III Analysis of PDEs

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Based on Lectures by Prof. Clément Mouhot

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Contents

0 Introduction

Email cm612, in E1.12. Notes are on wordpress, or by Warnick typed by Minter.

Books include Evans, Brézis, John and Lieb-Loss.

0.1 Overview

The field proceeds from works on differential calculus, and trying to turn laws of physics into equations.

We are focused on the modern approach: finding estimates, limits and the function space (using topology). We are not looking at finding explicit formulas.

The course is structured as follows.

Chapter 1. Introduction (2 lectures). This is focused on turning an ODE into a PDE.

Chapter 2. The Cauchy Kovalevskoya Theory (4-5 lectures). Here we look at a PDE with analytic function, where we want to solve for analytic solutions. This lets us construct locally a solution.

Chapter 3. Functional toolbox (4 lectures). Here we introduce Hölder and Lebesgue spaces, as well as weak derivatives, Sobolev spaces, inequalities, approximations by convolution, and extensions or traces of functions.

Chapter 4. Elliptic PDEs (6-7 lectures). Here we look at the Laplace equation and its variants $\Delta u = 0$ on U, and $u|_{\partial U} = g$. We are most interested in Lax-Milgram theory, and may look at Fredholm theory, and spectral theory.

Chapter 5. Hyperbolic PDEs (7 lectures). The main equations are the scalar transport equation (where we look at the Burgers equation), and the wave equation.

1 From ODEs to PDEs

In *differential equations*, the unknown is a function. In an ODE (ordinary differential equation), we first fix a function

$$F = F(x, y_1, \dots, y_{k+1}).$$

Here $k \geq 1$. We solve for $u : U \subseteq \mathbb{R} \to \mathbb{R}$ the relation, for all $x \in U$,

$$F(x, u(x), u'(x), \dots, u^{(k)}(x)) = 0. \tag{$*$}$$

Here the domain U is an open, connected, regular set in $\mathbb{R}$.

Example 1.1.

Consider

$$F = F(x, z, y) = f(x, z) - y.$$

Then the equation $(*)$ becomes

$$u'(x) = f(x, u(x)).$$

This can be solved by Picard-Lindelöf, with certain restrictions on f.

In a PDE, we no loner have x in $\mathbb{R}$, but in $\mathbb{R}^n$. Therefore the relation $(*)$ must be modified to include:

$$u(x) = u(x_1, \dots, x_n), \qquad \frac{\partial u}{\partial x_i}(x), \qquad \frac{\partial^2 u}{\partial x_i \partial x_j}(x), \qquad \dots$$

Definition 1.1. Give $n \geq 2$, $U \subseteq \mathbb{R}^n$ a domain, a *partial differential equation* of rank or order $k \geq 1$ is a relation of the form, for all $x \in U$,

$$F(x, u(x), Du(x), \dots, D^k u(x)) = 0, \tag{$**$}$$

where $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \times \dots \times \mathbb{R}^{n^k} \to \mathbb{R}$.

We solve for $u : U \subseteq \mathbb{R}^n \to \mathbb{R}$. If $u \in C^k(U)$, and satisfies $(**)$ identically as an equality between continuous functions, we say that u is a *classical solution* to the PDE.

Remark.

1. When possible (but not for elliptic PDEs) it is useful to identify one of the components of x, say x_1, as a time $x_1 = t$. We then say that the PDE takes the form of an *evolution problem*.

   Finding such a 'time variable' can be a difficulty in itself.

2. We can also consider the more general case $u(x) \in \mathbb{R}^m$, for $m \geq 1$, and F values in $\mathbb{R}^N$, for $N \geq 1$. When $m \geq 2$, we say it is a *system* of PDEs.

