III Functional Analysis

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Based on Lectures by Dr. András Zsák

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Contents

0 Introduction

Allen has good notes.

Books include Bollobás, Rudin, S.J. Taylor (measure theory), Rudin again and Murphy.

0.1 Overview

The course is structured as follows.

Chapter 1. Hahn-Banach extension theorems.

Chapter 2. Dual spaces of $L_p(\mu)$ and $C(K)$.

Chapter 3. Weak topologies.

Chapter 4. Convexity and Krein-Milman theorem.

Chapter 5. Banach algebras.

Chapter 6. Holomorphic functional calculus.

Chapter 7. C^*-algebras.

Chapter 8. Borel functional calculus and spectral theory.

1 Hahn-Banach Extension Theorems

Let X be a normed space. The *dual space* X^* of X is

$$X^* = \{f : X \to \text{scalars} \mid f \text{ linear, continuous (or bounded)}\}.$$

This is a normed space in the operator norm. For $f \in X^*$,

$$\|f\| = \sup\{|f(x)| \mid x \in B_X\},$$

where B_X is the unit ball in X, i.e. $\{x \in X \mid \|x\| \leq 1\}$. We also have $S_X = \{x \in X \mid \|x\| = 1\}$, the unit sphere.

Recall that X^* is a Banach space.

Example 1.1.

$\ell_p^* \cong \ell_q$, for $1 \leq p < \infty$, $1 < q \leq \infty$, and $1/p + 1/q = 1$.

We also have $c_0^* \cong \ell_1$.

Also if H is a Hilbert space, then $H^* \cong H$, by the Riesz representation theorem. This is conjugate linear in the complex case.

Definition 1.1. We write $X \sim Y$ if NVS's X and Y are isomorphic, so there exists a linear bijection $T : X \to Y$ where T and T^{-1} are bounded.

If X, Y are both Banach spaces, and $T : X \to Y$ is a continuous linear bijection, then T^{-1} is continuous by the open mapping theorem.

Write $X \cong Y$ if X and Y are isometrically isomorphic, i.e. there exists a surjective linear map $T : X \to Y$ such that T is isometric, i.e. $\|Tx\| = \|x\|$.

Note this automatically implies T is a linear bijection, and T^{-1} is isometric.

For a normed space X, and $x \in X$, $f \in X^*$ we write

$$\langle x, f \rangle = f(x).$$

This is bilinear, and $|\langle x, f \rangle| = |f(x)| \leq \|f\| \cdot \|x\|$. When X is a Hilbert space, X^* is identified with X, and $\langle \cdot, \cdot \rangle$ is the inner product.

Definition 1.2. Let X be a real vector space. A functional $p : X \to \mathbb{R}$ is:

(i) *positive homogeneous* if $p(tx) = tp(x)$ for all $x \in X$, $t \geq 0$.

(ii) *subadditive* if $p(x + y) \leq p(x) + p(y)$.

Theorem 1.1 (Hahn-Banach). *Let X be a real vector space, and $p : X \to \mathbb{R}$ be a positive homogeneous, subadditive functional on X. Let Y be a subspace of X, and $g : Y \to \mathbb{R}$ be linear such that $g(y) \leq p(y)$ for all $y \in Y$.*

Then there exists linear $f : X \to \mathbb{R}$ such that $f|_Y = g$, and $f(x) \leq p(x)$ for all $x \in X$.

To prove this, we need Zorn's lemma, and the theory of posets. Let $(P, \leq)$ be a poset.

For $A \subseteq P$, $x \in P$, say x is an *upper bound* for A if $a \leq x$ for all $a \in A$. For $C \subseteq P$, say C is a *chain* if $\leq$ is a linear order on C. Say $x \in P$ is a *maximal element* if, for all $y \in P$, $x \leq y$ implies $y = x$.

Theorem 1.2 (Zorn's lemma). *If P is a non-empty poset and every non-empty chain in P has an upper bound, then P has a maximal element.*

Proof: Consider the poset given by pairs (Z, h), where Z is a subspace of X containin Y, and $h : Z \to \mathbb{R}$ linear, with $h|_Y = g$, and $h(z) \leq p(z)$.

Here $(Z_1, h_1) \leq (Z_2, h_2)$ if $Z_1 \subseteq Z_2$ and $h_2|_{Z_1} = h_1$. This can be checked to be a partial order.

Now we check our conditions. First $P \neq \emptyset$ as $(Y, g) \in P$. Moreover, given a non-empty chain $C = \{(Z_i, h_I) \mid i \in I\}$ in P, we can set $Z = \bigcup_{i \in I} Z_i$, and define $h : Z \to \mathbb{R}$ by $h|_{Z_i} = h_i$. Then $(Z, h) \in P$ and is an upper bound for C.

Thus by Zorn's, P has a maximal element (W, f). Now we need to show that $W = X$, and we will be done.

Assume not. Fix $z \in X \setminus W$, and a real number $\alpha \in \mathbb{R}$. Define $f_1 : W_1 = W + \mathbb{R} \cdot z \to \mathbb{R}$ by

$$f_1(w + \lambda z) = f(w) + \lambda\alpha.$$

Then f_1 is linear, and $f_1|_W = f$. To be done, we need to choose α so that $f_1(w_1) \leq p(w_1)$ for all $w_1 \in W_1$.

Thus we need

$$\begin{aligned} & f(w) + \lambda\alpha \leq p(w + \lambda z) \\ \iff & f(w) + \alpha \leq p(w + z) \\ & f(w) - \alpha \leq p(w - z), \end{aligned}$$

for all $w \in W$. This means

$$f(x) - p(x-z) \leq \alpha \leq p(y+z) - f(y),$$

which is true if and only if

$$f(x) - p(x-z) \leq p(y+z) - f(y),$$

for all $x, y \in W$, by taking α to be the supremum of the left hand side as x ranges over W. But this is true as

$$f(x) + f(y) = f(x+y) \leq p(x+y) = p(x-z+y+z) \leq p(x-z) + p(y+z),$$

for all $x, y \in W$.

Definition 1.3. A *seminorm* on a real or complex vector space X is a functional $p : X \to \mathbb{R}$ such that:

(i) $p(x) \geq 0$, for all $x \in X$.

(ii) $p(\lambda x) = |\lambda| p(x)$, for all scalars λ, and for all $x \in X$.

(iii) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$.

This is the definition of the norm, without requiring $p(x) = 0 \implies x = 0$.

Of course, any seminorm is positive heterogeneous, and subadditive.

Theorem 1.3 (Hahn-Banach)**.** *Let X be a real or complex vector space, and p a seminorm on X. Let Y be a subspace of X, and g be a linear functional on Y such that $|g(y)| \leq p(y)$, for all $y \in Y$.*

Then there exists linear functional f on X, such that $f|_Y = g$, and $|f(x)| \leq p(x)$ for all $x \in X$.

Proof: We split into two cases, the real and the complex case.

In the real case, we have $g(y) \leq |g(y)| \leq p(y)$ for all $y \in Y$, so by the first version of Hahn-Banach, there exists a linear map $f : X \to \mathbb{R}$ such that $f|_Y = g$ and $f(x) \leq p(x)$.

We are almost done, except we need $|f(x)| \leq p(x)$. Here we use the fact that p is a seminorm, so

$$-f(x) = f(-x) \leq p(-x) = p(x).$$

Hence $|f(x)| \leq p(x)$.

Now we start with the complex case. Splitting into real and imaginary parts does not work, as f, g real linear does not imply $f + ig$ complex linear. To do this, we show the following claim:

Claim: For any real-linear $h_1 : X \to \mathbb{R}$, there is a unique complex linear $h : X \to \mathbb{C}$ such that $\Re(h) = h_1$.

We start with uniqueness. If $h_1 = \Re(h)$, then for $x \in X$,

$$\begin{aligned} h(x) &= h_1(x) + i\Im(h(x)) \\ &= -ih(ix) = -i(h_1(ix) + i\Im(h(ix))). \end{aligned}$$

So, $\Im(H(x)) = -h_1(ix)$, and thus

$$h(x) = h_1(x) - ih_1(ix).$$

For existence, we just check this h defined above works, and it does (clearly real-linear, just need to check multiplication by i is correct).

We return back to our proof. Let $g_1 = \Re(g) : Y \to \mathbb{R}$, which is real-linear. For $y \in Y$, note

$$|g_1(y)| \leq |g(y)| \leq p(y).$$

By the real case, there exists a real linear $f_1 : X \to \mathbb{R}$ such that $f_1|_Y = g_1$, and $|f_1(x)| \leq p(x)$ for all $x \in X$.

By the claim, $f_1 = \Re(f)$ for unique complex-linear functions $f : X \to \mathbb{C}$, and note

$$\Re(f|_Y) = f_1|_Y = g_1 = \Re(g).$$

Therefore by uniqueness, $f|_Y = g$. We are almost done apart form domination. Note that for $x \in X$, $|f(x)| = \lambda f(x)$, for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Then,

$$\begin{aligned} |f(x)| &= f(\lambda x) = f_1(\lambda x) + i\Im(f(\lambda x)) \\ &= f_1(\lambda x) \leq p(\lambda x) = |\lambda| p(x) = p(x). \end{aligned}$$

Remark. For a complex vector space X, let $X_{\mathbb{R}}$ be the real vector space obtained from X by restricting scalar multiplication to the reals.

If X is a complex normed space, then $f \mapsto \Re(f)$ on $(X^*)_{\mathbb{R}} \to (X_{\mathbb{R}})^*$ is an isometric isomorphism.

Corollary 1.1. *Let X be a real or complex vector space, and let p be a seminorm* *on X. Then for any $x_0 \in X$, there exists a linear functional f on X such that $f(x_0) = p(x_0)$, and $|f(x)| \leq p(x)$, for all $x \in X$.*

Proof: Let $Y = \operatorname{span}\{x_0\}$, and define g on Y be

$$g(\lambda x_0) = \lambda p(x_0).$$

Then g is linear on Y, and

$$|g(\lambda x_0)| = |\lambda| p(x_0) = p(\lambda x_0),$$

for all scalars λ. Thus by Hahn-Banach, there exists a linear functional f on X such that $f|_Y = g$, and $|f(x)| \leq p(x)$. So $f(x_0) = g(x_0) = p(x_0)$.

Theorem 1.4 (Hahn-Banach)**.** *Let X be a real or complex normed space.*

(i) *Given a subspace Y of X and $g \in Y^*$, here exists $f \in X^*$ such hat $f|_Y = g$, and $\|f\| = \|g\|$.*

(ii) *For $x_0 \in X \setminus \{0\}$, here exists $f \in S_{X^*}$ such that $f(x_0) = \|x_0\|$.*

Proof:

(i) Apply previous Hahn-Banach with $p(x) = \|g\|\|x\|$. Then for $y \in Y$,

$$|g(y)| \leq \|g\| \cdot \|y\| = p(y).$$

Hence there exists a linear functional f on X such that $f|_Y = g$, and

$$|f(x)| \leq p(x) = \|g\| \cdot \|x\|.$$

Therefore, $f \in X^*$, and $\|f\| = \|g\|$. Since f extends g, $\|f\| = \|g\|$.

(ii) Let $p = \|\cdot\|$. By the previous corollary, there exists a linear functional f on X such that $f(x_0) = \|x_0\|$, and $|f(x) \leq \|x\|$.

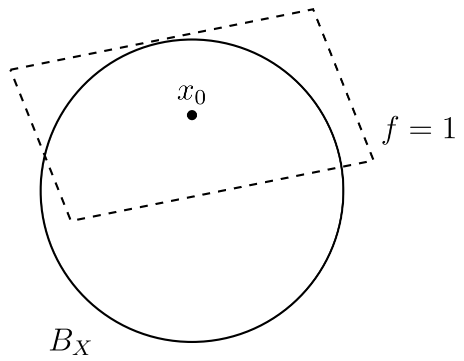
So $f \in X^*$, $\|f\| \leq 1$, but by equality at x_0, $\|f\| = 1$.

Remark.

1. We can think of this as a linear version of Tietze's extension theorem. Recall:

   If L is a closed subset of a compact Hausdorff space K and $g : L \to \mathbb{R}$ or $\mathbb{C}$ is continuous, then there exists continuous $f : K \to \mathbb{R}$ or $\mathbb{C}$ such that $f|_L = g$, and $\|f\|_\infty = \|g\|_\infty$.

2. Part (ii) implies that X^* separates points of X, i.e. if $x \neq y$ in X, then there exists $f \in X^*$ such that $f(x) \neq f(y)$, by taking $x_0 = x - y$.
3. The f in (ii) is called the *norming functional* at x_0. Therefore,
$$\|x_0\| = \max\{|g(x)| \mid g \in B_{X^*}\}.$$
Another name is the *support functional* at x_0. We can think of where $f = 1$ as the "tangent plane at x_0".



1.1 Bidual

Let X be a normed space. Then $X^{**} = (X^*)^*$ is the *bidual* or *second dual* of X.

For $x \in X$, define $\hat{x}$ on X^* by $f \mapsto f(x)$, i.e. evaluation at x.

Then $\hat{x}$ is linear, and
$$|\hat{x}(f)| = |f(x)| \leq \|f\|\|x\|,$$
for all $f \in X^*$. So $\hat{x} \in X^{**}$, and $\|\hat{x}\| \leq \|x\|$. The map $x \mapsto \hat{x}$ is the *canonical embedding* of X into X^{**}.

Theorem 1.5. *The canonical embedding is an isometric isomorphism of X into X^{**}.*

Proof: Linearity: note
$$\begin{aligned}\widehat{\lambda x + \mu y}(f) &= f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)\\ &= (\lambda \hat{x} + \mu \hat{y})(f).\end{aligned}$$

Isometric: for $x \in X$,
$$\|\hat{x}\| = \sup\{|f(x)| \mid f \in B_{X^*}\} = \|x\|,$$
by Hahn-Banach.

