MATH 323 - Tutorial 2 Solutions

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1. a) A lottery ticket is comprised of 6 unique numbers from the set $\{1, 2, \ldots 44\}$. Suppose that the winning ticket is drawn randomly without replacement. If you buy a single ticket, what is the probability that it is the winning one.

b) Now suppose that the lottery is ordered. It is not enough to get the right numbers, but must get the correct ordering. Find the probability of a single ticket being the winning one.

c) In addition to being ordered, the now suppose that the lottery is drawn with replacement. Find the probability of a single ticket being the winning one.

Solutions

a)

$$P(Winning) = \frac{\#\text{ ways to win}}{\#\text{ ways to draw a ticket}} = \frac{1}{|\Omega|} = \frac{1}{\binom{44}{6}} = \frac{6!38!}{44!} \tag{1}$$

As we discussed in the tutorial, the fact that this was a sampling with replacement and we were looking for an unordered outcome, we actually could have equivalently gotten the answer using a sample space defined by the pick operator. In that case we would say that Ω is the set of all the ways to get 6 ordered number. The size of omega would then be $|\Omega| = P^4_6 4 = \frac{44!}{38!}$. But now the number of ways to win in this sample space would be much bigger. Here the even winning would be any combination that gets us the right 6 numbers. How many such outcomes are there? Well the number of ways we can permute 6 objects, 6!.

In that case we would get the following:

$$\begin{aligned} P(Winning) &= \frac{\#\text{ ways to win}}{\#\text{ ways to draw a ticket}} \\ &= \frac{6!}{\frac{44!}{38!}} \\ &= \frac{6!38!}{44!} \end{aligned}$$

Which gives us the same answer as above.

b)

$$P(Winning)\frac{\text{\# ways to win}}{\text{\# ways to draw a ticket}} = \frac{1}{|\Omega|} = \frac{1}{\frac{44!}{38!}} = \frac{38!}{44!} \tag{2}$$

c)

$$P(Winning)\frac{\text{\# ways to win}}{\text{\# ways to draw a ticket}} = \frac{1}{|\Omega|} = \frac{1}{44 \times 44 \times 44 \times 44 \times 44 \times 44} = \frac{1}{44^6} \tag{3}$$

You will notice that this question covered 3 variations: Unordered without replacement, ordered without replacement, ordered with replacement. There is a fourth option that I did not ask, unordered with replacement. It turns out this is a much more difficult question. For those interested, the solution in this casse to the probability of winning on a single ticket (44 numbers total, 6 drawn) unordered with replacement would be $\frac{1}{\binom{44+(6-1)}{44}}$. For those intested in learning more, this is sometimes called the Stars and Bars problem popularized by William Feller in a famous probability text. It turns out this problem is equivalent to a few interesting problems which may be of interest to a few of you particularly the discrete mathematicians and computer scientists amoung us. But don't worry, this kind of a counting problem is beyond the scope of this course and you are not expected to know it. See `https://cp-algorithms.com/combinatorics/stars_and_bars.html` for more information.

2. The letters in the word lollipop are randomly rearranged. What is the probability that after the rearrangement, it still spells lollipop?

Solution

As with many counting problems, there are many ways to approach this question. I will show two. The first is the pure counting approach.

solution i) $\Omega :=$ { All the unique ways to rearrange the letters in lollipop}. Let A be the event that after we rearrange the letters that it still spells lollipop.

$$\begin{aligned}
|\Omega| &= \text{\# of ways to rearrange letters in lollipop uniquely} && (4)\\
&= \frac{8!}{3!2!2!1!1!} && (5)\\
P(A) &= \frac{\text{\# of ways to uniquely spell lollipop}}{|\Omega|} && (6)\\
&= \frac{1}{\frac{8!}{3!2!2!1!1!}} && (7)\\
&= \frac{3!2!2!}{8!} && (8)
\end{aligned}$$

There are a few ways to think about this formula. In your textbook, they call $\binom{8}{3,2,2,1} = \frac{8!}{3!2!2!1!}$ the multinomial coefficients. In tutorial we said that we can think about this in the following way. We start with the number of ways to shuffle (permute) 8 letters, 8!, on the numerator. Then we will divide out all of the redundent permutations that do not spell a different word. There are 3! ways to shuffle the L's around, and shuffling them around will not change the word we spell so we divide them out. In the same way, there are 2! ways to shuffle around the O's and the P's. This dividing out idea follows directly from the sausage rule (the sausage rule gives us a way to multiply and this is the corresponding way to divide).

