III Statistical Field Theory

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Based on Lectures by Prof. Harvey Reall

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Contents

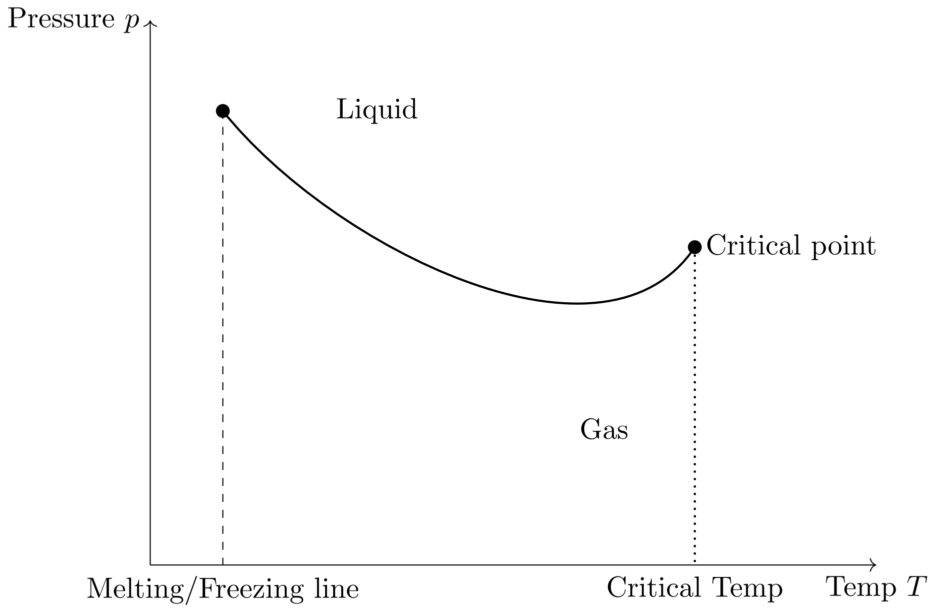
0 Introduction

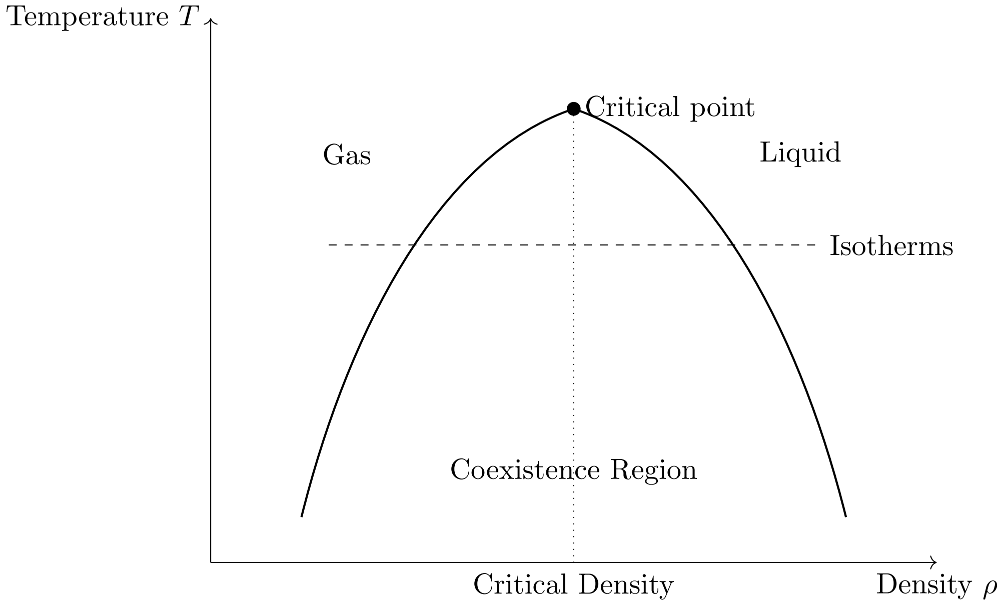
Office hours: Friday 2-3pm, in B2.09. We are following Tong's notes and example sheets.

Books include the one by Goldenfeld, and Kardar.

0.1 Motivation

Universality: sometimes very different physical systems exhibit the same behaviour, for example a liquid-gas system.





Experimentally,

$$|\rho_t - \rho_c| \propto |T - T_c|^{\beta},$$

for $\beta \simeq 0.327$.

Now consider a ferromagnet, where T_c is the Curie temperature. For $T > T_c$, the magnetization is $M = 0$, however for $T < T_c$, $T \sim T_c$, we find

$$M \propto (T_c - T)^{\beta},$$

where the coefficient β is experimentally also 0.327. We want to know why this is, and also why β is this constant.

In this course, we are looking at the classical statistical mechanics of fields.

1 From Spins to Fields

1.1 The Ising Model

This is a simple model for a magnet. In d spatial dimensions, consider a lattice with N sites. On the i'th site, we have a 'spin' $s_i \in \{-1, 1\}$.

A configuration of spins $\{s_i\}$ has energy

$$E = -B \sum_i s_i - J \sum_{\langle i,j \rangle} s_i s_j.$$

An important question is, how does the physics depend on the parameters B, J and T?

If $J > 0$, then the spins prefer to align, as $\uparrow\uparrow$ or $\downarrow\downarrow$. This is a ferromagnet. If $J < 0$, the spins prefer to antialign, as $\uparrow\downarrow$ or $\downarrow\uparrow$. This is an antiferromagnet.

Assume that $J > 0$. If $B > 0$, then the spins prefer to be $\uparrow$, and for $B < 0$, the spins prefer to be $\downarrow$.

Now let's consider changing temperature. Intuitively, for low temperature T, the system prefers to minimize E, which gives an ordered state. For high T, we want to maximize S, the entropy, which gives a disordered state.

In the canonical ensemble, we have

$$p[s_i] = \frac{e^{-\beta E[s_i]}}{Z},$$

where we recall $\beta = 1/T$. Moreover we always assume $k_B = 1$. Here Z is the partition function

$$Z(T, B) = \sum_{\{s_i\}} e^{-\beta E[s_i]}.$$

The *thermodynamic free energy* is

$$F_{\text{thermo}}(T, B) = \langle E \rangle - TS = -T \log Z.$$

Another observable is the *magnetization*

$$m = \frac{1}{N} \left\langle \sum_{i=1}^{n} s_i \right\rangle \in [-1, 1].$$

This distinguishes ordered phases, where $m \neq 0$, and disordered phases, where $m = 0$. Using the partition function,

$$m = \sum_{\{s_i\}} \frac{e^{-\beta[s_i]}}{Z} \cdot \frac{1}{N} \sum_i s_i = \frac{1}{N\beta} \frac{\partial}{\partial B} \log Z.$$

Therefore it suffices to find the partition function. For $d = 1$, this is easy. For $d = 2$ there is no analytic solution except for the square lattice with $B = 0$.

For the other cases there is no exact solution. Our aim is to approximate in a way that correctly captures long-distance behaviour. We define m for any $\{s_i\}$ by

$$m = \frac{1}{N} \sum s_i.$$

Then write

$$Z = \sum_m \sum_{\{s_i\}|m} e^{-\beta E[s_i]} := \sum_m e^{-\beta F(m)}.$$

Notice changing s_i changes m by $2/N$ so m is quantized into distances of $2/N$. For large N, we can approximate this as continuous, so

$$Z \approx \frac{N}{2} \int_{-1}^{1} \mathrm{d}m\, e^{-\beta F(m)}.$$

$F(m)$ is the *effective free energy*. This depends on T, B and m. It contains more information than F_{thermo}. If $f(m) = F(m)/N$, then

$$Z \propto \int_{-1}^{1} \mathrm{d}m\, e^{-\beta N f(m)}.$$

For N large, $\beta f(m) = \mathcal{O}(1)$, as it is intensive, so this integral will be dominated by the minimum of f, where

$$\left.\frac{\partial f}{\partial m}\right|_{m=m_{\text{min}}} = 0.$$

Here m_{min} is the equilibrium value of the magnetization. By the saddle-point approximation,

$$Z \propto e^{-\beta N f(m_{\text{min}})} = e^{-\beta F(m_{\text{min}})}.$$

Thus,

$$F_{\text{thermo}}(T, B) = F(m_{\text{min}}(T, B), T, B).$$

However computing $F(m)$ is hard. For a first attempt, we use the "mean field approximation". Here we replace s_i with m, so

$$E = -B \sum_i m - J \sum_{\langle i,j \rangle} m^2 = -BNm - \frac{1}{2} NJqm^2.$$

In this, q is the number of nearest neighbours. For $d = 1$, $q = 2$. In general it is $2d$. In this approximation,

$$Z \approx \sum_m \Omega(m) e^{-\beta E[m]},$$

where $\Omega(m)$ is the number of configurations with average value m.

Let $N_\uparrow$ be the number of up spins, and $N_\downarrow = N - N_\uparrow$ the number of down spins. Then,

$$m = \frac{N_\uparrow - N_\downarrow}{N} = \frac{2N_\uparrow - N}{N},$$

so

$$\Omega(m) = \frac{N!}{N_\uparrow!(N-N_\uparrow)!} = \frac{N!}{N_\uparrow! N_\downarrow!}.$$

By Stirling's approximation,

$$\log n! = n \log n - n$$

for large n, so

$$\log \Omega \approx N \log N - N_\uparrow \log N_\uparrow - N_\downarrow \log N_\downarrow.$$

Dividing by N and substituting in m for $N_\uparrow, N_\downarrow$,

$$\frac{\log \Omega}{N} \approx \log 2 - \frac{1}{2}(1+m)\log(1+m) - \frac{1}{2}(1-m)\log(1-m).$$

Since

$$e^{-\beta N f(m)} = \Omega(m) e^{-\beta E(m)}$$

in the mean field approximation, taking the logarithm we find

$$f(m) = -Bm - \frac{1}{2}Jqm^2 - T\left[\log 2 - \frac{1}{2}(1+m)\log(1+m) - \frac{1}{2}(1-m)\log(1-m)\right].$$

We minimize

$$\begin{aligned}\frac{\partial f}{\partial m} = 0 &\implies \beta(B + Jqm) = \frac{1}{2}\log\left(\frac{1+m}{1-m}\right)\\ &\implies m = \tanh[\beta(B+Jqm)],\end{aligned}$$

where $B_{\text{eff}} = B + Jqm$. The intuition is that each spin feels an effective magnetic field, given by the actual magnetic field, and the overall spin.

1.2 Landau Theory of Phase Transitions

At a phase transition, some quantity (an *order parameter*) is not smooth. For us, this is m. For small m,

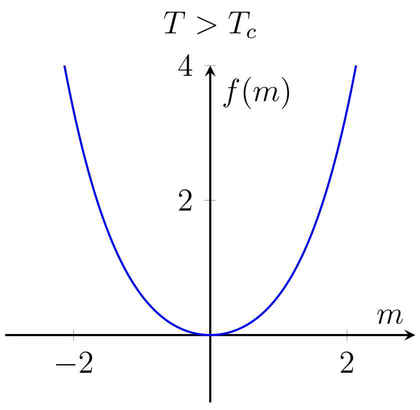
$$f(m) \approx -T\log 2 - Bm + \frac{1}{2}(T - Jq)m^2 + \frac{1}{12}Tm^4 + \cdots.$$

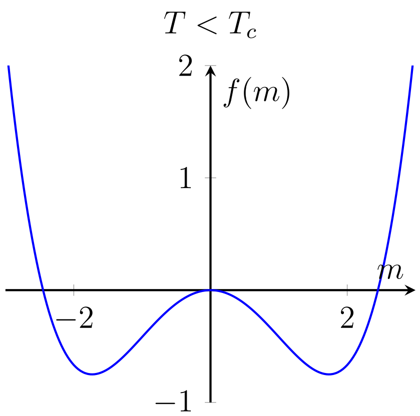
In equilibrium, $m = m_{\text{min}}$: how does this behave as we vary T and B?

First we look at $B = 0$. Note the first part does not change the minimization, so

$$f(m) \approx \frac{1}{2}(T - T_c)m^2 + \frac{1}{12}Tm^4 + \cdots ,$$

where $T_c = Jq$.

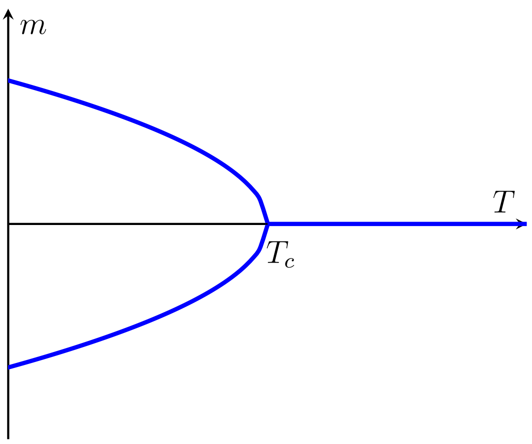




For $T < T_c$, $m_{\text{min}} = 0$. For $< T_c$, $m_{\text{min}} = \pm m_0$, where

$$m_0 = \sqrt{\frac{3(T_c - T)}{T}}.$$

Hence here is a phase transition at $T = T_c$. For $T > T_c$, we have $m = 0$, which is a disordered phase, and for $T < T_c$, $m \neq 0$, giving an ordered phase.



Here, m is continuous at $T = T_c$, giving a continuous phase transition, or second order phase transition.

Note that F is invariant under $\mathbb{Z}_2$ symmetry: if we swap $m \to -m$, and $B \to -B$. For $T < T_c$, either $m = +m_0$ or $m = -m_0$, the $\mathbb{Z}_2$ symmetry does not preserve the ground state, known as 'spontaneous symmetry breaking' (SSB).

At finite N, Z is *analytic* in T, B. Therefore the phase transition only occurs for $N \to \infty$, and SSB also only occurs for $N \to \infty$: in this case

$$m = \lim_{B \to 0} \lim_{N \to \infty} \frac{1}{N} \sum \langle s_i \rangle.$$

Note the order of the limits are important, otherwise we get 0.

For finite N, the $\mathbb{Z}_2$ symmetry tells us F_{thermo} is even in B, so

$$\langle m \rangle = -\frac{1}{N} \frac{\partial F_{\text{thermo}}}{\partial B}\bigg|_{B=0} = 0.$$

The heat capacity is

$$C = \frac{\partial \langle E \rangle}{\partial T}, \qquad \langle E \rangle = -\frac{\partial \log Z}{\partial \beta}.$$

Thus we find

$$C = \beta^2 \frac{\partial^2 \log Z}{\partial \beta^2},$$

and notice that

$$\log Z = -\beta N f(m_{\text{min}}) = \begin{cases} \text{const} & T > T_c, \\ \frac{3N}{4} \frac{(T_c - T)^2}{T^2} + \text{const} & T < T_c, \end{cases}$$

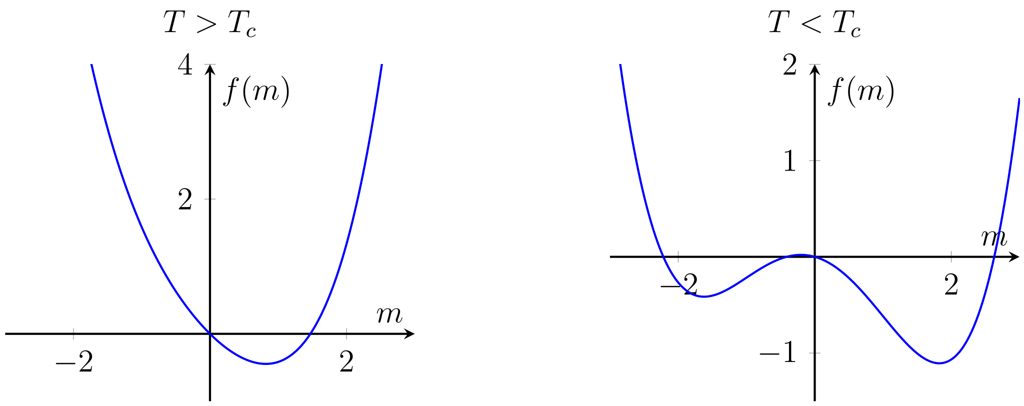
so

$$c = \frac{C}{N} \to \begin{cases} 0 & T \to T_c^+, \\ 3/2 & T \to T_c^-. \end{cases}$$

Therefore c is discontinuous at $T = T_c$.

