Spectral exterior calculus

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Abstract

A spectral approach to building the exterior calculus in manifold learning problems is developed. The spectral approach is shown to converge to the true exterior calculus in the limit of large data. Simultaneously, the spectral approach decouples the memory requirements from the amount of data points and ambient space dimension. To achieve this, the exterior calculus is reformulated entirely in terms of the eigenvalues and eigenfunctions of the Laplacian operator on functions. The exterior derivatives of these eigenfunctions (and their wedge products) are shown to form a frame (a type of spanning set) for appropriate L^2 spaces of k-forms, as well as higher-order Sobolev spaces. Formulas are derived to express the Laplace-de Rham operators on forms in terms of the eigenfunctions and eigenvalues of the Laplacian on functions. By representing the Laplace-de Rham operators in this frame, spectral convergence results are obtained via Galerkin approximation techniques. Numerical examples demonstrate accurate recovery of eigenvalues and eigenforms of the Laplace-de Rham operator on 1-forms. The correct Betti numbers are obtained from the kernel of this operator approximated from data sampled on several orientable and non-orientable manifolds, and the eigenforms are visualized via their corresponding vector fields. These vector fields form a natural orthonormal basis for the space of square integrable vector fields, and are ordered by a Dirichlet energy functional which measures oscillatory behavior. The spectral framework also shows promising results on a non-smooth example (the Lorenz 63 attractor), suggesting that a spectral formulation of exterior calculus may be feasible in spaces with no differentiable structure.

Keywords: Exterior calculus, frame theory, Galerkin methods, graph Laplacians

1. Introduction

The field of manifold learning has focused significant attention recently on consistently estimating the Laplacian operator,

$$\Delta = -\operatorname{div}\operatorname{grad} = \delta d \tag{1}$$

(in this paper we use the positive definite Laplacian, which we also refer to as the 0-Laplacian) [1–8]. Given data $\{x_i\} \subset \mathcal{M} \subset \mathbb{R}^n$, these manifold learning methods build a graph with weights given by a kernel function $k(x_i, x_j)$, and then approximate the Laplacian operator with the graph Laplacian

$$\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{K}, \tag{2}$$

where $\boldsymbol{K}$ and $\boldsymbol{D}$ are the kernel and degree matrices associated with k, respectively. In Table 1 we briefly summarize the current state-of-the-art results.

The Laplacian-based approach to manifold learning is justified by the fact that the Laplace-Beltrami operator encodes all the geometric information about a Riemannian manifold. A simple demonstration of this fact arises from the product formula for the Laplacian

$$\Delta(fh) = f\Delta h + h\Delta f - 2\operatorname{grad} f \cdot \operatorname{grad} h, \tag{3}$$

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Table 1: Summary of results on manifold learning

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1. For uniform data on a compact manifold, the Gaussian kernel is a consistent pointwise estimator [1].
2. For nonuniform data on a compact manifold, any isotropic kernel with exponential decay can be normalized to give a consistent pointwise estimator [3].
3. For nonuniform data on a compact manifold, any symmetric kernel with super-polynomial decay can be normalized to give a consistent pointwise estimator with respect to a geometry determined by the kernel function [5].
4. The bias-variance tradeoff implies error which is exponential in the dimension of the manifold [2, 7].
5. The above results can be generalized to non-compact manifolds by assuming appropriate lower bounds on the injectivity radius and either the curvature or the ratio between the intrinsic and extrinsic distances [6–8]
6. For smooth manifolds with finite volume (in the geometry imposed by the kernel) the graph Laplacian converges spectrally to the Laplacian operator [4, 8].
7. The bias-variance tradeoff implies that only the leading eigenfunctions and eigenvalues are well approximated (error is linear in the eigenvalue up to $\mathcal{O}(N^{\frac{d-2}{d+6}})$ and quadratic thereafter) [8].

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where the dot-product above is actually the Riemannian inner product $g_x : T_x\mathcal{M} \times T_x\mathcal{M} \to \mathbb{R}$,

$$g_x(\operatorname{grad} f(x), \operatorname{grad} h(x)) = \operatorname{grad} f \cdot \operatorname{grad} h = \frac{1}{2}(f(x)\Delta h(x) + h(x)\Delta f(x) - \Delta(fh)(x)). \tag{4}$$

Specifically, given any vectors $v, w \in T_x\mathcal{M}$, there must exist functions f, h with $\operatorname{grad} f(x) = v$ and $\operatorname{grad} h(x) = w$ and then the metric $g_x(v, w) = g_x(\operatorname{grad} f(x), \operatorname{grad} h(x))$ can be computed as above.

Since the geometry of a Riemannian manifold is completely determined by the Riemannian metric, the above formulas show that metric is completely recoverable from Laplacian, so learning the Laplacian is "sufficient" for manifold learning. Of course, this is a theoretical rather than pragmatic notion of sufficiency. If one asks certain geometric questions, such as "What is the 0-homology of the manifold?" (i.e., the number of connected components)m this can be easily answered as the dimension of the kernel (nullspace) of the Laplacian. However, if one asks for the higher homology of the manifold, or the harmonic vector fields, or the closed or exact forms, the above formulas do not suggest any practical approach. What is needed is not merely the Laplacian, but a consistent representation of the entire exterior calculus on the manifold.

In this paper, we introduce the Spectral Exterior Calculus (SEC) as a consistent representation of the exterior calculus based entirely on the eigenfunctions and eigenvalues of the Laplacian on functions. In essence, we will follow through on the above analysis and reformulate the entire exterior calculus in terms of these eigenfunctions and eigenvalues.

There are currently two alternative methods of discretely representing the exterior calculus, the Discrete Exterior Calculus (DEC) [9, 10], and the Finite element Exterior Calculus (FEC) [11, 12]. In Table 2, we compare the features of the SEC to the DEC and FEC. For manifold learning applications, we focus on three requirements: consistency, applicability to raw data, and amount of data and memory required.

The first requirement is that the method should be consistent, meaning that discrete analogs of objects from the exterior calculus should converge to their continuous counterparts in the limit of large data. In this paper we will focus on the pointwise and spectral consistency, in an appropriate Hilbert space, of the construction of the Laplace-de Rham operators on k-forms (or k-Laplacians, denoted Δ_k). We chose to focus on these operators because their eigenforms form a natural ordered basis for the space of k-forms. Moreover, in the case of the eigenforms of the 1-Laplacian, the Riemannian duals form a natural basis for smooth vector fields. While the DEC formulates a discrete analog to Δ_k, currently it has not been proven to be consistent, except for Δ_0 using the cotangent formula for the special case of surfaces in $\mathbb{R}^3$. A recent preprint shows consistency of the DEC when used to solve Poisson problems for the 1-Laplacian [13], but spectral

Table 2: Comparison of the Spectral Exterior Calculus (SEC) introduced in this manuscript to the Discrete Exterior Calculus (DEC) and the Finite element Exterior Calculus (FEC).

Feature	DEC	FEC	SEC
Pointwise consistency	Yes	Yes	Yes
Spectral consistency	No	Yes	Yes
Works on raw data	No	No	Yes
Decouples memory from data	No	N/A	Yes
Exterior Calculus structure	Partial	Partial	Partial

convergence is not addressed. The FEC has convergence results for estimating Laplace-de Rham and related operators, including eigenvalue problems. In Section, 5, we prove spectral convergence results for the SEC-approximated 1-Laplacian using a Galerkin technique. More generally, many of the operators encountered in exterior calculus, including the k-Laplacians, are unbounded, and the requirement of consistency must necessarily address domain issues for such operators. One of the advantageous aspects of the SEC is that the Sobolev regularity appropriate for differential operators such as k-Laplacians can be naturally enforced using the eigenvalues of the 0-Laplacian. This allows us to construct spectrally convergent Galerkin schemes using classical results from the spectral approximation theory for linear operators. This approach generalizes Galerkin approximation schemes for a class of unbounded operators on functions (generators of measure-preserving dynamical systems) [14–16] to operators acting on vector fields and higher-order k-forms.

Our second requirement is that the method should only require raw data, as the assumption of an auxiliary structure such as a simplicial complex is unrealistic for many data science applications. The FEC makes strong use of the known structure of the manifold to build their finite element constructions, which makes the FEC inappropriate for manifold learning. Indeed, it is instead targeted at solving PDEs on manifolds where the manifold structure is given explicitly. Based on this requirement we will not consider further comparison to the FEC. The DEC also makes strong use of a simplicial complex in their formulation and in the consistency results. It is conceivable that one could apply the DEC to an abstract simplicial complex based on an ϵ-ball or k-nearest neighbor construction, however there are no consistency results for such constructions.

Our third requirement is that the memory requirements should be decoupled from the data requirements, since data sets may be very large, rendering any method requiring memory that is even quadratic in the data impractical. In the DEC, discrete k-forms are encoded as weights on all combinations of k-neighbors of each data point. For a data set with N data points, each having ℓ neighbors, functions would be represented as $N \times 1$ vectors, 1-forms as $N \times \ell$ matrices, and general k-forms as $N \times \ell^k$ matrices. Thus, operators such that the k-Laplacian are represented as $N\ell^k \times N\ell^k$ matrices. The SEC provides an alternative which is much more memory efficient.

It has been shown that the error in the eigenfunctions of the 0-Laplacian is proportional to the eigenvalues [8], which by Weyl's law grow according to $\lambda_j \propto n^{2/d}$ where d is the dimension of the manifold. Moreover, for larger eigenvalues and eigenfunctions the error ultimately becomes quadratic in the eigenvalue. The idea of the SEC is to formulate the exterior calculus entirely in terms of the eigenfunctions of the 0-Laplacian, approximated through graph-theoretic kernel methods, and to discretize the exterior calculus by projecting onto the first $M \ll N$ eigenfunctions. Thus, functions would be represented at $M \times 1$ vectors, 1-forms as $M \times J$ matrices, and k-forms as $M \times J^k$. As we will explain in Sections 2.3 and 4.1, J is the number of eigenfunctions required to form an embedding of the manifold. Notice that highly redundant data sets may introduce extremely large values of N, but since M is decoupled from N this would not present a problem for the SEC. Also, high-dimensional manifolds which require a large data set N to obtain a small number M of accurate eigenfunctions, and the SEC could then proceed using only these accurate eigenfunctions potentially yielding very efficient representations of higher-dimensional manifolds. Another advantageous aspect of SEC representations is that their memory cost is independent of the ambient data space dimension

n (which can be very large in real-world applications). In fact, the only parts of the SEC framework with an n-dependent memory and computation cost are the initial graph-Laplacian construction and the spectral representation of the pushforward maps on vector fields, all of which depend linearly on n. In contrast, the cost of building simplicial complexes and other auxiliary constructs required by DEC and FEC approaches can be very high in large ambient space dimensions.

Finally, it is also desirable that a data-based exterior calculus should capture as much as possible of the structure of the exterior calculus from differential geometry, meaning that discrete analogs of continuous theorems should hold. While no method captures discrete analogs of all the continuous theorems, each method has some partial results. For example, the DEC beautifully captures a discrete analog of Stokes' theorem and the Leibniz rule for holds exactly closed forms, however the product rule for the Laplacian fails. In the SEC, the product rule for the Laplacian will hold exactly, however this leads inevitably to the failure of the Leibniz rule as shown in Appendix A.

The central challenge of the SEC approach is obtaining the representation of the exterior calculus in the spectral basis of eigenfunctions of the 0-Laplacian. In the next section we overview how vector fields, k-forms, and the central operators of the exterior calculus can all be represented spectrally. Since the gradients of these functions do not span the set of vector fields (otherwise every vector field would be a gradient field), we instead build a *frame* (overcomplete set) [17] consisting of products of Laplacian eigenfunctions and their gradient, and we represent vector fields in this frame. We proceed analogously for k-forms, using products of Laplacian eigenfunctions and k-fold wedge products of their differentials to construct frames.

We begin in Section 2 with an overview of the spectral exterior calculus approach, including the fundamental idea of our approach and tables which overview key formulas. Computation of the more complex formulas can be found in Appendix B. In Section 3, we briefly review the necessary background material and introduce our key definitions. Our central contribution to the theory of the exterior calculus is proving that our construction yields frames for L^2 and Sobolev spaces of vector fields and k-forms in Section 4. Then, in Section 5, we employ this frame representation to construct a Galerkin approximation scheme for the eigenvalue problem of the 1-Laplacian, which is shown to converge spectrally. Section 6 establishes the consistency of the data-driven SEC representation of the exterior calculus. In Section 7, we present numerical results demonstrating the consistency of the SEC on a suite of numerical examples involving orientable and non-orientable smooth manifolds, as well as the fractal attractor of the Lorenz 63 system. We conclude with a summary discussion and future perspectives in Section 8. A Matlab code reproducing the results in Section 7 is included as supplementary material.

2. Overview of the Spectral Exterior Calculus (SEC)

As mentioned above, many manifold learning techniques are based on the ability to approximate the Laplacian operator on a manifold (1) via a graph Laplacian (2) defined on a graph of discrete data points sampled from the manifold. When this convergence is spectral, we may use the eigendecomposition of an appropriately constructed graph Laplacian. In Section 2.1 below we briefly summarize the Diffusion Maps approach to the construction. The eigendecomposition from Diffusion Maps is the only input required to generate the entire SEC construction.

The SEC represents vector fields and differential forms using Laplacian eigenfunctions, and then reformulates the exterior calculus of Riemannian geometry in terms of these representations. This reformulation is described in Sections 2.2–2.5, and will be made rigorous in Sections 3 and 4. The motivation for this reformulation is that it allows us to define an exterior calculus using only the eigendecomposition of Laplacian. In other words, the eigendecomposition of the graph Laplacian provides all the necessary inputs to formulas which generate the entire exterior calculus formalism. Of course, this implies a "low-pass" or truncated representation, and in Sections 5 and 6 we will prove that these truncated representations converge in the limit, as the number of eigenvectors increases.

2.1. Summary of Diffusion Maps construction of the 0-Laplacian

Following the Diffusion Maps approach, we define a kernel matrix

$$K_{ij} = k_\epsilon(x_i, x_j) := \exp\left(-\frac{\|x_i - x_j\|^2}{4\epsilon}\right),$$

where $\{x_i\}_{i=1}^N \subset \mathcal{M} \subset \mathbb{R}^n$ is a data set sampled from the embedded manifold $\mathcal{M}$. We then normalize $\boldsymbol{K} = [K_{ij}]$ to a new kernel matrix $\hat{\boldsymbol{K}}$ to remove the sampling bias,

$$\hat{\boldsymbol{K}} = \boldsymbol{Q}^{-1}\boldsymbol{K}\boldsymbol{Q}^{-1}, \quad \boldsymbol{Q} = \operatorname{diag}[Q_{ii}], \quad Q_{ii} = \sum_{j=1}^N K_{ij},$$

and finally normalize $\hat{\boldsymbol{K}}$ into a Markov matrix $\boldsymbol{P}$,

$$\boldsymbol{P} = \boldsymbol{D}^{-1}\hat{\boldsymbol{K}}, \quad \boldsymbol{D} = \operatorname{diag}[D_{ii}], \quad D_{ii} = \sum_{j=1}^N \hat{K}_{ij}, \tag{5}$$

which approximates the heat semigroup $e^{-\epsilon\Delta_0}$; see [3] for details on this procedure. The normalized kernel matrix has the generalized eigendecomposition

$$\hat{\boldsymbol{K}} = \boldsymbol{D}\boldsymbol{\Phi}\boldsymbol{\Lambda}\boldsymbol{\Phi}^\top,$$

which can be computed by solving the fully symmetric generalized eigenvalue problem $\hat{\boldsymbol{K}}\vec{\phi} = \Lambda\boldsymbol{D}\vec{\phi}$. The eigenvalues satisfy $\Lambda = e^{-\epsilon\hat{\lambda}}$ where the values $\hat{\lambda} \geq 0$ approximate the eigenvalues of the Laplacian, so we define $\hat{\lambda} = -\log(\Lambda)/\epsilon$. An asymptotically equivalent (in the limit $N \to \infty$ and $\epsilon \to 0$) approach is to form the graph Laplacian $\boldsymbol{L} = \boldsymbol{D} - \hat{\boldsymbol{K}}$, and solve the generalized eigenvalue problem $\boldsymbol{L} = \boldsymbol{D}\boldsymbol{\Phi}\hat{\boldsymbol{\Lambda}}\boldsymbol{\Phi}^\top$. In either approach, we approximate the eigenfunctions $\Delta_0\phi_i = \lambda_i\phi_i$ of the Laplacian operator, and sort the columns of $\boldsymbol{\Phi}$ so that the eigenvalues are increasing. The diagonal matrix $\boldsymbol{D}$ represents the Riemannian L^2 inner product on the manifold, in the sense that if $\vec{f}_i = f(x_i)$ and $\vec{h}_i = h(x_i)$ are vector representations of (complex-valued) continuous functions, then

$$\vec{f}^\dagger \boldsymbol{D}\vec{h} \approx \int_{\mathcal{M}} f^*(x)h(x)\, d\mu(x) = \langle f, g\rangle_{L^2}$$

up to a constant proportionality factor, where $\dagger$ denotes complex-conjugate transpose, and μ is the Riemannian measure of the manifold. Thus, we can compute the generalized Fourier transform of the function f by

$$\hat{f} = \boldsymbol{\Phi}^\top \boldsymbol{D}\vec{f}, \qquad \hat{f}_j = \sum_{i=1}^N \phi_j(x_i)D_{ii}f(x_i) \approx \langle \phi_j, f\rangle_{L^2}.$$

We can then reconstruct the values of the function f on the data set by $\vec{f} = \boldsymbol{\Phi}\hat{f}$, which holds exactly since $\boldsymbol{\Phi}\boldsymbol{\Phi}^\top = \boldsymbol{D}^{-1}$. If a smaller number of eigenvectors are used, then $\boldsymbol{\Phi}\boldsymbol{\Phi}^\top$ is not full rank, and the result is a low-pass filter.

Note that in applications (including those presented in this paper), one is frequently interested in real-valued functions and self-adjoint operators, so complex conjugation is not included in L^2 inner products as above. However, applications with complex-valued functions can also be of interest (e.g., in dynamical systems modeling [15]), so in what follows we work with complex-valued functions to maintain generality.

2.2. Functions, multiplication, and the Riemannian metric

In Table 3, we show the basic elements of the exterior calculus and their SEC formulations. For example, functions are represented in the SEC by their generalized Fourier transform $\hat{f}_i = \langle f, \phi_i\rangle_{L^2}$, which is justified

Table 3: The SEC reformulation of the basic elements of the exterior calculus from Riemannian geometry.

Object	Symbolic	Spectral
Function	f	$\hat{f}_k = \langle \phi_k, f \rangle_{L^2}$
Laplacian	Δf	$\langle \phi_k, \Delta f \rangle_{L^2} = \lambda_k \hat{f}_k$
L^2 Inner Product	$\langle f, h \rangle_{L^2}$	$\sum_i \hat{f}_i^* \hat{h}_i$
Dirichlet Energy	$\langle f, \Delta f \rangle_{L^2} = \int_{\mathcal{M}} \|\mathrm{grad}\, f\|^2 \, d\mu$	$\sum_i \lambda_i |\hat{f}_i|^2$
Multiplication	$\phi_i \phi_j$	$c_{ijk} = \langle \phi_i \phi_j, \phi_k \rangle_{L^2}$
Function Product	fh	$\sum_{ij} c_{kij} \hat{f}_i^* \hat{h}_j$
Riemannian Metric	$g(\mathrm{grad}\, \phi_i, \mathrm{grad}\, \phi_j)$	$g_{kij} \equiv \langle g(\mathrm{grad}\, \phi_i, \mathrm{grad}\, \phi_j), \phi_k \rangle_{L^2}$
		$= \frac{1}{2}(\lambda_i + \lambda_j - \lambda_k) c_{kij}$
Gradient Field	$\mathrm{grad}\, f(h) = \mathrm{grad}\, f \cdot \mathrm{grad}\, h$	$\langle \phi_k, \mathrm{grad}\, f(h) \rangle_{L^2} = \sum_{ij} g_{kij} \hat{f}_i^* \hat{h}_j$
Exterior Derivative	$df(\mathrm{grad}\, h) = df \cdot dh$	$\sum_{ij} g_{kij} \hat{f}_i \hat{h}_j$
Vector Field (basis)	$v(f) = v \cdot \mathrm{grad}\, f$	$\sum_j v_{ij} \hat{f}_j$
Divergence	$\mathrm{div}\, v$	$\langle \phi_i, \mathrm{div}\, v \rangle_{L^2} = -v_{0i}$
Frame Elements	$b_{ij}(\phi_l) = \phi_i \,\mathrm{grad}\, \phi_j(\phi_l)$	$G_{ijkl} \equiv \langle b_{ij}(\phi_l), \phi_k \rangle_{L^2} = \sum_m c_{mik} g_{mjl}$
Vector Field (frame)	$v(f) = \sum_{ij} v^{ij} b_{ij}(f)$	$\langle \phi_k, v(f) \rangle_{L^2} = \sum_{ijl} G_{ijkl} v^{ij} \hat{f}_l$
Frame Elements	$b^{ij}(v) = b^i \, db^j(v)$	$\langle \phi_k, b^{ij}(v) \rangle_{L^2} = \sum_{nlm} c_{kmi} G_{nlmj} v^{nl}$
1-Forms (frame)	$\omega = \sum_{ij} \omega_{ij} b^{ij}$	$\langle \phi_k, \omega(v) \rangle_{L^2} = \sum_{ij} \omega_{ij} \langle \phi_k, b^{ij}(v) \rangle_{L^2}$

since the Hodge theorem shows that the eigenfunctions ϕ_i form a smooth orthonormal basis for square-integrable functions on the manifold. It should be noted that, as with all L^2 expansions, f may differ from the reconstructed function $\sum_i \hat{f}_i \phi_i$ on sets of measure zero. Similarly, all frame representations of vector fields and k-forms in Table 3 and the ensuing discussion should be interpreted in an L^2 sense.

The two key elements of Table 3 are the representation of function multiplication and the Riemannian metric.

First, function multiplication will be represented by the fully symmetric three-index tensor

$$c_{ijk} = \langle \phi_i \phi_j, \phi_k \rangle_{L^2}, \tag{6}$$

which will be a key building block of the SEC. Note that here we use the term "tensor" to represent a general multi-index object such as c_{ijk} derived from inner products of Laplacian eigenfunctions. While these objects are not geometrical tensors on the manifold, they nevertheless transform via familiar tensor laws under changes of L^2 basis preserving the Laplacian eigenspaces.

Next, the Riemannian metric is represented based on the product formula (3), and is given by (4). The power of the SEC is that we will only need to represent the metric for gradients of eigenfunctions $\mathrm{grad}\, \phi_i$ and $\mathrm{grad}\, \phi_j$, where we find that

$$g(\mathrm{grad}\, \phi_i, \mathrm{grad}\, \phi_j) = \frac{1}{2}(\phi_i \Delta \phi_j + \phi_j \Delta \phi_i - \Delta(\phi_i \phi_j)) = \frac{1}{2}((\lambda_i + \lambda_j)\phi_i \phi_j - \Delta(\phi_i \phi_j)).$$

We can further reduce this by writing the product $\phi_i \phi_j = \sum_k c_{ijk} \phi_k$, so that

$$g(\operatorname{grad} \phi_i, \operatorname{grad} \phi_j) = \frac{1}{2} \sum_k (\lambda_i + \lambda_j - \lambda_k) c_{ijk} \phi_k,$$

meaning that the k-th Fourier coefficient of the Riemannian metric is

$$g_{kij} \equiv \langle g(\operatorname{grad} \phi_i, \operatorname{grad} \phi_j), \phi_k \rangle_{L^2} = \frac{1}{2} (\lambda_i + \lambda_j - \lambda_k) c_{ijk}.$$

Notice that g_{kij} is symmetric in i and j but not in k. These first two simple formulas are the key to the SEC.

2.3. Vector fields

We will need two different ways of representing vector fields. The first method is called the operator representation and is based on the interpretation of a vector field as a map from smooth functions to smooth functions, defined by

$$v(f) = v \cdot \operatorname{grad} f = g(v, \operatorname{grad} f),$$

where g is the Riemannian metric. Since we have a smooth basis $\{\phi_i\}$ for functions, we can represent the vector field v in this basis by a matrix

$$v_{ij} = \langle \phi_i, v(\phi_j) \rangle_{L^2} = \langle \phi_i, v \cdot \operatorname{grad} \phi_j \rangle_{L^2_{\mathfrak{X}}} = \langle \phi_i \operatorname{grad} \phi_j, v \rangle_{L^2_{\mathfrak{X}}} = \langle b_{ij}, v \rangle_{L^2_{\mathfrak{X}}},$$

where the first two inner products appearing above are the L^2 inner products on functions, the last two inner products are the Hodge inner products induced on vector fields,

$$\langle v, w \rangle_{L^2_{\mathfrak{X}}} = \int_{\mathcal{M}} g(v^*, w) \, d\mu,$$

and $b_{ij} = \phi_i \operatorname{grad} \phi_j$ are smooth vector fields. Note that the Hodge inner product defines the space $L^2_{\mathfrak{X}}$ of square-integrable vector fields.

The second method of representing a vector field will be as a linear combination of the vector fields b_{ij} just introduced, with coefficients v^{ij} so that

$$v = \sum_{ij} v^{ij} b_{ij}.$$

As we will show in Section 4, the vector fields $\{b_{ij}\}$ where $i = 1, \ldots, \infty$ and $j = 1, \ldots, J < \infty$ spans the space of L^2 vector fields. However, instead of a basis, this set is only a *frame* for L^2. Since $\{b_{ij}\}$ is not a basis, this representation will generally not be unique, although frame theory ensures that there is a unique choice of coefficients v^{ij} which minimizes the ℓ^2 norm.

As we will see in the Section 2.5, there is a natural choice of basis for $L^2_{\mathfrak{X}}$, and constructing this basis will be a central goal of the SEC approach, however doing so requires using the frame $\{b_{ij}\}$. To motivate this choice of frame elements, note that given a fixed point x on the manifold and a sufficient (finite) number of eigenfunctions ϕ_j, the gradients of these eigenfunctions $\operatorname{grad} \phi_j(x)$ will span the tangent space $T_x \mathcal{M}$ (see Section 3 for details). In fact, for a d-dimensional compact manifold, we should be able to find d eigenfunctions whose gradients form a basis for $T_x \mathcal{M}$ for a fixed x. However, in general any choice of d eigenfunctions will not span $T_x \mathcal{M}$ for every x simultaneously, meaning that the choice of eigenfunctions which span depends on x. This is easily demonstrated by the example of the sphere S^2. That is, on S^2 every smooth vector field vanishes at some point $x \in S^2$, so at that point the collection of d gradient fields will at most span a $(d-1)$-dimensional subspace of $T_x S^2$. Intuitively, given a collection of sufficiently many gradients of eigenfunctions $\{\operatorname{grad} \phi_j\}_{j=1}^J$, and if the manifold is not too "large" (i.e., it is compact), we can span all the tangent spaces simultaneously with $J < \infty$, but of course we no longer have a basis. Given an

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arbitrary smooth vector field v, we can then represent v at each point $x \in \mathcal{M}$ as a linear combination of gradients of eigenfunctions,

$$v_x = \sum_{j=1}^{J} c_{v,j}(x) \operatorname{grad} \phi_j.$$

If we can choose the coefficients $c_{v,j}(x)$ in this linear combination to be smooth functions on the manifold, then these functions can be represented in the basis $\{\phi_i\}$ of eigenfunctions, so that

$$c_{v,j}(x) = \sum_{i=0}^{\infty} \langle \phi_i, c_{v,j} \rangle_{L^2} \phi_i(x),$$

which means that we can represent the vector field v as

$$v = \sum_{j=1}^{J} \sum_{i=0}^{\infty} \langle \phi_i, c_{v,j} \rangle_{L^2} \phi_i \operatorname{grad} \phi_j.$$

We now consider how to move between the operator representation and frame representation of a vector field. Substituting the frame representation into the operator representation, we find that

$$v_{ij} = \langle b_{ij}, v \rangle_{L_{\mathfrak{x}}^2} = \sum_{kl} v^{kl} \langle b_{kl}, b_{ij} \rangle_{L_{\mathfrak{x}}^2} = \sum_{kl} G_{ijkl} v^{kl}, \tag{7}$$

where $G_{ijkl} = \langle b_{ij}, b_{kl} \rangle_{L_{\mathfrak{x}}^2}$ is the Grammian matrix of the frame elements with respect to the Hodge inner product. Thus, we see that the Hodge Grammian is the linear transformation which maps from the frame representation v^{kl} to the matrix representation v_{ij}. Since the frame is overcomplete, the matrix G is necessarily rank deficient and thus there is no unique inverse transformation. However, if we also specify the minimum ℓ^2 norm then we can map from the matrix representation v_{ij} to the frame coefficients (with minimum norm) v^{kl} via the pseudo-inverse $\boldsymbol{G}^+$ of the Hodge Grammian.

