Linear theory for filtering nonlinear multiscale systems with model error

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In this paper, we study filtering multiscale dynamical systems with model error. This is a prototypical situation for many applications due to limitations in resolving the smaller scale processes. In the linear and Gaussian setting, we prove the existence and uniqueness of an optimal reduced filter based on a reduced stochastic prior model which only models the slow variables. By optimal, we mean the first two moments of the reduced filter solutions agree with the true filter solutions from the perfect model. This result relies on a notion of *consistency* for filtering with model error, which requires the error covariance estimate to equal the actual error covariance. We extend this linear theory to an exactly solvable, multiscale nonlinear filtering problem, in which the true filter solutions are practically inaccessible. In this situation, we found that the best reduced stochastic filter model accounts for the model error with a damping term and a combined, additive and multiplicative (in the Stratonovich sense), stochastic forcing.

To verify the applicability of this linear theory to general nonlinear filtering problems, we introduce an empirical measure of consistency that can be easily computed when the true filter posterior distribution is not accessible, which is typical for nonlinear problems. Motivated by the linear theory and the results from a nonlinear example, we propose a practical stochastic parameterization strategy to account for model error in filtering higher-dimensional nonlinear problems. This stochastic parameterization simultaneously accounts for both the mean model error and the model error covariance statistics. We demonstrate our stochastic parameterization in a numerical example by filtering a chaotic 81-dimensional model, which exhibits many of the characteristics seen in practical applications, using a 9-dimensional reduced model. We show that, with this parameterization, the reduced model slightly overestimates the true covariance, whereas many practical techniques tend to underestimate the covariance.

Keywords: data assimilation; filtering; multi-scale systems; covariance inflation; stochastic parameterization; additive noise; multiplicative noise; model error; averaging

1. Introduction

Model error is a fundamental barrier to state estimation (or filtering). This problem is attributed to incomplete understanding of the underlying physics and our lack of computational resources to resolve physical processes in various time and length scales. While many numerical approaches have been developed to cope with state estimation in the presence of model errors, most of these methods were designed to estimate only one of the model error statistics, either the mean or covariance, imposing various assumptions on the other statistics that are not estimated. For example, classical approaches proposed in [12, 3] estimate mean model error (which is also known as the forecast bias), assuming that the model error covariance is proportional to the prior error covariance from the imperfect model. An alternative popular approach is to inflate the prior error covariance statistics, either with an empirical choices of inflation factor [2, 42, 47, 27, 44] or with an adaptive inflation factor [39, 40, 5, 11, 1, 32, 41, 37, 6, 22]. All of these covariance inflation methods assume unbiased forecast error (meaning that there is no mean model error).

Recently, reduced stochastic filtering approaches to mitigate model errors in multiscale complex turbulent systems were introduced in [23, 19, 20, 18, 17]; see also [38, 36] for a complete treatment of filtering complex turbulent systems. While these computationally cheap methods produce relatively accurate mean estimates, they tend to underestimate the error covariance statistics that characterizes the uncertainty of the mean estimate. Similar conclusions were also reported in a comparison study of various approximate filtering methods [31]. In the authors' view, one significant challenge remains; finding the *optimal* statistical estimates in the presence of model error. By optimal here, we mean the expected state estimate and the error covariance matrix are accurate when compared to the true posterior estimates obtained with the perfect model. This optimality condition is only a minimum requirement for accurate uncertainty quantification.

In this paper, we develop a mathematical theory for optimal filtering of multiscale dynamical systems in the presence of model error through reduced filter models. In order to make a rigorous investigation of state estimation in the presence of model error, we consider the following prototype continuous-time filtering problem,

$$\begin{aligned} dx &= f_1(x,y;\theta)dt + \sigma_x(x,y;\theta)\,dW_x, \\ dy &= \frac{1}{\epsilon} f_2(x,y;\theta)dt + \frac{\sigma_y(x,y;\theta)}{\sqrt{\epsilon}}\,dW_y, \\ dz &= x\,dt + \sqrt{R}dV. \end{aligned} \tag{1.1}$$

Intuitively, the x variables represent the part of the state which we wish to estimate and predict, while the y variables represent a faster time scale (characterized by small ϵ) which is either unknown or impractical to estimate. In (1.1), W_x, W_y, and V are i.i.d. Wiener processes and θ denotes the true model parameters, which may be partially unknown in real application. In this article, we restrict ourself to observations z of only the resolved variables x, contaminated by unbiased noise with positive definite covariance matrix, R. For general observation models that involve x and y variables such as those considered in [23, 19, 20], the analysis will be more involved and we will defer this complicated situation to future research.

While there are many results concerning the convergence of (1.1) as $\epsilon \to 0$ to an averaged reduced filter for x (such as [24], which also develops a nonlinear theory), we are interested in the case where ϵ may be $\mathcal{O}(10^{-1})$ or even $\mathcal{O}(1)$. We wish to estimate parameters Θ in a reduced filtering problem,

$$\begin{aligned} dX &= F(X;\Theta)\,dt + \sigma_X(X;\Theta)\,dW_X, \\ dz &= X\,dt + \sqrt{R}dV, \end{aligned} \tag{1.2}$$

such that the posterior mean and covariance estimates of the reduced filtering problem in (1.2) are close to the posterior mean and covariance estimates of the true filtering problem with the perfect model in (1.1). In this reduced filtering problem, the observations z in (1.2) are noisy observations of the solutions of the true model in (1.1). We assume that there are no errors in the observation model of the reduced filtering problem, which will allow direct comparison of the filtered estimates from (1.1) and (1.2). The parameters Θ will depend on the scale gap ϵ and the unknown true dynamics, including the true parameters θ.

In Section 2, a linear theory for the existence and uniqueness of Θ is developed. This result relies on a notion of *consistency* in filtering with model errors. In Section 3, an empirical measure of consistency is introduced to quantify the 'validity' of the filter covariance estimates, especially when the mean and covariance estimates from the true filter are not available. In Section 4, we develop an accurate and empirically consistent reduced filter in a simple, yet challenging nonlinear setting, where the optimal filter is not available, which is typically the case in practical applications. Based on these theoretical findings, we propose a practical stochastic parameterization strategy to account for model errors in filtering higher-dimensional nonlinear problems in Section 5. Unlike the classical numerical methods mentioned above, this stochastic parameterization simultaneously accounts for both the mean model error and the model error covariance statistics; in principle, this method allows for accounting higher-order model error statistics, which can be important for uncertainty quantification. In Section 6, we demonstrate our stochastic parameterization in a numerical example by filtering an 81-dimensional model which exhibits many of the characteristics seen in practical applications using a 9-dimensional reduced model. We conclude the paper with a short summary and discussion in Section 7. We accompany this article with an electronic supplementary material that discusses the detailed calculations.

2. Linear Theory

The goal in this section is to develop a linear theory for filtering multiscale dynamical systems with model errors. Note that in the presence of model error, even for a linear system, we must carefully differentiate between the *actual error covariance* of the filtered mean estimate and the *error covariance estimate* produced by the filtering scheme. The actual error covariance is simply the expected mean squared error of the state estimate produced by the filter, on the other hand, the linear Kalman-Bucy filter [26] produces an estimate of error covariance that solves a Riccati equation. For a linear filter based on the perfect model, these two error covariances are identical. In this situation, we say the filtered estimate is *consistent*. However, when the model used by the filter does not match the true model, finding a consistent filtered estimate is nontrivial since the covariance solutions of the Riccati equation will typically differ from the actual error of the state estimate.

In the discussion below, we will first show that there are infinitely many choices of parameters, Θ, for the reduced model in (1.2), such that the filter covariance estimate matches the optimal covariance estimate of the true filter in (1.1). However, most of these parameters will not give accurate estimates of the mean and therefore the covariance estimate will be inconsistent with the actual error covariance. By enforcing the consistency condition on the filter covariance estimate, we find unique parameters for the reduced model which produce the optimal state and covariance estimates.

(*a*) *Two-Variable Linear System*

Consider a linear model where $f_1 = a_{11}x + a_{12}y$ and $f_2 = a_{21}x + a_{22}y$ with a linear observation G. For this particular case the full filtering problem in (1.1) becomes

$$\begin{aligned} dx &= (a_{11}x + a_{12}y)\,dt + \sigma_x dW_x, \\ dy &= \frac{1}{\epsilon}(a_{21}x + a_{22}y)\,dt + \frac{\sigma_y}{\sqrt{\epsilon}} dW_y, \\ dz &= x\,dt + \sqrt{R}\,dV \equiv G(x,y)^\top\,dt + \sqrt{R}\,dV, \end{aligned} \tag{2.1}$$

where we define observation operator $G = (1, 0)$ for convenience. We assume that the matrix $A = (a_{ij})$ is negative definite and $\sigma_x, \sigma_y > 0$ are constants of $\mathcal{O}(1)$. We also assume that $\tilde{a} = a_{11} - a_{12}a_{22}^{-1}a_{21} < 0$, which guarantees the existence of the averaged dynamics in (1.2) for $\epsilon \to 0$; in this case, $F(X) = \tilde{a}X$ and $\sigma_X = \sigma_x$ (see e.g., [21] for detailed derivation).

(*b*) *Expansion of the Optimal Filter*

For the continuous time linear filtering problem in (2.1), the optimal filter estimates (in the sense of minimum variance estimator), are the first and second order statistics of a Gaussian posterior distribution that can be completely characterized by the solutions of Kalman-Bucy equations [26]. For this linear and Gaussian filtering problem, the covariance solutions of the filter will converge to a steady state covariance matrix $\hat{S} = \{\hat{s}_{ij}\}_{i,j=1,2}$, which solves the following algebraic Riccati equation,

$$A_\epsilon \hat{S} + \hat{S}A_\epsilon^\top - \hat{S}G^\top R^{-1} G\hat{S} + Q_\epsilon = 0, \tag{2.2}$$

where,

$$A_\epsilon = \begin{pmatrix} a_{11} & a_{12} \\ a_{21}/\epsilon & a_{22}/\epsilon \end{pmatrix}, \qquad Q_\epsilon = \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2/\epsilon \end{pmatrix}.$$

We can rewrite the first diagonal component of the algebraic Riccati equation (2.2) for $\hat{s}_{11} \equiv \mathbb{E}((x - \hat{x})^2)$ as follows (see Appendix A in the electronic supplementary material):

$$-\frac{\hat{s}_{11}^2}{R} + 2\tilde{a}\,(1 - \epsilon\hat{a})\,\hat{s}_{11} + \sigma_x^2(1 - 2\epsilon\hat{a}) + \epsilon\sigma_y^2\frac{a_{12}^2}{a_{22}^2} = \mathcal{O}(\epsilon^2) \tag{2.3}$$

where $\tilde{a} = a_{11} - \frac{a_{12}a_{21}}{a_{22}}$ and $\hat{a} = \frac{a_{12}a_{21}}{a_{22}^2}$.

Our goal is to find a one-dimensional model for the slow variable x which still gives the optimal state estimate. Motivated by the results in [21], and the fact that (2.3) has the form of a one-dimensional Riccati equation, we consider the following one-dimensional linear filtering problem,

$$dX = aX\,dt + \sigma_X\,dW_X, \tag{2.4}$$

$$dz = X\,dt + \sqrt{R}\,dV.$$

The corresponding steady state covariance solution for the reduced filter in (2.4) satisfies the following algebraic Riccati equation,

$$-\frac{\tilde{s}^2}{R} + 2a\tilde{s} + \sigma_X^2 = 0. \tag{2.5}$$

Substracting equation (2.3) from (2.5), we have the following result (see the detailed proof in Appendix B in the electronic supplementary material),

Theorem 2.1. *Let $\hat{s}_{11}$ be the first diagonal component of the algebraic Riccati equation in* (2.2) *and let $\tilde{s}$ be the solution of* (2.5)*. Then $\lim_{\epsilon\to 0}\frac{\tilde{s}-\hat{s}_{11}}{\epsilon} = 0$ if and only if*

$$\sigma_X^2 = 2(a - \tilde{a}(1-\epsilon\hat{a}))\hat{s}_{11} + \sigma_x^2(1-2\epsilon\hat{a}) + \epsilon\sigma_y^2\frac{a_{12}^2}{a_{22}^2} + \mathcal{O}(\epsilon^2). \tag{2.6}$$

Theorem 2.1 says that there is a manifold of parameters $\Theta = \{a, \sigma_X\}$ for which the equilibrium covariance estimate $\tilde{s}$ produced by the reduced model agrees with the equilibrium covariance estimate of the optimal filter $\hat{s}_{11}$, obtained with perfect model. So, for any parameters on the manifold (2.6), the reduced filter mean estimate solves,

$$d\tilde{x} = a\tilde{x}\,dt + \frac{\tilde{s}}{R}(dz - \tilde{x}\,dt), \tag{2.7}$$

while the true filter mean estimate for x-variable solves,

$$d\hat{x} = GA_\epsilon(\hat{x},\hat{y})^\top\,dt + \frac{\hat{s}_{11}}{R}(dz - \hat{x}\,dt). \tag{2.8}$$

While the true filter estimate in (2.8) is consistent, meaning that $\hat{s}_{11} = \mathbb{E}[(x-\hat{x})^2]$, as shown in the derivation of the Kalman-Bucy equations [26], the reduced filter estimate $\tilde{x}$ from (2.7) is not always consistent in the presence of model error. This is because the actual steady state error covariance, $E = \lim_{t\to\infty}\mathbb{E}[e(t)^2]$, where $e(t) = x(t)-\tilde{x}(t)$, is not necessarily equal to the steady state filter covariance estimate $\tilde{s} = \hat{s}_{11}+\mathcal{O}(\epsilon^2)$. In fact, most choices of parameters on the manifold in (2.6) lead to poor filter performance, despite the optimality of $\tilde{s}$ (in the sense of minimum variance), due to the inconsistency of the reduced filter.

By requiring the reduced filter estimates to be consistent, we find a unique choice of parameters $\Theta = \{a, \sigma_X\}$ on the manifold in (2.6) which produces optimal filter solutions (see Appendix B the electronic supplementary material). To obtain these parameters, we apply the following procedures: First, apply an asymptotic expansion up to order-ϵ^2 to the actual error, $e = x - \tilde{x}$. Subsequently, we use the Lyapunov equation to find the steady state actual error covariance, $E = \lim_{t\to\infty}\mathbb{E}(e(t)^2)$, of the state estimate $\tilde{x}$. Finally, we enforce the consistency condition, $E = \tilde{s}$, which yields a unique choice of parameters on the manifold in (2.6). Using the parameters derived by the formal asymptotic expansion, we can then prove that the reduced filter mean state and covariance estimates agree with those of the true filter up to order-ϵ^2 uniformly for all time. These results are summarized in the following theorem.