For $B > 0$, we find

$$f(m) = -Bm + \frac{1}{2}(T - T_c)m^2 + \frac{1}{12}Tm^4 + \cdots$$

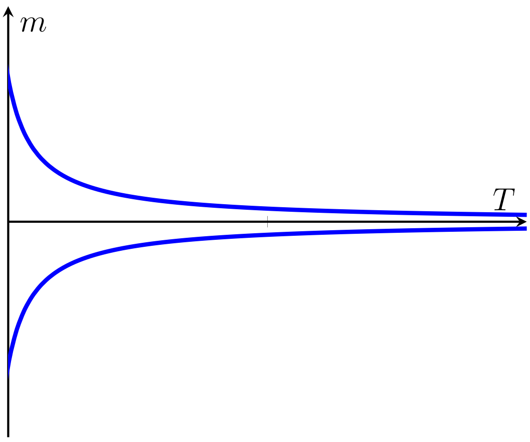


In this case,

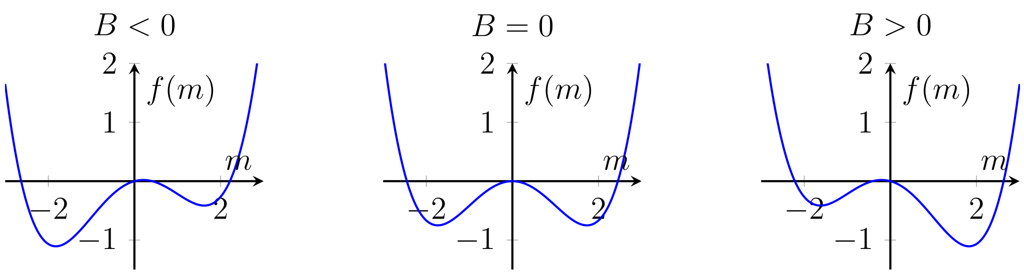
$$m_{\text{min}} = \frac{B}{T},$$

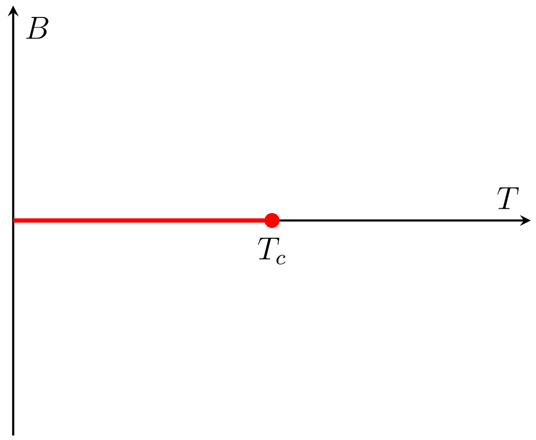
for $T \to \infty$.

In this case m_{min} depends smoothly on T, so there is no phase transition if T is varied at fixed $B \neq 0$.



But, if we vary B at fixed $T < T_c$, then m jumps discontinuously from m_0 to $-m_0$ as B decreases from positive to negative, which is an example of a first order phase transition.





We can draw the phase transition points as above.

Consider the behaviour near the critical point. If we fix $T = T_c$, then

$$f \approx -Bm + \frac{1}{12}Tm^4 + \cdots ,$$

so minimizing, we see that $m^3 \sim B$, and $m \sim B^{1/3}$.

We can also define the *magnetic susceptibility* as

$$\chi = \left(\frac{\partial m}{\partial B}\right)_T .$$

For $T > T_c$,

$$f(m) = -Bm + \frac{1}{2}(T - T_c)m^2 + \cdots ,$$

so

$$m \approx \frac{B}{T - T_c} \implies \chi = \frac{1}{T - T_c}.$$

For $T < T_c$, we write $m = m_0 + \delta m$, and solving for δm to leading order, we find

$$m = m_0 + \frac{B}{2(T_c - T)} \implies \chi = \frac{1}{2(T_c - T)}.$$

Hence,

$$\chi \sim \frac{1}{|T - T_c|}.$$

We have been using the MFT approximation. Does this give the correct results?

- $d = 1$. No, there is no phase transition.

- $d = 2, 3$. The phase diagram is qualitatively correct, but the qualitative predictions at the critical point are incorrect.
- $d \geq 4$. Yes.

Similarly for other systems, MFT gets the phase structure wrong for $d \leq d_l$, the 'lower critical dimension', and correct for $d \geq d_c$, the 'upper critical dimension'. In the Ising model, $d_l = 1$ and $d_c = 4$.

For $d_l < d < d_c$, the theory is interesting.

1.3 Critical Exponents

Near the critical point, MFT predicts the following:

- If $B = 0$, then as $T \to T_c^-$,

$$\begin{aligned} m &\sim (T_c - T)^\beta && \text{with } \beta = \frac{1}{2}, \\ c &\sim c_\pm |T - T_c|^{-\alpha} && \text{with } \alpha = 0, \\ \chi &\sim \frac{1}{|T - T_c|^\gamma} && \text{with } \gamma = 1. \end{aligned}$$

- As $B \to 0$,

$$m \sim B^{1/\delta} \qquad \text{with } \delta = 3.$$

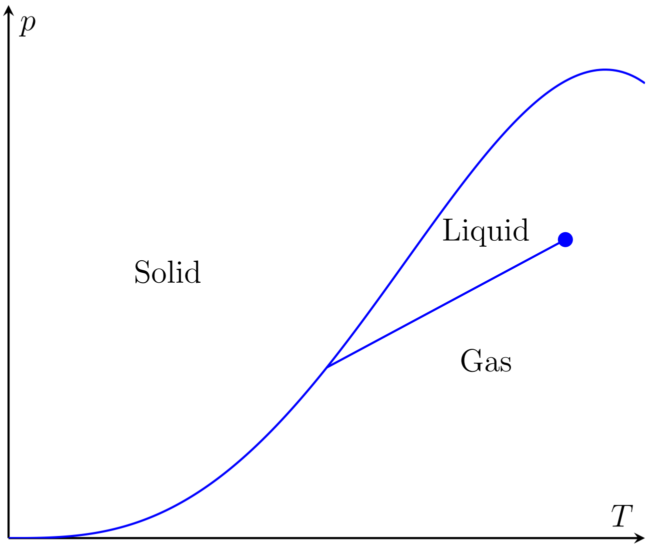
Here, $\alpha, \beta, \gamma, \delta$ are the *critical exponents*.

These values are not correct for small d.

	MFT	$d = 2$	$d = 3$
α	0 (disc)	0 (log)	0.1101..
β	$\frac{1}{2}$	$\frac{1}{8}$	0.3264..
γ	1	$\frac{7}{4}$	1.2371..
δ	3	15	4.7898..

Table 1: Predictions and Theoretical Critical Exponents

1.4 Universality



In a normal material, we have liquid-gas phase transition similar to the Ising model; a line of first order phase transitions, ending at a critical point.

If we replace B with p the pressure, and m with $v = V/N$ as our orde parameter, then using an equation of state (e.g. the van der Waals) to calculate the behaviour near the critical point:

- As $T \to T_c$,

$$v_{\text{gas}} - v_{\text{liquid}} \sim (T_c - T)^{\beta} \qquad \text{where } \beta = \frac{1}{2}.$$

- If $T = T_c$ is fixed, and $p \to p_c$,

$$v_{\text{gas}} - v_{\text{liquid}} \sim (p - p_c)^{1/\delta} \qquad \text{where } \delta = 3.$$

- The isothermal compatibility is

$$\kappa = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T \sim \frac{1}{|T - T_c|\gamma} \qquad \text{where } \gamma = 1.$$

- The heat capacity is

$$c_v \sim c_{\pm}|T - T_c|^{-\alpha} \qquad \text{where } \alpha = 0,$$

  i.e. the heat capacity is discontinuous.

These are the same predictions as for MFT for the Ising model. As is probably is expected, these are incorrect; but the correct values are the same as the correct values for $d = 3$ Ising model!

This is an example of *universality*: different physical systems can exhibit the same behaviour at the critical points.

This suggests that the microscopic physics is unimportant at a critical point. Systems governed by the same critical point belong to the same *universality class*.

1.5 Landau-Ginzberg Theory

Our aim is to find a model that correctly describes long-distance physics near the critical point, which can be used to calculate critical exponents for all theories in the same universality class.

LG theory generalizes MFT, to allow for spatial variation in m.

Here, $m(\mathbf{x})$ is a field, produced from a microscopic model by coarse-graining. In the Ising model, we divide the lattice into boxes, each with $N' \ll N$ sites, with size a. Then we can define

$$m(\mathbf{x}) = \text{average of spins in box with centre } \mathbf{x}.$$

Take $N' \gg 1$ the discreteness of $m(\mathbf{x})$ can be ignored, and $m \in [-1, 1]$.

Assume that $a \ll \xi$, the length scale over which the physics varies. Treat $m(\mathbf{x})$ as a smooth function, which does not vary on scales less than a. Then,

$$Z = \sum_{m(\mathbf{x})} \sum_{\{s_i\}|m(\mathbf{x})} e^{-\beta E[s_i]} = \sum_{m(\mathbf{x})} e^{-\beta F[m(\mathbf{x})]}.$$

Here, $F[m(\mathbf{x})]$ is a *functional*, which depends on a functional. This is the *Landau-Ginzberg free energy*.

We write

$$Z = \int \mathcal{D}m(\mathbf{x})\, e^{-\beta F[m(\mathbf{x})]},$$

which is a *functional integral*, a sum over all $m(\mathbf{x})$ that do not vary on scales less than a.

We can interpret this as a probability of a field configuration $m(\mathbf{x})$ as

$$p[m(\mathbf{x})] = \frac{e^{-\beta F[m(\mathbf{x})]}}{Z}.$$

The form of $F[m(\mathbf{x})]$ is constrained by the following:

- Locality: spins only influence nearby spins, so we may write

$$F[m(\mathbf{x})] = \int \mathrm{d}^d x\, f(m(\mathbf{x}), \nabla_i m(\mathbf{x}), \nabla_i \nabla_j m(\mathbf{x}), \ldots),$$

  not for example a function in x and y.
- Translational symmetry is inherited from discrete translational symmetry of the lattice.
- Can also inherit rotational symmetry.
- We also have a $\mathbb{Z}_2$ symmetry, by $s_i \to -s_i$, and $B \to -B$. So we can assume that F is invariant under $m(\mathbf{x}) \to -m(\mathbf{x})$, and $B \to -B$.
- Analyticity. We assume $F[m(\mathbf{x})]$ is defined by coarse graining over a finite number of spins. This lets us suppose f is *analytic*, i.e. we can Taylor expand near $m = 0$.

From dimensional analysis,

$$a\nabla m \sim \frac{am}{\xi}, \qquad a^2 \nabla\nabla m \sim \frac{a^2 m}{\xi^2}.$$

The fact $\xi \gg a$ suggests that ∇m is more important than $\nabla\nabla m$, etc.

From the above assumptions, for $B = 0$ we can write

$$F[m(\mathbf{x})] = \int \mathrm{d}^4 x \left(\frac{1}{2}\alpha_2(T) m^2 + \frac{1}{4}\alpha^4 m^4 + \frac{1}{2}\gamma(T)(\nabla m)^2 + \cdots \right).$$

If $B \neq 0$, we could include Bm, Bm^3 terms.

The coefficients $\alpha_2(T), \alpha_4(T)$ are hard to compute from first principles. From MFT, we expect that $\alpha_2(T) \sim T - T_c$, and $\alpha_4(T) \sim T/3$. But all we will assume is that the coefficients are analytic in T, with $\alpha_4(T) > 0$, $\gamma(T) > 0$, and $\alpha_2(T) > 0$ for $T > T_c$, with a simple zero at $T = T_c$.

Still, this field integral is hard to evaluate. Hence we will use a saddle point approximation.

Assume that the integral is dominated by the saddle point; the $m(\mathbf{x})$ that minimizes $F[m(\mathbf{x})]$. We can vary $m(\mathbf{x}) \to m(\mathbf{x}) + \delta m(\mathbf{x})$, with

$$\begin{aligned} \delta F &= \int \mathrm{d}^4 x (\alpha_2 m \delta m + \alpha_4 m^3 \delta m + \gamma \nabla m \cdot \nu \delta m + \cdots) \\ &= \int \mathrm{d}^4 x (\alpha_2 m + \alpha_4 m^3 - \gamma \nabla^2 m + \cdots) \delta m, \end{aligned}$$

by using integration by parts. The part in the brackets is often written as

$$\frac{\delta F}{\delta m(\mathbf{x})},$$

the *functional derivative*. If $m(\mathbf{x})$ minimizes F, then $\delta F = 0$ for all $\delta m(\mathbf{x})$, so

$$\gamma \nabla^2 m = \alpha_1 m + \alpha_4 m^3 + \cdots .$$

This is just the Euler-Lagrange equaions, I think? The simplest solution is when m is constant, which corresponds to MFT/Landau theory.

For $T > T_c$, $\alpha_2 > 0$, so $m = 0$, and for $T < T_c$, $\alpha_2 < 0$, so

$$m = \pm m_0 \sqrt{\frac{-\alpha_2}{\alpha_4}}.$$

1.6 Domain Walls

If $T < T_c$, then there are two ground states $m = \pm m_0$. We could have $m \to \pm m_0$ as $x \to \pm\infty$.

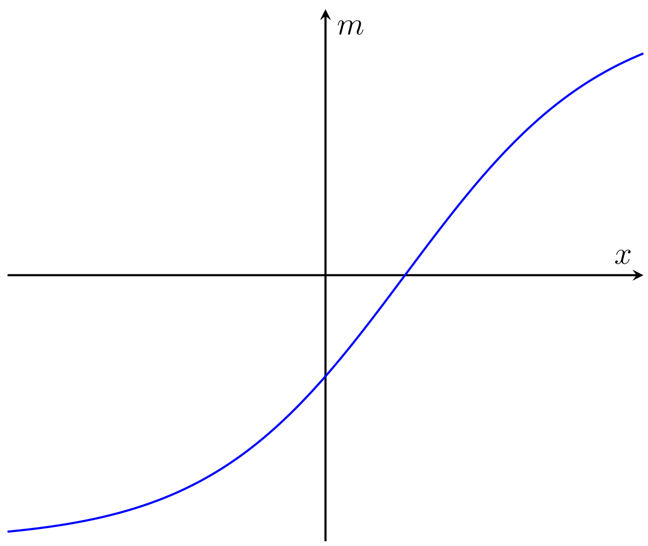
Consider a field that varies only in x, so $m(\mathbf{x}) = m(x)$. Then we need to solve

$$\gamma \frac{\mathrm{d}^2 m}{\mathrm{d}x^2} = \alpha_2 m + \alpha_4 m^3 + \cdots .$$

This is solved by

$$m = m_0 \tanh\left(\frac{x - X}{W}\right),$$

where X is some constant, and $W = \sqrt{-2\gamma/\alpha_2}$.



This is a *domain wall* with position X and width W.

