III Black Holes

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0 Introduction

Our conventions are:

- $c = G = 1$.
- The signature is $(-, +, \dots, +)$.
- $\Lambda > 0$ corresponds to de-Sitter, $\Lambda < 0$ corresponds to anti de-Sitter.
- Specific coordinate systems correspond to Greek indices.
- Things true in any coordinate system are written in latin indices.
- $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$.

1 Spherical Stars

Gravitational attraction wins when there is no fuel. Eventually, the Pauli exclusion principle takes over, leading to degeneracy pressure.

We have some scales of star.

- If a star has mass less than around $1.4M_0$, then we believe it will form a white dwarf.
- If it has mass less than M_0, it will form a neutron star due to neutron degeneracy pressure.
- For larger mass, it will form a black hole.

In general, the set of isometries of a manifold with a metric forms a group. A normal two sphere is invariant under $\mathsf{SO}(3)$:

$$\mathrm{d}\Omega_2^2 = \mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2.$$

It is also invariant under reflections: $\theta \to \pi - \theta$, giving $\mathsf{O}(3)$.

Definition 1.1. A spacetime is *spherically symmetric* if it has the same group of isometries as a normal sphere. More precisely, a spacetime s is spherically symmetric if its isometry group contains a $\mathsf{SO}(3)$ subgroup whose orbits are two-spheres.

In a spherically symmetric spacetime, we can define an area radius: $r : \mathcal{M} \to \mathbb{R}^2$ by

$$r(p) = \sqrt{\frac{A(p)}{4\pi}},$$

where $A(p)$ is the area of the orbit through p.

1.1 Time Independence

Definition 1.2. A spacetime is *stationary* if it admits a Killing vector field K^μ which is everywhere timelike:

$$K^a g_{ab} K^b < 0.$$

We pick a hypersurface Σ nowhere tangent to K. We assign a coordinate (t, x^i) to the point parametrized at distance t along the integral curves of K that start on Σ at x^i.

In this coordinate system,

$$K^a = \left(\frac{\partial}{\partial t}\right)^a.$$

In these nice coordinate,

$$\mathcal{L}_K g = 0 \implies \mathrm{d}s^2 = g_{tt}(x^k)\,\mathrm{d}t^2 + 2g_{ti}(x^k)\,\mathrm{d}t\,\mathrm{d}x^i + g_{ij}(x^k)\,\mathrm{d}x^i\,\mathrm{d}x^j.$$

This is still too complicated. We need another simplification.

1.2 Hypersurface Orthogonality

Let Σ be defined by $f(x) = 0$. Then the one form $\mathrm{d}f$ is orthogonal to Z. Let Z^a be tangent to Σ. Then,

$$(\mathrm{d}f)(Z) = Z(f) = Z^\mu \partial_\mu f = 0,$$

as the derivatives of f on Σ are 0.

Take a generic normal

$$m = g\,\mathrm{d}f + fm',$$

where m' is a smooth one form. Then

$$\mathrm{d}m = \mathrm{d}g \wedge \mathrm{d}f + \mathrm{d}f \wedge m' + f\,\mathrm{d}m' \implies \mathrm{d}m|_\Sigma = (\mathrm{d}g - m') \wedge \mathrm{d}f|_\Sigma.$$

And so, $(m \wedge \mathrm{d}m)_\Sigma = 0$.

Conversely, if n is a non-zero one-form such that $n \wedge \mathrm{d}n = 0$ everywhere, then

$$n = g\,\mathrm{d}f.$$

Definition 1.3. A spacetime is *static* if it admits a hypersurface orthogonal timelike Killing vector field.

Since the spacetime is hypersurface orthogonal, choose Σ to be orthogonal to K. Take for instance Σ to be $t = t_0$. Then $K_\mu \propto (1, 0, 0, 0)$. Indeed, $n = g\,\mathrm{d}f = g\,\mathrm{d}t$.

Hence $K_i = 0$, and $K^a = (\partial/\partial t)^a$. This implies $g_{ti} = 0$. Hence for a static metric,

$$\mathrm{d}s^2 = g_{tt}(x^k)\,\mathrm{d}t^2 + g_{ij}(x^k)\,\mathrm{d}x^i\,\mathrm{d}x^j.$$

1.3 Static and Spherical Symmetry

Since the spacetime is static, we have a Killing vector field K^a, and we can facilitate our spacetime with surfaces Σ_t, orthogonal to K.

Any $\mathsf{SO}(3)$ orbits of $p \in \Sigma_t$ will lie in Σ_t. Define polar coordinates (θ, ϕ) on this S^2 orbit.

