# MATH 323 - Tutorial 6 Questions

Tyrel Stokes

October 2020

1. Stonks!

Consider two different stocks, $X_1$ and $X_2$. Suppose that there are three different states of the world, say for example there is a state where the economy crashes ($S_b$), another where status quo is maintained ($S_m$), and the state of the world where the economy booms $S_g$. The stocks will have different values depending on the state of the world. You can assume that the value for both stocks will be lowest in $S_b$, at least as great as the bad state in $S_m$, and at least as great as that in the boom economy $S_b$. It is known that $P(S_b) = \frac{1}{2}$, $P(S_m) = \frac{1}{6}$, $P(S_g) = \frac{1}{3}$.

Suppose we know the CDF of $X_1$ is:

$$P(X_1 \leq x_1) = \begin{cases} = 0, x_1 < 0 \\ = \frac{1}{2}, 0 \leq x_1 < 4 \\ = \frac{2}{3}, 4 \leq x_1 < 7 \\ = 1, x \geq 7 \end{cases}$$

For $X_2$ we know that if the economy is bad then it will be worth 1 and it will be worth 5 if the economy either stays the same or booms.

a) This is not a real probability question, but imagine you were forced to buy one of the two stocks (without being able to predict the state of the economy, i.e only knowing the marginal probabilities above). Which would you prefer?

b) Find the PMF of $X_1$ and the CDF of $X_2$ (draw it).

c) Find the mean and variance of $X_1$ and $X_2$.

d)Now which stock would you buy? Why? Again, there is no absolute right answer.

e) Challenge Question [Return to after learning about continuous distributions, specifically learning more about the normal distribution and CDF] :Now consider just $X_2$ and following states, $A_1 = S_b$, $A_2 = S_m \cup S_g$. Suppose we have an index, $Y$ that gives us information about whether the economy will be in state $A_1$ or $A_2$ in the near future.

Let
$a* = \alpha + y + \epsilon$, where $y$ is a particular value of the index $Y$, $\alpha$ is some constant, and $\epsilon \sim N(0, 1)$ is a standard normal variable.

Let $P(A_2|Y = y) = P(a^* > 0)$. Find $P(X_2 = x_2|Y = y)$ for all admissible values $x_2$ in terms of $y, \alpha$ and the normal CDF.

f) Bonus Question (Less directly related to lecture material, but may be of interest to some of you). Sometimes in statistical decision theory we use utility function (or alternatively loss functions or cost functions) to try and map outcomes to their value. This can take into account our preferences around uncertainty in a given application. An example of a utility function is $U : \mathcal{R}^+ \to \mathcal{R}^+$, $U(x) = \frac{x^{1-\alpha}}{1-\alpha}$ where $\alpha > 0, \alpha \neq 1$ is a parameter indicating risk tolerance. If $\alpha$ is close to 0, this indicates that a person does not negatively weight uncertainty very much, they make decisions based on what happens on average. If $\alpha$ is very large, say $\alpha > 3$ this indicates that they strongly prefer low amounts of variation to high variation and they are willing to sacrifice gains on average for extra certainty. $E[U(X_1)]$ is the expected utility of stock 1 and $E[U(X_2)]$ is the expected utility of stock two. Suppose $\alpha = \frac{1}{2}$, which stock would someone with preferences represented by the above utility curve prefer? I.e, which stock has higher expected utility? Think about how this relates to the information we learned about the stocks through their first two moments.

2. Consider a telephone operator who, in expectation, handles 5 calls every three minutes. Assume that the number of calls in disjoint intervals of time is independent and identical and that as a time interval gets infinitely small the probability of more than one call (simultaneous calls) goes to 0. (Modified question from Cassella and Berger)

a) What distribution would be a good modelling choice for the number of calls in a minute?

b) Supposing the distribution in a, what is the probability of getting exactly 0 calls in a minute?

c) What is the probability of at least 2 calls in a minute?

d) Suppose I knew in a particular minute that there would be at least 2 phone calls, what is the probability that there will be at least four?

e) Challenge Question: Again assuming that the number of calls from minute to minute are independent, consider the following experiment. The boss in the next room feels like most minutes there aren't any calls (doesn't know much about probability, just has a gut feeling). Using the calculations from part c, the telephone operator says that almost half of the time there at least 2 calls in any given minute, but the boss isn't convinced by "fancy maths". The boss and the telephone operator make a bet. They monitor the phones and count the number of calls in each minute. If there is a minute with exactly 0 calls before a minute with at least 2 calls then the boss wins and the telephone operator agrees to do an extra overtime shift. If there is a minute with at least 2 calls before a minute with exactly zero calls then the operator wins the bet, then the operator gets to go home early and the boss with handle the incoming calls. What is the probability that the operator wins the bet?

f) Taking calls is tiring, the more calls the operator takes the more tired they get. Suppose $U : \mathcal{R}^+ \to [0,1]$ is a function which maps the number of calls in a particular minute $X$ to the interval $[0,1]$ which represents how rested or tired the operator is (1 being rested, 0 being exhausted) due to that minute of calls.

Let $U(x) = e^{-\alpha x}, \alpha > 0$.

What is the expected level of rest in a minute, i.e $E[U(X)]$?

g) Suppose the operator has a rule that determines how they take breaks. They take a break as soon as they have had 10 tiring minutes, that is 10 minutes where $U(X) < .5$.

i) What is the probability of a given minute being tiring according to the above definition when $\alpha = \frac{1}{2}$.

ii) What is the probability that the break occurs exactly on the 20th minute?

iii) What is the probability that it has been 30 minutes and they still haven't taken a break?