If the system has size L, then the free energy is $F[m_0] \propto L^d$. The cost of the domain wall is

$$\Delta F = F[m(\mathbf{x})] - F[m_0] \sim L^{d-1}\sqrt{\frac{-\gamma\alpha_2^3}{\alpha_4^2}},$$

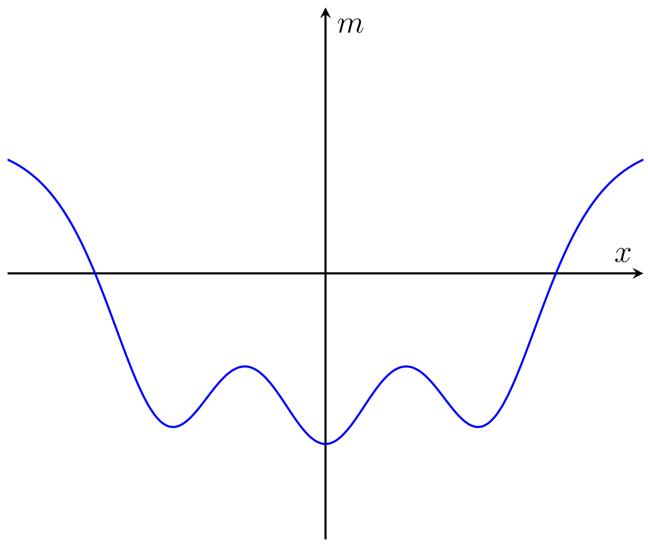
which is proportional to the area of the domain wall.

Near the critical point $\alpha_2 \to 0$ and $W \to \infty$, so $\Delta F \to 0$.

Domain walls explain why $d_l = 1$ for the Ising model. Let $d = 1$, and $-L/2 < x < L/2$, and assume $\alpha_2(T) < 0$.

The boundary conditions are $m(\pm L/2) = m_0$, and the probability of the configuration is

$$\frac{e^{-\beta F[m(\mathbf{x})]}}{Z}.$$



Now consider a field with two domain walls. Assume $L \gg W$. For well separated wells, $\Delta F_{\text{2wells}} = 2\Delta F$. Then,

$$\frac{\mathbb{P}(\text{2 domain walls at given } X, Y)}{p(m = m_0)} = e^{-2\beta\Delta F}.$$

Summing over all X, Y,

$$\frac{\mathbb{P}(\text{2 domain walls})}{\mathbb{P}(m = m_0)} \sim \int_{-L/2}^{L/2} \frac{\mathrm{d}X}{W} \int_X^{L/2} \frac{\mathrm{d}Y}{W} e^{-2\beta\Delta(F)} \sim \left(\frac{L}{W}\right)^2 e^{-2\beta\Delta(F)}.$$

For $d = 1$, the RHS goes to 0 as $L \to \infty$, showing that energy beats entropy. But for $d = 1$, the RHS tends to ∞, as entropy beats energy.

Hence for $d = 1$, it is much more probably to see two domain walls than a constant field, so any region with constant $m = \pm m_0$ is unstable to the formation of domain walls, so the ordered phase does not exist.

2 My First Path Integral

We want to go beyond the saddle point approximation to calculate

$$Z = \int \mathcal{D}m(\mathbf{x})\, e^{-\beta F[m(\mathbf{x})]}.$$

We make a couple of changes: first we change from $m(\mathbf{x})$ to $\phi(\mathbf{x})$, and also we assume $B = 0$, so

$$F[\phi(\mathbf{x})] = \int \mathrm{d}^d x \left(\frac{1}{2}\alpha_2(T)\phi^2 + \frac{1}{4}\alpha_4\phi^4 + \frac{1}{2}\gamma(T)(\nabla\phi)^2 + \cdots \right).$$

Evaluating the path integral is:

- Easy if F is quadratic in ϕ.
- Possibly if the higher order terms are small.
- Very hard otherwise.

For $T > T_c$, let $\mu^2 = \alpha_2(T) > 0$. Consider the quadratic approximation to F:

$$F[\phi(\mathbf{x})] = \frac{1}{2}\int \mathrm{d}^d x(\mu^2\phi^2 + \gamma(\nabla\phi)^2).$$

For $T < T_c$, $\alpha_2(T) < 0$, so $\langle\phi\rangle = \pm m_0$. Let $\tilde{\phi} = \phi - \langle\phi\rangle$. Then,

$$F = F[m_0] + \frac{1}{2}\int \mathrm{d}^d x(\alpha_2(T)\tilde{\phi}^2 + \gamma(T)(\nabla\tilde{\phi})^2 + \cdots).$$

Here

$$\alpha_2'(T) = \alpha_2(T) + 3m_0^2\alpha_4(T) = -2\alpha_2(T) > 0.$$

This quadratic approximation gives the same equation with ϕ replaced with $\tilde{\phi}$, and $\mu^2 = \alpha_2'(T) = 2|\alpha_2(T)| > 0$.

2.1 Thermodynamic Free Energy

We aim to compute the corrections to F_{thermo} from the fluctuation in $\phi(\mathbf{x})$. It is useful to take the Fourier transform

$$\phi_{\mathbf{k}} = \int \mathrm{d}^d x\, e^{-i\mathbf{k}\cdot\mathbf{x}}\phi(\mathbf{x}),$$

where since ϕ is real, $\phi_{\mathbf{k}}^* = \phi_{-\mathbf{k}}$. Here the wavevector $\mathbf{k}$ is often called the *momentum*. Since ϕ doesn't vary on scales less than a, we get $\phi_{\mathbf{k}} = 0$ for all $|\mathbf{k}| = \Lambda$, where $\Lambda = \pi/a$ is called the ultraviolet cutoff.

The inverse Fourier transform is given by

$$\phi(\mathbf{x}) = \int \frac{\mathrm{d}^d x}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{x}} \phi_{\mathbf{k}}.$$

In finite volume, consider a system occupying a cubic region with $V = L^d$. Then $\mathbf{k}$ takes discrete values, $\mathbf{k} = \frac{2\pi}{L}\mathbf{n}$, for some $\mathbf{n} \in \mathbb{Z}^d$. Thus, the integral becomes

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \cdots = \left(\frac{1}{L}\right)^d \sum_{\mathbf{n}} \cdots = V^{-1} \sum_{\mathbf{n}} \cdots ,$$

hence the inverse Fourier transform can be written as

$$\phi(\mathbf{x}) = \frac{1}{V} \sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} \phi_{\mathbf{k}}.$$

Substituting this into the form for F,

$$F[\phi_{\mathbf{k}}] = \frac{1}{2} \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \frac{\mathrm{d}^d k_2}{(2\pi)^d} \int \mathrm{d}^d x (\mu^2 - \gamma \mathbf{k}_1 \cdot \mathbf{k}_2) \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} e^{i(\mathbf{k}_1 + \mathbf{k}_2)\cdot\mathbf{x}}.$$

Using the fact the latter integral is a δ function, this simplifies to

$$\begin{aligned}
F &= \frac{1}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} (\mu^2 + \gamma \mathbf{k}^2) \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \\
&= \frac{1}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} (\mu^2 + \gamma \mathbf{k}^2 |\phi_{\mathbf{k}}|^2 \\
&= \int^{+} \frac{\mathrm{d}^d k}{(2\pi)^d} (\mu^2 + \gamma \mathbf{k}^2) |\phi_{\mathbf{k}}|^2 \qquad \text{integral over } k_x > 0 \\
&= \frac{1}{V} \sum_{k}^{+} (\mu^2 + \gamma \mathbf{k}^2) |\phi_{\mathbf{k}}|^2.
\end{aligned}$$

The measure is

$$\int \mathcal{D}\phi(\mathbf{x}) = \mathcal{N} \prod_{\mathbf{k}}^{+} \int \mathrm{d}(\Re\phi_{\mathbf{k}})\, \mathrm{d}(\Im\phi_{\mathbf{k}}),$$

so the partition function becomes

$$\begin{aligned}
Z &= \mathcal{N} \left(\prod_{\mathbf{k}} \int \mathrm{d}(\Re\phi_{\mathbf{k}})\, \mathrm{d}(\Im\phi_{\mathbf{k}}) \right) \exp\left[-\frac{\beta}{V} \sum_{\mathbf{k}}^{+} (\mu^2 + \gamma \mathbf{k}^2) |\phi_{\mathbf{k}}|^2 \right] \\
&= \mathcal{N} \prod_{\mathbf{k}}^{+} \int \mathrm{d}(\Re\phi_{\mathbf{k}})\, \mathrm{d}(\Im\phi_{\mathbf{k}}) \exp\left[-\frac{\beta}{V} (\mu^2 + \gamma \mathbf{k}^2)((\Re\phi_{\mathbf{k}})^2 + (\Im\phi_{\mathbf{k}})^2) \right].
\end{aligned}$$

Recall that
$$\int_{-\infty}^{\infty} e^{-x^2/a} = \sqrt{\pi a},$$
so we find that
$$e^{-\beta F_{\text{thermo}}} = Z = \mathcal{N} \prod_{\mathbf{k}}^{+} \left[\sqrt{\frac{\pi V T}{\mu^2 + \gamma \mathbf{k}^2}} \right]^2 = \mathcal{N} \prod_{\mathbf{k}}^{+} \frac{\pi V T}{\mu^2 + \gamma \mathbf{k}^2},$$
hence
$$\frac{F_{\text{thermo}}}{V} = -\frac{T}{V} \log Z = -\frac{T}{V} \sum_{\mathbf{k}}^{+} \log \left(\frac{\pi V T}{\mu^2 + \gamma \mathbf{k}^2} \right) - \frac{T}{V} \log N.$$
Now we compute the contribution of fluctuations to the heat capacity. Recall that
$$\langle E \rangle = -\frac{\partial \log Z}{\partial \beta} = \frac{\partial (\beta F_{\text{thermo}})}{\partial \beta},$$
and the heat capacity is
$$C = \frac{\partial \langle E \rangle}{T} = -\beta^2 \frac{\partial \langle E \rangle}{\partial \beta} = -\beta^2 \frac{\partial^2 (\beta F_{\text{thermo}})}{\partial \beta^2},$$
from which we get
$$\frac{C}{V} = -\beta^2 \frac{\partial^2}{\partial \beta^2} \left[-\frac{1}{V} \sum_{\mathbf{k}} \log \left(\frac{\pi V T}{\mu^2 + \gamma \mathbf{k}^2} \right) \right].$$
Take $\mu^2 = T - T_c$, and V a constant for simplicity. Taylor expanding,
$$\frac{C}{V} = \frac{1}{V} \sum_{\mathbf{k}}^{+} \left[1 + \frac{2T}{\mu^2 + \gamma \mathbf{k}^2} + \frac{T^2}{(\mu^2 + \gamma \mathbf{k}^2)^2} \right] = \frac{1}{2} \int \frac{\mathrm{d}^d \mathbf{k}}{(2\pi)^d} \left[\cdots \right].$$
The first term gives $\frac{1}{2} k_0$ per degree of freedom; this is equipartition.

The other terms may diverge as $T \to T_c$, and the integral may not converge at $\mathbf{k} = 0$, an example of infrared divergence. The final term is proportional to
$$\int_0^{\Lambda} \frac{\mathrm{d}k \, k^{d-1}}{(\mu^2 + \gamma k^2)^2} = \frac{\mu^{d-4}}{\gamma^{d/2}} \int_0^{\Lambda \sqrt{x/\mu^2}} \frac{\mathrm{d}x \, x^{d-1}}{(1 + x^2)^2} \sim \begin{cases} \Lambda^{d-4} & d > 4, \\ \mu^{d-4} & d < 4, \end{cases}$$
as $T \to T_c$. If $d = 4$, then this tends to $\log \Lambda$. Similarly, the second term is proportional to
$$\int_0^{\Lambda} \frac{\mathrm{d}k \, k^{d-1}}{\mu^2 + \gamma k^2} \sim \begin{cases} \Lambda^{d-2} & d > 2, \\ \mu^{-1} & d = 1. \end{cases}$$

as $T \to T_c$. Hence for $d \geq 4$, the contributions of fluctuations is finite as $T \to T_c$, and for $d < 4$ the contributions diverges. In general,

$$C \sim \mu^{d-4} \sim |T - T_c|^{-\alpha},$$

where $\alpha = 2 - d/2$. These fluctuations explain why the MFT value $\alpha = 0$ is incorrect. But this value is also wrong.

2.2 Correlation Functions

We know that

$$\langle \phi(\mathbf{x}) \rangle = \begin{cases} 0 & T > T_c, \\ \pm m_0 & T < T_c. \end{cases}$$

The fluctuations about this are captured by correlation functions:

$$\langle (\phi(\mathbf{x}) - \langle \phi(\mathbf{x}) \rangle)(\phi(\mathbf{y}) - \langle \phi(\mathbf{y}) \rangle) \rangle = \langle \phi(\mathbf{x}) \phi(\mathbf{y}) \rangle - \langle \phi(\mathbf{x}) \rangle \langle \phi(\mathbf{y}) \rangle.$$

We can compute this by including the magnetic field (B-field) in F:

$$F[\phi, B] = \int \mathrm{d}^d x \left(\frac{1}{2} \mu^2 \phi^2 + \frac{1}{2} \gamma (\nabla \phi)^2 - B(\mathbf{x}) \phi(\mathbf{x}) \right).$$

This becomes a functional in terms of B:

$$Z[B(\mathbf{x})] = \int \mathcal{D}\phi \, e^{-\beta F[\phi, B]}.$$

This contains a lot of information! For example,

$$\begin{aligned} \frac{1}{\beta} \frac{\delta \log Z}{\delta B(\mathbf{x})} &= \frac{1}{\beta Z} \frac{\delta Z}{\delta B(\mathbf{x})} = \frac{1}{2} \int \mathcal{D}\phi \, \phi(\mathbf{x}) e^{-\beta F} = \langle \phi(\mathbf{x}) \rangle_B, \\ \frac{1}{\beta^2} \frac{\delta^2 \log Z}{\delta B(\mathbf{x}) \delta B(\mathbf{y})} &= \frac{1}{\beta^2 Z} \frac{\delta^2 Z}{\delta B(\mathbf{x}) \delta B(\mathbf{y})} = \frac{1}{\beta^2 Z^2} \frac{\delta Z}{\delta B(\mathbf{x})} \frac{\delta Z}{\delta B(\mathbf{y})} \\ &= \langle \phi(\mathbf{x}) \phi(\mathbf{y}) \rangle_B - \langle \phi(\mathbf{x}) \rangle_B \langle \phi(\mathbf{y}) \rangle_B, \end{aligned}$$

hence at $B = 0$, this gives our correlation function. We can calculate $Z[B(\mathbf{x})]$ in Fourier space, as

$$F[\phi, B] = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left[\frac{1}{2} (\mu^2 + \gamma^2 \mathbf{k}^2) |\phi_{\mathbf{k}}|^2 - B_{-\mathbf{k}} \phi_{\mathbf{k}} \right].$$

We complete the square, by setting

$$\hat{\phi}_{\mathbf{k}} = \phi_{\mathbf{k}} - \frac{B_{\mathbf{k}}}{\mu^2 + \gamma \mathbf{k^2}}.$$

Then we find

$$F = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left[\frac{1}{2}(\mu^2 + \gamma \mathbf{k}^2)|\hat{\phi}_{\mathbf{k}}|^2 - \frac{1}{2}\frac{|B_{\mathbf{k}}|^2}{(\mu^2 + \gamma \mathbf{k}^2)} \right].$$

Shift $\phi_{\mathbf{k}} \to \hat{\phi}_{\mathbf{k}}$ in the path integral, so we are now integrating over $\hat{\phi}_{\mathbf{k}}$. So,

$$\begin{aligned} Z &= N \int \prod_k \mathrm{d}(\Re \hat{\phi}_{\mathbf{k}})\, \mathrm{d}(\Im \hat{\phi}_{\mathbf{k}}) e^{-\beta F} \\ &= e^{-\beta F_{\text{thermo}}} \exp\left(\frac{\beta}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{|B_{\mathbf{k}}|^2}{\mu^2 + \gamma \mathbf{k}^2} \right) \\ &= e^{-\beta F_{\text{thermo}}} \exp\left(\frac{\beta}{2} \int \mathrm{d}^d x\, \mathrm{d}^d y B(\mathbf{x}) G(\mathbf{x} - \mathbf{y}) B(\mathbf{y}) \right), \end{aligned}$$

where

$$G(\mathbf{x}) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{\mu^2 + \gamma \mathbf{k^2}}.$$

So now we can use this to find the correlation coefficient:

$$\langle \phi(\mathbf{x})\phi(\mathbf{y})\rangle - \langle \phi \rangle^2 = \frac{1}{\beta} G(\mathbf{x} - \mathbf{y}).$$

Let $\xi^2 = \gamma/\mu^2$. Then we claim that

$$G(\mathbf{x}) \sim \begin{cases} 1/\Gamma^{d-2} & r \ll \xi, \\ e^{-r/\xi}/(\xi^{(d-2)/2}\Gamma^{(d-1)/2}) & r \gg \xi. \end{cases}$$

This shows fluctuations occur over $|\mathbf{x}| < \xi$, and decay rapidly for $|\mathbf{x}| > \xi$. In general, this property defines the *correlation length* ξ.