Theorem 2.2. *There exists a unique choice of parameters given by $a = \tilde{a}(1-\epsilon\hat{a})$ and σ_X^2 according to Theorem 2.1, such that the steady state reduced filter* (2.4) *is both consistent and optimal up to order-ϵ^2. This means that $\tilde{s}$, the steady state covariance estimate of the reduced filter, is consistent with the steady state actual error covariance $E_{11} = \lim_{t\to\infty}\mathbb{E}[(x(t)-\tilde{x}(t))^2]$ so that $\tilde{s} = E_{11} + \mathcal{O}(\epsilon^2)$, and also $\tilde{s}$ agrees with the steady state covariance $\hat{s}_{11}$ from the optimal filter $\tilde{s} = \hat{s}_{11} + \mathcal{O}(\epsilon^2)$. Furthermore, the reduced filter mean and covariance estimates are uniformly optimal for all time in the following sense. Given identical initial statistics, $\hat{x}(0) = \tilde{x}(0), \hat{s}_{11}(0) = \tilde{s}(0) > 0$, there are time-independent constants C_1, C_2, such that:*

$$|(\hat{s}_{11}(t) - \tilde{s}(t))| \le C_1\epsilon^2,$$
$$\mathbb{E}\Big(|\hat{x}(t) - \tilde{x}(t)|^2\Big) \le C_2\epsilon^4.$$

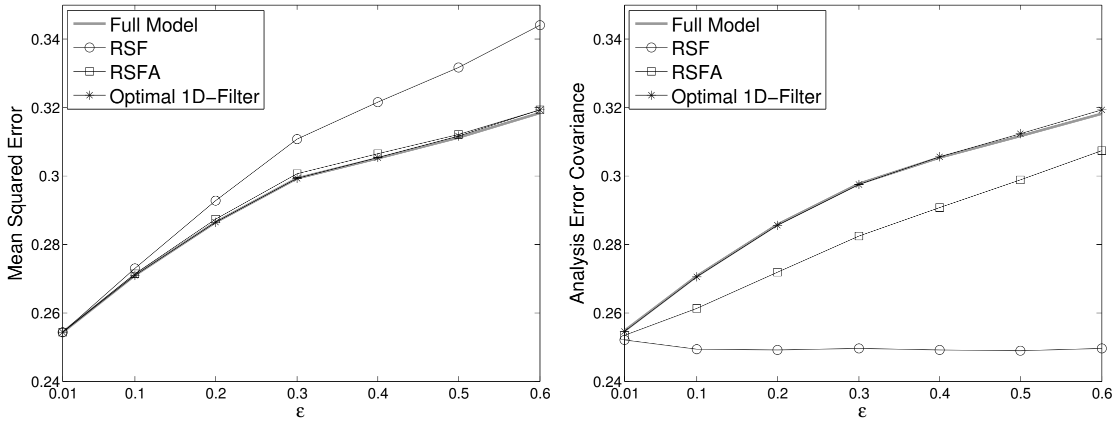

Figure 1. Comparison mean square error of the filtered mean (left) and covariance estimates (right) for an observation series produced by (2.1). The filter uses either the full model (2.1) or various reduced models given by parameter choices in (2.4). RSF: $a = \tilde{a}$ and $\sigma_X = \sigma_x$; RSFA: $a = \tilde{a}$ and $\sigma_X^2 = \sigma_x^2 + \epsilon\sigma_y^2 a_{12}^2/a_{22}^2$; optimal one-dimensional filter: $a = \tilde{a}(1-\epsilon\hat{a})$ and $\sigma_X^2 = \sigma_x^2(1-2\epsilon\hat{a}) + \epsilon\sigma_y^2 a_{12}^2/a_{22}^2$. The observation noise covariance is $R = 0.5$ and observations are at time intervals $\Delta t = 1$. Results are averaged over 100,000 observations.

Comparing this result to the reduced stochastic filter with an additive noise correction (RSFA) computed in Gottwald and Harlim [21], Theorem 2.2 imposes additional order-ϵ corrections in the form of linear damping, $-\epsilon\tilde{a}\hat{a}\tilde{x}$, and additive stochastic forcing, $-2\epsilon\sigma_x^2\hat{a}\,dW_x$. This additive noise correction term was also found in the formal asymptotic derivation of [21] (they denoted the covariance estimate associated with this additive noise correction by Q_2), but the absence of the order-ϵ linear damping correction term in their calculation makes it impossible to represent Q_2 with a Langevin equation. They dropped this additional additive noise term and, subsequently, underestimated the true error covariance (as shown in Figure 1). We now verify the accuracy of the filter covariance estimate suggested by Theorem 2.2 in the numerical simulation described below.

In Figure 1, we show numerical results comparing the true filter using the perfect model with approximate filter solutions based on three different one-dimensional reduced models of the form (2.4). All the filters are applied to discrete time observations at $\Delta t = 1$ with observation noise covariance $R = 0.5$ and the dynamics are solved analytically between observations. The three reduced models include: (1) the simple averaging model (RSF) where $a = \tilde{a}$ and $\sigma_X^2 = \sigma_x^2$; (2) the order-ϵ reduced model (RSFA) introduced in [21] with $a = \tilde{a}$ and $\sigma_X^2 = \sigma_x^2 + \epsilon\sigma_y^2 a_{12}^2/a_{22}^2$; and (3) the order-$\epsilon^2$ optimal reduced filter described in Theorem 2.2. Notice that only the order-ϵ^2 optimal reduced filter produces mean and covariance estimates that match the true filter solutions. Furthermore, its covariance estimate is consistent, that is, the mean square error, $\tilde{E} = \langle (x-\tilde{x})^2 \rangle$, where $\langle \cdot \rangle$ denotes temporal average (which equals E for ergodic posterior distribution) matches the asymptotic covariance estimates $\tilde{s}$ (compare the starred data points in the left and the right panels in Figure 1).

From this linear example, we learn that there exists a unique optimal reduced filter model such that the reduced filter covariance estimate, $\tilde{s}$, agrees with the true filter covariance estimate, $\hat{s}_{11}$, and the actual error covariance, E. We will see in Section 4 that these conditions are not trivially attainable even in a simple nonlinear filtering problem.

3. Filter Consistency: Quantifying Variance Estimation in the Presence of Model Error

The linear model examined in Section 2 showed the existence of one-dimensional optimal filter estimates which match the true statistical estimates of the Kalman-Bucy solutions from the full two-dimensional model, up to an error term depending on the time-scale separation parameter. The advantage of analyzing a linear problem is that the posterior distribution of the true filter is Gaussian and therefore can be described using only two moments given by the Kalman-Bucy equations. These filter estimates are optimal in the sense of minimum variance. For the nonlinear filtering problems discussed in the remainder of the paper, the posterior distribution of the true filter, $p(x(t)|z(\tau), 0 \leq \tau \leq t)$, is governed by a stochastically forced partial differential equation, known as the Kushner equation [25]. The Kushner equation is extremely difficult to solve explicitly or even numerically for general high-dimensional problems. Therefore, we have no access to the posterior distribution of the true filter in practical situation. Moreover, the posterior covariance estimate no longer has a fixed steady state solution, such as those used in (2.2) and (2.5) above, which further complicates any comparison between a suboptimal posterior estimate and the true posterior estimate. In this section we propose a new metric called *consistency* to measure the skill of the posterior covariance estimate.

In practical application with nonlinear systems, we often measure the filter performance by computing the mean squared error, $\tilde{E} \equiv \langle (x - \tilde{x})^2 \rangle$, which is a temporal average of the square difference of a single realization of the signal $x(t)$ and the estimate $\tilde{x}(t)$ produced by a filter. For practical consideration, we would like to have a measure for the covariance estimate, analogous to $\tilde{E}$, that can be computed by temporal averaging over a single realization of the filtered solutions. This is not a trivial problem when the true filter statistics, $\hat{x}(t)$ and $\hat{S}(t)$ are not available. Notice that for any fixed time t, the mean $\hat{x}(t)$ and covariance $\hat{S}(t)$ estimates of the true filter satisfy $\hat{S}(t) = \mathbb{E}[(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))^\top]$, where the expectation is with respect to the solutions of the Kushner equation, $p(x(t) \,|\, z(\tau), 0 \leq \tau \leq t)$. Thus if x is n-dimensional, at each t we have

$$\mathbb{E}[(x(t) - \hat{x}(t))^\top \hat{S}^{-1}(x(t) - \hat{x}(t))] = \mathrm{Trace}\left(\hat{S}(t)^{-1/2}\mathbb{E}[(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))^\top]\hat{S}(t)^{-1/2}\right) = n.$$

In general, given any filtered statistical estimates, $\tilde{x}(t)$ and $\tilde{S}(t)$, we can define a norm on the state estimate error as follows,

$$\|x(t) - \tilde{x}(t)\|^2_{\tilde{S}(t)} \equiv \frac{1}{n}(x(t) - \tilde{x}(t))^\top \tilde{S}(t)^{-1}(x(t) - \tilde{x}(t)),$$

where n is the dimension of the state vector x. Assuming that the posterior distribution has an ergodic invariant measure, we have

$$1 = \mathbb{E}[\|x - \hat{x}\|^2_{\hat{S}}] = \langle \|x - \hat{x}\|^2_{\hat{S}} \rangle,$$

where $\langle \cdot \rangle$ denotes temporal average. Motivated by this property, we propose the following metric to measure the quality of a filter covariance estimate.

Definition 3.1. *Consistency (of Covariance). Let $\tilde{x}(t)$ and $\tilde{S}(t)$ be a realization of the solution to a filtering problem for which the true signal of the realization is $x(t)$. The* consistency *of the realization is defined to be, $\mathcal{C}(x, \tilde{x}, \tilde{S})$ where*

$$\mathcal{C}(x, \tilde{x}, \tilde{S}) = \langle \|x - \tilde{x}\|^2_{\tilde{S}} \rangle. \tag{3.1}$$

We say that a filter is consistent *if $\mathcal{C} = 1$ almost surely (independent of the realization).*

With this definition, it is obvious that the true filter is consistent. However it is not the only consistent filter and not every consistent filter is accurate. For example, in the case of fully observed filtering problems, the observation sequence itself is a trivial filtered solution which is consistent if we take the covariance estimate to be the covariance of the observation noise. Second, assuming the dynamical system is ergodic, we can define a consistent filter based purely on the equilibrium statistics by using a constant prediction $\tilde{x} = \mathbb{E}[x(t)]$ and covariance $\tilde{S} = \mathbb{E}[(x(t) - \tilde{x}(t))(x(t) - \tilde{x}(t))^\top]$. These two examples are the trivial extremes of filtering, the first simply takes the observations as solutions, while the second completely ignores the observations. However, while these examples are both consistent, neither is doing a good job of estimating the state compared to the true filter. Therefore, this consistency measure is only a necessary condition for the covariance to be meaningful. It should be used together with the mean squared error measure. However, this measure has the following nice property: a consistent filter which produces posterior mean estimates close to the true posterior mean estimates also has a covariance close to the true posterior covariance, which we will verify now.

As above, let $\hat{x}(t)$ and $\hat{S}(t)$ be the posterior statistical estimates of the true filter and $\tilde{x}(t)$ and $\tilde{S}(t)$ be the estimates from a suboptimal filter. For convenience we will drop the letter t when no confusion is possible. Then, assuming $\mathcal{C}(x,\tilde{x},\tilde{S})$ exists, we can write

$$\begin{aligned}\mathcal{C}(x,\tilde{x},\tilde{S}) &= \frac{1}{n}\langle (x-\hat{x}+\hat{x}-\tilde{x})^\top \tilde{S}^{-1}(x-\hat{x}+\hat{x}-\tilde{x})\rangle,\\ &= \langle \|x-\hat{x}\|^2_{\tilde{S}}\rangle + \langle \|\tilde{x}-\hat{x}\|^2_{\tilde{S}}\rangle + \frac{2}{n}\langle (x-\hat{x})^\top \tilde{S}^{-1}(\hat{x}-\tilde{x})\rangle,\\ &= \mathcal{C}(x,\hat{x},\tilde{S}) + \langle \|\tilde{x}-\hat{x}\|^2_{\tilde{S}}\rangle + \frac{2}{n}\langle (x-\hat{x})^\top \tilde{S}^{-1}(\hat{x}-\tilde{x})\rangle.\end{aligned} \tag{3.2}$$

Note also that,

$$\begin{aligned}\Big|\mathcal{C}(x,\hat{x},\hat{S}) - \mathcal{C}(x,\hat{x},\tilde{S})\Big| &\le \Big|\mathcal{C}(x,\hat{x},\hat{S}) - \mathcal{C}(x,\tilde{x},\tilde{S})\Big| + \Big|\mathcal{C}(x,\tilde{x},\tilde{S}) - \mathcal{C}(x,\hat{x},\tilde{S})\Big|,\\ &= \Big|1-\mathcal{C}(x,\tilde{x},\tilde{S})\Big| + \langle \|\tilde{x}-\hat{x}\|^2_{\tilde{S}}\rangle + \frac{2}{n}\langle (x-\hat{x})^\top \tilde{S}^{-1}(\hat{x}-\tilde{x})\rangle,\\ &\le \Big|1-\mathcal{C}(x,\tilde{x},\tilde{S})\Big| + c\,\langle \|\tilde{x}-\hat{x}\|^2\rangle.\end{aligned} \tag{3.3}$$

where the first line is due to the standard triangle inequality, the equality in the second line is based on the fact that the true filter is consistent, $\mathcal{C}(x,\hat{x},\hat{S})=1$ and the algebraic expression in (3.2), and the inequality in the third line is due to Cauchy-Schwarz inequality and the constant c depends on the smallest eigenvalue of $\tilde{S}$ and $\langle\|x-\hat{x}\|^2\rangle$.

The inequality in (3.3) suggests that if the state estimate $\tilde{x}$ is close to the true posterior estimate $\hat{x}$ and the consistency measure is close to one, then the covariance estimate $\tilde{S}$ is close to the true posterior covariance $\hat{S}$ in the sense that

$$\Big|\mathcal{C}(x,\hat{x},\tilde{S}) - \mathcal{C}(x,\hat{x},\hat{S})\Big| = \frac{1}{n}\langle (x-\hat{x})^T(\tilde{S}^{-1}-\hat{S}^{-1})(x-\hat{x})\rangle,$$

is small. Thus, a consistent filter with a good estimate of the posterior mean has a good estimate of the posterior covariance. In practice, many approximate filter mean estimates are quite accurate, in the sense that they are close to the true posterior estimate [31]. Therefore the consistency measure on the approximate filter solutions, $\mathcal{C}(x,\tilde{x},\tilde{S})$, is relevant for quantifying the skill of $\tilde{S}$, when the true filter covariance estimate $\hat{S}$ is not available.

4. Nonlinear Theory

In this section we give an example of a simple nonlinear multiscale system with only additive noise and a reduced model for the slow variable which requires multiplicative noise. In contrast to linear filtering problems, the optimal solution to a nonlinear filtering problem is not available in practice, since it requires solving infinite-dimensional systems. For this test problem, the best practically available posterior estimate is given by a Gaussian closure on the posterior mean and covariance. Given discrete-time observations, these estimates were shown to have good performance in [18, 17, 21]. In this section, we show that there exists a reduced filter model with equivalent mean and covariance estimates to those of the slow variable from the full filter. Furthermore, the reduced filtered solutions are also consistent in the sense of Definition 3.1.

