The Nefarious Physicist:
(A fair coin, but not counting space)

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This is a small document that might give you a way to think about Question 6 from assignment 1. I give an example of how one might be able to construct a coin that is fair, but which does not have an equiprobable outcome space. You can think of this as a set of abstract instructions that you could give someone to build such a coin or coin flipping machine. I call it the nefarious physicist because I can only imagine a physicist trying to build on these machines. There is a small history of physicists or probabilists with physics backgrounds trying to do all sorts of things like predicting whether a coin is heads or tails and deconstructing the probabilistic process into a measured process which becomes more deterministic. No offence to the physicists amoung us, I admire many of these nefarious physicists and the insights they have brought us.

Suppose again that the experiment is flipping a coin twice and thus

$$\Omega = \{\{H, T\}, \{T, H\}, \{H, H\}, \{T, T\}\}$$

is the sample space.

Equiprobability would imply that $P(\{H, T\}) = P(\{T, H\}) = P(\{H, H\}) = P(\{T, T\}) = \frac{1}{4}$.

Like the question, we only make the assumption that the coin is fair. Explicitly, this would mean that on any given coin toss the probability that it is a head is equal to the probability of a tails and thus $\frac{1}{2}$.

To be very clear, I will define the following events:

$$\begin{aligned}
H_1 &= \{\text{The event of getting a heads on the first flip}\} = \{\{H, T\} \cup \{H, H\}\} \\
H_2 &= \{\text{The event of getting a heads on the 2nd flip}\} = \{\{T, H\} \cup \{H, H\}\} \\
T_1 &= \{\text{The event of getting a tails on the first flip}\} = \{\{T, T\} \cup \{T, H\}\} \\
T_2 &= \{\text{The event of getting a tails on the 2nd flip}\} = \{\{H, T\} \cup \{T, T\}\}
\end{aligned}$$

A way of expressing the fair coin assumption is that $P(H_1) = P(H_2) = P(T_1) = P(T_2) = \frac{1}{2}$. We will now give a particular example where the fair coin assumption is satisfied, but the sample space is not equiprobable.

Consider the following probabilities:

$$\begin{aligned} P(H_2|H_1) &= 0.6 \quad , P(H_2|T_1) = 0.4 \\ P(T_2|T_1) &= 0.6 \quad , P(T_2|H_1) = 0.4 \end{aligned}$$

Here we impose a dependence. In this example, a heads is more likely to follow a heads and a tail more likely to follow a tails. We will further impose that $P(T_1) = P(H_1) = \frac{1}{2}$ as given. With this information, we can solve for the remaining probabilities $P(H_2)$ and $P(T_2)$ as well as the probabilities of the sample outcomes.

$$\begin{aligned} P(H_2) &\overset{TLP}{=} P(H_2|T_1)P(T_1) + P(H_2|H_1)P(H_1) \\ &= (0.4) * (0.5) + (0.6) * (0.5) \\ &= \frac{1}{2} \\ P(T_2) &= 1 - P(H_2) \\ &= \frac{1}{2} \end{aligned}$$

Thus we have satisfied the fair coin assumption. Now we will find the sample outcome probabilities.

$$\begin{aligned} P(\{H,T\}) &= P(H_1 \cap T_2) \\ &= P(T_2|H_1)P(H_1) \\ &= (0.4) * (0.5) = 0.2 \\ P(\{T,H\}) &= P(T_1 \cap H_2) \\ &= P(H_2|T_1)P(T_1) \\ &= (0.4) * (0.5) \\ &= 0.2 \\ P(\{H,H\}) &= P(H_1 \cap H_2) \\ &= P(H_2|H_1)P(H_1) \\ &= (0.6) * (0.5) \\ &= 0.3 \\ P(\{T,T\}) &= 1 - P(\Omega \setminus \{T,T\}) \\ &\overset{\text{Disjoint}}{=} 1 - P(\{H,T\}) - P(\{T,H\}) - P(\{H,H\}) \\ &= 1 - 0.2 - 0.2 - 0.3 \\ &= 0.3 \end{aligned}$$

Clearly, the sample outcomes are not equally likely despite marginally the coins being fair. Nothing about fairness implies independence. Independence is an additional, different assumption even if in practice it is often a natural one for coin flips. In the abstract however, be wary of the nefarious physicist and be explicit with the assumptions you make. In particular, try to avoid unstated assumptions like the textbook did. If you would like to assume independence (or as learned in the course, sometimes approximate independence) and feel it is reasonable to do so, state in clearly and if necessary justify the assumption. This is even more important when working with real data than it is in the context of a course where we often study idealized settings.