2.4. Differential forms

In order to build a formulation of the exterior calculus we need to first move from vector fields to differential k-forms. First, 0-forms are equivalent to C^∞ functions defined on the manifold, which we represent in the basis of eigenfunctions of the Laplacian ϕ_i. Since each eigenfunction can also be thought of as a 0-form, we will sometimes also denote the eigenfunctions by

$$b^i = \phi_i,$$

since the superscript notation is oftentimes used for basis elements of spaces of differential forms. Our primary focus in this paper will be 1-forms, which are are duals to vector fields. That is, a 1-form takes in a vector field as its argument and returns a function. On a Riemannian manifold, we can move back and forth between vector fields and 1-forms with the $\sharp$ and $\flat$ operators. Locally, these operators map 1-forms and vector fields, respectively, to their Riesz representatives with respect to the Riemannian metric. In particular, if ω is a 1-form and v is a vector field, then

$$\omega(v) = g(\omega^\sharp, v) = g^{-1}(\omega, v^\flat).$$

A fundamental operator on differential forms is the exterior derivative, d, which maps k-forms to $(k+1)$-forms, so that the exterior derivative of a 0-form f is defined by the 1-form df, acting on a vector field v by

$$df(v) = v(f) = g(v, \operatorname{grad} f).$$

We will sometimes use the notation d_k to explicitly exhibit the order of differential forms on which a given exterior derivative acts.

Since 1-forms are dual to vector fields, we will use a similar frame representation to that in Section 2.3, based on the eigenfunctions, $\{b^{ij} = b^i\, db^j\}$. As we will show in Section 4, these 1-forms span the space L_1^2 of square-integrable 1-forms. We also note that the Riemannian metric lifts to k-forms (see Section 3 for details), and takes two k-forms and returns a function. Integrating the Riemannian inner product of two k-forms,

$$\langle \psi, \omega \rangle_{L_k^2} = \int_{\mathcal{M}} g^{-1}(\psi^*, \omega)\, d\mu,$$

defines the Hodge inner product, which then defines the Hilbert space of square integrable k-forms. Finally, since $df^\sharp = \operatorname{grad} f$ and $\operatorname{grad} f^\flat = df$, we have $b_{ij}^\flat = b^{ij}$ and $(b^{ij})^\sharp = b_{ij}$, so the coefficients of a vector field in the frame representation can also be used to represent the corresponding 1-form and vice versa.

2.5. The Laplacian on forms

The Laplacian k-forms is defined via the exterior derivative d and its Hodge dual, the codifferential δ, by

$$\Delta_k = d_{k-1}\delta_k + \delta_{k+1}d_k$$

In order to represent the eigenvalue problem for the operator Δ_1 in the frame $\{b^{ij}\}$, we need to compute the inner products

$$G_{ijkl} = \left\langle b^{ij}, b^{kl} \right\rangle_{L_1^2}, \qquad E_{ijkl} = \left\langle b^{ij}, \Delta_1 b^{kl} \right\rangle_{L_1^2} = \langle db^{ij}, db^{kl} \rangle_{L_2^2} + \langle \delta b^{ij}, \delta b^{kl} \rangle_{L^2}, \tag{8}$$

representing the Gramm matrix of Hodge inner products (which we will call the Hodge Grammian) and Dirichlet form matrix, respectively. We derive the expressions for these tensors in Appendix B below, and the formulas are summarized in Table 4. Note that both $\boldsymbol{G}$ and $\boldsymbol{E}$ can be written as symmetric matrices by numbering the frame elements. Moreover, we can easily represent the Gramm matrix with respect to the Sobolev H^1 inner product on 1-forms,

$$\langle \psi, \omega \rangle_{H_1^1} = \langle \psi, \omega \rangle_{L_1^2} + E_{1,1}(\psi, \omega), \quad E_{1,1}(\psi, \omega) = \langle d\omega, d\nu \rangle_{L_2^2} + \langle \delta\omega, \delta\nu \rangle_{L^2},$$

as

$$\boldsymbol{G}^1 = \boldsymbol{G} + \boldsymbol{E}.$$

The importance of the Sobolev Grammian $\boldsymbol{G}^1$ is that H_1^1 is a natural domain for weak (variational) formulations of the eigenvalue problem of the Laplacian on 1-forms.

As we will show in Section 5, the key to solving the eigenproblem of the Laplacian on 1-forms is to first express the problem in a weak sense, i.e., replace $\Delta_1 \varphi = \nu\varphi$ by

$$E_{1,1}(\psi, \varphi) = \nu\langle \psi, \varphi \rangle_{L_1^2}, \quad \forall \psi \in H_1^1,$$

which is equivalent to the minimization problem

$$\nu = \min_{\varphi \in H_1^1 \setminus \{0\}} \left\{ \frac{E_{1,1}(\varphi, \varphi)}{\langle \varphi, \varphi \rangle_{L_1^2}} \right\}.$$

Intuitively, the ratio $E_{1,1}(\varphi, \varphi)/\langle \varphi, \varphi \rangle_{H_1^1}$ is a measure of "roughness", or oscillatory behavior, of a given eigenform φ, much like the eigenvalues of the 0-Laplacian measure the roughness of the corresponding eigenfunctions. Thus, ordering eigenforms in order of increasing eigenvalue, as we will always do by convention, is tantamount to ordering them in order of increasing complexity that they exhibit on the manifold. As with functions, given finite amounts of data, the approximation error for eigenforms increases with the corresponding eigenvalue.

In SEC, we represent the eigenform in the frame, $\varphi = \sum_{ij} \varphi_{ij} b^{ij}$. The above variational problem can then in principle be written in matrix form as

$$E_{1,1}(\psi, \varphi) = \nu\langle \psi, \varphi \rangle_{L_1^2}, \quad \forall \psi \in H_1^1, \qquad \implies \qquad \boldsymbol{E}\vec{\varphi} = \nu\boldsymbol{G}\vec{\varphi}.$$

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Table 4: The SEC representation of the Laplacian on 1-forms and the Dirichlet and Sobolev energy forms. Derivations can be found in Appendix B. Pairs of integers in parentheses indicate symmetries under permutations of tensor indices; e.g., $(1,2)$ in c^n_{ijkl} indicates that $c^n_{ijkl} = c^n_{jikl}$.

Operator	Tensor	Symmetries
Quadruple Product	$c^0_{ijkl} = \langle \phi_i \phi_j, \phi_k \phi_l \rangle_{L^2} = \sum_s c_{ijs} c_{skl}$	Fully symmetric
Product Energy	$c^p_{ijkl} = \langle \Delta^p(\phi_i \phi_j), \phi_k \phi_l \rangle_{L^2} = \sum_s \lambda^p_s c_{ijs} c_{skl}$	(1,2), (3,4), (1,3), (2,4)
Hodge Grammian	$G_{ijkl} = \langle b^{ij}, b^{kl} \rangle_{L^2_1} = \frac{1}{2} \big[(\lambda_j + \lambda_l) c^0_{ijkl} - c^1_{ijkl} \big]$	(1,3), (2,4)
Antisymmetric	$\hat{G}_{ijkl} = \langle \hat{b}^{ij}, \hat{b}^{kl} \rangle_{L^2_1} = G_{ijkl} + G_{jilk} - G_{jikl} - G_{ijlk}$	(1,3), (2,4)
Dirichlet Energy	$E_{ijkl} = \frac{1}{4} \big[(\lambda_i + \lambda_j + \lambda_k + \lambda_l)(c^1_{iljk} - c^1_{ikjl})$	(1,3), (2,4)
$E_{ijkl} = \langle b^{ij}, \Delta_1(b^{kl}) \rangle_{L^2_1}$	$+ (\lambda_j + \lambda_l - \lambda_i - \lambda_k) c^2_{ijkl} + (c^2_{ijkl} + c^2_{ikjl} - c^2_{iljk}) \big]$	
Antisymmetric	$\hat{E}_{ijkl} = \langle \hat{b}^{ij}, \Delta_1 \hat{b}^{kl} \rangle_{L^2_1}$	(1,3), (2,4)
	$= (\lambda_i + \lambda_j + \lambda_k + \lambda_l)(c^1_{iljk} - c^1_{ikjl}) + (c^2_{ikjl} - c^2_{iljk})$	
Sobolev H^1 Grammian	$G^1_{ijkl} = E_{ijkl} + G_{ijkl}, \ \ \hat{G}^1_{ijkl} = \hat{E}_{ijkl} + \hat{G}_{ijkl}$	(1,3), (2,4)

However, the above eigenvector problem is not well conditioned because $\boldsymbol{G}$ is not full rank in general (since the frame is overcomplete, meaning there can be multiple representations of the same 1-form). In order to find an appropriate basis, we first solve the eigenvalue problem for the Sobolev Grammian,

$$\boldsymbol{G}^1 = \boldsymbol{U} \boldsymbol{H} \boldsymbol{U}^\top, \qquad \boldsymbol{H} = \text{diag}[h_{ii}],$$

and select the columns $\tilde{\boldsymbol{U}}$ of $\boldsymbol{U}$ corresponding to the largest eigenvalues. For example, in our implementation we choose $h_{ii} > h_{11} \times 10^{-3}$. Notice that the columns of $\tilde{\boldsymbol{U}}$ contain the frame coefficients of unique orthogonal 1-forms. In other words, the matrix $\tilde{\boldsymbol{U}}$ is a choice of basis for H^1_1 represented in the frame. Thus, we can project the eigenvalue problem onto this basis by writing

$$\boldsymbol{L} \vec{w} = \nu \boldsymbol{B} \vec{w}, \qquad \boldsymbol{L} = \tilde{\boldsymbol{U}}^\top \boldsymbol{E} \tilde{\boldsymbol{U}}, \qquad \boldsymbol{B} = \tilde{\boldsymbol{U}}^\top \boldsymbol{G} \tilde{\boldsymbol{U}}. \tag{9}$$

An eigenvector $\vec{w}$ of this generalized eigenvalue problem contains the coefficients of a frame representation for an eigenform φ of Δ_1.

We should note that in practice we found somewhat better results using the antisymmetric elements $\hat{b}^{ij} = b^i \, db^j - b^j \, db^i$, likely due to the fact that these forms are less redundant. All of the formulas for the antisymmetric formulation of the 1-Laplacian are given in Table 4. The one change in the antisymmetric formulation is that in order to move from the frame representation to the operator representation, we need the additional tensor

$$H_{ijkl} = \left\langle \phi_k, \hat{b}^{ij}(\text{grad} \, \phi_l) \right\rangle_{L^2} = \left\langle \phi_k, b^i \, db^j(\text{grad} \, \phi_l) \right\rangle_{L^2} - \left\langle \phi_k, b^j \, db^i(\text{grad} \, \phi_l) \right\rangle_{L^2} = G_{ijkl} - G_{jikl}.$$

With this tensor, given the frame representation of a 1-form $\varphi = \sum_{ij} \hat{\varphi}_{ij} \hat{b}^{ij}$, the operator representation of the corresponding vector field $v = \varphi^\sharp$ becomes $v_{kl} = \sum_{ij} H_{ijkl} \varphi_{ij}$.

In order to visualize these eigenforms, we will visualize the corresponding vector field, $v = \varphi^\sharp$, which has the same frame coefficients as shown in Section 2.4. In particular, it follows from (7) that simply multiplying $\vec{w}$ by the matrix $\boldsymbol{G}$, leads to $\boldsymbol{G}\vec{w}$, which contains the operator representation of v. By reshaping $\boldsymbol{G}\vec{w}$ into a $M \times M$ matrix $\boldsymbol{V}$, we have $V_{ij} = \langle \phi_i, v(\phi_j) \rangle_{L^2}$. To visualize this vector field, we need to map it back into the original data coordinates. This "pushforward" operation on vector fields can also be represented spectrally

([15], Proposition 6). In particular, let $\boldsymbol{X}$ be the $n \times N$ matrix of N data points in $\mathbb{R}^n$. We first compute the Fourier transform of these coordinates by computing the $\boldsymbol{D}$ inner product with the $N \times M$ matrix $\boldsymbol{\Phi}$ (see Section 2.1). Thus, $\hat{\boldsymbol{X}} = \boldsymbol{X}\boldsymbol{D}\boldsymbol{\Phi}$ is the $n \times M$ matrix containing the M Fourier coefficients of each of the n coordinate functions. We can now apply the vector field to each of these functions by multiplying $\hat{\boldsymbol{X}}\boldsymbol{V}^\top$, which now contains the Fourier coefficients of the pushforward of the coordinate functions. Finally, we can reconstruct the coordinates of the arrows by computing the inverse Fourier transform $\tilde{\boldsymbol{V}} = \hat{\boldsymbol{X}}\boldsymbol{V}^\top\boldsymbol{\Phi}^\top$, which is a $n \times N$ matrix containing the n-dimensional vectors which can plotted at each data point. This method is used to visualize the SEC eigenforms in Section 7.

3. Hilbert spaces and operators in the exterior calculus

Consider a closed (compact and without boundary), smooth, orientable, d-dimensional manifold $\mathcal{M}$, equipped with a smooth Riemannian metric g. Without loss of generality, we assume that g is normalized such that its associated Riemannian measure, μ, satisfies $\mu(\mathcal{M}) = 1$. We denote the associated metric tensor on dual vectors by $\eta = g^{-1}$, and use the notation $v^\flat = g(v, \cdot)$, $v \in T_x\mathcal{M}$, and $\alpha^\sharp = \eta(\alpha, \cdot)$, $\alpha \in T_x^*\mathcal{M}$, for the Riemannian duals of tangent vectors and dual vectors at a point $x \in \mathcal{M}$, respectively. In what follows, we introduce the spaces of functions, vector fields, and differential forms that will be employed in the SEC framework.

3.1. Function spaces

Let $\Delta : C^\infty(\mathcal{M}) \mapsto C^\infty(\mathcal{M})$ be the (positive-semidefinite) Laplace-Beltrami operator on smooth, complex-valued functions associated with the Riemannian metric g. It is a fundamental result in analysis on closed Riemannian manifolds (e.g., [18–20] that Δ extends to a unique self-adjoint operator $\bar{\Delta} : D(\bar{\Delta}) \mapsto L^2(\mathcal{M}, \mu)$ with a dense domain $D(\bar{\Delta}) \subset L^2(\mathcal{M}, \mu)$ in the L^2 space associated with the Riemannian measure, and a pure point spectrum of eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ with no accumulation points and an associated smooth orthonormal basis $\{\phi_j\}_{j=0}^\infty$ consisting of corresponding eigenfunctions. By smoothness of the Laplace-Beltrami eigenfunctions, the products $\phi_i\phi_j$ are in $L^2(\mathcal{M}, \mu)$; thus, we have

$$\phi_i\phi_j = \sum_{k=0}^\infty c_{ijk}\phi_k, \quad c_{ijk} = \langle \phi_k, \phi_i\phi_j \rangle_{L^2(\mathcal{M},\mu)}, \tag{10}$$

where the limit in the first equation is taken in the L^2 sense. As discussed in Section 2, our objective is to build a framework for tensor calculus on $\mathcal{M}$ that is defined entirely through the spectral properties of the Laplacian on functions, encoded in the eigenvalues λ_j, the corresponding eigenfunctions ϕ_j, and the coefficients c_{ijk} representing the algebraic relationships between the eigenfunctions.

We use the notation L^p, $1 \leq p \leq \infty$, to represent the standard Banach spaces of complex-valued functions on $\mathcal{M}$ associated with the Riemannian measure μ, equipped with the standard norms, $\|\cdot\|_{L^p}$. In the case $p = 2$, we use the shorthand notation $H = L^2$, and denote the corresponding Hilbert space inner product $\langle \cdot, \cdot \rangle_H$, which we take to be conjugate-linear on its first argument. We also consider Sobolev spaces of higher regularity, defined for $p \geq 0$ by

$$H^p = \left\{ \sum_{j=0}^\infty c_j\phi_j \in L^2 : \sum_{j=0}^\infty \lambda_j^p |c_j|^2 < \infty \right\}. \tag{11}$$

These spaces are closed with respect to the norms $\|f\|_{H^p} = \langle f, f \rangle_{H^p}^{1/2}$ associated with the inner products

$$\langle f, h \rangle_{H^p} = \sum_{q=0}^p \sum_{j=0}^\infty \lambda_j^q \hat{f}_j^* \hat{h}_j, \quad \hat{f}_j = \langle \phi_j, f \rangle_{L^2}, \quad \hat{h}_j = \langle \phi_j, h \rangle_{L^2}. \tag{12}$$

Among these, the space H^2 is precisely the domain of the self-adjoint Laplacian $\bar{\Delta}$.

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We equip each H^p space with a Dirichlet form $E_p : H^p \times H^p \mapsto \mathbb{C}$, defined as the bounded sesquilinear form

$$E_p(f, h) = \sum_{j=0}^{\infty} \lambda_j^p \hat{f}_j^* \hat{h}_j, \tag{13}$$

with f and h as in (12). This form induces a positive-semidefinite Dirichlet energy functional $E_p(f) = E_p(f, f)$. Given $f \in H^p \cap H$, the quantity $E_p(f)/\|f\|_H^2$ can be thought of as a measure of roughness of f. If f and h are smooth, $E_p(f, h)$ can be expressed in terms of the Laplace-Beltrami operator as $E_p(f, h) = \langle f, \Delta^p h \rangle_H$. Evidently, $E_p(f) = \langle f, \Delta^p f \rangle_H$ can be arbitrarily large for highly oscillatory functions.

In general, the $\{\phi_j\}$ basis of H is not a basis of H^p for $p \geq 1$; this is manifested from the fact that the Dirichlet energies $E_p(\phi_j) = \lambda_j^p$ of the basis elements are unbounded in j. On the other hand, the normalized eigenfunctions $\phi_j^{(p)}$, defined by

$$\phi_j^{(p)} = \frac{\phi_j}{\|\phi_j\|_{H^p}}, \tag{14}$$

where $\|\phi_j^{(p)}\|_{H^p} = 1$ by construction, form orthonormal bases of the respective H^p spaces.

3.2. Spaces of vector fields

We consider the space $\mathfrak{X}$ of C^∞ complex vector fields on $\mathcal{M}$ (that is, the space of derivations on the ring of smooth, complex-valued functions on $\mathcal{M}$), where we recall that $\mathfrak{X}$ can be viewed either as a vector space over the field of complex numbers, or as a $C^\infty(\mathcal{M})$-module. In the former case, it can be endowed with the structure of a Lie algebra with the vector field commutator, $[\cdot, \cdot] : \mathfrak{X} \times \mathfrak{X} \mapsto \mathfrak{X}$ acting as the algebraic product. We denote the gradient and divergence operators associated with g by $\mathrm{grad} : C^\infty(\mathcal{M}) \mapsto \mathfrak{X}$ and $\mathrm{div} : \mathfrak{X} \mapsto C^\infty(\mathcal{M})$. Note that these operators are related to the positive-semidefinite Laplacian via $\Delta = -\mathrm{div} \circ \mathrm{grad}$. As with functions, we consider the standard Banach spaces $L_{\mathfrak{X}}^p$, $1 \leq p \leq \infty$, of vector fields associated with the norms

$$\|v\|_{L_{\mathfrak{X}}^p} = \left(\int_{\mathcal{M}} (g(v^*, v))^{p/2} \, d\mu \right)^{1/p}, \quad 1 \leq p < \infty, \quad \text{and} \quad \|v\|_{L_{\mathfrak{X}}^\infty} = \underset{x \in \mathcal{M}}{\mathrm{ess\,sup}} \sqrt{g(v^*, v)_x},$$

where v^* is the complex-conjugate vector field to v. In the case $p = 2$, we set $H_{\mathfrak{X}} = L_{\mathfrak{X}}^2$ and use the notation $\langle v, w \rangle_{H_{\mathfrak{X}}} = \int_{\mathcal{M}} g(v^*, w) \, d\mu$ for the Hodge inner product inducing the $L_{\mathfrak{X}}^2$ norm.

With these definitions, the closure $\overline{\mathrm{grad}} : D(\overline{\mathrm{grad}}) \mapsto H_{\mathfrak{X}}$ of the gradient operator has domain $D(\overline{\mathrm{grad}}) = H^1$, and is bounded as an operator from H^1 to $H_{\mathfrak{X}}$. Similarly, the closure $\overline{\mathrm{div}} : D(\overline{\mathrm{div}}) \mapsto H_{\mathfrak{X}}$ of the divergence operator has as its domain $D(\overline{\mathrm{div}}) \subset H_{\mathfrak{X}}$ the Sobolev space $H_{\mathfrak{X},\mathrm{div}}$, which is defined as the closure of $\mathfrak{X}$ with respect to the norm $\|v\|_{H_{\mathfrak{X},\mathrm{div}}} = \langle v, v \rangle_{H_{\mathfrak{X},\mathrm{div}}}^{1/2}$ induced by the inner product

$$\langle v, w \rangle_{H_{\mathfrak{X}}^1} = \langle v, w \rangle_{H_{\mathfrak{X}}} + \langle \mathrm{div}\, v, \mathrm{div}\, w \rangle_H.$$

Also, for $p \geq 0$, we introduce the Sobolev spaces

$$H_{\mathfrak{X},\mathrm{div}}^p = \left\{ v \in H_{\mathfrak{X}} : \overline{\mathrm{div}}\, v \in H^p \right\},$$

which are equipped with the inner products

$$\langle v, w \rangle_{H_{\mathfrak{X},\mathrm{div}}^p} = \langle v, w \rangle_{H_{\mathfrak{X}}} + \langle \overline{\mathrm{div}}\, v, \overline{\mathrm{div}}\, w \rangle_{H^p},$$

and the corresponding norms $\|v\|_{H_{\mathfrak{X},\mathrm{div}}^p} = \langle v, v \rangle_{H_{\mathfrak{X},\mathrm{div}}^p}^{1/2}$. As in the case of functions, we define the Dirichlet forms $E_{p,\mathfrak{X},\mathrm{div}} : H_{\mathfrak{X},\mathrm{div}}^p \times H_{\mathfrak{X},\mathrm{div}}^p \mapsto \mathbb{C}$ by

$$E_{p,\mathfrak{X},\mathrm{div}}(u, v) = E_p(\overline{\mathrm{div}}\, u, \overline{\mathrm{div}}\, v).$$

The corresponding energy functionals, $E_{p,\mathfrak{X},\mathrm{div}}(f) = E_{p,\mathfrak{X},\mathrm{div}}(f,f)$ assign measures of roughness of vector fields in $H^p_{\mathfrak{X},\mathrm{div}} \cap H_{\mathfrak{X}}$ through the quantities $E_{p,\mathfrak{X},\mathrm{div}}(f)/\|f\|^2_{H_{\mathfrak{X}}}$.

An important subspace of $H_{\mathfrak{X}}$ is the closed subspace of gradient vector fields, $H_{\mathfrak{X},\mathrm{grad}} = \overline{\mathrm{ran}(\mathrm{grad})}$. This leads to the orthogonal decomposition $H_{\mathfrak{X}} = H_{\mathfrak{X},\mathrm{grad}} \oplus H^\perp_{\mathfrak{X},\mathrm{grad}}$, and it can be readily checked that any vector field in $H^\perp_{\mathfrak{X},\mathrm{grad}} \cap H_{\mathfrak{X},\mathrm{div}}$ has vanishing divergence. A natural smooth orthonormal basis $\{u_j\}^\infty_{j=1}$ for $H_{\mathfrak{X},\mathrm{grad}}$ is given by the normalized gradients of the Laplace-Beltrami eigenfunctions,

$$u_j = \mathrm{grad}\,\phi_j/\lambda_j^{1/2}. \tag{15}$$

The following two lemmas characterize the behavior of vector fields as operators on functions.

Lemma 1 (vector fields as conjugate-antisymmetric operators). *To every vector field $v \in H^1_{\mathfrak{X},\mathrm{div}}$ there corresponds a unique operator $A_v : C^\infty(\mathcal{M}) \mapsto H$ with the property*

$$\langle f, A_v h\rangle_H = -\langle A_{v*}f, h\rangle_H, \quad \forall f, h \in C^\infty(\mathcal{M}). \tag{16}$$

This operator is given by $A_v = v + D_v$, where $D_v : C^\infty(\mathcal{M}) \mapsto L^2$ is defined as $D_v f = \mathrm{div}(fv)$, and we also have

$$v(f) = \frac{A_v f - f A_v 1}{2}. \tag{17}$$

Proof. It follows from the Leibniz rule for the divergence, $\mathrm{div}(fv) = v(f) + f\,\mathrm{div}\,v$, and the fact that $\int_{\mathcal{M}} \mathrm{div}(fv)\,d\mu$ vanishes on closed manifolds that

$$\langle f, v(h)\rangle_H = -\langle D_{v*}f, h\rangle_H.$$

The claim in (16) follows from the definition of A_v and the last equation. Note that the restriction $v \in H^1_{\mathfrak{X},\mathrm{div}}$ is important for the well-definition of D_v and A_v. To show that A_v is unique, suppose that $u \in H^1_{\mathfrak{X},\mathrm{div}}$, $u \neq v$, and consider $(A_u - A_v)f = 2(u-v)f + f\,\mathrm{div}(u-v)$. If $\mathrm{div}\,u = \mathrm{div}\,v$, then $(A_u - A_v)f = 2(u-v)f$, which is non-vanishing for some $f \in C^\infty(\mathcal{M})$. On the other hand, if $\mathrm{div}\,u \neq \mathrm{div}\,v$, we have $(A_u - A_v)1 = \mathrm{div}(u-v)$, which is again non-vanishing. Equation (17) follows from the definition of A_v and the fact that $A_v 1 = \mathrm{div}\,v$. $\square$

Lemma 2 (vector fields as bounded operators). *Every vector field $v \in L^\infty_{\mathfrak{X}}$ extends uniquely to a bounded operator $L_v : H^1 \mapsto H$ with operator norm $\|L_v\| \leq \|v\|_{L^\infty_{\mathfrak{X}}}$.*

Proof. Consider a vector field $v \in L^\infty_{\mathfrak{X}}$, and let f be a C^∞ function. Then, we have

$$\|v(f)\|^2 = \int_{\mathcal{M}} |g(v, \mathrm{grad}\,f)|^2\,d\mu \leq \int_{\mathcal{M}} g(v^*, v)g(\mathrm{grad}\,f^*, \mathrm{grad}\,f)\,d\mu \leq \|v\|^2_{L^\infty_{\mathfrak{X}}}\|\mathrm{grad}\,f\|^2_{L^2_{\mathfrak{X}}} \leq \|v\|^2_{L^\infty_{\mathfrak{X}}}\|f\|^2_{H^1},$$

where the first inequality in the above follows from the Cauchy-Schwarz inequality. Thus, v is a densely-defined, bounded operator from $C^\infty(\mathcal{M}) \cap H^1$ to H, and thus can be uniquely extended to $L_v : H^1 \mapsto L^2$ by the bounded linear transformation theorem. The fact that $\|L_v\| \leq \|v\|_{L^\infty_{\mathfrak{X}}}$ follows from the inequality $\|v(f)\|/\|f\|_{H^1} \leq \|v\|_{L^\infty_{\mathfrak{X}}}$. $\square$

An implication of Lemma 2 is that for any $v \in L^\infty_{\mathfrak{X}}$ and every sequence $f_n \in C^\infty(\mathcal{M})$, converging to f in H^1 norm, $v(f) = \lim_{n\to\infty} v(f_n)$ despite that v is unbounded (and therefore discontinuous) on $C^\infty(\mathcal{M})$. It also follows from Lemma 2 that the operator A_v in Lemma 1 associated with $v \in H^1_{\mathfrak{X},\mathrm{div}} \cap L^\infty_{\mathfrak{X}}$ also extends uniquely to a bounded operator $\tilde{A}_v : H^1 \mapsto H$.

