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K-theory and the bridge from motives to noncommutative motives

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A B S T R A C T

In this work we present a new approach to the theory of noncommutative motives and use it to explain the different flavors of algebraic K-theory of schemes and dg-categories. The work is divided into three main parts. In the first part we use the techniques of higher algebra developed in [63] to provide a universal characterization for the symmetric monoidal $(\infty, 1)$-category underlying the motivic stable $\mathbb{A}^1$-homotopy theory of Morel–Voevodsky [107,67]. More precisely, given a symmetric monoidal model category $\mathcal{V}$ together with an object $X \in \mathcal{V}$, we characterize the underlying symmetric monoidal $(\infty, 1)$-category of the symmetric monoidal model category $Sp^{\Sigma}(\mathcal{V}, X)$ introduced by Hovey in [43], by means of a universal property amongst symmetric monoidal $(\infty, 1)$-categories. This characterization trivializes the problem of finding motivic monoidal realizations.

In the second part we introduce a new approach to the theory of noncommutative motives by constructing a stable motivic homotopy theory for the noncommutative spaces of Kontsevich [56,55,54]. The key ingredient is a notion of Nisnevich topology in the noncommutative setting, compatible with the classical notion. This compatibility, together with the universal property proved in the first part, ensures the existence of a canonical monoidal map from the

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stable motivic theory of Morel–Voevodsky towards these new noncommutative motives that allow us to compare the two theories.

In the last part of this paper we explain how this bridge can be used to explain the various flavors of algebraic K-theory of dg-categories. More precisely, we prove that the non-connective K-theory of dg-categories introduced by Schlichting [82] is the (non-commutative) Nisnevich sheafification of connective algebraic K-theory. Then we prove that its further (non-commutative) $\mathbb{A}^1$-localization is a tensor unit in our noncommutative motives. As a corollary we obtain a precise proof for an original conjecture of Kontsevich claiming that K-theory gives the correct mapping spaces in noncommutative motives. Our major application is the discovery of a canonical factorization of our motivic bridge through the $(\infty, 1)$-category of modules over the commutative algebra object representing homotopy invariant algebraic K-theory of schemes. The results in [77] imply that this bridge is fully faithful over a field k with resolutions of singularities, so that, at the motivic level, no information (below K-theory) is lost by passing to the noncommutative world.

Contents

1. Introduction

This paper is extracted from the author's Phd Thesis [79] where the author aims to compare classical algebraic geometry with the new noncommutative algebraic geometry in the sense of Kontsevich [55,56,54]. More precisely, the motivic levels of both theories.

1.1. Background

1.1.1. Motives

In the original program envisioned by Grothendieck, the motive of a geometric object X (e.g. X a projective smooth variety) was a new mathematical object designed to express "the arithmetical content of X".[2] More precisely, in the sixties, Grothendieck and his collaborators started a quest to construct examples of the so-called Weil cohomology theories, designed to capture different arithmetic information about X. In the presence of many such theories he envisioned the existence of a universal one, which would gather all the arithmetic information. At that time, cohomology theories were formulated in a rather artificial way using abelian categories as the basic input. The notion of triangulated categories appeared as an attemptive to provided a new, more natural setting for cohomology theories. Of course, the subject of motives followed these innovations [7] and finally, in the 90's, V. Voevodsky [105] constructed what became known as "motivic cohomology". Many good introductory references to this arithmetic program are now available [68,1,65], together with the historical background given in the introduction of [23] as well as the recent course notes by B. Kahn [46].

In the late 90's, Morel and Voevodsky [67] developed a more general theory of motives. In their theory, the motive of X is designed to be the cohomological skeleton of X, not only in the eyes of a Weil cohomology theory, but for all the generalized cohomology theories for schemes (like K-theory, algebraic cobordism and motivic cohomology) at once. The inspiration comes from the stable homotopy theory of spaces where all generalized cohomology theories (of spaces) become representable. Such a setting would provide easier definitions for the motivic cohomology, algebraic K-theory, algebraic cobordism, and so on, by merely providing their representing spectra. Their construction has two main steps: the first part mimics the homotopy theory of spaces and its stabilization; the second part forces the "Tate motive" to become invertible with respect to the monoidal multiplication. The final result is known as the *motivic stable homotopy theory of schemes*. Our first main goal in this work is to formulate a precise universal property for their construction.

1.1.2. Noncommutative algebraic geometry

In Algebraic Geometry, and specially after the works of Serre and Grothendieck, it became a common practice to study a scheme X via its abelian category of quasi-coherent

[2] For instance, it should capture the information underlying the L-function of X.

sheaves $Qcoh(X)$. The reason for this is in fact purely technical for at that time, abelian categories were the only formal background to formulate cohomology theories. In fact, the object $Qcoh(X)$ turns out to be a very good replacement for the geometrical object X: thanks to [36,80] we know that X can be reconstructed from $Qcoh(X)$. However, it happens that abelian categories do not provide a very natural framework for homological algebra. It was Grothendieck who first noticed that this natural framework would be, what we nowadays understand as, the homotopy theory of complexes in the abelian category. At that time, the standard way to deal with homotopy theories was to consider their homotopy categories – the formal strict inversion of the weak-equivalences. This is how we obtain the derived category of the scheme $D(X)$. For many reasons, it was clear that the passage from the whole homotopy theory of complexes to the derived category loses too much information. The answer to this problem appeared from two different directions. First, from the theory of dg-categories [16,14,15]. More recently, an ultimately, with the theory of ∞-categories [2,11,59,63,86,98]. The first subject became very popular specially with all the advances in [17,19,28,29,49,89,96]. The second, although initiated in the 80s with the famous manuscript [40], only in the last ten years reached a state where its full potential can be explored. This is specially due to the tremendous efforts of [59,63]. Both subjects provide an appropriate way to encode the homotopy theory of complexes of quasi-coherent sheaves. In fact, the two approaches are related and, for our purposes, should give equivalent answers (see the recent results in [26] and our discussion in [79, Section 6.2]). Every scheme X (over a ring k) gives birth to a k-dg-category $L_{qcoh}(X)$ – *the dg-derived category of* X – whose "zero level" recovers the classical derived category of X. For reasonable schemes, this dg-category has an essential property deeply related to its geometrical origin – it has a compact generator and the compact objects are the perfect complexes (see [19] and [94]). It follows that the smaller sub-dg-category $L_{pe}(X)$ spanned by the compact objects is "affine", and enough to recover the whole $L_{qcoh}(X)$.

In his works [56,55,54], Kontsevich initiated a systematic study of the dg-categories with the same formal properties of $L_{pe}(X)$, with the observation that many different examples of such objects exist in nature: if A is an associative algebra then A can be considered as a dg-category with a single object and we consider $L(A)$ the dg derived category of complexes of A-modules and take its compact objects. The same works with a differential graded algebra. The *Fukaya* category of a symplectic manifold is another example [57]. There are also examples coming from complex geometry [72], representation theory, matrix factorizations (see [32]), and also from the techniques of deformation quantization. This variety of examples with completely different origins motivated the understanding of dg-categories as natural *noncommutative spaces*. The study of these dg-categories can be systematized and the assignment $X \mapsto L_{pe}(X)$ can be properly arranged as a functor

$$L_{pe} : \text{Classical Schemes}/k \longrightarrow \text{Noncommutative Spaces}/k \tag{1.1}$$

In fact, the functor L_{pe} is defined not only for schemes but for a more general class of geometrical objects, so-called *derived stacks* (see [100,9,60]). They are the natural geometric objects in the theory of derived algebraic geometry of [102,103,62,99]. For the purposes of noncommutative geometry, this fact is crucial: thanks to the results of Toën–Vaquié in [100], at the level of derived stacks, L_{pe} admits a right adjoint, providing a canonical mechanism to construct a geometric object out of a noncommutative one.

Kontsevich proposes also that similarly to schemes, these noncommutative spaces should admit a motivic theory. Our second main goal in this work is to provide a natural candidate for this theory, that extends in a natural way the theory of Morel–Voevodsky. The bridge between the two theories is a canonical extension of the map L_{pe} given by our universal characterization of the theory for schemes.

1.2. In this work

The motivic construction of Morel–Voevodsky was originally obtained using the techniques of model category theory. Nowadays we know that a model category is merely strict presentation of a more fundamental object – an $(\infty, 1)$-category. Every model category has an underlying $(\infty, 1)$-category and the later is what really matters. It is important to say that the need for this passage overcomes the philosophical reasons and that thanks to the techniques of [59,63] we now have the ways to do and prove things which would remain out of range only with the highly restrictive techniques of model categories.

