MATH 323 - Tutorial 3 Partial Solutions

Tyrel Stokes

September 2020

1. Two six-sided fair die are tossed. Assume that the die are independent of each other.

a) If at least one 3 was rolled, what is the probability that both were rolled 3's?
b) Suppose this time that the die sum to 7. What is the probability that a 3 was rolled?
c) If it is known that at least one of the dice is 3, what is the probability that their faces sum to 7?

Solutions
$\Omega = \{(x, y) : x, y \in \{1, 2, ..., 6\}\} = \{x : x \in \{1, 2, .., 6\}\} \times \{y : y \in \{1, 2, ..., 6\}\}$
$\Omega = \{(1, 1), (1, 2), \ldots, (1, 6), (2, 1), (2, 2), \ldots, (2, 6), \ldots, (6, 1), (6, 2), \ldots, (6, 6)\}$

Let X denote roll for the first dice, and Y denote the roll of the second dice.

a)

$$P(\{X = 3\} \cap \{Y = 3\} | \{X = 3\} \cup \{Y = 3\}) \tag{1}$$
$$= \frac{P((\{X = 3\} \cap \{Y = 3\}) \cap (\{X = 3\} \cup \{Y = 3\})}{P(\{X = 3\} \cup \{Y = 3\})} \tag{2}$$
$$= \frac{P(\{X = 3\} \cap \{Y = 3\})}{P(\{X = 3\} \cup \{Y = 3\})} \quad \text{[Since } (\{X = 3\} \cap \{Y = 3\}) \subseteq (\{X = 3\} \cup \{Y = 3\})\text{]} \tag{3}$$
$$\frac{P(X = 3)P(Y = 3)}{P(\{X = 3\} \cup \{Y = 3\})} \quad \text{[Independence]} \tag{4}$$
$$= \frac{\frac{1}{6}\frac{1}{6}}{\frac{11}{36}} \tag{5}$$
$$= \frac{1}{11} \tag{6}$$

There are a few ways to get $P(\{X = 3\} \cup \{Y = 3\})$. One would be to appeal to the theorem about unions learned in class.

$$
\begin{aligned}
P(\{X=3\} \cup \{Y=3\}) &= P(X=3) + P(Y=3) - P(\{X=3\} \cap \{Y=3\}) \\
&\overset{Ind.}{=} \frac{1}{6} + \frac{1}{6} - P(X=3)P(Y=3) \\
&= \frac{2}{6} - \frac{1}{36} \\
&= \frac{11}{36}
\end{aligned}
$$

Another way would be to partition this event into 3 cases:

$$
\begin{aligned}
E_1 &= \{X = 3 \cap Y \neq 3\} \\
E_2 &= \{Y = 3 \cap X \neq 3\} \\
E_3 &= \{X = 3 \cap Y = 3\}
\end{aligned}
$$

Then we simply need to find the size of each set. $|E_1| = 5$, since the value of X is fixed, and Y could be the numbers $1, 2, 4, 5, 6$. Similarly $|E_2| = 5$. There is only one element in E_3. Thus:

$$
\begin{aligned}
P(\{X=3\} \cup \{Y=3\}) &= \frac{|(\{X=3\} \cup \{Y=3\})|}{|S|} \qquad \text{[By counting space]} \\
&= \frac{|E_1| + |E_2| + |E_3|}{|S|} \\
&= \frac{5+5+1}{36} \\
&= \frac{11}{36}
\end{aligned}
$$

And lastly one could simply count up the elements in S directly which belong to the set of interest. In general, the first way will be the most useful in this course and it is the most probabilistic approach, but there are of course many ways to attack counting problems.

b)

$$
\begin{aligned}
&P(\{X=3\} \cup \{Y=3\} | \{X+Y=7\}) && (7) \\
&= \frac{P((\{X=3\} \cup \{Y=3\}) \cap \{X+Y=7\})}{P(\{X+Y=7\})} && (8) \\
&= \frac{P(\{(3,4)\} \cup \{(4,3)\})}{P(\{X+Y=7\})} && (9) \\
&= \frac{\frac{2}{36}}{\frac{6}{36}} && (10) \\
&= \frac{1}{3} && (11)
\end{aligned}
$$