Remark.

1. Note that
$$\langle f, \hat{x} \rangle = \langle x, f \rangle,$$
for $x \in X$, $f \in X^*$.
2. $\hat{X} = \{\hat{x} \mid x \in X\} \cong X$. Therefore,
$$\hat{X} \text{ is closed in } X^{**} \iff X \text{ is complete.}$$
3. In general, the closure in X^{**} of $\hat{X}$ is a Banach space containing an isometric copy of X as a dense subspace.

Definition 1.4. A normed space X is *reflexive* if the canonical embedding $X \to X^{**}$ is surjective.

Example 1.2.

1. Any finite-dimensional space is reflexive.
2. ℓ_p for $1 < p < \infty$ is reflexive.
3. Any Hilbert space is reflexive.
4. $L_p(\mu)$ for $1 < p < \infty$ is reflexive.
5. $c_0, \ell_1, \ell_\infty, L_1([0,1])$ are not reflexive.

Remark. If X is reflexive, then X is a Banach space, and $X \cong X^{**}$.

However, there exists a Banach space X such that $X \cong X^{**}$, but X is not reflexive. So even though $\ell_p^{**} \cong \ell_q^* \cong \ell_p$, this is not enough to show ℓ_p is reflexive.

1.2 Dual Operators

Let X, Y be normed spaces. Then,
$$\mathcal{B}(X, Y) = \{T : X \to Y \mid T \text{ linear, bounded}\}.$$
Recall that $\mathcal{B}(X, Y)$ is a normed space with the operator norm:
$$\|T\| = \sup\{\|Tx\| \mid x \in B_X\}.$$
If Y is complete, then $\mathcal{B}(X, Y)$ is complete.

For $T \in \mathcal{B}(X, Y)$, its *dual operator* $T^* : Y^* \to X^*$ is given by
$$T^*(g) = g \circ T.$$

This is well-defined, and in the bracket notation

$$\langle x, T^* g\rangle = \langle Tx, g\rangle.$$

It is easy to see that T^* is linear, and moreover it is bounded. Note

$$\begin{aligned}\|T^*\| &= \sup_{g\in B_{Y^*}} \|T^* g\| = \sup_{g\in B_{Y^*}} \sup_{x\in B_X} |\langle x, T^* g\rangle| = \sup_{x\in B_X} \sup_{g\in B_{Y^*}} |\langle Tx, g\rangle| \\ &\overset{HB}{=} \sup_{x\in B_x} \|Tx\| = \|T\|.\end{aligned}$$

Remark. If X, Y are Hilbert spaces, and we identify X, Y with X^*, Y^* respectively, then T^* becomes the *adjoint* of T.

Example 1.3.

If $1 \le p < \infty$, and $R : \ell_p \to \ell_p$ is the right-shift, then $R^* : \ell_q \to \ell_q$ is the left-shift.

We have the following properties:

- $(\mathrm{id}_X)^* = \mathrm{id}_{X^*}$.
- $(\lambda S + \mu T)^* = \lambda S^* + \mu T^*$.
- $(ST)^* = T^* S^*$.
- $T \mapsto T^* : \mathcal{B}(X, Y) \to \mathcal{B}(Y^*, X^*)$ is an into isometric isomorphism.
- The following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d] & Y \arrow[d] \\ X^{**} \arrow[r, "T^{**}"] & Y^{**} \end{tikzcd}$$

  In other words $\widehat{Tx} = T^{**}\hat{x}$, for all $x \in X$.

Indeed, for all $x \in X$, $g \in Y^*$,

$$\begin{aligned}\langle g, T^{**}\hat{x}\rangle &= \langle T^* g, \hat{x}\rangle = \langle x, T^* g\rangle \\ &= \langle Tx, g\rangle = \langle g, \widehat{Tx}\rangle.\end{aligned}$$

1.3 Quotient spaces

Let X be a NVS and Y be a closed subspace. Then X/Y is a normed space in the *quotient norm*:

$$\|x+Y\| = \inf\{\|x+y\| \mid y \in Y\} = d(x,Y).$$

Here closed is important, so that $\|x+Y\| = 0 \implies x \in Y$.

The quotient map $q : X \to X/Y$ is linear, surjective and bounded with $\|q\| = 1$, since for $x \in X$

$$\|q(x)\| \leq \|x\|.$$

Letting D_X be the open unit ball of X, we can show $q(D_X) = D_{X/Y}$. Indeed if $x \in D_X$, then $\|q(x)\| \leq \|x\| < 1$. If $\|x+Y\| < 1$, then there exists $y \in Y$ with $\|x+y\| < 1$. So $x+y \in D_X$ and $q(x+y) = x+Y$.

So $\|q\| = 1$, unless $Y = X$. Also, q is an open map.

Assume $T : X \to Z$ is a bounded linear map, and $Y \subseteq \ker T$. Then there exists a unique map $\tilde{T} : X/Y \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "q"'] & & Z \\ & X/Y \arrow[ur, "\tilde{T}"'] & \end{tikzcd}$$

Moreover, $\tilde{T}$ is linear and bounded, and $\|\tilde{T}\| = \|T\|$, since

$$\tilde{T}(D_{X/Y}) = \tilde{T}(q(D_X)) = T(D_X).$$

Theorem 1.6. *Let X be a normed space. If X^* is separable, then so is X.*

Remark. The converse is false in general, by taking $X = \ell_1$, then $X^* = \ell_\infty$.

Proof: Since X^* is separable, so is S_{X^*}. Let (f_n) be a dense sequence in S_{X^*}. For all $n \in \mathbb{N}$, choose $x_n \in B_X$ such that $|f_n(x_n)| > 1/2$.

Set $Y = \overline{\text{span}}\{x_n \mid n \in \mathbb{N}\}$, the closed linear span of x_n Then we claim $Y = X$.

Assume not. Then we first find $f \in S_{X^*}$ such that $f|_Y = 0$. Since $X/Y \neq \{0\}$, we have $(X/Y)^* \neq \{0\}$, by Hahn-Banach. Choose any $g \in S_{(X/Y)^*}$.

Let $f = g \circ q$. Then $\|f\| = \|g\| = 1$, so $f \in S_{X^*}$, and $f|_Y = 0$.

Choose $n \in \mathbb{N}$ such that $\|f - f_n\| < 1/10$. Now,

$$\frac{1}{2} < |f_n(x_n)| = |(f_n - f)(x_n)| \leq \|f_n - f\| \cdot \|x_n\| < \frac{1}{10},$$

a contradiction.

Theorem 1.7. *Let X be a separable normed space. Then X is isometrically isomorphic to a subspace of ℓ_∞.*

Consider a map $T : X \to \ell_\infty$. The n'th coordinate is then a linear function of x, that is bounded, hence is a functional. So we can think of

$$Tx = (f_n(x)).$$

We also want $\|Tx\|_\infty = \|x\|$, which we can do by choosing a norming functional (or an appropriate approximate).

Proof: Let (x_n) be a dense sequence in X. For each $n \in \mathbb{N}$, choose $f_n \in S_{X^*}$ such that $f_n(x_n) = \|x_n\|$.

Define $T : X \to \ell_\infty$ by

$$T(x) = (f_1(x), f_2(x), \ldots).$$

Note that $|f_n(x)| \leq \|x\|$, so T is well-defined, linear and bounded with norm at most 1.

But for each n,

$$\|Tx_n\|_\infty \geq |f_n(x_n)| = \|x_n\|,$$

so $\|Tx_n\|_\infty = \|x_n\|$. Since (x_n) is dense, and continuity of T, we have $\|Tx\| = \|x\|$ for all $x \in X$.

Remark. We say that ℓ_∞ is *isometrically universal* for the class $\mathcal{SB}$ of all separable Banach spaces.

Theorem 1.8 (Vector-valued Liouville's Theorem)**.** *Let X be a complex Banach space, and $f : \mathbb{C} \to X$ bounded and holomorphic. Then f is constant.*

Proof: Since f is bounded, there is $M \in \mathbb{R}$ such that for all $z \in \mathbb{C}$, $\|f(z)\| \leq M$.

f is holomorphic means that

$$\lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

exists, and is denoted by $f'(z)$, for all $z \in \mathbb{C}$.

Fix $\phi \in X^*$. Since ϕ is linear and continuous,

$$\lim_{w \to z} \frac{\phi(f(w)) - \phi(f(z))}{w - z} = \phi\left(\lim_{w \to z} \frac{f(w) - f(z)}{w - z}\right).$$

So $\phi \circ f : \mathbb{C} \to \mathbb{C}$ is entire.

Also, for all $z \in \mathbb{C}$, $|\phi(f(z))| \leq \|\phi\| \cdot \|f(z)\| \leq M\|\phi\|$. So by Liouville, $\phi \circ f$ is constant, hence $\phi(f(z)) = \phi(f(0))$ for all $z \in \mathbb{C}$.

Fix $z \in \mathbb{C}$. Since X^* separates the points of X, $f(z) = f(0)$.

1.4 Locally Convex Spaces

Definition 1.5. A *locally convex space* (LCS) is a pair $(X, \mathcal{P})$ where X is a real or complex vector space, and $\mathcal{P}$ is a family of seminorms on X such that $\mathcal{P}$ separates the points of X, i.e. for all $x \in X \setminus \{0\}$, there exists $p \in \mathcal{P}$ with $p(x) \neq 0$.

The family $\mathcal{P}$ defines a topology on X as follows: $U \subseteq X$ is open if and only if, for all $x \in U$, there are seminorms $p_1, \ldots, p_n \in \mathcal{P}$ and $\varepsilon > 0$ such that

$$\{y \in X \mid p_k(y - x) < \varepsilon \text{ for } k = 1, \ldots, n\} \subseteq U.$$

So the open balls form a base of the topology.

Remark.

1. Addition and scalar multiplication are continuous.
2. This is Hausdorff, as $\mathcal{P}$ separates the points.
3. $x_n \to x$ in X if and only if $p(x_n - x) \to 0$ for all $p \in \mathcal{P}$.
4. Let Y be a subspace of X. Let $\mathcal{P}_Y = \{p|_Y \mid p \in \mathcal{P}\}$. Then $(Y, \mathcal{P}_Y)$ is a LCS, and the topology of $(Y, \mathcal{P}_Y)$ is the subspace topology induced by the topology of the LCS $(X, \mathcal{P})$.
5. Let $\mathcal{P}, \mathcal{Q}$ be two families ofseminorms on X, both separating points of X. Say $\mathcal{P}, \mathcal{Q}$ are *equivalent*, and we write $P \sim Q$, if they generate the same topology on X.

The topology of a LCS $(X, \mathcal{P})$ is metrizable if and only if there is a countable $Q \sim P$.

Definition 1.6. A *Fréchet space* is a complete metrizable LCS.

Example 1.4.

1. A normed space $(X, \|\cdot\|)$ is a LCS with $\mathcal{P} = \{\|\cdot\|\}$.
2. Let $U \subseteq \mathbb{C}$ be a non-empty open set, and

$$\mathcal{O}(U) = \{f : U \to \mathbb{C} \mid f \text{ holomorphic}\}.$$

For $K \subseteq U$, K compact, let

$$p_K(f) = \sup_{z \in K} |f(z)|,$$

for $f \in \mathcal{O}(U)$. Let $\mathcal{P} = \{p_K \mid K \subseteq U, K \text{ compact}\}$. Then $(\mathcal{O}(U), \mathcal{P})$ is a LCS. The topology is the topology of local uniform convergence.

Note that there exists (K_n) of compact subsets of U such that $K_n \subseteq \text{int}K_{n+1}$ for all n, and $\bigcup K_n = U$, and

$$\{p_{K_n} \mid n \in \mathbb{N}\} \sim \mathcal{P}.$$

So $(\mathcal{O}(U), \mathcal{P})$ is metrizable, and in fact a Fréchet space. This topology is not normable, i.e. there is no norm on $\mathcal{O}(U)$ inducing the same topology (can use Montel's theorem).

3. Take $d \in \mathbb{N}$, and $\Omega \subseteq \mathbb{R}^d$ non-empty and open. Take

$$C^\infty(\Omega) = \{f : \Omega \to \mathbb{R} \mid f \text{ infinitely differentiable}\}.$$

For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$, we have a differential operator D^α given by

$$D^\alpha f = \left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x^n}\right)^{\alpha_n}.$$

For $\alpha \in (\mathbb{Z}_{\geq 0})^d$, $K \subseteq \Omega$ compact, define

$$p_{K,\alpha}(f) = \sup\{|(D^\alpha)f(x)| \mid x \in K\}.$$

Let $\mathcal{P} = \{p_{K,\alpha} \mid \alpha \text{ multiindex}, K \text{ compact}\}$. Then $(C^\infty(\Omega), \mathcal{P})$ is a LCS, which is a Fréchet space that is not normable.

Lemma 1.1. *Let $(X, \mathcal{P})$ and $(Y, \mathcal{Q})$ be LCS, and $T : X \to Y$ a linear map. Then the following are equivalent:*

(i) *T is continuous.*

(ii) *T is continuous at 0.*

(iii) *For all $q \in \mathcal{Q}$, there are seminorms $p_1, \dots, p_n \in \mathcal{P}$ and $C \geq 0$ such that for all x,*

$$q(Tx) \leq C \max_{1 \leq k \leq n} p_k(x).$$

Proof: It is easy to see (i) $\iff$ (ii), since translations are a homeomorphism.

We show (ii) $\implies$ (iii). Let $q \in \mathcal{Q}$, and $V = \{y \in Y \mid q(y) < 1\}$ a neighbourhood of 0 in Y. As T is continuous at 0, there exists a neighbourhood of 0 in X such that $T(U) \subseteq V$. Without loss of generality,

$$U = \{x \in X \mid p_k(X) \leq \varepsilon, k = 1, \dots, n\}$$

for some $n \in \mathbb{N}$, and $p_1, \dots, p_n \in \mathcal{P}$, $\varepsilon > 0$.

