III Symmetries, Particles and Fields

Ishan Nath, Michaelmas 2024

Based on Lectures by Prof. Matthew Wingate

October 19, 2024

Contents

1 Introduction to Symmetries

Recall Newton's second law:

$$m\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = \mathbf{F}(\mathbf{x}).$$

This simplifies if we know F is rotationally symmetric, i.e. $\mathbf{F}(\mathbf{x}) = F(r)\hat{\mathbf{r}}$. Then $\mathbf{L} = \mathbf{x} \times \mathbf{p}$ is conserved, and trajectories lie in planes containing the origin.

Now consider Lagrangian mechanics, with Lagrangian $L(q_i, \dot{q}_i, t)$. The principle of least action says

$$S = \int_{t_1}^{t_2} \mathrm{d}t\, L(q_i(t), \dot{q}_i(t), t)$$

is minimized by classical trajectories. Hence Euler-Lagrange gives

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0.$$

Nöether's theorem says that invariance of L under some coordinate transform corresponds to an associated conserved quantity.

Example 1.1.

Consider a particle in three dimension, with a potential:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z),$$

which is independent of t, hence invariant under $t \mapsto t + \delta t$. This implies that the Hamiltonian $H = T + U$ is conserved. If we transform into canonical momenta $p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i$, then

$$H(x_i, p_i, t) = \sum \dot{x}_i p_i - L = \sum \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} - L$$

is invariant by Euler-Lagrange:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \sum \ddot{x}_i \frac{\partial L}{\partial \dot{x}_i} - \sum x_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \dot{x}_i\frac{\partial L}{\partial x_i} - \ddot{x}_i \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial t} = 0.$$

If L is invariant under $x \mapsto x + \delta x$, then

$$\frac{\partial L}{\partial x} = 0 \implies \frac{\partial L}{\partial \dot{x}} = p_x = \text{constant}.$$

If L is invariant under rotations about the z-axis, then the z-component of angular momentum, $xp_y - yp_x$, is constant. The best way to see is transform L into cylindrical coordinates:

$$L = \frac{1}{2}m(\dot{\rho}^2 - \rho^2\dot{\theta} + \dot{z}^2) - U(\rho, z).$$

So the invariance under rotations means

$$\frac{\partial L}{\partial \theta} = 0 \implies \frac{\partial L}{\partial \dot{\theta}} = 0 = m\rho^2\dot{\theta} = xp_y - yp_x$$

is constant.

1.1 Symmetries in Quantum Mechanics

Given a system whose states are element of a Hilbert space $\mathcal{H}$, a symmetry means there exists some invertible operator U acting on $\mathcal{H}$ which preserves inner products, up to an overall phase $e^{i\phi}$.

Definition 1.1. Let $|\psi\rangle, |\phi\rangle$ be any normalized vectors in $\mathcal{H}$. Denote $|U\psi\rangle = U|\psi\rangle$, and $|U\phi\rangle = U|\phi\rangle$.

U is a *symmetry transformation* if

$$|\langle U\phi|U\psi\rangle| = |\langle\phi|\psi\rangle|.$$

Proposition 1.1 (Wigner's Theorem)**.** *Symmetry transformation operators are either linear and unitary, or antilinear and antiunitary.*

Antilinear and antiunitary means

$$U(a|\psi\rangle + \beta|\phi\rangle) = a^*U|\psi\rangle + b^*U|\phi\rangle,$$

$$\langle U\phi|U\psi\rangle = \langle\phi|\psi\rangle^*.$$

Suppose we have a system with a time-independent Hamiltonian. Then we can write down

$$\langle\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle,$$

by Schrödinger's equation with $\hbar = 1$. In the first case, note

$$\begin{aligned}\langle U\phi|U\psi(t)\rangle &= \langle\phi|\psi(t)\rangle \\ &= \langle\phi|e^{-iHt}|\psi(0)\rangle.\end{aligned}$$

We should find the same result by transforming $|\psi(0)\rangle$ before time evolution:

$$\begin{aligned}\langle U\phi|U\psi(t)\rangle &= \langle U\phi|e^{-iHt}|U\psi(0)\rangle \\ &= \langle\phi|U^\dagger e^{-iHt}U|\psi(0)\rangle\,.\end{aligned}$$

Equating these, we find

$$U^\dagger e^{-iHt}U = e^{-iHt} \implies [U, H] = 0.$$

Example 1.2.

