III Topological Quantum Matter

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Based on Lectures by Prof. Benjamin Béri

March 5, 2025

Contents

0   Overview

The **themes** in this course are:

- Quantum matter, and topological order (TO).

- Quantum computing (QC), quantum error correction (QEC) and topological QC.

The **methods** are:

- Mostly operator algebra (Pauli operators, fermion operators).

- Some field theory (second quantization and path integrals).

- A little band theory.

For notation, we have N qubits, so N sites. X_j, Y_j, Z_j are the X, Y, Z Pauli operators on the site j.

1   Quantum Ising Model

To recover a quantum Ising model, we can consider Hamiltonian

$$H = -J \sum_{\langle i,j \rangle} Z_i Z_j - h \sum_j Z_j,$$

for some $J > 0$. Then for an eigenstate $|\{z_j\}\rangle$,

$$H |\{z_j\}\rangle = E(\{z_j\}) |\{z_j\}\rangle.$$

Also $|\{z_j\}\rangle$ are eigenstates of Z_i:

$$Z_i |\{z_j\}\rangle = z_i |\{z_j\}\rangle.$$

This is quite boring; we just need to compute $E(\{z_j\})$. The *transverse field Ising model* is given by replacing the last Z_j with an X_j:

$$H = -J \sum_{\langle i,j \rangle} Z_i Z_j - h \sum_j X_j.$$

This Hamiltonian has a $\mathbb{Z}_2$ symmetry $P = \prod_j X_j$. This is because X_j and Z_j anticommute, giving $HP = PH$, and $P^2 = 1$. This corresponds to flipping all the spins, $P |\{z_j\}\rangle = |\{-z_j\}\rangle$.

If $J = 0$, then we have a ground state given by

$$|\text{GS}\rangle = \bigotimes_{j=1}^{N} |+\rangle_j = |\mathbf{X}\rangle,$$

where $X_j |\mathbf{X}\rangle = |\mathbf{X}\rangle$. Recall that

$$|\pm\rangle = \frac{|\uparrow\rangle \pm |\downarrow\rangle}{\sqrt{2}}.$$

If $h = 0$, we have ground states

$$|\Uparrow\rangle = \bigotimes_{j=1}^{n} |\uparrow\rangle_j, \qquad |\Downarrow\rangle = \bigotimes_{j=1}^{n} |\downarrow\rangle_j,$$

or any linear combination of these.

We want to analyse these ground states. For the first, note $P |\mathbf{X}\rangle = |\mathbf{X}\rangle$, and $\langle \mathbf{X}|Z_j|\mathbf{X}\rangle = 0$, since $Z_j |+\rangle_j = |-\rangle_j$. This can be thought of as a *paramagnet*.

For the other ground state, $P \ket{\Uparrow} = \ket{\Downarrow}$, and $\bra{\Uparrow} Z_j \ket{\Uparrow} \neq 0$. This can be thought of as a *ferromagnet*.

Since $[H, P] = 0$, there exists a basis $\ket{\psi_{E,p}}$ with

$$H \ket{\psi_{E,p}} = E_P \ket{\psi_{E,p}}, \qquad P \ket{\psi_{E,p}} = p \ket{\psi_{E,p}}.$$

Consider the following ground state:

$$\ket{\mathrm{GS}_\pm} = \frac{\ket{\Uparrow} \pm \ket{\Downarrow}}{\sqrt{2}}.$$

Then this satisfies $P \ket{\mathrm{GS}_\pm} = \pm \ket{\mathrm{GS}_\pm}$, and $\bra{\mathrm{GS}_\pm} Z_j \ket{\mathrm{GS}_\pm} = 0$.

To analyse this, we let $H = H_0 + \delta H$, where

$$H_0 = -J \sum_{\langle i,j \rangle} Z_i Z_j, \qquad \delta H = -h \sum_j X_j.$$

1.1   Brillouin-Wigner Perturbation Theory

Let $H = H_0 + \delta H$, where $H_0 \ket{n} = E_n \ket{n}$ and $H \ket{\tilde{n}} = E_{\tilde{n}} \ket{\tilde{n}}$, the eigenstates.

For $h = 0$ there is a energy gap $\Delta > 0$ between the ground state and the first excited state. Let the ground state subspace be S, and $d = \dim S$, the levels or the ground state degeneracy.

We can write our particle number or projection operators as

$$P = \sum_{n \in S} \ket{n} \bra{n}.$$

Then we can take $Q = \mathbb{1} - P$.

Let $E_{\tilde{m}}$ be the perturbed GS energies, and $\ket{\tilde{m}^{(u)}}$ the unnormalised perturbed eigenstates. So $H \ket{\tilde{m}^{(u)}} = E_{\tilde{m}} \ket{\tilde{m}^{(u)}}$, and $P \ket{\tilde{m}^{(u)}} = \ket{\psi_{\tilde{m}}}$, which is normalized.

Now write out

$$(H_0 + \delta H) \ket{\tilde{m}^{(u)}} = E_{\tilde{m}} \ket{\tilde{m}^{(u)}},$$
$$(E_{\tilde{m}} - H_0) \ket{\tilde{m}^{(u)}} = \delta H \ket{\tilde{m}^{(u)}} \implies (E_{\tilde{m}} - E_{\tilde{n}}) \langle n | \tilde{m}^{(u)} \rangle = \langle n | \delta H | \tilde{m}^{(u)} \rangle.$$

This lets us write

$$\sum_{n \in S^\perp} \ket{n} \langle n | \tilde{m}^{(u)} \rangle = \sum_{n \in S^\perp} \frac{\ket{n} \bra{n}}{E_{\tilde{m}} - E_n} \delta H \ket{\tilde{m}^{(u)}}.$$

So

$$Q \left| \tilde{m}^{(u)} \right\rangle = G\delta H \left| \tilde{m}^{(u)} \right\rangle,$$

which we can invert and write in terms of ψ to get

$$\left| \tilde{m}^{(u)} \right\rangle = (\mathbb{1} - G\delta H)^{-1} \left| \psi_{\tilde{m}} \right\rangle.$$

For $\left| n \right\rangle \in S$,

$$(E_{\tilde{m}} - E_0) \left\langle n | \tilde{m}^{(u)} \right\rangle = \left\langle n | \delta H (\mathbb{1} - G\delta H)^{-1} | \psi_{\tilde{m}} \right\rangle$$
$$= \sum_{n' \in S} \left\langle n | A^{(\tilde{m})} | n' \right\rangle \left\langle n' | \tilde{m}^{(u)} \right\rangle.$$

We can think of the first term as $H_{nn'}^{\text{eff}}$, and we can write $v_n^{(\tilde{m})} = \left\langle n | \tilde{m}^{(u)} \right\rangle$.

Once again, the perturbation is such that the ground state energy, once perturbed, is still away from the first excited energy $E_0 + \Delta$. This lets us write

$$\sum_{n' \in S} H_{nn'}^{\text{eff}}(E_{\tilde{m}}) v_{n'} = (E_{\tilde{m}} - E_0) v_n.$$

When we write $H_{nn'}^{\text{eff}}$, we introduce

$$G^{(\tilde{m})} = \sum_{n \in S^{\perp}} \frac{\left| n \right\rangle \left\langle n \right|}{E_{\tilde{m}} - E_n}.$$

Note that

$$E_{\tilde{m}} - E_n = (E_0 - E_n) + (E_{\tilde{m}} - E_0),$$

and the latter is small for small perturbations, so we can neglect this.

1.2   Applications to Transverse Field Ising Model

Recall that

$$H = -J \sum_{\langle i,j \rangle} Z_i Z_j - h \sum_j X_j.$$

Recall P, the product of X_j, is a symmetry, and we have ground states S for the Hamiltonian H_0 spanned by

$$\left| \text{GS}_{\pm} \right\rangle = \frac{1}{\sqrt{2}} (\left| \Uparrow \right\rangle \pm \left| \Downarrow \right\rangle),$$

where $P \left| \text{GS}_{\pm} \right\rangle = \pm \left| \text{GS}_{\pm} \right\rangle$. Then we can write

$$\delta E_{\pm} = \left\langle \text{GS}_{\pm} | \delta H (\mathbb{1} - G\delta H)^{-1} | \text{GS}_{\pm} \right\rangle$$
$$= \left\langle \text{GS}_{\pm} | \delta H | \text{GS}_{\pm} \right\rangle + \left\langle \text{GS}_{\pm} | \delta H G \delta H | \text{GS}_{\pm} \right\rangle$$
$$+ \left\langle \text{GS}_{\pm} | \delta H G \delta H G \delta H | \text{GS}_{\pm} \right\rangle + \cdots.$$

First we look at $\delta H \left| \text{GS}_\pm \right\rangle$. Note this is

$$-h \sum_j X_j \left| \text{GS}_\pm \right\rangle .$$

This has no overlap with $\left| \text{GS}_\pm \right\rangle$, as the first has some spins flipped, and the latter has no spins flipped.

Consider now what G does to $\delta H \left| \text{GS}_\pm \right\rangle$. The latter is a sum of states, exactly one of which has a flipped spin. Hence when we compute it with G, it picks out the excited state and gives a term of

$$\frac{1}{E_0 - E_n} = \frac{-1}{cJ},$$

for some constant c. Hence we get

$$G \delta H \left| \text{GS}_\pm \right\rangle \sim \frac{h}{J} \sum_j X_j \left| \text{GS}_\pm \right\rangle .$$

The second δH may undo the spin flip, giving

$$\left\langle \text{GS}_\pm | \delta H G \delta H | \text{GS}_\pm \right\rangle \sim -N \frac{h^2}{J} \left\langle \text{GS}_\pm | \text{GS}_\pm \right\rangle .$$

This perturbation is independent of p. Moreover, the perturbation is of order N, which is scary. However it is fine after a bit of consideration; our ground states have energy at order N. To be more precise, we can start with N finite and then send N to infinity in our theory.

