Prelims Probability extra questions

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1 Functional representations

1.1 Moment generating functions

Sheet 3 Question 5 asks you to prove that,

$$\mathbf{E}\left[\exp(tX)\right] = \exp(\lambda(e^t - 1))$$

for X a Poisson random variable of parameter λ.

The expression on the left is called a moment generating function(mgf) and is a powerful analytical tool useful in probability to get concentration inequalities and uniqueness of distributions.

We write $M_X(t) = \mathbf{E}\left[\exp(tX)\right]$ for the mgf at t.

Calculate the mgf of the following:

1. Geo(p)
2. Ber(p)
3. Uniform on $\{1, ..., n\}$
4. Bin(n,p)
5. Exponential rate λ
6. Normal random variable mean μ, variance σ^2

Let a, b be some constants, then what is the mgf of $Z_1 = aX + b$ (i.e. $M_{Z_1}(t)$?

Now suppose X, Y are independent, then what is the mgf of $Z_2 = X + Y$ (i.e. $M_{Z_2}(t)$)?

You might have noticed that the mgf doesn't always exist, as sometimes the expectation of $\exp(tX)$ is infinite. Can you construct an example of a random variable where $\mathbf{E}\left[\exp(tX)\right]$ is infinite for all $t \neq 0$?

By differentiating (and not worrying about the important analysis that needs to be done in the background but that you will only do later), can you relate the moments of a distribution with the mgf?

Note: a moment is an expression of the form $\mathbf{E}\left[X^k\right]$. First moment is mean $\mathbf{E}\left[X\right]$ and second moment is $\mathbf{E}\left[X^2\right]$.

Use this to find expressions for the mean and variance of a random variable X using the mgf. Then check this with the distributions considered above.

Prove that for all t:

$$\mathbf{P}\{X > x\} \leq M_X(t)\exp(-tx)$$

Consider X is Ber(p),Geo(p),Bin(n,p),Poi(λ),Exp(λ) and for each of those optimise the right hand side in t to get optimal tail bounds.

1.2 Probability generating functions

Sheet 5 Question 4 asks you to calculate the probability generating function of a Geometric distribution.

Calculate the following pgf:

1. Ber(p)
2. Bin(n,p)
3. Poisson(λ)
4. Uniform $\{1, ..., n\}$
5. Negative Binomial (r,p) with pdf $p_k = \binom{k+r-1}{k}(1-p)^k p^r$

Let a, b be some constants, then what is the pgf of $Z_1 = aX + b$ (i.e. $M_{Z_1}(t)$?

Now suppose X, Y are independent, then what is the pgf of $Z_2 = X + Y$ (i.e. $M_{Z_2}(t)$)?

Can we recover the distribution of a random variable from the pgf?

2 Sums of random variables

Suppose $X_1, ..., X_n$ are independent and identically distributed (i.i.d.) with $\mathbf{E}[X_1] = 0$. Let $S_n = X_1 + ... + X_n$.

Suppose $\mathbf{E}\left[|X_1|^k\right] < \infty$ for $k \in \mathbb{N}$. What is $\mathbf{E}\left[S_n^k\right]$? What is $M_{S_n}(t)$ in terms of $M_{X_1}(t)$?

3 Fun questions

1. For sheet 8 question 7, what happens if $p = a$ or $p = b$? What does the limit approach, why?
2. Let $U_1, ..., U_n$ i.i.d Uniform [0,1] random variables. What is $\mathbf{P}\{U_1 + ... + U_n \leq 1\}$?

3.1 Sheet 3 Question 6

Sheet 3 Question 6 asks us to find out how long we have to wait on average to see 2 heads in a row, then Sheet 5 Question 5 asks us to calculate the exact distribution.

1. How long do you have to wait on average to get k heads in a row where $k \in \mathbb{N}$?
2. What is $r_n = \mathbf{P}\{X > n\}$ where X is the first time you see k heads in a row?
3. Let Z be the first time you see HTH. What is $\mathbf{E}[Z]$ and $\mathbf{P}\{Z > n\}$?
4. Let Z be the first time you see $HTTH$. What is $\mathbf{E}[Z]$?
5. If the challenge doesn't prove enough, try to extend your methods to any sequence.