Next, as discussed in Section 2.2, we introduce a spectral representation of pointwise Riemannian inner products between gradient vector fields. For that, we first consider the product rule for the positive-definite Laplacian on smooth functions,

$$\Delta(fh) = (\Delta f)h + f(\Delta h) - 2g(\mathrm{grad}\,f, \mathrm{grad}\,h), \quad f, h \in C^\infty(\mathcal{M}). \tag{18}$$

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It follows by definition of the H^2 norms that the self-adjoint Laplacian $\bar{\Delta}$ is bounded as an operator from H^2 to H. As a result, given a sequence $f_j \in C^\infty(\mathcal{M})$ converging to f in H^2 norm, we have

$$\bar{\Delta} f = \bar{\Delta} \left(\lim_{j \to \infty} f_j \right) = \lim_{j \to \infty} \Delta f_j, \tag{19}$$

despite the fact that Δ is unbounded as an operator on $C^\infty(\mathcal{M})$. Now, the fact that the Laplace-Beltrami eigenfunctions ϕ_j are smooth implies that given any $f \in C^\infty(\mathcal{M})$, the sequence $f_0, f_1, \dots$ with

$$f_j = \sum_{i=0}^{j} \hat{f}_i \phi_i, \quad \hat{f}_i = \langle \phi_i, f \rangle_H$$

is Cauchy in H^p for all $p \geq 0$, and hence (19) holds. As a result, we can use (18) in conjunction with (10) to obtain:

Lemma 3 (spectral representation of Riemannian inner products). *The Riemannian inner product between the gradient vector fields associated with two smooth functions $f, h \in C^\infty(\mathcal{M})$ can be expressed as*

$$g(\operatorname{grad} f, \operatorname{grad} h) = \frac{1}{2} \sum_{l=0}^{\infty} \hat{f}_j \hat{h}_k (\lambda_j + \lambda_k - \lambda_l) c_{jkl} \phi_l, \quad \hat{f}_j = \langle \phi_j, f \rangle_H, \quad \hat{h}_k = \langle \phi_k, h \rangle_H,$$

where the sum over l in the right-hand side converges in H norm.

3.3. Spaces of differential forms

We will use the symbols $\Lambda_x^k \mathcal{M}$, $\Lambda^k \mathcal{M}$, and Ω^k to represent the vector space of k-forms at $x \in \mathcal{M}$, the associated k-form bundle, and the space of smooth k-form fields on $\mathcal{M}$ (totally antisymmetric, k-multilinear maps on $\mathfrak{X}^k$, taking values in $C^\infty(\mathcal{M})$). As with vector fields, the spaces Ω^k can be viewed either as vector spaces over $\mathbb{C}$, or as $C^\infty(\mathcal{M})$-modules. As usual, we identify Ω^0 with $C^\infty(\mathcal{M})$. We also let $\eta_x^k : \Lambda_x^k \mathcal{M} \times \Lambda_x^k \mathcal{M} \mapsto \mathbb{R}$ be the canonical metric tensor on Λ_x^k, satisfying

$$\eta_x^k(\alpha^1 \wedge \cdots \wedge \alpha^k, \beta^1 \wedge \cdots \wedge \beta^k) = \det[\eta_x(\alpha^i, \beta^j)]_{ij}, \quad \forall \alpha^i, \beta^j \in T_x^* \mathcal{M}.$$

The metric induces a Hodge star operator $\star : \Lambda_x^k \mathcal{M} \mapsto \Lambda_x^{d-k} \mathcal{M}$, defined uniquely through the requirement that

$$\alpha \wedge \star \beta = \eta_x^k(\alpha, \beta) \mu, \quad \forall \alpha, \beta \in \Lambda_x^k \mathcal{M}.$$

The Hodge star has the useful property

$$\star \star \alpha = (-1)^{k(d-k)} \alpha, \quad \forall \alpha \in \Lambda_x^k \mathcal{M}. \tag{20}$$

As in the case of vector fields, we introduce the Banach spaces L_k^p, $1 \leq p \leq \infty$, defined as the completion of Ω^k with respect to the norms

$$\|\alpha\|_{L_k^p} = \left(\int_{\mathcal{M}} \eta^k(\alpha^*, \alpha)^{p/2} \, d\mu \right)^{1/p}, \quad 1 \leq p \leq \infty, \quad \text{and} \quad \|\alpha\|_{L_k^\infty} = \operatorname{ess\,sup}_{x \in \mathcal{M}} \sqrt{\eta^k(\alpha, \alpha)_x},$$

where α^* is the complex conjugate form to $\alpha \in \Omega^k$. The case $p = 2$ is a Hilbert space, $H_k = L_k^2$, with norm $\|\cdot\|_{H_k} = \|\cdot\|_{L_k^2}$ induced from the inner product

$$\langle \alpha, \beta \rangle_{H_k} = \int_{\mathcal{M}} \alpha^* \wedge \star \beta = \int_{\mathcal{M}} \eta^k(\alpha^*, \beta) \, d\mu.$$

A fundamental aspect of the spaces Ω^k is that they are linked by the exterior derivative and codifferential operators, $d_k : \Omega^k \mapsto \Omega^{k+1}$ and $\delta_k : \Omega^k \mapsto \Omega^{k-1}$, respectively. We recall that $d_0, \dots, d_{d-1}$ are the unique linear maps with the properties:

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1. d_0 is the differential of functions.
2. $d_{k+1}d_k = 0$.
3. The Leibniz rule,

$$d_{k+l}(\alpha \wedge \beta) = d_k\alpha \wedge \beta + (-1)^k\alpha \wedge d_l\beta, \tag{21}$$

holds for all $\alpha \in \Omega^k$ and $\beta \in \Omega^l$.

The codifferential operators are defined uniquely through the requirement that

$$\langle \alpha, \delta_k\beta \rangle_{H_{k-1}} = \langle d_{k-1}\alpha, \beta \rangle_{H_k};$$

i.e., δ_k is a formal adjoint of d_{k-1}. This definition of δ_k is equivalent to

$$\delta_k\alpha = (-1)^{d(k+1)+1} \star d_{d-k} \star \alpha, \tag{22}$$

and it also implies $\delta_k\delta_{k-1} = 0$. In the case $k = 1$, the codifferential operator is related with the divergence on vector fields via $\operatorname{div} v = -\delta_1 v^\flat$, $v \in \mathfrak{X}$. Note that despite its relationship with the exterior derivative in (22), the codifferential does not satisfy a Leibniz rule.

Another important class of operators on differential forms are the interior product and Lie derivative associated with vector fields. Given a vector field $v \in \mathfrak{X}$, these are defined as the maps $\iota_v : \Omega^k \mapsto \Omega^{k-1}$ and $\mathcal{L}_v : \Omega^k \mapsto \Omega^k$, respectively, such that

$$\iota_v\alpha = \alpha(v, \cdot), \quad \mathcal{L}_v = d\iota_v + \iota_v d.$$

Both ι_v and $\mathcal{L}_v$ satisfy Leibniz rules,

$$\iota_v(\alpha \wedge \beta) = (\iota_v\alpha) \wedge \beta + (-1)^k\alpha \wedge (\iota_v\beta), \quad \mathcal{L}_v(\alpha \wedge \beta) = (\mathcal{L}_v\alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_v), \quad \forall\alpha \in \Omega^k, \quad \forall\beta \in \Omega^l.$$

Moreover, they have the properties

$$\mathcal{L}_v d = d\mathcal{L}_v, \quad v^\flat \wedge \star\alpha = (-1)^k \star \iota_v\alpha. \tag{23}$$

The following lemma can be viewed as a generalization of Lemma 2 to spaces of differential forms.

Lemma 4. *For every $v \in \mathfrak{X}$, the operators $\iota_v d$, $d\iota_v$, and $\mathcal{L}_v$ extend to unique bounded operators $D_v : H_k^1 \mapsto H_k$, $\tilde{D}_v : H_k^1 \mapsto H_k$, and $L_v : H_k^1 \mapsto H_k$, respectively.*

Proof. First, establish that $\|\iota_v d\omega\|_{H_k}^2 / \|\omega\|_{H_k^1}^2$ is bounded using local Cauchy-Schwartz inequalities as in the proof of Lemma 2. This, in conjunction with the bounded linear transformation theorem implies the existence and uniqueness of D_v, as claimed. The results for $\tilde{D}_v$ and L_v follow similarly. $\square$

The exterior derivative and codifferential operators lead to the Hodge Laplacian $\Delta_k : \Omega^k \mapsto \Omega^k$ on k-forms, defined as

$$\Delta_k = \delta_{k+1}d_k + d_{k-1}\delta_k.$$

As with the Laplacian $\Delta = \Delta_0$ on functions, the Laplacian on k-forms on closed manifolds has a unique self-adjoint extension $\bar{\Delta}_k : D(\bar{\Delta}_k) \mapsto H_k$, with a pure point spectrum of eigenvalues $0 = \lambda_{0,k} \leq \lambda_{1,k} \leq \cdots$ with no accumulation points and an associated smooth orthonormal basis $\{\phi_{j,k}\}_{j=0}^\infty$ of eigenforms [18, 20].

A central result in exterior calculus on manifolds is the Hodge decomposition theorem, which states that Ω^k admits the decomposition

$$\Omega^k = \operatorname{ran} d_{k-1} \oplus \operatorname{ran} \delta_{k+1} \oplus \mathcal{H}_0^k, \quad \mathcal{H}_0^k = \ker \Delta_k, \tag{24}$$

into subspaces of closed ($\operatorname{ran} d_{k-1}$), coclosed ($\operatorname{ran} \delta_{k+1}$), and harmonic ($\mathcal{H}^k$) forms, all of which are invariant under Δ_k. On a a compact manifold, $\mathcal{H}_0^k = \ker d_k \cap \ker \delta_k$, and the dimension of this space is finite. The Hodge decomposition in (24) has an L^2 extension,

$$H_k = \mathcal{H}_d^k \oplus \mathcal{H}_\delta^k \oplus \mathcal{H}_0^k, \quad \mathcal{H}_d^k = \overline{\operatorname{ran} d_{k-1}}, \quad \mathcal{H}_\delta^k = \overline{\operatorname{ran} \delta_{k+1}},$$

where the closed spaces $\mathcal{H}_d^k$, $\mathcal{H}_\delta^k$, and $\mathcal{H}^k$ are mutually orthogonal.

It follows directly from the definition of Δ_k that $d_k\Delta_k = \Delta_{k+1}d_k$ and $\delta_k\Delta_k = \Delta_{k-1}\delta_k$. This implies that every eigenform of Δ_k lies in one of the $\mathcal{H}_d^k$, $\mathcal{H}_\delta^k$, or $\mathcal{H}_0^k$ subspaces. Moreover, for every k-eigenform $\psi \in \mathcal{H}_d^k$ there exists a $(k-1)$-eigenform $\varphi \in \mathcal{H}_\delta^k$ such that $\psi = d_{k-1}\varphi$, and similarly for every $\psi \in \mathcal{H}_\delta^k$ there exists as $(k+1)$-eigenform $\omega \in \mathcal{H}_d^{k+1}$ such that $\psi = \delta_{k+1}\omega$.

Besides providing orthonormal bases for the invariant subspaces in the Hodge decomposition of H_k, the eigenfunctions of Δ_k and the corresponding eigenvalues are also useful for constructing Sobolev spaces analogous to the H^p function spaces in (11). Given $p \geq 0$, we define

$$H_k^p = \left\{ \sum_{j=0}^\infty c_j\phi_{j,k} \in H_k : \sum_{j=0}^\infty \lambda_{k,j}^p |c_j|^2 < \infty \right\}.$$

These spaces are Hilbert spaces with inner products

$$\langle \alpha, \beta \rangle_{H_k^p} = \sum_{q=0}^p \sum_{j=0}^\infty \lambda_{j,k}^q \hat{\alpha}_j^* \hat{\beta}_j, \quad \hat{\alpha}_j = \langle \phi_{j,k}, \alpha \rangle_{L_k^2}, \quad \hat{\beta}_j = \langle \phi_{j,k}, \beta \rangle_{L_k^2} \tag{25}$$

and norms $\|\alpha\|_{H_k^p} = \langle \alpha, \alpha \rangle_{H_k^p}^{1/2}$. As in the case of functions and vector fields, we equip these spaces with positive-semidefinite Dirichlet forms $E_{p,k} : H_k^p \times H_k^p \mapsto \mathbb{C}$, given by

$$E_{p,k} = \sum_{j=0}^\infty \lambda_{j,k}^p \hat{\alpha}_j^* \beta_j$$

for α, β from (25). Note that $E_{p,k}$ can be equivalently expressed using the exterior derivative and codifferential operators; in particular,

$$E_{1,k}(\alpha, \beta) = \langle \bar{d}_k\alpha, \bar{d}_k\beta \rangle_{H_{k+1}} + \langle \bar{\delta}_k\alpha, \bar{\delta}_k\beta \rangle_{H_{k-1}}, \tag{26}$$

where overbars denote operator closures. Moreover, if β is smooth, $E_{p,k}(\alpha, \beta)$ can be expressed in terms of the Laplacian via

$$E_{p,k}(\alpha, \beta) = \langle \alpha, \Delta_k^p\beta \rangle_{H_k},$$

with an analogous expression holding if $\alpha \in \Omega^k$. The Dirichlet forms defined above induce the energy functionals $E_{p,k}(f) = E_{p,k}(f, f)$ measuring the roughness of forms in $H_k^p \cap H_k$ through $E_{p,k}(f)/\|f\|_{H_k}^2$.

For notational simplicity, henceforth we will drop the overbars from our notation for the closed differential, codifferential, and Laplacian on k-forms. We will also drop k superscripts and subscripts from η^k, d_k, δ_k, and Δ_k.

4. Spectral exterior calculus (SEC) on smooth manifolds

In this section, we introduce our representation of vector fields and forms using frame constructed from Laplace-Beltrami eigenfunctions and their derivatives. Besides rigorously satisfying the appropriate frame conditions for a number of Sobolev spaces of interest in exterior calculus, an advantage of this representation is that it is fully spectral, and thus can also be applied in the discrete case with little modification.

4.1. Frame representation of vector fields and forms

We begin by recalling the definition of a frame of an inner product space [17].

Definition 5 (frame of an inner product space). Let $(V, \langle \cdot, \cdot \rangle_V)$ be an inner product space and $u_0, u_1, \ldots$ a sequence of elements $u_k \in V$. We say that the set $\{u_k\}$ is a frame if there exist positive finite constants C_1 and C_2 such that the following frame conditions hold for all $v \in V$:

$$C_1\|v\|_V^2 \leq \sum_k |\langle u_k, v \rangle_V|^2 \leq C_2\|v\|_V^2. \tag{27}$$

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The frame $\{u_k\}$ induces a linear operator $T : V \mapsto \ell^2$, called analysis operator, such that $Tv = \hat{v} = (\hat{v}_k)_k$ with $\hat{v}_k = \langle u_k, f \rangle_V$. This operator is bounded by above and below via the same constants as in (27); that is,

$$C_1 \|v\|_V^2 \leq \|Tv\|_{\ell^2} \leq C_2 \|v\|_V^2.$$

The adjoint, $T^* : \ell^2 \mapsto V$, is called synthesis operator.

The analysis and synthesis operators induce a positive-definite, self-adjoint, bounded operator $S : V \mapsto V$ with bounded inverse, called frame operator, which is given by $S = T^*T$. This operator satisfies the bounds

$$C_1 \|v\|_V^2 \leq \langle v, Sv \rangle_V \leq C_2 \|v\|_V^2.$$

The fact that S has bounded inverse implies that the set $\{u_k'\}$ with $u_k' = S^{-1}u_k$ is also a frame, called dual frame. This frame has the important property

$$v = \sum_k \langle u_k', v \rangle_V u_k = \sum_k \langle u_k, v \rangle_V u_k', \quad \forall v \in V. \tag{28}$$

This means that the inner products $\langle u_k', v \rangle_V$ between v and the dual frame elements correspond to expansion coefficients in the original frame that reconstruct v, and conversely, the coefficients $\langle u_k, v \rangle_V$ reconstruct v in the dual frame. Denoting the analysis operator associated with the dual frame by $T' : V \mapsto \ell^2$, (28) can be equivalently expressed as

$$v = T^*T'v = T'^*Tv, \quad \forall v \in V.$$

Clearly, in a separable Hilbert space, every basis is also a frame. For example, in the case $V = H = L^2(\mathcal{M}, \mu)$, a natural frame is provided by the Laplace-Beltrami eigenfunction basis $\{\phi_k\}_{k=0}^\infty$. In that case, the analysis operator T is unitary, $T^*T = I_H$, $TT^* = I_{\ell^2}$, by orthonormality of the basis. In the setting of vector fields, a natural orthonormal set of smooth fields in $H_{\mathfrak{X}}$ is given by the normalized gradient fields e_j from (15). However, this only provides a basis for the space of gradient fields, $H_{\mathfrak{X},\mathrm{grad}}$. To construct a representation of arbitrary vector fields in $H_{\mathfrak{X}}$, we can take advantage of the $C^\infty(\mathcal{M})$-module structure of smooth vector fields to augment this set by multiplication of gradient fields by smooth functions. Doing so will result in an overcomplete spanning set of $H_{\mathfrak{X}}$, which will turn out to meet the frame conditions in Definition 5. We will follow a similar approach to construct frames for the H_k spaces of differential forms, where we will also construct frames for higher-order Sobolev spaces through eigenvalue-dependent normalizations of the frame elements as in (14).

We begin by introducing the vector fields and forms which will be employed in our frame construction and Galerkin schemes below. In the case of vector fields, we define

$$b_{ij} = \phi_i \operatorname{grad} \phi_j, \quad \tilde{b}_{ij} = e^{-\lambda_j} b_{ij}, \quad b_{ij}^{(p)} = b_{ij}/\|\phi_i\|_{H^p}, \quad \tilde{b}_{ij}^{(p)} = \hat{b}_{ij}/\|\phi_i\|_{H^p}, \tag{29}$$

and

$$\hat{b}_{ij} = b_{ij} - b_{ji}, \quad \check{b}_{ij} = e^{-(\lambda_i+\lambda_j)}\hat{b}_{ij}, \tag{30}$$

all of which are smooth vector fields in $\mathfrak{X}$. We also define the smooth forms

$$b^i = \phi_i \in \Omega^0, \quad b^{ij_1\cdots j_k} = b^i \, db^{j_1} \wedge \cdots \wedge db^{j_k} \in \Omega^k, \quad \tilde{b}^{ij_1\cdots j_k} = e^{-(\lambda_{j_1}+\ldots+\lambda_{j_k})}b^{ij_1\cdots j_k},$$
$$b_p^{ij_1\cdots j_k} = b^{ij_1\cdots j_k}/\|\phi_i\|_{H^p}, \quad \tilde{b}_p^{ij_1\cdots j_k} = \tilde{b}^{ij_1\cdots j_k}/\|\phi_i\|_{H^p}, \tag{31}$$

and

$$\hat{b}^{ij_1\cdots j_k} = b^{[ij_1\cdots j_k]}, \quad \check{b}^{ij_1\cdots j_k} = e^{-(\lambda_i+\lambda_{j_1}+\ldots+\lambda_{j_k})}\hat{b}^{[ij_1\cdots j_k]}, \tag{32}$$

where the square brackets $[ij_1\cdots j_k]$ denote total antisymmetrization with respect to the enclosed indices; e.g.,

$$b^{[ij_1j_2]} = b^{ij_1j_2} - b^{ij_2j_1} + b^{j_1j_2i} - b^{j_1ij_2} + b^{j_2ij_1} - b^{j_2j_1i}.$$

With these definitions, our main results on frames for Hilbert spaces of vector fields and forms are as follows.

Theorem 6 (frames for L^2 spaces of vector fields and forms)**.** *The sets*

$$B_{\mathfrak{X}}^J = \{b_{ij} : i \in \{0, 1, \ldots\}, \ j \in \{1, \ldots, J\}\}, \quad \tilde{B}_{\mathfrak{X}} = \{\tilde{b}_{ij} : i \in \{0, 1, \ldots\}, \ j \in \{1, 2, \ldots\}\},$$

where J is a finite integer greater than or equal to $d = \dim \mathcal{M}$, are frames for $H_{\mathfrak{X}}$. Moreover, the sets

$$B_k^J = \{b^{ij_1 \cdots j_k} : i \in \{0, 1, \ldots,\}, \ j_1, \ldots, j_k \in \{1, \ldots, J\}\}, \quad \tilde{B}_k = \{\tilde{b}^{ij_1 \cdots j_k} : i \in \{0, 1, \ldots\}, \ j_1, \ldots, j_k \in \{1, 2, \ldots\}\},$$

with $k \in \{1, \ldots, m\}$, are frames for H_k.

Theorem 7 (frames for order-1 Sobolev spaces of 1-forms)**.** *For the same integer J as in Theorem 6, the sets*

$$B_{1,1}^J = \{b_1^{ij} : i \in \{0, 1, \ldots\}, \ j \in \{1, \ldots, J\}\}, \quad \tilde{B}_{1,1} = \{\tilde{b}_1^{ij} : i \in \{0, 1, \ldots\}, \ j \in \{1, 2, \ldots\}\}.$$

are frames for H_1^1.

We will prove Theorems 6 and 7 in Sections 4.2 and 4.3, respectively. In addition, while we have not explicitly verified this, it should be possible to show via a similar approach to that in Section 4.3 that frames for H_k^p, $k, p > 1$, can be constructed using $b_p^{[ij_1 \cdots j_k]}$ or $\tilde{b}_p^{[ij_1 \cdots j_k]}$. It may also be possible to establish such results inductively with respect to the Sobolev order p, using the results in Appendix Appendix B. Based on these considerations, we conjecture the following:

Conjecture 8 (frames for Sobolev spaces of forms)**.** *For the same integer J as in Theorem 6, the following sets are frames for H_k^1, $k \in \{1, \ldots, m\}$, $p \geq 0$:*

$$B_{k,p}^J = \{b_p^{ij_1 \cdots j_k} : i \in \{0, 1, \ldots\}, \ j_1, \ldots, j_k \in \{1, \ldots, J\}\},$$
$$\tilde{B}_{k,p} = \{\tilde{b}_1^{ij_1 \cdots j_k} : i \in \{0, 1, \ldots\}, \ j_1, \ldots, j_k \in \{1, 2, \ldots\}\}.$$

Remark 9. It should also be noted that we have not established frame conditions for the antisymmetric elements, $\hat{b}_{ij}$ and $\hat{b}^{ij_1 \cdots j_k}$, or their rescaled counterparts, $\check{b}_{ij}$ and $\check{b}^{ij_1 \cdots j_k}$. In fact, to fully span $H_{\mathfrak{X}}$ and H_k using $\hat{b}_{ij}$ and $\hat{b}^{ij_1 \cdots j_k}$, respectively, one would have to use infinitely many i and j indices, leading to violations of the upper frame condition. Nevertheless, as demonstrated by the formulas derived in Appendix B and listed in Table 4, due to cancellation of terms by antisymmetrization, $\hat{b}_{ij}$ and $\hat{b}^{ij_1 \cdots j_k}$ can sometimes lead to considerable simplification of the representation of operators of interest in exterior calculus (e.g., the 1-Laplacian). Moreover, in the applications presented in Section 7 with available analytical results for the eigenvalues and eigenforms of the 1-Laplacian (i.e., the circle and flat torus), we found that SEC formulations based on the antisymmetric elements actually exhibit a moderate performance increase over those based on the non-symmetric elements. These facts motivate further exploration of the construction of frames based on antisymmetric elements. For example, the exponentially scaled $\check{b}^{ij_1 \cdots j_k}$ might provide frames for reproducing kernel Hilbert spaces associated with the heat kernel on k-forms.

For the remainder of this section, we discuss the basic properties of the vector fields and forms just defined. We begin by establishing that, while they may not form a basis, finitely many gradient vector fields of Laplace-Beltrami eigenfunctions are sufficient to generate arbitrary smooth vector fields on closed manifolds.

Lemma 10. *There exists a finite integer $J \geq \dim \mathcal{M}$ such that $\{\mathrm{grad}\,\phi_1|_x, \ldots, \mathrm{grad}\,\phi_J|_x\}$ is a spanning set of $T_x \mathcal{M}$ at every $x \in \mathcal{M}$.*

Proof. The claim will follow if it can be shown that the there exists an integer J such that the map $\vec{\phi}_j : \mathcal{M} \mapsto \mathbb{R}^J$ with $\vec{\phi}_J(x) = (\phi_1(x), \ldots, \phi_J(x))$ is an embedding of $\mathcal{M}$, which is proved in Theorem 4.5 in [21]. Note that B_J being an embedding implies that $J \geq \dim \mathcal{M}$. To see why it implies that $\{\mathrm{grad}\,\phi_1|_x, \ldots, \mathrm{grad}\,\phi_J|_x\}$ is a spanning set, fix a point $x \in \mathcal{M}$, and consider the tangent vector pushforward map $\vec{\phi}_{J*}|_x : T_x \mathcal{M} \mapsto T_{\vec{\phi}_J(x)} \mathbb{R}^J$. Since $T_{\vec{\phi}_J(x)} \mathbb{R}^J$ is canonically isomorphic to $\mathbb{R}^J$, in a coordinate chart $u : N_x \mapsto \mathbb{R}^J$ defined on a neighborhood

N_x of x, the pushforward map is represented by an $d \times J$ matrix $\mathbf{\Xi}(x)$, $d = \dim \mathcal{M}$, with elements $\Xi_{ij} = \left.\frac{\partial \phi_j}{\partial u_i}\right|_x$, and because $\vec{\phi}_J$ is an embedding, that matrix has full rank, $\operatorname{rank} \mathbf{\Xi}(x) = d$. In this coordinate basis, the components β_{ij} of $\operatorname{grad} \phi_j|_x = \sum_{i=1}^d \beta_{ij} \left.\frac{\partial}{\partial u_i}\right|_x$ are given by $\beta_{ij} = \sum_{k=1}^d \eta_{ik} \Xi_{kj}$, where η_{ik} are the components of the dual metric $\eta_x : T^*\mathcal{M}_x \times T^*\mathcal{M}_x \mapsto \mathbb{R}$. The η_{ik} form an $d \times d$ invertible matrix, and thus the $d \times J$ matrix with elements β_{ij} has rank d. This implies that $\operatorname{span}\{\operatorname{grad} \phi_1|_x, \ldots, \operatorname{grad} \phi_J|_x\} = T_x\mathcal{M}$. $\qquad\square$

Corollary 11. The set $\{\operatorname{grad} \phi_1, \ldots, \operatorname{grad} \phi_J\}$ is a generating set for $\mathfrak{X}$ viewed as a $C^\infty(\mathcal{M})$-module. That is, for every smooth vector field $v \in \mathfrak{X}$, there exist (not necessarily unique) smooth functions $f_1, \ldots, f_J$ such that $v = \sum_{j=1}^J f_j \operatorname{grad} \phi_j$.

Corollary 12. The collection of smooth k-form fields $d\phi_{j_1} \wedge \cdots \wedge d\phi_{j_k}$ with $j_1, \ldots, j_k \in \{1, \ldots, J\}$ spans Λ_x^k at every $x \in \mathcal{M}$. As a result, this set is a generating set for Ω^k, which means that for every $\omega \in \Omega^k$ there exist smooth functions $f_{j_1 \cdots j_k}$ such that $\omega = \sum_{j_1, \ldots, j_k=1}^J f_{j_1 \cdots j_k} d\phi_{j_1} \wedge \cdots \wedge d\phi_{j_k}$.

