# MATH 323 - Tutorial 1 Solutions

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1. Let $A, B, C$ be arbitrary sets such that $A, B, C \subseteq \Omega$. Using the sets themselves and the operations $\cap, \cup, \setminus$ and set complement write the following events:

i) Precisely 2 events occur
ii) At least 1 event occurs
iii) No more than 2 events occur
iv) Only $A$ occurs

**Solution**

i) $(A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C)$
ii) $A \cup B \cup C$
iii) $(A \cap B \cap C)^c$
iv) $A \cap B^c \cap C^c$

NB: There are often other ways to write the correct answers, they are not unique. It is a useful skill in probability to be able to be able to write these same events in different ways.

2. Let $A, B$ be arbitrary sets such that $A, B \subseteq \Omega$. Are the following relations true or false, justify your answer with a brief proof:

a) $(A \cup B) = (A \cap B^c) \cup B$
b) $(A^c \cap B) \cup (A \cap B^c) = (A \cup B) \cap (A \cap B)^c$

**Solution**

a)

$$\begin{aligned}
RHS &= (A \cap B^c) \cup B && (1)\\
&= (A \cup B) \cap (B \cup B^c) \quad [\text{ By the distributive property }] && (2)\\
&= (A \cup B) \cap \Omega && (3)\\
&= (A \cup B) \quad [\text{ Since } A \cap B \subseteq \Omega] && (4)\\
&= LHS && (5)
\end{aligned}$$

True

$B \cup B^c = \Omega$ for arbitrary $B \subseteq \Omega$. This is a useful fact to remember. In the course we will see this sort of thing applied often but in a different context. $B, B^c$ are an example of a **partition** for $\Omega$. A partition for the sample space is a set of **disjoint** sets (the intersection of any two sets is empty, i.e $B \cap B^c = \emptyset$) and each element in the sample space is in one of the partition sets (i.e $B \cup B^c = \Omega$). We will use partitions a lot as this course goes on, so it is worth thinking about them a bit if the idea is new to you.

b)

$$\begin{aligned}
RHS &= (A \cup B) \cap (A \cap B)^c \\
&= (A \cup B) \cap (A^c \cup B^c) \quad \text{[ By DeMorgan's Law ]} \\
&= ((A \cup B) \cap A^c)) \cup ((A \cup B) \cap B^c))\text{[ By the distributive Property ]} \\
&= ((A \cap A^c) \cup (A^c \cup B)) \cup ((A \cap B^c) \cup (B \cap B^c)) \quad \text{[ By the distributive property again ]} \\
&= ((\emptyset) \cup (A^c \cup B)) \cup ((A \cap B^c) \cup (\emptyset)) \quad \text{[ Since } A \cap A^c = B \cap B^c = \emptyset\text{]} \\
&= (A^c \cup B) \cup (A \cap B^c) \\
&= LHS
\end{aligned}$$

TRUE

The third line here uses the distributive property, but in a way that might not be immediately obvious. Remember, $A \cup B$ is just some event and a subset of $\Omega$. If we want we can temporarily define set $D := A \cup B$. Now we can rewrite line two as:

$$\begin{aligned}
(A) \cap (A^c \cup B^c) &= D \cap (A^c \cup B^c) \\
&= (D \cap A^c) \cup (D \cap B^c) \quad \text{[By distributive Property]} \\
&= ((A \cup B) \cap A^c) \cup ((A \cup B) \cap B^c) \quad \text{[ Substituting D for it's definition]}
\end{aligned}$$

NB: When reviewing question 2, the most important part of the proof is recalling and properly applying the correct set algebra rules. Getting a feel for partitions and DeMorgan's law especially is the most important outcome of doing a question like this, so please do not be discouraged if this is difficult.