If $\xi \to \infty$ as $T \to T_c$, then there are fluctuations on all length scales at the critical point. Moreover,

$$(-\gamma \nabla^2 + \mu^2) G(\mathbf{x}) = \delta^{(d)}(\mathbf{x}).$$

In other words, $G(\mathbf{x})$ is a Green's function for $-\gamma\nabla^2 + \mu^2$.

We can define another critical exponent

$$\xi \sim \frac{1}{|T - T_c|^\nu},$$

and at the critical point ($\xi = \infty$),

$$\langle \phi(\mathbf{x})\phi(0)\rangle \sim \frac{1}{\Gamma^{d02+\eta}}.$$

	MFT	$d=2$	$d=3$
η	0	1/4	0.0363...
ν	1/2	1	0.6300...

Table 2: Predictions and Theoretical Critical Exponents

Our predictions are $\eta = 0$, and $\nu = 1/2$.

We can also consider the susceptibility:

$$\chi = \left.\frac{\partial \langle \phi \rangle}{\partial B}\right|_{B=0}.$$

This is defined when there is no spatial variation. We can generalize to

$$\begin{aligned}\chi(\mathbf{x}, \mathbf{y}) = \left.\frac{\delta \langle \phi(\mathbf{x} \rangle_B}{\delta B(\mathbf{y})}\right|_{B=0} &= \frac{\delta}{\delta B(\mathbf{y})} \left.\left[\frac{1}{\beta} \frac{\delta \log 2}{\delta B(\mathbf{x})}\right]\right|_{B=0} \\ &= \beta(\langle \phi(\mathbf{x}) \phi(\mathbf{y}) \rangle - \langle \phi \rangle^2) = G(\mathbf{x} - \mathbf{y}).\end{aligned}$$

If there is no spatial variation,

$$\begin{aligned}\langle \phi \rangle &= \frac{1}{V} \int_V \mathrm{d}^d x \langle \phi(\mathbf{x}) \rangle \\ \delta \langle \phi \rangle &= \frac{1}{V} \mathrm{d}^d x \, \mathrm{d}^d y \frac{\delta \langle \phi(\mathbf{x}) \rangle}{\delta B(\mathbf{y})} \delta B(\mathbf{y}) = \frac{1}{V} \int \mathrm{d}^d x \, \mathrm{d}^d y \chi(\mathbf{x}, \mathbf{y}) \delta B,\end{aligned}$$

hence

$$\chi = \frac{1}{V} \int \mathrm{d}^d x \, \mathrm{d}^d y G(\mathbf{x} - \mathbf{y}) = \mathrm{d}^d x(\mathbf{x}).$$

This integral diverges at $|\mathbf{x}| \to \infty$ as $T \to T_c$, by substituting in the large r behaviour of G, as the integral is

$$\int r^{d-1} \, \mathrm{d}r \frac{e^{-r/\xi}}{\xi^{(d-2)/2} r^{(d-1)/2}} \sim \xi^2 \sim \frac{1}{|T - T_c|}.$$

The upper critical dimension is, for $T < T_c$, that

$$\langle \phi(\mathbf{x}) \rangle = \pm m_0.$$

The Ginzburg criterion says that MFT cannot be trusted when fluctuations in ϕ are large compared to $\langle \phi \rangle$. Define by

$$R = \int_{|x|,|y|<\xi} \mathrm{d}^d x \, \mathrm{d}^d y \, \langle (\phi(\mathbf{x}) - \langle \phi(\mathbf{x}) \rangle)(\phi(\mathbf{y}) - \langle \phi(\mathbf{y}) \rangle) \rangle \Big/ \left(\int_{|x|<\xi} \mathrm{d}^d x \langle \phi(\mathbf{x}) \rangle \right)^2.$$

This is dimensionless. Moreover by changing the coordinates $\mathbf{x}' = \mathbf{x} - \mathbf{y}$, this integral is approximated by

$$\begin{aligned}\frac{\int_{|\mathbf{x}|,|\mathbf{y}|<\xi} \frac{1}{\beta} G(\mathbf{x}) \,\mathrm{d}^d x \,\mathrm{d}^d y}{(\xi^d m_0)^2} &\sim \frac{\xi^d}{\xi^{2d} m_0^2} \int_0^\xi r^{d-1} \,\mathrm{d}r \frac{1}{r^{d-2}} \sim \frac{\xi^{2-d}}{m_0^2} \\ &\sim \frac{(|T-T_c|^{-1/2})^{2-d}}{(|T-T_c|^{1/2})^2} \sim |T-T_c|^{(d-4)/2}.\end{aligned}$$

Hence MFT is bad as $T \to T_c$ for $d < 4$.

3 The Renormalization Group

Now we include non-quadratic terms in F, for example φ^4.

3.1 The Key Idea

Suppose that

$$F[\varphi] = \int \mathrm{d}^d x \left[\frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}\mu^2\varphi^2 + g\varphi^4 + \cdots\right]$$

We allow all possible terms in F consistent with analyticity and the symmetries. For example, we could take φ^{12} or $\varphi^5(\nabla\varphi)^5\nabla^2\varphi$, but not φ^{17} due to $\mathbb{Z}_2$ symmetry, or φ^{-2} due to analyticity.

Each term has a coupling constant (e.g. μ^2, g in the above). These can be viewed as coordinates in ∞-dimensional "theory space". Then,

$$Z = \int \mathcal{D}\varphi\, e^{-F[\varphi]},$$

where we absorb β into F. This is well-defined because of the V cutoff: we say $\varphi_{\mathbf{k}} = 0$ for $|\mathbf{k}| > \Lambda$.

Suppose that we only care about the physics on a long-distance scale L. This corresponds to not caring about $\varphi_{\mathbf{k}}$ for $|\mathbf{k}| \gg L^{-1}$. Hence we try to construct a new theory with cutoff

$$\Lambda' = \Lambda/\zeta,$$

for some $\zeta > 1$. Write $\varphi_{\mathbf{k}} = \varphi^-_{\mathbf{k}} + \varphi^+_{\mathbf{k}}$, where

$$\varphi^-_{\mathbf{k}} = \begin{cases} \varphi_{\mathbf{k}} & |\mathbf{k}| < \Lambda', \\ 0 & |\mathbf{k}| > \Lambda', \end{cases} \qquad \varphi^+_{\mathbf{k}} = \begin{cases} \varphi_{\mathbf{k}} & \Lambda' < |\mathbf{k}| < \Lambda, \\ 0 & \text{else.} \end{cases}$$

Here $\varphi^-_{\mathbf{k}}$ corresponds to the long wavelength (infrared) modes, and $\varphi^+_{\mathbf{k}}$ are the short wavelength (ultraviolet) modes. We can decompose

$$F[\varphi_{\mathbf{k}}] = F_0[\varphi^-_{\mathbf{k}}] + F_0[\varphi^+_{\mathbf{k}}] + F_I[\varphi^-_{\mathbf{k}}, \varphi^+_{\mathbf{k}}].$$

Our partition function can then be written

$$\begin{aligned} Z &= \int \prod_{|\mathbf{k}|<\Lambda} \mathrm{d}\varphi_{\mathbf{k}}\, e^{-F[\varphi_{\mathbf{k}}]} \\ &= \int \prod_{|\mathbf{k}|<\Lambda'} \mathrm{d}\varphi_{\mathbf{k}}^-\, e^{-F_0[\varphi_{\mathbf{k}}^-]} \int \prod_{\Lambda'<|\mathbf{k}|<\Lambda} \mathrm{d}\varphi_{\mathbf{k}}^+\, e^{-F_0[\varphi_{\mathbf{k}}^+]-F_I[\varphi_{\mathbf{k}}^-,\varphi_{\mathbf{k}}^+]} \\ &= \int \prod_{|\mathbf{k}|<\Lambda'} \mathrm{d}\varphi_{\mathbf{k}}^-\, e^{-F'[\varphi_{\mathbf{k}}^-]}. \end{aligned}$$

Here we pretended to do the latter integral, to get $F'[\varphi_{\mathbf{k}}]$. This is the *Wilsonian effective free energy*, obtained by integrating out the ultraviolet modes to obtain a theory with a lower cutoff.

We must take the same ground form as F, since we included all possible terms. So,

$$F'[\varphi] = \int \mathrm{d}^d x \left(\frac{1}{2}\gamma'(\nabla\varphi)^2 + \frac{1}{2}\mu'^2\varphi^2 + g'\varphi^4 + \cdots \right).$$

Here the values γ', μ', g' of the coupling constants will differ. We want to compare with the original theory, but they have different cutoffs. So we rescale $\mathbf{k}' = \zeta\mathbf{k}$, such that

$$|\mathbf{k}'| < \zeta\Lambda' = \Lambda.$$

The similarly set $\mathbf{x}' = \mathbf{x}/\zeta$, so $\mathbf{k}\cdot\mathbf{x} = \mathbf{k}'\cdot\mathbf{x}'$. This corresponds to zooming out. Our integrals become, for example

$$\int \mathrm{d}^d x\, \frac{1}{2}\gamma'(\nabla\varphi)^2 = \int \zeta^d\, \mathrm{d}^d x'\, \frac{1}{2}\gamma'\zeta^{-2}(\nabla'\varphi)^2.$$

If we also rescale so that

$$\varphi'(\mathbf{x}') = \zeta^{(d-2)/2}\sqrt{\gamma'}\varphi(\mathbf{x}),$$

then our form of F' is

$$F'[\varphi'] = \int \mathrm{d}^d x' \left[\frac{1}{2}(\nabla'\varphi)^2 + \frac{1}{2}\mu(\zeta)^2\varphi'^2 + g(\zeta)\varphi'^4 + \cdots \right].$$

As ζ increases, we obtain a *flow* in theory space. The map $R(\zeta)$ from F to F' is a *renormalization group transformation*. It obeys

$$R(\zeta_1)R(\zeta_2) = R(\zeta_1\zeta_2).$$

But $R(\zeta)$ is not invertible (so not actually a group; instead a semigroup). This is because we lose information when we integrate out the higher energy modes.

In summary, there are three steps to the renormalization group:

- Integrate out the high momentum modes $\Lambda/\zeta < |\mathbf{k}| < \Lambda$.
- Rescale $\mathbf{k}' = \Lambda\mathbf{k}$, $\mathbf{x} = \mathbf{x}/\zeta$.
- Rescale our fields to get canonically normalized gradient terms.

This process ends by giving us a flow on couplings.

A key question is how RG flow behaves as $\zeta \to \infty$. There are a few possibilities.

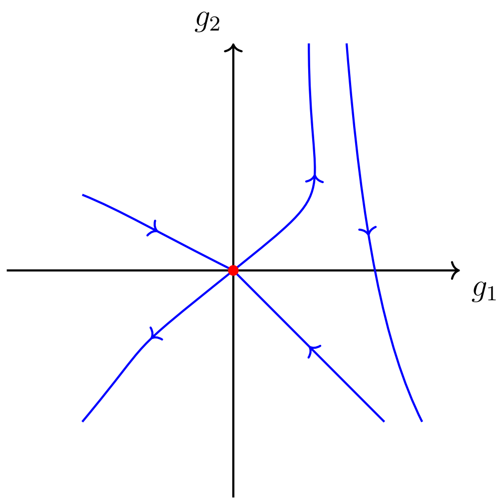
- The flow goes to ∞ in theory space, i.e. one of the couplings blows up.
- The flow approaches a fixed point.
- The flow goes into a limit cycle.
- The flow just wanders around.

The latter two do not tend to happen.

If we focus on the fixed point after, $\xi' = \xi/\zeta$. But this is a fixed point, so $\xi' = \xi = 0$ or ∞.

If we take the disordered phase ($T > T_c$), then $\xi = 0$ as $T \to \infty$. Similarly in the ordered phase ($T < T_c$), $\xi = 0$ as $T \to 0$.

But $\xi = \infty$ occurs at a critical point of a continuous phase transition.



All points on the RG trajectories that lead to the same fixed point have the same long-distance physics, controlled by the fixed point, i.e. these are different microscopic theories with the same long-distance behaviour, giving universality.

Note that $\xi' = \xi/\zeta = \infty$ as $\zeta \to \infty$ implies that $\xi = \infty$ for all points on the trajectory approaching a fixed point.

At a fixed point, we can classify couplings:

- If we turn on a (combination of) coupling(s), and the RG flow takes us back to the fixed point, this coupling is *irrelevant*.
- If turning on couplings takes us away from the fixed point, the coupling is *relevant*.

We could also have a line (or surface) of fixed point. A coupling that moves us along this line is *marginal*.

Typically, there exists just a few relevant directions or couplings, and infinitely many irrelevant directions. The basin of attraction of a fixed points is called its *critical surface*.

The *codimension* of such a surface is the number of relevant deformations of the fixed point.

3.2 Scaling

We use dimensional analysis, and measure dimensions in units of inverse length, so

$$[x] = -1, \qquad \left[\frac{\partial}{\partial x}\right] = 1.$$

Then we have $[F] = 0$, so e^{-F} makes sense. Notice that

$$F[\varphi] = \int \mathrm{d}^d x \left(\frac{1}{2}(\nabla\varphi)^2 + \cdots\right) \implies [\varphi] = \frac{d-2}{2}.$$

At the critical point, we have

$$\langle \varphi(\mathbf{x})\varphi(0)\rangle \sim \frac{1}{\Gamma^{d-2+\eta}}.$$

Dimensional analysis gives $\eta = 0$. However experimentally this is not true.