3. Can we consider a PDE as yet another ODE but in infinite dimensions, at least when it is in the form

$$\frac{\partial u}{\partial t} = G\left(\left(\frac{\partial u}{\partial x_i}\right)_{i=2}^{n}, \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)_{i,j=2}^{n}, \dots\right).$$

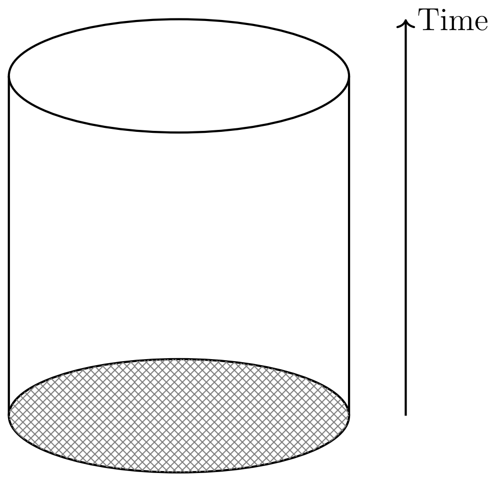
   No. First, losing the total order on the parameter x leads to some geometric phenomena. This is repsonsible for some differences (reversibility, or whether it is an evolution problem).

   Second, if we interpret this is an ODE $u'(t) = g(u)$, then u lives in functional space which is infinite-dimensional, whereas in an ODE we have a trajectory in $\mathbb{R}^N$. Even at a linear level, operators can be unbounded, and the topologies are no longer equivalent.

4. We also have boundary conditions. We know that just the condition $u'(t) = f(t, u(t))$ is not enough; we also need to specify, for example, $f(0) = u_0$.

   For PDEs in evolution form $\partial_t u = G$, then our boundary condition becomes $u(0, \cdot) = u_0(\cdot)$, where this is now a function. Moreover, we can consider boundary conditions on other variables.

5. Also PDEs come in so many different forms, that each structure must be understood.



Boundary condition at time $t = 0$

2 The Cauchy Problem

A basic question of mathematical analysis is to solve

$$u'(t) = F(t).$$

If F is continuous, then by FTC we get

$$u(t) = u(t_0) + \int_{t_0}^{t} F(z)\,\mathrm{d}z.$$

This is solved. We have shown there exists solutions, and there's a unique solution given $u(t_0) = u_0$, that depends continuously on boundary data u_0.

A more complicated ODE is where $F = F(t, u(t))$, so

$$u'(t) = F(t, u(t)), \qquad u(t_0) = u_0.$$

There are three main results for functions of this form:

Result 1. Cauchy-Kovalevskaya for ODEs. In the open region where F is real analytic (locally the sum of a Taylor series), there exists a unique local analytic solution: given (t_0, u_0) in this region, there is a neighbourhood around it so that a unique analytical solution u exists.

This has limited use: it is only for F analytic, it does not cover all PDEs, and it is rare to be able to continue the solution.

We can extend this to PDEs.

Result 2. Picard-Lindelöf. In the region where f' is continuous and Lipschitz in the second variable, there exist a local, unique solution C^1 solution u, which depends continuously on u_0.

This inspired the Cauchy problem and well-posedness. We can extend this to linear PDEs, known as Hille-Yosida theorem.

Result 3. Cauchy-Peano. In the region where f is merely continuous, there exists locally a C^1 solution u. In general, it is not unique.

This is done through an iterative scheme and compactness, and is the inspiration for theories of weak solutions in PDEs.

Note that in a larger space, existence is easier, but uniqueness is harder, and vice versa. Hence finding a sweet-spot is critical.

Example 2.1.

The ODE
$$u'(t) = \sqrt{u(t)}, \qquad u(0) = 0$$
has a solution which exists by Cauchy-Peano, but is not unique. Another example is
$$u'(t) = \frac{4u(t)t}{u(t)^2 + t^2}, \qquad u(0) = 0.$$

Another key question is local versus global solutions, i.e. finding a global solution to
$$u'(t) = F(t, u(t)), \qquad u(0) = u_0,$$
for all $t \geq 0$. We have a few criterion for when global solutions exist.

Criterion 1. F is uniform Lipschitz.

Here we can just apply Picard-Lindelöf to continue a solution. It is not easy to export this to PDEs.

Criterion 2. Assume the hypothesis of Picard-Lindelöf, as well as a growth condition on F:
$$|F(t, u)| \leq C(1 + |u|).$$
Then the solution can be continued globally.

The idea behind this is that, a priori, a solution C^1 has to satisfy
$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}|u(t)|^2 &\leq 2C(1 + |u(t)|^2), \\ u'(t) &= F(t, u(t)). \end{aligned}$$

This is similar to what we call an energy estimate in PDEs.