Let $p(x) = \max_{1 \leq k \leq n} p_k(x)$. We show that $q(Tx) \leq \frac{1}{\varepsilon} p(x)$ for all $x \in X$. Let $x \in X$. If $p(x) \neq 0$, then

$$p\left(\frac{\varepsilon x}{p(x)}\right) = \varepsilon,$$

so

$$\frac{\varepsilon x}{p(x)} \in U \implies T\left(\frac{\varepsilon x}{p(x)}\right) \in V.$$

Therefore,

$$q\left(T\left(\frac{\varepsilon x}{p(x)}\right)\right) < 1 \implies q(Tx) \leq \frac{1}{\varepsilon} p(x).$$

If $p(x) = 0$, then $\lambda x \in U$ for all scalars λ, hence $q(T(\lambda x)) < 1$ for all λ. So $q(Tx) = 0$.

Now we show (iii) $\implies$ (ii). Let V be an open neighbourhood of 0 in Y. We seek a neighbourhood U of 0 in X such that $T(U) \subseteq V$. Without loss of generality,

$$V = \{y \in Y \mid q_k(y) < \varepsilon, k = 1, \dots, m\}.$$

For each $k = 1, \dots, m$, there exist seminorms $p_{k,1}, \dots, p_{k,n_k} \in \mathcal{P}$ and $C_k > 0$ such that for all $x \in X$,

$$q_k(Tx) \leq C_k \max_{1 \leq j \leq n_k} p_{k,j}(x).$$

Then,

$$U = \{x \in X \mid p_{k,j}(x) \leq \frac{\varepsilon}{C_k}, k = 1, \dots, m, j = 1, \dots, n_k\}$$

is a neighbourhood of 0 in X, and for each $x \in U$,

$$q_k(Tx) \leq C_k \max_{1 \leq j \leq n_k} p_{k,j}(x) < \varepsilon$$

for each $k = 1, \dots, m$, so $Tx \in V$,

Definition 1.7. The *dual space* of a LCS $(X, \mathcal{P})$ is the space X^* of all linear functional of X which are continuous with respect to the topology of X.

Lemma 1.2. *Let f be a linear functional on a LCS X. Then,*

$$f \in X^* \iff \ker f \text{ is closed.}$$

Proof: One way is obvious: if f is continuous, then $\ker f = f^{-1}(\{0\})$ must be closed.

Now consider the other direction. We can assume without loss of generality that $f \neq 0$. Fix $x_0 \in X \setminus \ker f$. Since $\ker f$ is closed, there is a neighbourhood U of 0 in X, such that $x_0 + U$ is disjoint from $\ker f$.

Without loss of generality,

$$U = \{x \in X \mid p_k(x) < \varepsilon, k = 1, \dots, n\}$$

for seminorms $p_1, \dots, p_n \in \mathcal{P}$.

Note that U is convex and *balanced* (if $x \in U$, $|\lambda| = 1$ a scalar then $\lambda x \in U$) since p_i are seminorms.

As f is linear, $f(U)$ is also convex and balanced. Hence it is an interval or a disc.

But since $-f(x_0) \notin f(U)$, otherwise $0 \in f(x_0 + U)$, $f(U)$ is bounded. Hence

$$f(U) \subseteq \{\lambda \text{ a scalar} \mid |\lambda| < M\}.$$

Hence for any $\delta > 0$,

$$f\left(\frac{\delta}{M} U\right) \subseteq \{\lambda \text{ a scalar} \mid |\lambda| < \delta\},$$

and $\frac{\delta}{M} U$ is a neighbourhood of 0. Thus f is continuous at 0.

Theorem 1.9 (Hahn-Banach)**.** *Let $(X, \mathcal{P})$ be a LCS.*

(i) *If Y is a subspace of X and $g \in Y^*$, then there exists $f \in X^*$ such that $f|_Y = g$.*

(ii) *If Y is a closed subspace of X and $x_0 \in X \setminus Y$, then there exists $f \in X^*$ such that $f|_Y = 0$, and $f(x_0) \neq 0$.*

Proof:

(i) By lemma 1.1, there exists $p_1, \ldots, p_n \in \mathcal{P}$, and $C \geq 0$ such that for all $y \in Y$,

$$|g(y)| \leq C \max_{1 \leq k \leq n} p_k(y).$$

Define $p : X \to \mathbb{R}$ by

$$p(x) = C \max_{1 \leq k \leq n} p_k(x).$$

Then p is a seminorm on X, and on Y $|g(y)| \leq p(y)$ for all $y \in Y$.

By Hahn-Banach on seminorms, there exists a linear functional f on X such that $f|_Y = g$ and for all $x \in X$, $|f(x)| \leq p(x)$. Lemma 1.1 gives us that f is continuous.

(ii) Let $Z = \operatorname{span}(Y \cup \{x_0\})$. Define a linear functional g on Z by

$$g(y + \lambda x_0) = \lambda$$

for $y \in Y$, λ a scalar. Notice that $\ker g = Y$ is closed by supposition, so g is continuous, i.e. $g \in Z^*$. Then applying (i), we find $f \in X^*$ satisfying $f|_Z = g$, so in particular $f|_Y = 0$ and $f(x_0) = g(x_0) = 1$.

Remark. X^* separates the points of X: given $x \neq y$, apply (ii) to $Y = \{0\}$, and $x_0 = x - y$.

2 Dual Spaces of $L_p(\mu)$ and $C(K)$

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $1 \leq p < \infty$. Recall

$$L_p(\mu) = \left\{ f : \Omega \to \text{scalars} \;\middle|\; f \text{ measurable}, \int_\Omega |f|^p \,\mathrm{d}\mu < \infty \right\}.$$

This is a normed space in the L_p-norm,

$$\|f\|_p = \left(\int_\Omega |f|^p \,\mathrm{d}\mu \right)^{1/p}.$$

We identify functions f, g if $f = g$ almost everywhere. If $p = \infty$, then

$$L_\infty(\mu) = \{ f : \Omega \to \text{scalars} \mid f \text{ measurable, essentially bounded} \}.$$

Essentially bounded means f is bounded, up to a null set. This is a normed space in the L_∞ norm:

$$\|f\|_\infty = \operatorname{ess\,sup} |f| = \inf\{ \sup_{\Omega \setminus N} |f| \mid N \in \mathcal{F}, \mu(N) = 0 \}.$$

The infimum can be attained by taking N_i that limit to the infimum, and then taking their union.

Remark. If $\|\cdot\|$ is a seminorm on a vector space X, then

$$N = \{ x \in X \mid \|x\| = 0 \}$$

is a subspace of X, and $\|x + N\| = \|x\|$ defines a norm on the quotient.

We will not think like this for L_p.

Theorem 2.1. *$L_p(\mu)$ is a Banach space for $1 \leq p \leq \infty$.*

Our aim is to describe $L_p(\mu)^*$.

2.1 Complex Measures

Let Ω be a set, and $\mathcal{F}$ be a σ-algebra on Ω. A *complex measure* on $\mathcal{F}$ is a countably additive set function $\nu : \mathcal{F} \to \mathbb{C}$.

The *total variation measure* of ν, denoted by $|\nu|$, is defined as follows: for $A \in \mathcal{F}$,

$$|\nu|(A) = \sup \left\{ \sum_{k=1}^n |\nu(A_k)| \;\middle|\; A = \bigcup_{k=1}^n A_k \text{ is a measurable partition of } A \right\}.$$

Then $|\nu| : \mathcal{F} \to [0, \infty]$ is a positive measure, and is the smallest measure such that for all $A \in \mathcal{F}$,

$$|\nu(A)| \leq |\nu|(A).$$

In other words, if μ is a positive measure on $\mathcal{F}$ and for all $A \in \mathcal{F}$, $|\nu(A)| \leq \mu(A)$, then $|\nu|(A) \leq \mu(A)$.

The *total variation* of ν is

$$\|\nu\|_1 = |\nu|(\Omega).$$

As currently defined this could be infinite, but we will see that this is always finite.

ν satisfies the two continuity conditions:

- If $A_n \subseteq A_{n+1}$, then
$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \nu(A_n).$$
- If $A_n \supseteq A_{n+1}$, then
$$\nu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \nu(A_n).$$

Signed measures are complex measures that take real values, i.e. countably additive set functions $\mathcal{F} \to \mathbb{R}$.

Theorem 2.2. *Let $(\Omega, \mathcal{F})$ be as before, and ν a signed measure on $\mathcal{F}$.*

Then there exists a measurable partition $\Omega = P \cup N$ of Ω such that for all $A \in \mathcal{F}$ and $A \subseteq P$, then $\nu(A) \geq 0$, and if $A \subseteq N$ then $\nu(A) \leq 0$.

Remark.

1. $\Omega = P \cup N$ is the *Hahn decomposition* of Ω (or of ν).
2. Let $\nu^+(A) = \nu(A \cap P)$ and $\nu^-(A) = -\nu(A \cap N)$ for $A \in \mathcal{F}$.

   Then ν^+, ν^- are finite positive measures such that $\nu = \nu^+ - \nu^-$, and $|\nu| = \nu^+ + \nu^-$.

   These properties determine ν^+ and ν^- uniquely. This decomposition $\nu = \nu^+ - \nu^-$ is the *Jordan decomposition* of ν.
3. Let ν be a complex measure. Then $\Re(\nu)$ and $\Im(\nu)$ are signed measures with Jordan decompositions $\nu_1 - \nu_2$ and $\nu_3 - \nu_4$. Then,

$$\nu = \nu_1 - \nu_2 + i(\nu_3 - \nu_4).$$

This is the *Jordan decomposition* of ν. Note that $\nu_k \le |\nu|$, and

$$|\nu| \le \nu_1 + \nu_2 + \nu_3 + \nu_4.$$

So $|\nu|$ is a finite measure since $\nu_1, \nu_2, \nu_3, \nu_4$ are all finite, so $\|\nu\|_1 < \infty$.

4. Suppose the signed measure ν has Hahn decomposition $\Omega = P \cup N$ and Jordan decomposition $\nu^+ - \nu^-$. For $A, B \in \mathcal{F}$ with $B \subseteq A$,

$$\nu^+(A) \ge \nu^+(B) \ge \nu(B),$$

and $\nu^+(A) = \nu(B)$ if $B = P \cap A$. So,

$$\nu^+(A) = \sup\{\nu(B) \mid B \in \mathcal{F}, B \subseteq A\}.$$

Proof: This is a non-examinable sketch.

Define

$$\nu^+(A) = \sup\{\nu(B) \mid B \in \mathcal{F}, B \subseteq A\} \ge 0,$$

since we may always take $B = \emptyset$. It is clear that $\nu^+(\emptyset) = 0$, and ν^+ is finitely additive.

The main claim is that $\nu^+(\Omega) < \infty$. Assume not. Inductively construct $(A_n), (B_n)$ in $\mathcal{F}$ such that $A_0 = \Omega$, and if $\nu^+(A_{n-1}) = \infty$, pick $B_n \subseteq A_{n-1}$, with $\nu(B_n) > n$.

Then pick either $A_n = B_n$ or $A_{n-1} \setminus B_n$ such that $\nu^+(A_n) = \infty$.

We can then use continuity of ν to get a contradiction, by condition on whether $A_n = B_n$ eventually, or $A_n = A_{n-1} \setminus B_n$ infinitely often.

The next claim is that the supremum is achieved, so there exists $P \in \mathcal{F}$ such that

$$\nu^+(\Omega) = \nu(P).$$

Choose $A_n \in \mathcal{F}$, with $\nu(A_n) > \nu^+(\Omega) - 2^{-n}$, and we can check

$$P = \bigcup_m \bigcap_{n \ge m} A_n$$

works. Then letting $N = \Omega \setminus P$, we can check this works as a partition.

Definition 2.1. Fix a measure space $(\Omega, \mathcal{F}, \mu)$. A complex measure $\nu : \mathcal{F} \to \mathbb{C}$ is *absolutely continuous* with respect to μ, written $\nu \ll \mu$, if for all $A \in \mathcal{F}$,

$$\mu(A) = 0 \implies \nu(A) = 0.$$

Remark.

1. If $\nu \ll \mu$, then $|\nu| \ll \mu$. It follows that if $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$ is the Jordan decomposition of ν_1, then
$$\nu \ll \mu \iff \nu_k \ll \mu$$
for all k (note that ν_1, ν_2 are non-zero on different subsets of $\mathcal{F}$).
2. If $\nu \ll \mu$, then for all $\varepsilon > 0$, there exists $\delta > 0$ such that for any $A \in \mathcal{F}$,
$$\mu(A) < \delta \implies |\nu(A)| < \varepsilon.$$

Example 2.1.

If $f \in L_1(\mu)$, then
$$\nu(A) = \int_A f \, \mathrm{d}\mu,$$
for $A \in \mathcal{F}$, defines a complex measure on $\mathcal{F}$ (by dominated convergence), and $\nu \ll \mu$.

Definition 2.2. A set $A \in \mathcal{F}$ is σ-finite with respect to μ if there exists (A_n) in $\mathcal{F}$ such that
$$A = \bigcup_{n \in \mathbb{N}} A_n, \qquad \mu(A_n) < \infty.$$
We say that μ is σ-finite if Ω is a σ-finite set (so every $A \in \mathcal{F}$ is σ-finite).

Theorem 2.3 (Radon-Nikodym Theorem)**.** *Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure and $\nu : \mathcal{F} \to \mathbb{C}$ be a complex measure such that $\nu \ll \mu$.*

Then there exists a unique $f \in L_1(\mu)$ such that
$$\nu(A) = \int_A f \, \mathrm{d}\mu,$$
for all $A \in \mathcal{F}$. Moreover f takes values in $\mathbb{C}$ or $\mathbb{R}$ or $\mathbb{R}^+$ depending on whether ν is a complex/signed/positive measure.