We can continue this, and look at the perturbations, and see they are independent of p up to order $m < N$. However, eventually we get

$$\left\langle \text{GS}_\pm | \delta H G \delta H \cdots G \delta H | \text{GS}_\pm \right\rangle \to \frac{\delta E^{(m)}}{2} \left\langle \text{GS}_\pm \prod_{j=1}^N X_j | \text{GS}_\pm \right\rangle$$

$$= \pm \frac{\delta E^{(m)}}{2},$$

where our perturbation depends on p. So the splitting energy is

$$\delta E^{(\text{split})} \sim \frac{h^N}{J^{n-1}} \sim J \left(\frac{h}{J} \right)^N ,$$

which is exponentially suppressed in N.

So the ground state degeneracy is exponentially accurate, but it is linearly unstable; it depends on the $\mathbb{Z}_2$ symmetry of the perturbation. Indeed, if $\delta H = h'Z_j$, and if our ground states are $|n\rangle \rightarrow |\Uparrow\rangle, |\Downarrow\rangle$, then

$$\delta E_{\uparrow,\downarrow} = \pm h'.$$

A few things to glean from the transverse field Ising model:

- There is no more spontaneous symmetry breaking in the ground state. Let $|\psi\rangle$ be the exact ground state, for $N \gg 1$.

  Then $P|\psi\rangle = p|\psi\rangle$ is $\mathbb{Z}_2$ symmetric, hence since $PZ_i = -Z_i P$,

  $$\langle\psi|Z_i|\psi\rangle = 0.$$

- If $J \gg |h|$, we have a ferromagnet, with ground state degeneracy. If $J \ll |h|$, we have a paramagnet, with unique ground state.

- In the ferromagnetic case, if we let

  $$|\psi_{\Uparrow,\Downarrow}\rangle = \frac{|\psi_+\rangle \pm |\psi_-\rangle}{\sqrt{2}},$$

  these are approximate ground states, with

  $$P|\psi_\Uparrow\rangle = |\psi_\Downarrow\rangle \implies \langle\psi_\Uparrow|Z_j|\psi_\Uparrow\rangle = M_0 \neq 0.$$

  What we can do is include a
  $$h'Z_j$$
  term in the perturbation, such that $\delta\varepsilon_{\text{split}} \ll h' \ll h$. Then,

  $$\lim_{h'\to 0}\lim_{N\to\infty} \langle\psi|Z_j|\psi\rangle = M_0.$$

  If we swap the limits, we get 0, since we remove the symmetry breaking interaction. This can be thought of as the first definition of spontaneous symmetry breaking.

- Another way to think of symmetry breaking is through correlation functions,

  $$C^{(Z)}_{|i-j|} = \langle\psi|Z_iZ_j|\psi\rangle - \langle\psi|Z_i|\psi\rangle\langle\psi|Z_j|\psi\rangle \rightarrow \begin{cases} M_0^2 & \text{ferromagnetic,} \\ e^{-c|i=j|} & \text{paramagnetic.} \end{cases}$$

  Subtracting $\langle\psi|Z_i|\psi\rangle\langle\psi|Z_j|\psi\rangle$ just removes zero, but this is useful so that $C^{(O)}_{|i-j|} \rightarrow 0$ for large $|i-j|$, and $O = Z, X, Y$. This behaviour shows that $C^{(Z)}_{|i-j|}$ has long-range interactions in the ferromagnetic phase. In the paramagnetic there is no such order, giving a disordered phase.
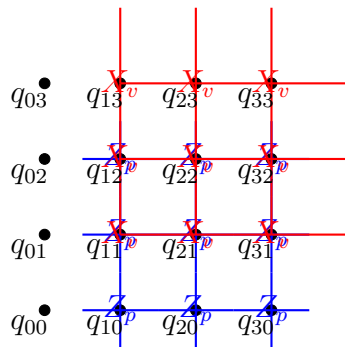
2   Surface Codes

Our motivation is a sort of master formula for Pauli operators:

$$\left(\prod_{j\in M} X_j\right)\left(\prod_{j\in M'} Z_j\right) = (-1)^{|M\cap M'|}\left(\prod_{j\in M'} Z_j\right)\left(\prod_{j\in M} X_j\right).$$

This can be proven by pulling each X_j through individually: it commutes with all $Z_{j'}$ except if $j = j'$, in which case it anticommutes.

To define surface codes, we consider a 2D square lattice, where qubits are placed on each edge of the lattice. We can represent operators by lines through these edges: if they are parallel to the edge, they correspond to Z_i, whereas if they are orthogonal they correspond to X_i.

The *plaquette operators* are the four lines around a face, corresponding to

$$\prod_{j\in p} Z_j = B_p.$$

The *vertex operators* are the four lines around a vertex, corresponding to

$$\prod_{j\in v} X_j = A_v.$$

A *surface code* is simply a system where the Hamiltonian is

$$H = -J_A \sum_v A_v - J_B \sum_p B_p.$$

Note that $[A_v, B_p] = 0$, because they have intersection either 0 or 2, and by our master formula. Hence also

$$[A_v, H] = [B_p, H] = 0.$$

Also $A_v^2 = \mathbb{1} = B_p^2$, so the eigenvalues are in $\{\pm 1\}$.

Our task is now to find more integrals of motion, and eigenstates $|n\rangle$. For example we can consider

$$\prod_{v \in V} A_v \prod_{p \in P} B_p,$$

which is an integral of motion but is not independent of the A_v, B_p we have already seen. One example is

$$\chi_M = \prod_{j \in M} X_j.$$

Can we pick M such that χ_M is an integral of motion? To commute with all of the B_p, then the M must represent a path without endpoints (insert picture).

One example of a loop is A_v. Moreover, we can multiply different A_v to find larger loops: if we multiply the A_v in a 3×3 region, we find a 3×3 loop.

We can now introduce *path deformation*. Consider taking a path, and multiplying it by an A_v. This gives us another path, e.g.

$$\prod_{j \in \tilde{\gamma}} X_j \cdot A_v = \prod_{j \in \tilde{\gamma}'} X_j.$$

We can now talk about *contractible loops*: we can multiply it by a set of A_v such that the path disappears.

We will now see where topology comes in.

Definition 2.1. A *manifold* is a space which is locally Euclidean, for example a sphere or a torus.

The *first Betti number* of $\mathcal{M}$, $\mathcal{B}_1$ is the number of noncontractible independent paths of $\mathcal{M}$.

Note that the Betti number is a topological invariant.

Consider our surface code on a manifold $\mathcal{M}$: and take $\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots$ the non-contractible independent paths. Then

$$\bar{X}_j = \chi_{\tilde{\gamma}_j} = \prod_{i \in \tilde{\gamma}_j} X_i \neq \prod_{v \in V} A_v,$$

for some v, hence this is independent. Note that $\bar{X}_j^2 = \mathbb{1}$, so it has eigenvalues in $\{\pm 1\}$.

Moreover, we can also consider

$$\bar{Z}_j = \prod_{i \in \tilde{\gamma}_i} Z_i.$$

Again $\bar{Z}_j^2 = \mathbb{1}$, but more importantly

$$\bar{X}_i \bar{Z}_j = (-1)^{\delta_{ij}} \bar{Z}_j \bar{X}_i,$$

like Pauli operators. We will find that our eigenstates are denoted by

$$|n\rangle = |\{a_v\}, \{b_p\}; \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_{\mathcal{B}_1}\rangle,$$

where we either take $\{\bar{X}_i\}$ or $\{\bar{Z}_i\}$. We have not shown yet that these are the complete quantum numbers. However what we do know is that the energy is independent of $\{\bar{Z}_j\}$, hence the ground state degeneracy is

$$2^{\mathcal{B}_1}.$$

2.1   Surface Code Excitations

The ground states are where $a_v = b_p = 1$, since $J_A, J_B > 0$. The excitations come in the form when some $a_v = -1$, or some $b_p = -1$, and combinations.

Denote one of the ground states as $|\psi\rangle$, and consider $Z_\alpha |\psi\rangle$, where α has some boundary. Consider what A_v evaluates on this state.

If v is the vertex for which α ends on, then we find

$$A_v(Z_\alpha |\psi\rangle) = -Z_\alpha(A_v |\psi\rangle) = -Z_\alpha |\psi\rangle,$$

so this state now has $a_v = -1$.

We can think of these lines as *particles*. In particular, if $a_v = -1$, we think of this as an 'e' particle, with energy $2J_A$. If we have M e particles, the energy is $2MJ_A$.

Moreover,

$$\prod_{p \in P} B_p Z_\alpha |\psi\rangle = Z_\alpha \prod_{p \in P} B_p |\psi\rangle = Z_\alpha |\psi\rangle.$$

So we only care about the endpoints of the strings, and not about the paths taken between the endpoints.

If $b_p = -1$, then we have 'm' particles, with energy $2J_B$.

2.2   Particle Exchange

We will look at particle exchange as a dynamical process, and we look at the exchange only.

The first thing we will focus on is identical particles, with initial configuration the same as the final configuration. We strip away:

- Interactions, so particles are "far" apart.

- Local details, such as potentials.

If particles are far apart, we are considering nonintersecting worldlines, and as we only care about local details we only think about topological invariant features.

We will sit in $2 + 1$ dimensions, and can view them in the following pictorial way, as braids (insert picture of braids somehow???).

These braids also form a group: we can compose exchanging particles. Moreover, we have inverses.