Example 2.2.

The ODE
$$u'(t) = u(t)^2, \qquad u(t_0) = u_0 > 0$$
has no global solutions. This is because when you square a big number it gets bigger. However if we swap the sign, the solution is global. This is because when you square a small number, it gets smaller.

The ODE
$$u'(t) = \sin(u(t)), \qquad u(0) = u_0$$

has global solutions, by criterion 1. Similarly,

$$u'(t) = \sin(u(t)^2), \qquad u(0) = u_0$$

has global solutions, this time by criterion 2.

2.1 Well-posedness for PDEs

Sometimes there is no explicit formula or even series for a solution to a PDE. In these cases we need to construct solutions abstractly.

Two breakthroughs happened when looking at when PDEs have solutions. The first is the definition of a Cauchy problem, and the second is looking at well-posedness.

Definition 2.1. A *Cauchy problem* is the combination of a PDE, and some boundary data; prescribing values of the unknown u, and possibly its derivatives, on parts of the domain.

Such a problem is said to be *well-posed* if:

- A solution exists (in some function space, e.g. $C^k(U), H^k(U)$, at least locally).
- The solution is unique among possible solutions in the function space.
- The solutions depends continuously on the boundary data.

2.2 Terminology and Examples

Definition 2.2. A PDE with vector field F is *linear* if F is a linear function of u and its derivative. So,

$$\sum_{|\alpha|\le k} a_\alpha(x)\partial^\alpha u(x) = f(x).$$

Here $f(x)$ is the *source*, or the RHS.

A PDE is *semilinear* when F is linear in the highest-order derivatives of u:

$$\sum_{|\alpha|=k} a_\alpha(x)\partial^\alpha u + a_0[x, u(x), Du(x), \dots, D^{k-1}u] = 0.$$

A PDE is *quasilinear* if F is linear in highest-order deriatives of u, but can depend nonlinearly on the lower-order derivatives:

$$\sum_{|\alpha|=k} a_\alpha[x, u(x), Du(x), \dots, D^{k-1}u]\partial^\alpha u(x) + a_0[x, u(x), Du(x), \dots, D^{k-1}u] = 0.$$

A PDE is *fully nonlinear* if it is none of the types above.

Example 2.3.

- Linear PDE: Take the Laplace,

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0.$$

- Semilinear PDE:

$$\Delta u = \left(\frac{\partial u}{\partial x_1}\right)^2.$$

- Quasilinear PDE:

$$u\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial x} \text{ on } \mathbb{R}^2.$$

- Fully nonlinear:

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 = 0.$$

We also have some examples from physics:

- Newtons' equations.
- Euler incompressibility equation.
- Navier-Stokes equation.
- Boltzmann equation.
- Vlasov equation.
- Schrödinger equation.
- Einstein equations.
- Dirac equation.

Moreover here are equations from math:

- Cauchy-Riemann equations.
- Ricci flow. $\partial_t g_{ij} = -2R_{ij}$.

3 The Cauchy-Kovalevskaya Theory

This is the only "general" theorem that can be salvaged from ODEs. Some concepts that arise are:

- Non-characteristic Cauchy data.
- Principal symbols.
- Basic classification of PDEs.

However the analyticity used in this theory is most often not satisfying, in the functional setting.

3.1 Real Analyticity

Definition 3.1. Given $U \subseteq \mathbb{R}^n$ open, a function $f : U \to \mathbb{R}$ is *real analytic* near $\tilde{x} \in U$ if there is $r > 0$ and real constants (f_α) so that the series

$$\sum_{\alpha \geq 0} f_\alpha (x - \tilde{x})^\alpha$$

converges for $x \in B(\tilde{x}, r)$ to $f(x)$.

If $f : U \to \mathbb{R}^n$, for $n \geq 2$, then it is real analytic if f_i is real analytic for $i = 1, \ldots, n$.

f is *real analytic* in U if it is real analytic near each point of U. This is sometimes denoted as

$$f \in C^\omega(U).$$

Example 3.1.

Simple examples of real analytic functions include polynomials, exponential functions, trigonometric functions.

The map $z \mapsto \overline{z}$, i.e. conjugation, is not $\mathbb{C}$-differentiable, but it is real analytic in $\mathbb{R}^2$.