Proof: This is a non-examinable sketch.

First we show uniqueness. This follows as if $f \in L_1(\mu)$ and $\int_A f \, \mathrm{d}\mu = 0$, then $f = 0$ almost everywhere.

For existence, first assume ν is a finite positive measure, by taking the Jordan decomposition.

We can also assume μ is finite: for each partition A_i we have function f_i, which we can glue together; this extends by monotone convergence, since we first assumed ν is finite and positive.

Now let

$$\mathcal{H} = \left\{ h : \Omega \to \mathbb{R}^+ \;\middle|\; \int_A h \,\mathrm{d}\mu \le \nu(A) \text{ for all } A \in \mathcal{F} \right\}.$$

Note $0 \in \mathcal{H}$, $h_1, h_2 \in \mathcal{H} \implies h_1 \vee H_2 \in \mathcal{H}$, and if $h_n \in \mathcal{H}$, then $h_n \uparrow h \implies h \in \mathcal{H}$. Let

$$\mathcal{L} = \sup \left\{ \int_\Omega h \,\mathrm{d}\mu \;\middle|\; h \in \mathcal{H} \right\}.$$

This sup is attained (by monotone convergence). Hence there exists $f \in \mathcal{H}$ which attains $\mathcal{L}$. We show that

$$\int_A f \,\mathrm{d}\mu = \nu(A),$$

for all $A \in \mathcal{F}$. The idea is that if there exists A with

$$\int_A f \,\mathrm{d}\mu < \nu(A),$$

then $f + \delta \mathbb{1}_A$ should be in $\mathcal{H}$ for some $\delta > 0$, contradicting the maximality. However this doesn't quite work as we may fail the condition for $B \subseteq A$.

For $n \in \mathbb{N}$, define

$$\nu_n(A) = \nu(A) - \int_A f \,\mathrm{d}\mu - \frac{1}{n}\mu(A) = \nu(A) - \int_A \left(f + \frac{1}{n} \right) \mathrm{d}\mu,$$

for all $A \in \mathcal{F}$. Now ν_n is a signed measure, so we get a Hahn decomposition

$$\Omega = P_n \cup N_n.$$

Then, $f + \frac{1}{n}\mathbb{1}_{P_n} \in \mathcal{H}$, so $\mu(P_n) = 0$ to not contradict maximality.

Let $P = \bigcup P_n$. Then $\mu(P) = 0$, so $\nu(P) = 0$ by absolute continuity.

Set $N = \bigcap N_n$. Then,

$$\begin{aligned} \nu(A) = \nu(A \cap N) &= \nu_n(A \cap N) + \int_{A \cap N} f \,\mathrm{d}\mu + \frac{1}{n}\mu(A \cap N) \\ &\le \int_A f \,\mathrm{d}\mu + \frac{1}{n}\mu(A \cap N). \end{aligned}$$

Then we let $n \to \infty$.

Remark.

1. The proof shows that every complex measure $\nu : \mathcal{F} \to \mathbb{C}$ has a decomposition $\nu = \nu_1 + \nu_2$, where $\nu_1 \ll \mu$ and $\nu_2 \perp \mu$. This is the *Lebesgue decomposition* of ν.
2. The unique $f \in L_1(\mu)$ in the Radon-Nikodym theorem is the *Radon-Nikodym derivative* of ν with respect to μ, denoted by $\mathrm{d}\nu / \mathrm{d}\mu$. For measurable g, then g is ν-integrable if and only if $g \cdot \mathrm{d}\nu / \mathrm{d}\mu$ is μ-integrable, and
$$\int_\Omega g \,\mathrm{d}\nu = \int_\Omega g \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \,\mathrm{d}\mu.$$

2.2 Duals of L_p

Fix a measure space $(\Omega, \mathcal{F}, \mu)$, and let $1 \leq p < \infty$. Let q be the conjugate index of p, and for $g \in L_q = L_q(\mu)$, define ϕ_g on L_p by
$$\phi_g(f) = \int_\Omega fg \,\mathrm{d}\mu.$$
By Hölder's, $fg \in L_1$, and
$$|\phi_g(f)| \leq \|f\|_p \|g\|_q.$$
So $\phi_g \in L_p^*$, and $\|\phi_g\| \leq \|g\|_q$. So $\phi : L_q \to L_p^*$ exists, given by $g \mapsto \phi_g$. This is linear and bounded, with $\|\phi\| \leq 1$.

Theorem 2.4. *Let $(\Omega, \mathcal{F}, \mu)$, and p, q, ϕ be as above.*

(i) *If $1 < p < \infty$, then ϕ is an isometric isomorphic, so $L_p^* \cong L_q$.*

(ii) *If $p = 1$ and μ is σ-finite, then $L_1^* \cong L_\infty$.*

Proof: What remains is to check that ϕ is isometric and onto. Fix $g \in L_q$. We need to check that $\|\phi_g\| = \|g\|_q$.

Let $\lambda : \Omega \to$ scalars be measurable, with $|\lambda| = 1$ and $\lambda \cdot g = |g|$, i.e. let $\lambda = \mathrm{sign}(g)$.

For $1 < p < \infty$, let $f = \lambda |g|^{q-1}$. Then,
$$\int_\Omega |f|^p \,\mathrm{d}\mu = \int_\Omega |g|^{pq-p} \,\mathrm{d}\mu = \int_\Omega |g|^q \,\mathrm{d}\mu < \infty,$$
so $f \in L_p$, and
$$\|f\|_p = \|g\|_q^{q/p} = \|g\|_q^{q-1}.$$

Then notice

$$\phi_g(f) = \int \lambda g|g|^{q-1}\,\mathrm{d}\mu = \|g\|_q^q = \|f\|_p \cdot \|g\|_q,$$

so $\|\phi_g\| \geq \|g\|_q$.

For $p = 1$, let $s < \|g\|_\infty$. Then $\mu(\{|g| > s\}) > 0$. Since $\mathcal{F}$ is σ-finite, there exists measurable $A \subseteq \{|g| > s\}$, such that $0 < \mu(A) < \infty$. Then $f = \lambda \mathbb{1}_A \in L_1$, and $\|f\|_1 = \mu(A)$. Now,

$$\|\phi_g\| \cdot \mu(A) \geq \phi_g(f) = \int_A |g|\,\mathrm{d}\mu \geq s\mu(A).$$

So $\|\phi_g\| \geq s$, and so $\|\phi_g\| \geq \|g\|_\infty$.

The hard part is showing ϕ is onto. Fix $\psi \in L_p^*$. We seek $g \in L_q$ such that $\psi = \phi_g$.

The idea is as follows: let $\psi(\mathbb{1}_A) = \int_A g\,\mathrm{d}\mu$. Then we can define $\nu(A) = \psi(\mathbb{1}_A)$ for $A \in \mathcal{F}$, with $\nu \ll \mu$, and apply Radon-Nikodym. But we have to split into cases to make this work.

First, consider when μ is finite. For $A \in \mathcal{F}$, $\mathbb{1}_A \in L_p$, so we can define $\nu(A) = \psi(\mathbb{1}_A)$. This is a measure, as if $A = \bigcup A_n$ is a measurable partition, then

$$\sum_{n=1}^N \mathbb{1}_{A_n} \to \mathbb{1}_A$$

in L_p, by DCT. So,

$$\sum_{n=1}^N \nu(A_n) = \psi\left(\sum_{n=1}^N \mathbb{1}_{A_n}\right) \to \psi(\mathbb{1}_A) = \nu(A).$$

So ν is a complex/signed measure. If $\mu(A) = 0$, then $\mathbb{1}_A = 0$ almost everywhere, so $\nu(A) = \psi(\mathbb{1}_A) = 0$. Thus $\nu \ll \mu$. Hence by Radon-Nikodym, there exists $g \in L_1(\mu)$ such that

$$\nu(A) = \int_A g\,\mathrm{d}\mu,$$

for all $A \in \mathcal{F}$. We show that $g \in L_q(\mu)$ and $\psi = \phi_g$, i.e.

$$\psi(f) = \int_\Omega fg\,\mathrm{d}\mu$$

for all $f \in L_p$. We have

$$\psi(\mathbb{1}_A) = \nu(A) = \int_A g \,\mathrm{d}\mu = \int_\Omega \mathbb{1}_A g \,\mathrm{d}\mu,$$

hence

$$\psi(f) = \int_\Omega fg \,\mathrm{d}\mu$$

for all simple functions f. Given $f \in L_\infty$, there is a sequence (f_n) of simple functions such that $f_n \to f$ in L_∞. Then $f_n g \to fg$ in L_1 by dominated convergence, and $f_n \to f$ in L_p, as μ is finite. Thus

$$\psi(f) = \lim_{n\to\infty} \psi(f_n) = \lim_{n\to\infty} \int_\Omega f_n g \,\mathrm{d}\mu = \int_\Omega fg \,\mathrm{d}\mu.$$

Next we deduce that $g \in L_q$. Fix a measurable function λ such that $|\lambda| = 1$ and $\lambda g = |g|$.

Split into cases. For $p \neq 1$, let $A_n = \{|g| \leq n\}$. Then $f = \lambda \mathbb{1}_{A_n} |g|^{q-1} \in L_\infty$, and

$$\int_{A_n} |g|^q \,\mathrm{d}\mu = \int_\Omega fg \,\mathrm{d}qm = \psi(f) \leq \|\psi\| \cdot \|f\| = \|\psi\| \left(\int_{A_n} |g|^q \,\mathrm{d}\mu \right)^{1/p},$$

so

$$\left(\int_{A_n} |g|^q \,\mathrm{d}\mu \right)^{1/q} \leq \|\psi\|.$$

Let $n \to \infty$, and use monotone convergence to get $g \in L_q$.

For $p = 1$, fix $s > \|\psi\|$ and let $A = \{|g| > s\}$. Then $f = \lambda \mathbb{1}_A \in L_\infty$, so

$$s\mu(A) = \int_A |g| \,\mathrm{d}\mu = \int_\Omega fg \,\mathrm{d}\mu = \psi(f) \leq \|\psi\| \|f\|_1 = \|\psi\| \mu(A).$$

The only way is if $\mu(A) = 0$, so $g \in L_\infty$.

Hence, ψ and ϕ_g are both in L_p^*, and $\psi = \phi_g$, on L_∞. Since L_∞ is dense in L_p, we get $\psi = \phi_g$.

Before we continue to our next cases when μ may not be finite, we need a few pieces notation.

Fix $A \in \mathcal{F}$. Then,

$$\mathcal{F}_A = \{B \in \mathcal{F} \mid B \subseteq A\}$$

is a σ-algebra on A. Define $\mu_A = \mu|_{\mathcal{F}_A}$. Then $(A, \mathcal{F}_A, \mu_A)$ is a measure space, with $L_p(\mu_A) \subseteq L_p(\mu)$. Let

$$\psi_A = \psi|_{L_p(\mu_A)}.$$

Let's continue.

Proof: Let $\psi_A = \psi|_{L_p(\mu_A)}$, the restriction onto a subset. Then $\psi_A \in L_p(\mu_A)^*$, and $\|\psi_A\| \leq \|\psi\|$.

Let $A, B \in \mathcal{F}$ with $A \cap B = \emptyset$. In the case $1 < p < \infty$,

$$\begin{aligned}
\|\psi_{A\cup B}\| &= \sup\{|\psi_{A\cup B}(h)| \mid h \in L_p(\mu_{A\cup B}), \|h\|_p \leq 1\} \\
&= \sup\{|\psi_A(f) + \psi_B(g)| \mid f \in L_p(\mu_A), g \in L_p(\mu_B), \|f\|_p^p + \|g\|_p^p \leq 1\} \\
&= \sup\{a|\psi_A(f)| + b|\psi_B(g)| \mid a, b \geq 0, a^p + b^p \leq 1, \\
&\qquad\qquad f \in B_{L_p}(\mu_A), g \in B_{L_p}(\mu_B)\} \\
&= \sup\{a\|\psi_A\| + b\|\psi_B\| \mid a, b \geq 0, a^p + b^p \leq 1\} \\
&= (\|\psi_A\|^q + \|\psi_B\|^q)^{1/q},
\end{aligned}$$

since $(\ell_p^2)^* =\cong \ell_q^2$.

The next case is when μ is σ-finite. We have a measurable partition $\Omega = \bigcup A_n$ of Ω, with $\mu(A_n) < \infty$ for all n. By the first case, there is $g_n \in L_q(\mu_{A_n})$ with

$$\psi_{A_n} = \phi_{g_n}.$$

Define g such that

$$g|_{A_n} = g_n.$$

When $p = 1$, then

$$\|g\|_\infty = \sup_n \|g_n\|_\infty = \sup_n \|\psi_{A_n}\| \leq \|\psi\|.$$

So $g \in L_q$. For $p \neq 1$, note

$$\sum_{n=1}^N \|g_n\|_q^q = \sum_{n=1}^N \|\psi_{A_n}\|^q = \|\psi_{A_1 \cup \dots \cup A_N}\|^q \leq \|\psi\|^q.$$

By monotone convergence, $g \in L_q$. In both cases, $g \in L_q$, so $\phi_g \in L_p(\mu)^*$, and so we have

$$\psi|_{L_p(\mu_{A_n})} = \psi_{A_n} = \phi_{g_n} = \phi_g|_{L_p(\mu_{A_n})}.$$

Since $\bigcup L_p(\mu_{A_n})$ has dense linear span in $L_p(\mu)$, we find that $\psi = \phi_g$ on $L_p(\mu)$.