Extend this definition to the rest of Σ_t by defining them to be constant along spacelike geodesics orthogonal to $S^2(p)$. We then use the area radius r:

$$\mathrm{d}s_{\Sigma_t}^2 = e^{2\psi(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_2^2.$$

We extend this definition:

$$\mathrm{d}s^2 = -e^{2\Phi(r)}\,\mathrm{d}t^2 + e^{2\psi(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_2^2.$$

Our stars will be modelled by fluids:

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab},$$

and $u_a u^a = -1$. Because the spacetime is spherically symmetric, so are p and ρ. As $(\mathcal{M}, g)$ it is static, we can immediately say that

$$u^a = e^{-\Phi}\left(\frac{\partial}{\partial t}\right)^a.$$

We had a quadratic constraint, so we could have picked a negative sign. We did not; from general relativity, we should have that u is in the same direction as $\partial/\partial t$, so particles flow in a timelike manner. Our convention results in a negative inner product, which want we want.

If the star has radius R, then

$$\rho(r) = p(r) = 0, \qquad r > R.$$

The equation $\nabla_a T^{ab} = 0$ corresponds to the fluid equation of motion. Einstein's equations say

$$R_{ab} - \frac{R}{2}g_{ab} = 8\pi T_{ab}.$$

Because of spherical symmetry and staticity, we only care about the tt, rr, $\theta\theta$ and $\phi\phi$ components, with $R_{\phi\phi} = \sin^2\theta R_{\theta\theta}$. We can show, on example sheet 1, that

$$\begin{aligned}
G_{tt} &= \frac{e^{2(\Phi-\psi)}}{r^2}(e^{2\psi} - 2r\psi' - 1),\\
G_{rr} &= \frac{1}{r^2}(1 - e^{2\psi} + 2r\Phi'),\\
G_{\theta\theta} &= e^{-2\psi}r[r\Phi'^2 + \psi' + \Phi'(1 - r\psi') + r\Phi''].
\end{aligned}$$

To make this equation a bit nicer, write

$$e^{2\psi} = \frac{1}{1 - \frac{2m(r)}{r}} \implies \psi(r) = \frac{1}{2}\log\left(1 - \frac{2m(r)}{r}\right).$$

Recall that

$$T_{tt} = e^{2\Phi}\rho, \qquad T_{rr} = e^{2\psi}\rho, \qquad T_{\theta\theta} = r^2 p.$$

So the equations boil down to:

$$\begin{aligned}
tt: \qquad & m'(r) = 4\pi r^2 \rho(r), \\
rr: \qquad & \Phi'(r) = \frac{m(r) + 4\pi r^3 p(r)}{r(r - 2m(r))}, \\
\theta\theta: \qquad & p'(r) = -[p(r) + \rho(r)]\frac{m(r) + 4\pi r^3 p(r)}{r(r - 2m(r))}.
\end{aligned}$$

This is three equations for four unknown variables: (ρ, m, p, Φ), hence we cannot solve.

What we are missing is the *equation of state*, which comes from chemistry:

$$p = p(\rho, T).$$

We want ultra-cold stars, so $T = 0$, and $p = p(\rho)$, and $\rho, p > 0$ as we are sensible. This is known as the TOV relation.

Outisde the star of radius R, $\rho = p = 0$. So $m'(r) = 0$, and hence $m(r) = M$, which gives

$$\psi(r) = -\frac{1}{2}\log\left(1 - \frac{2M}{r}\right) = -\Phi(r),$$

where we took $\lim \Phi(r) = 0$. So we get

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2M}{r}} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right),$$

where M is the total mass of the star. This is the *Schwarzschild metric*. This has far-reaching observations.

- The Schwarzschild solution appears singular at $r = 2M$.
- For stars, the solution is only meaningful for $r > 2M$.
- So we have a bound

  $$R > R_s = 2M.$$

  Reinstating units, we find

  $$\frac{GM}{c^2 R} < \frac{1}{2},$$

  which in the Newtonian limit gives $0 < 1/2$. Hence there is no analog in the Newtonian theory. Is this bound any good? For our sun,

  $$R_0 \simeq 7 \times 10^5\,\mathrm{km} \gg 2M_0 \simeq 3\,\mathrm{km}.$$

What about the solution inside of the star? From our equation for m,

$$m(r) = 4\pi \int_0^r \rho(r') r'^2 \, \mathrm{d}r' + m_*,$$

for m_* some constant. We need $m_* = 0$, so that we have a start that leads to smooth spacetime at $r = 0$. At the surface $r = R$,

$$m(R) = 4\pi \int_0^R \rho(r) r^2 \, \mathrm{d}r = M,$$

by continuity with the solution at $r > R$. If we assume that the speed of sound is well-defined, so

$$\frac{\mathrm{d}p}{\mathrm{d}\rho} = C_S^2 > 0.$$

then from the $\theta\theta$ component of the Einstein equation, $\rho' < 0$, and thus the condition means

$$\frac{m(r)}{r} < \frac{2}{9}\left[1 - 6\pi^2 r^2 p + (1_6 \pi r^2 p)^{1/2}\right].$$

At the surface, we get

$$R > \frac{9}{4} M.$$

This is the *Buchdahl limit*.