If H commutes with $\mathbf{p}$, then H cannot depend on $\mathbf{x}$, as

$$[x_i, p_j] = i\delta_{ij} \neq 0$$

generally. So H is invariant under translation $\mathbf{x} \mapsto \mathbf{x}+\mathbf{a}$, and this is generated by unitary operators $\exp(i\mathbf{p}\cdot\mathbf{a})$.

If H is rotationally symmetric, then any momentum operator $\mathbf{J}$ or $\mathbf{L}$ commutes with H.

2 Lie Groups and Algebras

2.1 Lie Groups

Recall the definition of a group: a set together with a relation which has an identity, inverses and is associative.

Also recall a group is abelian if $g \cdot h = h \cdot g$ for all $g, h \in G$.

Definition 2.1. A *manifold* is a space which looks Euclidean, like $\mathbb{R}^n$, on small scales, in small neighbourhoods.

A *differentiable manifold* is one which satisfies certain smoothness conditions.

Definition 2.2. A *Lie group* consists of a differentiable manifold G along with a binary operation $\cdot$, such that the group axioms hold, and that $\cdot$ and inverse are smooth operations.

2.2 Matrix Lie Groups

For example, the general linear group $\mathsf{GL}(n, \mathbb{F})$ is the group of invertible $n \times n$ matrices over a field $\mathbb{F}$. So,

$$\mathsf{GL}(n, \mathbb{F}) = \{M \in \mathrm{Mat}_n(\mathbb{F}) \mid \det M \neq 0\}.$$

The group operation is simply matrix multiplication.

The dimension of $\mathsf{GL}(n, \mathbb{R})$ is n^2, as there are n^2 free parameters. For $\mathsf{GL}(n, \mathbb{C})$, we have real dimension $2n^2$, and complex dimension n^2.

Important subgroups of $\mathsf{GL}(n, \mathbb{F})$ are:

- The special linear group

$$\mathsf{SL}(n, \mathbb{F}) = \{M \in \mathsf{GL}(n, \mathbb{F}) \mid \det M = 1\}.$$

- $\mathsf{SL}(n, \mathbb{R})$ has dimension $n^2 - 1$.
- The orthogonal group

$$\mathsf{O}(n) = \{M \in \mathsf{GL}(n, \mathbb{R}) \mid M^T M = I\}.$$

  This implies $\det M = \pm 1$. We can also define

$$\mathsf{SO}(n) = \{M \in \mathsf{O}(n) \mid \det M = 1\}.$$

- Pseudo-orthogonal group. Define an $(n+m)\times(n+m)$ matrix by

$$\eta = \begin{pmatrix} I_n & 0 \\ 0 & -I_M \end{pmatrix}.$$

Then we can define

$$\mathsf{O}(n,m) = \{M \in \mathsf{GL}(n+m,\mathbb{R}) \mid M^T \eta M = \eta\}.$$

Note $M \in \mathsf{SO}(n,m) \iff \det M = 1$.

- Unitary.

$$\mathsf{U}(n) = \{M \in \mathsf{GL}(n,\mathbb{C}) \mid M^\dagger M = I\}.$$

Similarly have $\mathsf{SU}(n)$.

- Pseudounitary.

$$\mathsf{U}(n,m) = \{M \in \mathsf{GL}(n,\mathbb{C}) \mid M^\dagger \eta M = \eta\}.$$

- Symplectic group. Define a fixed, antisymmetric $2n\times 2n$ matrix, e.g.

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Then,

$$\mathsf{Sp}(2n,\mathbb{R}) = \{M \in \mathsf{GL}(2n,\mathbb{R}) \mid M^T \Omega M = \Omega\}.$$

We can show that $\det M = 1$ using the Pfaffian.

