

# Prelims Probability extra questions

Leo Tyrpak

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## 1 Sheets extra questions

### 1.1 Sheet 1: Recap school probability

Notes:

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Extra:

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Do the following from Grimmett:

- Q 1.2 (all)
- Q 1.3.1, 1.3.6, 1.3.7

### 1.2 Sheet 2: Independence

Notes:

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Extra:

- Q1: Find 4 events which are 3-way independent but not 4-way independent, see if your construction generalises.
- Q5 extra: estimate the probabilities of (a), (b) replacing 13 by  $n$ , 26 by  $2n$  and 52 by  $4n$ . These are both related to the Central Limit Theorem and local Central Limit Theorem. Search them up if you want to find out more.

Q6 extra:

- Prove that if  $\{A_i : i \in I\}$  independent then  $\{A_i^c : i \in I\}$  independent
- Prove that if  $\{A_i : i \in I\}$  independent then  $\mathbf{P}\left\{\bigcap_{i \geq 1} A_i\right\} = \prod_{i \geq 1} \mathbf{P}\{A_i\}$

- Using equality  $\sum_{n \geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$  prove there are infinitely many primes

Do the following from Grimmett:

- Q 1.5.2-1.5.5, 1.5.7, 1.5.9

### 1.3 Sheet 3: Some analysis

Notes:

- Q1: One needs to state exactly how we represent a binomial as a sum of independent Bernoulli random variables.
- Q2 is about Fubini's theorem (interchanging order of integration) and a neat formula for expectation. Can be extended to continuous case.
- Q3 is about the memoryless property of Geometric random variables. This completely characterises the geometric distribution.
- Q4 is about approximating Poisson with suitably scaled Binomial, this should be done rigorously with the notion of convergence of sequences you learn in analysis. For part b you need to numerically compute what was asked.
- Q5 is about calculating moment generating function of Poisson distribution. The moment generating function completely characterises a random variable, see wiki.
- Q7c, here you need to give an asymptotic expansion of  $\sum_{i=1}^n \frac{1}{i}$ , you can learn more about it on wiki.

Extra:

- Q1: prove that if  $E_1, \dots, E_n$  are i.i.d. (independent and identically distributed)  $\text{Ber}(p)$  then  $X = \sum_{i=1}^n E_i$  is  $\text{Bin}(n, p)$ .
- Q5: calculate MGF of Bernoulli, Binomial, Geometric, Uniform
- Q3: prove that if a random variable on  $\mathbb{N}$  has the memoryless property then it is geometric
- Q6: expectation before seeing 3, 4, ...  $H$  in a row, what about a generic pattern of  $H, T$ ?

## 1.4 Sheet 4: Independence

Notes:

- Q2 shows covariance zero doesn't imply independence (converse is true)
- Q5 is about the thinning property of Poisson distributions
- Q6 is about different equivalent definitions of independence
- Q7: there is a nice way to solve the homogeneous problem using linear algebra, where if you have a recurrence relation of degree  $k$  for  $u_n$  then you get a matrix equation for  $v_n = (u_{n+k-1}, \dots, u_{n+1}, u_n)^T$

Extra:

- Q4 a,b: generalise  $X, Y$  to  $X$  is Geo( $p$ ) and  $Y$  is Geo( $q$ )
- Reprove Q6 carefully saying which axioms you use

Grimmett:

- All of Section 3.2 and 3.3

## 1.5 Sheet 5: Markov Chains

Notes:

- Questions 1,2,3,5,6 can all be put into the framework of Markov Chains. Markov chains are probabilistic objects of a system evolving over time where the evolution only depend on the current state, not on the whole past. We call states terminal/ absorbing if the chain stays at that state when it is there. An equilibrium distribution is one where if the chain has this distribution then next step it has that same distribution. This is discussed more thoroughly in Part A(2n year) Probability.
- Q1 is about finding the equilibrium of that Markov chain and finding out that we have exponential convergence to equilibrium.
- Q2 is about the expected time to reach the absorbing state of the Markov chain.
- Q3 is about the expected time to reach the absorbing states of the Markov chain.
- Q6 (a) is about finding the expected covering time of the Markov chain, i.e. the expected amount of steps required to visit every state of the Markov chain.