It follows from corollary 11 and the fact that $\mathfrak{X}$ is dense in $H_{\mathfrak{X}}$ that for every $v \in H_{\mathfrak{X}}$ there exist functions $f_1, \ldots, f_J \in H$ such that $v = \sum_{j=1}^J f_j \operatorname{grad} \phi_j$. Expanding these functions as $f_j = \sum_{i=0}^\infty c_{ij} \phi_j$ with $c_{ij} = \langle \phi_i, f_j \rangle_H$, we conclude that every vector field $v \in H_{\mathfrak{X}}$ is expressible in the form

$$v = \sum_{i=0}^\infty \sum_{j=1}^J c_{ij} \phi_{ij}, \tag{33}$$

for some (not necessarily unique) constants $c_{ij} \in \mathbb{C}$. Similarly, corollary 12 implies that every k-form field in H_k can be expanded as

$$\omega = \sum_{i=0}^\infty \sum_{j_1, \ldots, j_k=1}^J c_{ij_1 \cdots j_k} b^{ij_1 \cdots j_k}, \quad c_{ij_1 \cdots j_k} \in \mathbb{C}. \tag{34}$$

Example 13. As a simple example illustrating that $B_{\mathfrak{X}}^J$ may be a spanning set with linearly independent elements (as opposed to a basis), suppose that $(\mathcal{M}, g)$ is the circle equipped with the canonical arclength metric, normalized such that $\mu(\mathcal{M}) = 1$. Then, an orthonormal basis consisting of Laplace-Beltrami eigenfunctions is given by

$$\phi_i(\theta) = \begin{cases} \cos(i\theta/2), & i \text{ even}, \\ \sin((i+1)\theta/2), & i \text{ odd}, \end{cases}$$

where $\theta \in (0, 2\pi)$ is a canonical angle coordinate. Note that the coordinate basis vector field $\frac{\partial}{\partial \theta}$ extends to a globally defined harmonic vector field on $\mathcal{M}$, satisfying $\Delta \frac{\partial}{\partial \theta}^\flat = 0$. In this coordinate system, the metric and dual metric are given by $g = g_{11} d\theta \otimes d\theta$, $\eta = g'^{11} \frac{\partial}{\partial \theta} \otimes \frac{\partial}{\partial \theta}$, where $g_{11} = 1/g'^{11} = 1/(2\pi)^2$, and we have

$$\operatorname{grad} \phi_i = \eta(d\phi_i, \cdot) = \begin{cases} -2i\pi^2 \phi_{i-1} \frac{\partial}{\partial \theta}, & i \text{ even}, \\ 2(i+1)\pi^2 \phi_{i+1} \frac{\partial}{\partial \theta}, & i \text{ odd}, \end{cases} \qquad b_{ij} = \begin{cases} -2j\pi^2 \phi_i \phi_{j-1} \frac{\partial}{\partial \theta}, & i \text{ even} \\ 2(j+1)\pi^2 \phi_i \phi_{j+1} \frac{\partial}{\partial \theta}, & i \text{ odd}. \end{cases}$$

It therefore follows from standard trigonometric identities that for any odd $i \geq 1$,

$$\frac{b_{i+1,i} - b_{i,i+1}}{2(i+1)\pi^2} = \frac{\partial}{\partial \theta},$$

with an analogous relationship holding for i even. This shows that for $J \geq 2$, B_J contains linearly dependent elements. On the other hand, for $J = 1$, B_1 fails to be a spanning set as the harmonic vector field $\frac{\partial}{\partial \theta}$ does not lie in its span.

Remark 14. The circle example above might suggest that our representation of vector fields through linear combinations of elements of B_J is highly inefficient, since, after all, one could define $\xi_j = \phi_j \partial_\theta$, and $\{\xi_j\}_{j=0}^\infty$ would be an orthonormal basis of $H_{\mathfrak{X}}$. However, such a construction implicitly makes use of a special property of the circle, namely that it is a parallelizable manifold. Equivalently, as a $C^\infty(\mathcal{M})$-module, the space $\mathfrak{X}$ of smooth vector fields is free; that is, it contains a set $\{u_1, \ldots, u_d\}$ of $d = \dim(\mathcal{M})$ nowhere-vanishing linearly independent elements. Any such set would be a basis of $\mathfrak{X}$, meaning that for every $v \in \mathfrak{X}$ there would exist unique smooth function $f_1, \ldots, f_d \in C^\infty(\mathcal{M})$ such that $v = \sum_{j=1}^d f_j u_j$. In general, for non-parallelizable manifolds (e.g., the 2-sphere), $\mathfrak{X}$ does not have a basis, so any spanning set of $H_{\mathfrak{X}}$ such as B_J that makes use of a generating set of $\mathfrak{X}$ will necessarily be overcomplete.

We continue by stating a number of useful properties of the b_{ij} fields and their antisymmetric analogs, $\hat{b}_{ij}$. Many of these properties are also listed in tables in Figs. 3 and 4. In what follows, all equalities involving infinite sums hold in an L^2 sense.

1. *Relationship between antisymmetric and nonsymmetric frame elements.* Using the Leibniz rule, we compute

$$\phi_i \operatorname{grad} \phi_j + \phi_j \operatorname{grad} \phi_i = \operatorname{grad}(\phi_i \phi_j) = \sum_{k=0}^\infty c_{ijk} \operatorname{grad} \phi_k = \sum_{k=1}^\infty c_{ijk} \hat{b}_{0k},$$

   where and the last equality follows from the fact that ϕ_0 is a constant equal to 1. It therefore follows that

$$b_{ij} = \frac{\phi_i \operatorname{grad} \phi_j + \phi_j \operatorname{grad} \phi_i}{2} + \frac{\phi_i \operatorname{grad} \phi_j - \phi_j \operatorname{grad} \phi_i}{2} = \sum_{k=1}^\infty c_{ijk} \hat{b}_{0k} + \hat{b}_{ij}.$$

2. *Riemannian inner products.* Lemma 3 and (10) lead to the following expressions for the Riemannian inner products between the frame elements:

$$g(b_{ij}, b_{kl}) = \frac{\phi_i \phi_k}{2} (\lambda_j + \lambda_l - \Delta) (\phi_j \phi_l) = \frac{1}{2} \sum_{m,n,p=0}^\infty c_{ikm} c_{jlm} c_{mnp} (\lambda_j + \lambda_l - \lambda_n) \phi_p.$$

The next few results are for the 1-form fields b^{ij} and $\hat{b}^{ij}$. They will be employed in our proof of Theorem 7 in and the construction of Galerkin schemes for the Laplacian on 1-forms in Sections 4.3 and 5 ahead, respectively.

1. *Exterior derivative and codifferential.* It follows from the Leibniz rule for the exterior derivative in (21) and the definition of the codifferential in (22) that

$$db^{ij} = db^i \wedge db^j, \quad \delta b^{ij} = -\eta(db^i, db^j) + \lambda_j b^i b^j. \tag{35}$$

Similarly, we have

$$\hat{d}b^{ij} = 2db^i \wedge db^j, \quad \delta \hat{d}b^{ij} = (\lambda_j - \lambda_i) b^i b^j.$$

Observe, in particular, that if $\lambda_i = \lambda_j$ (i.e., b^i and b^j lie in the same eigenspace of the Laplacian), $\hat{b}^{ij}$ is co-closed, $\delta \hat{b}^{ij} = 0$.

4.2. Proof of Theorem 6

We will prove the theorem by establishing the upper and lower frame conditions in Definition 5 for $B_{\mathfrak{X}}^J$ and B_k^J, assuming that J is large-enough so that Lemma 10 holds. We begin from $B_{\mathfrak{X}}^J$.

Given any $v \in H_{\mathfrak{X}}$, we have

$$\sum_{i=0}^\infty \sum_{j=1}^J |\langle b_{ij}, v \rangle_{H_{\mathfrak{X}}}|^2 = \sum_{i=0}^\infty \sum_{j=1}^J |\langle b_{ij}, v \rangle_{H_{\mathfrak{X}}}|^2 = \sum_{i=0}^\infty \sum_{j=1}^J |\langle \phi_i, v(\phi_j) \rangle_H|^2 = \sum_{j=1}^J \|v(\phi_j)\|_H^2 = \sum_{j=1}^J \|g(\operatorname{grad} \phi_j, v)\|_H^2$$

$$\leq \sum_{j=1}^{J} \||g(\operatorname{grad}\phi_j, v)|\|_H^2 \leq \sum_{j=1}^{J} \left\| \sqrt{g(\operatorname{grad}\phi_j, \operatorname{grad}\phi_j)g(v^*, v)} \right\|_H^2$$

$$\leq \sum_{j=1}^{J} \|\operatorname{grad}\phi_j\|_{L_{\mathfrak{x}}^\infty}^2 \left\| \sqrt{g(v^*, v)} \right\|_H^2 = \sum_{j=1}^{J} \|\operatorname{grad}\phi_j\|_{L_{\mathfrak{x}}^\infty}^2 \|v\|_{H_{\mathfrak{x}}}^2,$$

so that the upper frame condition holds with $C_2 = \sum_{j=1}^{J} \|\operatorname{grad}\phi_j\|_{L_{\mathfrak{x}}^\infty}^2$. Note that the fact that J is finite is important in the derivation of this result. Next, to verify the lower frame condition, consider, at every $x \in \mathcal{M}$, the $J \times J$ Gramm matrix $\boldsymbol{\Psi}(x)$ with elements

$$\Psi_{ij}(x) = g(\operatorname{grad}\phi_i, \operatorname{grad}\phi_j)|_x,$$

and note that because the $\operatorname{grad}\phi_j|_x$ span $T_x\mathcal{M}$ that matrix has rank m. Therefore, writing $v = \sum_{j=1}^{J} f_j \operatorname{grad}\phi_j$, where the f_j are functions in H to be determined, the equation

$$g(\operatorname{grad}\phi_i, v)|_x = \sum_{j=1}^{J} f_j(x)\Psi_{ji}(x)$$

has a solution for μ-a.e. $x \in X$ given by

$$f_j(x) = \sum_{k=1}^{J} \Psi_{jk}^+(x)g(\operatorname{grad}\phi_k, v)|_x,$$

where $\boldsymbol{\Psi}^+(x) = [\Psi_{jk}^+(x)]$ is the Moore-Penrose pseudoinverse of $\boldsymbol{\Psi}(x)$.

Observe now that $\boldsymbol{\Psi}(x)$ can be expressed using a coordinate chart as $\boldsymbol{\Psi}(x) = \boldsymbol{\Xi}(x)^\top \boldsymbol{g}(x)\boldsymbol{\Xi}(x)$, where the matrices $\boldsymbol{\Xi}(x)$ and $\boldsymbol{g}(x)$ are as in the proof of Lemma 10. Thus, since $\boldsymbol{\Xi}(x)$ has full rank and $\boldsymbol{g}(x)$ is invertible, we have

$$\boldsymbol{\Psi}^+(x) = \boldsymbol{\Xi}^+(x)\boldsymbol{g}^{-1}(x)(\boldsymbol{\Xi}^+(x))^\top, \quad \boldsymbol{\Xi}(x) = \boldsymbol{\Xi}(x)^\top(\boldsymbol{\Xi}(x)\boldsymbol{\Xi}(x)^\top)^{-1},$$

where both $\boldsymbol{g}^{-1}(x)$ and $(\boldsymbol{\Xi}(x)\boldsymbol{\Xi}(x))^{-1}$ depend smoothly on x by compactness of $\mathcal{M}$ and smoothness of $\boldsymbol{g}(x)$ and $\boldsymbol{\Xi}(x)$, respectively. We therefore conclude that $\boldsymbol{\Psi}^+(x)$ depends smoothly on x, and thus that v admits an expansion of the form (33) with

$$c_{ij} = \langle \phi_i, f_j \rangle_H = \sum_{k=1}^{J} \langle \Psi_{jk}^+ b_{ik}, v \rangle_{H_{\mathfrak{x}}}. \tag{36}$$

We therefore obtain

$$\|v\|_{H_{\mathfrak{x}}}^2 = \sum_{i=0}^{\infty}\sum_{j=1}^{J} \langle c_{ij}b_{ij}, v \rangle_{H_{\mathfrak{x}}} = \sum_{i=0}^{\infty}\sum_{j,k=1}^{J} \langle \Psi_{jk}^+ b_{ik}, v \rangle_{H_{\mathfrak{x}}}^* \langle b_{ij}, v \rangle_{H_{\mathfrak{x}}} \leq \tilde{C}\sum_{i=0}^{\infty}\sum_{j,k=1}^{J} |\langle b_{ik}, v \rangle_{H_{\mathfrak{x}}}||\langle b_{ij}, v \rangle_{H_{\mathfrak{x}}}|,$$

where $\tilde{C} = \max_{j,k\in\{1,\ldots,J\}} \|\Psi_{jk}^+\|_{L^\infty}$. Defining now the vectors $\beta_i = (\beta_{i1}, \ldots, \beta_{iJ}) \in \mathbb{R}^J$ with with $\beta_{ij} = |\langle b_{ij}, v \rangle_{H_{\mathfrak{x}}}|$, it follows by equivalence of norms in finite-dimensional vector spaces that there exists a constant $\hat{C} \in \mathbb{R}$ such that

$$\sum_{j,k=1}^{J} |\langle b_{ik}, v \rangle_{H_{\mathfrak{x}}}||\langle b_{ij}, v \rangle_{H_{\mathfrak{x}}}| = \|\beta_i\|_1^2 \leq \hat{C}\|\beta_i\|_2^2,$$

where $\|\cdot\|_p$ is the canonical p-norm on $\mathbb{R}^J$. This leads to

$$\|v\|_{\mathfrak{x}}^2 \leq \tilde{C}\hat{C}\sum_{i=0}^{\infty}\|\beta_i\|_2^2 = \tilde{C}\hat{C}\sum_{i=0}^{\infty}\sum_{j=1}^{J} |\langle b_{ij}, v \rangle_{H_{\mathfrak{x}}}|^2,$$

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which proves the lower frame condition with $C_1 = 1/(\tilde{C}\hat{C})$. We have thus established that $B^J_{\mathfrak{x}}$ is a frame of $H_{\mathfrak{x}}$, as claimed.

Consider now the frame conditions for $\tilde{B}_{\mathfrak{x}}$. Using Cauchy-Schwartz inequalities as above, we can conclude that

$$\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}|\langle \tilde{b}_{ij}, v\rangle_{H_{\mathfrak{x}}}|^2 \leq \sum_{j=1}^{\infty}\frac{\|\operatorname{grad}\phi_j\|^2_{L^{\infty}_{\mathfrak{x}}}}{e^{\lambda_j}}\|v\|^2_{H_{\mathfrak{x}}}.$$

To bound the infinite sum in the right-hand side, we use the following estimates for the L^{∞} norms of Laplace-Beltrami eigenfunctions and their gradients on smooth, closed Riemannian manifolds:

$$\|\phi_j\|_{L^{\infty}} \leq C\lambda_j^{(m-1)/4}\|\phi_j\|_H, \quad \|\operatorname{grad}\phi_j\|_{L^{\infty}_{\mathfrak{x}}} \leq \tilde{C}\lambda_j^{1/2}\|\phi_j\|_{L^{\infty}}, \quad C, \tilde{C} \geq 0, \tag{37}$$

which hold for $j \leq 1$ and $j \geq 0$, respectively. The former is a classical result due to Hörmander [22]; the latter was proved by Shi and Xu in [23]. Combining these results with the Weyl estimate for the asymptotic distribution of Laplace-Beltrami eigenvalues as $j \to \infty$,

$$j = \hat{C}\lambda_j^{d/2} + o(\lambda_j^{(d-1)/2}), \tag{38}$$

we obtain

$$\|\operatorname{grad}\phi_j\|_{L^{\infty}_{\mathfrak{x}}} \leq \check{C}\lambda_j^{(d+1)/(2d)}\|\phi_j\|_H,$$

where $\hat{C}$ and $\check{C}$ in the last two equations are positive constants. Therefore, for any $l \geq 0$ there exists a finite constant C_l such that $e^{-\lambda_j}\|\operatorname{grad}\phi_j\|^2_{L^{\infty}_{\mathfrak{x}}} \leq C_l j^{-l}$. This implies that $C_2 = \sum_{j=0}^{\infty}e^{-\lambda_j}\|\operatorname{grad}\phi_j\|^2_{L^{\infty}_{\mathfrak{x}}}$ is finite, proving the upper frame bound.

To verify the lower frame bound, start from any expansion of v in the $B^J_{\mathfrak{x}}$ frame,

$$v = \sum_{i=0}^{\infty}\sum_{j=1}^{J}c_{ij}b_{ij},$$

and compute

$$\|v\|^2_{\mathfrak{x}} \leq C\sum_{i=0}^{\infty}\sum_{j=1}^{J}|\langle b_{ij}, v\rangle_{H_{\mathfrak{x}}}|^2 \leq Ce^{\lambda_J}\sum_{i=0}^{\infty}\sum_{j=1}^{J}|\langle \tilde{b}_{ij}, v\rangle_{H_{\mathfrak{x}}}|^2 \leq Ce^{\lambda_J}\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}|\langle \tilde{b}_{ij}, v\rangle_{H_{\mathfrak{x}}}|^2,$$

where C is a lower frame constant for $B^J_{\mathfrak{x}}$. This shows that the lower frame condition is satisfied for $C_1 = Ce^{-\lambda_J}$, and we thus conclude that $\tilde{B}_{\mathfrak{x}}$ is a frame.

We now turn to the frame conditions for B^J_k and $\tilde{B}_k$. These conditions follow by similar arguments as those just made to establish the frame conditions for vector fields.

First, we introduce for convenience an ordering $l \mapsto (j_1(l), \ldots, j_k(l))$ of the corresponding indices in $b^{ij_1\cdots j_k}$, where l is an integer ranging from 1 to J^k, and define $\alpha_l = d\phi^{j_1(l)} \wedge \cdots \wedge d\phi^{j_k(l)}$. Then, for any $\omega \in H_k$, we have

$$\sum_{i=0}^{\infty}\sum_{j_1,\ldots,j_k=1}^{\infty}|\langle b^{ij_1\cdots j_k}, \omega\rangle_{H_k}|^2 = \sum_{i=0}^{\infty}\sum_{l=1}^{J^k}|\langle \phi_i\alpha_l, \omega\rangle_{H_k}|^2 = \sum_{i=0}^{\infty}\sum_{l=1}^{J^k}|\langle \phi_i, \eta(\alpha_l, \omega)\rangle_H|^2 = \sum_{l=1}^{J^k}\|\eta(\alpha_l, \omega)\|^2_H$$

$$\leq \sum_{l=1}^{J^k}\left\|\sqrt{\eta(\alpha_l, \alpha_l)}\sqrt{\eta(\omega^*, \omega)}\right\|^2_H \leq \sum_{l=1}^{J^k}\|\alpha_l\|^2_{L^{\infty}_k}\|\omega\|^2_{H_k},$$

establishing the upper frame condition with $C_2 = \sum_{l=1}^{J^k}\|\alpha_l\|^2_{L^{\infty}_k}$. To verify the lower frame condition, we use (34) to expand $\omega = \sum_{i=0}^{\infty}\sum_{l=1}^{J^k}c_{il}\phi_i\alpha_l$, where the expansion coefficients c_{il} can be chosen as (cf. (36))

$$c_{il} = \sum_{m=1}^{J^k}\langle \Psi^+_{lm}\phi_i\alpha_m, \omega\rangle_{H_k},$$

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and in the above $\Psi_{lm}^+(x)$ are the elements of the pseudoinverse of the $J^k \times J^k$ Gramm matrix $\Psi_{lm}(x) = \eta(\alpha_l, \alpha_m)|_x$ (these depend smoothly on x as in the case of the corresponding Gramm matrix for vector fields). The calculation to establish the lower frame bound for $\|\omega\|_{H_k}^2 = \sum_{i=0}^{\infty} \sum_{l=1}^{J^k} \langle c_{il} \phi_i \alpha_l, \omega \rangle_{H_k}$ then proceeds analogously to that in the case of vector fields, leading to the conclusion that B_k^J is an H_k-frame. Similarly, that $\tilde{B}_k$ is a frame follows analogously to the vector field case. This completes our proof of Theorem 6.

4.3. Proof of Theorem 7

Let ω be an arbitrary 1-form field in H_1^1. We begin by stating two auxiliary results on the inner products between the exterior derivative (codifferential) of ω and the exterior derivative (codifferential) of the frame elements b_1^{ij} and $\tilde{b}_1^{ij}$.

Lemma 15. *(i) There exist finite constants U_J and V_J, independent of ω, such that*

$$\sum_{i=0}^{\infty} \sum_{j=1}^{J} |\langle db_1^{ij}, d\omega \rangle_{H_2}|^2 \leq U_J \|\omega\|_{H_1^1}^2, \quad \sum_{i=0}^{\infty} \sum_{j=1}^{J} |\langle \delta b_1^{ij}, \delta\omega \rangle_H|^2 \leq V_J \|\omega\|_{H_1^1}^2.$$

Moreover, there exist a positive real number $\hat{C}$ and a positive integer $q \geq 0$, both independent of J, such that U_J and V_J are both bounded above by $\hat{C} J^q$.

(ii) There exist finite constants $\tilde{U}$ and $\tilde{V}$, independent of ω, such that

$$\sum_{i=0}^{\infty} \sum_{j=1}^{\infty} |\langle d\tilde{b}_1^{ij}, d\omega \rangle_{H_2}|^2 \leq \tilde{U} \|\omega\|_{H_1^1}^2, \quad \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} |\langle \tilde{\delta b}_1^{ij}, \delta\omega \rangle_H|^2 \leq \tilde{V} \|\omega\|_{H_1^1}^2.$$

A proof of this lemma will be given below. Assuming, for now, that it is valid, leads to the following corollary:

Corollary 16. The frame elements b_1^{ij} and $\tilde{b}_1^{ij}$ satisfy the bounds

$$\sum_{i=0}^{\infty} \sum_{j=1}^{J} |E_{1,1}(b_1^{ij}, \omega)|^2 \leq C \|\omega\|_{H_1^1}, \quad \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} |E_{1,1}(\tilde{b}_1^{ij}, \omega)|^2 \leq \tilde{C} \|\omega\|_{H_1^1},$$

where $E_{1,1}$ is the Dirichlet form from (26), and C and $\tilde{C}$ are constants independent of ω.

Proof. We first verify the claim for b_1^{ij}. By Lemma 15, the sequences $(i,j) \mapsto |\langle db_1^{ij}, d\omega \rangle_{H_2}|^2$ and $(i,j) \mapsto |\langle \delta b_1^{ij}, \delta\omega \rangle_H|^2$ are in ℓ^2. Therefore, using the Cauchy-Schwartz inequality for ℓ^2 and Lemma 15(i), we obtain,

$$\sum_{i=0}^{\infty} \sum_{j=1}^{J} |E_{1,1}(b_1^{ij}, \omega)|^2 = \sum_{i=0}^{\infty} \sum_{j=1}^{J} |\langle db_1^{ij}, d\omega \rangle_{H_2} + \langle \delta b_1^{ij}, \delta\omega \rangle_H|^2$$

$$\leq \sum_{i=0}^{\infty} \sum_{j=1}^{J} |\langle db_1^{i,j}, d\omega \rangle_{H_2}|^2 + \sum_{i=0}^{\infty} \sum_{j=1}^{J} |\langle \delta b_1^{ij}, \delta\omega \rangle_H|^2 + 2 \sum_{i=0}^{\infty} \sum_{j=1}^{J} |\langle db_1^{i,j}, d\omega \rangle_{H_2} \langle \delta b_1^{ij}, \delta\omega \rangle_H|$$

$$\leq \sum_{i=0}^{\infty} \sum_{j=1}^{J} |\langle db_1^{i,j}, d\omega \rangle_{H_2}|^2 + \sum_{i=0}^{\infty} \sum_{j=1}^{J} |\langle \delta b_1^{ij}, \delta\omega \rangle_H|^2$$

$$+ 2 \left(\sum_{i=0}^{\infty} \sum_{j=1}^{J} |\langle db_1^{ij}, d\omega \rangle_{H_2}|^2 \right)^{1/2} \left(\sum_{k=0}^{\infty} \sum_{l=1}^{J} |\langle \delta b_1^{kl}, \delta\omega \rangle_H|^2 \right)^{1/2}$$

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$$\leq \left(U_J^{1/2} + V_J^{1/2}\right)^2 \|\omega\|_{H_1}^2,$$

and the first claim of the corollary follows with $C = \left(U_J^{1/2} + V_J^{1/2}\right)^2$. To verify the second claim, we proceed as above using Lemma 15(ii) to derive the bound

$$\sum_{i=0}^{\infty} \sum_{j=1}^{k} |E_{1,1}(\tilde{b}_1^{ij}, \omega)|^2 \leq \tilde{C} \|\omega\|_{H_1}^2, \quad \tilde{C} = \left(\tilde{U}^{1/2} + \tilde{V}^{1/2}\right)^2. \qquad \square$$

Having established corollary 16, the upper frame condition for $B_{1,1}^J$ follows from

$$\sum_{i=0}^{\infty} \sum_{j=1}^{J} |\langle b_1^{ij}, \omega \rangle_{H_1^1}|^2 = \sum_{i=0}^{\infty} \sum_{j=1}^{J} |\langle b_1^{ij}, \omega \rangle_{H_1} + E_{1,1}(b_1^{ij}, \omega)|^2$$

$$\leq \sum_{i=0}^{\infty} \sum_{j=1}^{J} |\langle b_1^{ij}, \omega \rangle_{H_1}|^2 + \sum_{i=0}^{\infty} \sum_{j=1}^{J} |E_{1,1}(b_1^{ij}, \omega)|^2$$

$$+ 2 \left(\sum_{i=0}^{\infty} \sum_{j=1}^{J} |\langle b_1^{ij}, \omega \rangle_{H_1}|^2\right)^{1/2} \left(\sum_{k=0}^{\infty} \sum_{l=1}^{J} |E_{1,1}(b_1^{kl}, \omega)|^2\right)^{1/2}$$

$$= \sum_{i=0}^{\infty} \sum_{j=1}^{J} \frac{|\langle b^{ij}, \omega \rangle_{H_1}|^2}{\|b^i\|_{H^1}^2} + \sum_{i=0}^{\infty} \sum_{j=1}^{J} |E_{1,1}(b_1^{ij}, \omega)|^2$$

$$+ 2 \left(\sum_{i=0}^{\infty} \sum_{j=1}^{J} \frac{|\langle b^{ij}, \omega \rangle_{H_1}|^2}{\|b^i\|_{H^1}^2}\right)^{1/2} \left(\sum_{k=0}^{\infty} \sum_{l=1}^{J} |E_{1,1}(b_1^{kl}, \omega)|^2\right)^{1/2}$$

$$\leq \sum_{i=0}^{\infty} \sum_{j=1}^{J} |\langle b^{ij}, \omega \rangle_{H_1}|^2 + \sum_{i=0}^{\infty} \sum_{j=1}^{J} |E_{1,1}(b_1^{ij}, \omega)|^2$$

$$+ 2 \left(\sum_{i=0}^{\infty} \sum_{j=1}^{J} |\langle b^{ij}, \omega \rangle_{H_1^1}|^2\right)^{1/2} \left(\sum_{k=0}^{\infty} \sum_{l=1}^{J} |E_{1,1}(b_1^{kl}, \omega)|^2\right)^{1/2}$$

$$\leq \left(C_2^{1/2} + C^{1/2}\right)^2 \|\omega\|_{H_1}^2,$$

where C_2 is an upper frame constant for B_1^J and C the constant in corollary 16. The upper frame condition for $\tilde{B}_{1,1}^J$ follows similarly.