A *one-parameter family of stars* is given by

$$\begin{aligned} m'(r) &= 4\pi r^2 \rho(r), \\ p' &= -[p(r) + \rho(r)] \frac{m(r) + 4\pi r^3 p(r)}{r(r - 2m(r))}. \end{aligned}$$

We can integrate these equation to $[m(r), p(r)]$. Recall that $m(0) = 0$, and specify $p(0) = \rho_C$. We integrate this outwards from $r = 0$.

As we integrate outwards, we find a value of R such that $p(R) = 0$: this is the radius of the star, hence $R = R(\rho_C)$. Recall that

$$M = 4\pi \int_0^R \rho(r) r^2 \, \mathrm{d}r \implies \Phi(R) = \frac{1}{2} \log\left(1 - \frac{2M}{R}\right).$$

Finally, we can integrate this equation of R inwards, using

$$\Phi'(r) = \frac{m(r) + 4\pi r^3 p(r)}{r(r - 2m(r))}.$$

1.4 Maximum Mass of a Cold Star

The maximum ought to depend on the equation of state. If ρ gets close to nuclear density, then that is bad. Remarkably, GR knows best, and shows the existence of such a bound.

Since $\rho(r)$ is a monotonically decreasing function of r, we can define a region $0 < r < r_0$, where $p(\rho)$ is not known. From this,

$$m'(r) = 4\pi r^2 \rho(r).$$

Assuming the right-hand side is constant,

$$m_0 \geq \frac{4}{3}\pi r_0^3 \rho_0.$$

But we also know that

$$\frac{m_0}{r_0} < \frac{2}{9}\left[1 - 6\pi r_0^2 p_0 + (1 + 6\pi r_0^2 p)^{1/2}\right],$$

evaluated at $r = r_0$. The right hand side is decreasing in p_0, so we can set $p_0 = 0$ to find

$$\frac{m_0}{r_0} < \frac{4}{9}.$$

This gives us both an upper and lower bound for m_0/r_0, giving

$$m_0 < m_0^* = \sqrt{\frac{16}{244\pi\rho_0}} \leq 5M_0.$$

We use atomic nuclei density, $\rho_0 \approx 5 \times 10^{14}\,\mathrm{g\,cm^{-3}}$.

This just gives a bound on the "core" mass m_0. We can do the numerics to find that $M < 5M_0$.

If we impose that $C_S < c$, then $M \leq 3M_0$. More assumptions give better constraints. For instance, if the core is an ideal Fermi gas, then $M \leq 1.4M_0$, the Chandrasekhar limit.

Recall our stars form a one-parameter family of solutions parametrized by M. We will see that M is the energy of the black hole, and we take $M > 0$. If $M < 0$, we see pathologies.

There is a special radius, the *Schwarzschild radius* given by $r = 2M$.

We derived the Schwarzschild solution assuming that it was static and spherically symmetric. But we actually only need one of these assumptions:

Theorem 1.1 (Birkhoff)**.** *Any spherically symmetric solution of the vacuum Einstein equation is isometric to the Schwarzschild solution.*

The proof is not long, but not enlightening either. See Hawking for a proof.

Remark.

- The theorem only assumes spherical symmetry, and then we are given $K = \partial/\partial t$.
- This Killing field K is timelike for $r > 2M$, and spacelike for $r < 2M$.

1.5 Gravitational Redshift

Consider two observers Alice (A), and Bob (B). The remain at fixed θ, ϕ in the Schwarzschild geometry, but with different radial positions $2M < r_a < r_B$.

Suppose that Alice carries a flashlight, which she turns on and off at intervals Δt in Shcwarzschild coordinate. Since $\partial/\partial t$ is a Killing vector field, each photon will follow the same trajectory, but there will be a delay.

From the perspective of the proper time τ between photons by Alice (or Bob) and measured by Bob (or Alice), we have

$$\Delta\tau_A = \sqrt{1 - \frac{2M}{r_a}}\Delta t,$$

and thus

$$\frac{\Delta\tau_B}{\Delta\tau_A} > 1,$$

for $r_B > r_A$. These two photons can be used as a proxy for two successive wavecrests of a light wave. Hence $\lambda_B > \lambda_A$, the wavelength of a light wave. So light undergoes a redshift when it climbs out of a gravitational field. Write

$$1 + z = \frac{\lambda_B}{\lambda_A} = \left(1 - \frac{2M}{r_a}\right)^{-1/2},$$

which is the limit as $r_B \to \infty$. If Alice goes to $2M$, then $z \to +\infty$. However, if $R = 9M/4$, the Buchdal limit, then $z = 2$.