Definition 2.3. Given a $2n\times 2n$ antisymmetric matrix A, its *Pfaffian* is given by

$$\mathrm{Pf} A = \frac{1}{2^n n!}\varepsilon_{i_1 i_2 \cdots i_{2n}} A_{i_1 i_2} A_{i_3 i_4} \cdots A_{i_{2n-1} i_{2n}}.$$

2.3 Group Elements as Transformations

We can define actions of group elements $g \in G$ on a set X.

Definition 2.4. The *left action* of G on X is a map $L : G \times X \to X$ such that $L(e,x) = x$, and

$$L(g_2, L(g_1, x)) = L(g_2 g_1, x),$$

for all $x \in X$ and $g_1, g_2 \in G$. In more usual notation, for all $g \in G$, we can associate a map $g : X \to X$ as $g(x) = gx$.

Definition 2.5. The *right action* of G on X is defined by $g : X \to X$ such that $g(x) = xg^{-1}$, for all $x \in X$, $g \in G$. The inverse preserves under composition, so

$$g_2(g_1(x)) = xg_1^{-1}g_2^{-1} = (g_2g_1)(x).$$

Definition 2.6. The action of *conjugation* by G on X is the action defined by

$$g(x) = gxg^{-1},$$

for $g \in G$, $x \in X$.

Definition 2.7. Given a group G and a set X, an *orbit* of an element $x \in X$ is the set of elements of X in the image of G.

Example 2.1.

If the action is a left action, then the orbit of $x \in X$ is

$$Gx = \{gx \mid g \in G\}.$$

It can be shown that the set of orbits under G partition X.

In $\mathbb{R}^n$, orthogonal matrices $\mathsf{O}(n)$ represent rotations and reflections, and preserve the inner product; similarly for $\mathsf{U}(n)$.

We can parametrize $\mathsf{SO}(2)$ as

$$\mathsf{SO}(2) = \left\{ R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \;\middle|\; \theta \in [0, 2\pi] \right\}.$$

cos, sin are smooth. We can show that $R(\theta_2)R(\theta_1) = R(\theta_1 + \theta_2)$.

$\mathsf{SO}(3)$ gives rotations of vectors in $\mathbb{R}^3$. The axis of rotation is given by a unit vector $\mathbf{n} \in S^2$, and we also have an angle θ.

Note that rotation by $\theta \in [-\pi, 0]$ about $\mathbf{n}$ is equivalent to rotation by $-\theta$ about $-\mathbf{n}$, so we can confine $\theta \in [0, \pi]$.

Hence we can depict the manifold of $\mathsf{SO}(3)$ as a ball of radius π in $\mathbb{R}^3$, where antipodal points are identified: $\pi\mathbf{n} = -\pi\mathbf{n}$.

The *pseudo-orthogonal group* $\mathsf{SO}(n, m)$ act on vectors in $\mathbb{R}^{n+m}$, and preserve the scalar product $v_2^T \eta v_1$ for $v_1, v_2 \in \mathbb{R}^{n+m}$.

For example,

$$\mathsf{SO}(1, 1) = \left\{ \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix} \;\middle|\; \psi \in \mathbb{R} \right\}.$$

$\mathsf{SO}(1, 1)$ is an example of a non-compact group.

2.4 Parametrization of Lie Groups

At least in small neighbourhoods, we can assign coordinates

$$x = (x^1, \ldots, x^n) \in \mathbb{R}^n,$$

such that $g(x) \in G$. Closure says that $g(y)g(x) = g(z)$, and smoothness says that the components of z are continuously differentiable functions of x and y, so

$$z^n = \phi^n(x, y).$$

We can choose the coordinates at the origin such that $g(0) = e$. Then $g(0)g(x) = g(x)$, so

$$\phi^r(x, 0) = x^r, \qquad \phi^r(0, y) = y^r.$$

From inverses, for each x there exists $\bar{x}$ such that $g(\bar{x}) = g(x)^{-1}$, hence

$$\phi^r(\bar{x}, x) = 0 = \phi^r(x, \bar{x}).$$

Finally, associativity means $g(z)[g(y)g(x)] = [g(z)g(y)]g(x)$, hence

$$\phi^r(\phi(x, y), z) = \phi^r(x, \phi(y, z)).$$

2.5 Lie Algebras

Lie groups are hard to quantify. Instead, we look at lie algebras, which are a linearization of the lie group.

