MATH 323 - Tutorial 10 Questions

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1. Let $Y \sim U(-1, 1)$. Let $X = Y^2$ be a transformation of Y.

a) Find the pdf of X.

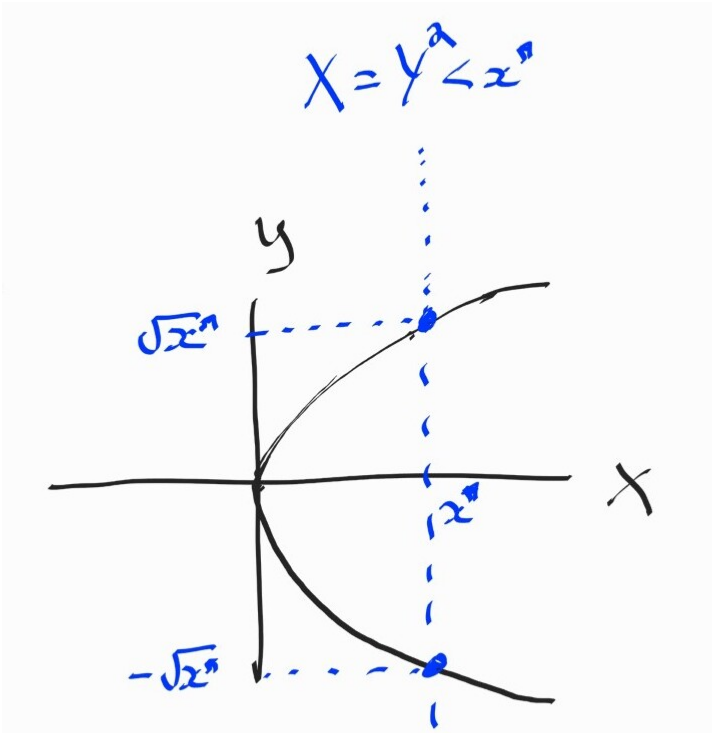
Solution:

We will use the CDF method. We know that $\frac{d}{dx}F_X(x) = f_X(x)$. To use this we will need the pdf (or alternatively cdf) of Y.

$$f_Y(y) = \frac{1}{2}, \quad -1 < y < 1$$

$$\begin{aligned}
\frac{d}{dx}F_X(x) &= \frac{d}{dx}P(X \leq x) \\
&= \frac{d}{dx}P(X \leq x) \\
&= \frac{d}{dx}P(Y^2 \leq x)
\end{aligned}$$

The goal then becomes to find $P(Y^2 \leq x)$. We can see how to re-express this in the following diagram.



$$y^2 = x$$
$$y = \begin{cases} \sqrt(x), y \geq 0 \\ -\sqrt(x), y < 0 \end{cases}$$

Thus from the above equation and diagram we can see that:

$$\begin{aligned} P(Y^2 \leq x) &= P(\{Y \geq -\sqrt{x}\} \cap \{Y \leq \sqrt{x}\}) \\ &= P(-\sqrt{x} \leq Y \leq \sqrt{x}) \\ &= F_Y(\sqrt{x}) - F_Y(-\sqrt{x}) \end{aligned}$$

Therefore

$$
\begin{aligned}
f_X(x) &= \frac{d}{dx}F_X(x) \\
&= \frac{d}{dx}(F_Y(\sqrt{x}) - F_Y(-\sqrt{x})) \\
&= f_Y(\sqrt{x})\frac{d}{dx}(\sqrt{x}) - f_Y(-\sqrt{x})\frac{d}{dx}(-\sqrt{x}) \\
&= \frac{1}{2} \times \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2} \times \frac{1}{2}x^{-\frac{1}{2}}, \quad -\sqrt{x}, \sqrt{x} \in (-1, 1) \\
&= \frac{1}{4}x^{-\frac{1}{2}} + \frac{1}{4}x^{-\frac{1}{2}}, \quad -\sqrt{x}, \sqrt{x} \in (-1, 1) \\
&= \begin{cases} \frac{1}{2}x^{-\frac{1}{2}}, \quad x \in (0, 1) \\ 0, x = 0 \end{cases}
\end{aligned}
$$

Where the case for zero follows from $x = 0$ $P(Y^2 \leq x) = P(Y^2 \leq 0) = P(Y^2 = 0) = P(X = 0) = 0$.

The support follows because the density for Y is only non-zero for inputs in $(-1, 1)$. Another way we could have figured this out would be to simply ask ourselves, if $X = Y^2$ and $Y \sim Unif(-1, 1)$, what is the support of X. Once again we arrive at the interval $[0, 1)$. A good habit in this kind of question is to check that our density integrates to 1. If not we must have made an error in defining the density function or the support.

b) What is the covariance of X and Y.