Let the set of *braids* be B_N, where we have N strands. Then,

- $\sigma_i, \sigma_j \in B_N \implies \sigma_i \sigma_j \in B_N$.

- $1 \in B_N$, $\sigma_i^{-1} \in B_N$.

These σ and their inverses form the *braid group*. This is not commutative. However what is true is that

$$\sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1},$$

$$\sigma_j \sigma_i = \sigma_i \sigma_j \text{ if } |i - j| = 2.$$

We can also talk about distinguishable particles, where particles need to end up as the same particle they started with. If σ_{a_i} is the swapping of two adjacent a's in position i and $i + 1$, this same action does not act appropriately on an a and a b in those positions.

In this case, we need to do $\sigma_{a_i}^2$, to end up with the same particles. This move is denoted as μ_{ab}.

Moreover in 3D, the overpass and underpass are topologically equivalent, so σ_i^2. In this case, how the particles move do not matter, only where they end up. Hence the underlying group is S_N.

To each exchange σ_j, we assign a unitary operator $U(\sigma_j)$. This can be thought of as the topological part of the time-evolution, during particle exchange.

Moreover, we require it to be a representation, so

$$U(\sigma_j)U(\sigma_i) = U(\sigma_j \sigma_i).$$

This gives a unitary representation of the braid group B_N. We may have *abelian representations*

$$U(\sigma_j) = e^{i\theta_j} \in \mathsf{U}(1),$$

If we use the first relation for the braid group,

$$e^{2i\theta_j}e^{i\theta_{j+1}} = e^{2i\theta_{j+1}}e^{i\theta_j} \implies e^{i\theta_j} = e^{i\theta_{j+1}} = e^{i\theta}.$$

In 3D, $(e^{i\theta})^2 = 1$, so $e^{i\theta} \in \{\pm 1\}$, giving the familiar bosons and fermions.

However in 2D, we have $\sigma_j^2 \neq \mathbb{1}$, so θ can be any value. This gives us *anyons*, or in this case *abelian anyons*.

Graphically, the clockwise exchange is $e^{i\theta}$ times doing nothing.

For distinguishable particles, $U(\mu_{ab}) = e^{i2\theta_{ab}}$.

We can also have non-abelian reprsentations, so

$$U(\sigma_i\sigma_j) = U(\sigma_i)U(\sigma_j) \neq U(\sigma_j)U(\sigma_i) = U(\sigma_j\sigma_i).$$

2.3   Particles Statistics in Surface Codes

Consider two anyons e and m. If the anyon is at vertex v, then we can successively move it by applying Z_j with $j \in v$. This moves $a_v = 1$ and the start of the anyon to an adjacent vertex.

Applying Z_{α_1} for successive applications of a path, the particles start point moves along the path.

We can consider a similar operation for m particles. Consider our starting point, which is an e and an m particle, separated.

Now consider encircling the m particle with the e particle clockwise, by doing a loop. Let $\mathcal{Z}_\alpha$ be the e particle, and $\mathcal{X}_\beta$ the m particle. Moreover Let $\mathcal{Z}_{\text{loop}}$ be the encircling loop. The initial state is

$$\mathcal{Z}_\alpha\mathcal{X}_\beta \ket{\psi}.$$

Consider $\mathcal{Z}_{\text{loop}}$ times the same state. Note that

$$\mathcal{Z}_{\text{loop}}\mathcal{X}_\beta = -\mathcal{X}_\beta\mathcal{Z}_{\text{loop}},$$

as they intersect at one point, and

$$\mathcal{Z}_{\text{loop}} \ket{\psi} = \ket{\psi}$$

since this is a loop. So

$$\mathcal{Z}_{\text{loop}}\mathcal{Z}_\alpha\mathcal{X}_\beta \ket{\psi} = -\mathcal{Z}_\alpha\mathcal{X}_\beta \ket{\psi}.$$

So encircling the e and the m gives us a sign change to -1. These are bosons, or *mutual semions*, as encircling gives $\pi/2$ rotation.

2.4   Charge-Flux Composites

Recall we said that $\mathsf{U}(g)$ is the 'topological part' of evolution, for some $g \in B_N$. We can write down

$$\langle\{\mathbf{x}_j(t_f)\}|T\exp(-i/h\int_{t_i}^{t_f}\mathrm{d}t\,\hat{H}(t)|\{\mathbf{x}_j(t_i)\}\rangle = \int_{\{\mathbf{x}_j(t_f)\}\leftarrow\{\mathbf{x}_j(t_i)\}}\mathcal{D}\{\mathbf{x}_j\}\,\exp(i/hS[\{\mathbf{x}_j\}]),$$

where

$$S = \int_{t_i}^{t_f}\mathrm{d}t\,L.$$

This is the *path integral*.

To prove this: If we denote the transition operator as $U(t_f, t_i)$, hen we introduce time slices Δt. Then,

$$U(t + \Delta t, t) = U(p)U(x) + \mathcal{O}(\Delta t^2).$$

Then we can insert the identity

$$\mathbb{1} = \int \mathrm{d}x\,\mathrm{d}p\,|x\rangle\,\langle x|\,|p\rangle\,\langle p|.$$

For our case, we are considering elements of the braid group, so $\{\mathbf{x}_j(t_f)\} = \{\mathbf{x}_j(t_i)\}$. So we can classify $\{\mathbf{x}_j(t)\}$ with the braids $g \in B_N$ for which they are topologically equivalent.

Suppose our action decouples as

$$S[\mathbf{x}_j(t)] = S_0[\{\mathbf{x}_j(t)\}] + \hbar W(g),$$

a local part plus a topological part. Then we can write the time evolution as

$$\sum_{g\in B_N} e^{iW(g)}\int_{\{\mathbf{x}_j(t)\}\in g}\mathcal{D}x\,\exp\left(\frac{i}{h}S[\{\mathbf{x}_k\}]\right).$$

2.5   Aharonov-Bohm and Aharonov-Casher

We work in two dimensions, with one type of particle of charge q and an electromagnetic field. The Lagrangian is given by

$$L = L_0 - q(\varphi - \dot{\mathbf{x}}\cdot\mathbf{A}).$$

Consider a flux Φ piercing the plane at a fixed point $\mathbf{X}$, with $\varphi = 0$.

Then, for a loop $\mathbf{x}$ encircling $\mathbf{X}$ n times, the action is

$$S[\mathbf{x}] = S_0 + q \int \mathrm{d}t\, \dot{\mathbf{x}} \cdot \mathbf{A} = S_0 + q \oint \mathrm{d}\mathbf{x} \cdot \mathbf{A} = S_0 + qn\Phi.$$

This is the *Aharonov-Bohm effect*.

Now suppose that the flux is situated at $\mathbf{X}(t)$. Then

$$\mathbf{A}(\mathbf{x}) = \mathbf{A}_f(\mathbf{x}_k - \mathbf{X}(t)),$$

where $\mathbf{A}_f$ is the potential for a flux at the origin, seems to give a good potential. However this is time-dependent, so

$$\mathbf{E} = -\partial_t \mathbf{A} = (\dot{\mathbf{X}}_f \cdot \nabla)\mathbf{A} \neq 0.$$

To remedy this, let $\phi = \dot{\mathbf{X}}_f \cdot \mathbf{A}$, which gives

$$\mathbf{E} = -\partial_t \mathbf{A} - \nabla\phi = -\dot{\mathbf{X}}_f \times \mathbf{B},$$

which vanishes at $\mathbf{x} \neq \mathbf{X}_f$. So,

$$L = L_0 - q(\dot{\mathbf{X}}_f - \dot{\mathbf{x}}) \cdot \mathbf{A}_f(\mathbf{x} - \mathbf{X}_f).$$

This also gives a phase $W[n] = qn\Phi/\hbar$ when circled n times. This is the *Aharonov-Casher effect*.

2.6   Charge-Flux Composites

Suppose a particle $\mathbf{x}_j$ has a charge q_j and flux Φ_j at some distance $\delta\mathbf{x}_j$ from the charge. The charge-flux interaction is

$$L_I = -\sum_{j,k} q_j(\dot{\mathbf{X}}_k - \dot{\mathbf{x}}_j) \cdot \mathbf{A}_k(\mathbf{x}_j - \mathbf{X}_k).$$

Suppose particle j encircles particle k. Then from the two effects together, the change in encircling phase is

$$e^{2i\theta_{jk}} = e^{i/\hbar(q_j\Phi_k + q_k\Phi_j)}.$$

If the particles are identical, the exchange phase is

$$e^{i\theta} = e^{i/\hbar(q\Phi)}.$$

Suppose we have a spin J operator, and rotate about z. Then the operator is

$$R(\alpha) = \exp(i/\eta \cdot \alpha J_z).$$

Note $J_z/\hbar$ is a diagonal matrix which has half-integer elements if the spin if a half-integer, and integer otherwise. So, $R(2\pi) = \pm 1$ if these particles are bosons or fermions, and $R(2\pi) = e^{i\theta}$.

In this model, doing a 2π rotation gives

$$R(2\pi) = e^{i/\hbar \cdot (q\Phi)} = e^{i\theta},$$

the exchange phase.

Suppose we have anyons (q_1, Φ_1) and (q_2, Φ_2), and then bring them close to each other. If we squint and only care about long-distance physics, then these become an anyon with state $(q_1 + q_2, \phi_1 + \phi_2)$.

In particular, bringing together (q_1, Φ_1) and $(-q_1, -\Phi_1)$ gives the *identity particle*, or nothing. We can tikz this out (do this later!!!!)

If we denote by $\times$ as fusion, then in surface codes, $e \times e = 1$.