The final case is for general μ, and $1 < p < \infty$. Choose (f_n) in B_{L_p} such that $\|\psi\| = \lim_n |\psi(f_n)|$. For all k, n, note that

$$\mu(|f_n| \geq 1/k) \leq k^p \|f_n\|_p^p < \infty,$$

by Markov's inequality. Hence

$$A = \bigcup_{n,k} \{|f_n| \geq 1/k\}$$

is σ-finite, and for all n, $f_n = 0$ on $\Omega \setminus A$. So $\|\psi_A\| = \|\psi\|$. So,

$$\|\psi_A\| = \|\psi\| = (\|\psi_A\|^q + \|\psi_{\Omega \setminus A}\|^q)^{1/q},$$

and hence $\psi_{\Omega \setminus A} = 0$. Hence we are done by case 2.

Corollary 2.1. *For $1 < p < \infty$, $L_p(\mu)$ is reflexive.*

Proof: Let $\phi \in L_p^{**}$. We seek $f \in L_p$ such that $\phi = \hat{f}$, i.e.

$$\phi(\psi) = \hat{f}(\psi) = \psi(f)$$

for all $\psi \in L_p^*$, i.e.

$$\phi(\phi_g) = \phi_g(f)$$

for all $g \in L_q$. The map $g \mapsto \phi(\phi_g)$ is in L_q^*, so by the previous theorem, there exists $f \in L_p$ such that

$$\phi(\phi_g) = \int_\Omega gf \, \mathrm{d}\mu = \phi_g(f).$$

2.3 $C(K)$ Spaces

Throughout, we assume that K is a compact Hausdorff space. Here we make a distinction on our base field:

$$C(K) = \{f : K \to \mathbb{C} \mid f \text{ continuous}\}.$$

This is a complex Banach space with the $\|\cdot\|_\infty$ norm. We also denote

$$C^{\mathbb{R}}(K) = \{f : K \to \mathbb{R} \mid f \text{ continuous}\},$$

and another important object is

$$C^+(K) = \{f : K \to \mathbb{R}^+ \mid f \text{ continuous}\},$$

which is a subset of $C^{\mathbb{R}}(K)$ (more specifically a cone). Let

$$M(K) = C(K)^* = \{\phi : C(K) \to \mathbb{C} \mid \phi \text{ linear, bounded}\},$$

and also we define

$$M^{\mathbb{R}}(K) = \{\phi \in M(K) \mid \phi(f) \in \mathbb{R} \text{ for all } f \in C^{\mathbb{R}}(K)\}.$$

We do not define $M^{\mathbb{R}} = (C^{\mathbb{R}})^*$, however we will show this is true. Also define

$$M^+(K) = \{\phi : C(K) \to \mathbb{C} \mid \phi \text{ linear, for all } f \in C^+(K), \phi(f) \in \mathbb{R}^+\}.$$

We do note assume continuity, however we will show any $f \in M^+$ is continuous, so $M^+ \subseteq M^{\mathbb{R}}$. The members of this set are the *positive linear functionals*.

Our aim is to describe $M(K), M^{\mathbb{R}}(K)$. We will show that it is enough to describe $M^+(K)$.

Lemma 2.1.

(i) *For all $\phi \in M(K)$, there is a unique $\phi_1, \phi_2 \in M^{\mathbb{R}}(K)$ such that*

$$\phi = \phi_1 + i\phi_2.$$

(ii) *The map $\phi \mapsto \phi|_{C^{\mathbb{R}}(K)}$, from $M^{\mathbb{R}} \to (C^{\mathbb{R}})^*$ is an isometric isomorphism.*

(iii) $M^+(K) \subseteq M^{\mathbb{R}}(K)$ *and*

$$M^+(K) = \{\phi \in M(K) \mid \|\phi\| = \phi(1_K)\}.$$

(iv) *For all $\phi \in M^{\mathbb{R}}(K)$, there exists a unique ϕ^+, ϕ^- such that*

$$\phi = \phi^+ - \phi^- \qquad \textit{and} \qquad \|\phi\| = \|\phi^+\| + \|\phi^-\|.$$

Proof:

(i) Define $\bar{\phi} : C(K) \to \mathbb{C}$, by

$$\bar{\phi}(f) = \overline{\phi(\bar{f})}.$$

Then $\bar{\phi} \in M(K)$, and

$$\phi \in M^{\mathbb{R}}(K) \iff \phi = \bar{\phi}.$$

First we show uniqueness. If $\phi = \phi_1 + i\phi_2$, then $\bar{f} = \phi_1 - i\phi_2$, so

$$\phi_1 = \frac{\phi + \bar{\phi}}{2}, \qquad \phi_2 = \frac{\phi - \bar{\phi}}{2i}.$$

This also shows existence, by defining ϕ_1, ϕ_2 in this way.

(ii) Let $\phi \in M^{\mathbb{R}}$, and $\psi = \phi|_{C^{\mathbb{R}}}$. Note that $\psi \in (C^{\mathbb{R}})^*$, and moreover $\|\psi\| \le \|\phi\|$.

First we show this map is isometric. Let $f \in C(K)$, and $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$ and $|\phi(f)| = \lambda\phi(f)$. Then,

$$\begin{aligned} |\phi(f)| = \lambda\phi(f) = \phi(\lambda f) &= \phi(\Re(\lambda f)) + i\,\underbrace{\phi(\Im(\lambda f))}_{0 \text{ as } \phi \in M^{\mathbb{R}}} \\ &= \phi(\Re(\lambda f)) = \psi(\Re(\lambda f)) \le \|\psi\|\|\Re(\lambda f)\|_\infty \le \|\psi\|\|f\|_\infty. \end{aligned}$$

So $\|\phi\| \le \|\psi\|$. To show this map is onto, say we have $\psi \in (C^{\mathbb{R}})^*$. Then the obvious way to define ϕ is

$$\phi(f) = \psi(\Re(f)) + i\psi(\Im(f)),$$

for $f \in C(K)$. Then $\phi \in M(K)$ and $\phi|_{C^{\mathbb{R}}(K)} = \psi$.

(iii) To show the inclusion, let $\phi \in M^+(K)$, and $f \in C^{\mathbb{R}}(K)$ with $-1 \le f \le 1$ on K. Then $1_K \pm f \ge 0$ on K. So,

$$\phi(1_K \pm f) = \phi(1_K) \pm \phi(f) \ge 0.$$

So $\phi(f) \in \mathbb{R}$, and $|\phi(f)| \le \phi(1_K)$. Hence, $\phi|_{C^{\mathbb{R}}(K)} \in (C^{\mathbb{R}})^*$, and $\|\phi|_{C^{\mathbb{R}}(K)}\| = \phi(1_K)$.

By (ii), this gives us that $\phi \in M^{\mathbb{R}}(K)$ and $\|\phi\| = \phi(1_K)$.

This immediately shows that $M^+(K) \subseteq \{\phi \in M(K) \mid \|\phi\| = \phi(1_K)\}$. To show the other inclusion, let $\phi \in M(K)$ with $\|\phi\| = \phi(1_K)$. Scale so that $\phi(1_K) = 1$.

Let $f \in C^{\mathbb{R}}(K)$, and $\phi(f) = a + ib$. For $f \in \mathbb{R}$,

$$\begin{aligned} |\phi(f + it1_K)|^2 = |a + i(b+t)|^2 &= a^2 + b^2 + 2bt + t^2 \\ &\le \|f + it1_K\|_\infty^2 \le \|f\|_\infty^2 + t^2. \end{aligned}$$

By looking at the degree 1 term, we get $b = 0$, so $\phi(f) \in \mathbb{R}$.

Now let $f \in C^+(K)$. Without loss of generality, let $0 \le f \le 1$ on K. Then

$$-1 \le 1_K - 2f \le 1,$$

so we get

$$\phi(1_K - 2f) = 1 - 2\phi(f) \le \|\phi\| \cdot \|1_K - 2f\|_\infty \le 1,$$

which gives $\phi(f) \ge 0$, as desired.

(iv) We begin by motivating uniqueness, but will sidetrack to prove existence.

Assume that $\psi^+, \psi^- \in M^+(K)$, $\phi = \psi^+ - \psi^-$, and $\|\phi\| = \|\psi^+\| + \|\psi^-\|$.

If $0 \le g \le f$ on K, then note

$$\phi(g) \le \psi^+(g) \le \psi^+(f).$$

So $\psi^+(f) \ge \sup\{\phi(g) \mid 0 \le g \le f\}$. This motivates our definition of ϕ^+, ϕ^-.

We segue into proving existence. Define for $f \in C^+(K)$

$$\phi^+(f) = \sup\{\phi(g) \mid 0 \le g \le f\}.$$

Then we have

$$|\phi(g)| \le \|\phi\| \cdot \|g\|_\infty \le \|\phi\| \cdot \|f\|_\infty,$$

so the supremum exists. Moreover $\phi^+(f) \ge 0$ since we may take $g = 0$, and $\phi^+(f) \ge \phi(f)$ by taking $g = f$.

Next it is easy to see that $\phi^+(\lambda f) = \lambda \phi^+(f)$ for all $f \in C^+(K)$ and $\lambda \in \mathbb{R}^+$. Moreover for $f_1, f_2 \in C^+(K)$,

$$\phi^+(f_1 + f_2) = \phi^+(f_1) + \phi^+(f_2).$$

Indeed, if $0 \le g_1 \le f_1$, and $0 \le g_2 \le f_2$, then $0 \le g_1 + g_2 \le f_1 + f_2$, showing that

$$\phi^+(f_1 + f_2) \ge \phi(g_1 + g_2) = \phi(g_1) + \phi(g_2).$$

Taking a supremum we get

$$\phi^+(f_1 + f_2) \ge \phi^+(g_1 + g_2).$$

For the other way around let $0 \le g \le f_1 + f_2$, and define

$$g_1 = g \wedge f_1, \qquad g_2 = g - g_1.$$

Then $0 \leq g_1 \leq f_1$, and $0 \leq g_2 \leq f_2$. And hence

$$\phi^+(f_1) + \phi^+(f_2) \geq \phi(g_1) + \phi(g_2) = \phi(g).$$

Taking a supremum over all g, gives the other inequality. So ϕ^+ is positively linear.

Next we want to extend ϕ to $C^{\mathbb{R}}$. Given $f \in C^{\mathbb{R}}$, we can write $f = f_1 - f_2$ for some $f_1, f_2 \in C^+(K)$ (splitting into positive and negative parts). Then define

$$\phi^+(f) = \phi^+(f_1) - \phi^+(f_2).$$

Then we can show ϕ^+ is well-defined and linear on $C^{\mathbb{R}}$, from the scaling and multiplicative properties we showed earlier.

Finally define $\phi^+ : C(K) \to \mathbb{C}$ by

$$\phi^+(f) = \phi^+(\Re f) + i\phi^+(\Im f).$$

Then ϕ^+ is in $M^+(K)$. Now we can define $\phi^- = \phi^+ - \phi$. For $f \in C^+(K)$, note that $\phi^+(f) \geq \phi(f)$, so $\phi^-(f) \leq 0$. Hence $\phi^- \in M^+(K)$ and $\phi = \phi^+ - \phi^-$.

Finally we show that the norms coincide. We have

$$\|\phi\| \leq \|\phi^+\| + \|\phi^-\| = \phi^+(1_K) - \phi^-(1_K) = 2\phi^+(1_K) - \phi(1_K).$$

Fox some $0 \leq g \leq 1$, so $-1 \leq 2g - 1_K \leq 1$ on K. So,

$$\phi(2g - 1_K) = 2\phi(g) - \phi(1_K) \leq \|\phi\|.$$

Taking a supremum over g, we find

$$2\phi^+(1_K) - \phi(1_K) \leq \|\phi\|.$$

Hence we get equality, and so

$$\|\phi\| = \|\phi^+\| + \|\phi^-\|.$$

Now we get back to uniqueness. Recall that if we have $\phi = \psi^+ - \psi^-$, then we have shown $\psi^+(f) \geq \phi^+(f)$ for all $f \in C^+(K)$.

This also implies that $\psi^-(f) \geq \phi^-(f)$. So we have

$$\psi^+ - \phi^+, \qquad \psi^- - \phi^- \in M^+(K).$$

But hence we have

$$\begin{aligned}\|\phi\| &= \|\phi^+\| + \|\phi^-\| = \phi^+(1_K) + \phi^-(1_K) \\ &\leq \psi^+(1_K) + \psi^-(1_K) = \|\psi^+\| + \|\psi^-\| \\ &= \|\phi\|.\end{aligned}$$

So we have equality. Hence $\phi^+(1_K) = \psi^+(1_K)$, but then

$$\|\psi^+ - \phi^+\| = (\psi^+ - \phi^+)(1_K) = 0.$$

So $\psi^+ = \phi^+$, and $\psi^- = \phi^-$.

Before starting the next discussion, we will need to go over a few topological preliminaries.

Recall that we have a fixed compact Hausdorff space K.

1. K is *normal* if given closed $E, F \subseteq K$, with $E \cap F = \emptyset$, then there exists open $U, V \subseteq K$ with $U \cap V = \emptyset$ such that $E \subseteq U, F \subseteq V$.

   Equivalently, given $E \subseteq U \subseteq K$, where E is closed and U is open, there exists open $V \subseteq K$ such that

   $$E \subseteq V \subseteq \overline{V} \subseteq U.$$

2. Urysohn's lemma: if $E, F \subseteq K$ are closed and disjoint in a normal space, then there exists continuous $f : K \to [0,1]$ such that $f = 0$ on E, and $f = 1$ on F.

3. $f \prec U$ means that U is open, and $f : K \to [0,1]$ is continuous with support of f $\operatorname{supp} f \subseteq U$.

   $E \prec f$ means that $E \subseteq K$ is closed, and $f : K \to [0,1]$ is a continuous function such that $f = 1$ on E.

   Urysohn is equivalent to: if $E \subseteq U \subseteq K$, with E closed, U open, then there exists f such that

   $$E \prec f \prec U.$$

Lemma 2.2.