We consider the following continuous-time nonlinear filtering problem,

$$\begin{aligned}du &= [-(\gamma+\lambda_u)u + b]\,dt + \sigma_u dW_u,\\ db &= -\frac{\lambda_b}{\epsilon} b\,dt + \frac{\sigma_b}{\sqrt{\epsilon}}dW_b,\\ d\gamma &= -\frac{\lambda_\gamma}{\epsilon}\gamma\,dt + \frac{\sigma_\gamma}{\sqrt{\epsilon}}dW_\gamma,\\ dz &= h(u,\beta,\gamma)\,dt + \sqrt{R}\,dV = u\,dt + \sqrt{R}\,dV.\end{aligned} \tag{4.1}$$

The discrete observation-time analog of this nonlinear filtering problem was introduced as SPEKF, which stands for "Stochastic Parameterized Extended Kalman Filter" in [17, 18], which solutions are attained by applying a Kalman update on the exactly solvable prior statistical solutions of the full model in (4.1). The nonlinear system in (4.1) has few attractive features as a test model. First, it has exactly solvable statistical solutions which are non-Gaussian, allowing to study non-Gaussian prior statistics conditional to the Gaussian posterior statistical solutions of Kalman filter and to verify uncertainty quantification method [9]. Second, recent study by [8] suggests that the system in (4.1) can reproduce signals in various turbulent regimes such as intermittent instabilities in a turbulent energy transfer range and in a dissipative range as well as laminar dynamics.

The true posterior distribution, $p(\vec{u},t) = P(\vec{u},t\,|\,z(\tau), 0\leq\tau\leq t)$ for $\vec{u}=(u,b,\gamma)$, evolves according to the Kushner equation [30],

$$dp = \mathcal{L}^* p\, dt + p(h-\mathbb{E}[h])^\top R^{-1} dw_{\hat{u}},$$

where $\mathcal{L}^*$ is the Fokker-Planck operator for the state variables $\vec{u}$. The term $dw_{\hat{u}} = dz - \mathbb{E}[h]dt$ is called the innovation process and represents the difference between the actual observation z and the expected observation $\mathbb{E}[h]$ with respect to p. As in the linear example above we will assume that $h(\vec{u}) = u$ so that only the slow variable is observed, thus allowing fair comparison with a reduced model for the slow variable.

The Kushner equation is a stochastic partial differential equation (SPDE) which is easily solved when both the dynamics and observation process are linear, in which case one recovers the Kalman-Bucy equations. Since our ultimate goal is to develop practical method for high-dimensional problems, we restrict our study on the first two moments, $\hat{u} = \int up\, d\vec{u}$ and $\hat{S} = \int (u-\hat{u})^2 p\, d\vec{u}$, for the slow variable u with respect to the conditional distribution p.

In particular, substituting the Kushner expression for dp in $d\hat{u} = \int udpd\vec{u}$ and applying integration by parts with the assumption that p has fast decay at infinity, we find that,

$$d\hat{u} = \left(-\lambda_u\hat{u} - \overline{u\gamma} + \overline{b}\right) dt + \hat{S}R^{-1}dw_{\hat{u}},$$

where $\overline{u\gamma} = \int u\gamma p\, d\vec{u}$ and $\overline{b} = \int bp\, d\vec{u}$. By differentiating these terms and applying the expansion $p = p_0 + \epsilon p_1$ we can explicitly estimate these terms up to order-ϵ^2. The full details of this expansion are found in in Appendix C (see the electronic supplementary material), where we find that evolution of $\hat{u}$ and $\hat{S}$ reduces to

$$\begin{aligned} d\hat{u} &= -\left(\lambda_u - \frac{\epsilon\sigma_\gamma^2}{2\lambda_\gamma(\lambda_u\epsilon+\lambda_\gamma)}\right)\hat{u}\, dt + \hat{S}R^{-1}dw_{\hat{u}} + \mathcal{O}(\epsilon^2), \\ d\hat{S} &= \left[-2\left(\lambda_u - \frac{\epsilon\sigma_\gamma^2}{\lambda_\gamma(\lambda_u\epsilon+\lambda_\gamma)}\right)\hat{S} + \frac{\epsilon\sigma_\gamma^2}{\lambda_\gamma(\lambda_u\epsilon+\lambda_\gamma)}\hat{u}^2 + \sigma_u^2 + \frac{\epsilon\sigma_b^2}{2\lambda_b(\lambda_b+\lambda_u\epsilon)} - \hat{S}R^{-1}\hat{S}\right] dt, \\ &\quad + \left[\int (u-\hat{u})^3 p d\vec{u}\right] R^{-1}dw_{\hat{u}} + \mathcal{O}(\epsilon^2). \end{aligned} \tag{4.2}$$

These equations give the exact solutions for the evolution of the first two statistics of the *posterior distribution*, p, up to order-ϵ^2, however they are not closed since the skewness $\int (u-\hat{u})^3 p\, d\vec{u}$ appears in the evolution of the covariance $\hat{S}$. We close these equations by assuming that the posterior distribution is Gaussian, or effectively, $\int (u-\hat{u})^3 pd\vec{u} = 0$. While the equilibrium statistics of the dynamics in (4.1) have zero skewness, this is not necessarily the case for the posterior distribution given a noisy observation sequence [9]. Note that this closure is significantly different from the Gaussian Closure Filter (GCF) introduced in [8], which applies a Gaussian closure on the prior dynamics before using a Kalman update to obtain posterior solutions.

Since we are interested in finding a one-dimensional reduced model for the slow variable u we only derive the moment estimates for u which is given by,

$$d\hat{u} = -\left(\lambda_u - \frac{\epsilon\sigma_\gamma^2}{2\lambda_\gamma(\lambda_u\epsilon+\lambda_\gamma)}\right)\hat{u}\, dt + \hat{S}R^{-1}dw_{\hat{u}} + \mathcal{O}(\epsilon^2),$$

$$\frac{d}{dt}\hat{S} = -2\left(\lambda_u - \frac{\epsilon\sigma_\gamma^2}{\lambda_\gamma(\lambda_u\epsilon+\lambda_\gamma)}\right)\hat{S} + \frac{\epsilon\sigma_\gamma^2}{\lambda_\gamma(\lambda_u\epsilon+\lambda_\gamma)}\hat{u}^2 + \sigma_u^2 + \frac{\epsilon\sigma_b^2}{2\lambda_b(\lambda_b+\lambda_u\epsilon)} - \hat{S}R^{-1}\hat{S} + \mathcal{O}(\epsilon^2). \quad (4.3)$$

We refer to these statistical estimates as the **continuous-time SPEKF solutions** for variable u. To obtain the full continuous-time SPEKF solution, one can compute the mean and covariance matrix of the full state $\vec{u}$, with similar Itô calculus. In this sense, the original SPEKF that was introduced in [18, 17] is a discrete-time analog of the continuous-time SPEKF since it implicitly truncates the higher-order moments of the posterior statistics through a discrete-time Kalman update.

Motivated by the results in [21, 9], we now propose the following reduced filter to approximate the filtering problem in (4.1),

$$\begin{aligned} dU &= -\alpha U dt + \beta U \circ dW_\gamma + \sigma_1 dW_u + \sigma_2 dW_b, \\ &= -\left(\alpha - \frac{\beta^2}{2}\right) U dt + \beta U dW_\gamma + \sigma_1 dW_u + \sigma_2 dW_b, \\ dz &= U\,dt + \sqrt{R}\,dV. \end{aligned} \quad (4.4)$$

The evolution of the first two moments of (4.4), $\tilde{u} = \int U\pi dU$ and $\tilde{S} = \int (U-\tilde{u})^2\pi\, dU$, where π is the posterior distribution governed by the Kushner equation for (4.4), is given by,

$$\begin{aligned} d\tilde{u} &= -\left(\alpha - \frac{\beta^2}{2}\right)\tilde{u}\,dt + \tilde{S}R^{-1}dw_{\tilde{u}}, \\ \frac{d}{dt}\tilde{S} &= -2\left(\alpha-\beta^2\right)\tilde{S} + \beta^2\tilde{u}^2 + \sigma_1^2 + \sigma_2^2 - \tilde{S}R^{-1}\tilde{S}, \end{aligned} \quad (4.5)$$

where $dw_{\tilde{u}} = dz - \tilde{u}\,dt$ denotes the innovation process and Gaussian closure is imposed by setting the skewness to zero (see Appendix C for detailed derivation). We can specify the coefficients in (4.4) by matching the estimates of the mean and covariance in the filters (4.3) and (4.5) which yields

$$\begin{aligned} &\alpha = \lambda_u, \quad \sigma_1^2 = \sigma_u^2, \\ &\sigma_2^2 = \frac{\epsilon\sigma_b^2}{2\lambda_b(\lambda_b+\epsilon\lambda_u)}, \quad \beta^2 = \frac{\epsilon\sigma_\gamma^2}{\lambda_\gamma(\lambda_u\epsilon+\lambda_\gamma)}. \end{aligned} \quad (4.6)$$

We refer to the solutions of (4.5) with parameters in (4.6) as the **continuous-time reduced SPEKF solutions** of the filtering problem (4.4). With this choice of coefficients we have the following result (see Appendix C for detailed proof).

Theorem 4.1. *Let $\lambda_u > 0$, and let z be noisy observations of the state variable u which solves the full model in* (4.1)*. Given identical initial statistics, $\tilde{u}(0) = \hat{u}(0)$ and $\tilde{S}(0) = \hat{S}(0) > 0$, the mean and covariance estimates of a stable continuous-time reduced SPEKF in* (4.4) *with parameters* (4.6) *agree with mean and covariance of a stable continuous-time SPEKF for variable u in the following sense. There exist time-independent constants, C_1, C_2, such that,*

$$\begin{aligned} |\hat{S}(t) - \tilde{S}(t)| &\le C_1\epsilon, \\ \mathbb{E}\left[|\hat{u}(t)-\tilde{u}(t)|^2\right] &\le C_2\epsilon^2. \end{aligned}$$

Furthermore, the reduced filtered solutions are also consistent, up to order-ϵ.

Theorem 4.1 shows that the continuous-time reduced SPEKF solutions in (4.5) are consistent up to order-ϵ, and match the first two moments of the continuous-time SPEKF solutions for the slow variable u up to order-ϵ. Notice that despite the formal asymptotic expansion of the filtered statistics up to order-ϵ^2, we cannot obtain the same uniform bounds on the errors $|\hat{S}(t) - \tilde{S}(t)|$ and $|\hat{x} - \tilde{x}|$ as in the linear case (see the Appendix C for details). This shows that in the context of Gaussian closure on the posterior distribution, accounting for truncation of fast time scales in a nonlinear model with only additive noise

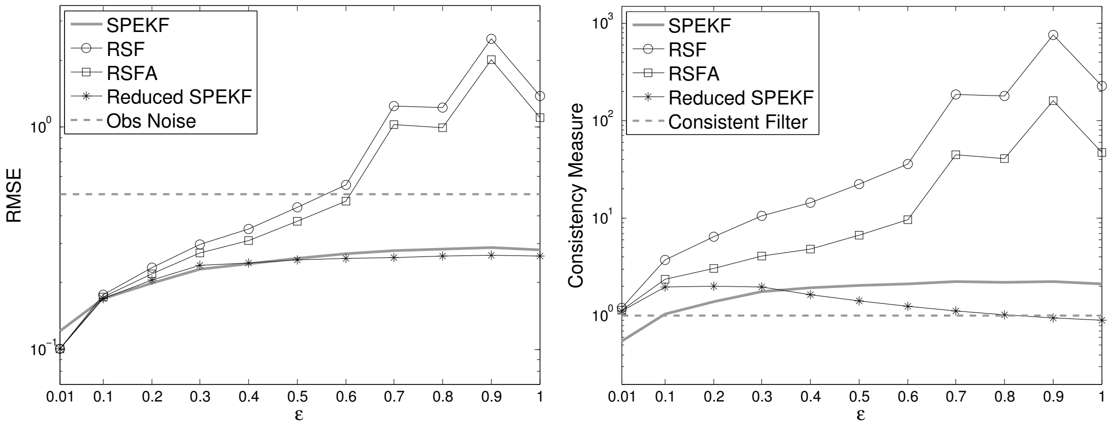

Figure 2. Filter performance measured in terms of root mean squared error (RMSE, left) and consistency measure (right) for various time-scale separations ϵ with an integration time step of $dt = 0.05$ and observations at time intervals $\Delta t = 0.5$ with observation noise $R = 0.25$. Results are overaged over 200,000 observations in Regime II from [21] where $\lambda_u = 0.55, \lambda_b = 0.4, \lambda_\gamma = 0.5, \sigma_u = 0.1, \sigma_b = 0.4, \sigma_\gamma = 0.5$. The *SPEKF* solution use the full model from (4.1), the *Reduced SPEKF* uses the model from (4.4) with the parameters (4.6) whereas *RSFA* sets $\beta = 0$ and *RSF* sets $\sigma_2 = \beta = 0$.

requires a multiplicative noise correction term in the reduced model. This fact, together with the linear theory from Section 2, will motivate our stochastic parameterization in Section 5. We note that the term $\epsilon\lambda_u$ appearing in the denominator of the parameters σ_2 and β in (4.6) is technically an order-ϵ^2 adjustment, however, this term arises naturally in the derivation of the continuous-time SPEKF solutions for (4.1) in Appendix C and is important for large ϵ as we will discussed below. We should point out that these order-ϵ^2 correction terms was not found in [21], which included corrections of only up to order ϵ.

In Figure 2 we show how the reduced SPEKF solutions of (4.4) match the solutions of the full SPEKF technique when only the slow variable is observed. The comparison is made in an extremely difficult regime of the dissipative range, known as Regime II in [9, 21], where the dynamics of $u(t)$ exhibits intermittent burst of transient instabilities, followed by quiescent phases. The parameters are: $\lambda_u = 0.55, \lambda_b = 0.4, \lambda_\gamma = 0.5, \sigma_u = 0.1, \sigma_b = 0.4, \sigma_\gamma = 0.5$. In this regime, u and γ have comparable decaying time scales. In this numerical example, we apply all the filters with discrete time observations at time $\Delta t = 0.5$ with observation noise covariance $R = 0.25$. Numerically, we simulate SPEKF and reduced SPEKF solutions exactly like in [18, 17, 21].