Another way to derive this form would be the following. We ask ourselves how many ways are there to get L's right after shuffling. Well there are 8 letters and 3 L's, so $\binom{8}{3}$, ways of shuffling and 1 to get it right in that sample space. Now suppose we fix the L's in there correct place. Given that, how many ways are there to get the O's right. There are 5 letters left and 2 Os, so 1 out of $\binom{5}{2}$. Then there are 1 of $\binom{3}{1}$ ways to get the I right and 1 of $\binom{2}{2}$ ways to get the P's right, fixing everything before that (In solution ii, we will be more explicit about a probabilistic way of thinking of this fixing, which is really just a conditioning). You can see that:

$$\begin{aligned}\frac{1}{\binom{8}{3}\binom{5}{2}\binom{3}{1}\binom{2}{2}} &= \frac{1}{\frac{8!}{3!5!}\frac{5!}{2!3!}\frac{3!}{1!2!}\frac{2!}{2!}} \\ &= \frac{3!2!2!1!}{8!}\end{aligned}$$

Just like before.

solution ii) This is a slightly more probabilistic approach that will become more and more useful as you progress through this course (and beyond in your statistical or probabilistic lives). This approach is sometimes called Backwards Conditioning or Conditioning Backwards and you should have seen it in your notes by now in the course.

First, define some events. Let X_i represent the event of drawing the letter X in the ith position, where X is some arbitrary letter and i is an integer such that $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$. For example L_1 represents the event that after the rearranging that the letter L is in the first position, and P_6 represents the event that the letter P is in the 6th position after the rearranging. The only way for the word lollipop to be spelled is if the event $L_1 \cap O_2 \cap L_3 \cap L_4 \cap I_5 \cap P_6 \cap O_7 \cap P_8$ occurs. Thus to answer the question we may find the probability $P(L_1 \cap O_2 \cap L_3 \cap L_4 \cap I_5 \cap P_6 \cap O_7 \cap P_8)$

$$
\begin{aligned}
P(L_1 \cap O_2 \cap L_3 \cap L_4 \cap I_5 \cap P_6 \cap O_7 \cap P_8) = &P(L_1)P(O_2|L_1)P(L_3|L_1 \cap O_2)P(L_4|L_1 \cap O_2 \cap L_3)\times \\
&P(I_5|L_1 \cap O_2 \cap L_3 \cap L_4)P(P_6|L_1 \cap O_2 \cap L_3 \cap L_4 \cap I_5)\times \\
&P(O_7|L_1 \cap O_2 \cap L_3 \cap L_4 \cap I_5 \cap P_6)P(P_8|L_1 \cap O_2 \cap L_3 \cap L_4 \cap I_5 \cap P_6 \cap O_7) \\
&= \frac{3}{8}\frac{2}{7}\frac{2}{6}\frac{1}{5}\frac{1}{4}\frac{2}{3}\frac{1}{2}\frac{1}{1} \\
&= \frac{(3\ 2 \times 1)(2 \times 1)(2 \times 1)}{(8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} \\
&= \frac{3!2!2!}{8!}
\end{aligned}
$$

Notice that both methods arrive at the same answers. Counting methods are good to know, but remember that we can only use counting methods if we are able to construct a sample space where all of the outcome are equally likely. This second method, while it looks a bit cumbersome is a more general technique that will become increasingly useful as the course goes on.

3. A gym class is made up of 20 students. In the gym class there are 4 students playing badminton, 8 people playing dodgeball, 5 playing basketball, and 3 sitting out. Suppose that each person in the class has 1 preferred group and in this class exactly 4 prefer badminton, 8 prefer dodgeball, 5 prefer basketball and 3 prefer sitting out. If the teacher randomly makes the groupings, what is the probability that each student gets their preferred option?

Solution

Again, we have several approaches possible and I will present 2.

First, pure counting. Notice that this is equivalent to the problem of partitioning n distinct objects into k groups, where in this case $k = 4$ and the group sizes are $4, 8, 5, 3$ respectively. On page 44 in the text, theorem 2.3 tells us that we can use the multinomial coefficients formula to count the number of ways of doing this. Let E : "The event that everyone gets there preferred group".

$$
\begin{aligned}
P(E) &= \frac{\#\text{ ways for everyone to get preferred group}}{\#\text{ ways to rearrange 20 people into groups of size 4,8,5,and 3}} = \frac{1}{|\Omega|} &\quad (9)\\
&= \frac{1}{\binom{20}{4,8,5,3}} = \frac{4!8!5!3!}{20!} &\quad (10)
\end{aligned}
$$

The backwards conditioning approach on the other hand. We can equivalently write $E = Bdm \cap Dodge \cap Bask \cap Sit$, where Bdm is understood to mean that all the people who prefer badminton get to be in the badminton group. In order for everyone to get the group they want each group must be filled with people preferring that group and vice versa since the preferences were unique.

$$
\begin{aligned}
P(E) &= P(Bdm \cap Dodge \cap Bask \cap Sit) && (11)\\
&= P(Bdm)P(Dodge|Bdm)P(Bask|Bdm, Dodge)P(Sit|Bask, Dodge, Bdm) && (12)\\
&= \frac{1}{\binom{20}{4}}\frac{1}{\binom{16}{8}}\frac{1}{\binom{8}{5}}\frac{1}{\binom{3}{3}} && (13)\\
&= \frac{4!16!}{20!}\frac{8!8!}{16!}\frac{5!3!}{8!}\frac{3!}{3!} && (14)\\
&= \frac{4!8!5!3!}{20!} && (15)
\end{aligned}
$$

We get the exact same thing. Convince yourself that the order of the backwards conditioning doesn't matter. [If you are careful about defining your events notice that you could also use the sausage rule to get a result exactly like the conditioning backwards approach. The reason I prefer to write this out probabilistically however is two-fold. The conditioning makes the counting problem more clear and as mentioned before this approach is useful later on in the course even once we have mostly left counting spaces behind.]