2.7   Ground State Degeneracy and Topology

We now claim that the ground state knows about what anyons are possible.

Consider the torus, with two non-contractible loops γ_1 and γ_2. We consider a process, starting with T_1, T_2 that create an anyon/anti-anyon pair, drag the anyon along $\gamma_1, \gamma)2$, and then reannihilate them.

Doing this, we end up back in the ground state. We now examine the operator

$$T_2^{-1} T_1^{-1} T_2 T_1.$$

Consider the braid lines of the anyons/anti-anyons for this operator. It consists of only one pair of anyons encircling, so the end result is the encircling phase $e^{i2\theta}$, multiplied by doing nothing.

Recall that T_i take the ground state to the ground state: hence they must have a ground-state diagonal block in their decomposition. Let $|\psi_\alpha\rangle$ be such that

$$T_2 |\psi_\alpha\rangle = e^{i\alpha} |\psi_\alpha\rangle.$$

Then from the fact about $T_2^{-1} T_1^{-1} T_2 T_1 = e^{i2\theta}$, we find that

$$T_2 T_1 |\psi_\alpha\rangle = e^{2i\theta} T_1 T_2 |\psi_\alpha\rangle = e^{2i\theta + i\alpha} T_1 |\psi_\alpha\rangle,$$

so $T_1 |\psi_\alpha\rangle$ is yet another eigenstate with eigenvalue $e^{2i\theta + i\alpha}$. If $e^{2i\theta} \neq 1$, these have different eigenvalues, so $|\psi_\alpha\rangle$ is perpendicular to $T_1 |\psi_\alpha\rangle$.

Moreover, we find $T_1^j \ket{\psi_\alpha}$ for $j = 0, \ldots, m-1$ are orthogonal if $e^{2i\theta j} \neq 1$ for $j \in \{1, 2, \ldots, m\}$. For a finite ground state degeneracy, we need n and m such that $2\theta m = 2\theta n$, so $\theta = \frac{n}{m}\pi$.

In this case, the ground state degeneracy is at least m. For a more complicated manifold with g handles, the ground state degeneracy is at least m^g.

We can do a similar thing when there are more than one particles, e.g. e and m particles.

3   Chern-Simons Theory

In this theory,

$$L = L_0 + \int \mathrm{d}^2 x \, \mathcal{L},$$

where

$$\mathcal{L} = \mathcal{L}_{\mathrm{CS}} - j^\alpha a_\alpha, \qquad \mathcal{L}_{\mathrm{CS}} = \frac{\kappa}{2} a^\alpha \varepsilon^{\alpha\beta\gamma} \partial_\beta a_\gamma = \frac{\kappa}{2} a(\varepsilon \partial a),$$

and

$$j^0 = \sum_{j=1}^N q_j \delta(\mathbf{x} - \mathbf{x}(t)), \qquad \mathbf{j} = \sum_{j=1}^N q_j \dot{\mathbf{x}}_j \delta(\mathbf{x} - \mathbf{x}_j(t)),$$

which gives $\partial_\alpha j^\alpha = 0$. We check that it is a gauge theory. Say $a \to a + \partial \Lambda$. Then note $\varepsilon \partial \partial = 0$, so

$$\delta \mathcal{L} = \left(\frac{\kappa}{2} \partial \Lambda (\varepsilon \partial a) - j \partial \Lambda \right) = \partial \left(\frac{\kappa}{2} \Lambda (\varepsilon \partial a) - j \Lambda \right) + (\partial j) \Lambda = 0.$$

3.1   CS Flux Attachment

Consider $a \to a + \delta a$. We find that

$$\delta a (\varepsilon \partial a) + (a \varepsilon \partial \delta a) = 2 \delta a (\varepsilon \partial a) + \partial(\cdots).$$

Hence

$$\partial S_{\mathcal{L}} = \int \mathrm{d}^3 x \, \delta a [\kappa(\varepsilon \partial a) - j] = 0 \implies j = \kappa \varepsilon \partial a.$$

Hence $j^0 = \kappa \varepsilon^{0\alpha\beta} \partial_\alpha a_\beta = \kappa b$. Here b is the CS magnetic field. Consider a particle at a position $\mathbf{x}_\ell$, and a charge q_ℓ. We integrate the charge over a space Ω_ℓ, to find

$$q_\ell = \kappa \Phi_\ell \implies \Phi_\ell = \frac{q_\ell}{\kappa}.$$

We claim that if we know $j^0 = \kappa b$ and $\partial_\alpha j^\alpha = 0$, then we get the equations of motion.

3.2   Quantum Flux Attachment

First we write

$$\begin{aligned}
\mathcal{L}_{\mathrm{CS}} &= \frac{\kappa}{2} \left[a^0 \varepsilon^{0\alpha\beta} \partial_\alpha a_\beta + a_\alpha \varepsilon^{\alpha\beta 0} \partial_\beta a_0 + a_\alpha \varepsilon^{\alpha 0 \beta} \partial_0 a_\beta \right] \\
&= \kappa a_0 b + \mathcal{L}_{\mathrm{CS}}(a_0 \to 0) + \partial(\cdots).
\end{aligned}$$

So quantum mechanically, we want to minimize a term involving

$$\int \mathcal{D}a_0 \, e^{\frac{i}{\eta} S[a_0]},$$

where

$$S[a_0] = \int \mathrm{d}^3x \, a_0(\kappa \varepsilon^{0\alpha\beta} \partial_\alpha a_\beta - j^0) = \int \mathrm{d}^3x \, a_0(\kappa b - j^0).$$

So using the fact that $\int \mathrm{d}a \, e^{iax} \propto \delta(x)$, we find $\delta(\kappa b - j^0)$. Hence $j^0 = \kappa b$ is enforced, and as charge is conserved we find $j = \kappa\varepsilon\partial a$.

Encircling two particles k and ℓ,

$$e^{i2\theta_{k\ell}} = e^{\frac{i}{\hbar}(q_k \Phi_\ell + q_\ell \Phi_k)} = e^{\frac{i}{\hbar}2\frac{q_k q_\ell}{\kappa}} \implies \theta_{k\ell} = \frac{q_k q_\ell}{\kappa}.$$

So we may think that we have found the encircling phase. But if we substitute back into our Lagrangian,

$$\mathcal{L} = a\frac{\kappa}{2}(\varepsilon\partial a) - j\alpha = -\frac{j\alpha}{2},$$

so the Lagrangian is half that which we found in our flux-charge model, hence in actuality

$$\theta_{k\ell} = \frac{q_k q_\ell}{2\hbar\kappa}.$$

3.3   Charge Quantisation

Instead of looking at a path integral, we consider a wavefunction ψ which, under a gauge transformation moves to $e^{\frac{i}{\hbar}q_0\Lambda}\psi$, for q_0 the particle charge.

We assume that any other charge (in terms of the wavefunction) is $\tilde{q} = nq_0$, for some $n \in \mathbb{Z}$.

Consider a system on a torus, and a particle that loops once around the torus. The action is

$$S_{\mathrm{curr}} = q \int \mathrm{d}t \, \dot{\mathbf{x}} \cdot \mathbf{a} = q \oint \mathrm{d}\boldsymbol{\gamma} \cdot \mathbf{a}.$$

Under a gauge transformation $\mathbf{a} \to \mathbf{a} + \nabla\Lambda$, then

$$\delta S_{\mathrm{curr}} = q \oint \mathrm{d}\boldsymbol{\gamma} \cdot \nabla\Lambda = q(\Lambda(L) - \Lambda(0)).$$

We do not need Λ to be single valued, just ψ. Hence we can have $\Lambda(L) - \Lambda(0) = 2\pi\hbar/q_0$, and find

$$\delta S_{\mathrm{curr}} = 2\pi\hbar\frac{q}{q_0}.$$

This is worrying, as we expect our action to be gauge invariance. But we only need $e^{\frac{i}{\hbar}S_{\text{curr}}}$ to be gauge invariant, showing that

$$2\pi\hbar\frac{q}{q_0} = 2\pi\hbar n \implies q = nq_0.$$

We now consider Chern-Simons theory on a closed manifold. Start with $\kappa b = j^0$, and say the theory lives on some 2-sphere Ω. Integrating this over Ω,

$$\int_\Omega \kappa b = \int_\Omega j^0 = mq_0,$$

since charge is quantized. Moreover, the LHS is also quantized in terms,

$$\int_\Omega b = n\Phi_0, \qquad \Phi_0 = \frac{2\pi\hbar}{q_0}.$$

So we find that

$$\kappa n\Phi_0 = mq_0.$$

For the minimum number of non-zero particles, we find

$$\kappa = \frac{q_0^2}{2\pi\hbar}m,$$

the minimum *Chern-Simons level*. Hence

$$q\theta = \frac{q_0^2}{2\pi\hbar} = \frac{\pi}{m}.$$

Earlier, we found that if $\theta = \frac{n}{m}\pi$, then the ground state degeneracy was at least m^g.

We can generalize to more particles. Say there are M particle types. We introduce M gauge fields $a_{I\alpha}$ for $I \in \{1, \ldots, M\}$, and also M charges q_I. Then the Chern-Simons Lagrangian is

$$\mathcal{L} = \mathcal{L}_{\text{CS}} - \sum_{I=1}^{M} j_I^\alpha a_{I\alpha},$$

where

$$\mathcal{L}_{\text{CS}} = \frac{1}{4\pi\hbar}\sum_{I,J} K_{IJ}a_I(\varepsilon\partial a_J),$$

where $K_{IJ} = K_{JI}$ is an integer and forms an invertible matrix.