The probabilities themselves follow from the fact that we have a counting space with all outcomes equally likely.

c)

$$P(\{X+Y=7\}|\{X=3\}\cup\{Y=3\}) \overset{(Bayes)}{=} \frac{P(\{X=3\}\cup\{Y=3\}|\{X+Y=7\})P(\{X+Y=7\})}{P(\{X=3\}\cup\{Y=3\})} \tag{12}$$

$$= \frac{\frac{1}{3}\frac{6}{36}}{\frac{11}{36}} \tag{13}$$

$$= \frac{2}{11} \tag{14}$$

Where we used the probabilities already worked out in parts a and b.

d) Challenge: Suppose we keep rolling a pair of dice and sum their faces. What is the probability that the faces sum to 3 before the faces sum to 7? Assume that the game stops as soon as the dice sum to either 3 or 7, but otherwise continues indefinitely. The rolls from one round to the next are assumed independent.

Solutions:

The first thing that we want to do is to partition each round of this game into 3 events. The first is that we roll a sum of 3. The next is we roll of 7. And finally, neither of those events occur, and thus we roll again. Now we can find the probability of those 3 events.

$P(X+Y=3) = \frac{|\{(1,2),(2,1)\}|}{|\Omega|} = \frac{2}{36}$
$P(X+Y=7) = \frac{6}{36}$, where we found this in the previous steps.
$P(continue) = 1 - P(X+Y=3) - P(X+Y=7) = \frac{28}{36}$

And notice that since we are rolling the same die and the rolls are independent from round to round, these probabilities are the same in each round.

Now the crux, or most difficult part, of this problem is to find a clever partition of the event of interest. We don't care directly about the probability of rolling a sum of 3, but of rolling a sum of 3 **BEFORE** a sum of 7. What we notice though, the event that we roll a sum of 3 before a roll of 7 either happens on the first time we roll, or the second time, or the third time, or, the fourth time and so on. And notice that rolling a sum of three on exactly the first time is disjoint to the event of rolling exactly on the second time and third time and so on. This is because if I roll a 3 on the first try, the game ends, I can't then roll a three for the first time in the next round.

A way to formulate this mathematically with set notation is the following. Let $E:$ event that we roll a sum of 3 before
And then further, let's say that $E_i:$ event we roll a 3 before a 7 in exactly the ith round.

There is a relationship between the two events. In fact, we can express E in terms of all the E_i's. Since in order to roll the 3 first it must happen in one of the rounds, i.e $E = \cup_{i=1}^{\infty} E_i$. And we already argued that all of the E_i's are disjoint. Therefore:

$$\begin{aligned} P(E) &= P(\cup_{i=1}^{\infty} E_i) \\ &\overset{disjoint}{=} \sum_{i=1}^{\infty} P(E_i) \end{aligned}$$

So now the goal is just to figure out the individual $P(E_i)$'s and to figure out this infinite sum. The easiest way to attack this is to start with the easiest case and see the pattern emmerge. Let X_i and Y_i denote the rolls of the first and second die respectively in the ith round.

$P(E_1) = P(X_1 + Y_1 = 3) = P(X + Y = 3) = \frac{2}{36}$

$P(E_2) = P(\{X_1 + Y_1 \neq 3, 7\} \cap \{X_2 + Y_2 = 3\}) \overset{independence}{=} P(continue)P(X + Y = 3) = \frac{28}{36}\frac{2}{36}$.

Now extrapolating, to win on exactly the ith time, the game must have continued on exactly i-1 times and then on the ith try we roll a 3. By independence we then get:

$P(E_i) = (\frac{28}{36})^{i-1}\frac{2}{36}$

Convince yourself that independence plays a crucial role in this result. In the next section of this course we will learn about a distribution called the geometric distribution which shares the essential properties with this problem, there too, independence will play a key role.