The first part of this work concerns the universal characterization of the $(\infty, 1)$-category underlying the stable motivic homotopy theory of schemes, as constructed by Voevodsky and Morel, with its symmetric monoidal structure. The characterization becomes relevant if we want to compare the motives of schemes with other theories. In our case, the goal is to conceive a theory of motives of noncommutative spaces and to relate it to the theory of Voevodsky–Morel. By providing such a universal characterization we will be able to ensure, for free, the existence of a (monoidal) dotted arrow at the motivic level

$$\begin{tikzcd} \text{Classical Schemes}/k \arrow[r] \arrow[d] & \text{NC-Spaces}/k \arrow[d] \\ \text{Stable Motivic Homotopy}/k \arrow[r, dashed] & \text{NC-Stable Motivic Homotopy}/k \end{tikzcd} \tag{1.2}$$

In general, monoidal maps such as the one here presented are extremely hard to obtain only by constructive methods and the techniques of model category theory. Other important advantage is that it allows us to work over any base scheme, not necessarily a field.

At this point we should also emphasize that a different approach to non-commutative motives already exists in the literature, due to D.-C. Cisinski and G. Tabuada (see [90, 25,93] and [91] for a pedagogical overview). Their approach is essentially of "'cohomological nature'" while our method could be said "'homological'" and follows the spirit of stable homotopy theory. In the appendix to this work (Appendix A) we systematize the comparison between the two approaches and unveal a form of duality between them. It is exactly this duality phenomenon that makes our new approach comparable to the theory of Morel–Voevodsky and allows the dotted monoidal map to exist in a natural way. The same duality blocks a direct comparison in their case. We should also mention that all our mathematical contents and proofs are independent of theirs.

To achieve the universal characterization we will need to rewrite the constructions of Morel–Voevodsky in the setting of ∞-categories. The dictionary between the two worlds is given by the techniques of [59] and [63]. In fact, [59] already contains all the necessary results for the characterization of the $\mathbb{A}^1$-*homotopy theory of schemes* and its *stable non-motivic* version. The problem concerns the description of the stable motivic world with its symmetric monoidal structure. This is our main contribution in this subject. The key ingredient is the following:

Insight 1.1. *(See Theorem 2.26 for the precise formulation.) Let $\mathcal{V}$ be a combinatorial simplicial symmetric monoidal model category with a cofibrant unit and let $\mathcal{C}^{\otimes}$ denote its underlying symmetric monoidal ∞-category. Let X be a cofibrant object in $\mathcal{V}$ satisfying the following condition:*

($*$) *the cyclic permutation of factors $\sigma = (123) : X \otimes X \otimes X \to X \otimes X \otimes X$ is equal to the identity map in the homotopy category of $\mathcal{V}$.*[3]

Then the underlying symmetric monoidal ∞-category of $Sp^{\Sigma}(\mathcal{V}, X)$ is the universal symmetric monoidal $(\infty, 1)$-category equipped with a monoidal map from $\mathcal{C}^{\otimes}$, sending X to an invertible object.

It is the goal of Section 2 to prove this theorem. This extra assumption on X is not new. It is already present in the works of Voevodsky [107] and it also appears in [43]. We must point out that we believe our result to be true even without this extra assumption on X. We will explain this in Remark 2.27.

In Section 2.4 we apply the general results of Section 2 to the motivic stable homotopy theory of schemes:

Corollary 1.2. *(See Corollary 2.39.) Let S be a base scheme and let $Sm^{ft}(S)$ denote the category of smooth separated schemes of finite type over S. The $(\infty, 1)$-category $\mathcal{SH}(S)$ underlying the stable motivic homotopy theory of schemes is stable, presentable and admits a canonical symmetric monoidal structure $\mathcal{SH}(S)^{\otimes}$. Moreover, the construction of*

[3] More precisely we demand the existence of a homotopy in $\mathcal{V}$ between the cyclic permutation and the identity.

Morel–Voevodsky provides a functor $Sm^{ft}(S)^{\times} \to \mathcal{SH}(S)^{\otimes}$ monoidal with respect to the cartesian product of schemes, and endowed with the following universal property:

(∗) *for any pointed presentable symmetric monoidal $(\infty,1)$-category $\mathcal{D}^{\otimes}$, the composition map*

$$Fun^{\otimes,L}\big(\mathcal{SH}(S)^{\otimes}, \mathcal{D}^{\otimes}\big) \to Fun^{\otimes}\big(Sm^{ft}(S)^{\times}, \mathcal{D}^{\otimes}\big) \tag{1.3}$$

is fully faithful and its image consists of those monoidal functors $Sm^{ft}(S)^{\times} \to \mathcal{D}^{\otimes}$ satisfying Nisnevich descent, $\mathbb{A}^1$-invariance and such that the cofiber of the image of the point at ∞, $S \xrightarrow{\infty} \mathbb{P}^1_S$ is an invertible object in $\mathcal{D}^{\otimes}$. Moreover, any pointed presentable symmetric monoidal $(\infty,1)$-category $\mathcal{D}^{\otimes}$ admitting a monoidal map in this image, is necessarily stable.

This result trivializes the problem of finding motivic monoidal realizations. The existence of these have deep consequences. See [47] for an overview.

Example 1.3. Let $S = Spec(k)$ be field of characteristic zero. The assignment $X \mapsto \Sigma^{\infty}(X(\mathbb{C}))$ provides a functor $Sm^{ft}(S) \to Sp$ with Sp the $(\infty,1)$-category of spectra (see below). This map is known to be monoidal, to satisfy all the descent conditions in the previous corollary and to invert $\mathbb{P}^1$ in the required sense. Therefore, it extends in an essentially unique way to a monoidal map of stable presentable symmetric monoidal $(\infty,1)$-categories $\mathcal{SH}(S)^{\otimes} \to Sp^{\otimes}$;

Example 1.4. Again, let $S = Spec(k)$. Another immediate example of a monoidal motivic realization is the Hodge realization. Properly constructed, the map $X \mapsto C_{DR}(X)$ sending a scheme to its De Rham complex provides a functor $Sm^{ft}(S) \to \mathcal{D}(k)$ with $\mathcal{D}(k)$ the $(\infty,1)$-derived category of k. This map is known to be monoidal with respect to the cartesian product of schemes (Kunneth formula), satisfies all the descent conditions and inverts $\mathbb{P}^1$ in the sense above. Because of the universal characterization, it extends in an essentially unique way to a monoidal motivic Hodge Realization $\mathcal{SH}(S)^{\otimes} \to \mathcal{D}(k)^{\otimes}$ (where on the left we have the monoidal structure induced by the derived tensor product of complexes). This example has recently been worked out in detail in the PhD thesis of B. Drew [27]. One of its corollaries is a new Riemann–Hilbert correspondence for holonomic D-modules (see [27, Thm. 3.4.1]).

Example 1.5. Our main theorem also provides a universal characterization for the G-equivariant version of motivic homotopy theory (in the sense of [45]). As proved in [45, Section 2.2, Lemma 2] we also fall in the situation of $\otimes$-inverting a symmetric object.

In the second part of this work we systematize the comparison between the commutative and noncommutative motivic worlds. After some preliminaries on dg-categories, we introduce the $(\infty,1)$-category of smooth noncommutative spaces $\mathcal{N}cS(k)$ as the opposite of the $(\infty,1)$-category of idempotent dg-categories of finite type $\mathcal{D}g(k)^{ft} \subseteq \mathcal{D}g(k)^{idem}$ introduced by Toën–Vaquié in [100]. By introducing an appropriate noncommutative analogue for the Nisnevich topology (Definition 3.12) and considering the noncommutative version of the affine line $L_{pe}(\mathbb{A}^1)$, we construct a new stable presentable symmetric monoidal $(\infty,1)$-category $\mathcal{SH}_{nc}(S)^{\otimes}$ encoding a stable motivic homotopy theory for these noncommutative spaces. This provides a new approach to noncommutative motives.