Extra:

- Q1,2,3,5,6: draw the associated Markov chain. What are the absorbing states for each Markov chain? What are the equilibrium distributions for each Markov chain? What is their transition matrix?
- Q1: consider the same question with a square, pentagon then  $n$ -gon, what do you get?
- Q3: consider the same questions with probability  $p$  of winning 1 and probability  $1 - p$  of winning -1. What about probability  $1/3$  of getting  $\{-1, 1, 2\}$  after each round?
- Q5: consider the same questions with a probability  $p$  of Heads. Can we characterise the distribution of  $Y$  easily.
- Q6: Suppose that we are given this question on an undirected graph, where at each step we travel along one of the edges of the graph to a neighbour. In this particular case the graph was a cycle, but we can consider other graphs. Prove that if the distribution of the last vertex to be visited is uniform then we are on a cycle or a complete graph.

Sheet 3 Question 6 asks us to find out how long we have to wait on average to see 2 heads in a row, then Sheet 5 Question 5 asks us to calculate the exact distribution. We will consider a biased coin with probability  $p$  of Heads.

- How long do you have to wait on average to get  $k$  heads in a row where  $k \in \mathbb{N}$ ?
- What is  $r_n = \mathbf{P}\{X > n\}$  where  $X$  is the first time you see  $k$  heads in a row?
- Let  $Z$  be the first time you see  $HTH$ . What is  $\mathbf{E}[Z]$  and  $\mathbf{P}\{Z > n\}$ ?
- Let  $Z$  be the first time you see  $HTTH$ . What is  $\mathbf{E}[Z]$ ?
- If the challenge doesn't prove enough, try to extend your methods to any sequence.

## 1.6 Sheet 6: Generating functions

Notes:

- Q6 (d):

Extra:

- Q2: prove that the sum of  $m$  independent Geometric random variables with parameter  $p$  has the distribution of a negative Binomial (e.g. prove both have same probability mass function).
- Q3: prove that the formulae in part (a) hold assuming only that  $\{N \geq k\}$  is independent of  $X_k$ .

- Q4: find an expression involving generating functions for the probability that  $n$  divides  $X \in \mathbb{Z}_+$ .
- Q5: find the probability that there is 1, 2, 3 individuals after 2 minutes.
- Q6: prove that the probability the process eventually dies out is the smallest non-negative solution to  $G(s) = s$  (Hint: consider  $\alpha_n = \mathbf{P}\{X_n = 0\}$  and let  $n \rightarrow \infty$ ).

## 1.7 Sheet 7: Continuous variables

Extra:

- Does there exist a function with a root at  $0, 1/2$  which is always positive. How can we tell if there is a sign change at a root. Give a complete characterisation and prove it is correct.
- Q2: is there normalising constant for  $x^\alpha(1-x)^\beta$  for  $x \in (0, 1)$ ? If so find it.
- Q3: Find  $\mathbb{E}[U^k], \mathbb{E}[\exp(\lambda U)]$

Grimmett:

- All 4.1, 4.3

## 1.8 Sheet 8: Continuous variables

Extra:

- Q2: give a few more terms of the asymptotic expansion of the quantity you get and give precise error bounds. (Hint: Taylor)
- Q4: use other means to get a more precise upper bound or an exact quantity. (Hint: MGF) What sort of bound would the central limit theorem tell us we should expect.
- Q6: Explain intuitively why we have a lower variance for  $d$  than for  $c$ . Does lower variance, same mean imply concentration around the mean? Why? Calculate higher order moments for the random variables in part  $c, d$ , at least to leading order of  $n$ . Then see if the scaling is correct so that we should expect a central limit theorem to hold.
- Q7: What happens when  $p = a, b$ ? Prove the law of large numbers for Bernoulli random variables. How would you start this question?