A lie group is homogeneous: any neighbourhood can be mapped to any other neighbourhood. We will linearize near the identity of G.

Definition 2.8. A *Lie algebra* is a vector space V, which additionally has a vector product, the *Lie bracket* $[\cdot, \cdot] : V \times V \to V$ possessing the following properties: for $X, Y, Z \in V$,

1. antisymmetry: $[X, Y] = -[Y, X]$.
2. Jacobi identity: $[X, [Y, Z]] + [Y, [X, Z]] + [Z, [X, Y]] = 0$.
3. linearity: for $\alpha, \beta \in \mathbb{F}$, $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$.

Remark. Any vector space which has a vector product $*$ can be made into a Lie algebra with Lie bracket

$$[X, Y] = X * Y - Y * X.$$

Given a Lie algebra V, choose a basis $\{T_a\}$. The basis vectors are called the *generators* of the Lie algebra.

Write their Lie brackets as

$$[T_a, T_b] = f^c{}_{ab} T_c,$$

with $f^c{}_{ab} \in \mathbb{F}$ called the *structure constants*. The properties imply:

- antisymmetry $\implies f^c{}_{ba} = -f^c{}_{ab}$.
- Jacobi $\implies f^e{}_{ad} f^d{}_{bc} + f^e{}_{cd} f^d{}_{ab} + f^e{}_{bd} f^d{}_{ca} = 0$.

General elements of Lie algebras are linear combinations of $\{T_a\}$. So $X \in V$ can be written as $X^a T_a$, where $X^a \in \mathbb{F}$, and

$$[X, Y] = X^a Y^b f^c{}_{ab} T_c.$$

2.6 Lie Groups and their Lie Algebras

We start with $\mathsf{SO}(2)$, where

$$g(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The identity is $e = I_2 = g(0)$. Near the identity, θ is small, and

$$\sin\theta = \theta - \frac{\theta^3}{3} + \cdots, \qquad \cos\theta = 1 - \frac{\theta^2}{2} + \cdots$$

Hence,

$$g(\theta) = I_2 + \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \frac{\theta^2}{2} I_2 + \mathcal{O}(\theta^3) = e + \theta \left.\frac{\mathrm{d}g}{\mathrm{d}\theta}\right|_0 + \mathcal{O}(\theta^2).$$

The linear term is the "tangent" to the manifold. We have a one-dimensional tangent space at e, and we claim that this is the Lie algebra of $\mathsf{SO}(2)$, i.e.

$$L(\mathsf{SO}(2)) = T_e(\mathsf{SO}(2)) = \left\{ \begin{pmatrix} 0 & -a & a & 0 \end{pmatrix} \,\middle|\, a \in \mathbb{R} \right\}.$$

For $\mathsf{SO}(n)$, we can show the dimension is $\frac{n(n-1)}{2} = d$. Choose coordinates $x_1, \ldots, x_d$, and consider a single-parameter family of $\mathsf{SO}(n)$ elements

$$M(t) = M(x(t)) \in \mathsf{SO}(n),$$

such that $M(0) = I_n$. Then,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(M^T(t)M(t)) = \frac{\mathrm{d}M^T}{\mathrm{d}t}M + M^T\frac{\mathrm{d}M}{\mathrm{d}t}.$$

Looking at $t = 0$, we find

$$\frac{\mathrm{d}M^T}{\mathrm{d}t} = -\frac{\mathrm{d}M}{\mathrm{d}t},$$

hence matrices in the tangent space are anti-symmetric. Moreover they are also traceless.

