

for any operator Θ . We also declare that at some time $t = t_0$, we can match the Hilbert space of H_0 to that of H , so

$$a_p(t) = e^{iH(t-t_0)} a_p e^{-iH(t-t_0)}.$$

Then our overall field becomes

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_p(t)e^{-ipx} + a_p^\dagger(t)e^{ipx}).$$

With this, we can write out states as, e.g.

$$\begin{aligned} |\text{initial}\rangle &= \sqrt{2\omega_1}\sqrt{2\omega_2}a_{p_1}^\dagger(t_i)a_{p_2}^\dagger(t_i)|\Omega\rangle, \\ |\text{final}\rangle &= \sqrt{2\omega_3}\sqrt{2\omega_4}a_{p_3}^\dagger(t_f)a_{p_4}^\dagger(t_f)|\Omega\rangle. \end{aligned}$$

In our definition of asymptotic states, we will want interactions to be turned off when $t_i \rightarrow -\infty$, and $t_f \rightarrow \infty$, so

$$\lim_{t \rightarrow \pm\infty} a_p^\dagger(t) = a_p^\dagger.$$

We can relate the states at $\pm\infty$ by

$$\begin{aligned} \langle f|S|i\rangle &= \langle \text{final}|\text{initial}\rangle \\ &= \prod_{i=1}^4 \sqrt{2\omega_i} \langle \Omega|T a_{p_3}(+\infty)a_{p_4}(+\infty)a_{p_1}^\dagger(-\infty)a_{p_2}^\dagger(-\infty)|\Omega\rangle. \end{aligned}$$

We want the S -matrix for other times. To achieve this, we prove the following:

Proposition 3.1. *We have*

$$\sqrt{2\omega_p}(a_p^\dagger(\infty) - a_p^\dagger(-\infty)) = -i \int d^4x e^{-ipx} (\square + m^2)\phi(x)$$

in the interacting theory, where

$$\omega_p = \sqrt{\mathbf{p}^2 + m^2}.$$

Proof: We have, the right side is

$$\begin{aligned}
 -i \int d^4x e^{-ipx} (\square + m^2) \phi(x) &= -i \int d^4x e^{-ipx} (\partial_t^2 - \vec{\nabla}^2 + m^2) \phi(x) \\
 &= -i \int d^4x e^{-ipx} (\partial_t^2 - \overleftarrow{\nabla}^2 + m^2) \phi(x) \\
 &= -i \int d^4x e^{-ipx} (\partial_t^2 + \mathbf{p}^2 + m^2) \phi(x) \\
 &= -i \int d^4x \partial_t (e^{-ipx} \partial_t \phi - (\partial_t e^{-ipx}) \phi).
 \end{aligned}$$

This only depends at times at $\pm\infty$. Recall from the free theory that

$$\begin{aligned}
 \sqrt{2\omega_p} \hat{a}_p &= i \int d^3x e^{ipx} \vec{\partial}_t \phi, \\
 \sqrt{2\omega_p} \hat{a}_p^\dagger &= -i \int d^3x e^{-ipx} \vec{\partial}_t \phi,
 \end{aligned}$$

where

$$f \vec{\partial}_t g = f \partial_t g - (\partial_t f) g.$$

Hence we can evaluate this as

$$\sqrt{2\omega_p} \int_{-\infty}^{\infty} dt \partial_t (a_p^\dagger(t)) = \sqrt{2\omega_p} (a_p^\dagger(\infty) - a_p^\dagger(-\infty)).$$

Analogously, we have

$$\sqrt{2\omega_p} (a_p(\infty) - a_p(-\infty)) = i \int d^3x e^{ipx} (\square + m^2) \phi(x).$$