## 2 Sums of random variables

Suppose  $X_1, \dots, X_n$  are independent and identically distributed (i.i.d.) with  $\mathbf{E}[X_1] = 0$ . Let  $S_n = X_1 + \dots + X_n$ .

Suppose  $\mathbf{E}[|X_1|^k] < \infty$  for  $k \in \mathbb{N}$ . What is  $\mathbf{E}[S_n^k]$ ? What is  $M_{S_n}(t)$  in terms of  $M_{X_1}(t)$ ?

Calculate  $\mathbf{E}[N^k]$  where  $N$  is a normal random variable with mean zero and variance 1. Now suppose  $\mathbf{E}[X_1] = 0, \mathbf{E}[X_1^2] = 1, \mathbf{E}[|X_1|^k] < \infty$ , then what is  $\alpha_{n,k} \mathbf{E}\left[\left(\frac{S_n}{\sqrt{n}}\right)^k\right]$ ? Prove that  $\lim_{n \rightarrow \infty} \alpha_{n,k} = \mathbf{E}[N^k]$ . This goes towards proving the Central Limit theorem by proving the moments converge, this is called the method of moments. This also appears in the 4'th year course random matrix theory.

## 3 Combinatorics

1. For sheet 8 question 7, what happens if  $p = a$  or  $p = b$ ? What does the limit approach, why?
2. Let  $U_1, \dots, U_n$  i.i.d Uniform  $[0,1]$  random variables. What is  $\mathbf{P}\{U_1 + \dots + U_n \leq 1\}$ ?
3. During a game of poker you are dealt a hand of 5 cards uniformly at random. With the convention that aces can be either high or low for a straight. What is the probability of having 1 pair, 2pairs, 3 of a kind, straight, flush, full house, 4 of a kind, straight flush. See Poker hands for a list of poker hands and what they mean.
4. 8 pawns are placed uniformly at random on a chess board, no more than one per square. What is the probability that no 2 are in the same row or column?
5. Players A, B, C throw a uniform dice at random in the order ABCAB-CABC... Each dice throw is independent. What is the probability that A throws a 6 first? What is the probability that A throws a 6 first and B throws a 6 the second and C is the third to throw a 6?
6. Each member of a group of  $n$  players roll a dice. For any pair of players who roll the same number, the group scores 1 point. Find the mean and variance of the total score of the group. Find the mean and variance of the total score if any pair of players who throw the same number scores that number.

## 4 Probabilistic method

This is also sometimes called probabilistic combinatorics.

Let  $G=(V,E)$  be a finite simple graph (so no loops or multiple edges between 2 vertices. Write  $d_v$  for the degree of an edge  $v$ . Prove the following:

1. An independent set is a set of vertices no 2 of which share an edge. Denote by  $\alpha(G)$  the largest independent set. Prove that:

$$\alpha(G) \geq \sum_v \frac{1}{d_v + 1} \quad (1)$$

2. For any set  $W \subseteq V$ , edge  $e \in E$ :

$$1_W(e) = 1 \text{ if } e \text{ connects } W \text{ and } W^c \quad (2)$$

If  $N_W = \sum_e 1_W(e)$  then prove that there exists  $W$  such that  $N_W \geq 1/2|E|$

We construct a random graph  $G'$  by including each edge independently with probability  $p \in (0, 1)$ .