(i) *Let $E, U_1, \ldots, U_n \subseteq K$, with E closed and $U_1, \ldots, U_n$ open, with $E \subseteq \bigcup U_j$. Then there exists open sets V_j such that $\overline{V_j} \subseteq U_j$, and $E \subseteq \bigcup V_j$.*

(ii) *There exists functions* f_j*, for* $j = 1, \ldots, n$ *, such that* $f_j \prec U_j$*,* $0 \le \sum f_j \le 1$*, and* $\sum f_j = 1$ *on* E*.*

Proof: Recall that
$$E \subseteq \bigcup_{j=1}^{n} U_j \subseteq K.$$

(i) By induction on n,
$$E \setminus U_n \subseteq \bigcup_{j=1}^{n-1} U_j.$$
By induction there exists open sets V_j such that $\overline{V_j} \subseteq U_j$ such that $E \setminus U_n \subseteq \bigcup_{j=1}^{n-1} V_j$. Then
$$E \setminus \bigcup_{j=1}^{n-1} V_j \subseteq U_n.$$
Since K is normal, there exists open V_n such that
$$E \setminus \bigcup_{j=1}^{n-1} V_j \subseteq V_n \subseteq \overline{V_n} \subseteq U_n.$$
Then $E \subseteq \bigcup_{j=1}^{n} V_j$.

(ii) We seek functions f_j, for $1 \le j \le n$ such that $f_j \prec U_j$ for all j, and
$$0 \le \sum_{j=1}^{n} f_j \le 1$$
on K, and which equals 1 on E. Let V_j be as in (i).

By Urysohn, for each $j = 1, \ldots, n$, there exists g_j such that
$$\overline{V_j} \prec g_j \prec U_j,$$
and there exists a function g_0 such that
$$K \setminus \bigcup_{j=1}^{n} V_j \prec g_0 \prec K \setminus E.$$
Define
$$g = \sum_{j=0}^{n} g_j.$$

Then $g \geq 1$ on K. Set $f_j = g_j/g$, for $1 \leq j \leq n$. Then $0 \leq f_j \leq 1$ on K, and f_j is continuous.

On K, note

$$\sum_{j=1}^{n} f_j = \sum_{j=1}^{n} \frac{g_j}{g} \leq \sum_{j=0}^{n} \frac{g_j}{g} = 1.$$

On E, $g_0 = 0$, so indeed

$$\sum_{j=1}^{n} f_j = 1.$$

2.4 Borel Measures and Regularity

Let X be a Hausdorff space, and let $\mathcal{G}$ be the subset of all open subsets of X. Then

$$\mathcal{B} = \sigma(\mathcal{G})$$

is the *Borel σ-algebra* on X, and the members are the *Borel sets* of X.

A *Borel measure* on X is a measure on $\mathcal{B}$. A Borel measure μ on X is *regular* if:

(i) $\mu(E) < \infty$ for all $E \subseteq X$ compact.

(ii) $\mu(A) = \inf\{\mu(U) \mid A \subseteq U \in \mathcal{G}\}$, for all $A \in \mathcal{B}$ (outer-regularity).

(iii) $\mu(U) = \sup\{\mu(E) \mid E \subseteq U, E \text{ compact}\}$ for all $U \in \mathcal{G}$ (inner-regularity).

An example is the Lebesgue measure on $\mathbb{R}$. A complex Borel measure ν is *regular* if $|\nu|$ is regular.

If X is compact, then a Borel measure μ on X is regular if and only if $\mu(X) < \infty$ and X is outer-regular.

Let Ω be a set, $\mathcal{F}$ be a σ-field on Ω, and ν a complex measure on $\mathcal{F}$. A measurable function $f : \Omega \to \mathbb{C}$ is *ν-integrable* if and only if

$$\int_\Omega |f| \,\mathrm{d}|\nu| < \infty.$$

Then we may define

$$\int_\Omega f \,\mathrm{d}\nu = \int_\Omega f \,\mathrm{d}\nu_1 - \int_\Omega f \,\mathrm{d}\nu_2 + i \int_\Omega f \,\mathrm{d}\nu_3 - i \int_\Omega f \,\mathrm{d}\nu_4,$$

where $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$ is the Jordan decomposition of ν. Note that f is ν-integrable if and only if f is ν_i integrable for all k. ν-integration has the following properties:

1. For all $A \in \mathcal{F}$, $\mathbb{1}_A$ is ν-integrable, and
$$\int_\Omega \mathbb{1}_A \, \mathrm{d}\nu = \nu(A).$$
2. Linearity: if f, g are ν-integrable, then so is $\alpha f + \beta g$ for any $\alpha, \beta \in \mathbb{C}$, and
$$\int_\Omega (\alpha f + \beta g) \, \mathrm{d}\nu = \alpha \int_\Omega f \, \mathrm{d}\nu + \beta \int_\Omega g \, \mathrm{d}\nu.$$
3. Dominated convergence: if $f_n \to f$ $|\nu|$-almost everywhere, and there exists $g \in L_1(|\nu|)$ such that $|f_n| \leq g$ ν-almost everywhere, then f, f_n are ν-integrable, and
$$\int_\Omega f_n \, \mathrm{d}\nu \to \int_\Omega f \, \mathrm{d}\nu.$$
This is true as it is true for all (ν_i).
4. If f is ν-integrable, then
$$\left| \int_\Omega f \, \mathrm{d}\nu \right| \leq \int_\Omega |f| \, \mathrm{d}|\nu|.$$

2.5 Dual of $C(K)$

Let ν be a complex Borel measure on K. If $f \in C(K)$, then
$$\int_K |f| \, \mathrm{d}|\nu| \leq \|f\|_\infty |\nu|(K) = \|f\|_\infty |\nu|_1,$$
so f is ν-measurable, and
$$\left| \int_K f \, \mathrm{d}\nu \right| \leq \|f\|_\infty \|\nu\|_1.$$
So we can define $\phi : C(K) \to \mathbb{C}$, by
$$\phi(f) = \int_K f \, \mathrm{d}\nu.$$
Then ϕ is linear, and $|\phi(f)| \leq \|f\|_\infty \|\nu\|_1$. So $\phi \in M(K)$, and $\|\phi\| \leq \|\nu\|_1$. If ν is a signed measure then $\phi \in M^{\mathbb{R}}(K)$, and if ν s a positive measure, then $\phi \in M^+(K)$.

Theorem 2.5 (Riesz Representation Theorem)**.** *For each $\phi \in M^+(K)$, there exists a unique regular Borel measure μ on K which represents ϕ, i.e.*
$$\phi(f) = \int_K f \, \mathrm{d}\mu.$$
Moreover $\|\phi\| = \|\mu\|_1 = \mu(K)$.

Proof: We begin with uniqueness. First, assume μ_1, μ_2 both represent ϕ. By Urysohn, there exists a function f with $E \prec f \prec U$. Then,

$$\mu_1(E) \leq \int_K f \,\mathrm{d}\mu_1 = \phi(f) = \int_K f \,\mathrm{d}\mu_2 \leq \mu_2(U).$$

Fix U, and take the supremum over E. Then by inner-regularity, $\mu_1(U) \leq \mu_2(U)$.

By symmetry, $\mu_1 = \mu_2$ on $\mathcal{G}$, and hence $\mu_1 = \mu_2$ on $\mathcal{B}$ by regularity.

For existence, define for $U \in \mathcal{G}$,

$$\mu(U) = \sup\{\phi(f) \mid f \prec U\}.$$

The supremum exists as for all $f \prec U$,

$$|\phi(f)| \leq \|\phi\|\|f\|_\infty \leq \|\phi\|,$$

and $0 \prec U$. Note $\mu(U) \geq 0$, $\mu(\emptyset) = 0$, and if $U \subseteq V$, then $\mu(U) \leq \mu(V)$. Moreover $\mu(K) = \phi(1_K)$.

We then define the outer-measure on $\mathcal{P}(K)$ by

$$\mu^*(A) = \inf\{\mu(U) \mid A \subseteq U \in \mathcal{G}\},$$

for all $A \subseteq K$. By μ increasing, $\mu^*(U) = \mu(U)$, so $\mu^*(\emptyset) = 0$, and $\mu^*(K) = \phi(1_K)$.

We prove that μ^* is indeed an outer measure. For $A \subseteq B \subseteq K$, then $\mu^*(A) \leq \mu^*(B)$. What we want to show is that if $A \subseteq \bigcup_n A_n$, then

$$\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

First suppose that $A = U \in \mathcal{G}$, and $A_n = U_n \in \mathcal{G}$. Fix $f \prec U$, and let $E = \operatorname{supp} f$. Then E is closed, and $E \subseteq U \subseteq \bigcup U_n$.

By compactness, there exists $E \subseteq \bigcup_{j=1}^n U_j$.

By lemma 2.2, there are functions g_j such that $g_j \prec U_j$, with $\sum g_j \leq 1$ on K, and with equality on E. Then $fg_j \prec U_j$ for all j, and $f = \sum_{j=1}^n fg_j$. Hence

$$\phi(f) = \sum_{j=1}^{n} \phi(fg_j) = \sum_{j=1}^{n} \mu(U_j) \leq \sum_{j=1}^{\infty} \mu(U_j).$$

Taking a supremum over all f,

$$\mu(U) \leq \sum_{j=1}^{\infty} \mu(U_j).$$

Now we tackle the general case. Fix $\varepsilon > 0$, and for each n choose open $U_n \supseteq A_n$ such that $\mu^*(U_n) \leq \mu^*(A_n) + \varepsilon 2^{-n}$. Then by the first case,

$$\mu^*(A) \leq \mu\left(\bigcup_{n\geq 1} U_n\right) \leq \sum_{n\geq 1} \mu(U_n) \leq \sum_{n\geq 1} \mu^*(A_n) + \varepsilon.$$

This holds for all ε, so

$$\mu^*(A) \leq \sum_{n\geq 1} \mu^*(A_n).$$

Therefore μ^* is an outer measure. By measure theory, the family $\mathcal{M}$ of μ^*-measurable subsets of K is a σ-algebra on K, and $\mu^*|_{\mathcal{M}}$ is a measure.

Next we show that $\mathcal{B} \subseteq \mathcal{M}$. It is enough to show that $\mathcal{G} \subseteq \mathcal{M}$.

Fix $U \in \mathcal{G}$. To show $U \in \mathcal{M}$, we need that for all $A \subseteq K$,

$$\mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A \setminus U).$$

First, if $A = V \in G$, then fix $f \prec V \cap U$, and then also fix $g \prec V \setminus \operatorname{supp} f$. Then $f + g \prec V$, so

$$\mu^*(V) \geq \phi(f+g) = \phi(f) + \phi(g).$$

Taking a supremum over g,

$$\mu^*(V) \geq \phi(f) + \mu^*(V \setminus \operatorname{supp} f) \geq \phi(f) + \mu^*(V \setminus U).$$

Then taking a supremum over f,

$$\mu^*(V) \geq \mu^*(V \cap U) + \mu^*(V \setminus U).$$

Now we consider general A. Take an open $V \supseteq A$, then $V \cap U \supseteq A \cap U$, and $V \setminus U \supseteq A \setminus U$. So,

$$\mu^*(V) \geq \mu^*(V \cap U) + \mu^*(V \setminus U) \geq \mu^*(A \cap U) + \mu^*(A \setminus U).$$

Taking an infimum over V,

$$\mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A \setminus U).$$

Hence $\mathcal{B} \subseteq \mathcal{M}$, so let $\mu = \mu^*|_{\mathcal{B}}$. Then μ is a Borel measure on K, with $\mu(K) = \phi(1_K) = \|\phi\|$, and μ is regular by definition.

We need to check that

$$\phi(f) = \int_K f \, \mathrm{d}\mu,$$

for all $f \in C(K)$. By our decomposition, it is enough to check this for $f \in C^{\mathbb{R}}(K)$, and in fact it is enough to check that

$$\phi(f) \leq \int_K f \, \mathrm{d}\mu$$

for all $f \in C^{\mathbb{R}}(K)$. Applying this to f and $-f$ gives equality. Note that for $f = 1_K$,

$$\phi(1_K) = \mu(K) = \int_K 1_K \, \mathrm{d}\mu,$$

so it is enough to show that

$$\phi(f) \leq \int_K f \, \mathrm{d}\mu$$

for $f > 0$ on K, as for $f \in C^{\mathbb{R}}$ general we may apply this to $f + \lambda 1_K$ for λ large.

Let $f > 0$ on K. Fix $b > a > 0$ such that $f(K) \subseteq [a, b]$ Fix $\varepsilon > 0$ and $0 \leq y_0 < a \leq y_1 \leq \cdots \leq y_n = b$ such that $y_i - y_{i-1} \leq \varepsilon$, and set

$$A_j = f^{-1}((y_{i-1}, y_j]).$$

Then these are Borel sets, and $K = \bigcup_j A_j$ is a Borel partition.

As μ is regular, we can pick $U_j \supseteq U_j$ with $\mu(U_j \setminus A_j) \leq \varepsilon/n$. By shrinking U_j, without loss of generality let

$$U_i = f^{-1}((y_{i-1}, y_i + \varepsilon)).$$

Since $K = \bigcup U_i$, by the previous lemma 2.2 there is $g_i \prec U_i$ with $\sum_i g_i = 1$ on K. Then $f = \sum_j fg_j$, and $fg_j \leq (y_j + \varepsilon)g_j$ for $1 \leq j \leq n$. So,

$$\begin{aligned}
\phi(f) = \sum_{j=1}^{n} \phi(fg_j) &\leq \sum_{j=1}^{n} (y_j + \varepsilon)\phi(g_j) \\
&\leq \sum_{j=1}^{n} (y_j + \varepsilon)\mu(U_j) \leq \sum_{j=1}^{n} (y_{j-1} + 2\varepsilon)(\mu(A_j) + \varepsilon/n) \\
&\leq \sum_{j=1}^{n} y_{j-1}\mu(A_j) + \varepsilon(b + 2\mu(K) + 2\varepsilon) \\
&= \int_K \sum_{j=1}^{n} y_{j-1}\mathbb{1}_{A_j}\,\mathrm{d}\mu + \varepsilon(b + 2\mu(K) + 2\varepsilon) \\
&\leq \int_K f\,\mathrm{d}\mu + \varepsilon(b + 2\mu(K) + 2\varepsilon).
\end{aligned}$$

Taking $\varepsilon \to 0$ gives the required result.