The first step to compare the commutative and the non-commutative world is to encode the map $X \mapsto L_{pe}(X)$ as a functor L_{pe} from smooth affine schemes towards $\mathcal{N}cS(k)$ (see Proposition 3.4). Our universal characterization of the stable motivic homotopy theory of schemes allows us to extend it to a monoidal colimit preserving functor

$$\begin{tikzcd}
N(\mathit{AffSm}^{ft}(k)) \arrow[r, "L_{pe}"] \arrow[d] & \mathcal{N}cS(k) \arrow[d] \\
\mathcal{SH}(k) \arrow[r, dashed] & \mathcal{SH}_{nc}(k)
\end{tikzcd} \tag{1.4}$$

Finally in the last part of this work we explain how this bridge can now be used to understand the different versions of algebraic K-theories of dg-categories and schemes. To explain our main results in this topic we need some technical background. First, and as the reader shall later see (Section 2.4.4 and Remark 3.28), both $\mathcal{SH}(k)^{\otimes}$ and $\mathcal{SH}_{nc}(k)^{\otimes}$ can be obtained as a sequence of monoidal reflexive localizations of $(\infty,1)$-categories of spectral presheaves, followed by the $\otimes$-inversion of the algebraic circle $\mathbb{G}_m$. With this in mind, the construction of the comparison map in the previous commutative diagram can be explained in a sequence of steps

$$\begin{tikzcd}
N(\mathit{AffSm}^{ft}(k))^{\times} \arrow[r, "L_{pe}^{\otimes}"] \arrow[d, hook, "(\Sigma^{\infty}_{+}\circ j)^{\otimes}"] & \mathcal{N}cS(k)^{\otimes} \arrow[d, hook, "(\Sigma^{\infty}_{+}\circ j_{nc})^{\otimes}"] \\
\mathit{Fun}(N(\mathit{AffSm}^{ft}(k))^{op}, \mathit{Sp})^{\otimes} \arrow[r, dashed, "\mathcal{L}_1^{\otimes}"] \arrow[d, "l_{Nis}^{\otimes}"] & \mathit{Fun}(\mathcal{D}g(k)^{ft}, \mathit{Sp})^{\otimes} \arrow[d, "l_{Nis}^{nc,\otimes}"'] \\
\mathit{Fun}_{Nis}(N(\mathit{AffSm}^{ft}(k))^{op}, \mathit{Sp})^{\otimes} \arrow[r, dashed, "\mathcal{L}_2^{\otimes}"] \arrow[d, "l_{\mathbb{A}^1}^{\otimes}"] & \mathit{Fun}_{Nis}(\mathcal{D}g(k)^{ft}, \mathit{Sp})^{\otimes} \arrow[d, "l_{\mathbb{A}^1}^{nc,\otimes}"'] \\
\mathit{Fun}_{Nis,\mathbb{A}^1}(N(\mathit{AffSm}^{ft}(k))^{op}, \mathit{Sp})^{\otimes} \arrow[r, dashed, "\mathcal{L}_3^{\otimes}"] \arrow[d, "\Sigma_{\mathbb{G}_m}^{\otimes}"] & \mathit{Fun}_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \mathit{Sp})^{\otimes} \arrow[d, "\sim"] \\
\mathcal{SH}(k)^{\otimes} \arrow[r, dashed, "\mathcal{L}^{\otimes}"] & \mathcal{SH}_{nc}(k)^{\otimes}
\end{tikzcd} \tag{1.5}$$

where each dotted map is induced by a universal property. By formal abstract nonsense these functors admit right-adjoints which we shall, respectively, denote as $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ and $\mathcal{M}$.

This mechanism allows us to restrict noncommutative invariants to the commutative world.

Example 1.6. An important example of a noncommutative invariant is the Hochschild homology of dg-categories. Thanks to the works of B. Keller in [52] and as explained in Remark 3.29 this invariant can be completely encoded by means of an ∞-functor $HH : \mathcal{D}g(k)^{ft} \to \widehat{Sp}$. Another important example is the so-called *periodic cyclic homology of dg-categories* HP. It follows from the famous HKR theorem that the restriction of HP to the commutative world recovers the classical de Rham cohomology of schemes. For more details see the discussion in [12, Section 3.1].

Example 1.7. Another important example recently introduced by A. Blanc in his thesis [12] is the topological K-theory of dg-categories. This is a candidate for the non-commutative version of the Betti realization.

In the last section we will be interested in the restriction of the various algebraic K-theories of dg-categories. As we shall explain, all of them live as objects in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$. There are two of primary relevance to us:

- K^c – encoding the *connective* K-theory given by Waldhausen's S-construction. See the discussion in Section 4.2.2 below.
- K^S – encoding the *non-connective* K-theory of dg-categories as defined in [24] using the adaptation of the Schlichting framework of [82] to the context of dg-categories (see the discussion in Section 4.2.3). By construction, this functor comes naturally equipped with a canonical natural transformation $K^c \to K^S$ which is an equivalence in the connective part.

For the first one, it follows immediately from the spectral version of Yoneda lemma (see Remark 2.41) and from the definition in [94, Section 3] that $\mathcal{M}_1(K^c)$ recovers the connective algebraic K-theory of schemes. The second one, by the comparison result [82, Theorem 7.1], recovers the non-connective K-theory of schemes of Bass–Thomason–Trobaugh of [94]. The construction of K^S in [24] using the methods of [82] is somehow ad-hoc. We explain how the non-connective version of K-theory K^S can be canonically obtained from the connective version K^c as a result of enforcing our noncommutative-world version of Nisnevich descent. The following theorem summarizes our main technical results in this topic:

Theorem 1.8.

(i) (Theorem 4.4) The canonical morphism $K^c \to K^S$ presents non-connective K-theory as the (noncommutative) Nisnevich sheafification of connective K-theory;
(ii) (Theorem 4.6) The further (noncommutative) $\mathbb{A}^1$-localization $l^{nc}_{\mathbb{A}^1}(K^S)$ is a unit 1_{nc} for the monoidal structure in $\mathcal{SH}_{nc}(k)^{\otimes}$;
(iii) (Theorem 4.7) The image of $l^{nc}_{\mathbb{A}^1}(K^S)$ along the right-adjoint $\mathcal{M}$ recovers the object KH in $\mathcal{SH}(k)$ representing $\mathbb{A}^1$-invariant algebraic K-theory of Weibel (also known as homotopy invariant K-theory) studied in [107] and in [22]. In particular, since $\mathcal{M}$ is lax monoidal (it is right-adjoint to a monoidal functor) it sends the trivial algebra structure in 1_{nc} to a commutative algebra structure in KH so that the monoidal map $\mathcal{L}^{\otimes}$ factors as

$$\mathcal{SH}(k)^{\otimes} \xrightarrow{-\otimes KH} Mod_{KH}(\mathcal{SH}(k))^{\otimes} \dashrightarrow \mathcal{SH}_{nc}(k)^{\otimes}$$

Let us emphasize that the part (i) of this theorem is not true if we restrict ourselves to the non-connective K-theory of schemes. The phenomenon that makes it possible in the noncommutative world is the fact the new notion of Nisnevich squares of noncommutative spaces combines at the same time coverings of geometrical origin (namely, those coming via L_{pe} from classical Nisnevich squares) and coverings of categorical origin, namely, the ones induced by exceptional collections.

The first part of this theorem is proved by showing that the Bass-construction $(-)^B$ given in Thomason's paper [94] is an explicit model for the (noncommutative) Nisnevich localization of presheaves with values in connective spectra and sending Nisnevich squares of noncommutative spaces to pullback squares in connective spectra. Recall that the inclusion of connective spectra in all spectra does not preserve pullbacks. More generally, we prove that the connective truncation functor induces an equivalence of $(\infty,1)$-categories between the $(\infty,1)$-category of Nisnevich local spectral presheaves and the $(\infty,1)$-category of spectral presheaves with values in connective spectra and satisfying connective Nisnevich descent. The $(-)^B$-construction is an explicit inverse to this truncation. The second result uses a fundamental result of A. Blanc in his Phd Thesis [12, Prop. 4.6], namely, that the split version of the Waldhausen S-construction is $\mathbb{A}^1$-homotopy equivalent to the full S-construction.

The following corollary provides a new version of a result understood by Kontsevich [55,54] long ago and also already satisfied by the formalism of Cisinski–Tabuada.

Corollary 1.9. *(See Corollary 4.8.) Let $\mathcal{X}$ and $\mathcal{Y}$ be two noncommutative smooth spaces and assume that $\mathcal{Y}$ is smooth and proper. Then we have an equivalence of spectra*

$$Map_{\mathcal{SH}_{nc}(k)}(\mathcal{X},\mathcal{Y}) \simeq l^{nc}_{\mathbb{A}^1}(K^S)(\mathcal{X}\otimes\mathcal{Y}^{op})$$

where $\mathcal{Y}^{op}$ is the dual of $\mathcal{Y}$ and we have identified $\mathcal{X}$ and $\mathcal{Y}$ with their images in $\mathcal{SH}_{nc}(k)$.