We can show that

$$\theta_{q,q'} = \pi q' K^{-1} q.$$

For the surface code, we can show

$$K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix},$$

and the ground state degeneracy is at least $|\det K|^g$.

4   Robustness of Topological Order

4.1   Perturbed Surface Code

Consider a Hamiltonian

$$H = H_0 + \delta H,$$

where H_0 is our familiar surface code,

$$H_0 = -J_A \sum_v A_v - J_B \sum_p B_p,$$

and

$$\delta H = \sum_j h O_j,$$

where O_j has local support, and the diameter of the support is at most $\xi < \infty$, where ξ is a contant and $j \in \operatorname{supp} O_j$.

Here we are saying that the lengths in the surface code are 1.

Moreover O_j is bounded, i.e. the matrix element are of order $\mathcal{O}(1)$.

For H and H_0 that satisfy these conditions, we call the pair a *local Hamiltonian*.

To study perturbations, we recall the effective Hamiltonian

$$H^{(\mathrm{eff})} = \langle n | \delta H (1 - G \delta H)^{-1} | n' \rangle, \qquad G = \sum_{n \in S^\perp} \frac{|n\rangle \langle n|}{E_{\tilde{n}} - E_n}.$$

For $h = 0$, the ground state degeneracy is 4^g, and the eigenbasis is

$$|n\rangle = |\{a_v, b_p\}, \{\bar{z}_j\}\rangle.$$

Here $\bar{Z}_j$ are the operators around independent noncontractible loops.

We set $O_j = Z_j$ for concreteness, so $[O_j, Z_i] = 0$, and hence $[\bar{Z}_i, H] = 0$. In this case, $H^{(\mathrm{eff})}$ is some diagonal matrix.

Consider expanding the expression for the effective Hamiltonian. We get

$$\langle n | \delta H G \delta H G \cdots G \delta H | n' \rangle \to \langle n | \prod_{i \in I} \bar{Z}_i \prod_{p \in P} B_p | n \rangle,$$

since each application creates a pair of e-particles, which must be cancelled out. If $I = \emptyset$, then we get something proportional to

$$J \left(\frac{h}{J} \right)^m \langle n | \prod_{p \in P} B_p | n' \rangle = J \left(\frac{h}{J} \right)^m \delta_{\{\bar{z}_j\}\{\bar{z}'_j\}}.$$

This is independent of $\bar{z}_i$.

If $I \neq \emptyset$, we get

$$J \left(\frac{h}{J}\right)^m \prod_{i \in I} \bar{z}_i \delta_{\{\bar{z}_j\}\{z'_j\}}.$$

For the last perturbation to occur, we need $m \geq L$, then length of the smallest loop. Hence

$$\delta E_{\text{split}} \leq J \left(hJ\right)^L.$$

This result generalizes to any δH local.

Comparing to the transverse field Ising model,

$$\delta E_{\text{split}} \leq J \left(\frac{h}{J}\right)^N.$$

This is order L^2. If $[\delta H, P] = 0$, then we get symmetry preserving behaviour.

If $[\delta H, P] \neq 0$, e.g. $\delta H = hZ_1$, then we may have $\delta E_{\text{split}} \sim h$.

4.2   Topological Order

A two-dimensional Hamiltonian is *topologically ordered* if:

- H is gapped.

- It has topological ground state degeneracy.

- For any locally supported A, and ground states $|\psi_j\rangle$,

$$\langle \psi_j | A | \psi_i \rangle = c_A \delta_{ij}.$$

These last two properties are required to hold up to e^{-cL} for some $c > 0$.

We can verify that these hold for the unperturbed surface code. For the perturbed surface code we have verified 1 and 2, but not yet 3.

4.3   Phases of Matter

Consider a Hamiltonian $H(g)$, where g are some parameters. Say $H(0) = H_0$, $H(1) = H_1$.

If we write out $H(g + \delta g) = H(g) + (H(g + \delta g) - H(g))$, and assume that $H(g)$ is local and gapped,

Topological ordering is a phase.

Definition 4.1. Suppose H_0, H_1 are gapped and local. They are in the same *topological phase* if there is a parametrization $g : [0,1] \to H(g)$ that is continuous, such that $H(g)$ is gapped and local for all g, and $H(0) = H_0$, $H(1) = H_1$.

This is a purely topological definition.

We say that $|\psi_0\rangle \sim |\psi_1\rangle$ if and only if $|\psi_j\rangle$ is a ground state of a gapped local H_j such that $H_0 \sim H_1$. Alternatively, $|\psi_1\rangle = U |\psi_0\rangle$, for U local unitary.

U is *local unitary* if it can be written as

$$U_i = \prod_j W_j^{(i)},$$

where the diameter of the support of $W_j^{(i)} < \ell < \infty$, and the supports are non-overlapping. Then $|U - U_1 U_2| \leq e^{-a\ell}$.

These two definitions are equivalent if we consider *quasilocal H*, where

$$H = \sum_j c_j O_j,$$

where O_j is bounded and $c_j \sim \exp(-\alpha_j \operatorname{diam} \operatorname{supp} O_j)$, where $a_j > 0$ is constant.

We can show that if O_j is instead local, then $U^\dagger O_j U$ is quasilocal, as is $U O_j U^\dagger$:

$$U O_j U^\dagger = \sum_i c_{ji} \tilde{O}_i,$$

where $\tilde{O}_i$ is bounded and $c_{ji} \sim \exp(-\alpha_{ji} \operatorname{diam}(j \cup \operatorname{supp} \tilde{O}_i))$. This is some sort of smearing.

Now we can show the equivalence: Let $|\psi_0\rangle$ be the ground state of some $H_0 = \sum_j O_j$, and $|\psi_1\rangle = U |\psi_0\rangle$ for U local unitary.

Consider a local unitary path $g \in [0,1] \mapsto U(g)$ continuous, with $U(0) = \mathbb{1}$, and $U(1) = U$. This is possible as the unitary group is a Lie group. Consider then

$$U(g) |\psi_0\rangle .$$

This will then be the ground state of the Hamiltonian

$$H(g) = U(g) H_0 U^\dagger(g) = \sum_j U(g) O_j U^\dagger(g).$$

But we know the latter sums are quasilocal, so $H(g)$ is a quasilocal Hamiltonian, and moreover gapped as it is a unitary transformation of gapped H_0.

If we let $H_0 = H(0), H_1 = H(1)$, then $|\psi_1\rangle$ is the ground state of H_1 since

$$H_1 |\psi_1\rangle = U H_0 U^\dagger U |\psi_0\rangle = U H_0 |\psi_0\rangle = E U |\psi_0\rangle = E |\psi_1\rangle.$$

Moreover $H_0 \sim H_1$, since we have constructed a given path. Hence $|\psi_0\rangle \sim |\psi_1\rangle$ in our first definition.

The other way is much harder. For this we need another definition of local unitary: U is local unitary if and only if

$$U = T \exp\left(\frac{i}{\hbar} \int_0^T dT \tilde{H}(t)\right),$$

where the initial T is time ordering, $\tilde{H}(t)$ is quasi local and T is a finite constant. This can be shown to be equivalent to the other definition (as shown by Osborne).

Chen Lou and Wen showed that if $|\psi_0\rangle \sim |\psi_1\rangle$, then there exists U local unitary with $\tilde{H}(t)$ build from $H(g)$ such that

$$|\psi_1\rangle = U |\psi_0\rangle.$$

In general, if we have a set of ground states $\{|\psi_0^{(i)}\rangle\}, \{|\psi_1^{(i)}\rangle\}$, then there exists a local unitary U such that

$$|\psi_1^{(i)}\rangle = |\psi_0^{(i)}\rangle,$$

where U is independent of i and the basis.

Indeed suppose that $U^{(i)}$ depends on i. Then by the definition of topological order,

$$O_A \delta_{ii'} = \langle \psi_1^{(i')} | A | \psi_1^i \rangle = \langle \psi_0^{(i')} | U^{(i')\dagger} A U^{(i)} | \psi_0^{(i)} \rangle \neq \delta_{ii'}.$$

We can also consider $H_0 \sim H_0$ and $|\psi_0^{(i)}\rangle = U |\psi_0^{(2)}\rangle$. This is not always possible; indeed in our statement we only have the existence of an U taking the set of ground states of H_0 to the set of ground states in H_1. But sometimes it can exist.

4.4   Robustness of Braiding Statistics

Let $H = H_0 + \delta H$, where

$$H_0 = -J \left(\sum_v A_v + \sum_p B_p \right),$$

and δH local, i.e.

$$\delta H = h \sum_j O_j.$$

Then if $|\psi\rangle$ is a ground state of H_0, and $|\tilde{\psi}\rangle$ is a ground state of H, we have

$$|\tilde{\psi}\rangle = U |\psi\rangle .$$

The parent Hamiltonian for $|\tilde{\psi}\rangle$ is either H, or $\tilde{H} = U H_0 U^\dagger$. However $H \neq \tilde{H}$.

For sake of illustration, take our base state $2e$ particles with energy $4J$, and $O_j = Z_j$. We have roughly N^2 of these unperturbed eigenstates, $|n_{2e}\rangle$. Thinking about matrix elements,

$$\langle n'_{2e}|\delta H|n_{2e}\rangle \neq 0 \implies \text{ the electrons are hopping.}$$

This has order h, so our eigenstates spread out into a band of order h.

