Towards an Analytic Description of a Laplacian on the Sierpinski Carpet

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by

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On my honor as a University student, on this assignment I have neither given nor received unauthorized aid as defined by the Honor Guidelines for papers in Science, Technology, and Society Courses.

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Approved ______________________________________ (Technical Advisor)
Lawrence Thomas (Signature)

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-*Ingrid Townsend* (Signature)

Foreword

Much of the work on this project was completed as part of a summer Research Experience for Undergraduates at Cornell University under the guidance of Dr. Robert Strichartz of Cornell University with the financial support of the National Science Foundation. The goals and methods of the experimental portions were initially set by Dr. Strichartz and evolved as new data was analyzed. All of the programming, data collection, and data analysis was performed by the author under the guidance of Dr. Strichartz. Theoretical ideas presented are due to both Dr. Strichartz and Dr. Lawrence Thomas (technical advisor) of the University of Virginia, but have been developed and made accessible by the author. I would like to thank both Dr. Strichartz and Dr. Thomas for their continued availability and assistance on all aspects of this research. The analysis of the social and ethical considerations and the development of the paper itself owes thanks to Dr. Laurie Thurneck and Dr. Ingrid Townsend of the Science, Technology, and Society Department of the School for Engineering and Applied Sciences at the University of Virginia.

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Key Terms

Fractal – A shape which, upon zooming in to view a subsection with finer detail, contains approximately the same amount of regularity and irregularity, or is just as "rough," as the original shape. For examples see Sierpinski Gasket and Sierpinski Carpet.

Sierpinski Gasket – A fractal formed from an equilateral triangle by removing the "center triangle" ad infinitum as shown in Figure 3 on page 11.

Sierpinski Carpet – A fractal formed from a square by removing the "center square" ad infinitum as shown below in Figure 1 on page 2.

Differential Equation – An invaluable tool for performing engineering calculations for heat and wave phenomena.

Laplacian – A mathematical operator which is used to develop differential equations.

Spectrum – A sequence of numbers which provides valuable information about an operator such as the Laplacian.

Analytical – A method of approaching differential equations which yield easy approximations and error estimates which are needed in engineering.

Outer Approximation – An analytical approach to developing a Laplacian on fractals.

Finite Element Method (FEM) – A computational technique for evaluating the classical Laplacian which can be used to perform Outer Approximation calculations.

Technical Terminology Specific to this Paper

Finite Element Method (FEM): Method of approximating the spectra and corresponding eigenfunctions of regions in the plane. In this project MATLAB's PDEToolbox (which employs FEM) is taken as a black box for producing spectra and eigenfunctions.

Level: We used Level to specify a domain from the sequence of domains approximating the fractal. Level zero refers to the initial domain, i.e. the triangle is level zero for SG. The eigenfunction displayed on the home page is on the level 3 domain approximating SG.

Localized Eigenfunction: An eigenfunction with small support (nonzero on small set).

Normalized Spectrum: The normalized spectrum refers to the result of dividing each eigenvalue by the first non-zero eigenvalue; this is meant to allow comparison of spectra up to a multiplicative constant.

Refinements: The finite element method requires that the domain be triangulated, this triangulation can then be refined as many times as the computer has memory to store (typically around 4-5 max). More refinements imply a finer triangulation which implies greater accuracy of the data. The triangles of the triangulations used in these experiments are not all the same size.

Spectral Gaps: A "gap" in the spectrum is an infinite sequence of non-overlapping pairs of eigenvalues such that there are no eigenvalues between each pair and the distance between the two eigenvalues in the pair becomes arbitrarily large.

Abstract

Since Benoit Mandelbrot first showed that fractals can be used to more efficiently approximate coastlines, fractals have been increasingly found in the physical world and are becoming more useful in engineering. In order to realize the benefits of these new fractal models of physical phenomena, it is necessary to develop a theory of differential equations on fractals so that traditional engineering problems can be solved. This project works towards a mathematical description of a Laplacian (a type of mathematical operator) which will allow easy approximation and error estimates for differential equations on the Sierpinski Carpet, a fractal for which this has not been achieved yet. This development will be a natural step forward in the larger context of the theory of differential equations on fractals by making the first step towards generalizing this theory to a larger class of fractals. The project combines traditional mathematical methods of analysis with a method of approximation, which is currently experimental, called the Outer Approximation method. The project succeeds in providing experimental evidence for the validity of the Outer Approximation method and using the method to learn about the Laplacian on the Sierpinski Carpet. Since the Outer Approximation method and modeling with fractals are such new developments, the project also addresses the social and ethical issues of the introduction and use of these new tools, with special attention given to avoiding misuse of the model.

Chapter 1: Introduction

This chapter presents the rationale and objectives of the project while explaining the key ideas and introducing the social and ethical context. It also explains the basic concept of the Outer Approximation method of developing a Laplacian on fractals.

In the spirit of applied mathematics, this project will develop an abstract mathematical theory to allow engineers to realize the benefits of a new model for physical phenomena. In 1983, Benoit Mandelbrot made the audacious claim that many physical phenomena could be modeled more accurately using *fractals*, as opposed to the traditional "smooth" models (Mandelbrot, 1983). Mandelbrot describes a fractal as a geometric shape which has the property that every subsection of the shape is just as "rough" as the whole shape. For example, in Figure 1 the fractal known as the Sierpinski Carpet is the result of repeating the illustrated process of removing central squares indefinitely and this fractal divides into eight smaller squares each of which looks the same as the entire Sierpinski Carpet. Evidence for this type of fractal behavior in physical phenomena is ubiquitous, and fractals are often used to create realistic pictures of natural phenomena such as the islands in Figure 2. Unfortunately, the mathematical theory of fractals is

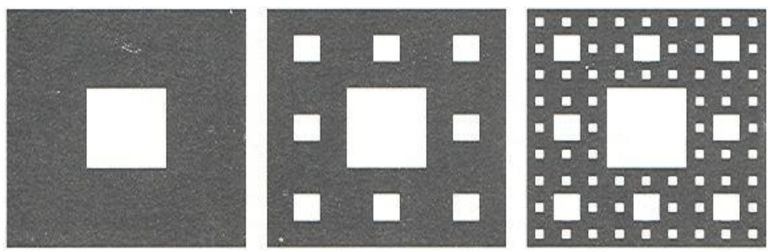

Figure SEQ Figure * ARABIC 1: Sierpinski Carpet (scanned from Mandelbrot, 1983)

