# MATH 323 - Tutorial 7 Solutions

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February 15, 2022

1. Waiting for the Bus. Suppose $Y$ is a random variable for the time it takes for the 80 bus on Parc to reach Milton Street at the top of the hour. Suppose the random variable $Y$ follows the following Probability Density Function (pdf):

$$
\begin{aligned}
f_Y(y) &= cy, \quad 0 \leq `y \leq 10, \\
&= 0, \quad \text{Otherwise}
\end{aligned}
$$

where $c$ is some constant.

a) If the above function is a proper pdf, what must the value of $c$ be?

**Solution:**

Here we exploit the fact that pdfs must intergrate to 1.

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty} f_Y(y)dy \\
&= \int_0^{10} cydy \\
&= c\frac{y^2}{2}|_0^{10} \\
&= 50c
\end{aligned}
$$

Therefore $c = \frac{1}{50}$
b) What is the expected time for the bus to arrive?

**Solution:**

$$\begin{aligned} E[Y] &= \int_0^{10} y f_Y(y) dy \\ &= \int_0^{10} \frac{y^2}{50} \\ &= \frac{y^3}{150}|_0^{10} \\ &= \frac{1000}{15} \\ &= \frac{20}{3} \end{aligned}$$

c) Find the CDF of Y?

For $y^\star \in (0, 10]$

$$\begin{aligned} F_Y(y^\star) &= P(Y \leq y^\star) \\ &= \int_0^{y^\star} f_Y(y) dy \\ &= \int_0^{y^\star} \frac{y}{50} dy \\ &= \frac{y^2}{2 \times 50}|_0^{y^\star} \\ &= \frac{y^{\star 2}}{100} \end{aligned}$$

thus:

$$F_Y(y) = \begin{cases} \frac{y^2}{100}, & y \in (0, 10] \\ 1, y > 10 \\ 0, y \leq 0 \end{cases}$$

Remember it is really important to specify all of the intervals for CDFs!!!

d) If you are waiting for the 80 at Milton at the top of the hour, what is the probability that you are waiting between 1 and 3 minutes.

See the bottom of part 3, similar calculation.

e) Suppose you have already been waiting 5 minutes, what is the probability that the 80 arrives in the next minute?

$$
\begin{aligned}
P(Y \leq 6 | Y \geq 5) &= \frac{P(\{Y \leq 6\} \cap \{Y \geq 5\})}{P(Y \geq 5)} \\
&= \frac{P(5 \leq Y \leq 6)}{P(Y \geq 5)}
\end{aligned}
$$

Now we will work towards the numerator and denominator separately.

$$
\begin{aligned}
P(Y \geq 5) &= 1 - P(Y < 5) \\
&= 1 - P(Y \leq 5) \quad \text{[Y is continuous]} \\
&= 1 - F_Y(5) \\
&= 1 - \frac{5^2}{100} \quad \text{[From part c]} \\
&= 1 - \frac{25}{100} \\
&= \frac{3}{4}
\end{aligned}
$$

$$
\begin{aligned}
P(5 \leq Y \leq 6) &= P(\{Y \leq 6\} \setminus \{Y < 5\}) \\
&= P(Y \leq 6) - P(\{Y < 5\} \cap \{Y \leq 6\}) \\
&= P(Y \leq 6) - P(Y < 5) \quad \text{[subset]} \\
&= P(Y \leq 6) - P(Y \leq 5) \quad \text{[Y is continuous]} \\
&= F_Y(6) - F_y(5) \\
&= \frac{36}{100} - 25100 \quad \text{[ part c]} \\
&= \frac{11}{100}
\end{aligned}
$$

2. Hydro-Quebec bills are a function of power consumption ($\epsilon$) and the price of Power ($X$). Let $Z$ be a random variable representing the power bill, and specifically let $Z = \epsilon X$. For now, let us assume that you consume a constant amount of energy per month, call this constant $\epsilon$. Suppose the price of electricity (in $c/kwh$) is a random variable with an exponential distribution and a mean of 5 c/kwh.

a) What is the probability that the bill will be more than \$100. Your answer should be an expression in terms of the constant $\epsilon$.

First notice that \$100 is $10,000$ cents.

$$
\begin{aligned}
P(Z \geq 10000) &= P(\epsilon X \geq 10000) \\
&= P(X \geq \frac{10000}{\epsilon}) \quad \text{since } \epsilon > 0 \\
&= 1 - P(X < \frac{10000}{\epsilon}) \\
&= 1 - P(X \leq \frac{10000}{\epsilon}) \quad \text{[X is continuous]} \\
&= 1 - (1 - \exp(\frac{-1}{5}(\frac{10000}{\epsilon}))) \quad \text{[Using the CDF of an exponential]} \\
&= \exp(\frac{-2000}{\epsilon})
\end{aligned}
$$

In general if $X \sim exponential(\beta)$, the $F_X(x) = 1 - \exp(\frac{-x}{\beta})$

b) Suppose you know the bill will be more than \$100, what is the probability that it will be more than \$200.

$$
\begin{aligned}
P(Z \geq 20000 | Z \geq 10000) &= \frac{P(Z \geq 20000)}{P(Z \geq 10000)} \quad \text{[Subset]} \\
&= \frac{1 - P(Z \geq 20000)}{1 - P(Z \geq 1000)} \quad \text{[ Z is continuous ]} \\
&= \frac{1 - (1 - \exp(\frac{-20000}{5\epsilon}))}{\exp(\frac{-4000}{\epsilon})} \quad \text{using X is exponential and part a} \\
&= \frac{\exp(\frac{-4000}{\epsilon})}{\exp(\frac{-2000}{\epsilon})} \\
&= \exp(\frac{-2000}{\epsilon})
\end{aligned}
$$

Notice that this is the same answer as in part a. This is because the exponential distribution enjoys the memoryless property, just like the geometric for discrete random variables.

**Email me or come to office hourse to discuss question 3**

3. Suppose that there are two mutually exclusive causes of machine failure, A and B, which occur with probabilities 0.3 and 0.7 respectively. If the cause of faiure is A, the repair-time random variable, $T$, has a Normal distribution with parameters $\mu_A$ and $\sigma = 45$, whereas if the cause is B, the $T$ has a normal distribution with parameters $\mu_B = 4$ and the same $\sigma = 45$. Writing your answer in terms of the standard normal CDF, what is the probabiltiy that:

a) a failure from cause A takes longer than 3 hours to prepare?

b) a machine which fails will take longer than 3 hours to repair?

c) There was a Type A failure, given that the repair time is longer than 3 hours?