For $\tilde{H}$, we can write it as

$$\tilde{H} = -J \left(\sum_v \tilde{A}_v + \sum_p \tilde{B}_p \right),$$

where

$$\tilde{A}_v = U A_v U^\dagger, \qquad \tilde{B}_p = U B_p U^\dagger,$$

for U local unitary. These are quasilocal. Now

$$\tilde{A}_v \tilde{B}_p = U A_v U^\dagger U B_p U^\dagger = U(A_v B_p)U^\dagger = \tilde{B}_p \tilde{A}_v,$$

so these commute, and are integrals of motion. Let their values be $\tilde{a}_v$, and $\tilde{b}_p$ in $\{-1, 1\}$. The same can be seen for the other integrals $\tilde{Z}$. Hence we find eigenstates

$$|\tilde{\psi}_{\{\tilde{a}_v, \tilde{b}_p\}, \tilde{z}_j}\rangle = U |\{a_v, b_p\}, \{z_j\}\rangle .$$

Then $\tilde{A}_v$, $\tilde{B}_p$ have these as eigenstates, with eigenvalues $\tilde{a}_v, \tilde{b}_p$. However these are now quasilocal, instead of the completely local A_v, B_p.

Suppose that in H_0, two electrons were created by

$$\mathcal{Z}_\alpha = \prod_{j \in \alpha} Z_j,$$

some path. Then in $\tilde{H}$,

$$\tilde{\mathcal{Z}}_\alpha = U \mathcal{Z}_\alpha U^\dagger.$$

In the unperturbed phase, we saw that the e and m particles were mutual semions. In $\tilde{H}$, we still find that e and m are mutual semions, as the entire systems goes about a unitary transformation. In our interpretation, we need the smeared e and m particles to not intersect, hence they must be far apart.

5   Surface Code with Boundaries

A motivation is thinking about realisations of quantum devices. A torus may be physically realizable, but more often we will have a 2D grid with boundaries.

For most of the lecture the surface code will be the usual, where Z_j is along the edges and X_j is transverse; A_v is around a vertex and B_p is around a plaquette. The Hamiltonian is

$$H = -J_A \sum_v A_v - J_B \sum_p B_p,$$

with $J_{A,B} > 0$. The ground states we have seen are those with

$$A_v \ket{\psi} = B_p \ket{\psi} = \ket{\psi},$$

i.e. $a_v = b_p = 1$. We also have path operators

$$\mathcal{Z}_\alpha = \prod_{j \in \alpha} Z_j, \qquad \mathcal{X}_{\tilde{\beta}} = \prod_{j \in \tilde{\beta}} X_j.$$

Then,

$$\mathcal{Z}_\alpha A_v = (-1)^{|\alpha \cap v|} A_v \mathcal{Z}_\alpha,$$

and similarly for $\mathcal{X}$ and B. How can we get integrals of motion in systems with boundaries?

The boundary may cut the vertices. For such vertices, again we let

$$A_v = \prod_{j \in v} X_v = X_1 X_2 X_3,$$

since there are only 3 paths surrounding the vertex. This gives away a key feature in this model: there are lots of paths given by $\mathcal{X}_{\tilde{\alpha}}$. However these paths still commute with plaquette operators, so

$$\mathcal{X}_{\tilde{\alpha}} B_p = B_p \mathcal{X}_{\tilde{\alpha}}$$

for all p. Does this give a new integral of motion? No; it can be written as the product of vertex operators. Moreover if we have a plaquette path $\mathcal{Z}_\alpha$, these do not commute with all A_v: for the vertices v the path start and ends at, A_v anticommutes with Z_α.

There is also an analogous model where the boundary cuts the plaquette operators.

The first kind of boundary is a *smooth boundary*, and the second is a *rough boundary*. These are defined up to some convention. Smooth can be thought of as a *X boundary*, and rough as a *Z boundary*.

Suppose we have two m particles in a smooth boundary. Their energy is $4J_B$. As we move the particles, the energy stays constant, until one of the particles hits the boundary, at which point the particle vanishes and the energy is $2J_B$. We say the boundary *condenses* m particles. Analogously, the rough boundary is an e condensing boundary.

If we have a connected boundary, then all loops happen to be contractible as above. But when this is not the case, we may find non-contractible paths.

Non-contractible meaning under deformation: either multiplying by plaquette or vertex operators. This lets us move the path, or change its endpoint.

We consider some examples in $[-2,2]^2 \setminus [-1,1]^2$, a square with a hole. A path from the $\partial[-2,2]^2$ to itself is not contractible, just by shifting the endpoint.

What about a path around the hole, $\tilde{\alpha}$ consisting of vertex paths? It is also contractible. One way to see this is by introducing a loop that joins our path to $\partial[-2,2]^2$, then moving the endpoints.

What about $\tilde{\gamma}$, going from $\partial[-2,2]^2$ to $\partial[-1,1]^2$? This is non-contractible. To remove it from the system, we must move the two endpoints to meet. Detaching an endpoint corresponds to creating a particle, as it anticommutes with some operator.

What if the path $\tilde{\gamma}'$ around the hole was of plaquette operators? Then this is non-contractible, as the plaquette operators cannot cut the path like with $\tilde{\alpha}'$.

Denote
$$\bar{X}_1 = \mathcal{X}_{\tilde{\gamma}}, \qquad \bar{Z}_1 = \mathcal{Z}_{\tilde{\gamma}'}.$$

These are integrals of motion which anticommute. We can label eigenstates with $|\{a_v, b_p\}, \bar{z}_1\rangle$. This gives a ground state degeneracy of 2.

This enumerates all possibilities up to deformation: anything is either a path or a loop.

What if the inner boundary was rough? Again $\tilde{\alpha}$ and $\tilde{\alpha}'$ are contractible. Moreover $\tilde{\gamma}'$ is also contractible, by moving it to the inner boundary, and we cannot connect the two boundaries. So here we have a ground state degeneracy of just 1.

What if we have $[-2,2]^2$, where the top and bottom boundaries are smooth, and the left and right boundaries are rough? Despite looking simply connected, there are some degeneracies, for example the path $\tilde{\gamma}$ from top to bottom or path $\tilde{\gamma}'$ from left to right. These anticommute, giving ground state degeneracy of 2.

We can generalize for more general manifolds. Let β_1 be the number of independent

non-contractible paths α_j with

$$\bar{Z}_j = \prod_{i \in \alpha_j} Z_i,$$

and with quantum number $\bar{z}_j \in \{-1, 1\}$. Then there exists $\bar{X}_j$ with

$$\bar{X}_j \bar{Z}_i = (-1)^{\delta_{ij}} \bar{Z}_i \bar{X}_j.$$

These are the dual lattice paths $\tilde{\gamma}_i$.

We can also ask about robustness of GSD, from $H = H_0 + \delta H$, where δH is again local. The order of splitting is

$$E_{\text{split}} \sim J \left(\frac{h}{J}\right)^L,$$

where L is the length of the shortest non-contractible path.

One tricky termination is the following. Consider again a square with top and bottom smooth, left and right rough.

6   Quantum Error Correction

6.1   Classical Error Correction

Our task is to store one bit $r \in \{0, 1\}$. For convenience we introduce $z = (-1)^r$, which can be thought of either $\{\uparrow, \downarrow\}$.

A logical operator is a *bit flip*: $z \mapsto -z$. This can be caused by environmental effects.

Error correction can be done via redundancy, for example the *classical repetition code*. In this case,

$$z \to (z_1, \ldots, z_n) = \begin{cases} \uparrow \cdots \uparrow & \uparrow \\ \downarrow \cdots \downarrow & \downarrow \end{cases}$$

For this we take n odd. The encoded logical operator is $z_i \mapsto -z_i$ for all i.

The first step is error detection: we measure all z_j. If they are not all equal, then an error has occurred.

For the error correction step, we flip the minority of spins. When does this procedure work? Let the number of errors be q. Then if $q < n/2$, error correction works. Otherwise, it gives a logical error.

The problem is that we do not know q. But we may know an error process, that is suspected. From this we can build an error model, for example a *bit flip channel*

$$z_j \mapsto \begin{cases} -z_j & \text{probability } p, \\ z_j & \text{probability } 1 - p, \end{cases}$$

independently for all j. Hence $q \sim \text{Bin}(n, p)$. Here p is known as the *error rate*. The *logical error probability* is $\text{Pr}_e = \text{Pr}(q > n/2)$.

We know that $\mu_q = np$, $\sigma_q = \sqrt{np(1-p)}$, and hence from CLT, we can estimate

$$\frac{n/2 - \mu_q}{\sigma_q} = \frac{n(1/2 - p)}{\sqrt{np(1-p)}} > \sqrt{n}(1 - 2p) > 0,$$

for $p < 1/2$. This is positive, and grows with n. So we can find

$$\text{Pr}_e = \text{Pr}(q > n/2) \leq e^{-2n(1/2 - p)^2}.$$

The *code distance* d is the number of bitflips between codewords. In our case $d = n$. In the $[n, k, d]$ notation, n is the length of the codewords, k is the number of logical bits, and d is the number of bitflips.

We can think of the *error correction phase*: this is the region of parameter space of the error model such that

$$\mathrm{Pr}_e \leq e^{-cd},$$

for some $c > 0$. The boundary of this region is the *error correction threshold*. In a one-parameter model, we have a threshold p^*. In our case, $p^* = 1/2$.

6.2   Quantum Repetition Code

Now what we want is to protect one qubit $|\psi\rangle = \alpha |\uparrow\rangle + \beta |\downarrow\rangle$ from bit flips.

Again the first step is encoding: $|\psi\rangle \to |\bar{\psi}\rangle = \alpha |\uparrow\rangle^{\otimes n} + \beta |\downarrow\rangle^{\otimes n}$. The logical operator we are taking is $\bar{X} = X_1 \cdots X_n$, and also $\bar{Z} = Z_j$ for any j.