For unitary groups, we again let $M(t)$ be a curve in $\mathsf{SU}(n)$ with $M(0) = I$. For small t, write

$$M(t) = I + tX + \mathcal{O}(t^2).$$

From unitarity, $I = M^\dagger M$, so looking at the expansion,

$$I = I + t(X + X^\dagger) + \mathcal{O}(t^2),$$

hence $X^\dagger = -X$, is anti-hermitian. We also claim X is traceless for $\mathsf{SU}(n)$. Indeed, looking at $\det M$, its expansion is

$$1 = \det M = 1 + t\,\mathrm{Tr}(X) + \mathcal{O}(t^2).$$

2.7 Lie Algebras of a Matrix Lie Group

Consider two curves through the identity e of some Lie group, $g_1(x(t))$ and $g_2(y(t))$, with $X_1 = \dot{g}_1|_0$, $X_2 = \dot{g}_2|_0$. The product is

$$g_3(z(t)) = g_2(y(t))g_1(x(t)) \in G.$$

Then,

$$\dot{g}_3|_0 = (\dot{g}_1 g_2 + g_1\dot{g}_2)|_0 = X_1 + X_2 \in T_e(G).$$

The Lie bracket arises from the group commutator.

Definition 2.9. The *group commutator* of $g_1, g_2 \in G$ is

$$[g_1, g_2] = g_1^{-1}g_2^{-1}g_1g_2 \in G.$$

Let $g_1(t), g_2(t)$ be two curves through the identity, and

$$g_i(t) = c + tX_i + t^2W_i + \mathcal{O}(t^3).$$

Then,

$$\begin{aligned}
g_1(t)g_2(t) &= e + t(X_1 + X_2) + t^2(X_1X_2 + W_1 + W_2) + \mathcal{O}(t^3),\\
g_2(t)g_1(t) &= e + e(X_1 + X_2) + t^2(X_2X_1 + W_1 + W_2) + \mathcal{O}(t^3).
\end{aligned}$$

Therefore,

$$h(t) = [g_2(t)g_1(t)]^{-1}g_1(t)g_2(t) = e + t^2[X_1, X_2] + \cdots .$$

So if $h(t) \in G$, then the tangent to $h(t)$ at e is $[X_1, X_2] \in L(G)$.

Now we can think of tangent spaces to $G \subseteq \mathsf{GL}(n, \mathbb{F})$ at a general element p, $T_p(G)$.

Let $g(t)$ be a curve in the manifold through p with $g(t_0) = p$, so

$$g(t_0 + \varepsilon) = g(t_0) + \varepsilon\dot{g}(t_0) + \mathcal{O}(\varepsilon^2).$$

Both $g(t_0)$ and $g(t_0 + \varepsilon)$ are in G, so there exists $h_p(\varepsilon) \in G$ such that

$$g(t_0 + \varepsilon) = g(t_0)h_p(\varepsilon),$$

where $h_p(0) = e$. For small ε,

$$h_p(\varepsilon) = e + \varepsilon X_p + \mathcal{O}(\varepsilon^2)$$

for some $X_p \in L(G) = T_e(G)$. Neglecting higher order terms,

$$\begin{aligned}
e + \varepsilon X_p = h_p(\varepsilon) &= g(h_0)^{-1}g(t_0 + \varepsilon)\\
&= g(t_0)^{-1}[g(t_0) + \varepsilon\dot{g}(t_0)]\\
&= e + \varepsilon g(t_0)^{-1}\dot{g}(t_0),
\end{aligned}$$

where $g(t)^{-1}\dot{g}(t) = X_p \in L(G)$.

Conversely, for any $X \in L(G)$, there exists a unique curve $g(t)$ with $g^{-1}(t)\dot{g}(t) = X$, and $g(0) = g_0$. This is a consequence of the existence and uniqueness of solutions to ODEs. The solution of this ODE is

$$g(t) = g_0 \exp tX,$$

where

$$\exp tX = \sum_{k=0}^{\infty} \frac{1}{k!}(tX)^k.$$

Given an $X \in L(G)$, the curve

$$g_X(t) = \exp tX,$$

which forms an abelian subgroup of G generated by X. Note $g_x(t)$ is isomorphic to $(\mathbb{R}, +)$ if only $g_x(0) = e$, and S^1 if $g_x(t_0) = e$ for some $t_0 \neq 0$.

Index