1.6 Geodesics of the Schwarzschild Geometry

Take an affinely parametrized geodesic with tangent $U^a = \mathrm{d}X^a/\,\mathrm{d}\tau$, and a spacetime $(\mathcal{M}, g)$ with Killing vector field K. Then $K \cdot U$ is conserved along the integral curves of U.

Indeed, since K is a Killing vector field and U is affinely parametrized,

$$\nabla K_b + \nabla_b K_a = 0, \qquad U^a \nabla_b U_b = 0.$$

Then we can check that $U^c \nabla_c (U^a K_a) = 0$.

For the Schwarzschild geometry, take $K = \partial/\partial t$, then

$$E = -K \cdot U = \left(1 - \frac{2M}{r}\right) \frac{\mathrm{d}t}{\mathrm{d}\tau},$$

and we can also take

$$h = m \cdot U = r^2 \sin^2 \theta \frac{\mathrm{d}\phi}{\mathrm{d}\tau},$$

where $m = \partial/\partial\phi$. For a timelike geodesic, E and h are interpreted as the energy and the angular momentum per unit mass, and τ is the proper time.

For null geodesics, τ is an affine parameter. So E and h have no absolute meaning for null geodesics. However, $h^2/E^2 = b^2$ is invariant, where b is the impact parameter of the trajectory.

The action for a geodesic is

$$\begin{aligned} S &= \int \mathrm{d}\tau \, L = \int \mathrm{d}\tau \, \dot{x}^a \dot{x}^b g_{ab} \\ &= \int \mathrm{d}\tau \, (g_{tt}\dot{t}^2 + g_{rr}\dot{r}^2 + \dot{\theta}^2 r^2 + r^2 \sin^2 \theta \dot{\phi}^2). \end{aligned}$$

Use the principle of least action to derive the Euler-Lagrange equations. We will only need one:

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = r^4 \ddot{\theta} + 2t^3 \dot{r}\dot{\theta} - h^2 \frac{\cos\theta}{\sin^3\theta}.$$

Because of $\mathsf{O}(3)$, we can always choose axes so that $\theta(0) = \pi/2$, $\dot{\theta}(0) = 0$.

From this, we see that $\ddot{\theta}(0) = 0$, so $\theta(\tau) = \pi/2$. Recall that from GR,

$$\dot{x}^a \dot{x}^b g_{ab} = -\sigma,$$

for $\sigma = 1, 0, -1$ depending on whether the geodesic is timelike, null or spacelike. Evaluating this at $\theta = \pi/2$,

$$\frac{\dot{r}^2}{2} + V(r) = \frac{E^2}{2},$$

where

$$V(r) = \frac{1}{2}\left(\sigma + \frac{h^2}{r^2}\right)\left(1 - \frac{2M}{r}\right).$$

1.7 Eddington-Finkelstein Coordinates

Consider the Schwarzschild metric with $r > 2M$, and the radial, null geodesics; $\dot{\theta} = \dot{\phi} = 0$, so $h = 0$, and choose a time parameter so that $E = 1$. With a bit of work, we find

$$\dot{t} = \left(1 - \frac{2M}{r}\right)^{-1}, \qquad \dot{r} = \pm 1.$$

There are two possible signs: if $\dot{r}/\dot{t} > 0$, we have outgoing null geodesics, otherwise these are ingoing.

Note that for $\dot{r} = -1$, we will reach $r = 2M$ in a finite affine parameter, and

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \frac{\dot{t}}{\dot{r}} = \pm\left(1 - \frac{2M}{r}\right)^{-1}.$$

We introduce the *tortoise coordinate* r_* as

$$\mathrm{d}r_* = \frac{\mathrm{d}r}{1 - \frac{2M}{r}} \implies r_* = r + 2M \log\left|\frac{r}{2M} - 1\right|.$$

At large r, $r_* \sim r$, but as $r \to 2M^+$, $r_* \to -\infty$. A radial null geodesic has

$$\frac{\mathrm{d}t}{\mathrm{d}r_*} = \pm 1 \implies t = \pm r_* + \tilde{c},$$

some constant.

The previous considerations suggest that we define

$$v = t + r_*,$$

so that v is constant along ingoing null radial geodesics. Moreover

$$\mathrm{d}t = \mathrm{d}v - \frac{\mathrm{d}r}{1 - \frac{2M}{r}}.$$

Putting this into the metric,

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + r^2\,\mathrm{d}\Omega_2^2.$$

This is amazing, as it is non-singular at $r = 2M$.

This metric is all good, until $r = 0$. At $r = 0$, you are stuck. We can compute

$$R^{abcd}R_{abcd} = \frac{48M^2}{r^6}.$$

Note that at $r = 0$, the spacetime is not well-defined. $r = 0$ does not belong to the spacetime.

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