Hence we can write $\langle f|S|i \rangle$ as

$$\begin{aligned}
 \langle f|S|i \rangle &= \prod_{i=1}^4 \sqrt{2\omega_i} \langle \Omega | T(a_{p_3}(+\infty) - a_{p_3}(-\infty))(a_{p_4}(+\infty) - a_{p_4}(-\infty)) \\
 &\quad \times (a_{p_1}^\dagger(-\infty) - a_{p_1}^\dagger(+\infty))(a_{p_2}^\dagger(-\infty) - a_{p_2}^\dagger(+\infty)) | \Omega \rangle \\
 &= \prod_{j=1}^4 \left(i \int d^4x_j \right) e^{-ip_1x_2} (\square_1 + m^2) e^{-ip_2x_2} (\square_2 + m^2) e^{ip_3x_3} (\square_3 + m^2) e^{ip_4x_4} (\square_4 + m^2) \\
 &\quad \times \langle \Omega | T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) | \Omega \rangle.
 \end{aligned}$$

The advantages of LSZ:

- This formula above is manifestly Lorentz invariant.
- It makes clear the relation between the correlation functions, and the S-matrix.

Note that

$$\langle \Omega | T(\square_x + m^2) \phi(x) \cdots | \Omega \rangle \neq (\square_x + m^2) \langle \Omega | T \phi(x) \cdots | \Omega \rangle$$

but we still pull these out anyway. The difference are contact terms.

3.3 Schwinger-Dyson Formula

We need a way to evaluate

$$\langle \Omega | T \phi(x_1) \cdots \phi(x_n) | \Omega \rangle .$$

The strategy is to present a Lagrangian approach to this. We make some assumptions.

1. At any given time, the Hilbert space of the interacting theory is equal to the Hilbert space of the free theory, so

$$\begin{aligned} [\phi(\mathbf{x}, t), \phi(\mathbf{x}', t)] &= 0, \\ [\phi(\mathbf{x}, t), \partial_t \phi(\mathbf{x}', t)] &= i\delta^3(\mathbf{x} - \mathbf{x}'). \end{aligned}$$

2. Our fields still comply with the Euler-Lagrange equations. In the free theory,

$$(\square + m^2)\phi(x) = 0.$$

In the interacting theory,

$$(\square + m^2)\phi(x) - \mathcal{L}'_{\text{int}}(\phi) = 0,$$

where $\mathcal{L}'_{\text{int}} = \partial \mathcal{L}_{\text{int}} / \partial \phi$.

In the Hamiltonian derivation, you would assume

$$\partial_t \phi = i[H, \phi].$$

For brevity, we remove the T :

$$\langle \Omega | \phi(x_1) \cdots \phi(x_n) | \Omega \rangle = \langle \phi_1 \cdots \phi_n \rangle .$$

We claim that

$$(\square_x + m^2) \langle \phi_x \phi_y \rangle = \langle (\square_x + m^2) \phi_x \phi_y \rangle - i\delta^4(x - y).$$

Proof: For our warm up, we take the free theory:

$$(\square_x + m^2) \langle \phi_x^0 \phi_y^0 \rangle = 0 - i\delta^4(x - y).$$

This is saying the Feynman propagator is a Green's function. For an interacting theory,

$$\begin{aligned} \partial_{x^0} \langle \phi_x \phi_y \rangle &= \partial_{x^0} (\langle \Omega | \phi_x \phi_y | \Omega \rangle \Theta(x^0 - y^0) + \langle \Omega | \phi_y \phi_x | \Omega \rangle \Theta(y^0 - x^0)) \\ &= \langle \partial_{x^0} \phi_x \phi_y \rangle + \langle \Omega | \phi_x \phi_y | \Omega \rangle \partial_x \Theta(x^0 - y^0) + \langle \Omega | \phi_y \phi_x | \Omega \rangle \partial_{x^0} \Theta(y^0 - x^0) \\ &= \langle \partial_{x^0} \phi_x \phi_y \rangle + \delta(x^0 - y^0) \langle \Omega | [\phi_x, \phi_y] | \Omega \rangle \\ \partial_{x^0}^2 \langle \phi_x \phi_y \rangle &= \langle \partial_{x^0}^2 \phi_x \phi_y \rangle + \delta(x^0 - y^0) \langle \Omega | [\partial_x \phi_x, \phi_y] | \Omega \rangle \\ &= \langle \partial_{x^0}^2 \phi_x \phi_y \rangle - i\delta^4(x - y). \end{aligned}$$