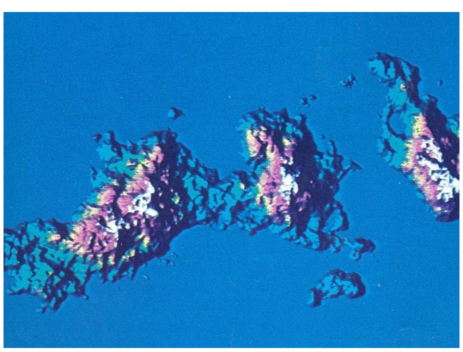

Figure 2: Realistic Islands Created with Fractals (Scanned from Mandelbrot, 1983)

underdeveloped from the standpoint of applied math. While a significant amount is known about the abstract mathematical properties of fractals, this rarely translates into a useful model which engineers can apply. A common method of solving engineering problems such as heat dispersion and wave phenomena (such as light and sound wave propagation) is to examine *differential equations*. The classical theory of differential equations gives powerful results which are extremely useful in engineering; unfortunately, the assumptions of this classical theory make it inapplicable to fractals. Recently, a theory of differential equations on fractals has been developed, but so far it only applies to a restricted class of fractals (Kigami, 2001). The key to this advancement was the definition of a mathematical operator called a *Laplacian* on the restricted class of fractals (Kigami, 2001). The goal of this project is to learn about a Laplacian on a less restricted class of fractals and thereby to learn about differential equations on these fractals.

A brief example will help indicate the need for these developments. Increasingly, engineering mathematics is being stretched to the limit to analyze complex designs. For example, a simple network containing a few heat generating elements (such as resistors or transistors) and connecting conductors (such as electrical wires) can easily be modeled in the classical theory of differential equations. This model allows an engineer to understand how heat will spread through the circuit, and he may analyze the stability and limits of the circuit's ability to handle heat loads. However, when billions of these elements are compacted into an area the size of a fingernail (such as on a modern microprocessor), modeling each element is infeasible even for supercomputers, so other approaches must be found. There are traditional mathematical methods of approaching such a problem, such as inhomogeneous models, but these are extremely complicated. Moreover, it has been shown that many useful shapes, such as those found in polymers, can be most accurately modeled as fractals (Kozlov, 2003). For this accurate model to be useful, the theory of differential equations must be extended to apply to these models so that engineering calculations can be made.

Benoit Mandelbrot has given a very broad definition of fractals, requiring only that they possess statistical regularities, irregularities, and a fractional dimension (Mandelbrot, 1983). This means that a fractal is any shape which, upon zooming in to view a subsection with finer detail, contains approximately the same amount of regularity and irregularity, or is just as "rough," as the original shape. This broad definition allowed Mandelbrot to find fractals everywhere in nature, his most explicit example being measurements of the coastlines of England (Mandelbrot, 1983). Unfortunately the requirements of mathematical formality make this definition difficult for a mathematician to work with, so the mathematical theories work with a more precise (and hence more restrictive) definition. Jun Kigami's class of post-critically-finite fractals is very restricted, and though it includes many common fractals, it is far from the generality of Mandelbrot's conception of fractals (Kigami, 2001). Thus the next step in the development of the theory of differential equations on fractals is to extend the theory to new fractals outside the p.c.f. class. The prototypical example of a non-p.c.f. fractal is the Sierpinski Carpet (Figure 1), and the key to developing a theory of differential equations on the Sierpinski Carpet is to develop a mathematical operator called a Laplacian for this fractal.

This project is part of the development of a mathematical description of a Laplacian which will allow easy approximation and error estimates for differential equations on the Sierpinski Carpet. One aspect that makes the Sierpinski Carpet especially appealing is that it has recently been shown with probabilistic techniques that a Laplacian exists on this fractal (Barlow and Bass, 1999). However, the proof uses non-constructive methods to show the existence; thus properties of this Laplacian are largely unknown. In particular it is not known whether there is more than one Laplacian on the Sierpinski Carpet; this development makes it very difficult to perform calculations. For the mathematics to be useful in engineering, a Laplacian must be developed directly using what are called *analytical* techniques.

The development of an analytic theory of the Laplacian on the Sierpinski Gasket will be an important step in the development of the theory of differential equations on non-p.c.f. fractal domains. The theory for p.c.f. fractals developed in a similar way. The first developments explored individual examples by probabilistic methods. These initial results were then obtained with analytical techniques which were then generalized to the p.c.f. class of fractals (Kigami, 2001). Thus, this project fits into the theoretical context by moving towards the analytical description, and the natural continuation would be the generalization of results to some larger class of fractals.

In the context of analytical techniques, one way to understand an operator is to understand the spectrum of the operator which is a sequence of numbers that are meant to capture the most important behavior of an operator. In many contexts the spectrum of an operator can encode all or most of the important information about the operator; thus to understand the Laplacian (an operator) on the Sierpinski Carpet it will help to examine its spectrum. To investigate this spectrum, Bob Strichartz of the Mathematics Department at Cornell University developed an experimental concept called the *Outer Approximation* method. The idea of the Outer Approximation method is to take a sequence of 'nice' shapes which approach the fractal, meaning that, as you move down the sequence, each shape better approximates the fractal. These shapes which approximate the fractal should be 'nice' in the sense that the classical theory of differential equations applies to each of them (for example, they should each admit an Energy Form on a Hilbert Space of real-valued functions). The idea of the Outer Approximation method is to find such a sequence and examine the spectrum of the classical Laplacian on each shape in the sequence. Fortunately, there is a clear choice of sequence for the Sierpinski Carpet (the first three shapes are shown in Figure 1); thus it remains to examine the classical spectrum on the shapes in this sequence. Since the classical theory of differential equations is so useful in engineering, an efficient method known as the *Finite Element Method* (FEM) has been developed for this task. This project will examine and test the Outer Approximation method using the Finite Element Method and then apply Outer Approximation to learn about the spectrum of the Laplacian on the Sierpinski Carpet.

Chapter 2: Social and Ethical Contexts of the Project

This chapter develops the key social and ethical concerns of the project and explains how this paper will attempt to alleviate these concerns.

In helping to make a new model useful in engineering and applied sciences, this work will be open to misuse and misinterpretation. In the larger context of differential equations on fractals it is important that the phenomena being studied actually display all the fractal properties which are necessary for the theory to apply. This is especially difficult because different results require different hypotheses on the properties of the application domain. If this theory is developed to the extent predicted by Mandelbrot and Barnsley, the potential applications will be extensive, as will the potential misapplications (Barnsley, 1988). Similarly, in the specific context of this project, the Outer Approximation method is especially vulnerable to misunderstanding. Since this method does not yet have a firm theoretical basis it is important that it be further studied before being applied. This is further complicated by certain results which show the Outer Approximation method is valid in certain specific cases; for example the interval [0,1] in the plane as explained in Chapter 6.2. These issues will be taken up formally in the discussion of results and conclusion sections.