Recall the Ising Hamiltonian

$$H = - \sum_j Z_j Z_{j+1}.$$

In this model, $\bar{X}$ is the $\mathbb{Z}_2$ symmetry, and Z_j are the order parameters.

The *codespace* is

$$\mathcal{L} = \mathrm{span}\{|\uparrow\rangle^{\otimes n}, |\downarrow\rangle^{\otimes n}\},$$

the groundstate of H.

Errors may be of the form

$$|\bar{\psi}\rangle \to \begin{cases} X_j |\bar{\psi}\rangle & \text{probability } p, \\ |\bar{\psi}\rangle & \text{probability } 1 - p, \end{cases}$$

independently.

Consider what happens when an error occurs in a ground state particle. In this case our state will look like

$$|\uparrow \cdots \uparrow \,|\, \downarrow \,|\, \uparrow \cdots \uparrow\rangle, \qquad |\downarrow \cdots \downarrow \,|\, \uparrow \,|\, \downarrow \cdots \downarrow\rangle.$$

We can do error detection by making use of $Z_j Z_{j+1} = S_j$, and if -1 is observed then an error occurs. This does not distinguish between the two above states, so we do not collapse our state.

Then, we can do error correction by flipping the minority spins: applying $\prod X_j$ for j in the minority set. The minority set can be identified just from the domain walls. In this case,

$$\mathrm{Pr}_e \leq e^{-cd},$$

for $p < 1/2$.

We can also think of phase flips, where X_j is replaced with Z_j in the above bit flips. The above error correction does not account for these errors.

6.3   Stabilizers

Consider the objects $S_j = Z_j Z_{j+1}$. These are:

- Hermitian Pauli: $S_j = S_j^\dagger$ and $S_j^2 = 1$.

- They mutually commute: $S_i S_j = S_j S_i$.

- Independent:

$$\prod_{j=1}^{m} S_j^{a_j} \neq \pm 1$$

for all $\mathbf{a} \neq \mathbf{0}$, where $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{Z}_2^m$.

If we denote the eigenvalues of S_j as $s_j \in \{-1, 1\}$, then the basis in $\mathcal{H} = \mathbb{C}_2^n$ is $|\{s_1, \ldots, s_m\}, a_1, \ldots, a_k\rangle$, where $a_j \in \{-1, 1\}$ are the eigenvalues of some independent integrals of motion.

We can write

$$H = -\sum_{j=1}^{m} S_j,$$

and our ground space will be the codespace $\mathcal{L}$, which satisfy $S_j |\varphi\rangle = |\varphi\rangle$ for all j.

We can choose k of these integrals of motion to be $\bar{Z}_j$, which are Pauli strings and Hermitian. These will supply some eigenvalues z_j. We can think of $\bar{X}_j$, other integrals of motion which flip $\bar{z}_j$:

$$\bar{X}_j \bar{Z}_i = (-1)^{\delta_{ij}} \bar{Z}_i \bar{X}_j.$$

Moreover $\bar{X}_j$, $\bar{Z}_i$ commute with the S.

We must have $k = n - m$, so the ground state degeneracy is 2^k. This gives k logical qubits.

The logical operators are not unique: note that $\bar{X}_j |\bar{\varphi}\rangle \in \mathcal{L}$. Let $\bar{X}' = S_l \bar{X}_j$, for some l. Then

$$\bar{X}' |\bar{\varphi}\rangle = S_l \bar{X}_j |\bar{\varphi}\rangle = \bar{X}_j (S_l |\bar{\varphi}\rangle) = \bar{X}_j |\bar{\varphi}\rangle.$$

6.4   Group Theoretic Stabilizer Codes

Consider an n-qubit Pauli group generated by the elements

$$\mathcal{P}_n = \langle i\mathbb{1}, X_1, Z_1, \ldots, X_n, Z_n \rangle.$$

Members of this group are *Pauli strings*. The stabilizer groups is $\mathcal{S} \leq \mathcal{P}_n$, which is abelian. Moreover $-\mathbb{1} \notin \mathcal{S}$. We can specify

$$\mathcal{S} = \langle S_1, \ldots, S_m \rangle.$$

The S_j are independent if $\mathcal{S}$ is not generated by a smaller subset of the $\{S_j\}$.

We can define the codespace

$$\mathcal{L} = \{|\psi\rangle \in \mathcal{H} \mid Q\,|\psi\rangle = |\psi\rangle \ \forall Q \in S\}.$$

The centralizer of $\mathcal{S}$ is

$$\mathcal{C}(\mathcal{S}) = \{P \in \mathcal{P}_n \mid PQ = QP \ \forall Q \in S\}.$$

Since $\mathcal{S}$ is abelian, $S \subseteq \mathcal{C}(\mathcal{S})$. The *logical operators* are

$$\mathrm{LOs} = \mathcal{C}(\mathcal{S}) \setminus \mathcal{S}.$$

We begin by looking at quantum error correction for $k = 1$. Consider encoding

$$|\psi\rangle = \alpha\,|\!\uparrow\rangle + \beta\,|\!\downarrow\rangle \to |\bar{\psi}\rangle = \alpha\,|\{s_i = 1\}, \bar{z} = 1\rangle + \beta\,|\{s_i = 1\}, \bar{z} = -1\rangle.$$

Here s_i are the eigenvalues of the S_i. Note that measuring by S_i reveals whether an error occurred, but does not change the underlying state.

We denote the encoded measurement of all S_j as $\mathbf{s} = (s_1, \ldots, s_m)$, the *syndrome*. If there exists $s_j = -1$, then an error has occurred. The post measurement state is $|\psi_{\mathbf{s}}\rangle$.

To do error correction, we take $\mathbf{s}$ and convert it to $C_{\mathbf{s}} \in \mathcal{P}_n$, the *decoder*. Then we apply $C_s\,|\psi_{\mathbf{s}}\rangle$ to get an element of $\mathcal{L}$.

Example 6.1.

Suppose we start with $|\psi\rangle \in \mathcal{S}$, and an error $\mathbf{E_s} \in \mathcal{P}_n$ occurs, giving syndrome $\mathbf{s}$. Then,

$$S_j E_{\mathbf{s}}\,|\bar{\psi}\rangle = \sigma_{E_{ij}} E_{\mathbf{s}}\,|\bar{\psi}\rangle.$$

So $s_j = \sigma_{E_{ij}}$, whether the operators commute or not. $E_{\mathbf{s}}\,|\bar{\psi}\rangle$ is an eigenstate

of S_j with eigenvalue $s_j = \sigma_{E_{ij}}$.

If $E_{\mathbf{s}}$ is known, then we can let $C_{\mathbf{s}} = E_{\mathbf{s}}^\dagger$, and apply it to find our original state. The problem is this is not known.

What is known are te equivalence classes $\mathbf{s} \to E_{\mathbf{s}}\mathcal{C}(\mathcal{S})$. Since the syndrome encodes $\sigma_{E_{ij}}$, if $E_{\mathbf{s}}' = E_{\mathbf{s}}P$, then $\sigma_{EP_{ij}} = \sigma_{E_{ij}} = s_j$. This means $P \in \mathcal{C}(\mathcal{S})$.

In fact, this is equivalent to $\sigma_{EP_{ij}} = \sigma_{E_{ij}}$.

Based on our information, we can pick $C_{\mathbf{s}}$ such that $C_{\mathbf{s}}E_{\mathbf{s}} |\psi\rangle \in \mathcal{L}$. Then since $s_j = 1$, we must have either $C_{\mathbf{s}}E_{\mathbf{s}} \in S$ or in LO, as $\mathcal{C}(\mathcal{S}) = \mathrm{LO} \cup \mathcal{S}$.

The decoder aims for $\mathcal{S}$, in which case the error is fixed. If not, we obtain a logical error.

The *weight* $\mathrm{wt}(O)$ for $O \in \mathcal{P}_n$ is $|\mathrm{supp}(O)|$, the number of qubits that O acts on nontrivially.

The *distance* is
$$d = \min_{O \in \mathrm{LO}} \mathrm{wt}(O).$$

We again have $[n, k, d]$ notation.

> **Example 6.2.**
>
> Consider the quantum repetition code where $\bar{Z} = Z_j$. Thus $d = 1$.
>
> For a surface code the weight is the length of the shortest non-contractible path.

We will write
$$E_{\mathbf{s}}\mathcal{C}(\mathcal{S}) = \bigcup_{\bar{P} \in \{\mathrm{LO} \cup \mathbb{1}\}} E_{\mathbf{s}}\bar{P}\mathcal{S},$$

where $\bar{P}$ is a representative of a logical operator in $\bar{P}\mathcal{S}$.

If $E_{\mathbf{s}}\bar{P}\mathcal{S}$ is known, then we can pick any $P \in E_{\mathbf{s}}\bar{P}\mathcal{S}$, say $P = \bar{P}E_{\mathbf{s}}^\dagger = C_{\mathbf{s}}$. Suppose the actual error is $A_s = E_{\mathbf{s}}\bar{P}Q$ for $Q \in S$. Then,

$$\bar{P}E_{\mathbf{s}}^\dagger A_{\mathbf{s}} |\bar{\psi}\rangle = Q |\bar{\psi}\rangle = |\bar{\psi}\rangle.$$

However we do not know the correct class. If the error model is known, then we can determine
$$\mathrm{Prob}(P|\mathbf{s}) = \frac{\mathrm{Prob}(E_{\mathbf{s}}\bar{P}\mathcal{S})}{\mathrm{Prob}(\mathbf{s})},$$

where $\text{Prob}(\mathbf{s})$ is the sum over the equivalence classes. We can choose the *maximum likelihood decoder*, which picks the P with maximal probability.