The reason is that there exists another length scale $\Lambda = 1a$, so writing

$$\langle \varphi(\mathbf{x})\varphi(0)\rangle \sim \frac{a^\eta}{\Gamma^{d-2+\eta}}$$

becomes dimensionally constant. Then $\eta \neq 0$ can come from the third stop of RG:

$$\varphi'(\mathbf{x}') = \zeta^{\Delta_\varphi}\varphi(\mathbf{x}).$$

Here Δ_φ is the *scaling dimension* of φ. Then we have

$$\langle \varphi'(\mathbf{x}')\varphi'(0)\rangle = \zeta^{2\Delta_\varphi}\langle \varphi(\mathbf{x})\varphi(0)\rangle \sim \zeta^{2\Delta_\varphi}/\Gamma^{d-2+\eta},$$

at the critical point. But the critical point is a fixed point, hence the left hand side scales as

$$\frac{1}{(\Gamma')^{d-2+\eta}} = \frac{\zeta^{d-2+\eta}}{\Gamma^{d-2+\eta}} \implies \Delta_\varphi = \frac{d-2}{2} + \frac{\eta}{2}.$$

Near the fixed point, $\xi' = \xi/\zeta$, so $\Delta_\xi = -1$. If $\xi \sim t^{-\nu}$, where $t = |T - T_c|/T_c$ is the reduced temperature, then

$$t' = \zeta^{\Delta_t} t,$$

where $\Delta_t = 1/\nu$. We have the following relations between critical exponents:

1. Write $F_{\text{thermo}}(t) = \int \mathrm{d}^d x\, f(t)$. Then RG doesn't change the value of $F_{\text{thermo}} = -\log Z$, as it does not change the physics, so

   $$F'_{\text{thermo}}(t') = F_{\text{thermo}}(t),$$

   giving

   $$\int \mathrm{d}^d x'\, f(t') = \int \mathrm{d}^d x\, f(t).$$

   At the fixed point, $f' = f$ near the fixed point, so the left hand side becomes

   $$\int \mathrm{d}^d x\, \zeta^{-d} f(\zeta^{\Delta_t} t).$$

   Hence we find $f(t) \sim t^{d/\Delta_t} = t^{dv}$. This can be thought of as spins correlating over blocks of size ξ, and by extensivity F_{thermo} is proportional to the number of blocks, which scales as $(L/\xi)^d \sim t^{dv}$.

   Then we can find that

   $$c \sim \frac{\partial^2 f}{\partial t^2} \sim t^{dv-2},$$

   but c scales as $t^{-\alpha}$, so $\alpha = 2 - dv$, a *hyperscaling relation*.

2. For $T < T_c$, and $T \approx T_c$, $\varphi \sim t^\beta$. Hence $\Delta_\varphi = \beta\Delta_t = \beta/\nu$. So

   $$\beta = \nu\Delta_\varphi = \frac{1}{2}(d - 2 + \eta)\nu.$$

3. With a magnetic field, F includes terms of the form

   $$\int \mathrm{d}^d x\, B(\mathbf{x})\varphi(\mathbf{x}).$$

   At the critical point, F is invariant under RG action, so $\Delta_B + \Delta_\varphi = d$. Hence

   $$\Delta_B = \frac{d + 2 - \eta}{2}.$$

Also the susceptibility scales as

$$\chi = \left(\frac{\partial \varphi}{\partial B}\right)_T \sim t^{-\gamma} \implies \Delta_\varphi - \Delta_B = -\frac{\gamma}{\nu} \implies \gamma = \nu(2-\eta).$$

4. Near the critical point, $\varphi \sim B^{1/\delta}$, so

$$\delta = \frac{\Delta_B}{\Delta_\varphi} = \frac{d+2-\eta}{d-2+\eta}.$$

Hence if we know η, ν, we are able to compute $\alpha, \beta, \gamma, \delta$.

This works well for $d = 2$ and 3 Ising model, and $d = 4$ mean field theory, with quadratic flows. However it disagrees with the MFT predictions for β and δ when $d > 4$. We will understand soon why this is true.

Scaling is used to characterize interactions:

$$F[\phi] \supset \int \mathrm{d}^d x \, g_{\mathcal{O}} \mathcal{O}(\mathbf{x}),$$

where $\mathcal{O} = \varphi^n$ or $\varphi^m(\nabla\varphi^2)$. Here $m\mathcal{O}$ is called an *operator*. Under RG by $\mathbf{x} \to \mathbf{x}'/\zeta$, if

$$\mathcal{O}(\mathbf{x}) \to \mathcal{O}'(\mathbf{x}') = \zeta^{\Delta_{\mathcal{O}}} \mathcal{O}(\mathbf{x}),$$

then $\mathcal{O}$ has scaling dimension $\Delta_{\mathcal{O}}$. But not all operators transform simply under RG; only certain combinations, analogous to the eigenvectors of RG. If $\mathcal{O}$ has scaling dimension $\Delta_{\mathcal{O}}$, then

$$\int \mathrm{d}^d x \, g_{\mathcal{O}} \mathcal{O}(\mathbf{x}) = \int \mathrm{d}^d x' \, \zeta^d g_{\mathcal{O}} \zeta^{-\Delta_{\mathcal{O}}} \mathcal{O}'(\mathbf{x}'),$$

hence $g'_{\mathcal{O}} = \zeta^{d-\Delta_{\mathcal{O}}} g_0$, giving $\Delta_{g_{\mathcal{O}}} = d - \Delta_{\mathcal{O}}$. The interaction is:

- relevant if $\Delta_{\mathcal{O}} < d$,
- irrelevant if $\Delta_{\mathcal{O}} > d$,
- marginal if $\Delta_{\mathcal{O}} = d$.

Note that $\Delta_{\mathcal{O}}$ depends on which fixed point we are studying.

3.3 The Gaussian Fixed Point

We apply RG starting from a very special point in theory space. Say

$$F_0[\varphi] = \int \mathrm{d}^d x \left(\frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}\mu_0^2\varphi^2 \right) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left(\frac{1}{2}(\mathbf{k}^2 + \mu_0^2)|\varphi_{\mathbf{k}}|^2 \right).$$

Break $\varphi_{\mathbf{k}}$ into $\varphi_{\mathbf{k}}^-$ and $\varphi_{\mathbf{k}}^+$, where the former is the low energy modes and the latter has the higher energy. Then the free energy splits:

$$F_0[\varphi] = F_0[\varphi^-] + F_0[\varphi^+].$$

The new free energy satisfies

$$e^{-F_0'[\varphi']} = \left(\int \mathcal{D}\varphi^+ \, e^{-F_0[\varphi^+]} \right) e^{-F_0[\varphi^-]} = \mathcal{N} e^{-F_0[\varphi^-]}.$$

Now we scale, letting $\mathbf{k}' = \zeta\mathbf{k}$, so $\varphi'_{\mathbf{k}'} = \zeta^{-w}\varphi_{\mathbf{k}}^-$, where w is to be determined. So

$$\begin{aligned} F_0[\varphi^{-1}] &= \int_{|\mathbf{k}|<\Lambda/\zeta} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{2}(\mathbf{k}^2 + \mu_0^2)|\varphi_{\mathbf{k}}^-|^2 \\ &= \int_{|\mathbf{k}'|<\Lambda} \int \frac{\mathrm{d}^d k'}{(2\pi)^d} \frac{1}{2\zeta^d} \left(\frac{\mathbf{k}'^2}{\zeta^2} + \mu_0^2 \right) \zeta^{2w} |\varphi'_{\mathbf{k}'}|^2. \end{aligned}$$

For the canonical kinetic terms, we need $w = \frac{d+2}{2}$. Relabelling,

$$F_0'[\varphi] = \int_{|\mathbf{k}|<\Lambda} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{2}(\mathbf{k}^2 + \mu(\zeta)^2)|\varphi_{\mathbf{k}}|^2,$$

where $\mu(\zeta)^2 = \zeta^2\mu_0^2$.

Interestingly the quadratic form of F is preserved by RG. For this theory, $\xi \sim 1/\mu$, then $\xi \to \xi/\zeta$ under RG, as expected. There are two fixed points:

- $\mu^2 = \infty$, when $T = \infty$.
- $\mu^2 = 0$. This is the Gaussian fixed point.

What happens near the fixed point? Since μ^2 grows under RG, this shows φ^2 is a relevant operator.

We can also include other quadratic terms such as $\alpha_0(\nabla^2\varphi)^2$, and find $\alpha(\zeta)(\nabla^2\varphi)^2$, where $\alpha(\zeta) = \zeta^{-2}\alpha_0$, hence $(\nabla^2\varphi)^2$ is irrelevant.

What about a general free energy

$$F_0[\varphi] = \int \mathrm{d}^d x \left(\frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}\mu_0^2\varphi^2 + \sum_{n=4}^{\infty} g_{0,n}\varphi^n \right).$$

This is non-quadratic, so the first step of integrating out higher order modes is harder. Integrating out φ^+ shifts $g_{0,n} \to g'_n = g_{0,n} + \delta g_n$.

We ignore this for now, and just look at applying the second and third steps. So say $\mathbf{x}' = \mathbf{x}/\zeta$, and $\varphi'(\mathbf{x}) = \zeta^{\Delta_\varphi}\varphi(\mathbf{x})$. Then

$$F_0'[\varphi'] = \int \mathrm{d}^d x' \, \zeta^d \left(\frac{1}{2}\zeta^{-2-2\Delta_\varphi}(\nabla'\varphi')^2 + \frac{1}{2}\mu_0^2 \zeta^{-2\Delta_\varphi}\varphi'^2 + \sum_{n \geq 0} g_{0,n}\zeta^{-n\Delta_\varphi}\varphi^{-n} \right).$$

As before we find

$$\Delta_\varphi = \frac{d-2}{2}, \qquad \mu(\zeta)^2 = \zeta^2 \mu_0^2.$$

Hence we must have

$$g_n(\zeta) = \zeta^{d-n\Delta_\varphi} g_{0,n} = \zeta^{(1-n/2)d+n} g_{0,n},$$

for example

$$g_4(\zeta) = \zeta^{4-d} g_{0,4}.$$

This shows φ^4 is irrelevant for $d > 4$, and is relevant for $d < 4$. This is why re normalization is only accurate for higher dimensions. $d = 4$ looks marginal, but we need to include boundary terms. Similarly,

$$g_6(\zeta) = \zeta^{6-2d} g_{0,6},$$

which says φ^6 is irrelevant for $d > 3$, and relevant for $d < 3$.

If $d = 2$, then all φ^n are relevant.

We will look at RG flow in the (μ^2, g_4) subspace of our theory space: (INSERT PICTURE)

To hit the GFP with $d < 4$, we need to tune at least two parameters μ^2 and g_4. But to hit the Ising CP with $B = 0$, we need to tune T. Hence the GFP does not describe the Ising critical point if $d < 4$. We will see there is another fixed point.

Previously, we saw that the saddle point approximation to the path integral lead to MFT with quadratic fluctutations. This is self-consistent only if the couplings of non-quadratic terms remain small, i.e. remain near the GFP. But if $d < 4$, g_4 does not remain small under RG, so the saddle point approximation cannot be trusted.

3.4 Dangerously Irrelevant Couplings

For $d > 4$ the GFP describes the Ising CP, but scaling gave us incorrect results for β and δ. Scaling $\varphi \sim t^\beta$, we get

$$\beta = \frac{\Delta_\varphi}{\Delta_t},$$

which is wrong for $d > 4$. Actually MFT told us that

$$\varphi \sim \left(\frac{t}{g_4}\right)^{1/2}.$$

For $d > 4$, φ^4 is irrelevant as $g_4 \to 0$ under RG. But we cannot ignore the denominator. If we assume that

$$\varphi \sim \left(\frac{t}{g_4}\right)^{\beta},$$

then we can show that $\beta = 1/2$.

Usually irrelevant operators can be ignored at long distances as coupling $\to 0$ under RG. But if a coupling multiplies a quantity of interest and changes scaling, then it is *dangerously irrelevant*.

3.5 Symmetries and Breaking

We assumed $\mathbb{Z}_2$ symmetry, i.e. $F[\varphi] = F[-\varphi]$. This excludes φ^n. Then this symmetry is preserved by RG.

Relax this assumption, so all odd n are possible.

- For $n = 1$, we get
  $$\int \mathrm{d}^d x \, g_1 \varphi,$$
  where $g_1 = B$, Then
  $$\Delta_B = \frac{d+2}{2}$$
  at the GFP. This is relevant, so we must tune B to tit the CP.
- If $n = 3$,
  $$\int \mathrm{d}^d x \, g_3 \varphi^3.$$
  This looks relevant for $d < 6$, but we can always set $g_3 = 0$ by redefining $\varphi \to \varphi + c$. Then the quartic term lets us eliminate the cubic term.

Moreover, we assumed that F is invariant under all rotations in $\mathsf{O}(d)$. But the original lattice is invariant only under a finite subgroup, i.e. in the $d = 2$ we are invariant only under $G = D_8$.

We should only demand that F is invariant under G. For a cubic lattice, the lowest dimension operator that is invariant with respect to G but not $\mathcal{O}(d)$ is

$$\mathcal{O} = \varphi \sum_{i=1}^{d} \partial_i^4 \varphi.$$

We expect such terms to be present in F. But at GFP, this has

$$\Delta_{\mathcal{O}} = 4 + 2\Delta_\varphi = d + 2,$$

which is irrelevant. Here $\mathcal{O}(d)$ is *emergent*: it is a symmetry of long-distance physics that is not present microscopically.

3.6 Renormalization with Interactions

We previously ignored $g_{0,n} \to g_{0,n} + \delta g_n$ under RG. Now we will do things properly. Start from

$$\begin{aligned} F[\varphi] &= \int \mathrm{d}^d x \left(\frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}\mu_0^2\varphi^2 + g_0\varphi^4 \right) \\ &= F_O[\varphi^-] + F_O[\varphi^+] + F_I[\varphi^-, \varphi^+]. \end{aligned}$$

Then the free energy ecomes

$$e^{-F'[\varphi^-]} = e^{-F_O[\varphi^-]} \int \mathcal{D}\varphi^+ \, e^{-F_O[\varphi^+]} e^{-F_I[\varphi^-,\varphi^+]}.$$

The latter integral can be thought of as

$$\left\langle e^{-F_I[\varphi^-,\varphi^+]} \right\rangle_+ .$$

Normalize $\mathcal{D}\varphi^+$ such that $\langle 1 \rangle_+ = 1$. Hence $\langle \cdot \rangle_+$ is an expectation value treating φ^+ as a random variable with probability density $e^{-F_0[\varphi^+]}$, i.e. as a Gaussian. So,

$$F'[\varphi^-] = F_O[\varphi^-] - \log \left\langle e^{-F_I[\varphi^-,\varphi^+]} \right\rangle_+ .$$

To compute this, we assume that g_0 is small, and use perturbation theory. Using the Taylor series for log,

$$\begin{aligned} \log\langle e^{-F_I} \rangle &= \log\langle 1 - F_I + F_I^2/2 - \cdots \rangle_+ \\ &= -\langle F_I \rangle_+ + \frac{1}{2}\left(\langle F_I^2 \rangle_+ - \langle F_I \rangle_+^2 \right) + \cdots . \end{aligned}$$

We then get

$$F_I = g_0 \int \mathrm{d}^d x \prod_{i=1}^{4} \int \frac{\mathrm{d}^d\, k_i}{(2\pi)^d} \varphi_{\mathbf{k}_i} e^{i\mathbf{k}_i \cdot \mathbf{x}}$$
$$= g_0 \int \left[\prod_{i=1}^{4} \frac{\mathrm{d}^d\, k_i}{(2\pi)^d} (\varphi^+_{\mathbf{k}_i} + \varphi^-_{\mathbf{k}_i}) \right] (2\pi)^d \delta^{(d)} \left(\sum \mathbf{k}_i \right).$$

There are five types of terms:

- $(\varphi^-)^4$. These expectations are trivial.
- $4(\varphi^-)^3\varphi^+$. Odd, so gives 0.
- $6(\varphi^-)^2(\varphi^+)^2$. These are interesting.
- $4\varphi^-(\varphi^+)^3$. Again odd, so gives 0.
- $(\varphi^+)^4$. Gives a constant term in F'.