We now return to the proof of Lemma 15. First, observe that

$$\langle db^{ij}, dv \rangle_{H_2} = \langle db^i \wedge db^j, dv \rangle_{H_2} = \int_{\mathcal{M}} db^i \wedge db^j \wedge \star dv = \int_{\mathcal{M}} db^i \wedge \star \star (db^j \wedge \star dv) = \langle db^i, \star(db^j \wedge \star dv) \rangle_{H_1},$$

where we have used (35) and (20) in the first and third equalities, respectively. By the Hodge decomposition theorem, there exists a unique function $f_j \in H^1$, a unique 2-form $\alpha_j \in H_2^1$, and a unique harmonic 1-form $\chi_j \in \mathcal{H}_0^1$ such that

$$\star(db^j \wedge \star d\omega) = df_j + \delta \alpha_j + \chi_j;$$

as a result,

$$\sum_{i=0}^{\infty} |\langle db_1^i, \star(db^j \wedge \star d\omega) \rangle_{H_1}|^2 = \sum_{i=0}^{\infty} |\langle db_1^i, df_j + \delta \alpha_j + \chi_j \rangle_{H_1}|^2 = \sum_{i=0}^{\infty} |\langle \delta db_1^i, f_j \rangle_H|^2 = \sum_{i=0}^{\infty} \lambda_i |\langle b^i, f_j \rangle_H|^2 = E_1(f_j).$$

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We therefore have,

$$\sum_{j=1}^{J}\sum_{i=0}^{\infty}|\langle db_1^{ij}, d\omega\rangle_{H_2}|^2 = \sum_{i=0}^{\infty}|\langle db_1^i, \star(db^j \wedge \star d\omega)\rangle_{H_1}|^2 = \sum_{j=1}^{J}E_1(f_j) = \sum_{j=1}^{J}\langle df_j, df_j\rangle_{H_1}$$

$$\leq \sum_{j=1}^{J}\|\star(db^j \wedge \star d\omega)\|_{H_1}^2 = \sum_{j=1}^{J}\|db^j \wedge \star d\omega\|_{H_{m-1}}^2$$

$$= \sum_{j=1}^{J}\|\star\iota_{\mathrm{grad}\,b^j}\star\star d\omega\|_{H_{m-1}}^2 = \sum_{j=1}^{J}\|\iota_{\mathrm{grad}\,b^j}d\omega\|_{H_1}^2,$$

where we have used (23) to obtain the first equality in the third line. It then follows from Lemma 4 that

$$\sum_{j=1}^{J}\sum_{i=0}^{\infty}|\langle db_1^{ij}, d\omega\rangle_{H_2}|^2 = \sum_{j=1}^{J}\|D_{\mathrm{grad}\,b^j}\omega\|_{H_1}^2 \leq U_J\|\omega\|_{H_1}^2, \quad U_J = \sum_{j=1}^{J}\|D_{\mathrm{grad}\,b^j}\|^2,$$

as claimed in part (i) of the lemma. It can further be shown via local Cauchy-Schwartz inequalities as in the proof of Lemma 4 that the operator norms $\|D_{\mathrm{grad}\,b^j}\| = \|D_{\mathrm{grad}\,\phi_j}\|$ can be bounded above by $\tilde{C}\|\phi_j\|_{L^\infty}^{\tilde{q}}$ for some positive constants $\tilde{C}$ and $\tilde{q}$ that do not depend on j. This, in conjunction with the Hörmander bound in (37) implies that that there exists a positive constant C_U and a positive integer q_U, both independent of J, such that

$$U_J \leq C_U J^{q_U}. \tag{39}$$

Moving on to the second claim of Lemma 15(i), consider

$$\sum_{j=1}^{J}\sum_{i=0}^{\infty}|\langle \delta b_1^{ij}, \delta\omega\rangle_{H}|^2 = \sum_{i=0}^{\infty}\sum_{j=1}^{J}|\langle -\eta(db_1^i, db^j) + \lambda_j b_1^i b^j, \delta\omega\rangle_{H}|^2,$$

where we have used the expression for δb^{ij} in (35). Expanding

$$\delta\omega\,db^j = df_j + \delta\alpha_j + \chi_j,$$

where $f_j \in H^1$, $\alpha_j \in H_2^1$, and $\chi_j \in \mathcal{H}_0^1$ are unique, we compute,

$$\sum_{j=1}^{J}\sum_{i=0}^{\infty}|\langle \eta(db_1^i, db^j), \delta\omega\rangle_{H}|^2 = \sum_{j=1}^{J}\sum_{i=0}^{\infty}\left|\int_{\mathcal{M}} db_1^i \wedge \star db^j\,\delta\omega\right|^2 = \sum_{j=1}^{J}\sum_{i=0}^{\infty}|\langle db_1^i, \delta\omega\,db^j\rangle_{H_1}|^2 = \sum_{j=1}^{J}E_1(f_j)$$

$$\leq \sum_{j=1}^{J}\|\delta\omega\,db^j\|_{H_1}^2 = \sum_{j=1}^{J}\left\|\sqrt{\eta(db^j, db^j)}\delta\omega\right\|_{H}^2 \leq \sum_{j=1}^{J}\|\mathrm{grad}\,\phi_j\|_{L_{\mathfrak{x}}^\infty}^2\|\delta\omega\|_{H}^2$$

$$\leq \sum_{j=1}^{J}\|\mathrm{grad}\,\phi_j\|_{L_{\mathfrak{x}}^\infty}^2\|\omega\|_{H_1}^2. \tag{40}$$

Moreover, we have

$$\sum_{j=1}^{J}\sum_{i=0}^{\infty}|\langle \lambda_j b_1^i b^j, \delta\omega\rangle_{H}|^2 = \sum_{j=1}^{J}\sum_{i=0}^{\infty}\frac{\lambda_j^2}{\|b^i\|_{H^1}^2}|\langle b^i b^j, \delta\omega\rangle_{H}|^2 \leq \sum_{j=1}^{J}\sum_{i=0}^{\infty}\lambda_j^2|\langle b^i, b^j\,\delta\omega\rangle_{H}|^2$$

$$= \sum_{j=1}^{J}\lambda_j^2\|b^j\,\delta\omega\|_{H}^2 \leq \sum_{j=1}^{J}\lambda_j^2\|b^j\|_{L^\infty}^2\|\omega\|_{H_1}^2. \tag{41}$$

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Equations (40) and (41) imply that the sequences $(i,j) \mapsto |\langle \eta(db_1^i, db^j), \delta\omega \rangle_H|$ and $(i,j) \mapsto |\langle \lambda_j b_1^i b^j, \delta\omega \rangle_H|$ are both in ℓ^2. Therefore, by the Cauchy-Schwarz inequality for that space we can conclude that

$$
\sum_{j=1}^{J} \sum_{i=0}^{\infty} |\langle \delta b_1^{ij}, \delta\omega \rangle_H|^2 = \sum_{i=0}^{\infty} \sum_{j=1}^{J} |\langle -\eta(db_1^i, db^j) + \lambda_j b_1^i b^j, \delta\omega \rangle_H|^2
$$

$$
\leq \sum_{j=1}^{J} \sum_{i=0}^{\infty} |\langle \eta(db_1^i, db^j), \delta\omega \rangle_H|^2 + \sum_{j=1}^{J} \sum_{i=0}^{\infty} |\langle \lambda_j b_1^i b^j, \delta\omega \rangle_H|^2
$$

$$
+ 2 \sum_{j=1}^{J} \sum_{i=0}^{\infty} |\langle \eta(db_1^i, db^j), \delta\omega \rangle_H \langle \lambda_j b_1^i b^j, \delta\omega \rangle_H|
$$

$$
\leq \sum_{j=1}^{J} \sum_{i=0}^{\infty} |\langle \eta(db_1^i, db^j), \delta\omega \rangle_H|^2 + \sum_{j=1}^{J} \sum_{i=0}^{\infty} |\langle \lambda_j b_1^i b^j, \delta\omega \rangle_H|^2
$$

$$
+ 2 \left(\sum_{j=1}^{J} \sum_{i=0}^{\infty} |\langle \eta(db_1^i, db^j), \delta\omega \rangle_H|^2 \right)^{1/2} \left(\sum_{j=1}^{J} \sum_{k=0}^{\infty} |\langle \lambda_l b_1^k b^l, \delta\omega \rangle_H|^2 \right)^{1/2}
$$

$$
\leq V_J \|\omega\|_{H_1^1},
$$

where

$$
V_J = \left(\left(\sum_{j=1}^{J} \|\operatorname{grad} \phi_j\|_{L_{\bar{x}}^\infty}^2 \right)^{1/2} + \left(\sum_{j=0}^{J} \lambda_j^2 \|\phi_j\|_{L^\infty}^2 \right)^{1/2} \right)^2.
$$

This establishes the existence of the ω-independent constants V_J claimed in the lemma. Invoking L^∞ and Weyl bounds as in the case of U_J, we can also deduce that there exist a real number C_V and a finite integer q_V such that

$$
V_J \leq C_V J^{q_V}. \tag{42}
$$

Combining (39) and (42) leads to $U_J \leq \hat{C} J^q$ and $V_J \leq \hat{C} J^q$ with $\hat{C} = \max\{C_U, C_V\}$ and $q = \max\{q_U, q_V\}$. This completes our proof of Lemma 15(i) and thus the upper frame condition for $B_{1,1}^J$.

To prove Lemma 15(ii), we proceed as above to establish that

$$
\sum_{j=1}^{k} \sum_{i=0}^{\infty} |\langle d\tilde{b}_1^{ij}, d\omega \rangle_{H_2}|^2 \leq \tilde{U}_k \|\omega\|_{H_1^1}^2, \quad \tilde{U}_k = \sum_{j=1}^{k} e^{-2\lambda_j} \|D_{\operatorname{grad} b^j}\|^2,
$$

and

$$
\sum_{j=1}^{k} \sum_{i=0}^{\infty} |\langle \tilde{\delta} b_1^{ij}, \delta\omega \rangle_H|^2 \leq \tilde{V}_k \|\omega\|_{H_1^1}^2, \quad \tilde{V}_k = \left(\left(\sum_{j=1}^{k} e^{-2\lambda_j} \|\operatorname{grad} \phi_j\|_{L_{\bar{x}}^\infty}^2 \right)^{1/2} + \left(\sum_{j=0}^{k} e^{-2\lambda_j} \lambda_j^2 \|\phi_j\|_{L^\infty}^2 \right)^{1/2} \right)^2.
$$

The L^∞ and Weyl estimates in (37) and (38), respectively, then again imply that $\tilde{U} = \lim_{k \to \infty} \tilde{U}_k$ and $\tilde{V} = \lim_{k \to \infty} \tilde{V}_k$ is finite, proving Lemma 15(ii), and completing our proof of the upper frame condition for $\tilde{B}_{1,1}$.

Next, to verify the lower frame conditions, we express ω as a linear combination $\omega = \sum_{j=1}^{J} f_j \, d\phi_j$, where $f_1, \ldots, f_J$ are H^1 functions satisfying

$$
f_j(x) = \sum_{k=1}^{J} \Psi_{jk}^+(x) \eta(d\phi_k, \omega)
$$

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for μ-a.e. $x \in \mathcal{M}$, and $\Psi_{jk}^+(x)$ are the elements of the Moore-Penrose pseudoinverse of the $J \times J$ Gramm matrix

$$\Psi_{jk}(x) = \eta(d\phi_i, d\phi_j) = g(\operatorname{grad} \phi_i, \operatorname{grad} \phi_j)$$

from Section 4.2. Note that the existence of such an expansion for ω follows from the fact that $\{d\phi_1, \dots, d\phi_J\}$ is a generating set of the space of smooth 1-forms Ω^1, and the latter is dense in H_1^1. The f_j functions can be expanded in the $\{\phi_i^{(1)}\}_{i=0}^\infty$ basis of H^1 from (14), viz.

$$f_j = \sum_{i=0}^\infty c_{ij} \phi_i^{(1)}, \quad c_{ij} = \langle \phi_i^{(1)}, f_j \rangle_{H^1} = \sum_{k=1}^J \langle \phi_i^{(1)}, \Psi_{jk}^+ \eta(d\phi_k, \omega) \rangle_{H^1}.$$

Therefore, setting $b^i = \phi_i$ and $b_1^i = \phi_i^{(1)}$ per our notational convention for frame elements, we obtain

$$\|\omega\|_{H_1^1}^2 = \langle \omega, \omega \rangle_{H_1^1} = \sum_{i=0}^\infty \sum_{j,k=1}^J \langle b_1^i, \Psi_{jk}^+ \eta(db^k, \omega) \rangle_{H^1} \langle b_1^{ij}, \omega \rangle_{H_1^1} \le \sum_{i=0}^\infty \sum_{j,k=1}^J |\langle b_1^i, \Psi_{jk}^+, \eta(db^k, \omega) \rangle_{H^1} \langle b_1^{ij}, \omega \rangle_{H_1^1}|$$

$$\le \left(\sum_{i=0}^\infty \sum_{j,k=1}^J |\langle b_1^i, \Psi_{jk}^+ \eta(db^k, \omega) \rangle_{H^1}|^2 \right)^{1/2} \left(\sum_{n=0}^\infty \sum_{p=1}^J |\langle b_1^{ij}, \omega \rangle_{H_1^1}|^2 \right)^{1/2}. \tag{43}$$

Note that to arrive at the inequality in the last line we used the ℓ^2 Cauchy-Schwartz inequality on the sequences $(i,j) \mapsto \langle b_1^i, \Psi_{jk}^+ \eta(db^k, \omega) \rangle_{H^1}$ and $(i,j) \mapsto \langle b_1^{ij}, \omega \rangle_{H_1^1}$, both of which can be verified to indeed lie in that space. We now proceed to bound the first term in the last line.

First, since $\{b_1^i\}_{i=0}^\infty$ is an orthonormal basis of H^1, we have

$$\sum_{i=0}^\infty \sum_{j,k=1}^J |\langle b_1^i, \Psi_{jk}^+ \eta(db^k, \omega) \rangle_{H^1}|^2 = \sum_{j,k=1}^J \|\Psi_{jk}^+ \eta(db^k, \omega)\|_{H^1}^2.$$

Moreover, observe that for any $f \in H^1$ and $h \in C^\infty(\mathcal{M})$, $\|hf\|_{H^1}$ can be bounded above by $\tilde{C}\|f\|_{H^1}$, where $\tilde{C}^2$ is a polynomial function of $\|h\|_{L^\infty}$ and $\|\operatorname{grad} h\|_{L_{\mathfrak{X}}^\infty}$. This implies that there exists a finite constant C such that

$$\sum_{j,k=1}^J \|\Psi_{jk}^+ \eta(db^k, \omega)\|_{H^1}^2 \le C \sum_{k=1}^J \|\eta(db_k, \omega)\|_{H^1}^2 = C \sum_{k=1}^J \left(\|\eta(db^k, \omega)\|_H^2 + \|d\eta(db^k, \omega)\|_{H_1}^2 \right). \tag{44}$$

In the above, the term $\|\eta(db^k, \omega)\|_H^2$ can be bounded above by $\|\operatorname{grad} b^k\|_{L_{\mathfrak{X}}^\infty}^2 \|\omega\|_{H_1^1}$ using local Cauchy-Schwartz inequalities and the fact that $\|\cdot\|_{H_1} \le \|\cdot\|_{H_1^1}$. To bound $\|d\eta(db^k, \omega)\|_{H_1}$, we use (23) to write down

$$d\eta(db^k, \omega) = d\iota_{\operatorname{grad} b^k} \omega.$$

It therefore follows from Lemma 4 that there exists a constant $\hat{C}$ such that

$$\|d\eta(db^k, \omega)\| \le C \|\omega\|_{H_1^1}. \tag{45}$$

Combining (43)–(45), we conclude that there exists a constant $\bar{C}$ such that

$$\|\omega\|_{H_1^1}^2 \le \bar{C} \|\omega\|_{H_1^1} \left(\sum_{i=0}^\infty \sum_{j=1}^J |\langle b_1^{ij}, \omega \rangle_{H_1^1}|^2 \right)^{1/2},$$

and thus the lower frame condition for $B_{1,1}^J$ holds with $C_1 = 1/\bar{C}^2$. To verify the lower frame condition for $\tilde{B}_{1,1}$, we use the results just established to compute

$$\|\omega\|_{H_1^1}^2 \le C_1 \sum_{i=0}^\infty \sum_{j=1}^J |\langle b_1^{ij}, \omega \rangle_{H_1^1}|^2 \le C_1 e^{\lambda_J} \sum_{i=0}^\infty \sum_{j=1}^J |\langle \tilde{b}_1^{ij}, \omega \rangle_{H_1^1}|^2 \le C_1 e^{\lambda_J} \sum_{i=0}^\infty \sum_{j=1}^\infty |\langle \tilde{b}_1^{ij}, \omega \rangle_{H_1^1}|^2.$$

This proves the lower frame condition for $\tilde{B}_{1,1}$, and completes our proof of Theorem 7.

5. Galerkin method for the 1-Laplacian

We now apply the framework developed in Section 4 to construct a Galerkin approximation scheme for the eigenvalues and eigenforms of the 1-Laplacian, Δ_1. Eigenvalue problems for other differential operators of interest in exterior calculus can be formulated analogously. For notational simplicity, in this section we will use the symbols (ν_k, φ_k) to denote a general eigenvalue-eigenfunction pair of Δ_1, as opposed to the multi-index notation $(\lambda_{j,1}, \phi_{j,1})$ from Section 3.3.

5.1. Variational eigenvalue problem for the 1-Laplacian and its Galerkin approximation

We begin by stating the eigenvalue problem for the 1-Laplacian in strong form. This amounts to finding $\varphi_k \in \Omega^1$ and $\nu_k \in \mathbb{R}$ such that

$$\Delta_1 \varphi_k = \nu_k \varphi_k. \tag{46}$$

As is well known [18, 20], for the class of smooth closed Riemannian manifolds studied here, Δ_1 has a unique self-adjoint extension $\bar{\Delta}_1 : D(\bar{\Delta}_1) \mapsto H_1$, with a dense domain $D(\bar{\Delta}_1) \simeq H_1^2 \subset H_1$ and a compact resolvent. As a result, we can obtain weak solutions to (46) by passing to a variational formulation, with an associated well posed Galerkin approximation scheme [24]. To construct this variational eigenvalue problem, we introduce the sesquilinear forms $L_\theta : H_1^1 \times H_1^1 \mapsto \mathbb{C}$, $\theta > 0$, and $B : H_1^1 \times H_1^1 \mapsto \mathbb{C}$, defined by

$$L_\theta(\psi, \omega) = E_{1,1}(\psi, \omega) + \theta \langle \psi, \omega \rangle_{H_1}, \quad B(\psi, \omega) = \langle \psi, \omega \rangle_{H_1},$$

where $E_{1,1}$ is the Dirichlet form on H_1 from (26). Note that $E_{1,1}(\psi, \omega)$ can be formally obtained by performing integration by parts on the expression $\langle \psi, \Delta_1 \omega \rangle_{H_1}$, taking ψ and ω to be smooth 1-form fields. The term in $L_\theta(\psi, \omega)$ proportional to θ is a regularization term, ensuring that L_θ has a coercivity property important to the well-posedness of our Galerkin scheme. Specifically, we seek to solve the following variational eigenvalue problem:

Definition 17 (eigenvalue problem for the 1-Laplacian, weak form). Find $\nu_k \in \mathbb{R}$ and $\varphi_k \in H_1^1$, such that for all $\psi \in H_1^1$, the equality

$$L_\theta(\psi, \varphi_k) = \nu_k B(\psi, \varphi_k)$$

holds.

We refer to the solutions (ν_k, φ_k) of the problem in Definition 17 as weak eigenvalues and eigenfunctions of the 1-Laplacian, respectively. Clearly, every classical eigenvalue (eigenfunction) from (46) is also a weak eigenvalue (eigenfunction).

We now discuss the well-posedness of the weak eigenvalue just formulated, and establish its Galerkin approximation using our frames for H_1^1. A data-driven analog of this Galerkin method, utilizing frame elements of H_1^1 approximated from data, will be presented in Section 6.

Lemma 18. *The sesquilinear forms L_θ and B obey the bounds*

$$|L_\theta(\psi, \omega)| \leq (1 + \theta) \|\psi\|_{H_1^1} \|\omega\|_{H_1^1}, \quad L_\theta(\omega, \omega) \geq \min\{\theta, 1\} \|\omega\|_{H_1^1}^2, \quad |B(\psi, \omega)| \leq \|\psi\|_{H_1^1} \|\omega\|_{H_1^1},$$

for all $\psi, \omega \in H_1^1$.

Proof. The upper bounds on $|L_\theta(\psi, \omega)|$ and $|B(\psi, \omega)|$ follow directly from application of Cauchy-Schwartz inequalities. To verify the lower (coercivity) bound on $L_\theta(\omega, \omega)$, note that if $\theta \leq 1$, we have

$$L_\theta(\omega, \omega) = \theta(\theta^{-1} E_{1,1}(\omega, \omega) + \|\omega\|_{H_1}^2) \geq \theta(E_{1,1}(\omega, \omega) + \|\omega\|_{H_1}^2) = \theta \|\omega\|_{H_1^1}^2.$$

The claim for $\theta > 1$ can be verified similarly. $\qquad \square$

According to classical results in spectral approximation theory [24], Lemma 18 implies that there exists a compact operator $A_\theta : H_1^1 \mapsto H_1^1$ such that

$$L_\theta(\psi, A_\theta\omega) = L_\theta(\psi, \omega), \quad \forall \psi, \omega \in H_1^1. \tag{47}$$

This implies in turn that (ν_k, φ_k) is a weak eigenvalue-eigenvector pair if and only if

$$A_\theta \varphi_k = \nu_k^{-1} \varphi_k.$$

Due to the above, A_θ can be thought of as a "solution operator" for the variational eigenvalue problem in Definition 17. In particular, the properties of spectral approximations to the solutions of that problem can be analyzed in terms of approximations of the eigenvalue problem for A_θ. This approach leads to a Galerkin approximation scheme, as follows.

Let $\Pi_0, \Pi_1, \ldots$ be a family of finite-rank projection operators on H_1^1, converging pointwise to the identity; that is, $\Pi_l^2 = \Pi_l$ and $\lim_{l\to\infty} \Pi_l \omega = \omega$ for every $\omega \in H_1^1$. Let also W_l be the closed subspaces of H_1^1 defined as $W_l = \operatorname{ran} \Pi_l$. These spaces, which will be constructed explicitly below, will be our Galerkin approximation spaces. In particular, we will solve:

Definition 19 (eigenvalue problem for the 1-Laplacian, Galerkin approximation). Find $\nu_{k,l} \in \mathbb{R}$ and $\varphi_{k,l} \in W_l$, such that for all $\psi \in W_l$, the equality

$$L_\theta(\psi, \varphi_{k,l}) = \nu_{k,l} B(\psi, \varphi_{k,l})$$

holds.

Given a basis $\{v_{0,l}, \ldots, v_{q_l-1,l}\}$ of W_l, where $q_l = \dim W_l$, the eigenvalue problem in Definition 19 is equivalent to a generalized matrix eigenvalue problem. That is, $(\nu_{k,l}, \varphi_{k,l})$ with $\varphi_{k,l} = \sum_{j=0}^{q_l-1} c_j v_{j,l}$ is a solution if and only if

$$\boldsymbol{L}_l \vec{w}_{k,l} = \nu_{k,l} \boldsymbol{B}_l \vec{w}_{k,l}, \tag{48}$$

where $\boldsymbol{L}_l$ and $\boldsymbol{B}_l$ are the $q_l \times q_l$ matrices with elements $[\boldsymbol{L}_l]_{ij} = L_\theta(v_{i,l}, v_{j,l})$ and $[\boldsymbol{B}_l]_{ij} = B(v_{i,l}, v_{j,l})$, respectively, and $\vec{w}_{k,l} = (w_0, \ldots, w_{q_l-1})^\top \in \mathbb{R}^l$ the l-dimensional column vector containing the expansion coefficients of $\varphi_{k,l}$ in the $\{v_{j,l}\}$ basis of W_l. In addition, one can verify that $(\nu_{k,l}, \varphi_{k,l})$ solves the eigenvalue problem in Definition 19 if and only if

$$A_{\theta,l} \varphi_{k,l} = \nu_{k,l}^{-1} \varphi_{k,l},$$

where $A_{\theta,l} : H_1^1 \mapsto H_1^1$ is the finite-rank operator given by $A_{\theta,l} = \Pi_l A_\theta$. Note that this operator satisfies (cf. (47))

$$L_\theta(\psi, A_{\theta,l}\omega) = L_\theta(\psi, \omega), \quad \forall \psi \in W_l, \quad \forall \omega \in H_1^1.$$

Now, because A_θ is compact, the fact that Π_l converges pointwise to the identity implies that $A_{\theta,l}$ converges to A_θ in norm. This implies in turn that for every eigenvalue ν_k of A_θ (which is nonzero by coercivity of L_θ, and thus isolated and with finite geometric multiplicity by compactness of A_θ) there exists a sequence $\nu_{k,l}$ of eigenvalues of $A_{l,\theta}$ converging as $l \to \infty$ to ν_k. Moreover, for every eigenfunction φ_k in the eigenspace of A_θ at eigenvalue ν_k, there exists a sequence $\varphi_{k,l}$ of eigenfunctions of $A_{\theta,l}$ at eigenvalue $\nu_{k,l}$ converging in H_1^1 norm to φ_k. This establishes convergence of the solutions of the the Galerkin scheme in Definition 19 to those of the eigenvalue problem in Definition 17.

5.2. Construction of the Galerkin approximation spaces

What remains is to construct the projection operators Π_l and the associated subspaces W_l. Here, we will construct these operators making use of the result established in Theorem 7 that $\{b_1^{ij}\}$ with $i \in \{0, 1, \ldots\}$ and $j \in \{1, \ldots, J\}$ is a frame of H_1^1. For notational simplicity, we set $\alpha_k = b_1^{i_k j_k}$, where $k \mapsto (i_k, j_k)$ is any ordering of the (i, h) indices with $k \in \{0, 1, \ldots\}$. A convenient choice of such ordering is a lexicographical ordering, i.e., $(i_0, j_0) = (0, 1)$, $\ldots$, $(i_{J-1}, j_{J-1}) = (0, J)$, $(i_J, j_J) = (1, 0)$, $\ldots$. We also let $\{e_k\}_{k=0}^\infty$ be the canonical orthonormal basis of ℓ^2, and $\pi_l : \ell^2 \mapsto \ell^2$ the orthogonal projection operators with range

$\tilde{W}_l = \text{span}\{e_1, \ldots, e_l\}$. As in Section 4.1, $T : H_1^1 \mapsto \ell^2$, $T^* : \ell^2 \mapsto H_1^1$, and $S = T^*T : \ell^2 \mapsto \ell^2$ will be the analysis, synthesis, and frame operators, respectively, associated with the $\{\alpha_k\}$ frame of H_1^1.

We begin by introducing the finite-rank (hence, compact) analysis and synthesis operators, $T_l = \pi_l T$ and $T_l^* = T^*\pi_l$, respectively, where

$$T_l(\omega) = (\langle \alpha_0, \omega \rangle_{H_1^1}, \ldots, \langle \alpha_l, \omega \rangle_{H_1^1}, 0, 0, \ldots), \quad T_l(c) = \sum_{k=0}^{l-1} c_k \alpha_k,$$

for $\omega \in H_1^1$ and $c = (c_0, c_1, \ldots) \in \ell^2$. We also define $S_l = T_l^*T_l$, $G_l = T_l T_l^*$, and $G = TT^*$. Since $\{e_k\}$ is a basis, the projection operators π_l converge pointwise to the identity as $l \to \infty$ (i.e., $\pi_l c \to c$ for any $c \in \ell^2$), which implies that S_l and G_l converge pointwise to S and G, respectively. Note that, unlike S, G may be non-invertible since the frame elements α_k can be linearly dependent (though G is bounded by boundedness of T). Moreover, neither G nor S are compact.

By construction, at each l, the operator G_l is a positive-semidefinite, self-adjoint, compact operator on ℓ^2. As a result, there exists an orthonormal basis $\{u_{k,l}\}_{k=0}^\infty$ of ℓ^2 consisting of eigenvectors of G_l. We denote the corresponding eigenvalues by $\eta_{k,l}$, and order the eigenpairs $(\eta_{k,l}, u_{k,l})$ in order of decreasing $\eta_{k,l}$. We let q_l be the number of nonzero eigenvalues, where $q_l = \text{rank}\, T_l \leq l$. The following lemma, whose proof is left to the reader, summarizes certain properties of the $(\eta_{k,l}, u_{k,l})$ eigenpairs.

Lemma 20. *The eigenpairs $(\eta_{k,l}, u_{k,l})$ have the following properties:*
  (i) $\eta_{k,l}$ is bounded above by the operator norm $\|G\|$ of G.
  (ii) $\eta_{k,l}$ is a nonzero eigenvalue of G_l if and only if it is a nonzero eigenvalue of S_l. Moreover, the vectors

$$v_{k,l} = \frac{1}{\eta_{k,l}^{1/2}} T^* u_{k,l}, \quad 0 \leq k \leq q_l - 1,$$

are orthonormal eigenvectors of S_l corresponding to the same eigenvalues as $u_{k,l}$.

By Lemma (20)(ii), we can approach the problem of constructing orthonormal sets in H_1^1, consisting of eigenvectors of S_l corresponding to nonzero eigenvalues, through the eigenvalue problem of G_l. This is advantageous, since the eigenvectors of G_l corresponding to nonzero eigenvalues can be computed via the eigenvalue problem of the $l \times l$ Grammian matrix for H_1^1 introduced in Section 2.5, which we denote here by $\boldsymbol{G}_l^1$ to make its dependence on $l = MJ$ explicit. In particular, observe that given any $c = (c_0, c_1, \ldots) \in \ell^2$, we have $d = G_l c = (d_0, \ldots, d_{l-1}, 0, 0, \ldots)$, where the first l elements of d are given by $d_j = \sum_{j=0}^{l-1} G_{ij} c_j$, with

$$G_{ij} = \langle e_i, G e_j \rangle_{\ell^2} = \langle T^* e_i, T^* e_j \rangle_{H_1^1} = \langle \alpha_i, \alpha_j \rangle_{H_1^1},$$

and the inner products $\langle \alpha_i, \alpha_j \rangle_{H_1^1}$ can be computed in closed form via the formulas in Table 4. The above implies that the nonzero eigenvalues $\eta_{k,l}$ of G_l are equal to the nonzero eigenvalues of $\boldsymbol{G}_l^1$, and the corresponding eigenvectors $\vec{u}_{k,l} = (u_{0,k,l}, \ldots, u_{l-1,k,l})^\top$ of that matrix yield $u_{k,l} = (u_{0,k,l}, \ldots, u_{l-1,k,l}, 0, 0, \ldots)$. We thus obtain

$$v_{k,l} = \frac{1}{\eta_{k,l}^{1/2}} \sum_{j=0}^{l-1} u_{j,k,l} \alpha_j.$$

Based on these considerations, we define our Galerkin approximation spaces as

$$W_l = \text{span}\{v_{0,l}, \ldots, v_{q_l-1,l}\}, \tag{49}$$

and the projection operators $\Pi_l : H_1^1 \mapsto H_1^1$ as orthogonal projectors onto those subspaces.