To finish then:

$$\begin{aligned} P(E) &= \sum_{i=1}^{\infty} P(E_i) \\ &= \sum_{i=1}^{\infty} (\frac{28}{36})^{i-1}\frac{2}{36} \\ &\overset{\text{geometric series}}{=} \frac{\frac{2}{36}}{1 - \frac{28}{36}} \\ &= \frac{\frac{2}{36}}{\frac{8}{36}} \\ &= \frac{1}{4} \end{aligned}$$

Notice that this can be re-expressed as $\frac{1}{4} = \frac{P(X+Y=3)}{P(X+Y=3)+P(X+Y=7)}$.

2. Workers and Shirkers:

A student is working on a group assignment in a class with 12 other students. Suppose that 2 of the other students are known to be shirkers (that is they neglect their responsibilities), but that the remaining 10 are known hard workers (called workers throughout). The teacher tells the student to reach into a hat containing the names of the twelve other students and randomly select 2 names which will make up the students work group.

- Suppose that if no shirkers are selected, the probability of getting an A on the assignment is 0.95.
- If 1 shirker is selected, the probability of getting an A is 0.6.
- If 2 shirkers are selected, the probability of getting an A is 0.1.

What is the probability that:

a) The student does not get an A on the assignment?
b) Given that the student didn't get an A, the probability that precisely 1 shirker was selected?
c) Given that the student didn't get an A, the probability that at least 1 shirker was selected?

Solutions

The first step is to write down the information given to us in the problem in terms of valid probability statements. Let A be the event that the group gets an A. Let $S_i, i \in \{0, 1, , 2\}$ be the events that 0,1, or 2 shirkers be placed into the group respectively.

- Suppose that **if** no shirkers are selected, the probability of getting an A on the assignment is 0.95. $P(A|S_0) = 0.95, \implies P(A^c|S_0) = 1 - P(A|S_0) = 1 - 0.95 = 0.05$
- **If** 1 shirker is selected, the probability of getting an A is 0.6. $\implies P(A|S_1) = 0.6. \implies P(A^c|S_1) = 1 - P(A|S_1) = 0.4$
- **If** 2 shirkers are selected, the probability of getting an A is 0.1. $\implies P(A|S_2) = 0.1, P(A^c|S_2) = 1 - P(A|S_2) = 0.9$

The word if was bolded because this was one of the main clues telling us that the probabilities being described were conditioning. Another common word you will see is given, or supposing this is true to denote conditional probabilities. Of course this list is not exhaustive.

a)

$$P(A^c) \overset{(TLP)}{=} P(A^c|S_0)P(S_0) + P(A^c|S_1)P(S_1) + P(A^c|S_2)P(S_2) \tag{15}$$

We are able to apply the Total Law of Probability since S_0, S_1, S_2 form a partition (i.e, they are disjoint events whose union has probability 1). We were given the conditional probabilities, but not the marginal probabilities $P(S_0), P(S_1)$, and $P(S_2)$. Since they form a partition, we only need to figure out two of them to get the third.

We can find these probabilities either using a conditioning backward approach or a counting argument. The counting argument uses the hyperbolic formula (See Fish tagging example in your notes). I will show both for $P(S_1)$ since it is possibly the trickiest and otherwise just give one solution.

We can break the total number of students into 10 workers, 2 shirks.

$$\begin{aligned} P(S_1) &= \frac{\text{\#ways to select 1 shirker (and implicitly 1 worker)}}{\text{\#ways to select 2 students}} && (16) \\ &\overset{\text{sausage rule}}{=} \frac{|\text{\#ways to select 1 shirker}| \times |\text{\#number of ways to select 1 worker}|}{|\Omega|} && (17) \\ &= \frac{\binom{2}{1}\binom{10}{1}}{\binom{12}{2}} && (18) \\ &= \frac{40}{132} && (19) \end{aligned}$$