Solution:

$$Cov(X, Y) = E[XY] - E[X]E[Y]$$

We need to find $E[XY]$ and we know this is an integral over the joint density of X and Y, but we don't know the joint density, only the marginals for X and Y. In general we can only easily construct the joint from the marginals with special assumptions like independence (which means the joint is the product of the marginals) and X and Y are certainly not independent since they are transformations of each other. This might seem like a problem - WON'T IT BE PAINFUL TO DERIVE THE JOINT??? - but we can exploit the fact these variables are transformations of each other, since $XY = (Y^2)Y = Y^3$ is just some function of Y and thus it's expectation can be evaluated with respect to the marginal of Y only!

$$
\begin{aligned}
E[XY] &= E[Y^3] \\
&= \int_{-1}^{1} y^3 \frac{1}{2} d_y \\
&= [\frac{y^4}{4}\frac{1}{2}]_{-1}^{1} \\
&= \frac{(1)^4}{8} - \frac{(-1)^4}{8} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
E[Y] &= \int_{-1}^{1} y \frac{1}{2} d_u \\
&= [y^2]_{-1}^{1} \\
&= 0
\end{aligned}
$$

Notice in the following that we don't even need to compute $E[X]$.

$$
\begin{aligned}
Cov(X, Y) &= E[XY] - E[X]E[Y] \\
&= 0 - E[X] * 0 \\
&= 0
\end{aligned}
$$

This is a useful example to remember. These two variables are certainly dependent, but their covariance is zero!!! Remember that independence implies covariance is 0, but NOT THE OTHER WAY AROUND.

2. Let $f_{Y_1,Y_2}(y_1, y_2) = c(y_1^2 + 2y_1y_2 + y_2^2), \quad 0 < y_1, y_2 < l$ for some positive constants c, l.

a) find the constant c in terms of l.

Solution:

$$
\begin{aligned}
\int_0^l \int_0^l c(y_1^2 + 2y_1y_2 + y_2^2 dy_1 dy_2 &= \int_0^l c * [\frac{y_1^3}{3} + y_1^2 y_2 + y_1 y_2^2]_0^l dy_2 \\
&= \int_0^l c[\frac{l^3}{3} + l^2 y_2 + l y_2^2] dy_2 \\
&= c * [\frac{l^3}{3} y_2 + l^2 \frac{y_2^2}{2} + l \frac{y^3}{3}]_0^l \\
&= c * [\frac{l^4}{3} + \frac{l^2}{2} + \frac{l^4}{3}] \\
&= c \frac{7l^4}{6} \\
c\frac{7l^4}{6} &\overset{\text{set to}}{=} 1 \\
c &= \frac{6}{7l^4}
\end{aligned}
$$

b) Let $l = 1$. What is $Cov(Y_1, Y_2)$.

Work ommitted

$$
\begin{aligned}
E[Y_1 Y_2] &= \frac{17}{42} \\
f_{Y_1}(y_1) &= \frac{2}{7}(1 + 3y_1 + 3y_1^2), 0 < y_1 < 1 \\
f_{Y_2}(y_2) &= \frac{2}{7}(1 + 3y_2 + 3y_2^2), 0 < y_2 < 1 \\
E[Y_1] = E[Y_2] &= \frac{9}{14} \\
Cov(Y_1, Y_2) &= \frac{-5}{588}
\end{aligned}
$$

3. Let $f_{Y_1,Y_2} = cy_1y_2, \quad 0 < y_1, y_2 < 2.$

a) Find c such that the above is a valid joint distribution,

$$
\begin{aligned}
\int\int f_{Y_1,Y_2}(y_1,y_2)d_{y_1}dy_2 &= c\int_0^2\int_0^2 y_1y_2dy_1dy_2 \\
&= c\int_0^2 [\frac{y_1^2}{2}y_2]_0^2 dy_2 \\
&= c\int_0^2 2y_2dy_2 \\
&= cy_2^2|_0^2 \\
&= 4c \\
&\overset{\text{set to}}{=} 1 \\
c &= \frac{1}{4}
\end{aligned}
$$

b) Find $f_{Y_2}(y_2)$

$$
\begin{aligned}
f_{Y_2}(y_2) &= \int f_{Y_1,Y_2}(y_1,y_2)dy_1 \\
&= \int_0^2 \frac{1}{4}y_1y_2dy_1 \\
&= \frac{1}{4}y_2\frac{y_1^2}{2}|_0^2 \\
&= \frac{1}{2}y_2, 0 < y_2 < 2
\end{aligned}
$$

c) Find $f_{Y_1|Y_2}(y_1|y_2)$

$$
\begin{aligned}
f_{Y_1|Y_2}(y_1|y_2) &= \frac{f_{Y_1,Y_2}(y_1,y_2)}{f_{Y_2}(y_2)} \\
&= \frac{\frac{1}{4}y_1y_2}{\frac{1}{2}y_2} \\
&= \frac{y_1}{2}, \quad 0 < y_1, y_2 < 2
\end{aligned}
$$

d) Show that Y_1 is independent of Y_2.