## 5 Analysis

1. Prove Bonferroni's inequality

$$\mathbf{P} \left\{ \bigcup_{r=1}^n A_r \right\} \geq \sum_{r=1}^n \mathbf{P} \{A_r\} - \sum_{r < k} \mathbf{P} \{A_r \cap A_k\} \quad (3)$$

2. Prove Kounia's inequality

$$\mathbf{P} \left\{ \bigcup_{r=1}^n A_r \right\} \leq \min_k \left\{ \sum_{r=1}^n \mathbf{P} \{A_r\} - \sum_{r \neq k} \mathbf{P} \{A_r \cap A_k\} \right\} \quad (4)$$

### 5.1 Moment generating functions

Sheet 3 Question 5 asks you to prove that,

$$\mathbf{E}[\exp(tX)] = \exp(\lambda(e^t - 1))$$

for  $X$  a Poisson random variable of parameter  $\lambda$ .

The expression on the left is called a moment generating function(mgf) and is a powerful analytical tool useful in probability to get concentration inequalities and uniqueness of distributions.

We write  $M_X(t) = \mathbf{E}[\exp(tX)]$  for the mgf at  $t$ .

Calculate the mgf of the following:

1. Geo(p)
2. Ber(p)
3. Uniform on  $\{1, \dots, n\}$
4. Bin(n,p)

5. Exponential rate  $\lambda$

6. Normal random variable mean  $\mu$ , variance  $\sigma^2$

Let  $a, b$  be some constants, then what is the mgf of  $Z_1 = aX + b$  (i.e.  $M_{Z_1}(t)$ )?

Now suppose  $X, Y$  are independent, then what is the mgf of  $Z_2 = X + Y$  (i.e.  $M_{Z_2}(t)$ )?

You might have noticed that the mgf doesn't always exist, as sometimes the expectation of  $\exp(tX)$  is infinite. Can you construct an example of a random variable where  $\mathbf{E}[\exp(tX)]$  is infinite for all  $t \neq 0$ ?

By differentiating (and not worrying about the important analysis that needs to be done in the background but that you will only do later), can you relate the moments of a distribution with the mgf?

Note: a moment is an expression of the form  $\mathbf{E}[X^k]$ . First moment is mean  $\mathbf{E}[X]$  and second moment is  $\mathbf{E}[X^2]$ .

Use this to find expressions for the mean and variance of a random variable  $X$  using the mgf. Then check this with the distributions considered above.

Prove that for all  $t \geq 0$ :

$$\mathbf{P}\{X > x\} \leq M_X(t) \exp(-tx)$$

Consider  $X$  is  $\text{Ber}(p), \text{Geo}(p), \text{Bin}(n, p), \text{Poi}(\lambda), \text{Exp}(\lambda)$  and for each of those optimise the right hand side in  $t$  to get optimal tail bounds.

We will specifically be interested in

$$\lim_{x \rightarrow \infty} \frac{\log(\mathbf{P}\{X > x\})}{x}$$

This is tail behaviour of  $X$  which is related to a subfield of probability called Large Deviation Theory (also a 4'th year course).

## 5.2 Probability generating functions

Sheet 5 Question 4 asks you to calculate the probability generating function of a Geometric distribution.

Calculate the following pgf:

1.  $\text{Ber}(p)$

2.  $\text{Bin}(n, p)$

3.  $\text{Poisson}(\lambda)$

4. Uniform  $\{1, \dots, n\}$

5. Negative Binomial  $(r, p)$  with pdf  $p_k = \binom{k+r-1}{k} (1-p)^k p^r$

Let  $a, b$  be some constants, then what is the pgf of  $Z_1 = aX + b$  (i.e.  $M_{Z_1}(t)$ )?

Now suppose  $X, Y$  are independent, then what is the pgf of  $Z_2 = X + Y$  (i.e.  $M_{Z_2}(t)$ )?

Can we recover the distribution of a random variable from the pgf?

## 6 General puzzles and brainteasers

- (From Joost) you are given a biased coin, but you do not know the heads probability. How do you use this to sample a fair coin? [Discuss optimality, expected number of tosses to get the sample, etc.]
- You are given a fair coin, and a number  $p \in (0, 1)$ . How can you use the fair coin to generate a biased coin with probability  $p$ ? Only finitely many flips allowed. [Discuss optimality, number of flips on average, etc.]