The function

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0, \\ 0 & x = 0 \end{cases}$$

is C^∞, but not real analytic. In fact any C_c^∞ function cannot be real analytic.

Liouville's theorem does not hold, by either sin or $1/(1 + x^2)$.

Real analyticity is local, meaning if f is real analytic near $\tilde{x}$, then f is real analytic in $B(\tilde{x}, r) \subseteq U$ for some $r > 0$.

Proposition 3.1. *Given $U \subseteq \mathbb{R}^n$ open and non-empty, then $f : U \to \mathbb{R}$ is real analytic on U if and only if $f \in C^\infty$, and for any $K \subseteq U$ compact, there are $C(K)$, $r(K) > 0$ so that the following growth conditions holds: for all $x \in K$, $\alpha \in \mathbb{N}^n$,*

$$|\partial^\alpha f(x)| \leq C(K) \frac{\alpha!}{r(K)^{|\alpha|}}.$$

Remark.

- When $U \subseteq \mathbb{R}$, another equivalent definition is, f is real analytic on U if it can be locally extended to a $\mathbb{C}$-differentiable function near each point of U.
- When $U = \mathbb{R}^n$, real analyticity is also equivalent to exponential decay in the Fourier variables.

Proof: Recall that, if

$$\sum_{\alpha \geq 0} f_\alpha (x - \tilde{x})^\alpha$$

converges at x such that $|x - \tilde{x}| = r$, then the general term is bounded by

$$|f\alpha| \leq Cr^{-|\alpha|}.$$

Hence for $|x - \tilde{x}| < r$, we have absolute convergence.

Recall for a function, the *radius of convergence* is the largest $r \geq 0$ so that we have a point of convergence at a distance r.

The easy implication is the forwards. Suppose that in $B(\tilde{x}, r) \subseteq U$, we have the power series

$$f(x) = \sum f_\alpha (x - \tilde{x})^\alpha,$$

with radius of convergence at least r. Then from a standard theorem, f is smooth in $B(\tilde{x}, r)$ with

$$\partial^\alpha f(\tilde{x}) = (f_\alpha)\alpha!.$$

We know that $|f_\alpha| \leq C\bar{r}^{-|\alpha|}$, for some $\tilde{r} < \bar{r} < r$. Then for all $x \in \bar{B}(\tilde{x}, \tilde{r})$, and $\beta \in \mathbb{N}^n$,

$$\begin{aligned}
|\partial^\beta f(x)| &= \left| \partial^\beta \left(\sum_{\alpha \geq 0} f_\alpha (x - \tilde{x})^\alpha \right) \right| \\
&\leq \sum_{\alpha \geq \beta} |f_\alpha| \frac{\alpha!}{(\alpha - \beta)!} |x - \tilde{x}|^{|\alpha - \beta|} \\
&\leq C \sum_{\alpha \geq \beta} \bar{r}^{-|\alpha|} \frac{\alpha!}{(\alpha - \beta)!} \tilde{r}^{|\alpha - \beta|} \\
&\leq C \bar{r}^{|\beta|} \sum_{\alpha \geq \beta} \left(\frac{\tilde{r}}{\bar{r}} \right)^{|\alpha - \beta|} \frac{\alpha!}{(\alpha - \beta)!}.
\end{aligned}$$

Let $\lambda = \tilde{r}/\bar{r} < 1$. Then by observation,

$$(1 - \lambda)^{-1} = \sum_{j \geq 0} \lambda^j.$$

Taking the m'th partial derivative,

$$\frac{m!}{(1 - \lambda)^{m+1}} = \sum_{j \geq m} \frac{j!}{(j - m)!} \lambda^{j - m}.$$

If we apply this, then

$$\begin{aligned}
|\partial^\beta f(x)| &\leq C r^{|\beta|} \sum_{\alpha \geq \beta} \frac{\alpha!}{(\alpha - \beta)!} \lambda^{|\alpha - \beta|} \\
&\leq C |r|^{|\beta|} \frac{\beta!}{(1 - \lambda)^{|\beta| + n}} \\
&\leq \frac{C \beta!}{(1 - \lambda)^n} \left(\frac{r}{1 - \lambda} \right)^{|\beta|}.
\end{aligned}$$