Here, we numerically compare the full SPEKF solutions with: the reduced stochastic filter (RSF) which assumes $\alpha = \lambda_u, \sigma_1^2 = \sigma_u^2, \sigma_2 = 0$ and $\beta = 0$; the reduced stochastic filter with additive correction (RSFA) which assumes the parameters in (4.6) except with $\beta = 0$, and the reduced SPEKF solutions with parameters in (4.6). Note that the consistency measures for RSF and RSFA are on the order of 10^2 which shows that these filters significantly underestimate the actual error covariance, which leads to insufficient Kalman gain and large mean squared errors (even larger than the observations error variance, R, for large ϵ). On the other hand, the reduced SPEKF is as accurate as the full SPEKF and even more consistent compared to the full SPEKF for large ϵ, based on the empirical measure in Definition 3.1. We suspect the loss of consistency of the full SPEKF solutions when ϵ is large is due to combinations of the following issues: (1) the full SPEKF has sparse observations of only u; (2) the prior statistical solutions of the full SPEKF involve quadrature approximation of various integral terms; (3) the sufficient conditions for the stability of the prior covariance of the full SPEKF [9, 21],

$$\Xi_2 = -2\lambda_u + \epsilon\frac{2\sigma_\gamma^2}{\lambda_\gamma^2} < 0, \tag{4.7}$$

is not satisfied. On the other hand, the order-ϵ^2 correction terms, $\epsilon\lambda_u$, that appears in both the denominator of β and σ_2 in (4.6), of the reduced SPEKF model, assures the stability of the prior covariance estimate, since

$$\tilde{\Xi}_2 = -2\lambda_u + \epsilon\frac{2\sigma_\gamma^2}{\lambda_\gamma(\lambda_\gamma + \epsilon\lambda_u)} < 0. \tag{4.8}$$

This clearly illustrates the need for the multiplicative noise correction in the reduced model to capture the actual error covariance of (4.1).

5. Stochastic parameterization for filtering with model errors

For general nonlinear multiscale problems as in (1.1), we define residuals for the deterministic and stochastic operators as the difference between the full model operators and the approximate model operators,

$$\begin{aligned} r_f(x,y) &= f_1(x,y) - \tilde{f}_1(x), \\ r_\sigma(x,y) &= \sigma_x(x,y) - \tilde{\sigma}_x(x). \end{aligned} \tag{5.1}$$

Obviously, there are many ways to choose an approximate reduced model. For example, classical averaging theory [29] suggests the following reduced model operators,

$$\begin{aligned} \tilde{f}_1(x) &= \lim_{\epsilon\to 0}\int f_1(x,y)p_\infty(y|x)\,dy, \\ \tilde{\sigma}_x(x)\tilde{\sigma}_x(x)^\top &= \lim_{\epsilon\to 0}\int \sigma_x(x,y)\sigma_x(x,y)^\top p_\infty(y|x)\,dy, \end{aligned} \tag{5.2}$$

assuming that the conditional invariant measure, $p_\infty(y|x)$, exists and these limits exist. If no explicit expression for $p_\infty(y|x)$ is available, but the small-scale dynamical operators, $f_2(x,y), \sigma_y(x,y)$ are known, one can apply the Heterogeneous Multiscale Method [13] to sample the conditional measure. Alternatively, one can also emulate the small-scale processes with Hidden Markov Models [34, 45, 10]. In the numerical simulation in Section 6, we will apply the crudest approximate model operators for (5.2) with a Dirac delta measure, $p_\infty(y|x) = \delta_0(y)$. This choice mimics the situation where nothing is known about the process y in the true dynamics.

Motivated by the theoretical results in Sections 2 and 4, we propose to model the residuals in (5.1) with a linear damping term and combined, additive and Stratonovich-type multiplicative, stochastic forcings,

$$r_f(x,y)\,dt + r_\sigma(x,y)dW_x \approx -\alpha X\,dt + \sigma\,dW_2 + \beta X \circ dW_3. \tag{5.3}$$

With this approximation, the resulting reduced prior filter model is given by,

$$\begin{aligned} dX &= (\tilde{f}_1(X;\theta) - \alpha X)\,dt + \tilde{\sigma}_x(X;\theta)\,dW_x + \sigma\,dW_2 + \beta X \circ dW_3, \\ &= F(X;\Theta)\,dt + \sigma_X(X;\Theta)dW_X, \end{aligned} \tag{5.4}$$

where we have rewritten the stochastic reduced prior model in (5.4) in the Itô sense with,

$$\begin{aligned} F(X;\Theta) &= \tilde{f}_1(X;\theta) - (\alpha - \frac{\beta^2}{2})X, \\ \sigma_X(X;\Theta) &= (\tilde{\sigma}_x(X;\theta), \sigma, \beta X), \\ dW_X &= (dW_x, dW_2, dW_3)^\top. \end{aligned}$$

In this context, the reduced model parameters are $\Theta = (\theta, \alpha, \beta, \sigma)$. We suspect that the approximation in (5.3) may not be sufficient for optimal reduced filtering in general complex nonlinear problems since more complicated correction terms may be needed. Nevertheless, we will numerically show that this simple approximation will significantly improve the filtered estimates in terms of accuracy and consistency (see Section 6). There are obviously many open questions regarding applications of this approximation in practice, such as whether α should vary at different grid points, etc. We will discuss some of these issues in Section 7.

In the remainder of this section, we will discuss an optimal stochastic parameter estimation method for filtering the linear problem in Section 2. Subsequently, we will briefly review some existing methods for estimating Θ in general nonlinear problems.

(a) An optimal stochastic parameter estimation method for filtering linear problems

For the linear problem discussed in Section 2, Theorem 2.2 suggests that an optimal filtering can be achieved with a reduced, one-dimensional prior model,

$$dX = F(X)\,dt + \sigma_X dW_X, \tag{5.5}$$

with the following operators,

$$\begin{aligned} F(X) &= aX = \tilde{a}(1-\epsilon\hat{a})X \\ \sigma_X &= \Big(\sigma_x(1-\epsilon\hat{a}), \sqrt{\epsilon}\sigma_y \frac{a_{12}}{a_{22}}\Big), \end{aligned}$$

where $dW_X = (dW_x, dW_y)^\top$ and $\Theta = \{a, \sigma_X\}$. In this case, we have no multiplicative noise correction, $\beta = 0$. On the other hand, recall that the dynamics of true signal, x, in (2.1) can be approximated (up to order ϵ^2) by the same reduced one-dimensional linear stochastic differential equation (see Appendix B in the electronic supplementary material). Recall that solutions of the linear SDE in (5.5) are known as the Ornstein-Uhlenbeck processes [16]. Furthermore, the two parameters, namely the linear damping coefficient, a, and the noise amplitude σ_X, can be characterized by two equilibrium statistics, variance and correlation time [16]. Therefore, an ideal parameter estimation scheme for the reduced, one-dimensional linear filter model in (5.5), is based on these two equilibrium statistics, which was introduced as the Mean Stochastic Model in [35] (see also [36] for different stochastic parameterization strategy for the linear SDE in (5.5)). More formally, Theorem 2.2 suggests the following result,

Corollary 5.1. *Given the equilibrium variance and the correlation time statistics of the true signal, $x(t)$, that evolves based on the linear SDE in (2.1), the reduced Mean Stochastic Model filter is an optimal filter, in the sense that the posterior mean and covariance estimates are differed by an order of ϵ^2 from the true filter estimates obtained with the perfect model.*

(b) Stochastic parameter estimation method for filtering high dimensional problems

For the nonlinear test filtering problem in Section 4, an equivalent offline parameter estimation scheme can be designed since the parameters of (4.4) can be uniquely defined by matching the equilibrium variance, kurtosis, and correlation time statistics. Unfortunately, this parameter estimation scheme will not produce optimal filtered solutions, since the mean and covariance estimates of the SPEKF solutions to (4.1) is only consistent up to order-ϵ (see Theorem 4.1). Therefore, it is even more difficult to design an optimal stochastic parameter estimation method as in Corollary 5.1.

Classical online parameter estimation method for α was proposed in [14, 15]. The main idea is to apply the standard Kalman filter on an augmented state-parameter space, assuming that the parameters evolve in time, satisfying an empirically chosen dynamical equations. Popular choices for the parameter dynamics are the persistent model, $d\alpha/dt = 0$, and white noise model, $d\alpha = \kappa\, dW$, with an empirically chosen noise amplitude, κ. In our numerical experiment below, we will apply the persistent model for estimating α. Classical approaches to estimate β were also proposed by Mehra [40] and Belanger [5] for linear Kalman filtering problems. Both methods utilize the information from the innovations and the correlation of lagged innovations to estimate β, but rely on many assumptions which fail for nonlinear models (such as stationarity of the innovation sequence). Recently, extension of these two methods in an ensemble Kalman filtering setting were independently developed; see Berry and Sauer [6] for the extension on Mehra's method [6] and Harlim et al., [22] for the extension on Belanger's scheme. In our numerical experiment below, we will adopt the Mehra's method on ensemble Kalman filter setup (see [6] for the algorithm details). Some methods for estimating the multiplicative noise amplitude β were proposed in [43, 46]. In our numerical application below, we will not estimate β and will simply set $\beta = 0$. We plan to incorporate the appropriate multiplicative noise estimation method in our future research.

6. Application to Two Layer Lorenz Model

In this section, we demonstrate the stochastic parameterization strategy proposed in (5.3) with a numerical example on the two-layer Lorenz96 model [33], given below in (6.1). In particular, we will use the adaptive parameter estimation method discussed in Section 5.(b) to obtain α and σ, ignoring the multiplicative noise correction by setting $\beta = 0$.

The two-layer Lorenz96 model is an $N(M+1)$-dimensional ODE given by,

$$\begin{aligned}\frac{dx_i}{dt} &= x_{i-1}(x_{i+1} - x_{i-2}) - ax_i + F + \frac{h_x}{M}\sum_{j=1}^{M} y_{i,j},\\ \epsilon\frac{dy_{i,j}}{dt} &= y_{i,j+1}(y_{i,j-1} - y_{i,j+2}) - y_{i,j} + h_y x_i,\end{aligned} \tag{6.1}$$

where $x = x(t)$ and $y_i = y(t)$ are vectors in $\mathbb{R}^N$ and $\mathbb{R}^M$ respectively and the subscript i is taken modulo N and j is taken modulo M. We integrate this model using the Runge-Kutta method (RK4) for the system (6.1) with a time step dt and observe every Δt time steps. In this case, we take $\Delta t = \min\{0.01, \epsilon/10\}$ and $dt = \frac{\Delta t}{10}$, the ϵ dependence in the time step is necessary due to the stiffness of the problem as $\epsilon \to 0$. At each discrete observation-time t_m the filter observes $z_m = h(x(t_m), y(t_m)) + \sqrt{R}v_m$ where v_m is standard i.i.d. white noise. The goal of this example is to show what can happen when an unresolved and possibly unknown time scale is neglected, thus in all examples we consider an observation function $h(x, y) = x$ which is the identity on the slow scales but does not observe the small scales at all. As in previous sections we note that the problem of mixed observations presents unique difficulties which we defer to future research. We generate a truth and observation time series using the parameters from Regime 2 of [28] where $N = 9$, $M = 8$, $a = 1$, $F = 10$, $R = 0.1$, $h_x = -0.8$ and $h_y = 1$. Note that there are 81 total variables, only 9 of which are observed. We will vary $2^{-7} \leq \epsilon \leq 1$.

We first consider the idealized case when the full model (6.1) and all the parameters are known exactly, we apply an ensemble Kalman filter based on the full ensemble transform [7], which uses 162 ensemble members each of which must be integrated 10 times between observations. For comparison purposes we will consider the resulting RMSE, shown as *Full Model* in Figure 3, to be the "gold standard" for a filter based on a Gaussian update.

Having established our "gold standard" given the full perfect model, we now attempt to filter using a class of reduced models, which neglects all of the fast processes. The reduced model

$$\frac{dx_i}{dt} = x_{i-1}(x_{i+1} - x_{i-2}) - ax_i + F + \Big(-\alpha x_i + \sum_{j=1}^{N}\sigma_{ij}dw_j\Big) \tag{6.2}$$

is simply the one layer Lorenz96 model augmented with an additional damping term αx_i and an additive noise term which is given by the matrix $\sigma = (\sigma_{ij})$. Numerically, the reduced model was integrated with $dt = \Delta t$, thus using 10 times fewer integration steps than the full model, since the numerical stiffness dissapears when the fast processes are removed. Essentially we are attempting to replace the 72 unmodeled fast variables with a linear feedback and an additive noise term (the terms in parentheses in (6.2)). Moreover, we will estimate the parameters α and σ as part of the filter procedure. To estimate α we use the standard approach of state augmentation, making our augmented state variable 10 dimensional ($N = 9$ state variables plus the single parameter α) and we use $\frac{d}{dt}\alpha = 0$ for the dynamics of the parameter. To estimate the full matrix σ we used the estimation technique of [6]. Of course, Section 4 suggests that we should also include a multiplicative noise term $\beta x_i \circ d\hat{w}_i$, however we do not include this term because of difficulty in estimated the β parameter. In our numerical experiments below, the reduced filter solutions are obtained by applying ETKF [7] with 18 ensemble members when the damping parameter α in (6.2) is ignored and with 20 ensemble members when α is estimated.

To evaluate the effectiveness of the additional damping and additive noise in the reduced model we consider four separate cases. First, we set $\alpha = 0$ and $\sigma = 0$, which we call the *reduced deterministic filter (RDF)* since the slow variables are unchanged and the fast variables have simply been truncated. As shown in Figure 3, RDF has very poor performance for all but extremely small values of ϵ. In fact for $\epsilon \geq 0.125$

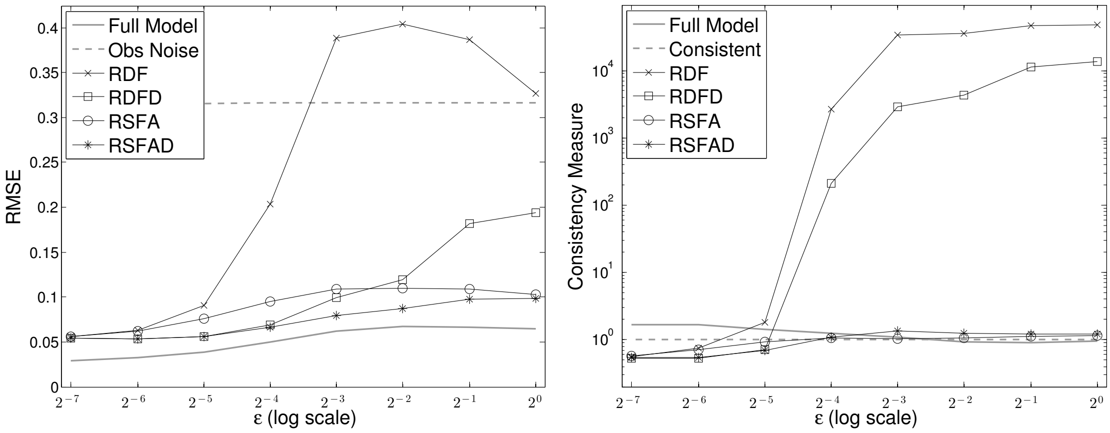

Figure 3. Filter performance measured in terms of root mean squared error (RMSE, left) and consistency measure (right) for various time-scale separations ϵ with an integration time step of $dt = \frac{\Delta t}{10}$ and observations at time intervals $\Delta t = \min\{0.01, \epsilon/10\}$ with observation noise $R = 0.1$. Results are overaged over $60,000$ observations in Regime 2 from [28] where $N = 9$, $M = 8$, $a = 1$, $F = 10$, $R = 0.1$, $h_x = -0.8$ and $h_y = 1$. All filters use the ensemble transform Kalman filter with dynamics integrated with the Runge-Kutta (RK4) method. The *Full Model* filter uses (6.1), the same model used to generate the data. The remaining models use (6.2) where *RDF* sets $\sigma_{ij} = \alpha = 0$, *RDFD* sets $\sigma_{ij} = 0$, *RSFA* sets $\alpha = 0$ and *RSFAD* fits both parameters simultaneously.

the truncated model's filtered estimate is actually worse than the observation. Next we consider the *reduced deterministic filter with an additional damping correction (RDFD)* where $\sigma = 0$ and the *reduced stochastic filter with an additive noise correction (RSFA)* where $\alpha = 0$. As shown in Figure 3 the damping improves the filter accuracy for small ϵ whereas the additive noise stochastic forcing improves the filter accuracy for large ϵ. Finally we combine both damping and additive stochastic forcing in RSFAD, which shows the improvement that is achievable with our simple stochastic parameterization of model error compared to simply neglecting unresolved scales.