Hence we need to compute

$$\langle F_I \rangle_+ \supseteq 6g_0 \int \prod_{i=1}^{k} \frac{\mathrm{d}^d\, k_i}{(2\pi)^d} \varphi^-_{\mathbf{k}_1} \varphi^-_{\mathbf{k}_2} \langle \varphi^+_{\mathbf{k}_3} \varphi^+_{\mathbf{k}_4} \rangle (2\pi)^d \delta^{(d)} \left(\sum \mathbf{k}_i \right).$$

Here we get the correlation function

$$\langle \varphi^+_{\mathbf{k}} \varphi^+_{\mathbf{k}'} \rangle_+ = (2\pi)^d \delta^{(d)}(\mathbf{k} + \mathbf{k}') G_0(\mathbf{k}),$$

where

$$G_0 = \frac{1}{\mathbf{k}^2 + \mu_0^2}.$$

Hence we get

$$\langle F_I \rangle_+ \supseteq 6g_0 \int_0^{\Lambda/\zeta} \frac{\mathrm{d}^d\, \mathbf{k}}{(2\pi)^d} \varphi^-_{\mathbf{k}} \varphi^-_{-\mathbf{k}} \int_{\Lambda/\zeta}^{\Lambda} \frac{\mathrm{d}^d\, q}{(2\pi)^d} \frac{1}{q^2 + \mu_0^2}.$$

This generates a correction to the $(\varphi^-)^2$ term in F'.

We get a movement of coefficients

$$\mu_0^2 \to \mu'^2 = \mu_0^2 + 12 g_0 \int_{\Lambda/\zeta}^{\Lambda} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{q^2 + \mu_0^2}.$$

The coefficient of $(\nabla\varphi^-)^2$ and $(\varphi^-)^4$ is unchanged at $\mathcal{O}(g_0)$. So under $\mathbf{k}' = \zeta\mathbf{k}$, we have φ' changes as

$$\mu^2(\zeta) = \zeta^2 \left(\mu_0^2 + 12 g_0 \int_{\Lambda/\zeta}^{\Lambda} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{q^2 + \mu_0^2} \right),$$

and $g(\zeta) = \zeta^{4-d} g_0$. Note μ^2 does not scaled as a power of ζ, so there is no well-defined scaling dimension. In general, he dimensions of the operators and couplings are defined by linearising around a fixed point.

Consider linearising around the GFP, so μ_0^2, g_0 are both small. Hence we find

$$\mu^2(\zeta) = \zeta^2(\mu_0^2 + 12 g_0 (I(\Lambda) - I(\Lambda/\zeta))) + \cdots ,$$

and $g(\zeta) = \zeta^{4-d} g_0$, where

$$I(\Lambda) = \int_0^\Lambda \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{q^2} = \frac{\Omega_{d-1}}{(2\pi)^d (d-2)} \Lambda^{d-2},$$

where Ω_n is the area of the n-sphere. We can rewrite this as

$$\mu^2 = \zeta^2(\mu_0^2 + c\Lambda^{d-2} g_0) - c\Lambda^{d-2} \zeta^{4-d} g_0,$$

for c some positive constant. Hence we find

$$\begin{aligned} \mu^2 + c\Lambda^{d-2} g &\propto \zeta^2, \\ g &\propto \zeta^{4-d}. \end{aligned}$$

This has well-defined dimensions $(2, 4-d)$ at the GFP. We can write $F'[\varphi]$ in terms of these couplings:

$$\frac{1}{2}\mu^2\varphi^2 + g\varphi^4 = \frac{1}{2}\left(\mu^2 + c\Lambda^{d-2} g\right)\varphi^2 + g\left(\varphi^4 - \frac{1}{2} c\Lambda^{d-2}\varphi^2\right).$$

These have well-defined dimensions at the Gaussian fixed point. Since $\mu^2 + c\Lambda^{d-2} g$ is relevant, we need to tune this to reach the GFP. We identify this with $T - T_c$.

If $d > 4$, g is irrelevant, and so we get RG flow into the GFP. For $d < 4$, g is relevant and perturbation theory breaks down.

For $d = 4$, g is marginal, so we need to got to $\mathcal{O}(g_0^2)$. This means looking at

$$\frac{1}{2}(\langle F_I^2 \rangle_+ - \langle F_I \rangle_+^2).$$

The first term gives two types of terms:

$$\left(\int \mathrm{d}^d x\, f(x)\right)^2,$$

which are cancelled by $\langle F_I \rangle_+^2$, and also

$$\int \mathrm{d}^d x\, f(x),$$

which are not cancelled. We focus on terms quartic in φ^-. Then we see

$$\frac{1}{2}\langle F_I^2\rangle_+ \supseteq \frac{1}{2}\binom{4}{2}^2 g_0^2 \int_0^{\Lambda/\zeta}\left(\prod_{i=1}^4 \frac{\mathrm{d}^d k_i}{(2\pi)^d}\varphi^-_{k_i}\right)\int_{\Lambda/\zeta}^{\Lambda}\prod_{j=1}^4 \frac{\mathrm{d}^d q_j}{(2\pi)^d}\langle \varphi^+_{q_1}\varphi^+_{q_2}\varphi^+_{q_3}\varphi^+_{q_4}\rangle_+$$
$$\times (2\pi)^{2d}\delta^{(d)}(k_1+k_2+q_1+q_2)\delta^{(d)}(k_3+k_4+q_3+q_4).$$

We claim that

$$\langle \varphi^+_{q_1}\varphi^+_{q_2}\varphi^+_{q_3}\varphi^+_{q_4}\rangle_+ = \langle \varphi^+_{q_1}\varphi^+_{q_2}\rangle_+\langle \varphi^+_{q_3}\varphi^+_{q_4}\rangle_+ + \langle \varphi^+_{q_1}\varphi^+_{q_3}\rangle_+\langle \varphi^+_{q_2}\varphi^+_{q_4}\rangle_+ + \langle \varphi^+_{q_1}\varphi^+_{q_4}\rangle_+\langle \varphi^+_{q_2}\varphi^+_{q_3}\rangle_+.$$

This is Wick's theorem. We use the fact hat

$$\langle \varphi^+_q \varphi^+_{q'}\rangle = (2\pi)^d\delta^{(d)}(q+q')G_0(q).$$

The first term in the expansion cancels. The second and third terms are

$$\binom{4}{2}^2 g_0^2 \int_{)}^{\Lambda/\zeta}\left(\prod_{i=1}^4 \frac{\mathrm{d}^d k_i}{(2\pi)^d}\varphi^-_{k_i}\right) f(k_1+k_2)(2\pi)^d\delta^{(d)}\left(\sum_{i=1}^4 k_i\right),$$

where

$$\begin{aligned} f(k_1+k_2) &= \int_{\Lambda/\zeta}^{\Lambda}\frac{\mathrm{d}^d q}{(2\pi)^d}\frac{1}{q^2+\mu_0^2}\frac{1}{(k_1+k_2+q)^2+\mu_0^2}\\ &= \int_{\Lambda/\zeta}^{\Lambda}\frac{\mathrm{d}^d q}{(2\pi)^d}\frac{1}{(q^2+\mu_0^2)}(1+\mathcal{O}((k_1+k_2)^2)).\end{aligned}$$

The latter term generates terms of order $k^2\varphi^{-4} = (\varphi^-)^2(\nabla\varphi^-)^2$, which is irrelevant. So we get

$$g_0 \to g_0' = g_0 - 36 g_0^2 \int_{\Lambda/\zeta}^{\Lambda}\frac{\mathrm{d}^d q}{(2\pi)^d}\frac{1}{(q^2+\mu_0^2)^2}.$$

The coefficient of $(\nabla\varphi^-)^2$ changes at $\mathcal{O}(g_0^2)$, and

$$\varphi'_{k'} = \zeta^{-(d+2)/2}(1+\mathcal{O}(g_0^2))\varphi^-_k.$$

Hence we see

$$g(\zeta) = \zeta^{4-d}\left(g_0 - 36 g_0^2\int\cdots\right).$$

When $d=4$, then at $\mathcal{O}(g_0)$ g was marginal. But at order $\mathcal{O}(g_0^2)$, g decreases under RG. We say that g is *marginally irrelevant* for $d=4$.

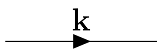
3.7 Feynman Diagrams

Each term in $\langle F_I^p \rangle_+$ can be represented by a diagram. Terms with an integrand of the form $g_0^p(\varphi^-)^n(\varphi^d)^d$ can be represented by diagrams with:

- n external solid lines labelled by k_i.
- d internal solid lines labelled by q_i.
- p vertices where 4 lines meet.

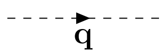
Each diagram corresopnds to an integral.

- A solid line corresponds to an integral over momentum:



$$\leftrightarrow \int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\varphi^-_{\mathbf{k}}.$$

- A dotted line corresponds to a propagator:



$$\leftrightarrow G_0(\mathbf{q}).$$

- Each loop corresponds to

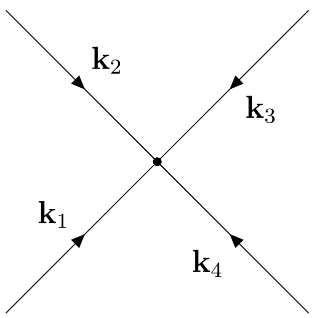
$$\int_{\Lambda/\zeta<|\mathbf{q}|<\Lambda} \frac{\mathrm{d}^d q}{(2\pi)^d}.$$

- Each vertex corresponds to

$$g_0(2\pi)^d\delta^{(d)}\left(\sum \text{momenta into vertex}\right).$$

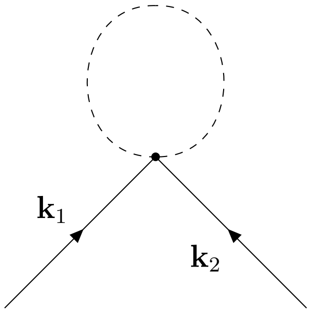
- We multiply each diagram by the symmetry factor: the number of topologically identical diagrams.

For example, at order $\mathcal{O}(g_0)$, we have



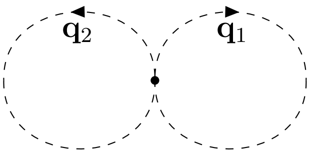
$$= \int_0^{\Lambda/\zeta} \prod_{i=1}^{4} \frac{\mathrm{d}^d k_i}{(2\pi)^d}\varphi^-_{\mathbf{k}_i} g_0(2\pi)^d\delta^{(d)}\left(\sum \mathbf{k}_i\right).$$

Also we have



$$= \int_0^{\Lambda/\zeta} \prod_{i=1}^2 \frac{\mathrm{d}^d k_i}{(2\pi)^d} \varphi^-_{\mathbf{k}_i} \int_{\Lambda/\zeta}^{\Lambda} \frac{\mathrm{d}^d q}{(2\pi)^d} G_0(q) g_0 (2\pi)^d \delta^{(d)}(\mathbf{k}_1 + \mathbf{k}_2).$$

The symmetry factor in this case is 6, as there are six ways of choosing which two of the four lines emerging from the vertex will be dotted. So we need to compute this integral, and multiply by 6. We also have



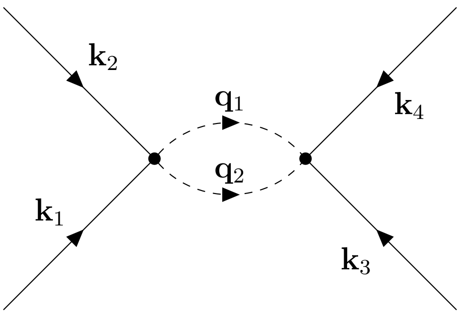
$$= g_0 \int_{\Lambda/\zeta}^{\Lambda} \frac{\mathrm{d}^d \mathbf{q}_i}{(2\pi)^d}$$

Here we have $\delta^{(d)}(0)$. We interpret this as

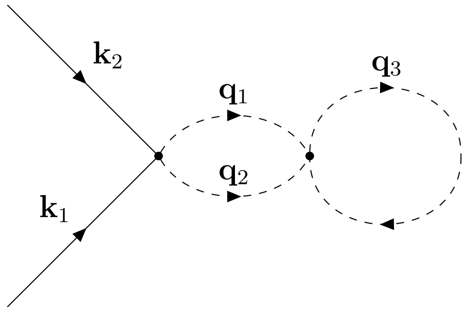
$$(2\pi)^d \delta^{(d)}(0) = \int \mathrm{d}^d x \, e^{i\mathbf{x}\cdot(0)} = V,$$

which forms a term in F. This has symmetry factor 3.

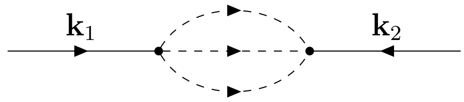
To order $\mathcal{O}(g_0)^2$, we have for example



This has symmetry factor $\binom{4}{2} \times 2 \times \binom{4}{2}$. We also have disconnected diagrams, but these always cancel out in F'. We also have two-loop diagrams, for example



$$= g_0^2 \int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} \frac{1}{2} C(\zeta, \Lambda) \varphi^-_{\mathbf{k}} \varphi^-_{-\mathbf{k}}.$$



$$= g_0^2 \int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} \frac{1}{2} A(\mathbf{k}, \zeta, \Lambda) \varphi^-_{\mathbf{k}} \varphi^-_{-\mathbf{k}}.$$

These generate $\mathcal{O}(g_0)^2$ corrections to μ'^2:

$$\mu'^2 = \mu^2 + 12 g_0 \int_{\Lambda/\zeta}^{\Lambda} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{q^2 + \mu_0^2} + g_0^2 (C(\zeta, \Lambda) + A(0, \zeta, \Lambda)).$$

But this also shifts the coefficients of $(\nabla\varphi)^2 \sim \mathbf{k}^2 \varphi^2$:

$$F' \subset \int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} \frac{1}{2} (1 + g_0^2 A''(0, \zeta, \Lambda)) \mathbf{k}^2 \varphi^-_{\mathbf{k}} \varphi_{-\mathbf{k}}.$$

Hence

$$\varphi'_{\mathbf{k}'} = \zeta^{-(d+1)/2} \left(1 + \frac{1}{2} g_0^2 A''(0, \zeta, \Lambda)\right)^{1/2} \varphi^-_{\mathbf{k}} = \zeta^{-(d+1)/2} \left(1 + \frac{1}{4} g_0^2 A''(0, \zeta, \Lambda)\right) \varphi^-_{\mathbf{k}}.$$

This is the origin of anomalous dimensions. Recall that

$$\begin{aligned} \Delta_f = \frac{d - 2 + \eta}{2} &\leftrightarrow \varphi'(\mathbf{x}') = \zeta^{(d-2+\eta)/2} \varphi(\mathbf{x}) \\ &\leftrightarrow \varphi'_{\mathbf{k}'} = \zeta^{-(d+2)/2 + \eta/2} = \zeta^{-(d+2)/2} e^{\log\zeta/2} \varphi^-_{\mathbf{k}} \\ &\leftrightarrow \varphi'_{\mathbf{k}'} = \zeta^{-(d+2)/2} \left(1 + \frac{1}{2}\log\zeta + \cdots\right) \varphi^-_{\mathbf{k}}. \end{aligned}$$

This gives $\eta = \mathcal{O}(g_0)^2$ at the fixed point (μ_0^2, g_0), so $\eta = 0$ at the GFP.