Now conditioning backwards. Let's define some notation. Let W_1 be the event that we pull a worker out of the hat on the first draw and W_2 be the event that we pull a worker out of the hat on the second draw. In order to pull out exactly 1 shirker, we need to either only draw a worker on the first draw or only draw a worker on the second draw.

$$\begin{aligned} P(S_1) &= P(\{W_1 \cap W_2^c\} \cup \{W_1^c \cap W_2\}) && (20) \\ &= P(W_1 \cap W_2^c) + P(W_1^c \cap W_2) \quad \text{[Disjoint]} && (21) \\ &= P(W_1)P(W_2^c|W_1) + P(W_1^c)P(W_2|W_1^c) && (22) \\ &= \frac{10}{12}\frac{2}{11} + \frac{2}{12}\frac{10}{11} && (23) \\ &= \frac{20}{132} + \frac{20}{132} && (24) \\ &= \frac{40}{132} && (25) \end{aligned}$$

You should now be able to prove yourself that:

$$\begin{aligned} P(S_0) &= \frac{90}{132} && (26) \\ P(S_2) &= \frac{2}{132} && (27) \end{aligned}$$

As an aside, convince yourself that the easiest path to figuring out these 3 probabilities would be to solve explicitly for $P(S_0)$ and $P(S_2)$ and then let $P(S_1) = 1 - P(S_0) - P(S_2)$.

Putting this all together.

$$P(A^c) \overset{(TLP)}{=} P(A^c|S_0)P(S_0) + P(A^c|S_1)P(S_1) + P(A^c|S_2)P(S_2) \tag{28}$$
$$= (0.05)(\frac{90}{132}) + (0.4)(\frac{40}{132}) + (0.9)(\frac{2}{132}) \tag{29}$$
$$\approx 0.1689 \tag{30}$$

b)

$$P(S_1|A^c) \overset{\text{(Bayes)}}{=} \frac{P(A^c|S_1)P(S_1)}{P(A^c)} \tag{31}$$
$$= \frac{(0.4)(\frac{40}{132})}{(0.05)(\frac{90}{132}) + (0.4)(\frac{40}{132}) + (0.9)(\frac{2}{132})} \tag{32}$$
$$\approx 0.7175 \tag{33}$$

Where the probabilities we used were derived in part a.

[NB: When faced with an unknown conditional probability like in part b, we often have 2 main strategies. First, we can apply the definition of a conditional probability (i.e $P(A|B) = \frac{P(A\cap B)}{P(B)}$] or we can apply Bayes rule. The route which is easiest will depend on the specifics of the problem. In this case, it made sense to apply Bayes rule because we already had the 'inverse probability' given in the problem. That is we wanted $P(S_1|A^c)$ and we already knew $P(A^c|S_1)$. This is a hint that bayes rule might be a useful strategy.]

c) I will not give the solution to c, but start you off.

$$P(S_1 \cup S_2|A^c) = P(S_1|A^c) + P(S_2|A^c) \quad \text{[Disjoint]} \tag{34}$$

We found $P(S_1|A^c)$ in part b and can find $P(S_2|A^c)$ analagously.

3. Athletes are routinely tested for the use of performance-enhancing drugs (PEDs). When a testis performed, an athlete provides 2 blood samples (an A sample and a B sample). If the A sample tests positive, then the B sample is also tested. If the B sample is also positive,the athlete is considered to have failed the drug test.

Suppose an athlete is selected at random and two identical, random blood samples (A and B) are obtained. Let $T_i, i \in \{A, B\}$ be the event that the ith sample tests positive. Let C be the event that their truly are PEDs present in the samples. Suppose the test is quite accurate, in that it correctly indicates the **presence** of drugs in 99.5% of samples and corrrectly indicates the absence of drugs in 98% of samples. Suppose that only 1 athlete per 1000 is actually taking PEDs which would show up in the samples.

a) What is the probability that the first test is positive?

Solutions:

From the question, we are given several conditional probabilities. The first step of these kinds of probabilities is to identify the probability of interest and take stock of the givens in the word problem. In this case $P(T_A)$ is the probability of interest. We are given that $P(T_A|C) = 0.995$ (correctly identifies positive cases) and that $P(T_A^c|C^c) = 0.98$ (correctly identifies negative cases). Finally, we are given that overall, $P(C) = \frac{1}{1000}$.