Notice that although I did not show it we could certainly use the dividing out method as well for this problem!

4. A committee of $n = 5$ students is to be selected, supposedly at random from a class of $N = 200$. The class is made up of 120 science students and 80 arts students.

a) What is the probability if the selection was random that all 5 members of the committee are sciences students?

b) What is the probability that at least 3 of the 5 members of the committee are science students?

Solutions

Notice that this problem is much like the fish tagging problem and as such we can use the hypergeometric formula (or equivalently the sausage rule or backwards conditioning to get the same thing). I will just show the hypergeometric formulation.

Let S_i : "The event that exactly i of the committee members are from the Faculty of Science."

a)

$$
\begin{aligned}
P(S_5) &= \frac{\binom{120}{5}\binom{80}{0}}{\binom{200}{5}} && (16)\\
&= \frac{\binom{120}{5}}{\binom{200}{5}} && (17)
\end{aligned}
$$

An alternative way to get this answer would be do condition backwards. Convince yourself this is true.

b) Let X be the variable which counts the number science students on the committee

$$
\begin{aligned}
P(X \geq 3) &= P(S_3 \cup S_4 \cup S_5) && (18)\\
&= P(S_3) + P(S_4) + P(S_5) \quad [\text{ Disjoint Sets}] && (19)\\
&= \frac{\binom{120}{3}\binom{80}{2}}{\binom{200}{5}} + \frac{\binom{120}{4}\binom{80}{1}}{\binom{200}{5}} + \frac{\binom{120}{5}\binom{80}{0}}{\binom{200}{5}} && (20)\\
&= \sum_{r=3}^{5} \frac{\binom{120}{r}\binom{80}{5-r}}{\binom{200}{5}} && (21)
\end{aligned}
$$

Similarly, it is possible to condition backwards for each of the probabilities above. Convince yourself that this would not be as easy or clean for say $P(S_3)$ compared to $P(S_5)$. Why is that the case? So although conditioning backwards is generally very useful, it will not always be the easiest way to attack a problem and it pays to have some basic grasp of counting methods.

5. Challenge Question: 2 players are playing cards from a standard deck of 52 (13 hearts, 13 spades, 13 diamonds, 13 clubs). Each player is randomly dealt exactly 13 cards. What is the probability that the first player is dealt exactly n_1 hearts ($n_1 \in \{n : n \in \mathcal{N}, 0 \leq n \leq 13\}$) and the second player is dealt exactly n_2 hearts ($n_2 \in \{n : n \in \mathcal{N}, 0 \leq n \leq 13\}$), where $n_1 + n_2 \leq 13$

Solution

I will just show the conditioning backwards approach since I believe it to be the easiest way to attack this problem.

Let H_i be the variable representing the number of hearts player i is dealt, where $i \in \{1, 2\}$.

$$
\begin{aligned}
P(\{H_1 = n_1\} \cap \{H_2 = n_2\}) &= P(\{H_1 = n_1\})P(H_2 = n_2 | H_1 = n_1) && (22)\\
&= \frac{\binom{13}{n_1}\binom{39}{13-n_1}}{\binom{52}{13}} \frac{\binom{13-n_1}{n_2}\binom{39-(13-n_1)}{13-n_2}}{\binom{39}{13}} && (23)\\
&= \frac{\binom{13}{n_1}\binom{39}{13-n_1}}{\binom{52}{13}} \frac{\binom{13-n_1}{n_2}\binom{26+n_1}{13-n_2}}{\binom{39}{13}} && (24)
\end{aligned}
$$

Notice that once we condition backwards we just have 2 hypergeometric problems. In the first one, there are 52 cards in total. 13 are hearts and 39 are not. In the second part, given the first player got exactly n_1 hearts and 13 cards overall there are only 39 cards left. $13 - n_1$ of those remaining 39 cards are hearts and $39 - (13 - n_1)$ are non-hearts.

This is a harder question that the others conceptually, but notice that the trick is not that the counting problem is hard, it is clearly defining the event of interest and then choosing the approach that is hard. Once you know what the correct probability is you are looking for and that a good strategy is conditioning backwards the individual steps are not much more difficult than in the previous question.

A trick for knowing how to condition backwards to simplify a problem is to ask yourself what do you which you knew? In this case counting both hand combinations simultaneously is super hard, but figuring out one hand is not so bad. And if you knew the one hand, the other wouldn't be so difficult either. Conditioning can be though of as "knowing" or "taking as given". Building some intuition around this will pay off in spades (pun very much intended) as you move forward in this course.