Finally, the following result is the main goal of this work. It follows from the previous theorem together with the results of J. Riou describing the compact generators in $\mathcal{SH}(k)$ over a field with resolutions of singularities (see [76]).

Corollary 1.10. *(See Corollary 4.12.) If k is a field admitting resolutions of singularities then the canonical factorization*

$$Mod_{KH}(\mathcal{SH}(k))^{\otimes} \dashrightarrow \mathcal{SH}_{nc}(k)^{\otimes}$$

is fully faithful.

This result has been expected and known to some people after a while. I think particularly of B. Toen, M. Vaquié and G. Vezzosi and also D.-C. Cisinski and G. Tabuada.

1.3. Further applications

Let us provide some further applications of our work.

The first application we would like to mention concerns the study of motives of Deligne–Munford stacks. If X is a Deligne–Munford stack over a field k of characteristic zero then our methods allow us to assign to it an object in $\mathcal{SH}(k)$. Namely, X has a naturally associated non-commutative motive – the non-commutative motive of the dg-category $L_{pe}(X)$. Our framework allows us to restrict this object to the commutative world and produce a module over K-theory, $\mathcal{M}(L_{pe}(X)) \in \mathcal{SH}(k)$. In particular, we hope that the decomposition of the inertia stack of X used in [95] can be also applied in this motivic context. This would provide a decomposition of the non-commutative motive of X in terms of pieces of geometric origin. The advantage of this decomposition is that it does not depend on any assumptions on the existence of semi-orthogonal decompositions for $L_{pe}(X)$. The important new ingredient is the fully-faithfulness of the bridge between the two motivic worlds (Corollary 4.12).

The second application concerns the construction of a non-commutative mixed Hodge realization functor. In the commutative case this was studied in the [27] already using the new universal property proved in this work (Corollary 2.39). In [51], the authors introduced the notion of a *noncommutative Hodge Structure*. Recall that thanks to the famous theorem HKR, the *Periodic Cyclic Homology* $HP_{\bullet}(X)$ provides the correct noncommutative analogue of the classical de Rham cohomology. They formulate the following conjecture:

(*) If X is a "good enough" noncommutative space then $HP_{\bullet}(X)$ carries a noncommutative Hodge Structure;

Said in a different way, $HP_{\bullet}$ should provide a functor from noncommutative spaces to noncommutative Hodge-structures. We should then expect this functor to factor through our new noncommutative version of the motivic stable homotopy theory because of its universal property. More generally, we expect our main commutative diagram to fit in a larger one

$$\begin{tikzcd}
\text{Classical Schemes}/k \arrow[rr, "L_{pe}"] \arrow[d] \arrow[ddr, "H_{DR}(-)"] & & \text{NC-Schemes}/k \arrow[d] \arrow[ddr, "HP_\bullet(-)"] & \\
\text{Stable Motivic Homotopy}/k \arrow[rr, dashed] \arrow[dr, dashed, "univ", "prop." '] & & \text{NC-Stable Motivic Homotopy}/k \arrow[dr, dashed, "univ", "prop." '] & \\
& \text{Classical Hodge-Structures} \arrow[rr] & & \text{NC-Hodge Structures}
\end{tikzcd} \tag{1.6}$$

where the map from the classical to the noncommutative Hodge structures was introduced in [51].[4] The diagonal maps are known as the *Hodge-realizations functors*: the commutative case is known to the experts (see [78] for a survey of the main results); the noncommutative case is given by the conjecture (∗). This conjecture can be divided in two parts: the first concerns the de Rham part (see [48]) and the second is related to the Betti part. In his thesis A. Blanc constructed a candidate for the second [12].

1.4. Acknowledgments and credits

This paper contains the results of Parts I and II of the author's Doctoral Thesis [79] under the direction of Bertrand Toën. I want to express and emphasize my sincere admiration, gratitude and mathematical debt to Bertrand. For accepting me as his student, for proposing me such an amazing quest, and for, so kindly, sharing and discussing his ideas and beautiful visions with me. Also, I want to thank him for all the comments and suggestions improving the text here presented.

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I also want to acknowledge the deep influence of the colossal works of Jacob Lurie in the subject of higher algebra. I have learned a lot from his writings and of course, this work depends heavily and continuously on his results and techniques.

I'm also thankful to the two anonymous referees for their valuable detailed comments and corrections.

[4] Of course we should only expected the part of the diagram concerning the Hodge Theory to work if we restrict to a good class of schemes over k.

In the process of preparing this work I have benefited from various conversations with several mathematicians. I'm very grateful to Denis-Charles Cisinski for answering some of my naive questions about motives and the $\mathbb{A}^1$-localization. I'm also deeply grateful to Anthony Blanc, Benjamin Hennion, Brad Drew, Claudia Scheimbauer, Damien Calaque, Dimitri Ara, Eduard Balzin, Francois Petit, Georges Maltsiniotis, Mathieu Anel, Mauro Porta, Nick Rozenblyum, Samuel Bach and Valerio Melani. For the mathematical discussions from which I have learned so much.

1.5. Notations and preliminaries

In this paper we will be using the theory of higher categories and the techniques of higher algebra as developed by J. Lurie in [59,63]. More precisely, the reader is assumed to be familiar with the theory of ∞-operads, symmetric monoidal $(\infty, 1)$-categories, their theories of algebras and modules. For a review of the techniques necessary along the paper we address the reader to our preliminaries in [79]. We will also be using many preliminary results from [79] whose proof will not be given here. We summarize them below.

1.5.1. Universes

We start this section of preliminaries with some set theoretical considerations. In order to deal with the matters of size, we will follow the approach of Universes[5] by fixing three of them $\mathbb{U} \in \mathbb{V} \in \mathbb{W}$ with $\mathbb{V}$ chosen conveniently large and $\mathbb{W}$ very large. We will denote by $\widehat{\Delta}$ the category of simplicial sets. More precisely, we will be working with $\mathbb{V}$-small simplicial sets and the $\mathbb{U}$-small objects will be referred to simply as *small*. In order to simplify the notations we write Cat_∞ (resp. $\mathcal{S}$, Sp) to denote the $(\infty, 1)$-category of small $(\infty, 1)$-categories (resp. spaces, spectra). With our convenient choice for $\mathbb{V}$, both of them are $\mathbb{V}$-small. The third universe $\mathbb{W}$ is assumed to be sufficiently large so that we have $\mathbb{W}$-small simplicial sets Cat_∞^{big} (resp. $\widehat{\mathcal{S}}$, $\widehat{Sp}$) to encode the $(\infty, 1)$-category of all the $\mathbb{V}$-small $(\infty, 1)$-categories (resp. spaces, spectra).

1.5.2. From model categories to $(\infty, 1)$*-categories*

For the bridge between the setting of model categories and the world of $(\infty, 1)$-categories we address the reader to [63, 1.3.4] and to our discussion in [79, Section 2.2]. We recall that the nerve functor $N : Cat \to \widehat{\Delta}$ provides a natural way to understand a category as a simplicial set and that if $\mathcal{M}$ is a model category with weak-equivalences W, the underlying $(\infty, 1)$-category associated to $\mathcal{M}$ is the quasi-category $N(\mathcal{M})[W^{-1}]$ obtained by localizing the nerve $N(\mathcal{M})$ along the weak-equivalences. This localization is performed in the setting of $(\infty, 1)$-categories and it is well-defined up to equivalence. More precisely, it can be obtained as a fibrant-replacement for the pair $(N(\mathcal{M}), W)$ in the model category of marked simplicial sets of [59, Chapter 3]. See the construction

[5] Our main reference being the Appendix by Nicolas Bourbaki in [3].

[63, 4.1.3.1] for more details. We recall also that in the case of a simplicial model category $\mathcal{M}$, the quasi-category obtained as the simplicial nerve (which we will denote as N_Δ) of the full simplicial subcategory of $\mathcal{M}$ spanned by the cofibrant–fibrant objects is a model for the underlying $(\infty, 1)$-category of $\mathcal{M}$ (see [63, 1.3.4.20]).

When restricted to combinatorial model categories, this bridge sends Quillen functors to adjunctions between the underlying $(\infty, 1)$-categories and Quillen equivalences to equivalences (see [63, 1.3.4.21]). More important to us is the link between combinatorial model categories and presentable $(\infty, 1)$-categories.