3.8 Beta Functions

Consider RG flow starting at an arbitrary point in theory space, and write $\zeta = e^s$ for $s \geq 0$. Write

$$\mathbf{g}(s) = \begin{pmatrix} \mu^2(s) \\ g(s) \\ \vdots \end{pmatrix},$$

an infinite-dimensional vector of all couplings after RG. RG satiafies

$$R[e^{s+\delta s}] = R[e^{\delta s}]R[e^s],$$

so

$$\mathbf{g}(s+\delta s) = \mathbf{g}(s) + \delta s \boldsymbol{\beta}(\mathbf{g}(s)),$$

where $\boldsymbol{\beta}$ is a vector. The components of $\boldsymbol{\beta}$ are called *beta functions*, and

$$\frac{\mathrm{d}\mathbf{g}}{\mathrm{d}s} = \boldsymbol{\beta}(\mathbf{g}(s)).$$

A fixed point $\mathbf{g}_*$ satisfies $\boldsymbol{\beta}(\mathbf{g}_*) = 0$. The dimensions of the couplings are defined by linearizing around $\mathbf{g}_*$. Say $\mathbf{g}(s) = \mathbf{g}_* + \delta\mathbf{g}(s)$. Then,

$$\frac{\mathrm{d}\delta g_a}{\mathrm{d}s} = \sum_i (\partial_b \beta_a)(\mathbf{g}_*)\delta g_b,$$

i.e.

$$\frac{\mathrm{d}\delta\mathbf{g}}{\mathrm{d}s} = M\delta\mathbf{g}.$$

Assume that M is diagonalizable, i.e. there exists a basis of eigenvector $\mathbf{v}_1, \mathbf{v}_2, \ldots$ with eigenvalues $\Delta_1 \leq \Delta_2 \leq \cdots$ and $M\mathbf{v}_i = \Delta_i \mathbf{v}_i$. Expand

$$\delta\mathbf{g} = \sum_i f_i(s)\mathbf{v}_i \implies f_i(s) = c_i e^{\Delta_i s},$$

so

$$\delta\mathbf{g} = \sum_i c_i e^{\Delta_i s}.$$

This describes RG flow near $\mathbf{g}_*$.

Let the row vector $\mathbf{w}_1$ be a left eigenvector of M with eigenvalue Δ_i, i.e. $\mathbf{w}_i M = \Delta_i \mathbf{w}_i$. We can choose $\mathbf{w}_i$ such that $\mathbf{w}_i \mathbf{v}_j = \delta_{ij}$. Then,

- $\mathbf{w}_i \cdot \delta\mathbf{g} = c_i e^{\Delta_i s}$.
- The Δ_i are the scaling dimensions of the couplings at $\mathbf{g}_*$, and $\mathbf{w}_i \delta\mathbf{g}$ is the combination that has scaling dimension Δ_i.

We found that

$$\mu^2(\zeta) = \zeta^2(\mu_0^2 + a g_0 + \mathcal{O}(g_0^2)), \qquad g(\zeta) = \zeta^{4-d}(g_0 - b g_0^2 + \mathcal{O}(g_0^3)),$$

where the coefficients are

$$a = 12 \int_{\Lambda/\zeta}^{\Lambda} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{\mu^2 + q^2} = \frac{12\Omega_{d-1}}{(2\pi)^d} \int_{\Lambda/\zeta}^{\Lambda} \frac{\mathrm{d}q\, q^{d-1}}{q^2 + \mu_0^2},$$

$$b = 36 \int_{\Lambda/\zeta}^{\Lambda} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{(q^2 + \mu_0^2)^2} = \frac{36\Omega_{d-1}}{(2\pi)^d} \int_{\Lambda/\zeta}^{\Lambda} \frac{\mathrm{d}q\, q^{d-1}}{(q^2 + \mu_0^2)^2}.$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}s} \int_{\Lambda e^{-s}}^{\Lambda} \mathrm{d}q\, f(q) = -f(\Lambda e^{-s}) \frac{\mathrm{d}}{\mathrm{d}s}(\lambda e^{-s}) = \Lambda e^{-s} f(\Lambda e^{-s}).$$

This means we can show:

$$\frac{\mathrm{d}\mu^2}{\mathrm{d}s} = 2\mu^2 + \frac{12\Omega_{d-1}}{(2\pi)^d} \Lambda^d \frac{g}{\Lambda^2 + \mu^2} + \mathcal{O}(g^2) = \beta_{\mu^2},$$

$$\frac{\mathrm{d}g}{\mathrm{d}s} = (4-d)g - \frac{36\Omega_{d-1}}{(2\pi)^d} \Lambda^d \frac{g^2}{(\Lambda^2 + \mu^2)^2} + \mathcal{O}(g^3) = \beta_g.$$

These are the β-functions. For the fixed points:

- If $d \geq 4$, then $\beta_g = 0$ means $g = 0$, so $\beta_{\mu^2} = 0$, hence $\mu^2 = 0$. This gives the GFP.
- If $d < 4$, then $4 - d > 0$. Hence $\beta_g = 0$ gives either $g = 0$, or $g = (4-d)\mathcal{O}(\Lambda^{4-d})$.

This sugests that there may be another fixed point, but this g is not small, hence it lies outside our perturbative approximation.

Linearizing around the GFP, we get

$$\frac{\mathrm{d}\delta\mu^2}{\mathrm{d}s} = 2\delta\mu^2 + \frac{12\Omega_{d-1}}{(2\pi)^d} \Lambda^{d-2} \delta g, \qquad \frac{\mathrm{d}\delta g}{\mathrm{d}s} = (4-d)\delta g.$$

This gives

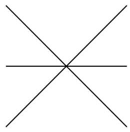
$$M = \begin{pmatrix} 2 & 2c\Lambda^{d-2} \\ 0 & 4-d \end{pmatrix} \implies \Delta_1 = 2, \Delta_2 = 4 - d.$$

Here we are dealing with only two couplings. But of course $\mathbf{g} = (\mu^2, g, \mathbf{h})$ for $\mathbf{h}$ the other couplings. We assume that $\mathbf{h}(0) = 0$, but we did not calculate $\mathbf{h}(s)$.

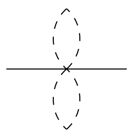
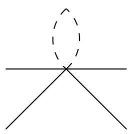
We can repeat the analysis allowing small $\mathbf{h}(0)$, and treat this perturbatively. This gives corrections to β_{μ^2}, β_g by terms involving $\mathbf{h}$.

Example 3.1.

We could include

$$h_6\phi^6 \leftrightarrow$$ 

This generates corrections to μ^2 and g, given by connecting appropriate pairs of lines of order $\mathcal{O}(h_0)$:

 , 

In general, we get

$$\beta_{h_a} = \tilde{\Delta}_a h_a + \mathcal{O}(h_{i>a}) + \mathcal{O}(h^2, hg, g^2),$$

where $\tilde{\Delta}_a$ is the *engineering dimension* of a. Hence at the GFP, we must have $\mu^2 = g = \mathbf{h} = 0$, and $\Delta_a = \tilde{\Delta}_a$ for all a, i.e. all couplings have scaling dimension equal to the engineering dimension.

3.9 Epsilon Expansion

In the above, we have $g = (4-d)\mathcal{O}(\Lambda^{4-d})$. This is not small.

The crazy idea is to let $d = 4 - \varepsilon$, where $\varepsilon > 0$ is small. The hope is that setting $\varepsilon = 1$ is valid and tells us something!

Let $\tilde{g} = \Lambda^{-\varepsilon} g$, which is dimensionless. Then

$$\begin{aligned}
\frac{\mathrm{d}\mu^2}{\mathrm{d}s} &= 2\mu^2 + \frac{12\Omega_{d-1}}{(2\pi)^d}\frac{\Lambda^4}{\Lambda^2+\mu^2}\tilde{g} + \mathcal{O}(\tilde{g}^2),\\
\frac{\mathrm{d}\tilde{g}}{\mathrm{d}s} &= \varepsilon\tilde{g} - \frac{36\Omega_{d-1}}{(2\pi)^d}\frac{\Lambda^4}{(\Lambda^2+\mu^2)^2}\tilde{g}^2 + \mathcal{O}(\tilde{g}^4).
\end{aligned}$$

Since $d - 1 \approx 3$, we get

$$\Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

hence

$$\frac{\Omega_{d-1}}{(2\pi)^d} = \frac{1}{8\pi^2} + \mathcal{O}(\varepsilon).$$

The fixed points are now:

- $\mu^2 = g = 0$ (GFP).
- μ_*^2, g_* with

$$\mu_*^2 = -\frac{3}{4\pi^2}\frac{\Lambda^4}{\Lambda^2+\mu_0^2}\tilde{g}_* \qquad \tilde{g}_* = \frac{2\pi^2}{9}\frac{(\Lambda^2+\mu_*^2)}{\Lambda^4}\varepsilon$$
$$\implies g_* = \frac{2\pi^2}{9}\varepsilon + \mathcal{O}(\varepsilon^2) \qquad \tilde{\mu}_*^2 = -\frac{1}{6}\Lambda^2\varepsilon + \mathcal{O}(\varepsilon^2).$$

This is the *Wilson-Fisher* fixed point, and is valid for $\varepsilon \ll 1$. We can show that $\mathbf{h} = \mathcal{O}(\varepsilon^2)$ or smaller.

Linearizing around the fixed point, we find

$$M = \begin{pmatrix} 2-\varepsilon/3 & M_{12} \\ 0 & -\varepsilon \end{pmatrix}, \qquad M_{12} = \frac{3}{2\pi^2}\lambda^2\left(1+\frac{\varepsilon}{6}\right).$$

Hence the scaling dimensions are

$$\Delta_1 = 2 - \frac{\varepsilon}{3} + \mathcal{O}(\varepsilon^2), \qquad \Delta_2 = -\varepsilon + \mathcal{O}(\varepsilon^2),$$

which eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} a \\ 1 \end{pmatrix},$$

where $a = M_{12}/(\Delta_2 - \Delta_1)$. Then

$$\mathbf{w}_1 = \begin{pmatrix} 1 & -a \end{pmatrix}, \qquad \mathbf{w}_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

Tying this all together,

$$\mathbf{w}_1 \begin{pmatrix} \delta\mu^2 \\ \delta\tilde{g} \end{pmatrix} = c_1 e^{\Delta_1 s} \qquad \implies \qquad \delta\mu^2 - a\delta\tilde{g} \sim e^{\Delta_1 s} = e^{(2-\varepsilon/3)s},$$
$$\mathbf{w}_2 \begin{pmatrix} \delta\mu^2 \\ \delta\tilde{g} \end{pmatrix} = c_2 e^{\Delta_2 s} \qquad \implies \qquad \delta\tilde{g} \sim e^{\Delta_2 s} = e^{-\varepsilon s}.$$

Hence $t = \mu^2 - a\tilde{g}$ is relevant, and $\tilde{g}$ is irrelevant. So we can reach this fixed point by tuning with a single parameter: we identify this with

$$\frac{T - T_c}{T_c}.$$

Here we get that

$$\Delta_t = 2 - \frac{\varepsilon}{3} + \mathcal{O}(\varepsilon^2).$$

We can draw this as a picture, with the GFP and WFFP. Note that the WFFP is a non-trivial fixed point with only one relevant parameter.

We assumed $\varepsilon \ll 1$. We expect qualitatively this to be the same for $d = 3$, i.e. $\varepsilon = 1$. We can now calculate the critical exponents:

$$\nu = \frac{1}{\Delta_t} = \frac{1}{2} + \frac{\varepsilon}{12} + \mathcal{O}(\varepsilon^2), \qquad \eta = \mathcal{O}(\varepsilon^2) = \frac{\varepsilon^2}{6} + \mathcal{O}(\varepsilon^3).$$

By scaling we find $\alpha, \beta, \gamma, \delta$. We can set $\varepsilon = 1$, and find that the exponents are significantly better than in MFT.

	α	β	γ	δ	η	ν
MFT	0	1/2	1	3	0	1/2
$\varepsilon = 1$	0.17	0.33	1.17	4	0	0.58
$d = 3$	0.11	0.33	1.24	4.79	0.04	0.63

Table 3: Predictions and Theoretical Critical Exponents

How do we know that there exists RG flow from the GFP to the WFFP, for $d < 4$? In WF, there is one relevant coupling, so the critical surface has codimension one.

In the GFP as s increases, the irrelevant couplings decay, but as s decreases the irrelevant coupling grow. So points that originate from the GFP at $s = -\infty$ for a two-dimensional surface in theory space.

Generically, a codimension one surface will intersect a two-dimensional surface in a line.

Return to the $d = 2$ case. Here

$$F = \int \mathrm{d}^2 x \, \frac{1}{2} (\nabla \varphi)^2 + \cdots .$$

Note that $\Delta_\varphi = 0$ at the GFP, so φ^n is relevant for all n. It turns out that there exist infinitely many fixed points: the rough idea is to flow to the n'th fixed point by turning on $\varphi^{2(n+1)}$.

$\varphi^2, \varphi^4, \ldots, \varphi^{2n}$ are all generated by RG; we need to tune these coefficients to hit the fixed point which has n relevant directions. Each fixed point has different operator dimensions, which are known and rational.

3.10 Conformal Symmetry

At a fixed point the theory is invariant under scale transformations $\mathbf{x}' = \mathbf{x}/\zeta$. However it turns out that the theory is also invariant under *conformal transformations*:

these are maps $\mathbf{x} \to \mathbf{x}'(s)$ such that

$$\delta_{ij} \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} = f(\mathbf{x}) \delta_{kl},$$

for some $f(\mathbf{x})$. These include translations, rotations, scale transformations and special conformal transforms:

$$x'^i = \frac{x^i - \mathbf{x}^2 a^i}{1 - 2\mathbf{x} \cdot \mathbf{a} + \mathbf{a}^2 \mathbf{x}^2}.$$

These form the *conformal group* $\mathsf{SO}(d+1, 1)$. In this framework, fields and correlators correspond to representations of this group. This is especially powerful for $d = 2$, where the group is infinite, and recent progress is happening in $d = 3$.

4 Continuous Symmetry

4.1 Symmetry

We can characterize phases of matter by two symmetry groups:

G: The symmetry of the affective free energy (symmetry of the theory).

H: The symmetry of the ground state.

Example 4.1.

If $B = 0$, then the Ising model has $\varphi \to -\varphi$ symmetry, so $G = \mathbb{Z}_2$. The disordered phase $\langle \varphi \rangle = 0$ as $H = \mathbb{Z}_2$, and the ordered phase $\langle \varphi \rangle = \pm m_0$ has $H = \{\mathrm{id}\}$.

If $B \neq 0$, then the Ising model has $G = \{\mathrm{id}\}$.

This is useful as:

- For fixed G, having different H tells us we are in different phases.
- The nature of critical points (i.e. the universality class) is determined by G and the number of different order parameters.

4.2 $\mathsf{O}(N)$ Models

Suppose we have N order parameters. Combine them into a vector

$$\varphi_{(*)} = (\varphi_1^{(*)}, \ldots, \varphi_n^{(*)}).$$

Assume that F is invariant under $G = \mathsf{O}(N)$, which is unrelated to $\mathsf{O}(d)$, with $\varphi_a(\mathbf{x}) \mapsto R_{ab}\varphi_b(\mathbf{x})$ for $a, b = 1, \ldots, N$, and $R \in \mathsf{O}(N)$. Again write

$$F[\varphi] = \int \mathrm{d}^d x \, \left(\frac{1}{2}\gamma(\nabla\varphi)^2 + \frac{1}{2}\mu^2\varphi^2 + g(\varphi^2)^2 + \cdots \right).$$

- If $N = 1$, this is just $\mathbb{Z}_2$ symmetry, which gives Ising again.
- If $N = 2$, then it is better to think of φ as complex:

  $$\psi(\mathbf{x}) = \varphi_1(\mathbf{x}) + i\varphi_2(\mathbf{x}).$$

  Then $\mathsf{O}(2) = \mathsf{U}(1)$ acts as $\psi \to e^{i\alpha}\psi$. So F must take the form

  $$F[\psi] = \int \mathrm{d}^d x \, \left(\frac{\gamma}{2}\nabla\psi^a \cdot \nabla\psi + \frac{\mu^2}{2}|\psi|^2 + g|\psi|^4 + \cdots \right).$$

This is the *XY model*, which arises by coarse-graining a microscopic model of spins free to rotate in a plane:

$$E = -J \sum_{\langle ij \rangle} \mathbf{s}_i \cdot \mathbf{s}_j,$$

where $\mathbf{s}_i$ are two-dimensional unit vectors. This describes certain types of magnets, superfluids and Bose-Einstein condensate.