As with any model, the key to proper application is thorough and accurate analysis of the phenomena being modeled. Since this project develops the Outer Approximation method it must provide specific criteria for the applicability of the method to insure that it is not misused. These notes should be clearly differentiated from the valid cases; however, it is also necessary to indicate potential applications for further study. Thus to avoid any misinterpretations the first solution is clarity and the second solution is prototyping the application. By clearly indicating a methodology for examining whether the method can be validly applied, the project will set a positive example so that people applying the model will remember to check all the hypotheses. Researchers such as Mandelbrot (1983), Li (2003), and Kozlov (2003) provided positive examples for modeling physical phenomena as fractals; however there is no corresponding prototype for applying differential equations on fractals to physical phenomena. The project addresses this issue by examining the validity and possible complications of such an application in Chapter 6.1. A full fledged case study, while useful, would be beyond the scope of the project; however, careful descriptions of hypotheses will go far towards preventing misuse.

Finally, since the project is part of the context of making the theory of differential equations on fractals more applicable, it should make the future direction of the research clear. The danger is that the experimental results will be taken as mere evidence rather than part of a process of developing an applicable theory. Thus it will be indicated, in Chapter 6.2, how the Outer Approximation could be extended to a large class of fractals. Chapter 6.2 also explores the potential of this method to lead to rigorous theoretical results. These experiments have guided the theoretical work and excluded certain possibilities which may otherwise have been attempted but which could not be successful.

Chapter 3: Review of Technical Literature

This chapter presents the key technical papers used in this research.

The broad historical and theoretical context for this project, as analyzed in the Rationale and Objectives section, draws mainly from the works of Mandelbrot and Kigami. The work of Barlow and Bass (1999) proves the existence of a Laplacian on the Sierpinski Carpet and thus is important in locating this project in the context of the development of the theory of differential equations on fractals. However, the techniques of their paper are probabilistic and thus fundamentally different from the techniques of the Outer Approximation method; thus the main technical use for this paper will be in attempting to compare similar results. Unfortunately, even this will be difficult as the non-constructive nature of the probabilistic methods makes computations for comparison difficult. Since this research uses Robert Strichartz's method of Outer Approximation, and lies in his vein of research, the pertinent technical works are mostly authored or co-authored by him.

The Outer Approximation method is an analytic approach to developing a Laplacian on the Sierpinski Carpet which gives it unique goals such as developing a calculus, which is a methodology and toolbox for performing calculations, on the Sierpinski Carpet. Although these goals will not be achieved in this project, it is important to understand these goals in order for this research to fit into the context of the analytical approach. This requires understanding the unique features of a calculus on a fractal, and, in 2004, Strichartz et al. wrote a paper which develops a calculus on the Sierpinski Gasket, a p.c.f. fractal for which much is known (Strichartz et al., 2004). The ideas and goals explored in that paper will help give direction to exploring the properties of a Laplacian on the Sierpinski Carpet. Similarly, papers by Strichartz (1999) and Strichartz et al. (1999) catalog properties of the Laplacian on other fractals and corresponding properties could be searched for on the Sierpinski Carpet.

Beyond the broad context for this project there is a specific context which draws on previous research by students at Cornell who also worked with Robert Strichartz. Most relevant is the work of Strichartz with Gibbons and Raj (2001), which applied the Finite Element Method (FEM) to the Sierpinski Gasket. Their paper attempts to perform calculations which would provide the proof of concept which is needed to confirm the validity of the Outer Approximation method. Unfortunately there was an extensive investment of programming required due to lack of easy to use tools. Moreover there were some computational complexity problems. The current availability of tools for performing FEM calculations helped circumvent some of the programming issues and in the future more powerful computers could overcome the computational complexity problems.

Since the Outer Approximation method has not been formalized it must be tested on a known fractal as a “proof of concept” to show that it returns valid results in a known case. Two well-understood fractals are the Sierpinski Gasket and the Pentagasket, and the papers by Strichartz (2003) and Strichartz et al. (2003) provide methods for calculating the spectra of these fractals. The proof of concept for the Outer Approximation will be comparing parts of the spectra of these fractals calculated using both Outer Approximation and the proven techniques from these papers.

Chapter 4: Methods and Theoretical Overview

This chapter is a more formal introduction to the experiments and theory, and addresses the technical difficulties encountered.

The goals of this project were to search for theoretical and experimental evidence for the validity of an "outer approximation" to the Laplacian on fractal domains. While the Laplacian on post critically finite (p.c.f) fractals is well understood, the outer approximation could provide an analytic description of the Laplacian on non-p.c.f. fractals such as the Sierpinski Carpet (SC). If theoretical justification of the outer approximation method were found, this could lead to a long sought after theory of analysis on some class of non-p.c.f. fractals.

The outer approximation method takes advantage of the natural embedding of fractal sets into Euclidean space. Since fractals are most naturally described as the fixed points of a set of contraction mappings, they are naturally the limit (by the Contraction Mapping Principle) of the iterative application of these contractions on a certain starting domain. For example, in the case of the Sierpinski Gasket (SG) the starting domain is a triangle in the plane and the contraction mappings are the three mappings with contraction ratio 1/2 fixing the corners of the triangle. In the outer approximation method, we take each iteration of the contractions and expand it slightly if necessary in order to make it a region (and open, bounded, connected set) in the plane. Some fractals, notably most non-p.c.f. fractals, do not need to be expanded since each iteration is already a region; however in the case of SG some

Figure 3: Outer Approximations to the Sierpinski Gasket

expansion was necessary. In the sequence in Figure 3 you can see how the first four iterations were expanded into regions by making the three contracted triangles overlap slightly.

Once we have a sequence of regions in the plane which approximate the fractal we can use the standard plane-Laplacian on each approximation and hope that this sequence of Laplacians will converge in some meaningful sense to a Laplacian on the fractal. For experimental purposes we decided to examine the eigenvalues and eigenfunctions of the sequence of plane-Laplacians in the hopes that these spectra would converge to the spectrum of the fractal Laplacian. As a proof of concept we examined SG, a p.c.f. fractal whose spectrum is well known and can be obtained with great accuracy by a method called spectral decimation. We found that the spectra of the plane-Laplacian on the domains at the top of the page did seem to approach that of SG. Thus we applied the method to several non-p.c.f. fractals to see what we could learn about them.

In order to obtain the spectra of these regions in the plane we used MATLAB's PDEToolbox which employs a method of estimation know as the Finite-Element-Method (FEM). For the purposes of this experiment the PDEToolbox was taken to be a black box for producing the spectra of the sequence of Laplacians on the plane, with computational tractability being the only limit on the accuracy of the data produced via the FEM.