Proposition 1.11. *(See [59, A.3.7.4, A.3.7.6].) Let $\mathcal{C}$ be a big $(\infty, 1)$-category. Then, $\mathcal{C}$ is presentable if and only if there exists a big $\mathbb{U}$-combinatorial simplicial model category $\mathcal{M}$ such that $\mathcal{C}$ is the underlying $(\infty, 1)$-category of $\mathcal{M}$. Moreover, if $\mathcal{M}$ is left-proper, left Bousfield localizations of $\mathcal{M}$[6] correspond bijectively to accessible reflexive localizations of $\mathcal{C}$.*

Thanks to the results of [30] we know that every combinatorial model category is Quillen equivalent (by a zig-zag) to a simplicial combinatorial model category. The proposition implies that the underlying $(\infty, 1)$-category of a combinatorial model category is always presentable. In particular, it admits all limits and colimits. By [63, 1.3.4.23, 1.3.4.24] we know that homotopy limits and homotopy colimits in the model category correspond to the notions of limit and colimit in the underlying $(\infty, 1)$-category. Moreover, combining [59, Thm. 4.2.4.4] again with the main result of [30] we find that for any combinatorial model category $\mathcal{M}$ and small discrete 1-category I, there is an equivalence

$$N(\mathcal{M}^I)\big[W_{levelwise}^{-1}\big] \simeq Fun\big(N(I), N(\mathcal{M})\big[W^{-1}\big]\big) \tag{1.7}$$

providing a strictification of diagrams. We will explicitly warn the reader everytime this strictification is being used.

To conclude, we ask the reader to recall the notion of a compactly generated model category and the fact (see [100, Prop. 2.2]) that in any such model category, any object is equivalent to a filtered colimit of strict finite I-cell objects. Moreover, if the (strict) filtered colimits in $\mathcal{M}$ are exact, an object X is homotopically finitely presented if and only if it is a retract of a strict finite I-cell object. This result, together with the results of [63] described above, implies that if $\mathcal{M}$ is a combinatorial compactly generated model category where (strict) filtered colimits are exact, then the compact objects in the presentable $(\infty, 1)$-category $N(\mathcal{M})[W^{-1}]$ are exactly the homotopically finitely presented objects in $\mathcal{M}$. In this case we have a canonical equivalence $N(\mathcal{M})[W^{-1}] \simeq Ind((N(\mathcal{M})[W^{-1}])^\omega)$.

We will also use the monoidal extension of this bridge. If $\mathcal{M}$ is symmetric monoidal model category then the underlying ∞-category of $\mathcal{M}$ inherits a canonical symmetric monoidal structure which we denote here as $N(\mathcal{M})[W^{-1}]^\otimes \to N(Fin_*)$. It can be ob-

[6] With respect to a class of morphisms of small generation.

tained as follows: in a monoidal model category $\mathcal{M}$, the collection of cofibrant objects is closed under the tensor structure and we assume by definition that the unit is cofibrant. In this case, the full subcategory $\mathcal{M}^c \subseteq \mathcal{M}$ spanned by the cofibrant objects inherits the structure of a symmetric monoidal category. Moreover, restricted to this subcategory, the tensor product preserves weak-equivalences. By taking its operadic nerve (see [63, Def. 2.1.1.23]) we obtain a trivial symmetric monoidal $(\infty, 1)$-category. The symmetric monoidal $(\infty, 1)$-category associated to $\mathcal{M}$ is the monoidal localization (see [63, Construction 4.1.3.1, Prop. 4.1.3.4]) of this operadic nerve along the class of weak-equivalences in $\mathcal{M}$. If $\mathcal{M}$ comes equipped with a *compatible simplicial enrichment*, then $\mathcal{M}^\circ$, although not a simplicial monoidal category (because the product of fibrant objects is not fibrant in general), can be seen as the underlying category of a simplicial colored operad $(\mathcal{M}^\circ)^\otimes$ where the colors are the cofibrant–fibrant objects in $\mathcal{M}$ and the operation space is given by

$$Map_{(\mathcal{M}^\circ)^\otimes}\big(\{X_i\}_{i\in I}, Y\big) := Map_{\mathcal{M}}\left(\bigotimes_i X_i, Y\right) \tag{1.8}$$

which is a Kan-complex because Y is fibrant and the product of cofibrant objects is cofibrant. With this, we consider the ∞-operad given by the operadic nerve $N_\Delta^\otimes((\mathcal{M}^\circ)^\otimes)$ (see [63, Def. 2.1.1.23]). By [63, 4.1.3.10], this ∞-operad is a symmetric monoidal $(\infty, 1)$-category and the product of cofibrant–fibrant objects X, Y is given by the choice of a trivial cofibration $X \otimes Y \to Z$ providing a fibrant replacement for the product in $\mathcal{M}$. Moreover, by [63, 4.1.3.16] the symmetric monoidal $(\infty, 1)$-category $N_\Delta^\otimes((\mathcal{M}^\circ)^\otimes)$ is monoidal equivalent to the underlying symmetric monoidal $(\infty, 1)$-category of $\mathcal{M}$. A particular instance of this is when $\mathcal{M}$ is a cartesian closed combinatorial simplicial model category with a cofibrant final object. In this case, it is a symmetric monoidal model category with respect to the product and we can consider its operadic nerve $N_\Delta^\otimes((\mathcal{M}^\circ)^\times)$. From [63, Example 2.4.1.10], this is equivalent to a cartesian structure in the underlying ∞-category of $\mathcal{M}$ – $N_\Delta(\mathcal{M}^\circ)^\times$. To conclude we remark also that monoidal left Quillen functors induce strong monoidal functors between the underlying symmetric monoidal $(\infty, 1)$-categories. See [63, Section 4.1.3] and our discussion in [79, Section 3.9].

1.5.3. Stable $(\infty, 1)$*-categories and compact generators*

We will also need some results about stable $(\infty, 1)$-categories and the existence of compact generators therein. First we recall that any stable $(\infty, 1)$-category $\mathcal{C}$ is naturally enriched over spectra. This can be made precise using the universal property of the stabilization [63, 1.4.2.2]: it provides for any object X an essentially unique factorization of the functor $Map_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathcal{S}$ as

$$\begin{tikzcd} \mathcal{C} \arrow[r, "Map_{\mathcal{C}}(X,-)"] \arrow[d, dashed, "Map^{Sp}_{\mathcal{C}}(X,-)"'] & \mathcal{S} \\ Sp \arrow[ur, "\Omega^\infty"'] & \end{tikzcd} \tag{1.9}$$

with Ω^∞ the ∞-looping functor.

Let κ be a regular cardinal. Recall that a presentable $(\infty,1)$-category $\mathcal{C}$ is said to be *κ-compactly generated* if there exists a small $(\infty,1)$-category $\mathcal{D}$ and an equivalence $\mathcal{C} \simeq Ind_\kappa(\mathcal{D})$. For stable $(\infty,1)$-categories there is another possible notion of compact generation: if $\mathcal{C}$ is stable, its homotopy category is triangulated and therefore it makes sense to ask for a family of κ-compact generators in the sense of Neeman [70]. These two notions of compact generation are related. More precisely, one can prove the following version of the Proposition [63, 1.4.4.2] relating the two notions of compact generation:

Proposition 1.12. *Let $\mathcal{C}$ be a stable $(\infty,1)$-category. Then, $\mathcal{C}$ is presentable if and only if the following conditions are satisfied:*

(i) *$\mathcal{C}$ has arbitrary small coproducts*[7]*;*
(ii) *the triangulated category $h(\mathcal{C})$ is locally small;*
(iii) *there exists a regular cardinal κ and a (classical) family $\mathcal{E}$ (indexed by a small set) of κ-compact generators in $h(\mathcal{C})$. In this case $\mathcal{C}$ is a presentable κ-compactly generated $(\infty,1)$-category.*

Proof. The proof follows essentially by the same arguments as in [63, 1.4.4.2] and by comparison of the notions of compact generating families. For a detailed proof see [79, Prop. 2.1.2]. □

To conclude this section we present another useful result whose proof can be found in [79, Prop. 2.1.7]:

Proposition 1.13. *Let $f : \mathcal{C} \to \mathcal{D}$ be colimit preserving functor between stable presentable $(\infty,1)$-categories. Assume that*

(i) *The $(\infty,1)$-category $\mathcal{C}$ has a family of ω-compact generators $\mathcal{E}$ in the sense of Proposition 1.12 (here we assume, without loss of generality, that $\mathcal{E}$ is closed under suspensions and loopings*[8]*) and f is fully-faithful when restricted to the objects in this collection;*
(ii) *for any object $E \in \mathcal{E}$, the object $f(E)$ is ω-compact in $\mathcal{D}$.*

[7] Since $\mathcal{C}$ is stable this is equivalent to ask for all small colimits.
[8] We can always assume this because, as discussed in the previous footnote, suspensions of compact objects are compact.