3. Suppose we are testing a patient for a disease. Each patient either has the disease, denoted $D$, or they do not, denoted $D^c$. Suppose we administer a blood test. The blood test will either come back positive, denoted $B$, or negative, $B^c$.

a) Use the sets $D, D^c, B, B^c$ to describe the event that someone was misclassified by the Blood Test. First consider the two ways that a person could be misclassified and then use set notation to write this down as an event.

b) Suppose in addition to the blood test ($B$), the patient consulted a Physician and took an x-ray. Let the event $P$ denote the physician *believes* that the patient has the disease and the event

$X$ denote that the x-ray came back positive. Suppose the hospital uses the following classification rule to decide overall if they believe the patient has the disease or not. In order to be classified as having the disease, the Physician and at least one of the tests needs to come back positive. Write the event that the patient was correctly classified

(NB: There is a difference between a patient actually having the disease $D$ and a test indicating that the patient has it or not ($P$, $X$, and/or $B$). The difference is a test can be wrong. $D$ represents the truth which we cannot observe, but that we can still describe)

**Solutions**

a) $(D \cap B^c) \cup (D^c \cap B)$

$(D \cap B^c)$ is a false negative. That is our classification rule suggests the patient is negative, but the rule is incorrect. Similarly $(D^c \cap B)$ is a false positive. Classification rules are really common in things like statistics and machine learning. The nice thing is that in a case where the outcome is binary - that is the outcome is either true or false, or yes or no, or in this case the outcome is that the patient has the disease or they don't - then there are really only 4 possibilities. Either the classification rule says positive and we are right, or classification is positive and we are wrong (false positive), or classification is negative and we are right, or finally the classification is negative and we are wrong. This is true, as we will see in part b), even if the classification rule gets much more complicated.

b)

$$(D \cap (P \cap (X \cup B))) \cup (D^c \cap ((P \cap (X \cup B))^c)) \tag{6}$$

$$= (D \cap (P \cap (X \cup B))) \cup (D^c \cap ((P^c \cup (X^c \cap B^c)))) \quad \text{[ By Demorgan's Law]} \tag{7}$$

NB: Question 3 is likely the most important question in this tutorial in terms of the rest of this course and future statistics courses. It is also the first word problem you may have encountered in probability. Many of the more difficult problems in this course will not necessarily feature tricky maths, but rather the difficulty lies in correctly translating the word problem into the correct sets and probabilities.

4. Show that for arbitrary $A, B \subseteq \Omega$ that:

a) $P(A \cap B) \geq P(A) + P(B) - 1$
b) $P(A \cap B) \leq min(P(A), P(B))$

(Ask yourself when the bound in a) is a useful one, that is when it tells you something that you don't already know from the basic rules and axioms of probability. What does this look like (draw it if it helps you)? Sometimes in math we use the word non-trivial to describe a useful bound, you may see this in futures questions. A trivial probablity bound would be a bound from below less than or equal to 0 since we know that all probablities must be greater than or equal to 0 from Kolmogorov's axioms already. Similarly, a trivial bound from above would be greater than or equal to 1.)

**Solution**

a)

$$\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A \cap B) \text{ [ From Class ]} && (8)\\
P(A \cap B) &= P(A) + P(B) - P(A \cup B) && (9)\\
&\geq P(A) + P(B) - \max(P(A \cup B)) && (10)\\
&= P(A) + P(B) - 1 \text{ [ Since } P(A \cup B) \leq 1] && (11)
\end{aligned}$$

b)

$$\begin{aligned}
A \cap B \subseteq A && (12)\\
\implies P(A \cap B) \leq P(A) \text{ [ Consequence of axioms ]} && (13)\\
A \cap B \subseteq B && (14)\\
\implies P(A \cap B) \leq P(B) && (15)\\
P(A \cap B) \leq \min(P(A), P(B)) && (16)
\end{aligned}$$

To be more explicit about the last step, we can split things into 2 possible cases. Either $P(A) \geq P(B)$ (case 1) or $P(A) < P(B)$ (case 2).

Suppose we are in case 1, $P(A) \geq P(B)$. Then:

$$\begin{aligned}
min(P(A), P(B)) &= P(B) && (17)\\
&\geq P(A \cap B) \quad \text{[From above reasoning]} && (18)
\end{aligned}$$

So the proposition holds under case 1. Now in case 2

$$\begin{aligned}
min(P(A), P(B)) &= P(A) && (19)\\
&\geq P(A \cap B) && (20)
\end{aligned}$$

So the proposition holds in all possible cases!