Chapter 5: Results

This chapter presents the results of experimentally applying the Outer Approximation method to the Sierpinski Gasket and the Sierpinski Carpet.

5.1: Results for the Sierpinski Gasket (SG)

Since the contractions of the triangle (which, when iterated, converge to SG) are not regions in the plane, they had to be modified. We decided to modify them by increasing the contraction ratio thus resulting in a slight overlap between the triangles at each iteration (as opposed to meeting at their vertices as in SG, see Figure 3). This overlap is controlled by two variables. The first is called the Initial Overlap, and indicates the percentage of the total side length at level 1 which is overlap; in Figure 4, the Initial Overlap is length "a" as a percentage of the total bottom length (which is always 1). The second variable is called Overlap Rate, which determines at what rate the overlaps shrink as you move from level n to level n+1; in Figure 4, the Overlap rate is the length "b" divided by the length "a". The overlap rate must be less than 1/2 or eventually the structure of region will no longer give finer approximations. The MATLAB code takes four input variables and appears in Appendix A along with some relevant data collected on the spectra of these regions (more data is available on the project website: "http://www.math.cornell.edu/~thb9d/" which is maintained by the author). The important result of this experiment is best understood through a

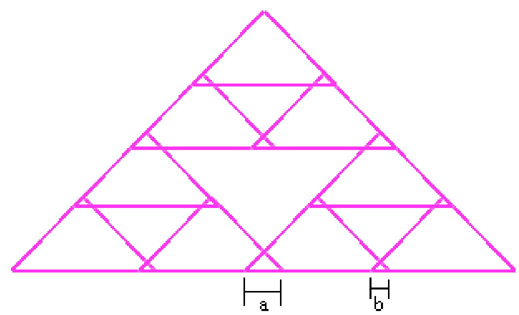

Figure 4: Explanation of Initial Overlap (a) and Overlap Rate (b/a)

graph of the eigenvalues shown in Figure 5, which clearly shows the convergence of the spectra of the regions to the known spectrum of the Laplacian on SG.

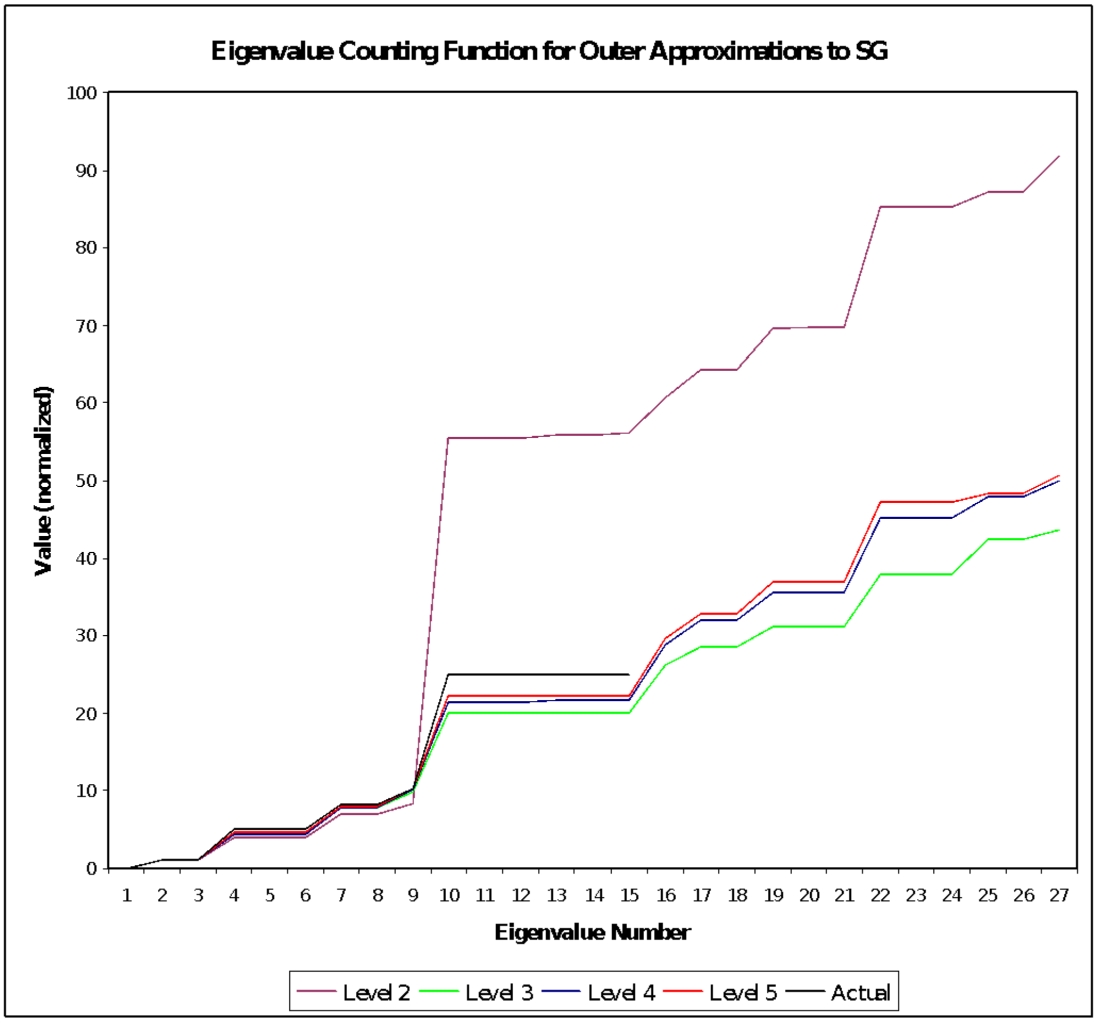

Figure 5: Eigenvalues for Outer Approximations to SG

As the outer approximations approach the actual Sierpinski Gasket the eigenvalues approach the known values. The multiplicities also converge experimentally, indicating that the approximate eigenfunctions have the same symmetries as the actual eigenfunctions which is further evidence for the validity of the outer approximation approach.