For the other direction, consider our assumption on $K = \bar{B}(\tilde{x}, r) \subseteq U$, there exists $\tilde{C}, \tilde{r} > 0$ such that for all $x \in K$, $\alpha \in \mathbb{N}^n$,

$$|\partial^\alpha f(x)| \leq \tilde{C} \tilde{r}^{-|\alpha|} \alpha!.$$

Choose $x \in B(\tilde{x}, \tilde{r}/2)$, and Taylor expand, so

$$f(x) = \sum_{|\alpha| \leq k} \partial^\alpha f(x) \frac{(x - \tilde{x})^\alpha}{\alpha!} + \sum_{|\alpha| = k+1} R_\alpha(x) (x - \tilde{x})^\alpha.$$

If $n = 1$, we have

$$R_\alpha(x) = \frac{|\alpha|}{\alpha!}\int_0^1 (1-t)^{|\alpha|-1}\partial^\alpha f(\tilde{x} + t(x-\tilde{x}))\,\mathrm{d}t.$$

From the growth condition, the main part of the expansion is a partial sum of an absolute series, and

$$\begin{aligned}\left|\sum_{|\alpha|=k+1} R_\alpha(x)(x-\tilde{x})^\alpha\right| &\le \sum_{|\alpha|=k+1} |R_\alpha(x)|\left(\frac{\tilde{r}}{2}\right)^{k+1}, \\ |R_\alpha(x)| &\le \tilde{C}\frac{|\alpha|}{\alpha!}\int_0^1 (1-t)^{|\alpha|-1}\tilde{r}^{-(k+1)}\,\mathrm{d}t \\ &\le \tilde{C}\frac{\tilde{r}^{-(k+1)}}{\alpha!}.\end{aligned}$$

So,

$$\begin{aligned}I = \left|\sum_{|\alpha|=k+1} R_\alpha(x)(x-\tilde{x})^\alpha\right| &\le \tilde{C}\tilde{r}^{-(k+1)}\left(\frac{\tilde{r}}{2}\right)^{k+1}\cdot\binom{k+n}{n-1} \\ &\le C'(k+n)^{n-1}2^{-(k+1)} \to 0\end{aligned}$$

as $k \to \infty$, which shows the convergence of the Taylor series.

Definition 3.2. Let

$$f = \sum_{\alpha\ge 0} f_\alpha x^\alpha, \qquad g = \sum_{\alpha\ge 0} g_\alpha x^\alpha$$

be two formal power series. Then g *majorizes* f, or g is a *majorant* of f, written $g \gg f$ if $g_\alpha \ge |f_\alpha|$ for all $\alpha \in \mathbb{N}^\alpha$.

If f, g are $\mathbb{R}^m$-valued, then each component $g_j \gg f_j$, for $j = 1, \ldots, m$.

Proposition 3.2. *Given f, g formal power series:*

(i) *If $g \gg f$ and g converges for $\|x\| < r$, then f converges for $\|x\| < r$ as well.*

(ii) *If f converges for $\|x\| < r$, and $\tilde{r} \in (0, r/\sqrt{n})$, there is a majorant $g \gg f$ which converges in $\|x\| < \tilde{r}$.*

Proof:

(i) Let $x \in B(0, r)$, then

$$\begin{aligned}\sum_{\alpha \le k} |f_\alpha x^\alpha| &\le \sum_{|\alpha| \le k} |f_\alpha| |x_1|^{\alpha_1} \cdots |x_n|^{\alpha_n} \\ &\le \sum_{|\alpha| \le k} g_\alpha |x_1|^{\alpha_1} \cdots |x_n|^{\alpha_n}.\end{aligned}$$

If $y = (|x_1|, \dots, |x_n|)$, then $\|y\| = \|x\| < r$, and since g converges for $\|x\| < r$ it converges for y.