Lemma 21. *The sequence Π_l of projection operators converges pointwise to the identity; that is, for any $\omega \in H_1^1$, $\lim_{l \to \infty} \Pi_l \omega = \omega$.*

Proof. Since the frame operator S has a bounded inverse, it suffices to show that $S\Pi_l\omega$ converges to $S\omega$ for any $\omega \in H_1^1$. To verify this, observe first that

$$T\Pi_l\omega = \sum_{k=0}^{q_l-1} \langle v_{k,l}, \omega \rangle_{H_1^1} T v_{k,l} = \sum_{k=0}^{q_l-1} \langle u_{k,l}, T\omega \rangle_{\ell^2} u_{k,l} = \pi_l \sum_{k=0}^{q_l-1} \langle u_{k,l}, T\omega \rangle_{\ell^2} u_{k,l} = \pi_l \sum_{k=0}^{\infty} \langle u_{k,l}, T\omega \rangle_{\ell^2} u_{k,l} = \pi_l T\omega.$$

We therefore have

$$S\Pi_l\omega = T^* T\Pi_l\omega = T^* \pi_l T\omega = S_l\omega,$$

which converges to $S\omega$ by the pointwise convergence of S_l to S. $\qquad\square$

Lemma 21 implies that with the choice of approximation spaces in (49), the Galerkin scheme in Definition 19 converges. Moreover, all of the matrix elements of the associated sesquilinear forms can be evaluated using Lemma 20 in conjunction with the formulas listed in Section 4.1 and Appendix Appendix B. Explicitly, the $q_l \times q_l$ matrices appearing in the generalized eigenvalue problem in (48) are given by

$$\boldsymbol{L}_l = \boldsymbol{H}_l^{-1/2} \boldsymbol{U}_l^\dagger (\boldsymbol{E}_l - \theta \boldsymbol{G}_l^1) \boldsymbol{U}_l \boldsymbol{H}_l^{-1/2}, \quad \boldsymbol{B}_l = \boldsymbol{H}_l^{-1/2} \boldsymbol{U}_l^\dagger \boldsymbol{G}_l \boldsymbol{U}_l \boldsymbol{H}_l^{-1/2}, \tag{50}$$

where $\boldsymbol{H}_l$ is a $q_l \times q_l$ diagonal matrix with $[\boldsymbol{H}_l]_{ii} = \eta_{i,l}$, $\boldsymbol{U}_l$ an $l \times q_l$ matrix with $[\boldsymbol{U}_l]_{ij} = u_{i,j,l}$, $\boldsymbol{E}_l$ an $l \times l$ matrix with $[\boldsymbol{E}_l]_{ij} = E_{1,1}(\alpha_i, \alpha_j)$ determined from Table 4, and $\boldsymbol{G}_l$ the $l \times l$ Grammian matrix for H_1 with $[\boldsymbol{G}_l]_{ij} = \langle \alpha_i, \alpha_j \rangle_{H_1}$ determined from Table 4. Note that the matrices $\boldsymbol{L}_l$ and $\boldsymbol{B}_l$ in (50) differ from the corresponding matrices appearing in the generalized eigenvalue problem in (9) in that they include $\boldsymbol{H}_l$- and θ-dependent terms, which do not appear in (9). Due to the absence of these terms, (9) represents the problem in Definition 19 in a basis of W_l that exhibits unbounded growth of H_1^1 norm with l (controlled by the $\boldsymbol{H}_l$ terms in (50)), and is also not compatible with the coercivity condition in Lemma 18 (enforced by the θ-dependent terms). While both of these issues could potentially affect the numerical conditioning of (9), especially at large spectral orders l, in the examples studied in Section 7 we found that (9) and (50) perform comparably.

6. Data-driven approximation

All of the schemes in Sections 4 and 5 can be implemented given knowledge of the eigenvalues and eigenfunctions of the Laplacian on functions. The problem of approximating these objects from finite sets of points in a purely data-driven manner (that is, without requiring explicit knowledge of the manifold $\mathcal{M}$ and/or its embedding in data space) has been studied extensively in recent years [1–5, 7, 8], leading to the development of approximation techniques with well-established spectral convergence guarantees. In this section, we summarize the main properties of these techniques, and describe the analogs of the methods of Sections 4 and 5 in a data-driven, discrete setting.

6.1. Assumptions for data-driven approximation

We consider that the Riemannian manifold $(\mathcal{M}, g)$ is embedded in n-dimensional Euclidean space by means of a smooth, isometric embedding $F : \mathcal{M} \mapsto \mathbb{R}^n$. Using the notation $y \cdot z = \sum_{i=1}^n y^i z^i$ to represent the canonical Euclidean inner product between two vectors $y = (y^1, \ldots, y^n)$ and $z = (z^1, \ldots, z^n)$ in $\mathbb{R}^n$, we thus have $g_x(u, v) = F_{*,x} u \cdot F_{*,x} v$ for any point $x \in \mathcal{M}$ and tangent vectors $u, v \in T_x\mathcal{M}$, where $F_{*,x} : T_x\mathcal{M} \mapsto T_{F(x)}\mathbb{R}^n$ is the pushforward map on tangent vectors associated with F, and we have used the canonical isomorphism $T_{F(x)}\mathbb{R}^n \simeq \mathbb{R}^n$. If the embedding F is not isometric, then the method described below can be modified via the techniques developed in [8] to yield approximations of Laplacian eigenvalues and eigenfunctions with respect to any (known) metric g.

We also assume that we have access to a dataset consisting of N samples $y_1, \ldots, y_N$ in $\mathbb{R}^n$ with $y_j = F(x_j)$ taken on a sequence of points $x_1, x_2, \ldots$ in $\mathcal{M}$, which is equidistributed with respect to a smooth sampling measure σ supported on $\mathcal{M}$. By that, we mean that given any continuous function $f : \mathcal{M} \mapsto \mathbb{C}$, the result

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^N f(x_j) = \int_{\mathcal{M}} f \, d\sigma \tag{51}$$

holds, and moreover σ has a smooth density $\rho = d\sigma/d\mu$ with respect to the Riemannian measure μ (bounded away from zero by compactness of $\mathcal{M}$). Such an equidistributed sequence can be provided, e.g., by i.i.d. points on $\mathcal{M}$ (as is commonly assumed in machine learning applications), or by an orbit of an ergodic dynamical system (in which case, the x_j are not independent). The requirement in (51) is equivalent to assuming that the sequence of sampling measures $\sigma_N = N^{-1} \sum_{n=0}^{N-1} \delta_{x_j}$ weak-* converges to σ; that is,

$$\lim_{N\to\infty} \int_{\mathcal{M}} f \, d\sigma_N = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} f(x_j) = \int_{\mathcal{M}} f \, d\sigma, \quad \forall f \in C(\mathcal{M}).$$

In this data-driven setting, we will be working with the N-dimensional Hilbert space $L^2(\mathcal{M}, \sigma_N)$ associated with the discrete sampling measure σ_N, equipped with the inner product

$$\langle f, h \rangle_{L^2(\mathcal{M}, \sigma_N)} = \int_{\mathcal{M}} f^* h \, d\sigma_N = \frac{1}{N} \sum_{j=1}^{N} f^*(x_j) h(x_j).$$

Note that $L^2(\mathcal{M}, \sigma_N)$ consists of equivalence classes of functions on $\mathcal{M}$ which are equal up to sets of zero σ_N measure; that is, two functions $f : \mathcal{M} \mapsto \mathbb{C}$ and $h : \mathcal{M} \mapsto \mathbb{C}$ satisfying $f(x_j) = h(x_j)$ for all $j \in \{1, \ldots, N\}$, but taking arbitrarily different values at other points, lie in the same $L^2(\mathcal{M}, \sigma_N)$ equivalence class. Clearly, $L^2(\mathcal{M}, \sigma_N)$ is isomorphic as a Hilbert space to $\mathbb{C}^N$ equipped with the normalized dot product $f \cdot g/N$, but here we prefer to work with $L^2(\mathcal{M}, \sigma_N)$ to emphasize the fact that our data-driven approximation spaces contain equivalence classes of functions on the same underlying manifold as the equivalence classes comprising $L^2(\mathcal{M}, \mu)$.

6.2. Kernel method for the eigenvalues and eigenfunctions of the Laplacian on functions

Following the approach introduced in the diffusion maps algorithm [3], and further generalized in [5], we compute data-driven approximations of the eigenvalues and eigenfunctions of the Laplacian on functions through the eigenvalues and eigenfunctions of a kernel integral operator approximating, in a suitable sense, the heat operator $\mathcal{P}^\tau = e^{-\tau\bar\Delta}$, $\tau \geq 0$, on $L^2(\mathcal{M}, \mu)$. This kernel kernel integral operator is constructed from a smooth, exponentially decaying kernel $k_\epsilon : \mathcal{M} \times \mathcal{M} \mapsto \mathbb{R}_+$, bounded away from zero. Here, as a concrete example, we work with a Gaussian kernel,

$$k_\epsilon(x, x') = \exp\left(-\frac{\|F(x) - F(x')\|_{\mathbb{R}^n}^2}{\epsilon}\right),$$

where ϵ is a positive bandwidth parameter. Approximation techniques based on other classes of kernels, including kernels with variable bandwith functions [7, 8], have equivalent asymptotic properties while generally achieving higher performance in terms of approximation accuracy and noise robustness, particularly in applications with large variations in the sampling density ρ.

Having specified an appropriate kernel, we introduce the associated kernel integral operators $\hat{K}_\epsilon : L^2(\mathcal{M}, \mu) \mapsto C(\mathcal{M})$ and $\hat{K}_{\epsilon,N} : L^2(\mathcal{M}, \sigma_N) \mapsto C(\mathcal{M})$, where

$$\hat{K}_\epsilon f = \int_{\mathcal{M}} k_\epsilon(\cdot, x) f(x) \rho(x) \, d\mu(x), \quad \hat{K}_{\epsilon,N} h = \int_{\mathcal{M}} k_\epsilon(\cdot, x) h(x) \, d\sigma_N(x).$$

Composing $\hat{K}_\epsilon$ with the canonical inclusion operator $\iota : C(\mathcal{M}) \mapsto L^2(\mathcal{M}, \mu)$, we also define $\tilde{K}_\epsilon : L^2(\mathcal{M}, \mu) \mapsto L^2(\mathcal{M}, \mu)$ and $K_\epsilon : C(\mathcal{M}) \mapsto C(\mathcal{M})$, where $\tilde{K}_\epsilon = \iota\hat{K}_\epsilon$ and $K_\epsilon = \hat{K}_\epsilon\iota$. Similarly, we define $\tilde{K}_{\epsilon,N} : L^2(\mathcal{M}, \sigma_N) \mapsto L^2(\mathcal{M}, \sigma_N)$ and $K_{\epsilon,N} : C(\mathcal{M}) \mapsto C(\mathcal{M})$, where $\tilde{K}_\epsilon = \iota_N\hat{K}_{\epsilon,N}$, $K_{\epsilon,N} = \hat{K}_{\epsilon,N}\iota_N$, and $\iota_N : C(\mathcal{M}) \mapsto L^2(\mathcal{M}, \sigma_N)$ is the canonical restriction operator from $C(\mathcal{M})$ to $L^2(\mathcal{M}, \sigma_N)$.

Proposition 22. *The operators $\tilde{K}_\epsilon$, K_ϵ, $\tilde{K}_{\epsilon,N}$, and $K_{\epsilon,N}$ have the following properties.*
(i) They are all compact.
(ii) As $N \to \infty$, $K_{\epsilon,N}$ converges pointwise to K_ϵ; that is, for any $f \in C(\mathcal{M})$, we have $\lim_{N\to\infty} K_{\epsilon,N}f = K_\epsilon f$ in uniform norm.

Proof. (i) That $\tilde{K}_{\epsilon,N}$ and $K_{\epsilon,N}$ are compact follows immediately from the fact that they have finite rank. The compactness of $\tilde{K}_\epsilon$ follows from the facts that k_ϵ is a Hilbert-Schmidt kernel on $L^2(\mathcal{M} \times \mathcal{M}, \sigma \times \sigma)$ (i.e., $\int_\mathcal{M} \int_\mathcal{M} k_\epsilon(x, x') \, d\sigma(x) \, d\sigma(x') < \infty$), and $L^2(\mathcal{M}, \mu)$ and $L^2(\mathcal{M}, \sigma)$ are isomorphic Hilbert spaces (by smoothness of ρ and compactness of $\mathcal{M}$). The compactness of K_ϵ can be verified using the Arzelà-Ascoli theorem in conjunction with the continuity of k_ϵ; see, e.g., [16].

(ii) The claim is a direct consequence of (51) and the fact that $\int_\mathcal{M} f \rho \, d\mu = \int_\mathcal{M} f \, d\sigma$ for any $f \in L^2(\mathcal{M}, \mu)$. $\square$

Proposition 22(ii) shows that, on $C(\mathcal{M})$, we can approximate kernel integral operators with respect to the Riemannian measure by kernel integral operators with respect to the sampling measure. However, the pointwise convergence established there does not, in general, imply spectral convergence for these operators. Moreover, in applications we work in the finite-dimensional Hilbert space $L^2(\mathcal{M}, \sigma_N)$, as opposed to the infinite-dimensional Banach space $C(\mathcal{M})$, which necessitates establishing connections between the spectral properties of $\tilde{K}_{\epsilon,N}$ and $K_{\epsilon,N}$. Another issue that must be addressed is that of approximating the heat operator $\mathcal{P}^\tau$ by a suitable modification of $\tilde{K}_\epsilon$.

Following [3, 5], we proceed by normalizing k_ϵ to construct a smooth Markov kernel $p_\epsilon : \mathcal{M} \times \mathcal{M} \mapsto \mathbb{R}_+$, satisfying $\int_\mathcal{M} p_\epsilon(\cdot, x) \, d\mu(x) = 1$. For that, we introduce the normalization functions $r_\epsilon = \hat{K}_\epsilon 1$ and $l_\epsilon = \hat{K}_\epsilon(1/r_\epsilon)$, which are both smooth, positive, and bounded away from zero, and define

$$p_\epsilon(x, x') = \frac{k_\epsilon(x, x')}{l_\epsilon(x) r_\epsilon(x')}.$$

The Markov property of p_ϵ then follows by construction. In the data-driven case, we define

$$r_{\epsilon,N} = \hat{K}_{\epsilon,N} 1, \quad l_{\epsilon,N} = \hat{K}_{\epsilon,N}(1/r_{\epsilon,N}), \quad p_{\epsilon,N}(x, x') = \frac{k_\epsilon(x, x')}{l_{\epsilon,N}(x) r_{\epsilon,N}(x')},$$

and $p_{\epsilon,N}$ is a smooth Markov kernel satisfying $\int_\mathcal{M} p_{\epsilon,N}(\cdot, x) \, d\sigma_N(x) = 1$. As in the case of the kernel k_ϵ, we define the kernel integral operators $\hat{P}_\epsilon : L^2(\mathcal{M}, \mu) \mapsto C(\mathcal{M})$ and $\hat{P}_{\epsilon,N} : L^2(\mathcal{M}, \sigma_N) \mapsto C(\mathcal{M})$ via

$$\hat{P}_\epsilon f = \int_\mathcal{M} p_\epsilon(\cdot, x) f(x) \rho(x) \, d\mu(x), \quad \hat{P}_{\epsilon,N} h = \int_\mathcal{M} p_{\epsilon,N}(\cdot, x) h(x) \, d\sigma_N(x),$$

and also introduce the operators $\tilde{P}_\epsilon : L^2(\mathcal{M}, \mu) \mapsto L^2(\mathcal{M}, \mu)$, $P_\epsilon : C(\mathcal{M}) \mapsto C(\mathcal{M})$, $\tilde{P}_{\epsilon,N} : L^2(\mathcal{M}, \sigma_N) \mapsto L^2(\mathcal{M}, \sigma_N)$, and $P_{\epsilon,N} : C(\mathcal{M}) \mapsto C(\mathcal{M})$, where $\tilde{P}_\epsilon = \iota \hat{P}_\epsilon$, $P_\epsilon = \hat{P}_\epsilon \iota$, $\tilde{P}_{\epsilon,N} = \iota_N \hat{P}_{\epsilon,N}$, and $P_{\epsilon,N} = \hat{P}_{\epsilon,N} \iota_N$. These operators have the analogous properties to those stated in Proposition 22. Among them, $\tilde{P}_{\epsilon,N}$ is represented by the Markov matrix $\boldsymbol{P}$ from (5) (note that, due to cancellation of terms, the normalization of $k_\epsilon(x, x')$ by $r_{\epsilon,N}$ and $l_{\epsilon,N}$ to construct $p_{\epsilon,N}$ is equivalent to the normalization procedure used to construct $\boldsymbol{P}$; see [5] for details). The following theorem summarizes how the nonzero eigenvalues and corresponding eigenvectors of $\tilde{P}_\epsilon$ can be approximated by the corresponding eigenvalues and eigenvectors of $\tilde{P}_{\epsilon,N}$, which are accessible from data as described in Section 2.1.

Theorem 23. *The following hold:*

(i) $\Lambda_{j,\epsilon}$ is a nonzero eigenvalue of $\tilde{P}_\epsilon$ if and only if it is a nonzero eigenvalue of P_ϵ. Similarly, $\Lambda_{j,\epsilon,N}$ is a nonzero eigenvalue of $\tilde{P}_{\epsilon,N}$ if and only if it is a nonzero eigenvalue of $P_{\epsilon,N}$. Moreover, the $\Lambda_{j,\epsilon,N}$ and $\Lambda_{j,\epsilon,N}$ are all real, and thus lie in the interval $[-1, 1]$ by the Markov property of p_ϵ and $p_{\epsilon,N}$.

(ii) If $\tilde{\phi}_{j,\epsilon} \in L^2(\mathcal{M}, \mu)$ and $\vec{\phi}_{j,\epsilon,N} \in L^2(\mathcal{M}, \sigma_N)$ are eigenvectors of P_ϵ and $P_{\epsilon,N}$ at eigenvalues $\Lambda_{j,\epsilon}$ and $\Lambda_{j,\epsilon,N}$, respectively, then the smooth functions

$$\phi_{j,\epsilon} = \frac{1}{\Lambda_{j,\epsilon}} \hat{P}_\epsilon \tilde{\phi}_{j,\epsilon}, \quad \phi_{j,\epsilon,N} = \frac{1}{\Lambda_{j,\epsilon,N}} \hat{P}_{\epsilon,N} \vec{\phi}_{j,\epsilon,N}$$

are eigenvectors of P_ϵ and $P_{\epsilon,N}$, respectively, at the same eigenvalues.

(iii) For every nonzero eigenvalue $\Lambda_{j,\epsilon}$ of P_ϵ, the sequence of eigenvalues $\Lambda_{j,\epsilon,N}$ of $P_{\epsilon,N}$ converges as $N \to \infty$ to $\Lambda_{j,\epsilon}$. Moreover, if $\phi_{j,\epsilon}$ is an eigenvector of P_ϵ corresponding to eigenvalue $\Lambda_{j,\epsilon}$, then there exists a sequence of eigenvectors $\phi_{j,\epsilon,N}$ of $P_{\epsilon,N}$ at eigenvalues $\Lambda_{j,\epsilon,N}$, converging as $N \to \infty$ to $\phi_{j,\epsilon}$ in uniform norm.

Proof. (i,ii) The claims on the relationships between the eigenvalues and eigenvectors of $\tilde{P}_\epsilon$ and P_ϵ (and those of $\tilde{P}_{\epsilon,N}$ and $P_{\epsilon,N}$) can be verified from the definition of these operators. In addition, it can be verified that $\tilde{P}_\epsilon$ and $\tilde{P}_{\epsilon,N}$ are related to self-adjoint operators by similarity transformations, which implies that their eigenvalues are real.

(iii) The convergence of the eigenvalues follows by showing that the operators $P_{\epsilon,N}$ converge compactly to P_ϵ (a stronger notion of convergence than pointwise convergence, but weaker than convergence in operator norm); see [4] for additional details. A proof of the convergence of the eigenvectors can be found in [16]. We also note that [4] establishes pointwise convergence of projection operators onto the corresponding eigenspaces. $\qquad\square$

That we can approximate eigenvalues and eigenvectors of $\tilde{P}_\epsilon$ through eigenvalues and eigenvectors of $\tilde{P}_{\epsilon,N}$ is important since the eigenvalue problem for the latter operator is equivalent to the numerically solvable $N \times N$ matrix eigenvalue problem for $\boldsymbol{P}$. What remains is to establish that the eigenvalues and eigenvectors of $\tilde{P}_\epsilon$ approximate in turn eigenvalues and eigenvectors of the heat operator (and thus the Laplacian).

Theorem 24. *For any $\tau > 0$ and as $\epsilon \to 0^+$, the operators $\tilde{P}_\epsilon^{\tau/\epsilon}$ converge to $\mathcal{P}^\tau$ in operator norm.*

Proof. The claim was proved in [3]. $\qquad\square$

For compact operators, convergence in operator norm implies convergence of their nonzero eigenvalues and the corresponding finite-dimensional eigenspaces. Thus, since the j-th eigenvalue of $\mathcal{P}^\tau$ is given by $e^{-\tau\lambda_j}$, we can conclude from Theorems 23 and 24 that the quantities

$$\lambda_{j,\epsilon} = \frac{1}{\epsilon}|\log|\Lambda_{j,\epsilon}||, \quad \lambda_{j,\epsilon,N} = \frac{1}{\epsilon}|\log|\Lambda_{j,\epsilon,N}||, \tag{52}$$

converge to the eigenvalues of the Laplacian, i.e.,

$$\lim_{\epsilon\to 0^+}\lim_{N\to\infty}\lambda_{j,\epsilon,N} = \lim_{\epsilon\to 0^+}\lambda_{j,\epsilon} = \lambda_j.$$

Moreover, for any eigenfunction $\phi_j \in C^\infty(\mathcal{M})$ of Δ at eigenvalue λ_j, there exist eigenfunctions $\tilde{\phi}_{j,\epsilon}$ and $\tilde{\phi}_{j,\epsilon,N}$ of P_ϵ and $P_{\epsilon,N}$, respectively, such that the smooth functions

$$\phi_{j,\epsilon} = \frac{1}{\Lambda_{j,\epsilon}}\hat{P}_\epsilon\tilde{\phi}_{j,\epsilon}, \quad \phi_{j,\epsilon,N} = \frac{1}{\Lambda_{j,\epsilon,N}}\hat{P}_{\epsilon,N}\vec{\phi}_{j,\epsilon,N} \tag{53}$$

satisfy

$$\lim_{\epsilon\to 0^+}\lim_{N\to\infty}\phi_{j,\epsilon,N} = \lim_{\epsilon\to 0^+}\phi_{j,\epsilon} = \phi_j,$$

where the limits are taken with respect to uniform norm.

6.3. Data-driven frame elements and approximation of sesquilinear forms

Using the approximate eigenvalues and eigenfunctions from (52) and (53), we can construct data-driven analogs of the various basis and frame elements for functions, vector fields, and forms introduced in Sections 3 and 4. For example,

$$\phi_{j,\epsilon,N}^{(p)} = \frac{\phi_{j,\epsilon,N}}{\lambda_{j,\epsilon,N}^{p/2}}, \quad b_{ij,\epsilon,N}^{(p)} = \frac{\phi_{i,\epsilon,N}\,\mathrm{grad}\,\phi_{j,\epsilon,N}}{\lambda_{i,\epsilon,N}^{p/2}}, \quad b_{p,\epsilon,N}^{ij_1\cdots j_k} = \frac{d\phi_{j_1,\epsilon,N} \wedge \cdots \wedge d\phi_{j_k,\epsilon,N}}{\lambda_{i,\epsilon,N}^{p/2}}$$

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are data-driven analogs of the basis functions $\phi_j^{(p)}$ in (14), the frame elements $b_{ij}^{(p)}$ in (30) for vector fields, and the frame elements $b_p^{i j_1 \cdots j_k}$ in (31) for forms, respectively. All of these objects are "concrete", i.e., the take values pointwise on $\mathcal{M}$ as opposed to being defined up to null sets. Moreover, their pointwise evaluation in practice relies on the ability to compute derivatives of the kernel k_ϵ.

In SEC, however, it oftentimes suffices to consider quantities that can be computed using only the "weak" counterparts $\tilde{\phi}_{j,\epsilon,N}$ of $\phi_{j,\epsilon,N}$ lying in $L^2(\mathcal{M}, \sigma_N)$. As a concrete example, consider the Galerkin method for the 1-Laplacian in Definition 19. To construct a data-driven analog of this scheme we compute the following quantities, using the shorthand notation $\alpha_{k,\epsilon,N} = b_{1,\epsilon,N}^{i_k j_k}$ as in Section 5:

1. Triple products:
$$c_{ijk,\epsilon,N} = \langle \vec{\phi}_{i,\epsilon,N}, \vec{\phi}_{j,\epsilon,N} \vec{\phi}_{k,\epsilon,N} \rangle_{L^2(\mathcal{M}, \sigma_N)};$$

2. Approximations $G_{ij,\epsilon,N}$ and $E_{ij,\epsilon,N}$ of the H_1 inner products $\langle \alpha_{i,\epsilon,N}, \alpha_{j,\epsilon,N} \rangle_{H_1}$ and Dirichlet energies $E_{1,1}(\alpha_{i,\epsilon,N}, \alpha_{j,\epsilon,N})$, computed via the formulas in Table 4, with the eigenvalues λ_i and triple products c_{ijk} replaced by $\lambda_{i,\epsilon,N}$ and $c_{ijk,\epsilon,N}$, respectively.

3. Approximations $u_{k,l,\epsilon,N} \in \ell^2$ of the eigenvectors of the operator G_l, computed by solving the eigenvalue problem of the $l \times l$ matrix $\boldsymbol{G}_{l,\epsilon,N}^1$ with elements $[\boldsymbol{G}_{l,\epsilon,N}]_{ij} = G_{ij,\epsilon,N} + E_{ij,\epsilon,N}$.

4. Approximation of the $q_l \times q_l$ matrices $\boldsymbol{L}_l$ and $\boldsymbol{B}_l$ by $q_{l,\epsilon,N} \times q_{l,\epsilon,N}$ matrices $\boldsymbol{L}_{l,\epsilon,N}$ and $\boldsymbol{B}_{l,\epsilon,N}$, respectively, where $q_{l,\epsilon,N}$ is the number of nonzero eigenvalues of $\boldsymbol{G}_{l,\epsilon,N}$, and $\boldsymbol{L}_{l,\epsilon,N}$ and $\boldsymbol{B}_{l,\epsilon,N}$ are computed via (50), using the results of steps 2 and 3 above as appropriate.

By Theorems 23 and 24, $\lim_{\epsilon \to 0^+} \lim_{N \to \infty} \boldsymbol{L}_{l,\epsilon,N} = \boldsymbol{L}_l$ and $\lim_{\epsilon \to 0^+} \lim_{N \to \infty} \boldsymbol{B}_{l,\epsilon,N} = \boldsymbol{B}_l$ in any matrix norm. Thus, for any solution $(\nu_{k,l}, \vec{c}_{k,l})$ of the generalized eigenvalue problem in (48) there exist solutions $(\nu_{k,l,\epsilon,N}, \vec{c}_{k,l,\epsilon,N})$ of the generalized eigenvalue problems

$$\boldsymbol{L}_{l,\epsilon,N} \vec{c}_{k,l,\epsilon,N} = \nu_{k,l,\epsilon,N} \boldsymbol{B}_{l,\epsilon,N} \vec{c}_{k,l,\epsilon,N},$$

such that $\lim_{\epsilon \to 0^+} \lim_{N \to \infty} \nu_{k,l,\epsilon,N} = \nu_{k,l}$ and $\lim_{\epsilon \to 0^+} \lim_{N \to \infty} \vec{c}_{k,l,\epsilon,N} = \vec{c}_{k,l}$. The latter, in conjunction with the convergence of the Laplacian eigenfunctions in Theorem 23(iii), implies that the reconstructed 1-forms

$$\varphi_{k,l,\epsilon,N} = \sum_{j=0}^{q_{l,\epsilon,N}} [\vec{c}_{k,l,\epsilon,N}]_j \alpha_{j,\epsilon,N}$$

converge, in H_1^1 norm, to the reconstructed form $\varphi_{k,l}$ associated with $\vec{c}_{k,l}$. By convergence of the Galerkin scheme in Definition 19, we therefore conclude that

$$\lim_{l \to \infty} \lim_{\epsilon \to 0^+} \lim_{N \to \infty} \nu_{k,l,\epsilon,N} = \nu_k, \quad \lim_{l \to \infty} \lim_{\epsilon \to 0^+} \lim_{N \to \infty} \varphi_{k,l,\epsilon,N} = \varphi_{k,l},$$

where (ν_k, φ_k) is a weak eigenvalue-eigenvector pair of the 1-Laplacian, solving the variational eigenvalue problem in Definition 17.