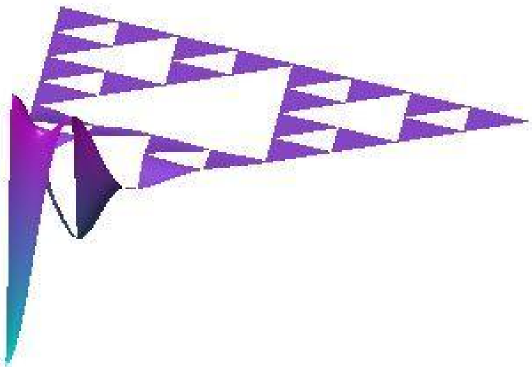

Figure 6: Example of an Eigenfunction which is approximately Localized

More striking evidence for the validity of the outer approximation is the appearance of eigenfunctions that are approximately "localized eigenfunctions." Localized eigenfunctions are eigenfunctions which take the value zero on a particular subset of the domain and are non-zero elsewhere in the domain. Localized eigenfunctions cannot occur in the classical theory of the Laplacian on the plane (technically this is because these eigenfunctions will be analytic, and hence if they are constant on an open subset of the domain they will be identically constant); it is a startling feature of the Laplacian on the Sierpinski Gasket that such eigenfunctions do exist. When examining plots of the eigenfunctions using the outer approximation method, eigenfunctions such as that in Figure 6, which is nearly localized, appeared. This is strong evidence that the convergence of the eigenvalues observed in Figure 5 is not merely coincidental, but a result of a deeper convergence of the eigenfunctions, and possibly, in some sense, the operator itself.

5.2: Results for the Sierpinski Carpet (SC)

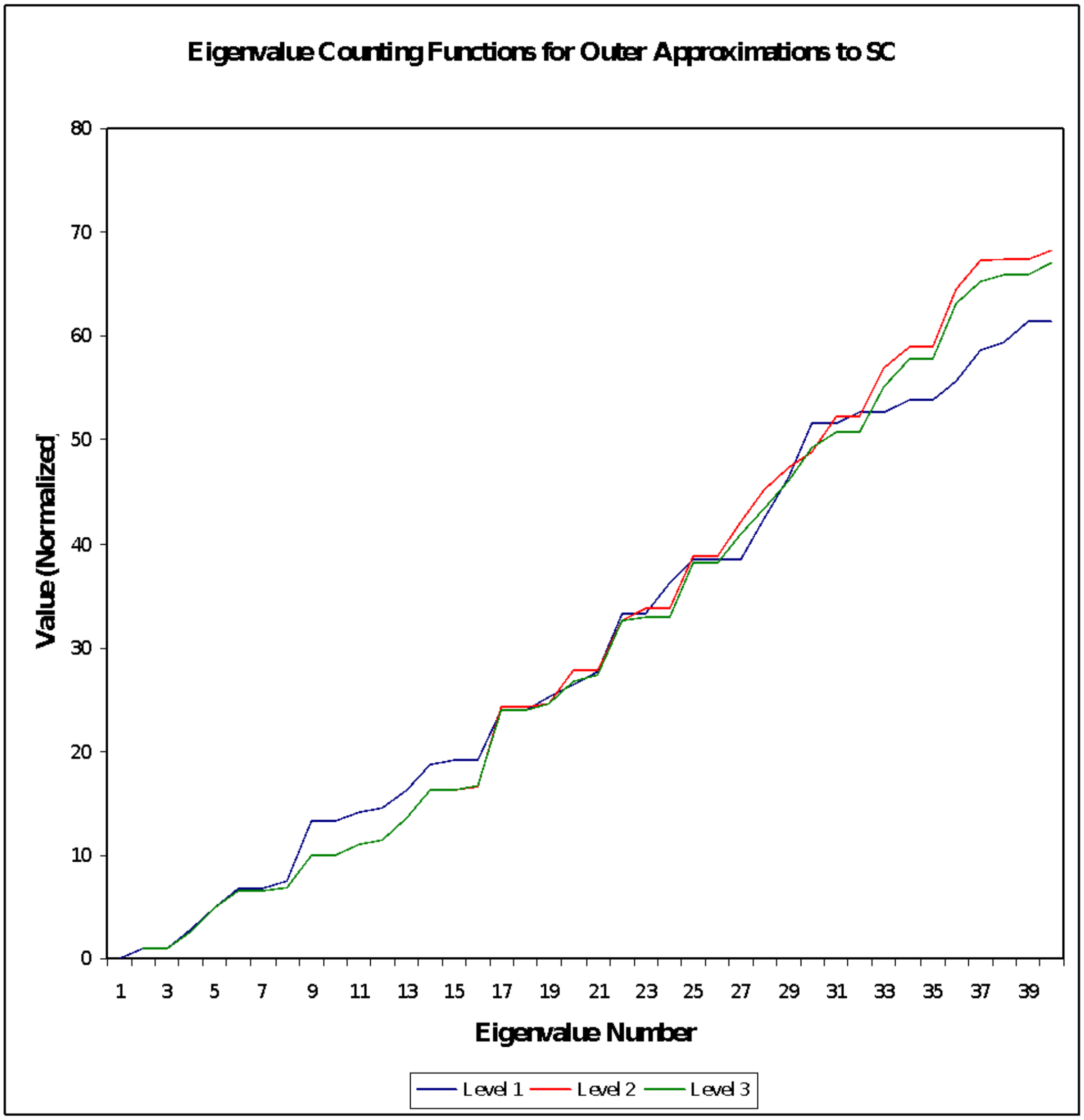

Figure 7: Eigenvalues for Outer Approximations to SC

Results on the Sierpinski Carpet have a similar nature. A detailed analysis of the results is included in the next section; here only the basic data is presented. It appears from the graph of eigenvalues (Figure 7), that the beginning of the spectrum is converging; recall that one of the limitations of the FEM method is that it is only accurate for small eigenvalues. As with the Sierpinski Gasket, computational tractability became an issue quickly and only levels 1-5 of the Sierpinski Carpet were able to be analyzed (although only levels 1-3 are presented here as levels four and five did not yield enough accurate eigenvalues to be useful for the eigenvalue counting function). Nevertheless, Figure 7 clearly shows that the level 2 and level 3 approximations to the Sierpinski Carpet agree much more closely than level 1 and level 2, which diverge fairly quickly. Such close agreement in structure is strong evidence for convergence; however the slight changes in each level indicate that any convergence will be somewhat complicated (see Chapter 6.2).

Because of the self-similarity of fractals there is usually a multiplicative rule for eigenvalues; this is true for the Sierpinski Carpet. For each eigenvalue in the spectrum of the Sierpinski Carpet, it has been shown by probabilistic methods that the spectrum contains the eigenvalue multiplied by a constant factor which is approximately 10.011. Thus, in the observed normalized spectra it was exciting to see eigenvalues which appeared close to 10.011.

No approximately localized eigenfunctions were observed on the Sierpinski Carpet, which indicates that the non-post-critically-finite nature of the Sierpinski Carpet is too restrictive to allow localized eigenfunctions.