- If $N = 3$, then we can obtain this theory by coarse-graining the *Heisenberg model*, which is the same energy but instead $\mathbf{s}_i$ is a three-dimensional unit vector.

Consider the ordered phase for $N \geq 2$. Then for $\mu^2 < 0$, F is minimized by

$$|\varphi| = M_0 = \sqrt{-\frac{\mu^2}{4g}}.$$

But the direction of φ is undetermined, so it can lie anywhere on S^{n-1}.

There are also configurations that minimize the potential energy

$$\frac{1}{2}\mu^2|\varphi|^2 + g|\varphi|^4 + \cdots ,$$

i.e. have correct magnitude but with varying direction in space. These get energy only from the derivative terms e.g, $|\nabla\varphi|^2$ in F. We can lower this energy by increasing the length scale over which the variation occurs.

These excitations, arising from spontaneous symmetry breaking of a continuous symmetry, are called *Goldstone bosons*. They are said to be *gapless* if the energy ends to 0 as the wavelength increases to ∞.

The choice of φ breaks $G = \mathsf{O}(N)$ to $H = \mathsf{O}(N-1)$. The space of all ground states is given by

$$S^{N-1} = \mathsf{O}(N)/\mathsf{O}(N-1) = G/H,$$

hence the number of Goldstone bosons is $\dim G - \dim H$.

Example 4.2.

1. Consider the XY-model, where

$$f(\psi) = \frac{1}{2}\mu^2|\psi|^2 + g|\psi|^4.$$

If $\mu^2 < 0$, then we have the Mexican hat potential. If $\psi(\mathbf{x}) = M(\mathbf{x})e^{i\theta(\mathbf{x})}$, for $M \in \mathbb{R}$, then the ground state is given by

$$M(\mathbf{x}) = M_0 = \sqrt{-\frac{\mu^2}{4g}}.$$

$\theta = \theta_0$ is constant. Consider flucutations with $M(\mathbf{x}) = M_0 + \tilde{M}(\mathbf{x})$, and $\theta = \theta(\mathbf{x})$. Then

$$F = \int \mathrm{d}^d x \left[\frac{\gamma}{2}(\nabla\tilde{M})^2 + |\mu|^2\tilde{M}^2 + g\tilde{M}^4 + \frac{\gamma}{2}M_0^2(\nabla\theta)^2 \right.$$
$$\left. + \gamma M_0\tilde{M}(\nabla\theta)^2 + \cdots\right].$$

Here θ is the Goldstone boson; it only has derivative interactions.

2. If $N = 3$, write

$$\varphi(\mathbf{x}) = M(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

Consider fluctuations around the ground state. Then

$$F = \int \mathrm{d}^d x \left[\frac{\gamma}{2}(\nabla\tilde{M})^2 + |\mu^2|\tilde{M}^2 + g\tilde{M}^4 \right.$$
$$\left. + \frac{\gamma}{2}M_0^2[(\nabla\theta)^2 + \sin^2\theta(\nabla\phi)^2] + \cdots\right].$$

Here θ, ϕ are the Goldstone bosons.

4.3 Critical Exponents

Include a term

$$\int \mathrm{d}^d x \, B_a(\mathbf{x})\varphi_a(\mathbf{x})$$

in F. For $B_a(\mathbf{x}) = Bn_a$ a constant, let

$$\chi = \frac{\partial(n_a\varphi_a)}{\partial B}.$$

Let $\alpha, \beta, \gamma, \delta$ be the critical exponents of c, φ with respect to T, χ with respect to T and φ with respect to B.

If $N = 1$, then $\alpha, \beta, \gamma, \delta$ have the same values as for $N = 1$. For correlation functions, we use a quadratic approximation. For $T > T_c$, take

$$Z = e^{-F_{\text{thermo}}} \exp\left[\frac{1}{2}\int \mathrm{d}^d x\, \mathrm{d}^d y\, B_a(\mathbf{x}) G_0(\mathbf{x}-\mathbf{y}) B_a(\mathbf{y})\right],$$

where

$$G_0(\mathbf{x}-\mathbf{y}) = \int \frac{\mathrm{d}^d \mathbf{k}}{(2\pi)^d} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{\gamma \mathbf{k}^2 + \mu^2},$$

then we find

$$\langle \varphi_a(\mathbf{x}) \varphi_b(\mathbf{y}) \rangle = \left.\frac{\delta \log Z}{\delta B_a(\mathbf{x}) \delta B_b(\mathbf{y})}\right|_{B=0} = \delta_{ab} G_0(\mathbf{x}-\mathbf{y}).$$

If $T < T_c$, then assume that $\langle \varphi \rangle = (M_0, 0, \ldots, 0)$ without loss of generality. Then take $\tilde{\varphi} = \varphi - \langle \varphi \rangle$. To quadratic order,

$$\frac{1}{2}\mu^2 \varphi^2 + g(\varphi^2)^2 = \text{const} + \frac{1}{2}\mu_1^2 \tilde{\varphi}_1^2.$$

We have $\mu_1^2 = -2\mu^2$, as for $N = 1$. Note $\tilde{\varphi}_a^2$ for $a \geq 2$ are absent: these are the Goldstone bosons. To compute $\langle \tilde{\varphi}\tilde{\varphi} \rangle$, we introduce

$$-\int \mathrm{d}^d x\, \tilde{B}_a(\mathbf{x}) \tilde{\varphi}_a(\mathbf{x}).$$

This gives

$$Z = e^{-F_{\text{thermo}}} \exp\left[\frac{1}{2}\int \mathrm{d}^d x\, \mathrm{d}^d y\, (\tilde{B}_1(\mathbf{x}) G_L(\mathbf{x}-\mathbf{y}) \tilde{B}_1(\mathbf{y}) + \sum_{a=2}^{N} \tilde{B}_a(\mathbf{x}) G_T(\mathbf{x}-\mathbf{y}) \tilde{B}_a(\mathbf{y}))\right]$$

Here

$$G_L(\mathbf{x}-\mathbf{y}) = \int \frac{\mathrm{d}^d \mathbf{k}}{(2\pi)^d} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{\gamma \mathbf{k}^2 + \mu_1^2}, \qquad G_T(\mathbf{x}-\mathbf{y}) = \int \frac{\mathrm{d}^d \mathbf{k}}{(2\pi)^d} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{\gamma \mathbf{k}^2},$$

so the correlation function is

$$\langle \tilde{\varphi}_a(\mathbf{x}) \tilde{\varphi}_b(\mathbf{y}) \rangle = \left.\frac{\delta \log Z}{\delta \tilde{B}_a(\mathbf{x}) \delta \tilde{B}_b(\mathbf{y})}\right|_{\tilde{B}=0} = \begin{cases} G_L(\mathbf{x}-\mathbf{y}) & a = b = 1, \\ G_T(\mathbf{x}-\mathbf{y}) & a = b \neq 1, \\ 0 & \text{else.} \end{cases}$$

The correlation function for $T > T_c$ and the correlation function for longitudinal modes for $T < T_c$ behaves just as for $N = 1$,

$$G_L(\mathbf{x}-\mathbf{y}) \sim \begin{cases} e^{-r/\xi} & r \gg \xi, \\ r^{-(d-2)} & r \ll \xi, \end{cases}$$

where

$$\xi^2 \sim \frac{1}{\mu^2} \sim \frac{1}{\mu_1^2} \sim \frac{1}{|T-T_c|}.$$

We can define ν by

$$\xi \sim \frac{1}{|T-T_c|^\nu} \implies \nu = \frac{1}{2},$$

the MFT prediction. Define η by

$$\langle \varphi_a(\mathbf{x})\varphi_b(\mathbf{x})\rangle \sim \frac{1}{r^{d-2+\eta}}\delta_{ab}$$

at $T = T_c$, $\eta = 0$ is the MFT prediction.

For $T < T_c$, G_T has the same form as $G_L|_{T=T_c}$, so the Goldstone modes have infinite correlation length. I.e. in the spontaneous symmetry breaking of a continuous symmetry, there are long-range correlations in the broken phase, and this is not just at the critical point.

The critical points with different values of N describe different universality classes. We will see that $d_c = 4$ as for $N = 1$.

Looking at numerical data for $d = 3$, we see the following:

	η	ν
MFT	0	1/2
$N = 1$	0.0363	0.6300
$N = 2$	0.0385	0.6719
$N = 3$	0.0386	0.7020

Table 4: Predictions and Theoretical Critical Exponents

Note $\alpha = 2 - 3\nu$, so $\alpha < 0$ for $N \geq 2$. Hence c exhibits a cusp, rather than a divergence. This is seen in liquid helium.

We showed $G_T \sim r^{-(d-2)}$; we can show that this is exact for all $T < T_c$, so there is no anomalous dimension.

Recall if $\varphi : \mathbb{R}^d \to \mathbb{R}^N$, then for $T < T_c$ we have $|\varphi| = M_0$. A change of coordinates induces a field redefinition $\varphi \to \varphi'(\varphi)$. Choosing coordinates, $(R, G_1, \ldots, G_{n-1})$, these G_i are the Goldstone bosons, and they describe fluctuations tangential to S^{n-1}.

With $\varphi_a = (M_0 + \tilde{\varphi}_1, \tilde{\varphi}_2, \ldots, \tilde{\varphi}_n)$, if $\tilde{\varphi}_a$ is small, then $\tilde{\varphi}_2, \ldots, \tilde{\varphi}_n$ are the Goldstone boson.

4.4 Mermin-Wagner Theorem

If $d = 2$, then

$$\int_0^\Lambda \frac{\mathrm{d}^d k}{k^2} \sim \int_0^\Lambda \frac{k^{d-1}\,\mathrm{d}k}{k^2} = \infty.$$

Hence $G_T = \infty$. For $d \leq 2$ mean field theory implies phase transitions, but quadratic fluctuations diverge in the ordered phase, hence mean field theory is not reliable.

Theorem 4.1 (Mermin-Wagner)**.** *There is no ordered phase for $d \geq 2$. Hence a spontaneous symmetry breaking of a continuous symmetry cannot occur for $d \leq 2$.*

Thus there are no Goldstone bosons for $d \leq 2$, and the lower critical dimensino for the $\mathsf{O}(N)$ model is

$$d_l = \begin{cases} 2 & N \geq 2, \\ 1 & N = 1. \end{cases}$$

4.5 Renormalization Group Flow for $\mathsf{O}(N)$

In the $\mathsf{O}(N)$ model,

$$F[\varphi] = \int \mathrm{d}^d x \left(\frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}\mu^2\varphi^2 + g(\varphi^2)^2 + \cdots \right).$$

The Gaussian fixed point has $\mu^2 = g = \cdots = 0$, and at the GFP,

$$\Delta_\varphi = \frac{d-2}{2}, \qquad \Delta_1 = 2, \qquad \Delta_2 = 4 - d.$$

We repeat what we did for $N = 1$, starting at $\mathbf{g}_0 = (\mu_0^2, g_0, 0, 0, \ldots)$, and use perturbation theory in g_0. Using Feynman diagrams again, note that for the order g_0 interaction we need a slightly different diagram, as there are now multiple fields. What we find is

$$\langle \varphi^+_{a\mathbf{k}} \varphi^+_{b\mathbf{k}'} \rangle_+ = (2\pi)^d \delta^{(d)}(\mathbf{k} + \mathbf{k}') G_0(\mathbf{k}) \delta_{ab}.$$

Hence at $\mathcal{O}(g_0)$, we have

$$\begin{aligned} 2g_0 &\int_0^{\Lambda/\zeta} \prod_{i=1}^2 \frac{\mathrm{d}^d k_i}{(2\pi)^d} \varphi^-_{a\mathbf{k}_1} \varphi^-_{b\mathbf{k}_2} \int_{\Lambda/\zeta}^\Lambda \frac{\mathrm{d}^d q}{(2\pi)^d} \delta_{cd} G_0(\mathbf{q}) \times (2\pi)^d \delta_{ab} \delta_{cd} \delta^{(d)}(\mathbf{k}_1 + \mathbf{k}_2) \\ &= 2Ng_0 \int_0^{\Lambda/\zeta} \frac{\mathrm{d}^d k}{(2\pi)^d} \varphi^-_{a\mathbf{k}} \varphi^-_{a-\mathbf{k}} \int_{\Lambda/\zeta}^\Lambda \frac{\mathrm{d}^d q}{(2\pi)^d} G_0(\mathbf{q}) = 2Ng_0 I. \end{aligned}$$

There is also another way to draw the Feynman diagram, which gives $4g_0 I$. Hence after RG,

$$\mu'^2 = \mu_0^2 + 4(N+2)g_0 \int_{\Lambda/\zeta}^{\Lambda} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{q^2 + \mu_0^2}.$$

We get a similar effect at $\mathcal{O}(g_0^2)$, to get

$$g_0' = g_0 - 4(N+8)g_0^2 \int_{\Lambda/\zeta}^{\Lambda} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{(q^2 + \mu_0^2)^2}.$$

The β functions can be read off from the $N = 1$ results by adjusting the coefficients.

If $d = 4 - \varepsilon$, then $\tilde{g} = \Lambda^{-\varepsilon} g$, and

$$\begin{aligned} \frac{\mathrm{d}\mu^2}{\mathrm{d}s} &= 2\mu^2 + \frac{N+2}{2\pi^2} \frac{\Lambda^4}{\Lambda^2 + \mu^2} \tilde{g} + \mathcal{O}(\tilde{g}^2), \\ \frac{\mathrm{d}\tilde{g}}{\mathrm{d}s} &= \varepsilon \tilde{g} - \frac{N+8}{2\pi^2} \frac{\Lambda^4}{(\Lambda^2 + \mu^2)^2} \tilde{g}^2 + \mathcal{O}(\tilde{g}^2). \end{aligned}$$

Linearizing around the GFP, $\Delta_1 = 1$ and $\Delta_2 = \varepsilon$. Hence there is one relevant direction for $\varepsilon < 0$, and this describes the critical point for $d > 4$.

For $\varepsilon = 0$ this is marginal. For positive ε, we have the WF fixed points,

$$\mu_*^2 = -\frac{1}{2} \frac{N+2}{N+8} \Lambda^2 \varepsilon + \mathcal{O}(\varepsilon^2), \qquad \tilde{g}_* = \frac{2\pi^2}{N+8} \varepsilon + \mathcal{O}(\varepsilon^2).$$

Linearizing around this,

$$\Delta_1 = 2 - \frac{N+2}{N+8} \varepsilon + \mathcal{O}(\varepsilon^2), \qquad \Delta_2 = -\varepsilon + \mathcal{O}(\varepsilon^2).$$

So there is one relevant direction. Note $\eta = \mathcal{O}(\varepsilon^2)$ as before, by looking at two loops. ν scales as

$$\nu = \frac{1}{\Delta_1} = \frac{1}{2} + \frac{N+2}{4(N+8)} \varepsilon + \mathcal{O}(\varepsilon^2).$$

By scaling, we find

ν	$N = 1$	$N = 2$	$N = 3$
$\mathcal{O}(\varepsilon)$	0.58	0.60	0.61
$d = 3$	0.63	0.67	0.71

Table 5: Predictions and Theoretical Values of ν

$$\alpha = \frac{4-N}{2(N+8)}\varepsilon \qquad \beta = \frac{1}{2} - \frac{3}{2(N+8)}\varepsilon$$
$$\gamma = 1 + \frac{N+2}{2(N+8)}\varepsilon \qquad \delta = 3 + \varepsilon$$

This predicts the incorrect sign of α when $N = 2, 3$.

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