Of course, estimating the state is only part of filter performance, the filter also quantifies the uncertainty in the mean state estimate $\tilde{x}(t)$ via the estimated covariance matrix $\tilde{S}(t)$. We would like to know if the filter is doing a good job of determining $\tilde{S}(t)$, however judging the accuracy of $\tilde{S}(t)$ is difficult since we do not have access to the optimal filter (even our "gold standard" with the *full model* simply uses a Gaussian update). Thus we apply the empirical measure of filter consistency introduced in Section 3. As shown in Section 3, the consistency quantifies the degree to which the actual error covariance of the suboptimal estimate $\tilde{x}(t)$ agrees with the filtered covariance estimate, $\tilde{S}(t)$. Moreover, if $\tilde{x}(t)$ is a good estimate of the true posterior mean, consistency close to one implies that $\tilde{S}(t)$ is close to the true posterior covariance. From the definition 3.1, we see that consistency greater than one implies that $\tilde{S}(t)$ underestimates the actual error covariance, and consistency less than one implies that $\tilde{S}$ overestimates the actual error covariance. From the standpoint of uncertainty quantification it is typically more useful to overestimate the actual error covariance, thus giving an upper bound on the uncertainty. In Figure 3 we show that consistency is most effected by the additive noise term defined by the parameters σ_{ij}. When these stochastic parameters are included, the reduced model has consistency on par with the full model; compared to the order 10^4 underestimation of the actual error covariance without this stochastic parameterization.

7. Summary and discussion

In this paper, we studied two simple examples to understand how model error from unresolved scales affects the state estimation and uncertainty quantification of multiscale dynamical systems. From the mathematical analysis of these simple test problems, we learned that:

- Model error creates inconsistency in the filtered statistical estimates, that is, the filtered covariance estimates are different than the actual error covariance of the filtered mean estimates.

- For linear and Gaussian filtering problems, there exists a unique reduced filter model involving only the slow variables that will produce optimal mean and covariance estimates. The reduced model includes correction terms in the form of a linear damping and an additive stochastic forcing. We found this model by imposing this consistency condition.

- For general nonlinear filtering problems, the true filter posterior solutions are characterized by a conditional distribution that solves a stochastically forced PDE (Kushner equation). Therefore, it is impractical to find the unique reduced model since it requires imposing consistency on all higher-order moments. To gain insight on nonlinear filtering, we compared the posterior mean and covariance estimates obtained by imposing a Gaussian closure on the posterior distribution on our simple test problem. This approximation turns out to be the continuous-time analog of the SPEKF [18, 17]. In this simple setup, we found a reduced filter model, with mean and covariance estimates which are close to the solutions with the perfect model. The reduced model includes an additional correction term in the form of multiplicative noise stochastic forcing (in the Stratonovich sense) in addition to a linear damping and an additive noise stochastic forcing.

Based on these theoretical results, we propose:

- A useful metric to quantify consistency of filtered solutions when the true posterior solutions are not accessible in real applications. The consistency measure is designed to be the analog of RMSE for measuring the performance of covariance estimates for complex simulations in the presence of model error. Consistency, together with the mean square error (MSE), quantify the performance of the filtered estimates of the first two moments.

- A simple stochastic parameterization that includes a linear damping term, as well as additive and multiplicative stochastic forcings, to account for unresolved scales in filtering general multiscale problems. Unlike the classical numerical methods discussed in the Introduction, this stochastic parameterization simultaneously accounts for both the mean model error and the model error covariance statistics.

While this stochastic parameterization will not be the optimal choice for a general complex model, it is theoretically justified on a linear and a simple nonlinear example, and produces encouraging statistical estimates on the slow variables of the two-layer Lorenz96 model after truncating 72 fast variables of a total of 81 variables. Furthermore, it improves the consistency of our estimates. All of these results were obtained by ignoring the multiplicative noise term , $\beta = 0$ in the stochastic parameterization (5.3).

To make this stochastic parameterization becomes useful in real application, many open problems remain to be solved. Naturally, one would ask the following questions:

- What will be an alternative stochastic parameterization if the one we proposed is inadequate?

- For linear problems, the optimal parameter estimation method is the Mean Stochastic Model (MSM) introduced in [35]. In general high-dimensional, nonlinear problems, how can we efficiently estimate the stochastic parameters?

- Should these parameters be spatially dependent (in a matrix form)?

- What effect will the stochastic parameterization have on medium to long term predictive skill?

We will address these questions in our future research.

Acknowledgments

The research of J.H. is partially supported by the Office of Naval Research Grants N00014-11-1-0310, N00014-13-1-0797, MURI N00014-12-1-0912 and the National Science Foundation. T. B. is supported by the ONR MURI grant N00014-12-1-0912.

Appendix A. Expansion of the True Filter Covariance

We consider the two-dimensional linear and Gaussian filtering problem (written in matrix form),

$$\begin{aligned} d\vec{x} &= A_\epsilon \vec{x}\, dt + \sqrt{Q_\epsilon} dW, \\ dz &= G\vec{x} dt + \sqrt{R} dV, \end{aligned} \tag{A 1}$$

where $\vec{x} = (x, y)^\top$, $G = [1, 0]$, $W = (W_x, W_y)^\top$ and V are standard i.i.d. Wiener processes,

$$A_\epsilon = \begin{pmatrix} a_{11} & a_{12} \\ a_{21}/\epsilon & a_{22}/\epsilon \end{pmatrix}, \qquad Q_\epsilon = \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2/\epsilon \end{pmatrix}.$$

Throughout this analysis, we assume that A_1 has negative eigenvalues, $a_{22} < 0$, and $\tilde{a} \equiv a_{11} - a_{12}a_{22}^{-1}a_{21} < 0$. The optimal filter posterior statistics, $\hat{\vec{x}} = \mathbb{E}\big[\vec{x}\big]$ and $\hat{S} = \mathbb{E}\big[(\vec{x} - \hat{\vec{x}})(\vec{x} - \hat{\vec{x}})^\top\big]$, are given by the Kalman-Bucy equations [26]. In this appendix we find the steady state covariance $\hat{s}_{11} = \mathbb{E}[(x - \hat{x})^2]$ for the slow variable x and we expand the solution in terms of ϵ. We show that up to order-ϵ^2, the covariance $\hat{s}_{11}$ solves a one-dimensional Riccati equation which will motivate the optimal reduced model in Appendix B.

The Kalman-Bucy solution implies that $\hat{S}$ has a steady state solution given by the algebraic Riccati equation,

$$0 = A_\epsilon \hat{S} + \hat{S} A_\epsilon^\top - \hat{S} G^\top R^{-1} G \hat{S} + Q_\epsilon. \tag{A 2}$$

For the case of the two variable system, by setting $\hat{S} = \begin{pmatrix} \hat{s}_{11} & \hat{s}_{12} \\ \hat{s}_{12} & \hat{s}_{22} \end{pmatrix}$ the steady state Riccati equation yields the following three equations,

$$\begin{aligned} 0 &= \sigma_x^2 - \hat{s}_{11}^2/R + 2a_{11}\hat{s}_{11} + 2a_{12}\hat{s}_{12}, \\ 0 &= a_{11}\hat{s}_{12} + a_{12}\hat{s}_{22} - \hat{s}_{11}\hat{s}_{12}/R + \hat{s}_{11}a_{21}/\epsilon + \hat{s}_{12}a_{22}/\epsilon, \\ 0 &= \sigma_y^2/\epsilon - \hat{s}_{12}^2/R + 2\hat{s}_{12}a_{21}/\epsilon + 2\hat{s}_{22}a_{22}/\epsilon. \end{aligned}$$

Solving the third equation for $\hat{s}_{22}$ and plugging the result into the second equation yields,

$$0 = \left(\frac{\epsilon a_{12}}{2Ra_{22}}\right)\hat{s}_{12}^2 + \left(a_{11} - \frac{a_{12}a_{21}}{a_{22}} - \frac{\hat{s}_{11}}{R} + \frac{a_{22}}{\epsilon}\right)\hat{s}_{12} + \left(\frac{\hat{s}_{11}a_{21}}{\epsilon} - \frac{a_{12}\sigma_y^2}{2a_{22}}\right).$$

Multiplying this expression by ϵ, for $a_{22} \neq 0$ we obtain,

$$\begin{aligned} \hat{s}_{12} &= \frac{-\hat{s}_{11}a_{21} + \epsilon\left(\frac{a_{12}\sigma_y^2}{2a_{22}}\right)}{a_{22} + \epsilon(a_{11} - a_{12}a_{21}/a_{22} - \hat{s}_{11}/R)} + \mathcal{O}(\epsilon^2), \\ &= -\hat{s}_{11}\frac{a_{21}}{a_{22}} + \epsilon\frac{a_{12}\sigma_y^2/2 + \hat{s}_{11}a_{21}(a_{11} - a_{12}a_{21}/a_{22} - \hat{s}_{11}/R)}{a_{22}^2} + \mathcal{O}(\epsilon^2), \end{aligned} \tag{A 3}$$

where we have used $\frac{a+b\epsilon}{c+d\epsilon} = \frac{a}{c} + \frac{bc-ad}{c^2}\epsilon + \mathcal{O}(\epsilon^2)$. Plugging this solution for $\hat{s}_{12}$ into the first Riccati equation above gives the following equation for $\hat{s}_{11}$,

$$\begin{aligned} 0 &= -\left(\frac{1}{R} + \epsilon\frac{2a_{12}a_{21}}{a_{22}^2 R}\right)\hat{s}_{11}^2 + 2\left(a_{11} - \frac{a_{12}a_{21}}{a_{22}} + \epsilon\left(\frac{a_{21}a_{12}a_{11}}{a_{22}^2} - \frac{a_{21}^2a_{12}^2}{a_{22}^3}\right)\right)\hat{s}_{11} + \left(\sigma_x^2 + \epsilon\frac{a_{12}^2\sigma_y^2}{a_{22}^2}\right) + \mathcal{O}(\epsilon^2), \\ &= -\left(\frac{1 + 2\epsilon\hat{a}}{R}\right)\hat{s}_{11}^2 + 2\tilde{a}\left(1 + \epsilon\hat{a}\right)\hat{s}_{11} + \left(\sigma_x^2 + \epsilon\sigma_y^2\frac{a_{12}^2}{a_{22}^2}\right) + \mathcal{O}(\epsilon^2), \end{aligned}$$

where $\tilde{a} = a_{11} - \frac{a_{12}a_{21}}{a_{22}}$ and $\hat{a} = \frac{a_{12}a_{21}}{a_{22}^2}$. Dividing both sides by $(1 + 2\epsilon\hat{a})$ and expanding in ϵ we have,

$$0 = -\frac{\hat{s}_{11}^2}{R} + 2\tilde{a}(1 - \epsilon\hat{a})\hat{s}_{11} + \left(\sigma_x^2(1 - 2\epsilon\hat{a}) + \epsilon\sigma_y^2\frac{a_{12}^2}{a_{22}^2}\right) + \mathcal{O}(\epsilon^2). \tag{A 4}$$

Equation (A 4) yields the variance $\hat{s}_{11}$ of the state estimate for the observed variable x based on the optimal filter using the full model. Note that by truncating terms that are order ϵ^2, equation (A 4) for $\hat{s}_{11}$ has the form of a Riccati equation for a one dimensional dynamical system. In particular, consider the linear one-dimensional filtering problem,

$$\begin{aligned} dX &= aX\,dt + \sigma_X\,dW_X, \\ dz &= X\,dt + \sqrt{R}\,dV. \end{aligned} \tag{A 5}$$

The steady state covariance $\tilde{s}$ from the Kalman-Bucy solution to (A 5) solves the Riccati equation,

$$-\frac{\tilde{s}^2}{R} + 2a\tilde{s} + \sigma_X^2 = 0. \tag{A 6}$$

Our goal is to find $\{a, \sigma_X\}$ such that the solution of (A 5) agrees with the solution for x of (A 1). In order to make $\tilde{s}$ agree with $\hat{s}_{11}$, we establish the following tradeoff between a and σ_X^2 in the limit as $\epsilon \to 0$.

