MATH 323 - Tutorial 9 Questions

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1. Let

$$P(X = x) = \begin{cases} = .2, & x = 1 \\ = .5, & x = 2, \\ = .3, & x = 3 \end{cases}$$

a) Find $M_X(t)$

$$\begin{aligned} E[e^{Xt}] &= \sum_{x=1}^{3} \exp(xt)P(X = x) \\ &= .2e^t + .5e^{2t} + .3e^{3t} \end{aligned}$$

b) Using part a, find $E[X]$ and $Var(X)$

$$\begin{aligned} E[X] &= \frac{d}{dt} M_X(t)|_{t=0} \\ &= \frac{d}{dt}(.2e^t + .5e^{2t} + .3e^{3t})|_{t=0} \\ &= (.2e^t + e^{2t} + .9e^{3t})|_{t=0} \\ &= .1 + 1 + .9 \\ &= 2.1 \end{aligned}$$

$$\begin{aligned} E[X^2] &= \frac{d^2}{dt^2}(M_X(t))|_{t=0} \\ &= \frac{d}{dt}(.2e^t + e^{2t} + .9e^{3t})|_{t=0} \\ &= (.2e^t + 2e^{2t} + 2.7e^{3t})|_{t=0} \\ &= .2 + 2 + 2.7 \\ &= 4.9 \end{aligned}$$

$$\begin{aligned} Var(X) &= E[X^2] - (E[X])^2 \\ &= 4.9 - (2.1)^2 \\ &= 4.9 - 4.41 \\ &= .49 \end{aligned}$$

2. Let $X_1 \sim N(\mu_1, \sigma)$ and $X_2 \sim N(\mu_2, \sigma)$.

a) What is the Moment Generating function of $Y = aX_1 + X_2$.

$$\begin{aligned} M_Y(t) &= E[e^{Yt}] \\ &= E[\exp((X_1 + aX_2)t)] \\ &= E[e^{X_1 t} e^{atX_2}] \\ &= E[e^{X_1 t}]E[e^{atX_2}] \quad \text{[independence]} \\ &= M_{X_1}(t)M_{X_2}(at) \\ &= \exp(t\mu_1 + \frac{\sigma^2 t^2}{2}) \exp(at\mu_2 + \frac{\sigma^2 a^2 t^2}{2} \quad \text{using normal MGFs} \\ &= \exp(t(\mu_1 + a\mu_2) + \frac{\sigma^2(1 + a^2)t^2}{2}) \end{aligned}$$

b) Using part a, what is the distribution of Y.

Notice that in part a the MGF of Y has the form of a normal random variable. Specifically a normal random variable with mean $\mu_1 + a\mu_2$ and a variance of $\sigma^2(1 + a^2)$. When the MGF exists (in a neighbourhood of 0) it is enough to characterize the random variable and thus $Y \sim N(\mu_1 + a\mu_2, \sigma^2(1 + a^2))$.

When we add independent normal random variables together (remember since the normal distribution is a location-scale family that if X_2 is normal then so is aX_2) then we get another normal random variable.

3. Consider the following joint pmf, for the random variables X_1, X_2.

		X_1		
		0	1	2
	0	0.1	0.05	0,2
X_2	1	0	0.1	0.2
	2	0.3	0.05	0

a) Find the marginal support (values x_j for which $P(X_i = x_j) > 0$ for $i = 1, 2$) of each of the random variables X_1, X_2. Find the marginal distributions for X_1 and X_2.

Solution:

$$P(X_1 = x_1) = \begin{cases} 0.4, & x_1 = 0 \\ 0.2, & x_1 = 1 \\ 0.4, & x_1 = 2 \end{cases}$$

$$P(X_2 = x_2) = \begin{cases} 0.35, & x_2 = 0 \\ 0.3, & x_2 = 1 \\ 0.35, & x_2 = 2 \end{cases}$$

To find the marginal supports we simply need to scan the rows and columns. Start with X_1 and thus the columns. If the column has at least one entry in it with positive probability, it much have marginal support on that value. Thus:

Support of $X_1 = \{0, 1, 2\}$
Support of $X_2 = \{0, 1, 2\}$

Similarly, to find the marginal distributions we simply need to sum up the columns and rows. This may be a fact known to you, but let's try to connect why we are able to do this to things we studied earlier in the course. For example

$$\begin{aligned} P(X_1 = 1) &= P(\{(X_1 = 1) \cap (X_2 = 0)\} \cup \{(X_1 = 1) \cap (X_2 = 1)\} \cup \{(X_1 = 1) \cap (X_2 = 2)\}) \\ &\overset{disjoint}{=} P(\{(X_1 = 1) \cap (X_2 = 0)\}) + P(\{(X_1 = 1) \cap (X_2 = 1)\}) + P(\{(X_1 = 1) \cap (X_2 = 2)\}) \\ &= 0.05 + 0.1 + 0.05 \\ &= 0.2 \end{aligned}$$

Above we used the fact that all the values that a random variable can take (i.e the support) form a partition to break $P(X_1 = 1)$ into 3 disjoint parts which we already have information about. Notice that we get back simply the entries in the second column. In fancier mathematical terms we can get back the marginal distribution by summing over the joint pmf with respect to the other variables, i.e $P(X_1 = x_1) = \sum_{x_2 \in \mathcal{X}_\in} P(X_1 = x_1, X_2 = x_2)$, where $x_2 \in \mathcal{X}_\in$ are the values that X_2 takes with positive probability (the support). This follows directly from the kinds of steps we did above to find $P(X_1 = x_1)$.