(ii) Take $\tilde{r} \in (0, r/\sqrt{n})$ and $y = (\tilde{r}, \dots, \tilde{r})$. Then $\|y\| = \sqrt{n}\tilde{r} < r$

Now f converges on y, so the general term is bounded:

$$|f_\alpha| \le C \tilde{r}^{-|\alpha|}.$$

Now consider

$$\bar{f}(x) = \frac{C}{1 - (x_1 + \cdots + x_n)/\tilde{r}},$$

for $x \in B(0, \tilde{r}/\sqrt{n})$. Then

$$\begin{aligned}\bar{f}(x) &= C \sum_{k \ge 0} \tilde{r}^{-k} \left(\sum_{|\alpha| = k} \frac{|\alpha|!}{\alpha!} x^\alpha \right) \\ &= C \sum_{\alpha \ge 0} \frac{\tilde{r}^{-|\alpha|} |\alpha|!}{\alpha!} x^\alpha.\end{aligned}$$

We can check that $\alpha! \le |\alpha|!$, so $f \ll \bar{f}$.

Another majorant we can consider is

$$\begin{aligned}\bar{f}(x) &= C \prod_{i=1}^{n} \left(\frac{1}{1 - (x_i/\tilde{r})} \right) = C \prod_{\alpha \ge 0} \left(\frac{x}{\tilde{r}} \right)^\alpha \\ &= C \sum_{\alpha \ge 0} \tilde{r}^{-|\alpha|} x^\alpha.\end{aligned}$$

4 Cauchy-Kovalevskaya for ODEs

Theorem 4.1. *Let $a, b > 0$, and $u_0 \in \mathbb{R}$. Consider $F : (u_0 - b, u_0 + b) \to \mathbb{R}$ real analytic, and $u : (-a, a) \to (u_0 - b, u_0 + b)$ a C^1 solution to*

$$u'(t) = F(u(t)).$$

Then u is real analytic on $(-a, a)$.

Proof: We prove this in many ways. The first is by Picard iteration.

Define

$$\begin{aligned} u_{l+1}(z) &= u_0 + \int_0^z F(u_l(z))\, \mathrm{d}z, \\ u_0(z) &= u_0, \end{aligned}$$

where we exited F to be homomorphic near the origin. We may prove that (u_l) is Cauchy in the $\|\cdot\|_\infty$ norm, on a small enough neighbourhood of $t = 0$. Here we use the bound on F'.

Now $u_l \to u$ converges uniform locally. By induction, we can show u_l is $\mathbb{C}$-differentiable, so by Morera's theorem, u is $\mathbb{C}$-differentiable.

The second proof is by separation of variables. If $F(0) = 0$, then $u = 0$, and there is nothing to prove.

Otherwise, $F \neq 0$ near 0, and we may write

$$G(y) = \int_0^y \frac{\mathrm{d}x}{F(x)},$$

for $y \in (-b', b')$ for some $0 < b' < b$ small enough. Then we find that

$$\frac{\mathrm{d}}{\mathrm{d}t} G(u(t)) = \frac{F(u(t))}{F(u(t))} = 1.$$

For $t \in (-a', a')$, $G(u(0)) = G(0) = 0$, and $G(u(t)) = t$, and

$$G'(0) = \frac{1}{F(0)} \neq 0.$$

So there exists a smaller $(-a'', a'') \subseteq (-a', a')$ such that G^{-1} is defined, and F is real analytic. Then since G is real analytic, G^{-1} is as well, so

$$u(t) = G^{-1}(t)$$

is real analytic on $(-a'', a'')$.

The third proof is by embedding the equation in a larger continuum of equations. For $z \in \mathbb{C}$, consider

$$\begin{aligned} u_z'(t) &= zF(u_z(t)), \\ u_1(0) &= 0, \end{aligned}$$

where the original equation is $z = 1$.

For $|z| < 2$, and $|t| < \varepsilon$ small enough, Picard-Lindelöf gives a solution uniformly in $|z| < 2$ by having Lipschitz constant on zF uniformly in $|z| < 1$.