Then, f is fully-faithful. Moreover, if the image of the collection $\mathcal{E}$ in $\mathcal{D}$ is a family of ω-compact generators, then f is an equivalence.

1.5.4. Exact sequences and localizations

Recall now that a sequence of triangulated categories $A \to C \to D$ is said to be exact if the composition is zero, the first map is fully-faithful and the inclusion from the Verdier quotient $C/A \hookrightarrow D$ is cofinal, meaning that every object in D is a direct summand of an object in B/A.

Following [13], we say that a sequence in $\mathcal{P}r^L_{Stb}$

$$\mathcal{A} \to \mathcal{C} \to \mathcal{D} \tag{1.10}$$

is exact if the composition is zero, the first map is fully-faithful and the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[d] \\ * \arrow[r] & \mathcal{D} \end{tikzcd} \tag{1.11}$$

is a pushout. Here we denote by $*$ the final object in $\mathcal{P}r^L_{Stb}$. As proved in [13, Prop. 4.5, Prop. 4.6], this notion of exact sequence can be reformulated using the language of localizations: if $\phi : \mathcal{A} \hookrightarrow \mathcal{C}$ is a fully-faithful functor, the cofiber of ϕ can be identified with the accessible reflexive localization

$$\begin{tikzcd} \mathcal{D} \arrow[r, hook] & \mathcal{C} \arrow[l, bend right] \end{tikzcd} \tag{1.12}$$

with local equivalences given by the class of edges f in $\mathcal{C}$ with cofiber in the essential image of ϕ. In particular, an object $x \in \mathcal{C}$ is in $\mathcal{D}$ if and only if for every object $a \in \mathcal{A}$ we have $Map_{\mathcal{C}}(a, x) \simeq *$.

Remark 1.14. Let $\mathcal{A} \hookrightarrow \mathcal{C} \to \mathcal{D}$ be an exact sequence of presentable stable $(\infty, 1)$-categories as above. If the homotopy category $h(\mathcal{A})$ has a compact generator in the sense of Neeman, say $k \in \mathcal{A}$, then for an object $x \in \mathcal{C}$ to be in $\mathcal{D}$ it is enough to have $Map_{\mathcal{C}}(k, a) \simeq *$. This follows from the arguments in the proof of the Proposition 1.12: every object in $\mathcal{A}$ can be obtained as a colimit of suspensions of k.

Thanks to [13, Prop. 5.9] and to the arguments in the proof of [13, Prop. 5.13], this notion of exact sequence extends the notion given by Verdier in [104]: a sequence $\mathcal{A} \hookrightarrow \mathcal{C} \to \mathcal{D}$ in $\mathcal{P}r^L$ is exact if and only if the sequence of triangulated functors $h(\mathcal{A}) \hookrightarrow h(\mathcal{C}) \to h(\mathcal{D})$ is exact sequence in the classical sense and the inclusion $h(\mathcal{C})/h(\mathcal{A}) \hookrightarrow h(\mathcal{D})$ is an equivalence of triangulated categories. In the compactly generated case we have the following

Proposition 1.15. *Let $\mathcal{A} \hookrightarrow \mathcal{C} \to \mathcal{D}$ be a sequence in $\mathcal{P}r^L_{\omega,Stb}$. The following are equivalent:*

1. *the sequence is exact;*
2. *the induced sequence of triangulated functors $h(\mathcal{A}) \hookrightarrow h(\mathcal{C}) \to h(\mathcal{D})$ is exact in the classical sense and the inclusion $h(\mathcal{C})/h(\mathcal{A}) \hookrightarrow h(\mathcal{D})$ is an equivalence;*
3. *the sequence of triangulated functors induced between the homotopy categories of the associated stable subcategories of compact objects $h(\mathcal{A}^\omega) \hookrightarrow h(\mathcal{C}^\omega) \to h(\mathcal{D}^\omega)$ is exact in the classical sense.*

Proof. The equivalence between 1) and 2) follows from the results of [13] discussed above. The equivalence between 2) and 3) follows from the results of B. Keller [52, Section 4.12, Corollary] and the fact that for any compactly generated stable $(\infty,1)$-category $\mathcal{C}$ we can identify $h(\mathcal{C}^\omega)$ with the full subcategory of compact objects (in the sense of Neeman) in $h(\mathcal{C})$. □

The following result will become important in the last section of our work. It follows as general form of the usual Bondal–Van den Bergh argument:

Proposition 1.16. *(See [79, Prop. 2.1.10].) Let*

$$\begin{tikzcd} & \mathcal{D} \arrow[d, "f"] \\ \mathcal{C} \arrow[r, "L"] & \mathcal{C}_0 \end{tikzcd} \tag{1.13}$$

be a diagram in $\mathcal{P}r^L_{\omega,Stb}$ such that

- *The homotopy triangulated category $h(\mathcal{D})$ has a compact generator in the sense of Neeman;*
- *The map $L : \mathcal{C} \to \mathcal{C}_0$ is an accessible reflexive localization of $\mathcal{C}$ obtained by killing a stable subcategory $\mathcal{A} \subseteq \mathcal{C}$ such that $h(\mathcal{A})$ has a compact generator (in the sense of Neeman) and the inclusion $\mathcal{A} \subseteq \mathcal{C}$ is a map in $\mathcal{P}r^L_{\omega,Stb}$.*

Then:

1. *the diagram admits a limit $\sigma =$*

$$\begin{tikzcd} \mathcal{T} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d, "f"] \\ \mathcal{C} \arrow[r, "L"] & \mathcal{C}_0 \end{tikzcd} \tag{1.14}$$

in $\mathcal{P}r^L_{\omega,Stb}$;

2. *the diagram σ remains a pullback after the (non-full) inclusion $\mathcal{P}r^L_{\omega,Stb} \subseteq \mathcal{P}r^L_{Stb}$;*
3. *the homotopy category $h(\mathcal{T})$ has compact generator in the sense of Neeman.*

Remark 1.17. The proof of Proposition 1.16 works mutatis mutandis if we replace the hypothesis of solo compact generators in $\mathcal{A}$ and $\mathcal{D}$ by the existence of compact generating families. More precisely, and using the same arguments and notations, if $\mathcal{E}_{\mathcal{D}} = \{d_i\}_{i\in I}$ and $\mathcal{E}_{\mathcal{A}} = \{k_j\}_{j\in J}$ are families of compact generators respectively in $\mathcal{D}$ and in $\mathcal{A}$, we can prove that the family $\{\tilde{k}_j \oplus v_i\}_{(i,j)\in I\times J}$ is a family of compact generators in $\mathcal{T}$.

In particular, we have the following immediate corollary, obtained as a degenerated case of Proposition 1.16 (together with Remark 1.17) where $\mathcal{D} = 0$:

Corollary 1.18. *Let $\sigma =$*

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[d] \\ * \arrow[r] & \mathcal{C}_0 \end{tikzcd} \tag{1.15}$$

be an exact sequence in $\mathcal{P}r^L_{\omega,Stb}$ such that $h(\mathcal{A})$ admits a family of compact generators in the sense of Neeman. Then, the diagram σ is a pullback in $\mathcal{P}r^L_{\omega,Stb}$.

1.5.5. Higher algebra over k

In the second and third parts of this paper we will be working over a base commutative ring k. We denote by $\mathcal{D}(k)$ the derived $(\infty,1)$-category of k. By definition, it is the underlying $(\infty,1)$-category of the combinatorial model structure on complexes of k-modules, with the weak-equivalences given by the quasi-isomorphisms and fibrations given by the levelwise surjections. This model category admits a compatible tensor product [42, 4.2.13] so that by the discussion above $\mathcal{D}(k)$ inherits a symmetric monoidal structure corresponding to the derived tensor product of complexes. Moreover, this monoidal structure is closed. We shall use that this $(\infty,1)$-category is stable and that it admits a natural both left and right complete t-structure where $\mathcal{D}(k)_{\geq 0}$ is the full subcategory spanned by those complexes with zero homology in negative degrees. Its heart $\mathcal{D}(k)^{\heartsuit}$ is the category of k-modules and the functors $\mathbb{H}_n := \tau_{\leq n} \circ \tau_{\geq n} : \mathcal{D}(k) \to \mathcal{D}(k)^{\heartsuit}$ correspond to the classical n-th-cohomology functors. Moreover, the general Kunneth formula for complexes implies that the monoidal structure is compatible with this t-structure.