I'm not just needlessly trying to make what seems like a simple question harder, the goal is to set up question 2. In question 2 we are dealing with a slightly funky joint pdf instead of a simple pmf. Students, in my experience, often have good intuition about joint pmfs, but much less so about pdfs. The goal is to try and make it clear how much the two share in common and to hopefully allow your intuition to jump from pmfs to pdfs. Notice in particular that summing over and integrating over the other variables (which we do in question 2) are accomplishing the same thing.

b) Fjnd the marginal variance of X_1 and X_2.

Solution:

$$
\begin{aligned}
E[X_1] &= \sum_{x_1 \in \mathcal{X}_\infty} x_1 P(X_1 = x_1) \\
&= 0 * 0.4 + 1 * 0.2 + 2 * 0.4 \\
&= 1 \\
E[X_2] &= \sum_{x_2 \in \mathcal{X}_\in} x_2 P(X_2 = x_2) \\
&= 0 * 0.35 + 1 * 0.3 + 2 * 0.35 \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
Var(X_1) &= \sum_{x_1 \in \mathcal{X}_\infty} (x_1 - E[X_1])^2 P(X_1 = x_1) \\
&= 0.4 * (0-1)^2 + 0.2 * (1-1)^2 + 0.4(2-1)^2 \\
&= 0.4 + 0.4 \\
&= 0.8 \\
Var(X_2) &= \sum_{x_2 \in \mathcal{X}_\in} (x_2 - E[X_2])^2 P(X_2 = x_2) \\
&= 0.35 * (0-1)^2 + 0.3 * (1-1)^2 + 0.35 * (2-1)^2 \\
&= 0.35 + 0.35 \\
&= 0.7
\end{aligned}
$$

c) Find the covariance between X_1 and X_2.

Solution:

$$Cov(X_1, X_2) = E[X_1 X_2] - E[X_1]E[X_2]$$

From part b, we have that $E[X_1]E[X_2] = 1 * 1 = 1$, so it remains only to find $E[X_1 X_2]$.

$$\begin{aligned}
E[X_1 X_2] &= \sum_{x_1 \in \mathcal{X}_\infty} \sum_{x_2 \in \mathcal{X}_\in} x_1 x_2 P(X_1 = x_1, X_2 = x_2) \\
&= \sum_{x_1=0}^{2} \sum_{x_2=0}^{2} x_1 x_2 P(X_1 = x_1, X_2 = x_2) \\
&= \sum_{x_1=1}^{2} \sum_{x_2=1}^{2} x_1 x_2 P(X_1 = x_1, X_2 = x_2) \\
&= \sum_{x_1=1}^{2} (x_1 * 1 * P(X_1 = x_1, X_2 = 1) + x_1 * 2 * P(X_1 = x_1, X_2 = 1) \\
&= (1 * 1 * P(X_1 = 1, X_2 = 1) + 1 * 2 * P(X_1 = 1, X_2 = 1) + (2 * 1 * P(X_1 = 2, X_2 = 1) + 2 * 2 * P(X_1 = 2, X_2 \\
&= (1 * 1 * (0.1) + 1 * 2 * (0.05) + 2 * 1 * (0.2) + 2 * 2 * (0) \\
&= 0.1 + 0.1 + 0.4 + 0 \\
&= 0.6
\end{aligned}$$

Where the third line follows from noticing that whenever either $x_1 = 0$ or $x_2 = 0$ the summand will be zero. This is just to save us from having to work out a larger sum.

$$\begin{aligned}
Cov(X_1, X_2) &= E[X_1 X_2] - EX_1 E[X_2] \\
&= 0.6 - 1 \\
&= -0.4
\end{aligned}$$

This tells us there is a negative linear relationship between X_1 and X_2. Look again at the joint pmf and see if you notice this pattern there directly.

d) Using the information in the previous parts, find the correlation between X_1 and X_2.

Solution:

$$\begin{aligned}
Cor(X_1, X_2) = \rho(X_1, X_2) &= \frac{Cov(X_1, X_2)}{\sqrt{Var(X_1)}\sqrt{Var(X_2)}} \\
&= \frac{-0.4}{\sqrt{0.8}\sqrt{0.7}} \\
&= -0.53452248382 \\
&\approx -0.53
\end{aligned}$$

Remember that this quantity must be between -1 and 1. This can help weed out some numerical errors by checking that your solution adheres to this bound.

e) Are X_1 and X_2 independent, justify your answer using information from the previous questions.