A spectral gap is formed by two different eigenvalues which are next to each other in the spectrum and are such that every multiple by the multiplicative factor (in the case of SC 10.011) are also next to each other. In other words this occurs when two eigenvalues a_n and a_{n+1} are such that the "gap" between $10.011*a_n$ and $10.011*a_{n+1}$ does not contain any eigenvalues. At the level three Outer Approximation of the Sierpinski Carpet the eigenvalues 10.88 and 11.41 occurred next to each other in the spectrum, and their multiples, 108.92 and 114.20, are also next to each other in the spectrum, which could be the beginning of a spectral gap. Unfortunately the spectrum could not be examined far enough to see if this gap persisted between the next multiples, namely 1090 and 1143. Regardless, this is important evidence for spectral gaps on the Sierpinski Carpet.

Chapter 6: Discussion of Results

This chapter analyzes the results of the previous chapter and presents a simple case where the formal mathematics can be easily explained.

6.1: Application Notes and Limitations of the Results

The results on the Sierpinski Gasket were very encouraging, and justified further theoretical study as well as the application to the Sierpinski Carpet. Nevertheless, it is important to remember that this evidence does not constitute mathematical proof, and thus the Outer Approximation method is not yet ready for engineering or non-research related applications. Moreover, it is important that uniqueness of the discovered Laplacian on the Sierpinski Carpet has not been proven. Thus, it is possible that the outer approximation may yield a different Laplacian than that found by probabilistic methods.

In any application of fractals to modeling physical phenomena it is important that the domain in question display the necessary fractal characteristics. The work of Mandelbrot in modeling physical phenomena simply involves approximating naturally occurring shapes with shapes of “fractal dimension” (Hausdorff dimension greater than their topological dimension), however, in order to use the theory of differential equations on fractals there are more stringent requirements. (Mandelbrot, 1983). For the theory of Kigami to be valid the domain in question must be well approximated by a post-critically-finite fractal. Here a simple meaning of "well approximated" would be the same up to a small Lebesgue measure set, although other meanings could be valid as well. The requirements for a post-critically finite fractal are quite stringent and are fully exposited in Jun Kigami's book Analysis on Fractals (Kigami, 2001). For the outer approximation to be applicable the domain in question must be well approximated (in some sense) by any fractal set in the sense of Kigami (meaning the fixed point of a set of contractions of the plane) although it may be possible to generalize in the future. Although the outer approximation will produce results in these cases, it is important to remember that the relation of these results to an actual Laplacian on the fractal set are unknown; moreover it is not even known if a Laplacian operator exists on some domain. Nevertheless, the results of these experiments will direct further research and thus are summarized below.

An interesting result is the absence of any observed localized eigenfunctions on the Sierpinski Carpet. However, since the Sierpinski Carpet is a non-post-critically-finite fractal, it can only be disconnected by removing an infinite set of points; thus if an eigenfunction were identically zero on a region it would have to be compatible along the boundary which would contain infinitely many points. This was thought to be too strict a requirement to hope for localized eigenfunctions on the Sierpinski Carpet; thus it is not surprising, although disappointing, that none were observed.

The most important result of the experiments on the Sierpinski Carpet is evidence for Spectral Gaps. Spectral Gaps play a key role in proving important facts on the Sierpinski Gasket. For example Dr. Robert Strichartz of Cornell used them to prove that every continuous function on the Sierpinski Gasket can be approximated by eigenfunctions (Strichartz, 2004). Thus, although the evidence here is sparse, it is very important.

6.2: Notes on Theoretical Aspects

The success of the Outer Approximation method on the Sierpinski Gasket led to further theoretical analysis of the Outer Approximation method. The most simple example of the Outer Approximation is the Outer Approximation of the interval [0,1] by squares of shrinking height ([0,1]x[0,h] as h approaches zero); see Figure 8 for an illustration. Considering both of these sets as natural subsets of the plane one easily finds the eigenvalues of the Laplacian on the interval (which is simply the second derivative) to be of the form $n^2*\pi^2$ where n is any integer. Similarly, the eigenvalues of the Laplacian on [0,1]x[0,h] are of the form $n^2*\pi^2 + m^2*\pi^2/h$ where n and m are integers. Note that there are many distinct operators here, namely for each h>0 there is a separate Laplacian for the appropriate rectangle in the plane, and there is also the Laplacian on the interval. Thus it is interesting that the spectra of the Laplacians on the rectangles converge to the spectrum on the interval in the following sense. As h approaches zero all the eigenvalues of the rectangle grow very large for m not equal to zero. Thus if h is made small enough the first part of the spectrum will just be the m=0 terms. In this case the eigenvalues are simply $n^2*\pi^2$ which is the same as the spectrum of the Laplacian on the interval. Thus, as the Outer Approximations improve (as h goes to zero), the eigenvalues which are not part of the interval spectrum go to infinity, so more of the lower part of the spectrum looks the same as that of the Laplacian on the interval.

To clarify this type of convergence, see Figure 8 which shows the eigenvalues of the Laplacian on the interval [0,1] and then the successive Outer Approximations by rectangles with shrinking heights. For each height the eigenvalues of the interval are highlighted in red, and the diagram clearly shows how the other eigenvalues leave the lower part of the spectrum as the height shrinks, so that by the time the height is .01, the first seven eigenvalues are identical.

Spectrum of the Interval	Spectrum of Rectangle with Height h			
	$n^2*\pi^2+m^2*\pi^2/h$			
$n^2*\pi^2$	h=.9	h=.5	h=.1	h=.01
0.000	0.000	0.000	0.000	0.000
9.870	9.870	9.870	9.870	9.870
39.478	10.966	19.739	39.478	39.478
88.826	20.836	29.609	88.826	88.826
157.913	39.478	39.478	98.696	157.913
246.740	43.865	59.218	108.565	246.740
355.305	50.445	78.957	138.174	355.305
483.610	53.734	88.826	157.913	394.784

Figure 8: Eigenvalues of Outer Approximations to the Interval

Unfortunately, based on experimental observations, any convergence of the Laplacians on the Sierpinski Gasket or Carpet could not be this simple since moving between successive Outer Approximations changes all the eigenvalues slightly. These contrasts with the above example where some eigenvalues will stay constant (although their positions in the spectrum will change) and other eigenvalues will grow and leave the lower spectrum.

Chapter 7: Conclusion

This chapter summarizes the results and analysis of the previous sections and indicates new directions for research.

This research paper has shown that the growing number of applications of fractals to engineering have justified further development of mathematics on fractals. In particular a theory of differential equations on fractals would be useful for modeling heat flow and wave phenomena. This project combined experimental mathematics methods and formal theory to attempt to learn about the Laplacian, an important differential operator, on the Sierpinski Carpet, a fractal for which little is currently known. The project had many successes, gathering experimental evidence, and examining simple cases formally. Furthermore the limitations of this evidence were clearly outlined to attempt to prevent misuse of the Outer Approximation method of modeling differential equations on fractals; thus this section provides the important conclusions and new directions for research.