There are several ways to do this. One shown here. First it can be shown through identical arguments as used in part b that $f_{Y_1}(y_1) = \frac{y_1}{2}, \quad 0 < y_1 < 2$.

Then we can show that:

$$
\begin{aligned}
f_{Y_1,Y_2}(y_1,y_2) &= \frac{1}{4}y_1y_2, \quad 0<y_1,y_2<2 \\
&= \frac{1}{2}y_1\frac{1}{2}y_2, \quad 0<y_1,y_2<2 \\
&= f_{Y_1}(y_1)F_{Y_2}(y_2)
\end{aligned}
$$

Thus Y_1, Y_2 are independent.

e) Find $E[Y_1|Y_2>1]$

Here we have a few options on how to solve this. The first would be to try and get the conditional distribution of $Y_1|Y_2>1$. This will likely be time-consuming. The second way is using the independence from part d. Since Y_1 and Y_2 are independent, so are Y_1 and the event $Y_2>1$ are also independent. Therefore $f_{Y_1|Y_2>1}(y1) = f_{Y_1}(y_1)$. Therefore:

$$
\begin{aligned}
E[Y_1|Y_2>1] &= E[Y_1] \\
&= \int y_1 f_{Y_1}(y_1) \\
&= \int_0^2 \frac{1}{2}y_1^2 \\
&= \frac{1}{6}y_1^3|_0^2 \\
&= \frac{4}{3}
\end{aligned}
$$

4. Let $f_Y(y) = \alpha\beta y^{\alpha-1}\exp(-\beta y^{\alpha}), y>0$. Y follows the Weibell Distribution. Show that the transformation $U = Y^{\alpha}$ follows an exponential distribution.
Solution:

The goal here is to find the pdf of U and we will show that it is in fact the pdf of an exponential distribution. THe

$$
\begin{aligned}
F_U(u) &= P(U \le u) \\
&= P(Y^{\alpha} \le u) \\
&= P(Y \le u^{\frac{1}{\alpha}}) \\
&= F_Y(u^{\frac{1}{\alpha}})
\end{aligned}
$$

Where the third line is only a monotonic increasing function because Y is restricted to be positive. Remember that for inequalities, applying montonic increasing functions keeps the inequality the same and applying monotonic decreasing functions flips it. Notice that in part 1, we implictly split the function in two, a part where it was monotonically increasing ($\sqrt{x}$) and a part that was monotonically decreasing $-\sqrt{x}$. This is a useful strategy to know if you go further in stats and probability and you will likely encounter it again.

$$
\begin{aligned}
f_U(u) &= \frac{d}{du} F_U(u) \\
&= \frac{d}{du} F_Y(u^{\frac{1}{\alpha}}) \\
&= f_y(u^{\frac{1}{\alpha}}) \frac{1}{\alpha} u^{\frac{1}{\alpha}-1} \\
&= (\alpha\beta(u^{\frac{1}{\alpha}})^{\alpha-1} \exp(-\beta(u^{\frac{1}{\alpha}})^{\alpha}) \frac{1}{\alpha} u^{\frac{1}{\alpha}-1} \\
&= \frac{\alpha\beta}{\alpha} \exp(-\beta u)(u^{\frac{1}{\alpha}})^{\alpha-1} u^{\frac{1}{\alpha}-1} \\
&= \beta \exp(-\beta u) u^{\frac{\alpha-1}{\alpha}} u^{\frac{1-\alpha}{\alpha}} \\
&= \beta \exp(-\beta u) u^{\frac{-(1-\alpha)}{\alpha}} u^{\frac{1-\alpha}{\alpha}} \\
&= \beta \exp(-\beta u) \frac{u^{\frac{1-\alpha}{\alpha}}}{u^{\frac{1-\alpha}{\alpha}}} \\
&= \beta \exp(-\beta u)
\end{aligned}
$$

Since $U = Y^{\alpha}$ and $y, \alpha > 0$, it must be that $u > 0$ thus:

$$f_U(u) = \beta \exp(-\beta u), \quad u > 0, \beta > 0$$

This is in fact an exponential distribution but you make not recognize this particular parametrization from class. Typically an exponential is writted $f_X(x) = \frac{1}{\lambda} \exp(\frac{-x}{\lambda}), \quad x > 0, \lambda > 0$. We can go from one to the other by simply letting $\beta = \frac{1}{\lambda}$ since if $\beta > 0$ then so must $\lambda > 0$ and vice versa. Thus we have shown that U is in fact an exponential since it's density has the same functional form and support as an exponential distribution!