There are many ways that we could see they are in fact dependent. First notice that $Cov(X_1, X_2) \neq 0$ which implies dependency. Remember that the converse is NOT TRUE, i.e showing that the covariance is 0 is not enough to prove independence except in a few very special cases. The same holds for correlation (notice that correlation is just a scaling of the covariance. If correlation is 0 then so is covariance and vice versa).

A few other things we could notice that would also show dependence. Notice that the conditional support of the variables depends on each other. For example, knowing that $X_2 = 2$ means that it is impossible for $X_2 = 2$. Another way to say this is that the support depends on each other. Similar to covariance, showing the support is depedent is enough for dependence, but the converse showing the supports do not depend on each other is not enough to prove independence.

In this same line of thinking we could show that $P(X_1 = x_1, X_2 = x_2) \neq P(X_1 = x_1)P(X_2 = x_2)$ for at least one set of constants. For example $P(X_1 = 1, X_2 = 1) = 0.1 \neq P(X_1 = 1)P(X_2 = 1) = 0.2 * 0.3 = 0.06$.

Any of these answers are sufficient to show dependence and thus answer the question. Support dependence was not mentioned in class to my knowledge so I would focus on the correlation/covariance or joint and marginal routes to answer this question for this course.

4. Let $f_{Y_1,Y_2}(y_1, y_2) = 3y_1 + cy_2, \quad 0 < y_1 < 1, 0 < y_2 < 1$

a) Find c

Solution:

$$
\begin{aligned}
1 &= \int\int f_{Y_1,Y_2}(y_1,y_2)dy_1dy_2 \\
&= \int_0^1\int_0^1 (3y_1+cy_2)dy_1dy_2 \\
&= \int_0^1 (\frac{3y_1^2}{2}+cy_1y_2)|_0^1 dy_2 \\
&= \int_0^1 (\frac{3}{2}+cy_2)dy_2 \\
&= \frac{3y_2}{2}+\frac{cy_2^2}{2}|_0^2 \\
&= \frac{3}{2}+\frac{c}{2}
\end{aligned}
$$

Therefore $1 = \frac{3}{2} = \frac{c}{2}$ which implies that $c = -1$

b) What is the covariance of Y_1 and Y_2

$$
\begin{aligned}
E[Y_1Y_2] &= \int_0^1\int_0^1 y_1y_2(3y_1-y_2)dy_1dy_2 \\
&= \int_0^1\int_0^1 (3y_!^2y_2 - y_1y_2^2)dy_1dy_2 \\
&= \int_0^1 (y_1^3y_2 - \frac{y_1^2y_2^2}{2})|_0^1 \\
&= \int_0^1 (y_2 - \frac{y_2^2}{2})dy_2 \\
&= \frac{y_2}{2} - \frac{y_2^3}{6}|_0^1 \\
&= \frac{1}{2} - \frac{1}{6} \\
&= \frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
f_{Y_1}(y_1) &= \int_0^1 f_{Y_1,Y_2}(y_1,y_2)dy_2 \\
&= \int_0^1 (3y_1-y_2)dy_2 \\
&= 3y_1y_2 - \frac{y_2^2}{2}|_0^1 \\
&= 3y_1 - \frac{1}{2}, \quad 0 < y_1 < 1
\end{aligned}
$$

$$
\begin{aligned}
E[Y_1] &= \int_0^1 y_1(3y_1 - \frac{1}{2}) \\
&= \int_0^1 (3y_1^2 - \frac{y_1}{2})dy_1 \\
&= y_1^3 - \frac{y_1^2}{4}|_0^1 \\
&= 1 - \frac{1}{4} \\
&= \frac{3}{4}
\end{aligned}
$$

$$
\begin{aligned}
f_{Y_2}(y_2) &= \int_0^1 (3y_1 - y_2)dy_1 \\
&= \frac{3y_1^2}{2} - y_! y_2|_0^1 \\
&= \frac{3}{2} - y_2, \quad 0 < y_2 < 1
\end{aligned}
$$

$$
\begin{aligned}
E[Y_2] &= \int_0^1 y_2(\frac{3}{2} - y_2)dy_2 \\
&= \int_0^1 (\frac{3y_2}{2} - y_2^2)dy_2 \\
&= \frac{3y_2^2}{4} - \frac{y_2^3}{3}|_0^1 \\
&= \frac{3}{4} - \frac{1}{3} \\
&= \frac{5}{12}
\end{aligned}
$$

$$
\begin{aligned}
COV(Y_1, Y_2) &= E[Y_1Y_2] - E[Y_1]E[Y_2] \\
&= \frac{1}{3} - \frac{3}{4}\frac{5}{12} \\
&= \frac{1}{3} - \frac{15}{48} \\
&= \frac{1}{48}
\end{aligned}
$$