Hence we see

$$(\square_x + m^2) \langle \phi_x \phi_y \rangle = \langle (\square_x + m^2) \phi_x \phi_y \rangle - i\delta^4(x - y).$$

Now that this has been shown, we can write down

$$(\square_x + m^2) \langle \phi_x \phi_y \rangle = \langle \mathcal{L}'_{\text{int}}(\phi_x) \phi_y \rangle - i\delta^4(x - y).$$

This generalizes.

$$(\square_x + m^2) \langle \phi_1 \cdots \phi_n \rangle = \langle \mathcal{L}'_{\text{int}}(\phi_x) \phi_1 \cdots \phi_n \rangle - i \sum_{j=1}^n \delta^4(x - x_j) \langle \phi_1 \cdots \phi_{j-1} \phi_{j+1} \cdots \phi_n \rangle.$$

Example 3.1. (Uses of Schwinger-Dyson)

1. Consider finding the 4-point function in the free theory. By Wick's theorem, we find

$$\begin{aligned} \langle \phi_1^0 \phi_2^0 \phi_3^0 \phi_4^0 \rangle &= \Delta_F(x_1 - x_2) \Delta_F(x_3 - x_4) + \cdots \\ &= \Delta_{12} \Delta_{34} + \Delta_{13} \Delta_{24} + \Delta_{14} \Delta_{23}. \end{aligned}$$

We can also derive this using Schwinger-Dyson:

$$\begin{aligned}
\langle \phi_1^0 \phi_2^0 \phi_3^0 \phi_4^0 \rangle &= \int d^4x \delta^4(x - x_1) \langle \phi_x \phi_2 \phi_3 \phi_4 \rangle \\
&= i \int d^4x ((\square_x + m^2) \Delta_{x_1}) \langle \phi_x \phi_2 \phi_3 \phi_4 \rangle \\
&= i \int d^4x \Delta_{x_1} ((\square_x + m^2) \langle \phi_x \phi_2 \phi_3 \phi_4 \rangle) \\
&= i \int d^4x \Delta_{x_1} (-i\delta(x - x_2) \langle \phi_3 \phi_4 \rangle - i\delta(x - x_3) \langle \phi_2 \phi_4 \rangle \\
&\quad - i\delta(x - x_4) \langle \phi_2 \phi_3 \rangle) \\
&= \Delta_{12} \Delta_{34} + \Delta_{13} \Delta_{24} + \Delta_{14} \Delta_{23}.
\end{aligned}$$

We can also write this pictorially.

2. Consider a cubic interaction term

$$\mathcal{L}_{\text{int}} = \frac{g}{3!} \phi^3,$$

and let us try to calculate the one-point function. Then,

$$\begin{aligned}
\langle \phi_x \rangle &= \int d^4y \delta(x - y) \langle \phi_y \rangle \\
&= i \int d^4y ((\square_y + m^2) \Delta_{xy}) \langle \phi_y \rangle \\
&= i \int d^4y \Delta_{xy} ((\square_y + m^2) \langle \phi_y \rangle) \\
&= i \int d^4y \Delta_{xy} \frac{g}{2} \langle \phi_y^2 \rangle \\
&= \frac{ig}{2} \int d^4y \Delta_{xy} \langle \phi_y^{02} \rangle + \mathcal{O}(g^3) \\
&= \frac{ig}{2} \int d^4y \Delta_{xy} \Delta_{yy} + \mathcal{O}(g^3).
\end{aligned}$$