The key results of this research were experimental in nature, but this data along with some theoretical results will reveal the direction for research to come. The most critical experimental result was strong evidence for the validity of the Outer Approximation method, via application to the known case of the Sierpinski Gasket.

The evidence came in the form of numerically evidence for the convergence of spectra on outer approximations to the known spectrum on the Sierpinski Gasket, and in the appearance of approximately localized eigenfunctions. These results justified application of the Outer Approximation method to the Sierpinski Carpet (an unknown case). The results on the Sierpinski Carpet provided evidence for spectral gaps, and evidence that localized eigenfunctions will not be found on the Sierpinski Carpet. Neither the presence of spectral gaps nor the absence of localized eigenfunctions is known for the Sierpinski Carpet so this evidence will help direct efforts in development of the theory. As far as pure mathematics goes, one possible mode of "convergence" of spectra was discovered by examining an Outer Approximation to the interval [0,1] in the plane. Although this mode of "convergence" will probably not be useful on the fractals, it gave some idea of how to approach the problem, and confirmed theoretical validity of the Outer Approximation method in a simple case.

This project opens up many new avenues of research in terms of further experiments, formal mathematical research, and engineering applications. In terms of further experimentation, one of the limitations of the PDEToolkit was poor error estimation tools; although this is probably due to several features which make the PDEToolkit computationally faster than other tools it would be good to get error bounds on the eigenvalues. The Outer Approximation method could be applied to other fractals, such as the Octagasket, about which little is known, and in fact preliminary research has begun (see http://www.math.cornell.edu/~thb9d/). Finally, better hardware could allow examination of more eigenvalues or deeper levels of the Outer Approximation, although unless the data was needed for some specific application this would probably be unnecessary. Thus experimentally there are still a few unexplored avenues but the important research has been done, the main need for continuation is in the formal theory.

There are many avenues for continuation of the formal theory. The most important result would be proving that the Outer Approximation yields either an operator or at least a valid spectrum in the limit as the Outer Approximations approach the fractal. The main hope would be to examine how the convergence works on the Sierpinski Gasket (the known case) and then try to generalize. Work on this has begun by examining a more complicated Outer Approximation to the interval which is called the Sawtooth region; for more information on this see http://www.math.cornell.edu/~thb9d/. Another possibility would be taking trying to prove the existence of spectral gaps on the Sierpinski Carpet or show that localized eigenfunctions could not exist on the Sierpinski Carpet. Evidence for these results was gathered in the experimentation, and proof of the existence of spectral gaps especially would be an important result. Thus as far as pure mathematics there remains much work to be done, which leads us to the extensions to engineering applications.

In some sense the Outer Approximation is valid for engineering applications; after all it is merely using the classical theory of differential equations, and if an Outer Approximation models a domain well enough it will be perfectly valid. Of course, this is nothing novel, the real benefit of the Outer Approximation method for engineering applications will come from the development of a calculus on the fractals which would simplify computations. Dr. Strichartz has developed such a calculus for the Sierpinski Gasket and extending these results to the Sierpinski Carpet, either probabilistic or analytic methods, is the most important goal for engineering applications.

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Appendix A: MATLAB Code and Data for the Sierpinski Gasket

```
function [p,t,v,l] =
SG(level,initialoverlap,overlaprate,numrefinements)
%solve the eigenvalue problem on SG

m = level;
o = initialoverlap;
r = overlaprate/2;

t = sqrt(3);

%calculate side length
l = (1+o*((1-.5^m)/.5))/(2^(m));

%build bottom left triangle
x1 = 0;
x2 = x1+l;
x3 = x1+l/2;
y1 = 0;
y2 = 0;
y3 = y1+t*l/2;

A = [2;3;x1;x2;x3;y1;y2;y3];

%construct the mth level using a recursive type algorigthm where
%each triangle is a column in a matrix and each iteration builds
%two triangles for each triangle already in the matrix (m
%iterations)
for i=1:m
    for j=1:3^(i-1)
        x1 = A(3,j)+2^(-i)-o*(r^(i-1))/2;
        y1 = A(6,j);
```

```
%describe the boundary conditions (Neumann in this case)
%the boundary conditions are described by a matrix with
%one column for each column in the decomposed geometry
B = [1;0;1;1;48;48;48;48;49;48];
C = [1;0;1;1;48;48;48;48;49;48];
for j=1:length(e)
    B = [B C];
end

%perform any refinements to the triangulation
for k=1:numrefinements
    [p,e,t]=refinemesh(g,p,e,t);
end
```

4 Refinements	3 Refs	0 Refs	0 Refs	continued	continued	continued	continued
0	0	0	0	408.4531	149.0089	150.7148	124.3641
4.7832	3.9473	3.3372	2.6327	417.2121	167.4636	159.7685	127.3438
4.7833	3.9473	3.3376	2.6333	417.2131	167.4646	159.7975	127.3581
19.508	17.2965	15.0372	12.213	439.5866	172.1788	166.8725	133.4426
19.5251	17.3134	15.0492	12.2253	804.9468	1051.938	335.7102	288.0482
19.5254	17.3135	15.0518	12.2256	834.2263	1051.941	335.7379	288.0994
33.1678	30.4394	26.2223	20.8931	834.2374	1051.975	335.7705	288.1694
33.1685	30.4394	26.2245	20.8956	918.222	1054.94	335.8927	288.1905
40.5037	39.0578	33.7524	26.8513	922.7715	1054.949	335.8962	288.2102
265.4017	78.7016	71.6959	58.5692	922.7812	1054.953	335.9219	288.2306
265.41	78.7019	71.698	58.5772	980.6827	1054.965	335.9374	288.251
265.4123	78.7031	71.7027	58.5823	980.6887	1054.969	335.9398	288.2655
267.1889	78.9054	71.9215	58.7675	1012.2129	1054.976	335.9538	288.27
267.1903	78.9064	71.9232	58.7823	1061.8501	1055.24	335.9606	288.2956
267.8062	78.9892	72.0256	58.867	1061.8796	1055.245	336.0023	288.3556
290.7663	103.9332	96.155	78.0084	1061.9241	1055.246	336.0368	288.3577
307.195	113.157	106.2227	86.4177	1068.7754	1070.462	338.6302	291.1745
307.1966	113.1582	106.2293	86.4276	1068.8069	1070.47	338.695	291.312
332.8849	123.0738	118.5657	97.4488	1070.9267	1071.463	338.8553	291.4274
333.6483	123.1212	118.6246	97.5097	1184.6079	1146.814	425.3762	371.3145
333.6491	123.1219	118.6412	97.515	1188.8129	1146.818	425.398	371.3541
408.4465	149.0087	150.705	124.3511	1188.8201	1146.825	425.4297	371.3888
408.4495	149.0087	150.7129	124.3577	1192.0944	1155.238	437.9033	379.8021