Let $Alg(\mathcal{D}(k))$ denote the $(\infty,1)$-category of associative algebra objects in $\mathcal{D}(k)$. It is equivalent to the underlying $(\infty,1)$-category associated to the model structure on strict algebra objects in the category of complexes of k-modules (also known as k-dg-algebras – see [63, 6.1.4.5]). We write $Alg(\mathcal{D}(k))^{cn}$ for the full subcategory of $Alg(\mathcal{D}(k))$ spanned by those algebra-objects whose underlying complex is in $\mathcal{D}(k)_{\geq 0}$. As the t-structure is compatible with the monoidal structure, $\mathcal{D}(k)_{\geq 0}$ is closed under tensor products and we have an equivalence $Alg(\mathcal{D}(k)_{\geq 0}) \simeq Alg(\mathcal{D}(k))^{cn}$. Finally, the left-completeness implies that for any ∞-operad $\mathcal{O}^{\otimes}$, we have $\tau_{\leq n} Alg_{\mathcal{O}}(\mathcal{D}(k))^{cn} \simeq Alg_{\mathcal{O}}(\mathcal{D}(k)_{\geq 0} \cap \mathcal{D}(k)_{\leq n})$ and that Postnikov towers converge (see [59, 5.5.6.23 and 5.5.6.26])

$$Alg_{\mathcal{O}}\big(\mathcal{D}(k)\big)^{cn} \simeq lim_n Alg_{\mathcal{O}}\big(\mathcal{D}(k)_{\geq 0} \cap \mathcal{D}(k)_{\leq n}\big) \tag{1.16}$$

See [63, Sections 2.2.1, 7.1.1 and 7.1.2].

1.5.6. Cotangent complexes and square-zero extensions

In the last part of the paper we construct a functor L_{pe} connecting the classical theory of schemes to the noncommutative world (see Proposition 3.4). The main step – Lemma 3.5 – is a noncommutative analogue of [103, Prop. 2.2.2.4] and [63, 7.4.3.18]. In order to prove this result we will need to say what is the cotangent complex of a connective dg-algebra. This is a particular instance of the more general notion of cotangent complex given in [34] which has sense for any $\mathcal{O}$-algebra object in a stable symmetric monoidal $(\infty, 1)$-category $\mathcal{C}^{\otimes}$ compatible with small colimits. More precisely, let $\mathcal{C}^{\otimes}$ be a stable symmetric monoidal $(\infty, 1)$-category compatible with colimits. Let $\mathcal{O}^{\otimes}$ be a κ-small coherent ∞-operad and let $A \in Alg_{\mathcal{O}}(\mathcal{C})$ be an algebra-object in $\mathcal{C}$. Given a module-object $M \in Mod_A^{\mathcal{O}}(\mathcal{C})$ and using the hypothesis that the monoidal structure is compatible with colimits, the direct sum $A \oplus M$ comes naturally equipped with the structure an $\mathcal{O}$-algebra-object in $\mathcal{C}$ where the multiplication is determined by the multiplicative structure on A, the module action of A on M and the zero map $M \otimes M \to M$. This new $\mathcal{O}$-algebra-object comes naturally equipped with a morphism of $\mathcal{O}$-algebras $A \oplus M \to A$ which we can informally describe via the formula $(a, m) \to a$. Its fiber can be naturally identified with the module M. This construction provides a functor (see [34]-Theorem 3.4.2)

$$Mod_A^{\mathcal{O}}(\mathcal{C}) \to Alg_{\mathcal{O}}(\mathcal{C})_{./A} \tag{1.17}$$

By definition, a derivation of A into M is the data of a morphism of $\mathcal{O}$-algebras $A \to A \oplus M$ over A. It is an easy exercise to see that this notion recovers the classical definition using the Leibniz rule. We set $Der(A, M) := Map_{Alg_{\mathcal{O}}(\mathcal{C})_{./A}}(A, A \oplus M)$ to denote the space of derivations with values in M. The formula $M \mapsto Der(A, M)$ provides a functor $(Mod_A^{\mathcal{O}}(\mathcal{C}))^{op} \to \mathcal{S}$ which, through the Grothendieck construction, corresponds to a left fibration over $Mod_A^{\mathcal{O}}(\mathcal{C})$. By definition, the (absolute) *cotangent complex of* A is an object $\mathbb{L}_A \in Mod_A^{\mathcal{O}}(\mathcal{C})$ which makes this left fibration representable. In other words, if it has the universal property

$$Map_{Mod_A^{\mathcal{O}}(\mathcal{C})}(\mathbb{L}_A, M) \simeq Map_{Alg_{\mathcal{O}}(\mathcal{C})_{./A}}(A, A \oplus M) \tag{1.18}$$

which allows us to understand the formula $A \mapsto \mathbb{L}_A$ as a left adjoint L_A to the functor in (1.17), evaluated in A. In particular, if $\mathcal{C}$ is presentable this left adjoint exists because of the adjoint functor theorem together with the fact that (1.17) commutes with limits [34, Lemma 3.1.3]. Moreover, under the equivalence between modules and the stabilization of algebras, L_A can be identified with the suspension functor Σ^∞.

Example 1.19. When applied to the example $\mathcal{C}^\otimes = \mathcal{D}(k)^\otimes$ and for $\mathbb{E}_1 \simeq \mathcal{A}ss$, this definition recovers the classical associative cotangent complex introduced by Quillen and studied in [58], given by the kernel of the multiplication map $A \otimes_k A^{op} \to A$ in the $(\infty, 1)$-category $Mod_A^{\mathcal{A}ss}(\mathcal{D}(k))$. Recall also that $Mod_A^{\mathcal{A}ss}(\mathcal{D}(k))$ is equivalent to ${}_A BMod_A(\mathcal{D}(k))$ which by the strictification results [63, 4.3.3.15, 4.3.3.17] is equivalent to the underlying $(\infty, 1)$-category of the model category of strict A-bimodules in the model category of complexes $Ch(k)$. This example will play an important role in the last section of this paper.

Remark 1.20. The construction of cotangent complexes is well-behaved with respect to base-change. If $f : A \to A'$ is a morphism of $\mathcal{O}$-algebras we can put together the functors $A \oplus -$ and $A' \oplus -$ in a diagram

$$\begin{tikzcd} Mod_A^{\mathcal{O}}(\mathcal{C}) \arrow[r, "A\oplus -"] & Alg_{\mathcal{O}}(\mathcal{C})_{./A} \\ Mod_{A'}^{\mathcal{O}}(\mathcal{C}) \arrow[u, "For"] \arrow[r, "A'\oplus -"] & Alg_{\mathcal{O}}(\mathcal{C})_{./A'} \arrow[u, "(-\times_{A'} A)"'] \end{tikzcd} \tag{1.19}$$

where For is the map that considers an A'-module as an A-modules via f and the map $(- \times'_A A)$ is obtained by computing the fiber product of a morphism $C \to A'$ with respect to f. The fact that this diagram commutes follows from the equivalence relating modules and the stabilization of algebras and from the definition of *tangent bundle* studied in [63, Section 7.3.1]. Moreover, the commutativity of this diagram implies the commutativity of the diagram associated to the left adjoints

$$\begin{tikzcd} Mod_A^{\mathcal{O}}(\mathcal{C}) \arrow[d, "A'\otimes_A -"] & Alg_{\mathcal{O}}(\mathcal{C})_{./A} \arrow[l, "L_A"'] \arrow[d, "f\circ -"] \\ Mod_{A'}^{\mathcal{O}}(\mathcal{C}) & Alg_{\mathcal{O}}(\mathcal{C})_{./A'} \arrow[l, "L_{A'}"'] \end{tikzcd} \tag{1.20}$$

where now $A' \otimes_A -$ is the base change with respect to f and the $(f \circ -)$ is the map obtained by composing with f. In particular, we find that $A' \otimes_A \mathbb{L}_A$ is equivalent to $L_{A'}$ evaluated at $f : A \to A'$.