3. Now let's look at the three-point function in the cubic theory:

$$\begin{aligned}
\langle \phi_1 \phi_2 \phi_3 \rangle &= \int d^4x \delta(x - x_1) \langle \phi_x \phi_2 \phi_3 \rangle \\
&= i \int d^4x \Delta_{x1} (\nabla_x + m^2) \langle \phi_x \phi_2 \phi_3 \rangle \\
&= \frac{ig}{2} \int d^4x \Delta_{x1} \langle \phi_x \phi_x \phi_2 \phi_3 \rangle + \int d^4x \Delta_{x1} (\delta(x - x_2) \langle \phi_3 \rangle + \delta(x - x_3) \langle \phi_2 \rangle).
\end{aligned}$$

Here we will use a couple of approximations, by relating the one-point and four-point functions to either their free-theory results, or results from previous calculation, to first order in g . This gives

$$\begin{aligned}
&= \frac{ig}{2} \int d^4x \Delta_{x1} (\Delta_{xx} \Delta_{23} + 2\Delta_{x3} \Delta_{x2}) \\
&\quad + \frac{ig}{2} \int d^4x \Delta_{x1} \delta(x - x_2) \int d^4y \Delta_{3y} \Delta_{yy} \\
&\quad + \frac{ig}{2} \int d^4x \Delta_{x1} \delta(x - x_3) \int d^4y \Delta_{2y} \Delta_{yy} + \mathcal{O}(g^2).
\end{aligned}$$

This gives

$$\begin{aligned}
\langle \phi_1 \phi_2 \phi_3 \rangle &= ig \int d^4x \Delta_{x1} \Delta_{x2} \Delta_{x3} \\
&\quad + \frac{ig}{2} \int d^4x \Delta_{xx} (\Delta_{x1} \Delta_{23} + \Delta_{12} \Delta_{3x} + \Delta_{13} \Delta_{2x}) + \mathcal{O}(g^2).
\end{aligned}$$

We can repeat our diagram drawing.

4. Finally we can find our two-point function:

$$\begin{aligned}
\langle \phi_1 \phi_2 \rangle &= i \int d^4x \Delta_{1x} \left(\frac{g}{2} \langle \phi_x^2 \phi_2 \rangle - i\delta(x - x_2) \right) \\
&= \Delta_{12} + \frac{ig}{2} \int d^4x \int d^4y \delta(y - x_2) \Delta_{1x} \langle \phi_x^2 \phi_y \rangle \\
&= \Delta_{12} + \frac{ig}{2} \int d^4x d^4y i \Delta_{1x} \Delta_{2y} \left(\frac{g}{2} \langle \phi_x^2 \phi_y^2 \rangle - 2i\delta(x - y) \langle \phi_x \rangle \right) \\
&= \Delta_{12} + \frac{(ig)^2}{4} \int d^4x d^4y (\Delta_{1x} \Delta_{2y} \Delta_{xx} \Delta_{yy} + 2\Delta_{1x} \Delta_{2y} \Delta_{xy} \Delta_{xy} \\
&\quad + 2\Delta_{1x} \Delta_{2x} \Delta_{xy} \Delta_{yy}) + \mathcal{O}(g^3).
\end{aligned}$$

3.4 Feynman Diagrams

What we learn from Schwinger-Dyson is that

$$\langle \phi_1 \cdots \phi_n \rangle = i \int d^4x \left(\langle \mathcal{L}'_{\text{int}}[\phi_x] \phi_2 \cdots \phi_n \rangle - i \sum_j \delta(x - x_j) \langle \phi_2 \cdots \phi_{j-1} \phi_{j+1} \cdots \phi_n \rangle \right).$$

We have two terms: the field propagation, and the contraction. Let us draw these as diagrams:

1. Start with x_i external points, and draw a line from each point.
2. A line can either:
 - Contract with an existing line. This gives $\Delta_f(x_i - x_j)$.
 - Split, where the split gives a new vertex. The coefficient will be $i\lambda_n$, for $\mathcal{L}_{\text{int}} = \frac{\lambda_n}{n!}\phi^n$, and the number of lines depends on $\mathcal{L}'_{\text{int}}$.
3. At any given order in $i\lambda_n$, the result is the sum of all diagrams with all lines contracted and integrated over vertices.
4. We have symmetry factors: the same combination can be obtained in different ways.

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