Appendix B: MATLAB Code and Data for the Sierpinski Carpet

```
function [p,t,v,l] = SC(level,numrefinements)
%sets up and solves eigenvalue problem on outer approximation of %SC

m = level;
r = overlaprate;

%calculate side length
l = 1/(3^m)

%build bottom left square
x1 = 0;
x2 = x1;
x3 = x1+l;
x4 = x1+l;
y1 = 0;
y2 = y1+l;
y3 = y1+l;
y4 = y1;

A = [2;4;x1;x2;x3;x4;y1;y2;y3;y4];

%construct the mth level using a recursive type algorigthm where
%each square is a column in a matrix and each iteration builds
%seven squares for each square already in the matrix
for i=1:m
    for j=1:8^(i-1)
        x1 = A(3,j)+1/(3^i);
        y1 = A(7,j);
        x2 = x1;
        x3 = x1+l;
        x4 = x1+l;
```

```
        x3 = x1+l;
        x4 = x1+l;
        y2 = y1+l;
        y3 = y1+l;
        y4 = y1;

        B = [2;4;x1;x2;x3;x4;y1;y2;y3;y4];
        A = [A B];

        x1 = A(3,j);
        y1 = A(7,j)+2/(3^i);
        x2 = x1;
        x3 = x1+l;
        x4 = x1+l;
        y2 = y1+l;
        y3 = y1+l;
        y4 = y1;

        B = [2;4;x1;x2;x3;x4;y1;y2;y3;y4];
        A = [A B];

        x1 = A(3,j)+2/(3^i);
        y1 = A(7,j)+1/(3^i);
```

```
    end
end

%tell pdetoolbox to build the 'decomposed geometry' and %triangulate
g=decsg(A);
[p,e,t]=initmesh(g);

%describe the boundary conditions (Neumann in this case)
%the boundary conditions are described by a matrix with
%one column for each column in the decomposed geometry
B = [1;0;1;1;48;48;48;48;49;48];
C = [1;0;1;1;48;48;48;48;49;48];
for j=1:length(e)
    B = [B C];
end

%perform any refinements to the triangulation
for k=1:numrefinements
    [p,e,t]=refinemesh(g,p,e,t);
end


%pdemesh(p,e,t);                          %display mesh

%solve eigenvalue problem, only looking for eigenvals between 0 %and
1000
[v,l]=pdeeig(B,p,e,t,1,0,1,[0,1000]);
```

6.7449	6.0999	5.4936	170.1977	149.9259	135.1289
6.7451	6.0999	5.4936	177.7272	169.4852	146.4968
18.8127	16.2607	14.5823	186.9544	169.6577	149.7908
32.7746	29.9997	27.0651	224.3803	198.6283	178.8197
45.4829	40.2428	36.1754	224.3886	205.9248	180.4537
45.4834	40.2434	36.1754	243.7824	205.9273	180.4537
50.8059	42.1209	37.8262	259.106	236.1448	209.5562
88.8425	60.9654	54.9598	259.8246	236.154	209.5562
88.8432	60.9688	54.9598	259.8329	256.7518	225.0039
94.3576	66.3865	59.7724	285.9648	275.2315	238.0328
97.8154	69.4964	62.6712	312.0478	288.0364	252.3353
109.7554	83.2379	74.7457	347.0751	297.352	270.3526
126.0367	99.0051	89.4182	347.0933	318.2773	278.7202
129.2825	99.0095	89.4182	355.5688	318.2931	278.7202
129.2854	100.5738	91.7984	355.5831	347.2696	302.8469
161.4947	147.5414	131.6229	363.1377	358.8827	317.6237

Normalized Spectrum on SC

Level 1	Level 2	Level 3	Level 1	Level 2	Level 3
4 Refinements	3 Refs	3 Refs	Contd.	Contd.	Contd.
0.00	0.00	0.00	73.83	79.84	77.30
1.00	1.00	1.00	73.83	79.85	77.30
1.00	1.00	1.00	75.33	81.45	78.60
2.79	2.67	2.65	79.10	85.05	83.42
4.86	4.92	4.93	79.14	89.06	85.05
6.74	6.60	6.59	84.95	89.07	85.10
6.74	6.60	6.59	84.96	90.67	85.10
7.53	6.91	6.89	91.77	103.53	96.12
13.17	9.99	10.00	91.78	103.55	97.06
13.17	10.00	10.00	93.21	107.53	97.40
13.99	10.88	10.88	94.33	108.09	97.40

14.50	11.39	11.41	95.81	113.23	100.00
16.27	13.65	13.61	95.82	115.86	100.00
18.69	16.23	16.28	95.82	116.12	100.21
19.17	16.23	16.28	105.52	116.14	100.63
19.17	16.49	16.71	106.04	117.82	100.79
23.94	24.19	23.96	106.33	124.50	100.79
23.94	24.19	23.96	109.55	124.55	101.11
25.23	24.58	24.60	109.55	126.25	102.59
26.35	27.79	26.67	112.29	131.69	103.76
27.72	27.81	27.27	118.71	131.71	104.47
33.27	32.56	32.55	118.72	132.82	104.47
33.27	33.76	32.85	119.77	132.99	105.12
36.14	33.76	32.85	119.86	138.31	108.83
38.42	38.71	38.15	121.09	138.34	108.83
38.52	38.71	38.15	121.10	139.94	108.90
38.52	42.09	40.96	125.29	141.37	114.02
42.40	45.12	43.33	125.30	145.18	119.70
46.26	47.22	45.93	126.73	145.77	120.65
51.46	48.75	49.21	126.78	153.59	124.70
51.46	52.18	50.74	131.92	153.64	124.70
52.72	52.18	50.74	131.94	156.91	136.52
52.72	56.93	55.13	132.74		136.68
53.84	58.83	57.82	138.96		136.68
53.84	58.84	57.82	139.68		137.49
55.56	64.40	63.08	139.69		140.94
58.61	67.19	65.27	141.76		140.94
59.33	67.42	66.00	141.78		142.01
61.31	67.44	66.00			147.95
61.32	68.21	67.12			148.93
65.91	74.75	69.08			151.48
65.91	74.76	71.07			151.48
69.94	75.20	71.07			152.80