We can define the *error-correction phase* as

$$\Delta_{\mathbf{s}} = 1 - \max_{\bar{P}} \text{Prob}(\bar{P}|\mathbf{s}),$$

and we can require that $\Delta_{\mathbf{s}} \sim e^{-cd}$ for $c > 0$.

Moreover we can define

$$\Delta = \sum_{\mathbf{s}} \text{Prob}(\mathbf{s})\Delta_{\mathbf{s}},$$

and again require $\Delta \sim e^{-c'd}$.

6.5   Knill-Laflamme Conditions

Consider what happens when our errors are $e^{i\phi Z_j} \notin \mathcal{P}_n$.

We begin by looking at *errors* given by error operators $\tilde{E}_a$ and an error set

$$|\bar{\psi}\rangle \to |\psi_a\rangle = \frac{1}{\sqrt{\text{Prob}(a)}} \tilde{E}_a |\psi\rangle,$$

with

$$\text{Prob}(a) = \langle \bar{\psi}| \tilde{E}_a^\dagger \tilde{E}_a |\bar{\psi}\rangle.$$

To formalise this, we consider an initial state

$$|\bar{\psi}\rangle_{\text{sys}} |\phi\rangle_{\text{env}} \to U[|\bar{\psi}\rangle_{\text{sys}} |\phi\rangle_{\text{env}}] = \sum_a (\tilde{E}_a |\bar{\psi}\rangle_{\text{sys}}) |a\rangle_{\text{env}}.$$

We do a measurement by the environment state, and get an outcome a:

$$\tilde{E}_a |\bar{\psi}\rangle_{\text{sys}} |a\rangle_{\text{env}},$$

with

$$\text{Prob}(a) = \langle \bar{\psi}| \tilde{E}_a^\dagger \tilde{E}_a |\bar{\psi}\rangle_{\text{sys}}.$$

Take $|\bar{j}\rangle$ to be an orthonormal basis of $\mathcal{L}$. For any $a \in \{1, \ldots, e\}$, we need that $\tilde{E}_a |\bar{j}\rangle = |j_a\rangle$, orthonormal states, so we can read out what the original state is after a unitary correction.

Moreover, we need $\text{Prob}_\psi(a) = \text{Prob}(a)$, independent of state, otherwise measuring the error would reveal information about the state, and therefore corrupt it. This means that

$$\langle \bar{j}| \tilde{E}_a^\dagger \tilde{E}_a |\bar{i}\rangle = c_a \delta_{ij}.$$

Hence $\tilde{E}_a$ is unitary, mapping $\mathcal{L}$ to some subspace $\tilde{E}_a\mathcal{L}$. If $\tilde{E}_a\mathcal{L} \perp \tilde{E}_b\mathcal{L}$ for $a \neq b$, then QEC works: via projective measurements we can read out which error occurred, then fix it, via the projection $\tilde{E}_a\Pi_\mathcal{L}\tilde{E}_a^\dagger$.

If this is not true, but

$$\tilde{E}_a\Pi_\mathcal{L}\tilde{E}_a^\dagger\tilde{E}_b\ket{\bar{\psi}} = c_{ab}\tilde{E}_a\ket{\bar{\psi}},$$

then this also works; in actuality error b occurred, but after measuring we can treat it as if error a occurred. This is guaranteed if

$$\Pi_\mathcal{L}\tilde{E}_a^\dagger\tilde{E}_b\Pi_\mathcal{L} = c_{ab}\Pi_\mathcal{L}.$$

This corresponds to

$$\bra{\bar{j}}\tilde{E}_a^\dagger\tilde{E}_b\ket{\bar{i}} = c_{ab}\delta_{ij}.$$

Suppose we have another error set

$$\tilde{F}_a = \sum_{a'} M_{a'a}\tilde{E}_{a'}.$$

Then,

$$\Pi_\mathcal{L}\tilde{F}_a\tilde{F}_b\Pi_\mathcal{L} = (M^+cM)_{ab}\Pi_\mathcal{L},$$

so the linear combination of errors can be identified. Hence without loss of generality, we can take Pauli errors. This is the *discretisation of errors*.

> **Example 6.3.**
>
> Take the bit flip channel on qubit j, where
>
> $$\tilde{E}_1 = \sqrt{1-p}\mathbb{1}, \qquad \tilde{E}_2\sqrt{p}X_j.$$
>
> We can also have the phase flip where instead $\tilde{E}_2 = \sqrt{p}Z_j$.
>
> We can also have depolarity, where $\tilde{E}_1$ is the same, and
>
> $$\tilde{E}_j = \sqrt{p/3}P_j,$$
>
> for $P_j \in X_j, Y_j, Z_j$.

6.6   KL Conditions for Stabilizer Codes

We claim that the KL conditions hold if and only if $E_a^\dagger E_b \notin$ LO, for all $a, b \in \{1, \ldots, e\}$.

Suppose that there is a, b such that $E_a^\dagger E_b \in \mathrm{LO}$. Then

$$\Pi_\mathcal{L} E_a^\dagger E_b \mathcal{L} \Pi_\mathcal{L} = E_a^\dagger E_b \Pi_\mathcal{L} \neq c_{ab} \Pi_\mathcal{L}.$$

If otherwise $E_a^\dagger E_b \in \{\mathcal{S}, i\mathbb{1}\}$ or $E_a^\dagger E_b \notin \mathcal{C}(\mathcal{S})$, then the left hand side will be proportional to $\Pi_\mathcal{L}$.

Hence if $\mathrm{wt}(E_a^\dagger E_b) < d$, then KL holds. This is implied if $\mathrm{wt}(E_a) < d/2$.

> **Example 6.4.**
>
> In the bit flip example,
> $$E_a \sim \prod_{j=1}^n X_j$$
> is in the error set, so KL does not hold.

7   QEC in the Surface Code

In the surface code, the stabilizer generators are A_v and B_p.

For error detection, if we do a measurement of S_j with post measurement $|\psi_{\mathbf{s}}\rangle$, if we measure $s_j = -1$, then there is an anyon.

For error correction, we have a syndrome $\mathbf{s}$, which corresponds to $C_{\mathbf{s}} \in \mathcal{P}_n$. This corresponds to dragging pairs of particles to annihilate them.

7.1   Surface Code with Bit Flip Channels

We saw that under some error with syndrome $\mathbf{s}$, we are in the class

$$E_{\mathbf{s}} \bar{P} \mathcal{S} \to \mathcal{X}_{\mathbf{s}} \bar{X} \mathcal{A} = \{ \mathcal{X}_{\mathbf{s}} \bar{X}^q \mathcal{X}_{V_i} \mid \mathcal{X}_{V_i} = \prod_{v \in V_i} A_v,\ V_i \subseteq V \},$$

where, since we know that the error are bit flips we know they must be of the form $\mathcal{X}_\alpha = \prod_{j \in \alpha} X_j$. Here $q \in \{0,1\}$, and V is the set of vertices. We want to find

$$\mathrm{Prob}(\mathcal{X}_{\mathbf{s}} \bar{X}^q \mathcal{A}) = \mathrm{Prob}(q, \mathbf{s}).$$

First, note that

$$\mathrm{Prob}(\mathcal{X}_\alpha) = \prod_{j=1}^{n} \sqrt{p(1-p)} \left(\sqrt{\frac{1-p}{p}} \right)^{\eta_j(X_\alpha)},$$

where

$$\eta_j(X_\alpha) = \begin{cases} -1 & j \in \mathrm{supp}(X_\alpha), \\ 1 & \text{otherwise.} \end{cases}$$

Consider $\mathcal{X}_{V_1} \in \mathcal{A}$. We can transform to Ising variables

$$\sigma(V_1) = (\sigma_1(V_1), \sigma_2(V_1), \ldots, \sigma_{|V|}(V_1))$$

where

$$\sigma_v(V_i) = \begin{cases} -1 & v \in V_1, \\ 1 & \text{otherwise.} \end{cases}$$

Then we claim that

$$\eta_j(\mathcal{X}_{V_1}) = \sigma_{v_j} \sigma_{v'_j},$$

where j is adjacent to vertices v and v'. This is because $X_j^2 = \mathbb{1}$. Hence also

$$\eta_j(\mathcal{X}_{\mathbf{s}} \bar{X}^q \mathcal{X}_{V_1}) = \eta(\mathcal{X}_{\mathbf{s}} \bar{X}^q) \sigma_{v_j} \sigma_{v'_j}.$$

So we can write

$$\mathrm{Prob}(\mathcal{X}_\mathbf{s} \bar{X}^q \mathcal{A}) = \mathcal{N} \sum_\sigma \left(\sqrt{\frac{1-p}{p}} \right)^{\eta_j(\mathbf{s},q) \sigma_{v_j} \sigma_{v'_j}} .$$

Letting $\sqrt{(1-p)/p} = e^K$, and letting the sum go to the top, this is

$$\mathcal{N} \sum_\sigma \exp\left(K \sum_{v \sim v'} \eta_{vv'}(\mathbf{s},q) \sigma_v \sigma_{v'} \right).$$

Note that as $p < 1/2$, $e^K > 1$. This probability is equivalent to the Ising model Hamiltonian:

$$Z_{\mathbf{s},q} = \sum_\sigma e^{-\beta H_{\mathbf{s},q}(\sigma)}, \qquad \beta H_{\mathbf{s},q}(\sigma) = -K \sum_{v \sim v'} \eta_{vv'}(\mathbf{s},q) \sigma_v \sigma_{v'}.$$

If $X_\mathbf{s} = 1$, then $\eta = 1$ and this corresponds to a ferromagnet. Otherwise if $q = -1$, then $\eta = -1$ and we get an antiferromagnet.

Index