Notice that $C \cup C^c$ form a partition, thus we can use the law of total probability.

$$\begin{aligned}
P(T_A) &\overset{TLP}{=} P(T_A|C)P(C) + P(T_A|C^c)P(C^c) \\
&= P(T_A|C)P(C) + (1 - P(T_A^c|C^c))(1 - P(C)) \\
&= .995 * .001 + 0.02 * .999 \\
&= 0.021
\end{aligned}$$

b) What is the probability that drugs are actually present in the sample if the A sample tests positive?

Solutions:

Here the probability of interest is $P(C|T_A)$. When faced with a conditional probability like this, we typically have two options. Bayes rule or use the direct definition. Which is easier will depend (usually) on whether it is easier to find (or access if it is given) the so-call "inverse probability" (in this case $P(C|T_A) \to P(T_A|C)$) or the intersection (in this case $P(C \cap T_A)$). In this case, due to the information given before, bayes rule will be easier.

$$
\begin{aligned}
P(C|T_A) &\overset{Bayes}{=} \frac{P(T_A|C)P(C)}{P(T_A)} \\
&= \frac{.995 * .001}{0.021} \\
&= 0.047
\end{aligned}
$$

Where we used the law of total probality to find the denominator in part a.

Think about what this answer means! Even though the test is very accurate in a sense, it still doesn't give us much information about whether or not the athlete is actually on banned substances. In this case, this is largely die to the fact that $P(C)$ is so small. If the rate of banned substance use was much higher, the information from a positive test would be much greater.

c) What is the the probability that both tests are positive? Suppose that the tests are independent.

Solutions:

$$
\begin{aligned}
P(T_A \cap T_B) &= P(T_A)P(T_B) \quad [\text{ Independence }] \\
&= P(T_A)^2 \quad [\text{ Identical }] \\
&= (0.21)^2 \quad [\text{ From part a) }] \\
&\approx 0.000441
\end{aligned}
$$

Challenge:

d) Suppose instead that the tests are not independent, but independent conditional on C. (This hasn't been covered in class so do not worry if you have not heard of this concept. For those looking to go further in statistics, this is a concept that will come up but again not something you need to know for this class.). For the purposes of this question, all you need to know is that conditional independence here implies that $P(T_A \cap T_B|C) = P(T_A|C)P(T_B|C)$. What is the probability under the conditional independence assumption that both tests are positive? (Think about how this compares to the case that the tests are fully independent and why)

Solutions:

$$
\begin{aligned}
P(T_A \cap T_B) &\overset{TLP}{=} P(T_A \cap T_B|C)P(C) + P(T_A \cap T_B|C^c)P(C^c) \\
&\overset{\text{cond. ind.}}{=} P(T_A|C)P(T_B|C)P(C) + P(T_A|C^c)P(T_B|C^c)P(C^C) \\
&\overset{ident.}{=} (P(T_A|C))^2 P(C) + (P(T_A|C^c))^2(1 - P(C)) \\
&= (.995)^2 \times (0.001) + (1 - .98)^2(1 - .001) \\
&\approx 0.0014
\end{aligned}
$$

Notice that compared to when we assumed independence that the probability of $P(T_A \cap T_B)$ is higher (i.e $0.014 > 0.000441$)

e) What is the probability that drugs are actually present in the sample if both the A and B samples test positive?

Solutions:

$$
\begin{aligned}
P(C|T_A \cap T_B) &\overset{Bayes}{=} \frac{P(T_A\ capT_B|C)P(C)}{P(T_A \cap T_B)} \\
&\overset{cond.ind.}{=} \frac{P(T_A|C)P(T_B|C)P(C)}{P(T_A \cap T_B)} \\
&= \frac{(0.995)^2(0.001)}{0.014} \\
&\approx 0.712
\end{aligned}
$$