7. Numerical examples

In this section, we apply the SEC to several smooth manifolds and a fractal set to verify and demonstrate the utility of our approach. In each example, we constructed the 1-Laplacian and its eigenvalue and eigenforms using the same procedure, which we describe here. First, we applied the diffusion maps algorithm described in Section 2.1 to the data in order to estimate the eigenvalues and eigenfunctions of the 0-Laplacian. Setting $M = 20$ eigenfunctions, we used 100 eigenfunctions to compute the $M \times M \times M$ tensor c which is the Fourier representation of function multiplication (see Table 3 in Section 2.2). Using the formulas in Table 4 in Section 2.5, we constructed the $M^2 \times M^2$ energy matrix $\hat{\boldsymbol{E}}$ and Hodge Grammian $\hat{\boldsymbol{G}}$ for the anti-symmetric formulation of the SEC. Following the procedure in Section 2.5, we then projected the eigenvalue problem onto the appropriate Sobolev H_1^1 basis, and computed the eigenvalues and eigenforms of the 1-Laplacian. Finally, we visualized the vector fields corresponding to the eigenforms by computing their
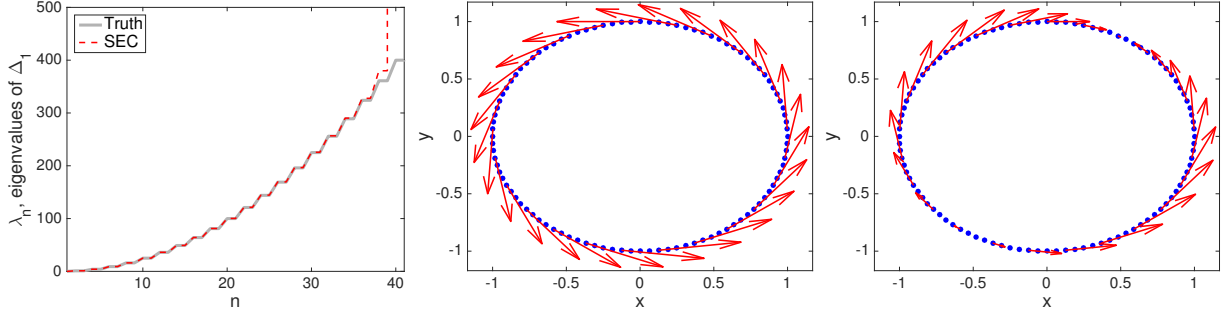
Figure 1: Left: Eigenvalues of the Laplacian on 1-forms estimated using the SEC (red, dashed) compared to the analytical spectrum (gray, solid) for a unit circle. Middle: The vector field representation of the 1-form corresponding to the smallest eigenvalue ($\nu_1 \approx 2.46e - 06$) clearly corresponds to the harmonic form $d\theta$. Right: The eigenform of the second smallest eigenvalue ($\nu_2 \approx 1.06$) is a linear combination of $\sin(\theta)d\theta$ and $\cos(\theta)d\theta$, which both have eigenvalue 1.

operator representation and pushing forward these vector fields into the original data space as described in Section 2.5.

In the supplementary material we have included Matlab code which implements the SEC 1-Laplacian construction along with a Diffusion Maps implementation. We also include code that generates all the data sets shown below and a simple "DEMO.m" file to replicate our results.

7.1. Validation of the SEC 1-Laplacian spectra

In this section we apply the SEC based construction of the 1-Laplacian to two examples where the spectrum of the 1-Laplacian can easily be worked out analytically.

We first consider the circle S^1, where there is a one-dimensional global coordinate θ so that every 1-form can be represented as $fd\theta$, and $d\theta$ is harmonic. Note that $d(fd\theta) = df \wedge d\theta = 0$, so that $\Delta_1(fd\theta) = \delta d(fd\theta) + d\delta(fd\theta) = d\delta(fd\theta)$, and $\delta d\theta = \Delta\theta = 0$ on S^1. Therefore,

$$\Delta_1(fd\theta) = d\delta(fd\theta) = d(-g(df, d\theta) + f\delta d\theta) = -d(g(df, d\theta)) = -d(d\theta(\nabla f)) = -d\left(\frac{df}{d\theta}\right) = -\frac{d^2f}{d\theta^2}d\theta,$$

meaning that $\Delta_1(fd\theta) = \Delta(f)d\theta$. Thus, the eigenforms of the 1-Laplacian on S^1 are simply $\sin(k\theta)d\theta$ and $\cos(k\theta)d\theta$, and the eigenvalues are the same as those of Δ which are simply $\{0, 1, 1, 4, 4, ..., k^2, k^2, ...\}$.

We apply the SEC by generating 101 uniformly spaced data points on the unit circle in $\mathbb{R}^2$, and using the method outlined at the beginning of the section. In the leftmost plot in Fig. 1, we compare the analytic eigenvalues of Δ_1 (gray, solid) to the eigenvalues estimated by the SEC (red, dashed). We also show the first two eigenforms of the 1-Laplacian in Fig. 1, and note that the first eigenform clearly approximates the harmonic form $d\theta$. Notice that a closed integral curve of a harmonic form should correspond to a unique representative of a nontrivial 1-homology class, as will be further demonstrated below.

We next consider the 1-Laplacian on the flat torus which has two nontrivial 1-homology classes. Every smooth 1-form on the flat torus can be written as $fd\theta + hd\phi$, where θ, ϕ are canonical angle coordinates and f, g smooth functions. Considering $fd\theta$, we compute

$$\delta d(fd\theta) = \delta(df \wedge d\theta) = \delta\left(\frac{df}{d\phi}d\phi \wedge d\theta\right) = \delta\left(-\frac{df}{d\phi}d\theta \wedge d\phi\right) = -*d*\left(-\frac{df}{d\phi}d\theta \wedge d\phi\right)$$

$$= *d\left(\frac{df}{d\phi}\right) = *\left(\frac{d^2f}{d\phi^2}d\phi + \frac{d^2f}{d\theta d\phi}d\theta\right) = -\frac{d^2f}{d\phi^2}d\theta + \frac{d^2f}{d\theta d\phi}d\phi, \tag{54}$$

where we choose the ordering $d\theta \wedge d\phi = -d\phi \wedge d\theta$ so that $*d\theta = d\phi$ and $*d\phi = -d\theta$. Next,

$$d\delta(fd\theta) = -d(*d*(fd\theta)) = -d*d(fd\phi) = -d*(df \wedge d\phi) = -d*\left(\frac{df}{d\theta}d\theta \wedge d\phi\right)$$
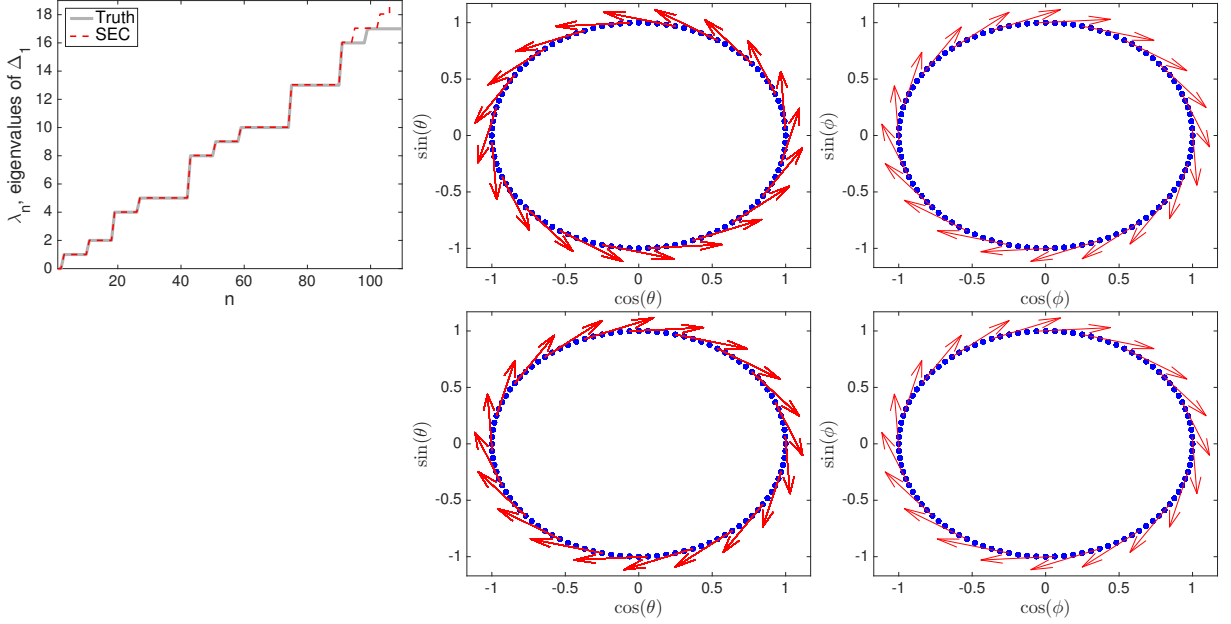
Figure 2: Top, Left: Eigenvalues of the Laplacian on 1-forms estimated using the SEC (red, dashed) compared to the analytical spectrum (gray, solid) for a flat torus. Top, Middle/Right: The vector field representation of the 1-form corresponding to the smallest eigenvalue ($\nu_1 \approx 1.20 \times 10^{-4}$) shown in the embedding coordinates $(\cos(\theta), \sin(\theta))$ (middle) and $(\cos(\phi), \sin(\phi))$ (right); this eigenform approximates $d\theta - d\phi$ (for clarity we only draw arrows every fifth data point). Bottom, Middle/Right: The eigenform of the second smallest eigenvalue ($\nu_2 \approx 2.43 \times 10^{-4}$) shown in the same coordinates as the first eigenform; this eigenform approximates $-d\theta - d\phi$. Notice that the two eigenforms together form a basis for the harmonic 1-forms which are the span of $d\theta$ and $d\phi$.

$$= -d\left(\frac{df}{d\theta}\right) = -\frac{d^2 f}{d\theta^2} d\theta - \frac{d^2 f}{d\phi d\theta} d\phi, \tag{55}$$

so that

$$\Delta_1(f d\theta) = \delta d(f d\theta) + d\delta(f d\theta) = \left(-\frac{d^2 f}{d\phi^2} - \frac{d^2 f}{d\theta^2}\right) d\theta = \Delta_0(f) d\theta,$$

and similarly $\Delta_1(h d\phi) = \Delta_0(h) d\phi$. Thus, for an eigenform with eigenvalue ν we have

$$\xi(f d\theta + g d\phi) = \Delta_1(f d\theta + h d\phi) = \Delta_0(f) d\theta + \Delta_0(g) d\phi$$

which implies that both f and g must be eigenfunctions of the 0-Laplacian with eigenvalue ν. This implies that the non-zero eigenvalues of the 1-Laplacian are the same as those of the 0-Laplacian up to multiplicity. Due to the two harmonic forms $d\theta$ and $d\phi$, each eigenvalue of the 1-Laplacian has double the multiplicity of the same eigenvalue for the 0-Laplacian.

To verify the SEC on this example, we generate 10000 data points on a flat torus in $\mathbb{R}^4$ with the embedding $(\cos\theta, \sin\theta, \cos\phi, \sin\phi)^\top$. In the top left of Fig. 2, we plot the analytic spectrum of the 1-Laplacian (grey, solid curve) along with the SEC approximation of the spectrum (red, dashed curve). We also show the vector fields corresponding to the first two SEC eigenforms. The latter approximate $d\theta - d\phi$ and $-d\theta - d\phi$, which span the space of harmonic forms which is the span of $d\theta$ and $d\phi$.

Notice that this example requires 100 times more data than the circle (data required grows exponentially in the intrinsic dimension) for the diffusion maps algorithm to yield the same accuracy for the 0-Laplacian eigenfunctions and eigenvalues. This also means that the diffusion maps algorithm takes significantly longer

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Table 5: First eight eigenvalues of the Laplacian on 1-forms estimated using the SEC for a Möbius band, torus (standard embedding in $\mathbb{R}^3$), a genus two surface (two-holed torus), a two-dimensional sphere in $\mathbb{R}^3$, and the Lorenz-63 attractor. Notice that the eigenvalues close to zero represent harmonic forms, and the number of the eigenvalues close to zero matches the first Betti numbers for the manifolds which are 1,2,4, and 0 respectively. While the Lorenz-63 attractor is not a manifold, a coarse approximation as a manifold would suggest a Betti number of 2 due to the two holes.

Möbius	Torus	Genus 2	Sphere	L63
0.0242	0.0040	0.0021	1.9349	0.0011
1.0415	0.0093	0.0026	1.9521	0.0017
1.0449	0.2574	0.0026	1.9781	0.0030
3.8684	0.2575	0.0041	1.9817	0.0072
3.8948	0.2575	0.0893	2.0042	0.0105
8.0352	0.2587	0.0901	2.0172	0.0109
8.1018	0.8061	0.2151	5.8001	0.0205
8.9369	0.8067	0.2175	5.8142	0.0262
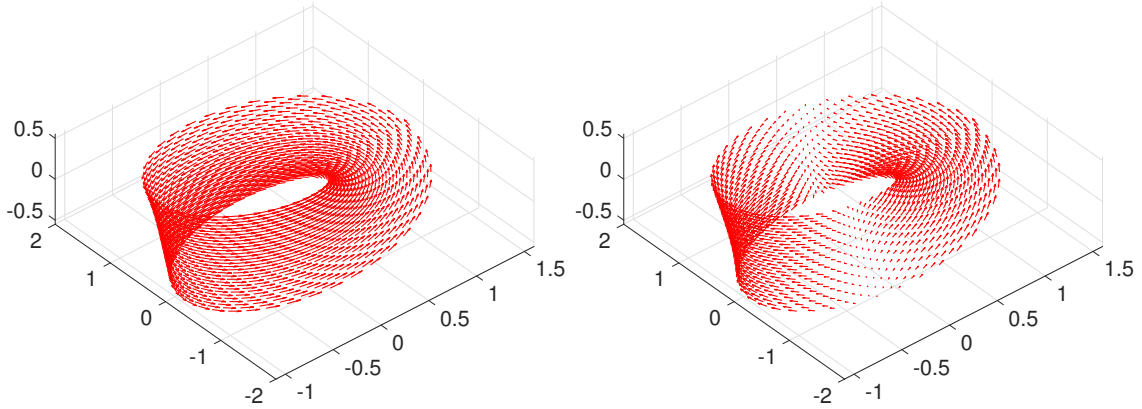
Figure 3: First two eigenforms on the Möbius band. The first represents the one 1-homology class.

to run on this larger data set. Crucially, the SEC still only uses $M = 20$ (and 100 eigenfunctions to compute the c tensor) so the matrices in the SEC were the same size for the torus example as for the circle example. Thus, following the initial diffusion maps step, the SEC algorithm runtime is the same in the torus example as for the circle example. This demonstrates how the SEC formulation allows us to decouple the representation of differential forms and their associated Laplacian operators from the amount of data. In particular, larger data sets are only needed in the initial diffusion maps step to obtain the best possible estimate of the 0-Laplacian eigenfunctions and eigenvalues. Since many of these eigenfunctions and eigenvalues are known to be poor estimates, we can set M much less than the number of data points to obtain high quality 1-Laplacian representations.

7.2. Topological features and eigenforms via the SEC

We now apply the SEC on several surfaces embedded in $\mathbb{R}^3$ in order to demonstrate its connection to the manifold homology. We also demonstrate the algorithm on a data set which is not sampled from a manifold to show that the SEC has potential applications even when the assumption of an underlying manifold does not hold. While the results on the circle and flat torus above used large uniformly distributed data sets to
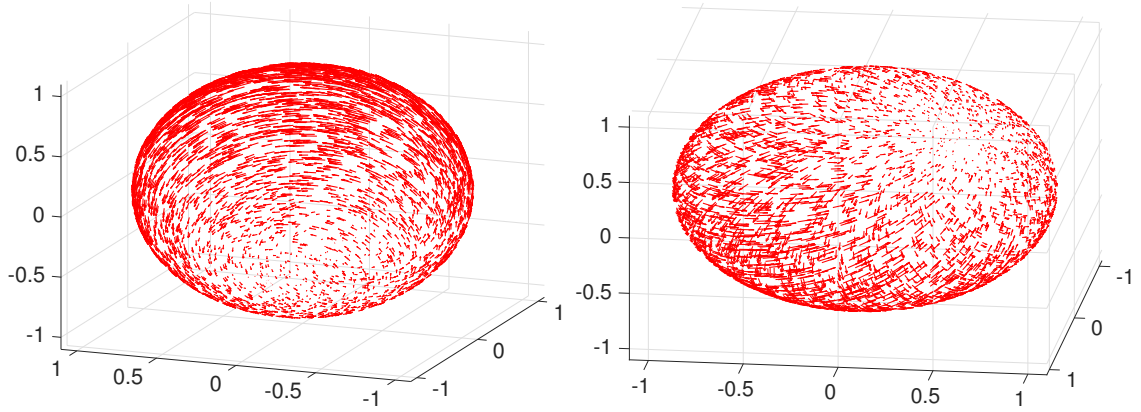
Figure 4: First two eigenforms on the sphere S^2. Notice that the SEC constructs valid smooth vector fields which must vanish on the sphere.
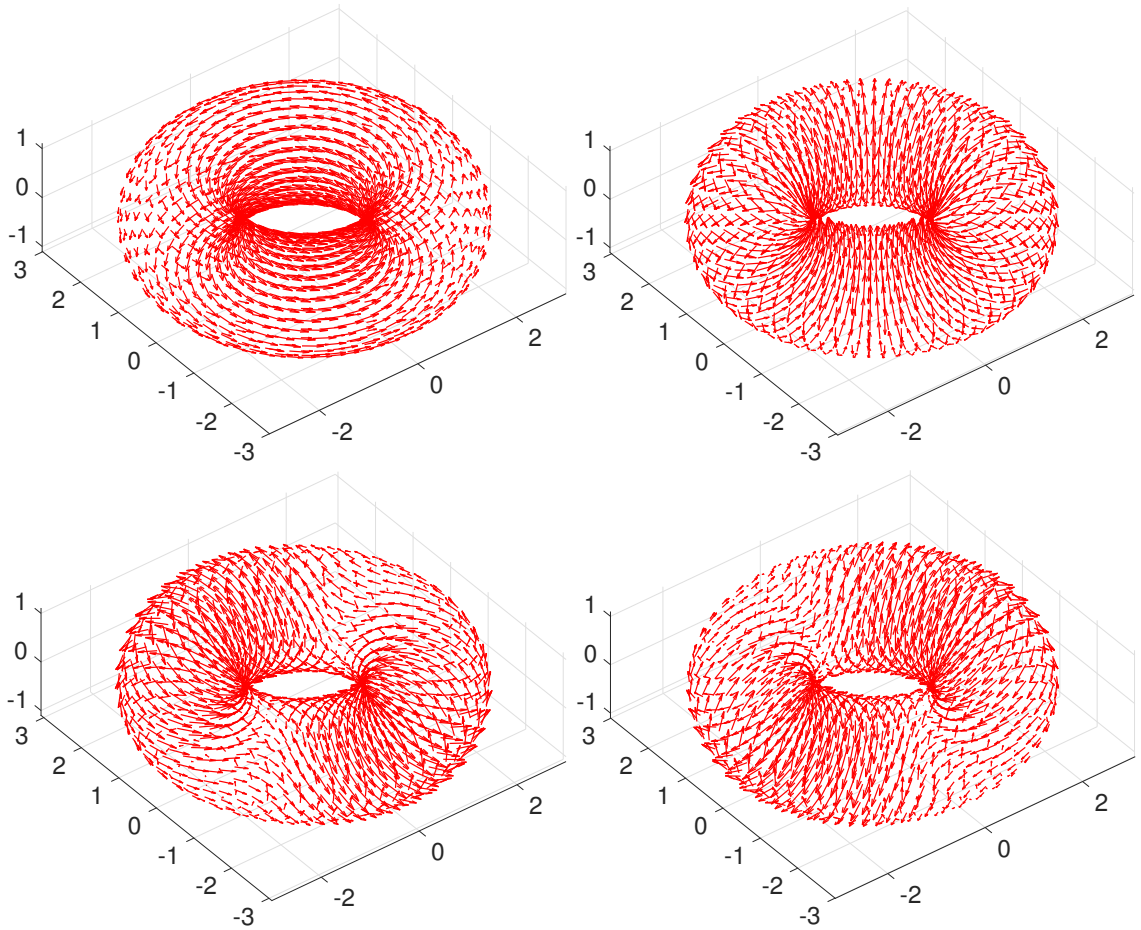

Figure 5: First four eigenforms on the standard embedding of the torus in $\mathbb{R}^3$. Notice that the first two represent the two 1-homology classes.
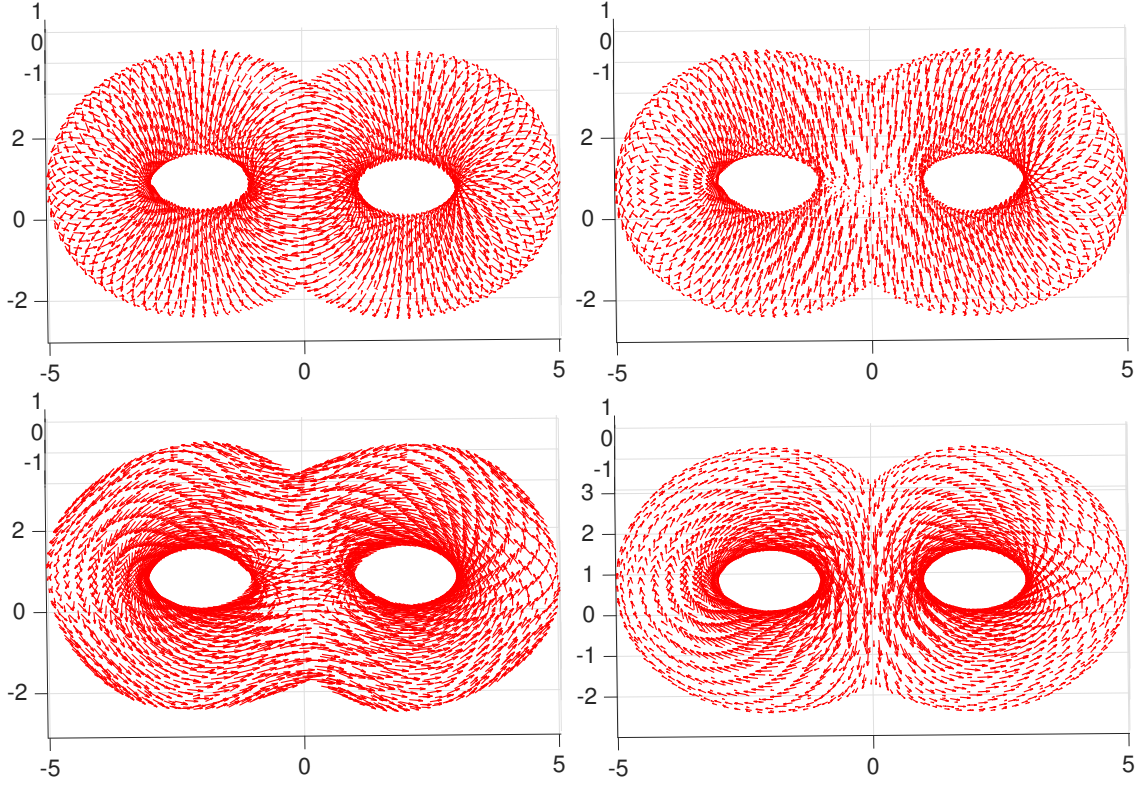
Figure 6: First four eigenforms on the genus two surface, representing the four 1-homology classes.

validate the algorithm, in this case we will work with much smaller data sets that may not be uniformly distributed with respect to the volume form.

We first consider four surfaces, namely a Möbius band, sphere, torus, and a genus-2 surface. Our goal for these examples is to show that the SEC correctly captures the coarse topological features of these smaller nonuniform data sets. The 1-homology of these manifolds corresponds to the kernel of the 1-Laplacian so that the first Betti number should be equal to the multiplicity of the zero eigenvalue of Δ_1. This is shown in Table 5 if we consider the eigenvalues closest to zero to represent 1-homology. We can also make this connection by visualizing the eigenforms via their corresponding vector fields as shown in Figs. 3–6. Following the vector field corresponding to a harmonic form (having eigenvalue zero) should generate a closed curve, which is a representative of a unique homology classes. We also show in Figs. 3–6 that the SEC approximations are smooth forms, demonstrated both by the smooth changes in the arrows, as well as the fact that the corresponding vector fields each vanish at some point on the sphere (since there are no smooth non-vanishing vector fields on a sphere).

Finally, we demonstrate the SEC on data sampled from the chaotic attractor of the Lorenz-63 dynamical system (we refer to this set as "L63''), a fractal set having no differential structure. While there is no exterior calculus defined on L63, it is a well-defined compact subset on $\mathbb{R}^3$ (with an induced metric topology), and exhibits certain coarse-grained topological features, most notably a hole in each of the two lobes of the attractor as shown in Fig. 7. Moreover, the diffusion maps and SEC algorithms can easily be applied to data sampled from this set (or indeed any data set in a metric space). The SEC spectrum for the 1-Laplacian is shown in Fig. 5, and the first two eigenvalues are very close to zero, while the corresponding eigenforms, shown in Fig. 7 seem to capture these two coarse topological features.

The examples in this section show that the SEC can generate a collection of 1-forms (equivalently, vector fields) which can be used as a basis for vector fields defined on the data set. These vector fields are
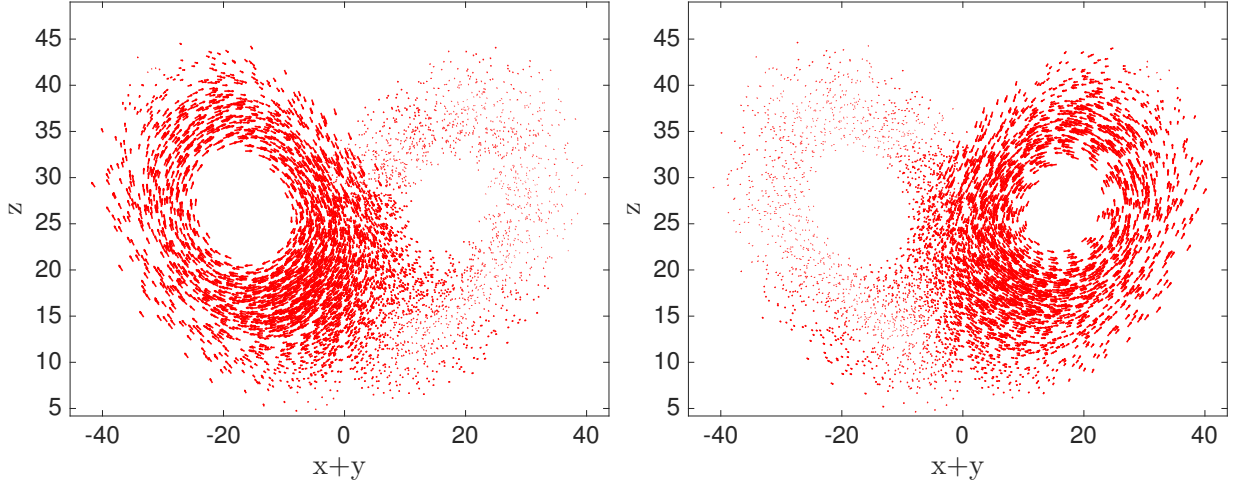
Figure 7: First two eigenforms of the SEC 1-Laplacian on 10000 data points sampled from the Lorenz-63 attractor. Notice that closed integral curves of these vector fields correspond to independent representatives of the two "holes" in the attractor.

ordered by smoothness based on their corresponding eigenvalue (Dirichlet energy; see Section 2.5), and higher eigenvalues correspond to more oscillatory vector fields. Moreover, our theory shows that in the limit of large data and large M (number of eigenfunctions of Δ_0 used), the SEC basis corresponds to the natural basis for square integrable 1-forms. However, even for small nonuniform data sets the SEC reflects coarse topological features of the underlying continuous space, which indicates that the SEC-derived approximations could be useful even outside of the large data limit.