Corollary 2.2. *For any $\phi \in M(K)$, there exists a unique regular complex Borel measure ν on K that represents ϕ:*

$$\int_K f\,\mathrm{d}\nu = \phi(f)$$

for all $f \in C(K)$. Moreover, $\|\phi\| = \|\nu\|_1$. If $\phi \in M^{\mathbb{R}}(K)$, then ν is a signed measure.

Proof: For existence, use lemma 2.1 then theorem 2.5. We only need to show that $\|\phi\| = \|\nu\|_1$.

This will follow from uniqueness: if ν_1, ν_2 both represent ϕ, then $\nu_1 - \nu_2$ represents 0, so $\|\nu_1 - \nu_2\|_1 = 0$, hence $\nu_1 = \nu_2$.

Say that ν represents ϕ. We have seen that $\|\phi\| \leq \|\nu\|_1$. Let

$$K = \bigcup_{i=1}^{n} A_i$$

be a Borel partition of K. We will show that

$$\sum_{i=1}^{n} |\nu(A_i)| \leq \|\phi\|,$$

then we are done.

Choose $\lambda_i \in \mathbb{C}$ such that $|\lambda_i| = 1$, and

$$|\nu(A_i)| = |\lambda_i|\nu(A_i).$$

Then we find

$$\sum_{i=1}^{n} |\nu(A_i)| = \sum_{i=1}^{n} \lambda_i \nu(A_i) = \int_K \sum_{j=1}^{n} \lambda_j \mathbb{1}_{A_j} \, \mathrm{d}\nu.$$

Fix $\varepsilon > 0$, and $E_j \subseteq A_j \subseteq U_j$ with E_j closed, U_j open such that $|\nu|(U_j \setminus E_j) < \varepsilon/n$. Without loss of generality,

$$U_j \subseteq K \setminus \bigcup_{\substack{i=1 \\ i \neq j}}^{n} E_i.$$

Let $E = \bigcup E_j$, so $E \subseteq \bigcup U_j$. So there exists $g_j \prec U_j$ such that $\sum g_j = 1$ on E, and $\sum g_j \leq 1$ on K.

But on E_j, $g_i = 0$ for $i \neq j$, so $g_j = 1$. Set

$$f = \sum_{j=1}^{n} \lambda_j g_j \in C(K),$$

and moreover $\|f\|_\infty \leq 1$. Then

$$\begin{aligned}
\left| \sum_{j=1}^{n} |\nu(A_j)| - \phi(f) \right| &= \left| \int_K \left(\sum_{j=1}^{n} \lambda_j \mathbb{1}_{A_j} - \sum_{j=1}^{n} \lambda_j g_j \right) \mathrm{d}\nu \right| \\
&= \sum_{j=1}^{n} \int_K \left| \mathbb{1}_{A_j} - g_j \right| \mathrm{d}\nu \\
&\leq \sum_{j=1}^{n} |\nu|(U_j \setminus E_j) < \varepsilon.
\end{aligned}$$

So

$$\sum_{j=1}^{n} |\nu(A_j)| \leq |\phi(f)| + \varepsilon \leq \|\phi\| + \varepsilon.$$

Corollary 2.3. *The space of regular complex Borel measures on K is a complex*

Banach space with $\|\cdot\|_1$*, and is isometrically isomorphic to* $M(K) = C(K)^*$*.*

The space of regular signed Borel measures on K *is a real Banach space with* $\|\cdot\|_1$*, and is isometrically isomorphic to* $M^{\mathbb{R}}(K) \cong C^{\mathbb{R}}(K)^*$*.*

3 Weak Topologies

Let X be a set, and $\mathcal{F}$ a family of functions, where each $f \in \mathcal{F}$ is $f : X \to Y_f$, with Y_f a topological space.

The *weak topology* $\sigma(X, \mathcal{F})$ on X generated by $\mathcal{F}$ is the smallest topology on X such that each $f \in \mathcal{F}$ is continuous.

Remark.

1. Let $S = \{f^{-1}(U) \mid f \in \mathcal{F}, U \subseteq Y_f \text{ open}\}$. Then this is a subbase of $\sigma(X, \mathcal{F})$, i.e. $\sigma(X, \mathcal{F})$ is generated by S. So the finite intersections of members of S is a base of $\sigma(X, \mathcal{F})$.

   So $V \subseteq X$ is open, if and only if for all $x \in V$, there exists $n \in \mathbb{N}$, and $f_1, \ldots, f_n \in \mathcal{F}$, and $U_j \subseteq Y_{f_j}$ open such that
   $$x \in \bigcap_{j=1}^{n} f_j^{-1}(U_j) \subseteq V.$$
   Hence $V \subseteq X$ is open if and only if for all $x \in V$, there exists $f_1, \ldots, f_n \in \mathcal{F}$ and neighbourhoods U_j of $f_j(x)$ in Y_{f_j} such that
   $$\bigcap_{j=1}^{n} f_j^{-1}(U_j) \subseteq V.$$

2. If S_f is a subbase for the topology of Y_f for each f, then
   $$S = \{f^{-1}(U) \mid f \in \mathcal{F}, U \in S_f\}$$
   is a subbase for $\sigma(X, \mathcal{F})$.

3. If all Y_f are Hausdorff, and $\mathcal{F}$ separates points, then $\sigma(X, \mathcal{F})$ are Hausdorff.

4. Let $Y \subseteq X$ and $\mathcal{F}_Y = \{f_Y \mid f \in \mathcal{F}\}$. Then $\sigma(X, \mathcal{F})|_Y = \sigma(Y, \mathcal{F}_Y)$.

5. There is a universal property: a function $g : Z \to X$ is continuous with respect to $\sigma(X, \mathcal{F})$ if and only if $f \circ g : Z \to Y_f$ is continuous, for each $f \in \mathcal{F}$.

Example 3.1.

1. Let X be a topological space, and $Y \subseteq X$, with $\iota : Y \hookrightarrow X$ the inclusion. Then $\sigma(Y, \{\iota\})$ is the subspace topology.

2. Let Γ be a set, and for each $\gamma \in \Gamma$ let X_γ be a topological space. Take
$$X = \prod_{\gamma \in \Gamma} X_\gamma,$$
and define $\pi_\gamma : X \to X_\gamma$. Then $\sigma(X, \{\pi_\gamma \mid \gamma \in \Gamma\})$ is the product topology.

Proposition 3.1. *Let X be a set, and for each $n \in \mathbb{N}$ let (Y_n, d_n) be a metric space and $f_n : X \to Y_n$ a function. Assume that $\mathcal{F} = \{f_n \mid n \in \mathbb{N}\}$ separates the points of X.*

Then $\sigma(X, \mathcal{F})$ is metrizable.

In general, in the non-separable case, we get a pseudo-metric.

Proof: For $x, y \in X$, set
$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \min\{d_n(f_n(x), f_n(y)), 1\}.$$

Then d is a metric on X, as if $x \neq y$ then since $\mathcal{F}$ separates points, there is n with $f_n(x) \neq f_n(y)$, and so $d(x, y) > 0$. Moreover if d_n is a metric, $\min(d_n, 1)$ is a metric, so d, as a sum of metrics, is a metric.

All that is required is to show that this induces the same topology as $\sigma(X, \mathcal{F})$, which amounts to showing the identity map is continuous. Let τ be the induced topology.

First, note $\tilde{f}_n$, which is the function f_n, is uniformly continuous on the metric topology. Indeed, for $\varepsilon > 0$, we want to show there is $\delta > 0$ such that $d(x, y) \leq \delta \implies d_n(f_n(x), f_n(y)) \leq \varepsilon$.

It suffices to show this for $\varepsilon < 1$. Set $\delta = 2^{-n}\varepsilon$. Then $d(x, y) \leq \delta \implies 2^{-n} \min\{d_n(f_n(x), f_n(y)), 1\} \leq 2^{-n\varepsilon}$, so since $\varepsilon < 1$, this means $d_n(f_n(x), f_n(y)) \leq \varepsilon$. So as each $\tilde{f}_n$ continuous, $\sigma(X, \mathcal{F}) \subseteq \tau$.

Conversely, we need to show id is continuous from $\sigma(X, \mathcal{F})$ to τ. This is equivalent to showing that d is continuous, as a map $d : X \times X \to \mathbb{R}$. Indeed, f_n is continuous, so $\tilde{d}_n : X \times X \to \mathbb{R}$ by $\tilde{d}_n = d_n \circ (f_n \times f_n)$ is a composition of continuous functions, so it continuous.

Then d is the uniform limit of continuous functions, and hence is continuous.

So for all $x \in X$, $B_r(x) = \{y \in X \mid d(y, x) < r\}$ is open in $\sigma(X, \mathcal{F})$, as this is the preimage of $(-\infty, r)$ under d.

Theorem 3.1 (Tychonov's Theorem)**.** *The product of compact topological spaces is compact in the product topology.*

Proof: TODO

3.1 Weak Topologies on Vector Spaces

Let E be a real or complex vector space, and let $F \subseteq E^*$ be a subspace that separates the points of E. We want to consider the topology $\sigma(E, F)$.

For $f \in F$, let $P_f : E \to \mathbb{R}$ be defined by $P_f(x) = |f(x)|$. Then $\mathcal{B} = \{P_f \mid f \in F\}$ is a family of seminorms that separates the points of E. Then the topology of the LCS $(E, \mathcal{B})$ is exactly the same as $\sigma(E, F)$, by the definition of the topology. In particular, $\sigma(E, F)$ makes E into a Hausdorff TVS.

Lemma 3.1. *Let E be a vector space and $f, g_1, \ldots, g_n$ be linear functions. If*

$$\bigcap_{i=1}^{n} \ker g_i \subseteq \ker f,$$

then $f \in \operatorname{span}\{g_1, \ldots, g_n\}$.

Proof: Define $T : E \to \mathbb{R}^n$ or $\mathbb{C}^n$ by

$$T(x) = (g_1(x), \ldots, g_n(x)).$$

Then T is linear, and

$$\ker T = \bigcap_{i=1}^{n} \ker g_i \subseteq \ker f.$$

So there is a map $\tilde{f}$ from $\operatorname{Im} T$ to the scalars such that

$$f = \tilde{f} \circ T,$$

and moreover $\tilde{f}$ is linear. We can extend $\tilde{f}$ to the entirety of $\mathbb{R}^n$ or $\mathbb{C}^n$ linearly, and so $\tilde{f}$ takes the form

$$\tilde{f}(y_1, \ldots, y_n) = a \cdot y = \sum_{i=1}^{n} a_i y_i.$$

Hence as $f = \tilde{f} \circ T$, we have

$$f(x) = \sum_{i=1}^{n} a_i g_i(x),$$

for all $x \in E$, as required.

Proposition 3.2. *Let E, F be as above, and f be a linear functional on E. Then f is continuous with respect to $\sigma(E, F)$ if and only if $f \in F$.*

In other words, $(E, \sigma(E, F))^* = F$.

Proof: The reverse direction is clear by definition, as $\sigma(E, F)$ is the smallest topology making all $f \in F$ continuous.

For the forwards, assume f is continuous with respect to $\sigma(E, F)$. Then $V = \{x \in E \mid |f(x)| < 1\}$ is a neighbourhood of 0 in E.

Hence there exists $g_1, \ldots, g_n \in F$, and $\varepsilon > 0$ such that

$$U = \{x \in E \mid |g_j(x)| \leq \varepsilon \text{ for all } 1 \leq j \leq n\} \subseteq V.$$

If $x \in \bigcap \ker g_j$, then for every scalar λ, $\lambda x \in U$, so $|f(\lambda x)| < 1$. Hence $f(x) = 0$. This shows that

$$\bigcap_{j=1}^{n} \ker g_j \subseteq \ker f.$$

This implies that $f \in \operatorname{span}\{g_1, \ldots, g_n\} \subseteq F$.

Example 3.2.

Let X be a normed space.

1. The *weak topology* on X is $\sigma(X, X^*)$. Note that X^* separates points of X by Hahn-Banach.

   We write (X, w) for $(X, \sigma(X, X^*))$. Open sets in $\sigma(X, X^*)$ are called *weakly open*. Note that $U \subseteq X$ is weakly open if and only if, for all $x \in U$, there is $f_1, \ldots, f_n \in X^*$ and $\varepsilon > 0$ such that

   $$\{y \mid |f_j(y - x)| < \varepsilon, 1 \leq j \leq n\} \subseteq U.$$

2. The *weak-$*$ topology* on X^* is $\sigma(X^{*,X)}$, regarding X as a subset of X^{**}. We write (X^*, w^*) for $(X^*, \sigma(X^*, X))$.

   Open sets in $\sigma(X^*, X)$ are *weak-$*$ open*. Then, $U \in X^*$ is weak-$*$ open if and only if, for all $f \in U$, there exists $x_1, \ldots, x_n \in X$, and $\varepsilon > 0$ such that

$$\{g \in X^* \mid |(g - f)(x_j)| < \varepsilon, 1 \leq j \leq n\} \subseteq U.$$

These topologies satisfy the following properties:

(i) (X, w) and (X^*, w^*) are LCSs, and hence Hausdorff with continuous functions of addition and scalar multiplication.

(ii) $\sigma(X, X^*)$ is a subset of the $\|\cdot\|$-induced topology of X, with equality if and only if $\dim X < \infty$.