Theorem 2.1. *Let $\hat{s}_{11}$ be the first component of the steady state solution to* (A 2) *and let $\tilde{s}$ solve* (A 6). *Then $\lim_{\epsilon\to 0} \frac{\tilde{s}-\hat{s}_{11}}{\epsilon} = 0$ if and only if*

$$\sigma_X^2 = 2(a - \tilde{a}(1-\epsilon\hat{a}))\hat{s}_{11} + \sigma_x^2(1-2\epsilon\hat{a}) + \epsilon\sigma_y^2\frac{a_{12}^2}{a_{22}^2} + \mathcal{O}(\epsilon^2).$$

Proof. Subtracting (A 6) from (A 4), we obtain

$$-\frac{\hat{s}_{11}^2 - \tilde{s}^2}{r} + 2\tilde{a}(1-\epsilon\hat{a})\hat{s}_{11} - 2a\tilde{s} + \left(\sigma_x^2(1-2\epsilon\hat{a}) + \epsilon\sigma_y^2\frac{a_{12}^2}{a_{22}^2}\right) - \sigma_X^2 = \mathcal{O}(\epsilon^2). \tag{A 7}$$

First, assuming the σ_X^2 has the form given in the statement, (A 7) reduces to

$$\mathcal{O}(\epsilon^2) = -\frac{\hat{s}_{11}^2 - \tilde{s}^2}{R} + 2a(\hat{s}_{11} - \tilde{s}) = (\hat{s}_{11} - \tilde{s})\left(-\frac{\hat{s}_{11}+\tilde{s}}{R} + a\right),$$

which shows that $\hat{s}_{11} - \tilde{s} = \mathcal{O}(\epsilon^2)$ so $\lim_{\epsilon\to 0}\frac{\hat{s}_{11}-\tilde{s}}{\epsilon} = 0$. Conversely, if we assume that $\lim_{\epsilon\to 0}\frac{\hat{s}_{11}-\tilde{s}}{\epsilon} = 0$ then we can rewrite (A 7) as

$$\begin{aligned} 0 &= -(\hat{s}_{11} - \tilde{s})\frac{\hat{s}_{11}+\tilde{s}}{R} + 2\tilde{a}(1-\epsilon\hat{a})\hat{s}_{11} - 2a\hat{s}_{11} + \left(\sigma_x^2(1-2\epsilon\hat{a}) + \epsilon\sigma_y^2\frac{a_{12}^2}{a_{22}^2}\right) - \sigma_X^2 + \mathcal{O}(\epsilon^2), \\ &= (2\tilde{a}(1-\epsilon\hat{a}) - 2a)\hat{s}_{11} + \left(\sigma_x^2(1-2\epsilon\hat{a}) + \epsilon\sigma_y^2\frac{a_{12}^2}{a_{22}^2}\right) - \sigma_X^2 + \mathcal{O}(\epsilon^2), \end{aligned} \tag{A 8}$$

and solving for σ_X^2 gives the desired identity. □

Appendix B. Existence and Uniqueness of the Optimal Reduced Model

In this appendix we consider using the one-dimensional filtering scheme (A 5) to filter noisy observations of x that solves the two-dimensional model in (A 1). We will show that there exists a unique choice of parameters $\{a, \sigma_X\}$ which gives the optimal filtered estimate of the slow variable x from (A 1) in the sense that both the mean and covariance estimates match the true filtered solutions. This optimal choice is determined by requiring the parameters to lie on the manifold defined by Theorem 2.1, and additionally, requiring the reduced filter to be consistent in the sense that the *actual error covariance* of the filtered mean estimate must equal to the *error covariance estimate* produced by the filtering scheme.

In order to find the optimal parameters for the reduced model in (A 5), we first expand the full model in (A 1), specifically we will expand the matrix exponential of the deterministic component in order to approximate the forward operator. We first compute the eigenvalues and eigenvectors of A_ϵ from (A 1) which have the following expansion

$$\begin{aligned}
\lambda_1 &= \frac{a_{22} + a_{11}\epsilon + \sqrt{a_{11}^2\epsilon^2 + a_{22}^2 + 4a_{12}a_{21}\epsilon - 2a_{11}a_{22}\epsilon}}{2\epsilon} = \tilde{a} + \mathcal{O}(\epsilon), \\
\lambda_2 &= \frac{a_{22} + a_{11}\epsilon - \sqrt{a_{11}^2\epsilon^2 + a_{22}^2 + 4a_{12}a_{21}\epsilon - 2a_{11}a_{22}\epsilon}}{2\epsilon} = \frac{a_{22}}{\epsilon} + \mathcal{O}(1), \\
u_1 &= (\epsilon\lambda_1 - a_{22}, a_{21})^\top = (-a_{22}, a_{21})^\top + \mathcal{O}(\epsilon), \\
u_2 &= ((\epsilon\lambda_2 - a_{22})/a_{21}, 1)^\top = (0,1)^\top + \mathcal{O}(\epsilon).
\end{aligned}$$

We will assume that $a_{22} < 0$ so that the fast variable equilibrates in the limit as the time scale parameters ϵ goes to zero. This can be seen in the expansion of λ_2 since exponentiating yields $e^{\lambda_2 dt} = \mathcal{O}(\epsilon^n)$ for any power n. Since $A_\epsilon = U_\epsilon \Lambda_\epsilon U_\epsilon^{-1}$, for a time step τ we can write the forward operator for the deterministic problem as,

$$\begin{aligned}
e^{A_\epsilon \tau} = U_\epsilon e^{\Lambda_\epsilon \tau} U_\epsilon^{-1} &= \begin{pmatrix} -a_{22} & 0 \\ a_{21} & 1 \end{pmatrix} \begin{pmatrix} e^{\tilde{a}\tau} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -a_{22} & 0 \\ a_{21} & 1 \end{pmatrix}^{-1} + \mathcal{O}(\epsilon), \\
&= e^{\tilde{a}\tau} \begin{pmatrix} 1 & 0 \\ -a_{21}/a_{22} & 0 \end{pmatrix} + \mathcal{O}(\epsilon). \qquad \text{(B 1)}
\end{aligned}$$

Since the solution to (A 1) is given by,

$$\vec{x}(t) = e^{A_\epsilon t}\vec{x}_0 + \int_0^t e^{A_\epsilon (s-t)} \sqrt{Q_\epsilon}\, dW_s, \qquad \text{(B 2)}$$

we can now use (B 1) to write,

$$\vec{x}(t) = e^{\tilde{a}t} \begin{pmatrix} 1 & 0 \\ -a_{21}/a_{22} & 0 \end{pmatrix} \vec{x}_0 + \int_0^t e^{\tilde{a}(s-t)} \begin{pmatrix} 1 & 0 \\ -a_{21}/a_{22} & 0 \end{pmatrix} \sqrt{Q_\epsilon}\, dW_s + \mathcal{O}(\epsilon),$$

and writing this in the derivative form gives the reduced model up to order-ϵ,

$$d\vec{x} = \tilde{a}\vec{x}\, dt + \begin{pmatrix} 1 & 0 \\ -a_{21}/a_{22} & 0 \end{pmatrix} \sqrt{Q_\epsilon}\, dW_t + \mathcal{O}(\epsilon). \qquad \text{(B 3)}$$

Solving the second equation in (A 1) for y and inserting the solution into the first equation yields,

$$dx = \tilde{a}x\, dt + \sigma_x\, dW_x - \sqrt{\epsilon}\frac{a_{12}}{a_{22}}\sigma_y dW_y + \epsilon\frac{a_{12}}{a_{22}} dy. \qquad \text{(B 4)}$$

Note that by (B 3) and (A 1) we have,

$$dy = \tilde{a}y\, dt - \frac{a_{12}}{a_{22}}\sigma_x dW_x + \mathcal{O}(\epsilon) = -\tilde{a}\frac{a_{21}}{a_{22}}x\, dt - \frac{a_{12}}{a_{22}}\sigma_x dW_x - \sqrt{\epsilon}\frac{\sigma_y}{a_{22}} dW_y + \mathcal{O}(\epsilon),$$

so recalling that $\hat{a} = a_{12}a_{21}/a_{22}^2$, we can write $\epsilon\frac{a_{12}}{a_{22}}dy$ as,

$$\epsilon\frac{a_{12}}{a_{22}} dy = -\epsilon\tilde{a}\hat{a}x\, dt - \epsilon\hat{a}\sigma_x dW_x - \epsilon^{3/2}\tilde{a}\frac{a_{12}}{a_{22}^2}\sigma_y dW_y + \mathcal{O}(\epsilon^2).$$

Finally substituting this expression for $\epsilon\frac{a_{12}}{a_{22}}dy$ into (B 4) we have the following expansion of the prior model for the slow variable x,

$$dx = \tilde{a}(1-\epsilon\hat{a})x\, dt + \sigma_x(1-\epsilon\hat{a})\, dW_x - \sqrt{\epsilon}\frac{a_{12}}{a_{22}}\sigma_y \left(1 - \epsilon\frac{\tilde{a}}{a_{22}}\right) dW_y + \mathcal{O}(\epsilon^2). \qquad \text{(B 5)}$$

We will compare the equation (B 5) for the true state with the Kalman-Bucy mean estimate [26] for the reduced filtering problem in (A 5), which solves,

$$d\tilde{x} = a\tilde{x}\,dt + K(dz - \tilde{x}\,dt), \tag{B 6}$$

where $K = \tilde{s}/R$ and $\tilde{s}$ is determined by $\{a, \sigma_X\}$ since $\tilde{s}$ solves (A 6) the time-dependent Riccati equation,

$$\frac{d\tilde{s}}{dt} = -\frac{\tilde{s}^2}{R} + 2a\tilde{s} + \sigma_X^2.$$

Our goal is to determine the parameters $\{a, \sigma_X\}$ which minimize the *actual error covariance* $\mathbb{E}[(x - \tilde{x})^2]$. Subtracting (B 6) from (B 5), we find that the actual error $e \equiv x - \tilde{x}$ evolves according to

$$\begin{aligned} de &= [(\tilde{a}(1-\epsilon\hat{a}) - K)e + (\tilde{a}(1-\epsilon\hat{a}) - a)\tilde{x}]\,dt, \\ &+ \sigma_x(1-\epsilon\hat{a})dW_x - \sqrt{\epsilon}\frac{a_{12}}{a_{22}}\left(1 - \epsilon\frac{\tilde{a}}{a_{22}}\right)\sigma_y dW_y - K\sqrt{R}\,dV + \mathcal{O}(\epsilon^2). \end{aligned} \tag{B 7}$$

To find the covariance of the actual error, e, first concatenate (B 7) with (B 6),

$$\begin{aligned} \begin{pmatrix} de \\ d\tilde{x} \end{pmatrix} &= F\begin{pmatrix} e \\ \tilde{x} \end{pmatrix} + P\begin{pmatrix} dW_x \\ dW_y \\ dV \end{pmatrix}, \\ &= \begin{pmatrix} \tilde{a}(1-\epsilon\hat{a}) - K & \tilde{a}(1-\epsilon\hat{a}) - a \\ 0 & a - K \end{pmatrix}\begin{pmatrix} e \\ \tilde{x} \end{pmatrix} \\ &+ \begin{pmatrix} \sigma_x(1-\epsilon\hat{a}) & \sqrt{\epsilon}\sigma_y\frac{a_{12}}{a_{22}}(1-\epsilon\frac{\tilde{a}}{a_{22}}) & -K\sqrt{R} \\ 0 & 0 & K\sqrt{R} \end{pmatrix}\begin{pmatrix} dW_x \\ dW_y \\ dV \end{pmatrix}. \end{aligned} \tag{B 8}$$

The steady state covariance matrix, E, of the system (B 8) is given by the Lyapunov equation, $FE + EF^\top + PP^\top = 0$ with F and P from (B 8). Note that $E_{11} = \lim_{t\to\infty}\mathbb{E}[e^2] = \lim_{t\to\infty}\mathbb{E}[(x(t) - \tilde{x}(t))^2]$ meaning that the $(1,1)$-entry of E, is the actual error covariance between the true signal x and the filter estimate $\tilde{x}$ from the reduced model. As we showed in Appendix A, by choosing the parameters $\{a, \sigma_X\}$ on the manifold from Theorem 2.1 we have $\tilde{s} = \hat{s}_{11} + \mathcal{O}(\epsilon^2)$. Therefore, if we require the filter to be consistent, in the sense that the covariance $\tilde{s}$ produced by the filter must agree with the actual error covariance E_{11}, then the actual error covariance will also equal the optimal covariance $\hat{s}_{11}$ of the full model. In other words, given that we are on the manifold of Theorem 2.1, requiring the reduced filter to be consistent implies that it is optimal. Requiring the reduced filter to be consistent implies that $K = \tilde{s}/R = E_{11}/R$, and plugging this into the Lyapunov equation for the system (B 8) yields the following three equations,

$$\begin{aligned} 0 &= 2E_{22}(a - E_{11}/R) + E_{11}^2/R, \\ 0 &= E_{12}(a - E_{11}/R) - E_{22}(a - \tilde{a}(1-\epsilon\hat{a})) - E_{11}^2/R - E_{12}(E_{11}/R - \tilde{a}(1-\epsilon\hat{a})), \\ 0 &= -E_{11}^2/R + 2\tilde{a}(1-\epsilon\hat{a})E_{11} + \sigma_x^2(1-\epsilon\hat{a})^2 + \epsilon\sigma_y^2\frac{a_{12}^2}{a_{22}^2} - 2E_{12}(a - \tilde{a}(1-\epsilon\hat{a})). \end{aligned} \tag{B 9}$$

Solving the first two equations we find that $E_{12} = -E_{11}^2/(2E_{11} - 2Ra)$ and substituting this expression into the last equation yields,

$$0 = -E_{11}^2/R + 2\tilde{a}(1-\epsilon\hat{a})E_{11} + \sigma_x^2(1-\epsilon\hat{a})^2 + \epsilon\sigma_y^2\frac{a_{12}^2}{a_{22}^2} + 2E_{11}^2\frac{(a - \tilde{a}(1-\epsilon\hat{a}))}{2E_{11} - 2Ra}.$$

Finally, since consistency requires $E_{11} = \tilde{s} = \hat{s}_{11} + \mathcal{O}(\epsilon^2)$, the error variance E_{11} must solve the algebraic Riccati equation (A 4) meaning that,

$$0 = -E_{11}^2/R + 2\tilde{a}(1-\epsilon\hat{a})E_{11} + \left(\sigma_x^2(1-\epsilon\hat{a})^2 + \epsilon\sigma_y^2\frac{a_{12}^2}{a_{22}^2}\right) + \mathcal{O}(\epsilon^2),$$

and combining the previous two equations, we must have,

$$0 = 2E_{11}^2 \frac{(a - \tilde{a}(1-\epsilon\hat{a}))}{2E_{11} - 2Ra} + \mathcal{O}(\epsilon^2),$$

which implies that $a = \tilde{a}(1-\epsilon\hat{a}) + \mathcal{O}(\epsilon^2)$.