8. Discussion

In this paper we have developed a spectral framework, called spectral exterior calculus (SEC), for the exterior calculus on Riemannian manifolds. A central underpinning of this approach is a family of frames for L^2 and higher-order Sobolev spaces of vector fields and differential forms, built entirely from the eigenvalues and eigenfunctions of the Laplacian on functions. By virtue of this construction, our framework lends itself well to data-driven approximation of the objects of interest in exterior calculus, such as vector fields and differential forms, as well as operators acting on these objects (e.g., the Laplacian on forms), requiring no additional information beyond point-clouds of data. In particular, SEC extends the applicability of the extensive array of graph-theoretic techniques for pointwise and spectral approximation of the Laplacian on functions [1–8] to learning problems involving higher-order objects, with rigorous convergence guarantees in the asymptotic limit of large data. Crucially, by relying solely on approximations of the eigenvalues and eigenfunctions of the Laplacian on functions, SEC decouples the computational complexity of approximation of vector fields, forms, and related operators from the number of samples and ambient data space dimension.

Another key aspect of SEC is its focus on L^2-convergent, as opposed to pointwise-convergent approximations, as the latter typically require additional structures such as simplicial complexes that are difficult to estimate from data alone. Here, we have shown that our frames for H^1 Sobolev spaces of 1-forms lead naturally to Galerkin approximation schemes for the eigenvalue problem of the Laplacian on 1-forms, which are provably well-posed by classical results on variational formulations of elliptic eigenvalue problems [24]. These techniques extend previously developed data-driven Galerkin approximation techniques for unbounded operators on functions [14–16] to the setting of 1-forms. These Galerkin methods for the Laplacian on one-forms, and more broadly approximation techniques for unbounded operators in exterior calculus, crucially depend on the availability of well-conditioned approximation spaces for H^p spaces of higher regularity than L^2. In the framework of SEC, such spaces are naturally constructed through singular value decomposition of sparse

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Grammian matrices associated with H^p frame elements, which are computable via closed-form expressions involving the eigenvalues of the Laplacian on functions, and inner-product relationships between products of the corresponding eigenfunctions. Once computed, the Galerkin-approximated Laplacian eigenforms can be visualized through their dual vector fields, reconstructed via a data-driven spectral approximation of the pushforward map. In addition to frame representations, SEC provides data-driven representations of vector fields as operators on functions, which is useful for tasks such as dynamical systems forecasting [15, 25], among other applications.

We have demonstrated the efficacy of SEC approximations to the eigenvalues and eigenforms of the 1-Laplacian, and their associated vector field duals, in a suite of examples involving orientable (circle, flat and curved 2-tori, 2-sphere, genus-2 surface) and non-orientable (Möbius band) smooth manifolds, as well as a fractal set having no differentiable structure (the L63 attractor). In the circle and flat-torus examples, where analytical expressions for the 1-Laplacian eigenvalues and eigenforms are available, we found that SEC accurately approximates the leading 50 to 100 eigenvalues using a modest number (20) of 0-Laplacian eigenfunctions to build the SEC frames, and a moderate dimension (40 for circle and 107 for the flat torus) of the corresponding Galerkin approximation spaces. In the curved torus, sphere, and Möbius band examples (where analytical expressions for the eigenvalues and eigenforms of the 1-Laplacian are not readily available), we demonstrated that the SEC results are consistent with with the 1-homology of these manifolds. In particular, the number of eigenvalues of the SEC-approximated 1-Laplacian numerically close to zero was found to be equal to the first Betti number of the manifold under study, and the corresponding eigenforms were found to generate closed curves (through the integral curves of their vector field duals), representing 1-homology classes. Moreover, even though our theory was developed in the smooth-manifold setting, the SEC-derived eigenforms for the L63 attractor were also found to be qualitatively consistent with 1-homology (which is well-defined for this topological space even though it does not have a differentiable structure). In this example, we obtained two eigenforms at small corresponding eigenvalue, generating closed curves around each of the "holes" in the lobes of the attractor (Fig. 7), which is what one would intuitively expect for the L63 topology.

There are several avenues of future research stemming from this work. First, it would be of interest to develop SEC approximation schemes for other operators of interest in exterior calculus, such as the Hodge star operator and the Lie derivative. Among other applications, such approximation schemes are likely to be of interest in dynamical systems modeling. For instance, for a dynamical system on a manifold generated by a vector field v, there is an associated dynamical system acting on the tangent bundle, which can be thought of as a pointwise linearization of the system. In this setting, the Lie derivative $\mathcal{L}_v$ on vector fields generates the action of this system on sections of the tangent bundle, and is known to have useful spectral properties characterizing growth rates of perturbations [26]. Approximating these spectra from data would thus provide a computationally efficient empirical method for characterizing the principal modes of instability in datasets generated by dynamical systems; a topic of considerable current interest that has spurred the development of powerful geometrical approaches such as the theory of covariant Lyapunov vectors [27].

Another research direction would involve extending SEC to non-smooth topological spaces. Our results for the L63 attractor suggest that a spectral formulation of exterior calculus, possibly with a very similar structure to the one presented here, may be possible provided one can construct an appropriate diffusion operator generalizing the notion of the Laplacian on Riemannian manifolds. To that end, it is worthwhile noting that the Markov integral operators employed here to approximate the heat semigroup and its associated Laplacian are, in fact, well defined as operators on C^0 and L^2 function spaces on Borel measure spaces without differentiable structure. Moreover, recent work [16] has shown that, at least in some scenarios, such operators can be consistently approximated from data under natural assumptions (e.g., data sampled from a dynamical system possessing a physical ergodic invariant measure, such as the L63 system studied here). Yet, a key step relevant to the SEC framework, namely the consistent approximation of the Laplacian from the heat semigroup, achieved by taking $\epsilon \to 0$ kernel bandwidth limits, has, to our knowledge, no known generalization to non-differentiable spaces. In such a setting, the challenge would be to approximate the generator of an appropriate C^0 diffusion semigroup based on kernel integral operators approximating that semigroup (akin to the operators $\tilde{P}_\epsilon$ in Section 6.2 approximating the heat semigroup). Such a diffusion operator would lead to an analog of the product rule for the 0-Laplacian on smooth functions through its

associated symmetric bilinear form [28], providing one of the necessary ingredients to building the SEC framework. However, at present the construction of this operator and its approximation remain elusive.

Finally, while the present SEC formulation has focused heavily on H^p Sobolev spaces and their associated notions of regularity based on Dirichlet energy, there is another class of Hilbert spaces with a natural notion of regularity, namely the class of reproducing kernel Hilbert spaces (RKHSs). These spaces are widely used in statistical learning of functions, but they also have analogs appropriate for spaces of differential forms. In addition to providing well-defined notions of regularity through the corresponding RKHS norms, the fact that RKHS spaces have continuous pointwise evaluation functionals allows for pointwise- (as opposed to L^2-) convergent approximations when working in such spaces, with well-known applications in smoothing and interpolation. These properties motivate the construction of frames for RKHSs of forms, and their use in Galerkin approximation schemes for operators on forms, analogous to the schemes presented here utilizing Sobolev spaces.

Acknowledgments

Appendix A. Discrete incompatibility of product rule and Leibniz rule

Let f, h, and b be arbitrary smooth functions on a Riemannian manifold. The product rule for the Laplacian operator can be written as

$$\Delta(fh) = f\Delta h + h\Delta f - 2\operatorname{grad} f(h),$$

where $\operatorname{grad} f(h) = \operatorname{grad} f \cdot \operatorname{grad} h$. The Leibniz rule for vector fields states that

$$v(fh) = fv(h) + hv(f),$$

and since $\operatorname{grad} f$ is a vector field, we can easily derive a triple product rule for the Laplacian

$$\begin{aligned}
\Delta(fhb) &= f\Delta(hb) + hb\Delta f - 2\operatorname{grad} f(hb) \\
&= f\Delta(hb) + hb\Delta f - 2h\operatorname{grad} f(b) - 2b\operatorname{grad} f(h) \\
&= f\Delta(hb) + hb\Delta f - hf\Delta b - hb\Delta f + h\Delta(fb) - bf\Delta h - bh\Delta f + b\Delta(hf).
\end{aligned}$$

Rearranging the triple product rule, we have

$$0 = \Delta(fhb) - f\Delta(hb) - h\Delta(fb) - b\Delta(fh) + fh\Delta b + fb\Delta h + hb\Delta f.$$

Now assume that L is a discrete Laplacian operator (meaning a matrix), without any assumption on the form of the representation of vector fields. If we assume that gradient fields are represented in such a way that both the product rule for the Laplacian and the the Leibniz rule hold, then the same derivation will produce the triple product rule for the discrete Laplacian. However, if the triple product rule holds on the standard basis vectors $\{e_i\}$, then we find that

$$0 = L(e_i e_j e_k) - e_i L(e_j e_k) - e_j L(e_i e_k) - e_k L(e_i e_j) + e_i e_j Le_k + e_i e_k Le_j + e_j e_k Le_i,$$

where vector-vector products are componentwise. Setting $i = j = k$, we find

$$0 = L(e_i) - 3e_i L(e_i e_i) + 3e_i e_i Le_i = Le_i$$

and since $Le_i = 0$ for all i, we conclude that the only matrix which satisfies the triple product rule is the zero matrix. This shows that no discrete representation of vector fields can simultaneously satisfy both the Laplacian product rule and the Leibniz rule.

Appendix B. Computations and derivations of formulas

In this appendix, we derive the formulas for the inner products and Dirichlet energies of the SEC frame elements for 1-forms listed in Table 4 (Appendix B.1). In addition, we discuss how these formulas can be extended to higher-order k-forms (Appendix B.2). Since the SEC frame elements are real, in these derivations we will consider that all Hilbert spaces of functions, vector fields, and k-forms are over the real numbers, i.e., there will be no complex conjugation in the corresponding inner products.

Appendix B.1. Inner products and Dirichlet forms for 1-forms

Recall that the Hodge inner product 1-forms ω, ν is defined by

$$\langle \omega, \nu \rangle_{L^2_{\mathfrak{X}}} = \int_{\mathcal{M}} \omega \wedge *\nu = \left\langle g^{-1}(\omega, \nu), 1 \right\rangle_{L^2} = \left\langle \omega^\sharp \cdot \nu^\sharp, 1 \right\rangle_{L^2},$$

where we abbreviate the Riemannian inner product for vector fields w, v by $w \cdot v \equiv g(w, v)$. Similarly for 1-forms ω, ν we will abbreviate the Riemannian inner product by $\omega \cdot \nu \equiv g^{-1}(\omega, \nu) = g(\omega^\sharp, \nu^\sharp)$. In particular, if $\omega = f dh$ and $\nu = \alpha d\beta$, then the Hodge inner product can be further simplified as

$$\langle f dh, \alpha d\beta \rangle_{L^2_{\mathfrak{X}}} = \langle dh \cdot d\beta, f\alpha \rangle_{L^2} = \frac{1}{2} \langle h\Delta\beta + \beta\Delta h - \Delta(h\beta), f\alpha \rangle_{L^2},$$

and substituting eigenfunctions b^i of Δ into this formula, we define

$$\begin{aligned}
G_{ijkl} &\equiv \left\langle b^i db^j, b^k db^l \right\rangle_{L^2(\mathfrak{X})} = \frac{1}{2} \left\langle b^j \Delta b^l + b^l \Delta b^j - \Delta(b^j b^l), b^i b^k \right\rangle_{L^2} \\
&= \frac{1}{2} \left\langle \lambda_l b^j b^l + \lambda_j b^l b^j - \Delta\left(\sum_s c_{ljs} b^s\right), b^i b^k \right\rangle_{L^2} \\
&= \frac{1}{2}(\lambda_l + \lambda_j) c_{ijkl} - \sum_s \lambda_s c_{ljs} c_{iks}, \\
&= \frac{1}{2} \sum_s (\lambda_l + \lambda_j - \lambda_s) c_{ljs} c_{iks},
\end{aligned} \tag{B.1}$$

where $c_{ijk} \equiv \left\langle b^i b^j, b^k \right\rangle_{L^2}$ and $c^0_{ijkl} \equiv \left\langle b^i b^j, b^k b^l \right\rangle_{L^2} = \sum_s c_{ijs} c_{kls}$ (note that these are invariant to permutations of indices). Notice that the Gramm matrix of the antisymmetric elements is easy to compute from G_{ijkl} since

$$\hat{G}_{ijkl} = G_{ijkl} + G_{jilk} - G_{ijlk} - G_{jikl},$$

so next we will consider the computation of the values E_{ijkl} of the Dirichlet form.

Recalling the formula for the 1-Laplacian $\Delta_1 = d\delta + \delta d$, for 1-forms ω, ν we can write

$$\langle \omega, \Delta_1 \nu \rangle_{L^2_1} = \langle \delta\omega, \delta\nu \rangle_{L^2} + \langle d\omega, d\nu \rangle_{L^2_2} \tag{B.2}$$

where d is the exterior derivative, and δ its adjoint, the codifferential. For 1-forms $\omega = f dh$ and $\nu = \alpha d\beta$, we have

$$\begin{aligned}
\langle d(f dh), d(\alpha d\beta) \rangle_{L^2_2} &= \langle df \wedge dh, d\alpha \wedge d\beta \rangle_{L^2_2} = \left\langle g^{-1}(df \wedge dh, d\alpha \wedge d\beta), 1 \right\rangle_{L^2_2} \\
&= \left\langle \det \begin{bmatrix} df \cdot d\alpha & df \cdot d\beta \\ dh \cdot d\alpha & dh \cdot d\beta \end{bmatrix}, 1 \right\rangle_{L^2} \\
&= \langle df \cdot d\alpha, dh \cdot d\beta \rangle_{L^2} - \langle df \cdot d\beta, dh \cdot d\alpha \rangle_{L^2}.
\end{aligned} \tag{B.3}$$

We recall that the codifferential acting on k-forms is $\delta = (-1)^{n(k-1)+1} * d *$, where d is the exterior derivative acting on $n - k$ forms. For a function f and 1-form ω, we have

$$\delta(f\omega) = -*d*(f\omega) = -*d(f*\omega) = -*(df \wedge *\omega + f d*\omega) = -*(df \wedge *\omega) - f*d*\omega$$

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$$= -g^{-1}(df, \omega) + f\delta\omega = -df \cdot \omega + f\delta\omega,$$

so in particular if $\omega = dh$ we find (using $\Delta = \delta d$)

$$\delta(fdh) = -df \cdot dh + f\Delta h = \frac{1}{2}(\Delta(fh) - f\Delta h - h\Delta f) + f\Delta h = \frac{1}{2}(\Delta(fh) + f\Delta h - h\Delta f).$$

Using this formula we can simplify the inner product

$$\langle \delta(fdh), \delta(\alpha d\beta) \rangle_{L^2} = \frac{1}{4} \left\langle \Delta(fh) + f\Delta h - h\Delta f, \Delta(\alpha\beta) + \alpha\Delta\beta - \beta\Delta\alpha \right\rangle_{L^2}$$

$$= \langle -df \cdot dh + f\Delta h, -d\alpha \cdot d\beta + \alpha\Delta\beta \rangle_{L^2}$$

$$= \langle df \cdot dh, d\alpha \cdot d\beta \rangle - \langle df \cdot dh, \alpha\Delta\beta \rangle_{L^2} - \langle d\alpha \cdot d\beta, f\Delta h \rangle_{L^2} + \langle f\Delta h, \alpha\Delta\beta \rangle_{L^2}. \quad \text{(B.4)}$$

Now that both summands in (B.2) have been written in terms of Δ, we can combine (B.3) and (B.4) and several cancellations yield,

$$\langle fdh, \Delta_1(\alpha d\beta) \rangle_{L_1^2} = \langle df \cdot d\alpha, dh \cdot d\beta \rangle_{L^2} - \langle df \cdot d\beta, dh \cdot d\alpha \rangle_{L^2} + \langle df \cdot dh, d\alpha \cdot d\beta \rangle_{L^2}$$

$$- \langle df \cdot dh, \alpha\Delta\beta \rangle_{L^2} - \langle d\alpha \cdot d\beta, f\Delta h \rangle_{L^2} + \langle f\Delta h, \alpha\Delta\beta \rangle_{L^2}. \quad \text{(B.5)}$$

When the functions are eigenfunctions of Δ, we find

$$db^i \cdot db^j = \frac{1}{2}\left(b^i\Delta b^j + b^j\Delta b^i - \Delta(b^ib^j)\right) = \frac{1}{2}\left((\lambda_i + \lambda_j)b^ib^j - \Delta(b^ib^j)\right) = \frac{1}{2}\sum_s (\lambda_i + \lambda_j - \lambda_s)c_{ijs}b^s, \quad \text{(B.6)}$$

so that we can define

$$F_{ijkl} = \left\langle db^i \cdot db^j, db^k \cdot db^l \right\rangle_{L^2} = \frac{1}{4}\sum_s (\lambda_i + \lambda_j - \lambda_s)(\lambda_k + \lambda_l - \lambda_s)c_{ijs}c_{kls}$$

$$= \frac{1}{4}\left[(\lambda_i + \lambda_j)(\lambda_k + \lambda_l)c_{ijkl} - (\lambda_i + \lambda_j + \lambda_k + \lambda_l)\sum_s \lambda_s c_{ijs}c_{kls} + \sum_s \lambda_s^2 c_{ijs}c_{kls}\right]. \quad \text{(B.7)}$$

Moreover, recalling that

$$G_{ijkl} = \frac{1}{2}\sum_s (\lambda_l + \lambda_j - \lambda_s)c_{ljs}c_{iks} = \frac{1}{2}(\lambda_l + \lambda_j)c_{ijkl}^0 - \frac{1}{2}\sum_s \lambda_s c_{ljs}c_{iks},$$

we simplify the above formulas as

$$\left\langle d(b^idb^j), d(b^kdb^l) \right\rangle_{L_2^2} = F_{ikjl} - F_{iljk}$$

and

$$\left\langle \delta(b^idb^j), \delta(b^kdb^l) \right\rangle_{L^2} = F_{ijkl} - \lambda_l G_{kilj} - \lambda_j G_{ikjl} + \lambda_j\lambda_l c_{ijkl}^0.$$

so that

$$E_{ijkl} = \left\langle b^idb^j, \Delta_1(b^kdb^l) \right\rangle_{L_2^2} = F_{ikjl} - F_{iljk} + F_{ijkl} - \lambda_l G_{kilj} - \lambda_j G_{ikjl} + \lambda_j\lambda_l c_{ijkl}$$

$$= c_{ijkl}^0 \left(\frac{1}{4}(\lambda_i + \lambda_k)(\lambda_j + \lambda_l) - \frac{1}{4}(\lambda_i + \lambda_l)(\lambda_j + \lambda_k) + \frac{1}{4}(\lambda_i + \lambda_j)(\lambda_k + \lambda_l)\right)$$

$$+ c_{ijkl}^0 \left(\lambda_j\lambda_l - \frac{1}{2}\lambda_l(\lambda_i + \lambda_j) - \frac{1}{2}\lambda_j(\lambda_k + \lambda_l)\right)$$

$$+ \frac{1}{4}(\lambda_i + \lambda_j + \lambda_k + \lambda_l)(c_{iljk}^1 - c_{ikjl}^1 - c_{ijkl}^1) + \frac{1}{2}(\lambda_j + \lambda_l)c_{ijkl}^1 + \frac{1}{4}(c_{ikjl}^2 + c_{ijkl}^2 - c_{iljk}^2)$$

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$$= \frac{1}{4}(\lambda_i + \lambda_j + \lambda_k + \lambda_l)(c^1_{iljk} - c^1_{ikjl} - c^1_{ijkl}) + \frac{1}{2}(\lambda_j + \lambda_l)c^1_{ijkl} + \frac{1}{4}(c^2_{ikjl} + c^2_{ijkl} - c^2_{iljk})$$

$$= \frac{1}{4}\left[(\lambda_i + \lambda_j + \lambda_k + \lambda_l)(c^1_{iljk} - c^1_{ikjl}) + (\lambda_j + \lambda_l - \lambda_i - \lambda_k)c^1_{ijkl} + (c^2_{ikjl} + c^2_{ijkl} - c^2_{iljk})\right]$$

A straightforward computation then shows that the c^0_{ijkl} coefficients exactly cancel, and we define

$$c^1_{ijkl} \equiv \sum_s \lambda_s c_{ijs} c_{skl} \qquad \text{and} \qquad c^2_{ijkl} \equiv \sum_s \lambda^2_s c_{ijs} c_{skl},$$

which also allows us to write $G_{ijkl} = \frac{1}{2}((\lambda_l + \lambda_j)c_{ijkl} - c^1_{ikjl})$. Finally, to compute the Dirichlet form for the antisymmetric elements, we note that elements,

$$E_{ijkl} - E_{jikl} = \frac{1}{2}\left[(\lambda_i + \lambda_j + \lambda_k + \lambda_l)(c^1_{iljk} - c^1_{ikjl}) + (\lambda_j - \lambda_i)c^1_{ijkl} + (c^2_{ikjl} - c^2_{iljk})\right]$$

and

$$E_{ijlk} - E_{jilk} = \frac{1}{2}\left[-(\lambda_i + \lambda_j + \lambda_k + \lambda_l)(c^1_{iljk} - c^1_{ikjl}) + (\lambda_j - \lambda_i)c^1_{ijkl} - (c^{\lambda^2}_{ikjl} - c^2_{iljk})\right],$$

leading to

$$\hat{E}_{ijkl} \equiv \left\langle b^i db^j - b^j db^i, \Delta_1(b^k db^l - b^l db^k)\right\rangle = E_{ijkl} - E_{jikl} - (E_{ijlk} - E_{jilk})$$

$$= (\lambda_i + \lambda_j + \lambda_k + \lambda_l)(c^1_{iljk} - c^1_{ikjl}) + (c^2_{ikjl} - c^2_{iljk}). \tag{B.8}$$

These formulas are summarized in Table 4 in Section 2.5.

Appendix B.2. Extension to k-forms

Next we briefly summarize how we can extend the SEC formulas derived in Appendix B.1 to higher-order forms. Let $b^I = b^{i_0} db^{\tilde{I}} \equiv b^{i_0} db^{i_1} \wedge \cdots db^{i_k}$ be a k-form frame element ($\tilde{I} = (i_1, ..., i_k)$ and $I = (i_0, ..., i_k)$). As we will see below, the Hodge inner product of the exterior derivatives is easily computed as a determinant of inner products of 1-forms. The more complex term is the Hodge inner product of the codifferential terms. To understand this, we need to generalize the product rule for the codifferential as

$$\delta(b^I) = (-1)^{n(k-1)+1} * d * (b^{i_0} db^{\tilde{I}}) = (-1)^{n(k-1)+1} * d(b^{i_0} * db^{\tilde{I}}) = (-1)^{n(k-1)+1} * (db^{i_0} \wedge *db^{\tilde{I}} + b^{i_0} d * db^{\tilde{I}})$$

$$= (-1)^{n(k-1)+1} * (db^{i_0} \wedge *db^{\tilde{I}}) + b^{i_0} \delta(db^{\tilde{I}}).$$

We can now reduce $\delta(db^{\tilde{I}})$ by rewriting $db^{\tilde{I}} = d(b^{i_1} db^{\hat{I}})$,where $\hat{I} = (i_2, ..., i_k)$, so that

$$\delta_k(db^{\tilde{I}}) = \delta_k d_{k-1}(b^{i_1} db^{\hat{I}}) = \Delta_{k-1}(b^{i_1} db^{\hat{I}}) - d_k \delta_{k-1}(b^{i_1} db^{\hat{I}}).$$

Notice that in the above formula we have reduced the problem of computing the k-codifferential to computing the $(k-1)$-Laplacian and the $(k-1)$-codifferential. Since we have shown how to compute the codifferential on 1-forms, this strategy can be used to lift the SEC formulas to higher-order forms, although the formulas become quite complicated. We carried this out for 2-forms, but the derivations are quite long and a closed formula for the general case remains elusive.

Thus, while in principle the iterative formula above allows us to lift our formulation to k-forms, we do not yet have a closed formula for the inner product

$$\left\langle b^I, \Delta_k b^J\right\rangle_{L^2_k} = \left\langle d(b^I), d(b^J)\right\rangle_{L^2_{k+1}} + \left\langle \delta(b^I), \delta(b^J)\right\rangle_{L^2_{k-1}}.$$

However, the first term above is simply the integral of the determinant of the matrix of pairwise inner products $db^{i_s} \cdot db^{j_r}$ for $s, r \in \{0, ..., k\}$. The key to computing this term is the formula

$$(db^i \cdot db^j)(db^k \cdot db^l) = \left(\sum_m g_{ijm} b^m\right)\left(\sum_n g_{kln} b^n\right) = \sum_{m,n} g_{ijm} g_{kln} b^m b^n,$$

Table B.6: Table of formulas for the SEC on k-forms. We abbreviate the multi-indices $I = (i_0, ..., i_k)$ and $\tilde{I} = (i_1, ..., i_k)$ and similarly $J = (j_0, ..., j_k)$ and $\tilde{J} = (j_1, ..., j_k)$. We use $[db^{i_m} \cdot db^{j_n}]$ to abbreviate the $k \times k$ matrix with (m, n)-th entry given by $db^{i_m} \cdot db^{j_n}$. Finally, note that $db^I = db^{i_0} \wedge \cdots \wedge db^{i_k}$.

Object	Symbolic	Spectral
Multiple Product	$c_I^0 = \left\langle b^{i_0} \cdots b^{i_k}, 1 \right\rangle_{L^2}$	$c_I^0 = \sum_s c_{i_0 i_1 s} c_{s i_2 \cdots i_k}^0$
Tensor	$H^{\tilde{I}\tilde{J}} = (db^{i_1} \cdot db^{j_1}) \cdots (db^{i_k} \cdot db^{j_k})$	$\hat{H}_l^{\tilde{I}\tilde{J}} \equiv \left\langle H^{\tilde{I}\tilde{J}}, b^l \right\rangle_{L^2}$
Evaluation	$= \operatorname{grad} b^{i_1} \otimes \cdots \otimes \operatorname{grad} b^{i_k}(b^{j_1}, ..., b^{j_k})$	$= \sum_{n=1}^{k^2} \prod_{s,r=1}^k g_{i_s j_r m_n} c_{l m_1 \cdots m_{k^2}}$
Tensor Product	$b_J = b^{j_0} \operatorname{grad} b^{j_1} \otimes \cdots \otimes \operatorname{grad} b^{j_k}$	$\left\langle b_J(b^{i_1}, ..., b^{i_k}), b^l \right\rangle = \sum_s \hat{H}_s^{\tilde{J}\tilde{I}} c_{s j_0 l}$
Frame Elements	$b^I = b^{i_0} db^{i_1} \wedge \cdots \wedge db^{i_k}$	$\left\langle b^I(b_J), b^l \right\rangle_{L^2} = \left\langle b^I \cdot b^J, b^l \right\rangle_{L^2}$
Riemannian Metric	$b^I \cdot b^J = b^{i_0} b^{j_0} \det([db^{i_m} \cdot db^{j_n}])$	$\left\langle b^I \cdot b^J, b_l \right\rangle_{L^2} = \sum_s \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) c_{s i_0 j_0 l} \hat{H}_s^{\tilde{I}\sigma(\tilde{J})}$
Hodge Grammian	$G_{IJ} = \left\langle b^I, b^J \right\rangle_{L_k^2} = \left\langle b^I \cdot b^J, 1 \right\rangle_{L^2}$	$\sum_s \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) c_{s i_0 j_0} \hat{H}_s^{\tilde{I}\sigma(\tilde{J})}$
d-Energy	$E_{IJ}^d = \left\langle db^I, db^J \right\rangle_{L_{k+1}^2}$	$\left\langle db^I \cdot db^J, 1 \right\rangle_{L_{k+1}^2} = \hat{H}_0^{IJ}$

and iterating the above formula we can expand the product as

$$\prod_{s,r=0}^k db^{i_s} \cdot db^{j_r} = \sum_{n=0}^{k^2} \prod_{s,r=0}^k g_{i_s j_r m_n} b^{m_n}.$$

The integral of these products can then be represented in terms of the integrals of products of eigenfunctions, which can be computed from the c tensor. We briefly summarize these formulas for the SEC on general k-forms in Table B.6.

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