(iii) $\sigma(X^*, X) \subseteq \sigma(X^*, X^{**})$, with equality if and only if X is reflexive.

(iv) Let Y be a subspace of X. Then

$$\sigma(X, X^*)|_Y = \sigma(Y, \{f|_Y \mid f \in X^*\}) = \sigma(Y, Y^*).$$

by Hahn-Banach. Similarly,

$$\sigma(X^{**}, X^*)|_X = \sigma(X, X^*).$$

So the canonical embedding $X \to X^{**}$ is a weak-to-weak-$*$ homeomorphism between X and $\hat{X}$.

Proposition 3.3. *Let X be a normed space.*

(i) *A linear function f on X is weakly-continuous $\iff$ $f \in X^*$, i.e. $(X, w)^* = X^*$.*

(ii) *A linear functional φ on X^* is weak-$*$-continuous $\iff$ $\varphi \in X$, i.e. $\varphi = \hat{x}$ for some $x \in X$. So $(X^*, w^*)^* = X$.*

Moreover, $\sigma(X^*, X^{**}) = \sigma(X^*, X) \iff X$ is reflexive.

Proof: (i) and (ii) are special cases of proposition 3.2.

If X is reflexive, then clearly $\sigma(X^*, X^{**}) = \sigma(X^*, X)$.

On the other hand, if $\sigma(X^*, X^{**}) = \sigma(X^*, X)$, then the duals are equal, in particular $X^{**} = X$.

Definition 3.1. Let X be a normed space.

(i) $A \subseteq X$ is *weakly bounded*if $\{f(x) \mid x \in A\}$ is bounded for all $f \in X^*$. This is true if and only if, for all weak-neighbourhoods U of 0, there exists $\lambda > 0$ with $A \subseteq \lambda U$.

(ii) $B \subseteq X^*$ is *weak-$*$ bounded* if $\{f(x) | f \in B\}$ is bounded for all $x \in X$. This is true if and only if, for all weak-$*$-neighbourhoods U of 0, there exists $\lambda > 0$ with $B \subseteq \lambda U$.

Recall the following:

Lemma 3.2 (Principle of Uniform Boundedness)**.** *Let X, Y be normed spaces with X complete, and $\mathcal{T} \subseteq \mathcal{B}(X, Y)$ be pointwise bounded, i.e. for all $x \in X$,*

$$\sup_{T \in \mathcal{T}} \|Tx\| < \infty.$$

Then

$$\sup_{T \in \mathcal{T}} \|T\| < \infty.$$

Proposition 3.4. *Let X be a normed space.*

(i) *If $A \subseteq X$ is weakly bounded, then A is norm bounded.*

(ii) *If $B \subseteq X^*$ is weak-$*$ bounded and X is complete, then B is norm bounded.*

Proof:

(i) Consider $\{\hat{x} \mid x \in A\} \subseteq X^{**} = \mathcal{B}(X^*)$. X^* is complete, so since by assumption $\hat{x}$ is pointwise bounded, it is uniformly bounded. Hence

$$\sup_{x \in A} \|x\| = \sup_{x \in A} \|\hat{x}\| < \infty.$$

(ii) $B \subseteq X^* = B(X)$. By assumption, B is pointwise bounded, hence by the principle of uniform boundedness, it is norm bounded.

Definition 3.2. We say that (x_n) *converges weakly* to x in some normed space X if $x_n \to x$ in $\sigma(X, X^*)$. We write

$$x_n \xrightarrow{w} x.$$

Then note $x_n \xrightarrow{w} x \iff \langle x_n, f \rangle \to \langle x, f \rangle$ for all $f \in X^*$.

We say that (f_n) *converges weak-$*$* to f in some X^* if $f_n \to f$ in $\sigma(X^*, X)$. We write

$$f_n \xrightarrow{w^*} f.$$

Again, $f_n \xrightarrow{w^*} f \iff \langle x, f_n \rangle \to \langle x, f \rangle$ for all $x \in X$.

Recall the following consequence of PUB: let X, Y are normed spaces, X complete and (T_n) in $\mathcal{B}(X, Y)$ which converges to T pointwise.

Then $T \in \mathcal{B}(X, Y)$, $\sup \|T_n\| < \infty$ and

$$\|T\| \leq \liminf \|T_n\|.$$

Proposition 3.5. *Let X be a normed space.*

(i) *If $x_n \xrightarrow{w} x$ in X, then $\sup \|x_n\| < \infty$, and $\|x\| \leq \liminf \|x_n\|$.*

(ii) *If $f_n \xrightarrow{w^*} f$ in X^* and X is complete, then $\sup \|f_n\| < \infty$, and $\|f\| \leq \liminf \|f_n\|$.*

Proof: Analogous to proposition 3.4 (look at $\{x_n\} \subseteq X^{**}$, use PUB and the above corollary, and similarly for $\{f_n\}$).

3.2 Hahn-Banach Separation Theorems

Let $(X, \mathcal{P})$ be a LCS. Let C be a convex subset of X with $0 \in \text{int}C$. For $x \in X$, let

$$\mu_C(x) = \inf\{t > 0 \mid x \in tC\}.$$

This is well-defined: $x/n \to 0$ as $n \to \infty$, so there exists $n \in \mathbb{N}$ with $x/n \in C$, i.e. $x \in nC$.

The function $\mu_C : X \to \mathbb{R}^+$ is the *Minkowski functional* of C.

Example 3.3.

In a normed space,

$$\mu_{B_X} = \|\cdot\|.$$

Lemma 3.3. *μ_C is positive homogeneous and subadditive. Moreover,*

$$\{x \in X \mid \mu_C(x) < 1\} \subseteq C \subseteq \{x \in X \mid \mu_C(x) \leq 1\},$$

with equalities on the left and right if C is open or closed, respectively.

Proof:

For positive homogeneity, $\mu_C(0) = 0$ since $0 \in C$ and hence $0 \in tC$ for all $t > 0$.

For $t > 0$ and $x \in X$, $x \in sC \iff tx \in stC$ so $\mu_C(tx) = t\mu_C(x)$.

If $\mu_C(x) < t$, then there is t' with $0 < t' < t$ such that $x \in t'C$. Then,

$$\frac{x}{t} = \frac{t'}{t}\frac{x}{t'} + \left(1 - \frac{t'}{t}\right) 0 \in C$$

as C is convex, so $x \in tC$.

Now let $x, y \in X$, and fix $s > \mu_C(x)$, $t > \mu_C(y)$. Then $x \in sC$, $y \in tC$, and

$$\frac{x+y}{s+t} = \frac{s}{s+t}\frac{x}{s} + \frac{t}{s+t}\frac{y}{t} \in C$$

by convexity, so $x + y \in (s+t)C$. This shows that $\mu_C(x+y) \leq s+$. It follows that $\mu_C(x + y) \leq \mu_C(x) + \mu_C(y)$.

If $\mu_C(x) < 1$, then $x \in C$ by the observation. Moreover if $x \in C$, then $\mu_C(x) \leq 1$.

If C is open, pick $x \in C$. As $(1 + \frac{1}{n})x \to x \in C$, for some n we must have $(1 + \frac{1}{n})x \in C$, and hence $\mu_C(x) \leq 1/(1 + \frac{1}{n}) < 1$.

If C is closed and $\mu_C(x) \leq 1$, then $\mu_C((1 - \frac{1}{n})x) = (1 - \frac{1}{n})\mu_C(x) < 1$, so $(1 - \frac{1}{n})x \in C$. Since $(1 - \frac{1}{n})x \to x$ and C is closed, $x \in C$.

Remark. If, in the real case C is symmetric, or in the complex case C is balanced, then μ_C is a seminorm.

If we further assume that C is bounded, then μ_C is a norm.

Theorem 3.2 (Hahn-Banach Separation Theorem)**.** *Let $(X, \mathcal{P})$ be a LCS, C an open convex set in X with $0 \in C$. Let $x_0 \in X \setminus C$. Then there exists $f \in X^*$ such that*

$$f(x_0) > f(x) \text{ for all } x \in C,$$

in the real case, or

$$\Re f(x_0) > \Re f(x) \text{ for all } x \in C,$$

in the complex case.

Proof: We show that the real case implies the complex case.

Think of the complex space X as a real space, and find a real-linear continuous functional g on X such that $g(x_0) > g(x)$ for all $x \in C$. Then $f(x) = g(x) - ig(ix)$ defines a complex-linear continuous functional on X such that $\Re f = g$.

So we can assume the scalar field is $\mathbb{R}$.

By lemma 3.3, $C = \{x \in X \mid \mu_C(x) < 1\}$, and $\mu_C(x_0) \geq 1$. Let $Y = \mathrm{span}\{x_0\}$, and define $g : Y \to \mathbb{R}$ by

$$g(\lambda x_0) = \lambda \mu_C(x_0).$$

For $\lambda \geq 0$, $g(\lambda x_0) = \mu_C(\lambda x_0)$. For $\lambda < 0$, $g(\lambda x_0) \leq 0 \leq \mu_C(\lambda x_0)$. So $g \leq \mu_C$ on Y.

By Hahn-Banach, there exists real-linear $f : X \to \mathbb{R}$ with $f|_Y = g$ and $f \leq \mu_C$ on X. For $x \in C$, $f(x) \leq \mu_C(x) < 1 \leq \mu_C(x_0) = f(x_0)$.

For $\varepsilon > 0$, $|f| < \varepsilon$ on the neighbourhood $\varepsilon C \cap (-\varepsilon)C$ of 0. So f is continuous at 0, and hence $f \in X^*$.

Theorem 3.3. *Let A, B be non-empty, disjoint convex subsets of a LCS.*

(i) *If A is open, then there exists $f \in X^*$ and $\alpha \in \mathbb{R}$ such that $f(a) < \alpha \leq f(b)$ for all $a, b \in A, B$.*

(ii) *If A is compact and B is closed, then there exists $f \in X^*$ such that $\sup_A f < \inf_B f$.*

Proof:

(i) Fix $a_0 \in A$, $b_0 \in B$. Set $x_0 = b_0 - a_0$, and $C = A - B + x_0$. Then C is convex,

$$C = \bigcup_{b \in B} (A - b + x_0)$$

is open, $0 \in C$ and $x_0 \notin C$, as $A \cap B \neq \emptyset$. By the previous theorem, there exists $f \in X^*$ such that $f(a - b + x_0) < f(x_0)$ for all $a \in A$, $b \in B$, i.e. $f(a) < f(b)$.

So if $\alpha = \inf_B f$, then $f(a) \leq \alpha \leq f(b)$ for all $a \in A$, $b \in B$. Note $f \neq 0$, so we can fix $u \ni X$ such that $f(u) > 0$. Then $a + u/n \to a$ as $n \to \infty$, so

$a + u/n \in A$ for some n This gives

$$f(a) < f\left(a + \frac{u}{n}\right) \leq \alpha.$$

(ii) We show that there exists an open convex neighbourhood V of 0 such that $(A+V) \cap B \neq \emptyset$. For $x \in A$ there exists a neighbourhood U_x of 0 such that $(x + U_x) \cap B \neq \emptyset$. Addition is continuous, so there exists a neighbourhood V_x of 0 such that $V_x + V_x \subseteq U_x$. Let V_x be open and convex.

A is compact, so there exists $x_1, \ldots, x_n \in A$ such that

$$A \subseteq \bigcup_{i=1}^{n} (x_i + V_{x_i}).$$

Let $V = \bigcap V_{x_i}$. Then this is a convex open neighbourhood of 0 and $(A + V) \cap B = \emptyset$. Indeed, $a \in x_n + V_{x_i}$, so $a + v \in x_i + V_{x_i} + V_{x_i} \subseteq x_i + U_{x_i}$ which is disjoint from B.

Now $A + V$ is open and convex, so by (i), there exists $f \in X^*$ and $\alpha \in \mathbb{R}$ such that

$$f(\alpha + v) < \alpha \leq f(b),$$

for all $a \in A, v \in V$ and $b \in B$. As $f \neq 0$, choose $z \in X$ with $f(z) > 0$. Then $z/n \to 0$, so for some n, $z/n \in V$. Then for all $a \in A$,

$$f(a) = f\left(a + \frac{z}{n}\right) - f\left(\frac{z}{n}\right) \leq \alpha - f\left(\frac{z}{n}\right) < \alpha.$$

Hence $\sup_A f < \inf_B f$.

Theorem 3.4 (Mazur's Theorem). *Let X be a normed space and C a convex subset of X. Then $\overline{C}^w = \overline{C}^{\|\cdot\|}$. So C is weakly-closed if and only if C is norm closed.*

Proof: Note that $\overline{C}^w \supseteq \overline{C}^{\|\cdot\|}$ since the weak topology is weaker than the norm topology.

If $x \notin \overline{C}^{\|\cdot\|}$, then applying theorem 3.3 part (ii) with $A = \{x\}$, $B = \overline{C}^{\|\cdot\|}$, and X with the $\|\cdot\|$-topology, then there is $f \in X^*$ such that $f(x) < \inf_B f \leq \inf_C f = \alpha$.

Then $\{y \in X \mid f(y) < \alpha\}$ is a weak neighbourhood of x disjoint from C. So

$x \notin \overline{C}^w$.

Corollary 3.1. *If $x_n \xrightarrow{w} 0$ in a normed space X, then*

$$0 \in \overline{\operatorname{conv}\{x_n \mid n \in \mathbb{N}\}}^{\|\cdot\|}.$$

Proof: Let $C = \operatorname{conv}\{x_n \mid n \in \mathbb{N}\}$. Then $0 \in \overline{C}^w = \overline{C}^{\|\cdot\|}$.

Remark. So there exists $p_1 < q_1 < p_2 < q_2 < \cdots$ in $\mathbb{N}$ and convex combinations

$$\sum_{i=p_n}^{q_n} t_i x_i \to 0$$

in $\|\cdot\|$.

Index