Defining

$$\partial z = \left(\frac{\partial x - i\partial y}{2}\right), \qquad \partial \bar{z} = \left(\frac{\partial x + i\partial y}{2}\right),$$

then a function f is complex differentiable if and only if

$$\partial \bar{z}(f) = 0,$$

Taking our function to be u_z', we find

$$\partial t \partial \bar{z}[u_z(t)] = zF'(u_z(t))\partial \bar{z}[u_z(t)].$$

We can integrate this to find

$$\partial \bar{z}[u_z(t)] = \exp\left[\int_0^t zF'(u_z(s))\,\mathrm{d}s\right]\partial \bar{z}[u_z(0)],$$

where the last term is 0, hence the entire thing is 0. So, for $|t|$ small enough and $|z| < 2$, $z \mapsto u_z(t)$ is $\mathbb{C}$-differentiable. Hence, we can write

$$u_1(t) = \sum_{n=0}^{\infty} \frac{1^n}{n!}\frac{\partial^n}{\partial z^n}[u_z(t)]\bigg|_{z=0}.$$

For $|z| < 2$ real,

$$\frac{\partial^n}{\partial z^n}[u(zt)] = t^n u^{(n)}(0),$$

where the latter is real differentiable. This implies convergence and equality,

$$u(t) = u_1(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} u^{(n)}(0).$$

Another proof is by majorant. If u, F are smooth with

$$u'(t) = F(u(t)),$$

then we will induct on $u \in C^k((-a,a))$, $k \geq 1$. Then note $F \circ u \in C^k(\cdot)$, so $u' \in C^k$, hence $u \in C^{k+1}$. For example,

$$\begin{aligned}
u^{(1)}(t) &= F^{(0)}(u(t)), \\
u^{(2)}(t) &= F^{(1)}(u(t)) \times u^{(1)}(t) = F^{(1)}(u(t)) \times F^{(0)}(u(t)), \\
u^{(3)}(t) &= F^{(2)}(u(t)) \times F^{(0)}(u(t))^2 + F^{(1)}(u(t))^2 F^{(0)}(u(t)),
\end{aligned}$$

and we can go on. By induction, we can show $u^{(k)}(t)$ is a polynomial in $F^{(0)}(u(t)), F^{(1)}(u(t)), \ldots, F^{(k-1)}(u(t))$, with non-negative integer coefficients, so write

$$u^{(k)}(t) = p_k(F^{(0)}(u(t)), F^{(1)}(u(t)), \ldots, F^{(k)}(u(t))).$$

For example,

$$\begin{aligned}
p_1(x_1) &= x_1, \\
p_2(x_1, x_2) &= x_1 x_2, \\
p_3(x_1, x_2, x_3) &= x_1^2 x_3 + x_1 x_2^2.
\end{aligned}$$

These polynomials are universal; they do not depend on F. If $G \gg F$, then $|G^{(k)}(0)| > |F^{(k)}(0)|$ for all k, and so

$$\begin{aligned}
p_k(F^{(0)}(0), \ldots, F^{(k-1)}(0)) &\leq p_k(|F^{(0)}(0)|, \ldots, |F^{(k-1)}(0)|) \\
&\leq p_k(G^{(0)}(0), \ldots, G^{(k-1)}(0)).
\end{aligned}$$

Assume that we have $G \gg F$, and that v is a solution to

$$\begin{aligned}
v'(t) &= G(v(t)), \\
v(0) &= 0,
\end{aligned}$$

and v is real analytic near 0. Then,

$$v^{(k)}(0) = p_k(G^{(0)}(0), \ldots, G^{(k-1)}(0)),$$

so that $v^{(k)}(0) > |u^{(0)}(0)|$, for all $k \geq 0$. Since v is real analytic,

$$v(t) = \sum_{k \geq 0} v^{(k)}(0) \frac{t^k}{k!},$$

which is absolutely convergent near 0. Define

$$\tilde{u}(t) = \sum_{k \geq 0} p_k(F^{(0)}(0), \ldots, F^{(k-1)}(0)) \frac{t^k}{k!},$$

ons the same disc of convergence. This $\tilde{u}$ is real analytic near 0, and since $\tilde{u}(t)$ and $F(\tilde{u}(t))$ are real analytic and all derivatives agree at $t = 0$, they are equal near $t = 0$.

Now all we need to do is construct G and v. This is possible since

$$|F^{(k)}(0)| \leq Ck!r^{-k},$$

for $k \geq 0$ and some $C, r > 0$. So we can define

$$G(x) = \frac{Cr}{r - x},$$

for $|x| < r$. Then the solution to

$$\begin{aligned} v'(t) &= G(v(t)), \\ v(0) &= 0 \end{aligned}$$

is

$$v(t) = r - r\sqrt{1 - \frac{2Ct}{r}}.$$

This is real analytic for $|t| < r/2C$.

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