With these coefficients, we now consider the difference $e_s(t) = \hat{s}_{11}(t) - \tilde{s}(t)$ between the time-dependent variances. Below, when the time variable does not explicitly appear in the covariance we are referring to the steady state covariance so that $\tilde{s} = \lim_{t\to\infty} \tilde{s}(t)$ and $\hat{s}_{11} = \lim_{t\to\infty} \hat{s}_{11}(t)$. Applying the Kalman-Bucy equation for the evolution of the covariance estimates we see that $e_s(t)$ satisfies,

$$\frac{de_s}{dt} = \big(2\tilde{a}(1-\epsilon\hat{a}) - \frac{1}{R}(\hat{s}_{11}(t) + \tilde{s}(t))\big)e_s + \mathcal{O}(\epsilon^2). \tag{B 10}$$

Notice that for the linear filtering problems in (A 1) and (A 5) with negative definite A_1 and $\tilde{a}(1-\epsilon\hat{a}) < 0$, respectively, the corresponding filtered solutions are stable [26], that is, the covariances are uniformly bounded, $0 < \hat{s}_{11}(t), \tilde{s}(t) < C$ and therefore,

$$e^{\int_0^t 2\tilde{a}(1-\epsilon\hat{a}) - R^{-1}(\hat{s}_{11}(t') + \tilde{s}(t'))dt'} \leq e^{2\tilde{a}(1-\epsilon\hat{a})t}. \tag{B 11}$$

With this inequality, for $e_s(0) = 0$, we have,

$$|e_s(t)| \leq C\epsilon^2 \Big| \int_0^t e^{2\tilde{a}(1-\epsilon\hat{a})t'} dt' \Big| \leq C_2\epsilon^2. \tag{B 12}$$

Applying the Kalman-Bucy equation for the evolution of the mean estimates, we find that the difference between the mean estimates satisfies,

$$\begin{aligned} d(\hat{x} - \tilde{x}) &= \tilde{a}(1-\epsilon\hat{a})(\hat{x} - \tilde{x})\,dt + \frac{\hat{s}_{11}(t) - \tilde{s}(t)}{R} dz + \frac{1}{R}(-\hat{s}_{11}(t)\hat{x} + \tilde{s}(t)\tilde{x})\,dt + \mathcal{O}(\epsilon^2), \\ &= \left(\tilde{a}(1-\epsilon\hat{a}) - \frac{\tilde{s}(t)}{R}\right)(\hat{x} - \tilde{x})\,dt + \frac{\hat{s}_{11}(t) - \tilde{s}(t)}{R} dz + \frac{1}{R}(-\hat{s}_{11}(t) + \tilde{s}(t))\hat{x}\,dt + \mathcal{O}(\epsilon^2). \end{aligned}$$

Using the fact that the variance is bounded, the inequality (B 11), and assuming that $\hat{x}(0) = \tilde{x}(0)$ we have,

$$|\hat{x}(t) - \tilde{x}(t)| \leq \frac{1}{R}\int_0^t e^{\tilde{a}(1-\epsilon\hat{a})t'} |e_s(t')| dz(t') + \frac{1}{R}\int_0^t e^{\tilde{a}(1-\epsilon\hat{a})t'} |e_s(t')| dt' + \mathcal{O}(\epsilon^2).$$

Since the covariance difference is bounded as shown in (B 12), by applying the Ito isometry we have,

$$\mathbb{E}(|\hat{x}(t) - \tilde{x}(t)|^2) \leq \frac{C\epsilon^4}{R^2}\left[\int_0^t e^{\tilde{2}a(1-\epsilon\hat{a})t'} dt' + \Big(\int_0^t e^{\tilde{a}(1-\epsilon\hat{a})t'})dt'\Big)^2\right] \leq C_2\epsilon^4. \tag{B 13}$$

This result is summarized in Theorem 2.2 below.

Theorem 2.2. *There exists a unique choice of parameters given by $a = \tilde{a}(1-\epsilon\hat{a})$ and σ_X^2 according to Theorem 2.1, such that the steady state reduced filter* (A 5) *is both consistent and optimal up to order-ϵ^2. This means that $\tilde{s}$, the steady state covariance estimate of the reduced filter, is consistent with the steady state actual error covariance $E_{11} = \lim_{t\to\infty} \mathbb{E}[(x(t) - \tilde{x}(t))^2]$ so that $\tilde{s} = E_{11} + \mathcal{O}(\epsilon^2)$, and also $\tilde{s}$ agrees with the steady state covariance $\hat{s}_{11}$ from the optimal filter $\tilde{s} = \hat{s}_{11} + \mathcal{O}(\epsilon^2)$. Furthermore, the reduced filter mean and covariance estimates are uniformly optimal for all time in the following sense. Given identical initial statistics, $\hat{x}(0) = \tilde{x}(0), \hat{s}_{11}(0) = \tilde{s}(0) > 0$, there are time-independent constants C_1, C_2, such that:*

$$\begin{aligned} |(\hat{s}_{11}(t) - \tilde{s}(t))| &\leq C_1\epsilon^2, \\ \mathbb{E}\Big(|\hat{x}(t) - \tilde{x}(t)|^2\Big) &\leq C_2\epsilon^4. \end{aligned} \tag{B 14}$$

Appendix C. Nonlinear Solvable Model

Consider the following continuous-time nonlinear problem,

$$\begin{aligned}
du &= [-(\gamma+\lambda_u)u + b]\,dt + \sigma_u dW_u,\\
db &= -\frac{\lambda_b}{\epsilon} b\,dt + \frac{\sigma_b}{\sqrt{\epsilon}} dW_b,\\
d\gamma &= -\frac{\lambda_\gamma}{\epsilon}\gamma\,dt + \frac{\sigma_\gamma}{\sqrt{\epsilon}} dW_\gamma,\\
dz &= h(u,\beta,\gamma)\,dt + \sqrt{R}\,dV = u\,dt + \sqrt{R}\,dV,
\end{aligned} \tag{C 1}$$

where W_u, W_b, W_γ, V are standard i.i.d. Wiener processes. We will call this filtering problem SPEKF, which stands for "Stochastic Parameterization Extended Kalman Filter", as introduced in [18, 17]. The posterior statistical solutions of SPEKF for discrete-time observations were obtained in [18, 17] by applying a Kalman update to the analytically solved prior mean and covariance of the stochastic model for (u, b, γ) appearing in (C 1). To avoid confusion, we refrained from the common practice of calling the dynamical model in (C 1) the SPEKF model.

Notice that the SPEKF posterior solutions obtained in [18, 17] are not the optimal filtered solutions. The true posterior solutions for (C 1), given noisy observations, z, are characterized by the conditional distribution, $p(u, b, \gamma, t \,|\, z(\tau), 0 \le \tau \le t)$. The evolution of p is described by the Kushner equation [30],

$$dp = \mathcal{L}^* p\,dt + p(h - \mathbb{E}[h])^\top R^{-1}(dz - \mathbb{E}[h]dt),$$

where,

$$\mathcal{L}^* p = -\nabla\cdot\left((-(\gamma+\lambda_u)u + b, \frac{-\lambda_b b}{\epsilon}, \frac{-\lambda_\gamma\gamma}{\epsilon})^\top p\right) + \frac{1}{2}\left(\sigma_u^2 p_{uu} + \frac{\sigma_b^2 p_{bb}}{\epsilon} + \frac{\sigma_\gamma^2 p_{\gamma\gamma}}{\epsilon}\right),$$

is the Fokker-Planck operator. For convenience we will write the innovation process as $dw_{\hat{u}} = dz - \mathbb{E}[h]dt$ which allows us to write the Kushner equation as,

$$dp = \mathcal{L}^* p\,dt + p(h - \mathbb{E}[h])^\top R^{-1} dw_{\hat{u}}. \tag{C 2}$$

In order to make a formal asymptotic expansion in terms of the time scale separation ϵ, we write the posterior as $p = p_0 + \epsilon p_1$. Notice that the Fokker-Planck operator can be written as $\mathcal{L}^* = \frac{1}{\epsilon}\mathcal{L}_0^* + \mathcal{L}_1^*$ where

$$\begin{aligned}
\mathcal{L}_0^* p &= \frac{\partial}{\partial b}(\lambda_b b p) + \frac{\partial}{\partial\gamma}(\lambda_\gamma\gamma p) + \sigma_b^2 p_{bb} + \sigma_\gamma^2 p_{\gamma\gamma},\\
\mathcal{L}_1^* p &= \frac{\partial}{\partial u}\left((\gamma+\lambda_u)up + bp\right) + \sigma_u^2 p_{uu}.
\end{aligned}$$

With this expansion the Kushner equation becomes,

$$dp_0 + \epsilon dp_1 = \frac{1}{\epsilon}\mathcal{L}_0^* p_0 + \mathcal{L}_0^* p_1 + \mathcal{L}_1^* p_0 + \epsilon\mathcal{L}_1^* p_1 + (p_0 + \epsilon p_1)(h - \mathbb{E}[h])^\top R^{-1} dw_{\hat{u}}.$$

The order-ϵ^{-1} term requires that $\mathcal{L}_0^* p_0 = 0$ which says that p_0 is in the null space of the operator $\mathcal{L}_0^*$. Since b, γ in (C 1) are ergodic processes, letting $p_\infty(b)$ and $p_\infty(\gamma)$ be the respective invariant measures, we can write,

$$p_0(u, b, \gamma, t) = \tilde{p}(u, t) p_\infty(b) p_\infty(\gamma). \tag{C 3}$$

We will use this fact repeatedly to complete the asymptotic expansion of the posterior distribution p.

Note that convergence results of the marginal true filter distribution to the reduced filter characterized by $\tilde{p}$ are on the order of $\sqrt{\epsilon}$ for general nonlinear problems (see [24]). Here, we consider a higher order correction and we will show that for this specific example, we obtain convergence of order ϵ for the first two-moments. From the Kalman filtering perspective, we are only interested in capturing the first two moments of the posterior distribution p. Using Ito's formula, we will compute the governing equations of the first two moments of the conditional distribution, p, which solves the Kushner equation in (C 2). Throughout this section we will assume that the posterior distribution, p, has fast decay at infinity allowing us to use integration by parts and neglect boundary terms. To simplify the presentation, we define $\vec{u} = (u, b, \gamma)$, and all the integrals with respect to $d\vec{u}$ are three-dimensional integrals.

For the first moment we have,

$$d\hat{u} = \int u\,dp\,d\vec{u},$$

and substituting the Kushner equation we note that up_{uu}, up_{bb} and $up_{\gamma\gamma}$ integrate to zero leaving,

$$\begin{aligned}
d\hat{u} &= \left(\int \Big(u\frac{\partial}{\partial u}((\gamma+\lambda_u)up + bp) - u\frac{\partial}{\partial b}(\lambda_b bp) + u\frac{\partial}{\partial \gamma}(\frac{\lambda_\gamma}{\epsilon}\gamma p)\Big)d\vec{u}\right)dt + \int up(u-\hat{u})^\top R^{-1}dw_{\hat{u}}\,d\vec{u},\\
&= \left(-\lambda_u\hat{u} - \int \gamma up\,d\vec{u} + \int bp\,d\vec{u}\right)dt + \int (u^2 - \hat{u}u)pR^{-1}dw_{\hat{u}}\,d\vec{u},\\
&= \left(-\lambda_u\hat{u} - \int \gamma up\,d\vec{u} + \overline{b}\right)dt + \hat{S}R^{-1}dw_{\hat{u}},\\
&= \left(-\lambda_u\hat{u} - \overline{u\gamma} + \overline{b}\right)dt + \hat{S}R^{-1}dw_{\hat{u}},
\end{aligned} \tag{C 4}$$

where we have used the fact that the innovation process $dw_{\hat{u}}$ is Brownian and uncorrelated with $\vec{u}$ [4]. To estimate $\overline{b} = \int bp\,d\vec{u}$ we again apply the Kushner equation to compute,

$$\begin{aligned}
d\overline{b} = \int bdp\,d\vec{u} &= \left(\int b\frac{\partial}{\partial b}\left(\frac{\lambda_b}{\epsilon}bp\right)\,d\vec{u}\right)dt + \int b(u-\overline{u})pR^{-1}dw_{\hat{u}}\,d\vec{u},\\
&= \frac{\lambda_b}{\epsilon}\overline{b}\,dt + \int b(u-\overline{u})pR^{-1}dw_{\hat{u}}\,d\vec{u},\\
&= \frac{\lambda_b}{\epsilon}\overline{b}\,dt + \mathcal{O}(\epsilon),
\end{aligned} \tag{C 5}$$

where the last equality comes from the expansion $p = p_0 + \epsilon p_1$ with p_0 satisfies (C 3). Equation (C 5) implies that,

$$\epsilon d\overline{b} = \lambda_b\overline{b}\,dt + \mathcal{O}(\epsilon^2),$$

which has solution $\overline{b}(t) = \overline{b}(0)e^{-\lambda_b t/\epsilon} + \mathcal{O}(\epsilon^2) \to \mathcal{O}(\epsilon^2)$ as $t \to \infty$. Thus we can rewrite (C 4) as

$$d\hat{u} = (-\lambda_u\hat{u} - \overline{u\gamma})\,dt + \hat{S}R^{-1}dw_{\hat{u}} + \mathcal{O}(\epsilon^2). \tag{C 6}$$

The term $\overline{u\gamma} = \int \gamma up\,d\vec{u}$ represents an uncentered correlation between the two variables. To find the evolution of $\overline{u\gamma}$ we again use the Kushner equation to expand

$$\begin{aligned}
d\,\overline{u\gamma} &= \left(\int \Big(u\gamma\frac{\partial}{\partial u}((\gamma+\lambda_u)up + bp) + u\gamma\frac{\partial}{\partial\gamma}(\frac{\lambda_\gamma\gamma}{\epsilon}p)\Big)\,d\vec{u}\right)dt + \int u\gamma p(u-\hat{u})R^{-1}dw_{\hat{u}}\,d\vec{u},\\
&= \left(-(\lambda_u + \frac{\lambda_\gamma}{\epsilon})\overline{u\gamma} - \int \gamma^2 up\,d\vec{u} + \int b\gamma p\,d\vec{u}\right)dt + \int u\gamma p(u-\hat{u})R^{-1}dw_{\hat{u}}d\vec{u},
\end{aligned}$$

where the second derivative terms p_{uu} and $p_{\gamma\gamma}$ are both zero. Applying the expansion $p = p_0 + \epsilon p_1$ with (C 3), we can write the integral

$$\int \gamma^2 up\,d\vec{u} = \int \gamma^2 p_\infty(\gamma)d\gamma \int u\tilde{p}(u,t)du + \mathcal{O}(\epsilon) = \mathrm{var}_\infty[\gamma]\hat{u} + \mathcal{O}(\epsilon) = \frac{\sigma_\gamma^2}{2\lambda_\gamma}\hat{u} + \mathcal{O}(\epsilon),$$

similarly $\int b\gamma p\,d\vec{u} = \mathcal{O}(\epsilon)$ which gives us the expansion,

$$\frac{d}{dt}\overline{u\gamma} = -(\lambda_u + \frac{\lambda_\gamma}{\epsilon})\overline{u\gamma} - \mathrm{var}_\infty[\gamma]\hat{u} + \mathcal{O}(\epsilon) = -(\lambda_u + \frac{\lambda_\gamma}{\epsilon})\overline{u\gamma} - \frac{\sigma_\gamma^2}{2\lambda_\gamma}\hat{u} + \mathcal{O}(\epsilon).$$

Multiplying by ϵ we have

$$\epsilon\frac{d}{dt}\overline{u\gamma} = -(\lambda_u\epsilon+\lambda_\gamma)\overline{u\gamma} - \epsilon\frac{\sigma_\gamma^2}{2\lambda_\gamma}\hat{u} + \mathcal{O}(\epsilon^2),$$

which has solution $\overline{u\gamma} = e^{-(\lambda_u+\lambda_\gamma/\epsilon)t}\overline{u\gamma}_0 - \frac{\epsilon\sigma_\gamma^2}{2\lambda_\gamma(\lambda_u\epsilon+\lambda_\gamma)}\hat{u}(1-e^{-(\lambda_u+\lambda_\gamma/\epsilon)t}) + \mathcal{O}(\epsilon^2)$. In the limit as $t\to\infty$ the correlation approaches a steady state $\overline{u\gamma}_\infty = -\frac{\epsilon\sigma_\gamma^2}{2\lambda_\gamma(\lambda_u\epsilon+\lambda_\gamma)}\hat{u}+\mathcal{O}(\epsilon^2)$. Applying this result to (C 6) gives the following evolution for the mean state estimate

$$d\hat{u} = -\left(\lambda_u - \frac{\epsilon\sigma_\gamma^2}{2\lambda_\gamma(\lambda_u\epsilon+\lambda_\gamma)}\right)\hat{u}\,dt + \hat{S}R^{-1}dw_{\hat{u}} + \mathcal{O}(\epsilon^2). \tag{C 7}$$

By Ito's lemma we have

$$d(\hat{u}^2) = \left(-2\left(\lambda_u - \frac{\epsilon\sigma_\gamma^2}{2\lambda_\gamma(\lambda_u\epsilon+\lambda_\gamma)}\right)\hat{u}^2 + \hat{S}R^{-1}\hat{S}\right)dt + 2\hat{u}\hat{S}R^{-1}dw_{\hat{u}}.$$

Following the same procedure for the second moment we have

$$\begin{aligned} d\hat{S} &= \int u^2 dp\,d\vec{u} - d(\hat{u}^2), \\ &= \int u^2\mathcal{L}^*p\,d\vec{u} + \int u^2 p(u-\hat{u})R^{-1}dw\,d\vec{u} + 2\left(\lambda_u - \frac{\epsilon\sigma_\gamma^2}{2\lambda_\gamma(\lambda_u\epsilon+\lambda_\gamma)}\right)\hat{u}^2 - \hat{S}R^{-1}\hat{S} - 2\hat{u}\hat{S}R^{-1}dw_{\hat{u}}. \end{aligned}$$

A straightforward computation shows that $\int u^2p(u-\hat{u})R^{-1}dw_{\hat{u}}d\vec{u} = \int(u-\hat{u})^3pR^{-1}dw_{\hat{u}}d\vec{u} + 2\hat{u}\hat{S}R^{-1}\,dw_{\hat{u}}$, so **assuming the p has zero skewness**, we have

$$\begin{aligned} \frac{d}{dt}\hat{S} &= \int u^2\mathcal{L}^*p d\vec{u} + 2\left(\lambda_u - \frac{\epsilon\sigma_\gamma^2}{2\lambda_\gamma(\lambda_u\epsilon+\lambda_\gamma)}\right)\hat{u}^2 - \hat{S}R^{-1}\hat{S}, \\ &= \int u^2\frac{\partial}{\partial u}((\gamma+\lambda_u)up + bp)\,d\vec{u} + \sigma_u^2 + 2\left(\lambda_u - \frac{\epsilon\sigma_\gamma^2}{2\lambda_\gamma(\lambda_u\epsilon+\lambda_\gamma)}\right)\hat{u}^2 - \hat{S}R^{-1}\hat{S}, \\ &= -2\int\left((\gamma+\lambda_u)u^2p - ub\right)d\vec{u} + \sigma_u^2 + 2\left(\lambda_u - \frac{\epsilon\sigma_\gamma^2}{2\lambda_\gamma(\lambda_u\epsilon+\lambda_\gamma)}\right)\hat{u}^2 - \hat{S}R^{-1}\hat{S}, \\ &= -2\lambda_u\hat{S} - 2\overline{u^2\gamma} + 2\overline{ub} - \frac{\epsilon\sigma_\gamma^2}{\lambda_\gamma(\lambda_u\epsilon+\lambda_\gamma)}\hat{u}^2 + \sigma_u^2 - \hat{S}R^{-1}\hat{S}. \end{aligned} \tag{C 8}$$

Simplifying this expression requires finding $\overline{ub} = \int bup\,d\vec{u}$ and $\overline{u^2\gamma} = \int u^2\gamma p\,d\vec{u}$. First $\overline{ub}$ has evolution

$$d\,\overline{ub} = -(\lambda_u\epsilon+\lambda_b)\overline{ub} + \epsilon\frac{\sigma_b^2}{2\lambda_b} + \mathcal{O}(\epsilon^2),$$

the solution of which approaches $\overline{ub}\to\overline{ub}_\infty = \frac{\epsilon\sigma_b^2}{2\lambda_b(\lambda_b+\lambda_u\epsilon)} + \mathcal{O}(\epsilon^2)$ as $t\to\infty$. Second $\overline{u^2\gamma}$ has the following evolution

$$\begin{aligned} \frac{d}{dt}\overline{u^2\gamma} &= -(2\lambda_u + \frac{\lambda_\gamma}{\epsilon})\overline{u^2\gamma} - 2\int u^2\gamma^2p\,d\vec{u}, \\ &= -(2\lambda_u + \frac{\lambda_\gamma}{\epsilon})\overline{u^2\gamma} - \frac{\sigma_\gamma^2}{\lambda_\gamma}\int u^2\tilde{p}\,du + \mathcal{O}(\epsilon). \end{aligned} \tag{C 9}$$

Multiplying this expression by ϵ we find the steady state solution $\overline{u^2\gamma}_\infty = -\frac{\epsilon\sigma_\gamma^2}{\lambda_\gamma(2\lambda_u\epsilon+\lambda_\gamma)}\int u^2\tilde{p}du + \mathcal{O}(\epsilon^2)$. Substituting the expressions for $\overline{ub}$, $\overline{u\gamma}_\infty$, and $\overline{u^2\gamma}_\infty$ into (C 8), we find,

$$\frac{d}{dt}\hat{S} = -2\lambda_u\hat{S} + \sigma_u^2 + \frac{\epsilon\sigma_b^2}{2\lambda_b(\lambda_b+\lambda_u\epsilon)} - \hat{S}R^{-1}\hat{S} - \frac{\epsilon\sigma_\gamma^2}{\lambda_\gamma(\lambda_u\epsilon+\lambda_\gamma)}\hat{u}^2 + 2\frac{\epsilon\sigma_\gamma^2}{\lambda_\gamma(2\lambda_u\epsilon+\lambda_\gamma)}\int u^2\tilde{p}du + \mathcal{O}(\epsilon^2),$$

$$= -2\left(\lambda_u - \frac{\epsilon\sigma_\gamma^2}{\lambda_\gamma(\lambda_u\epsilon+\lambda_\gamma)}\right)\hat{S} + \frac{\epsilon\sigma_\gamma^2}{\lambda_\gamma(\lambda_u\epsilon+\lambda_\gamma)}\hat{u}^2 + \sigma_u^2 + \frac{\epsilon\sigma_b^2}{2\lambda_b(\lambda_b+\lambda_u\epsilon)} - \hat{S}R^{-1}\hat{S} + \mathcal{O}(\epsilon^2). \quad \text{(C 10)}$$

Equations (C 7) and (C 10) describe the dynamics of the posterior mean and covariance estimates of the slow variable u from (C 1) up to order-ϵ^2, assuming that the skewness is zero. We refer to the solutions of (C 7) and (C 10) as the **continuous-time SPEKF solutions** for variable u. We do not find all of the cross-correlation statistics between the variables (u, b, γ) since our goal is to find a one-dimensional reduced stochastic filter model for the u variable. Motivated by the results in [21], we propose the following reduced filter to approximate the filtering problem in (C 1),

$$\begin{aligned} dU &= -\alpha U dt + \beta U \circ dW_\gamma + \sigma_1 dW_u + \sigma_2 dW_b, \\ &= -\left(\alpha - \frac{\beta^2}{2}\right) U dt + \beta U dW_\gamma + \sigma_1 dW_u + \sigma_2 dW_b, \\ dz &= U\,dt + \sqrt{R}\,dV. \end{aligned} \quad \text{(C 11)}$$

Our goal is to write the evolution for the first two moments of the corresponding conditional distribution $\pi(U,t\,|\,z_\tau, 0\le\tau\le t)$ for (C 11) and match coefficients with (C 7) and (C 10). The Fokker-Planck operator for (C 11) is $\mathcal{L}^*\pi = -\frac{d}{dU}(-\alpha U\pi) + \frac{1}{2}\frac{d^2}{dU^2}(\beta^2U^2\pi) + \frac{1}{2}(\sigma_1^2+\sigma_2^2)\frac{d^2\pi}{dU^2}$. By differentiating the first moment $\tilde{u} = \int U\pi\,dU$ and substituting the Kushner equation we have,

$$\begin{aligned} d\tilde{u} &= -\left(\alpha - \frac{\beta^2}{2}\right)\tilde{u}\,dt + \int U\pi(U-\tilde{u})R^{-1}dw_{\tilde{u}}\,dU, \\ &= -\left(\alpha - \frac{\beta^2}{2}\right)\tilde{u}\,dt + \tilde{S}R^{-1}dw_{\tilde{u}}, \end{aligned} \quad \text{(C 12)}$$

where $dw_{\tilde{u}} = dz - \tilde{u}\,dt$ is the innovation process. By Ito's formula, we have,

$$\frac{d}{dt}(\tilde{u}^2) = -2\left(\alpha - \frac{\beta^2}{2}\right)\tilde{u}^2 + \tilde{S}R^{-1}\tilde{S} + 2\tilde{u}\tilde{S}R^{-1}dw_{\tilde{u}}.$$

For the second moment, $\tilde{S} = \int U^2\pi\,dU - \tilde{u}^2$, we have,

$$\begin{aligned} \frac{d}{dt}\tilde{S} &= \int U^2\left(\frac{d}{dU}\Big(\Big(\alpha - \frac{\beta^2}{2}\Big)U\pi\Big) + \frac{1}{2}\frac{d^2}{dU^2}(\beta^2U^2\pi) + \frac{\sigma_1^2+\sigma_2^2}{2}\frac{d^2\pi}{dU^2}\right)dU + 2\Big(\alpha - \frac{\beta^2}{2}\Big)\tilde{u}^2 - \tilde{S}R^{-1}\tilde{S}, \\ &= -\int 2U^2\Big(\alpha - \frac{\beta^2}{2}\Big)\pi\,dU + \int(\beta^2U^2\pi + (\sigma_1^2+\sigma_2^2)\pi)dU + 2\Big(\alpha - \frac{\beta^2}{2}\Big)\tilde{u}^2 - \tilde{S}R^{-1}\tilde{S}, \\ &= -2\alpha\int U^2\pi\,dU + \beta^2\int U^2\pi\,dU + \beta^2\int U^2\pi\,dU + \sigma_1^2 + \sigma_2^2 + 2\Big(\alpha - \frac{\beta^2}{2}\Big)\tilde{u}^2 - \tilde{S}R^{-1}\tilde{S}, \\ &= -2(\alpha-\beta^2)\tilde{S} + \beta^2\tilde{u}^2 + \sigma_1^2 + \sigma_2^2 - \tilde{S}R^{-1}\tilde{S}, \end{aligned} \quad \text{(C 13)}$$

assuming the third moments are zero. We can specify the coefficients in (C 11) by matching the two mean estimates in (C 7) and (C 12) and the two covariance estimates, (C 13) and (C 10), in particular,

$$\begin{aligned} &\alpha = \lambda_u, \quad \sigma_1^2 = \sigma_u^2, \\ &\sigma_2^2 = \frac{\epsilon\sigma_b^2}{2\lambda_b(\lambda_b+\epsilon\lambda_u)}, \quad \beta^2 = \frac{\epsilon\sigma_\gamma^2}{\lambda_\gamma(\lambda_u\epsilon+\lambda_\gamma)}. \end{aligned} \quad \text{(C 14)}$$

We refer to the reduced stochastic filter in (C 11), with posterior mean and covariance estimates given by (C 12) and (C 13) with parameters (C 14), as the **continuous-time reduced SPEKF**. Notice that this reduced filter applies a Gaussian closure on the posterior solutions by truncating the third-order moments. This is different than the Gaussian Closure Filter (GCF) introduced in [8], which applies a Gaussian closure on the prior dynamics before using a Kalman update to obtain posterior solutions.

Assume that the parameters in SPEKF and in the reduced SPEKF yield stable filtered solutions, that is, there are constants such that the posterior mean and covariance statistics are uniformly bounded. Then for identical initial statistics, $\hat{u}(0) = \tilde{u(0)}$ and $\hat{S}(0) = \tilde{S}(0) > 0$, with the same argument as in the linear case (see Appendix B), we can show that,

$$|\hat{S}(t) - \tilde{S}(t)| \leq C_1\epsilon,$$
$$\mathbb{E}\left(|\hat{u}(t) - \tilde{u}(t)|^2\right) \leq C_2\epsilon^2.$$

Notice that despite the formal asymptotic expansion of the filtered statistics up to order-ϵ^2, we cannot obtain the same uniform bounds on the errors $|\hat{S}(t) - \tilde{S}(t)|$ and $|\hat{x} - \tilde{x}|$ as in the linear case. This is because the covariance estimates no longer converge to a constant steady state due to the term $\beta^2\tilde{u}^2$ in (C 13) which results from the multiplicative noise in (C 11). This also implies that we no longer have uniform bounds on the covariances without assuming that both filters are stable.

Finally, we show that the reduced SPEKF is consistent. Consider the actual error $e = u - \tilde{u}$, with evolution given by,

$$de = du - d\tilde{u} = -\lambda_u e + \sigma_u dW_u - \tilde{S}R^{-1}dw_{\tilde{u}} + \mathcal{O}(\epsilon).$$

The evolution of the actual error covariance, $E = \mathbb{E}[e^2]$, is given by the Lyapunov equation,

$$\frac{dE}{dt} = -2\lambda_u E + \sigma_u^2 - \tilde{S}R^{-1}\tilde{S} + \mathcal{O}(\epsilon),$$

which implies that the difference between the actual error covariance and the filtered covariance estimate evolves according to,

$$\frac{d}{dt}(E - \tilde{S}) = -2\lambda_u(E - \tilde{S}) + \mathcal{O}(\epsilon),$$

and by Gronwall's lemma, we have $|E - \tilde{S}| \leq \mathcal{O}(\epsilon)$, when $\lambda_u > 0$. We summarize these results in the following theorem.

Theorem 4.1. *Let $\lambda_u > 0$, and let z be noisy observations of the state variable u which solves the full model in* (C 1). *Given identical initial statistics, $\tilde{u}(0) = \hat{u}(0)$ and $\tilde{S}(0) = \hat{S}(0) > 0$, the mean and covariance estimates of a stable continuous-time reduced SPEKF in* (C 11) *with parameters* (C 14) *agree with mean and covariance of a stable continuous-time SPEKF for variable u in the following sense. There exist time-independent constants, C_1, C_2, such that,*

$$|\hat{S}(t) - \tilde{S}(t)| \leq C_1\epsilon,$$
$$\mathbb{E}\left[|\hat{u}(t) - \tilde{u}(t)|^2\right] \leq C_2\epsilon^2.$$

Furthermore, the reduced filtered solutions are also consistent, up to order-ϵ.

We remark that if we did not assume that the filtered solutions are stable, then the constants C_1, C_2 may be time-dependent.

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