We recall also that the notion of derivation can be presented using the idea of a *square-zero extension*. If $d : A \to A \oplus M$ is a derivation, we fabricate a new $\mathcal{O}$-algebra $\tilde{A}$ as the pullback in $Alg_{\mathcal{O}}(\mathcal{C})$

$$\begin{tikzcd} \tilde{A} \arrow[r, "f"] \arrow[d] & A \arrow[d, "d"] \\ A \arrow[r, "d_0 0"] & A \oplus M \end{tikzcd} \tag{1.21}$$

where $d_0 : A \to A \oplus M$ is the zero derivation $a \mapsto (a, 0)$. Since the functor $Alg_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$ preserves limits, the diagram (1.21) provides a pullback diagram in $\mathcal{C}$ and given a morphism $* \to A$ in $\mathcal{C}$, we can identify the fiber $\tilde{A} \times_A *$ in $\mathcal{C}$ with the loop $\Omega(M)$. Indeed, we have a pullback in $\mathcal{C}$

$$\begin{tikzcd} \tilde{A} \times_A * \arrow[r, "f"] \arrow[d] & (A \times_A *) \simeq * \arrow[d, "d"] \\ * \simeq (A \times_A *) \arrow[r, "d_0"] & (A \oplus M) \times_A * \end{tikzcd} \tag{1.22}$$

and since the fiber of the canonical map $A \oplus M \to A$ can be identified with M, we find $\tilde{A} \times_A * \simeq \Omega(M)$.

Recall that a morphism of algebras $B \to A$ is said to be a *square-zero extension of A by $\Omega(M)$* if there is a derivation d of A with values in $M \simeq \Sigma(\Omega(M))$ such that $B \simeq \tilde{A}$. Thanks to [63, Theorem 7.4.1.26] if $\mathcal{C}^{\otimes}$ is a stable presentable $\mathbb{E}_k$-monoidal $(\infty, 1)$-category with a compatible t-structure, then the formula $(A \to A \oplus M) \mapsto (f : \tilde{A} \to A)$ establishes an equivalence between the theory of derivations and the subcategory of $Fun(\Delta[1], Alg_{\mathbb{E}_k}(\mathcal{C}))$ spanned by the square-zero extensions (see [63, Section 7.4.1] for a precise formulation).

Remark 1.21. In the presence of a square-zero extension (1.21), every $\mathcal{O}$-algebra B induces a pullback diagram of spaces

$$\begin{tikzcd} Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B, \tilde{A}) \arrow[r] \arrow[d] & Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B, A) \arrow[d] \\ Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B, A) \arrow[r] & Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B, A \oplus M) \end{tikzcd} \tag{1.23}$$

Let $\phi : B \to A$ be a morphism of algebras. It follows that we can describe the fiber of the morphism $Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B, \tilde{A}) \to Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B, A)$ over the point corresponding to ϕ with the help of the cotangent complex of B. More precisely, we observe first that the mapping space $Map_{Alg_{\mathcal{O}}(\mathcal{C})_{./A}}(B, A \oplus M)$ (where B is defined over A via ϕ) fits in a pullback diagram

$$\begin{tikzcd}
Map_{Alg_{\mathcal{O}}(\mathcal{C})_{./A}}(B, A\oplus M) \arrow[r] \arrow[d] & Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B, A\oplus M) \arrow[d] \\
\Delta[0] \arrow[r, "\phi"] & Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B, A)
\end{tikzcd} \tag{1.24}$$

where the right vertical map is the composition with the canonical map $A \oplus M \to A$. By tensoring with $(- \times_{Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B,A)} \Delta[0])$ the diagram (1.23) produces a new pullback diagram

$$\begin{tikzcd}
Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B, \tilde{A}) \times_{Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B,A)} \Delta[0] \arrow[r] \arrow[d] & Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B, A) \times_{Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B,A)} \Delta[0] \simeq \Delta[0] \arrow[d] \\
Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B, A) \times_{Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B,A)} \Delta[0] \simeq \Delta[0] \arrow[r] & Map_{Alg_{\mathcal{O}}(\mathcal{C})_{./A}}(B, A\oplus M)
\end{tikzcd} \tag{1.25}$$

so that the fiber $Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B, \tilde{A}) \times_{Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B,A)} \Delta[0]$ becomes the space of paths in $Map_{Alg_{\mathcal{O}}(\mathcal{C})_{./A}}(B, A \oplus M)$ between the point $B \xrightarrow{\phi} A \xrightarrow{d} A \oplus M$ and the point $B \xrightarrow{\phi} A \xrightarrow{d_0} A \oplus M$. To conclude, we can use the adjunctions of Remark 1.20 to find equivalences

$$\begin{aligned} Map_{Alg_{\mathcal{O}}(\mathcal{C})_{./A}}(B, A\oplus M) &\simeq Map_{Mod_A^{\mathcal{O}}(\mathcal{C})}\big(L_A(\phi), M\big) \simeq Map_{Mod_A^{\mathcal{O}}(\mathcal{C})}(A \otimes_B \mathbb{L}_B, M) \\ &\simeq Map_{Mod_B^{\mathcal{O}}(\mathcal{C})}\big(\mathbb{L}_B, For(M)\big) \end{aligned} \tag{1.26}$$

so that we find an equivalence

$$Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B, \tilde{A}) \times_{Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B,A)} \Delta[0] \simeq \Omega_{0, d\circ\phi} Map_{Mod_B^{\mathcal{O}}(\mathcal{C})}\big(\mathbb{L}_B, For(M)\big) \tag{1.27}$$

We now collect the last ingredient to prove Lemma 3.5:

Theorem 1.22. *(See Lurie [63, Corollary 7.4.1.28].) Let $\mathcal{C}^{\otimes}$ be a stable presentable symmetric monoidal $(\infty,1)$-category equipped with a compatible t-structure. Then for every $k \geq 0$ and any algebra $A \in Alg_{\mathbb{E}_k}(\mathcal{C})^{cn}$ the morphisms in the Postnikov tower*

$$\ldots \to \tau_{\leq 2} A \to \tau_{\leq 1} A \to \tau_{\leq 0} A \tag{1.28}$$

are square-zero extensions. More precisely, for every $n \geq 0$ the truncation map $\tau_{\leq n} A \to \tau_{\leq n-1} A$ is a square-zero extension of $\tau_{\leq n-1} A$ by a module-structure in $\mathbb{H}_n(A)[n]$. This is equivalent to the existence of a derivation $d_n : \tau_{\leq n-1} A \to \tau_{\leq n-1} A \oplus \mathbb{H}_n(A)[n+1]$ and a pullback diagram of algebras

$$\begin{tikzcd}
\tau_{\leq n}A \arrow[r] \arrow[d] & \tau_{\leq n-1}A \arrow[d, "d_n"] \\
\tau_{\leq n-1}A \arrow[r] & \tau_{\leq n-1}A \oplus \mathbb{H}_n(A)[n+1]
\end{tikzcd} \tag{1.29}$$

2. A universal characterization of the motivic stable homotopy theory of schemes

In 2.1 we deal with the formal inversion of an object in a symmetric monoidal $(\infty, 1)$-category. First we deal with the situation for small $(\infty, 1)$-categories (Propositions 2.1 and 2.2) and then we extend the result to the presentable setting (Proposition 2.9). This method allow us to invert any object and the result is endowed with the expected universal property. In 2.2 we deal with the notion of spectrum-objects. Our main result (Corollary 2.22) is that if the object we want to invert satisfies a certain symmetry condition then the underlying $(\infty, 1)$-category of the formal inversion is nothing but the stabilization with respect to the chosen object. In 2.3 we prove our main theorem (see Theorem 2.26), which ensures that the familiar construction of symmetric spectrum objects with respect to a given symmetric object X together with the convolution product, is the "model category" incarnation of our ∞-categorical phenomenon of inverting X.

Finally, in Section 2.4 we use these results to provide a universal characterization for the Motivic Stable homotopy theory of Morel–Voevodsky.

2.1. Formal inversion of an object in a symmetric monoidal $(\infty, 1)$-category

Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal $(\infty, 1)$-category and let X be an object in $\mathcal{C}$. We will say that X is invertible with respect to the monoidal structure if there is an object X^* such that $X \otimes X^*$ and $X^* \otimes X$ are both equivalent to the unit object. Since the monoidal structure is symmetric, it is enough to have one of these conditions. It is an easy observation that this condition depends only on the monoidal structure induced on the homotopy category $h(\mathcal{C})$, because equivalences are exactly the isomorphisms in $h(\mathcal{C})$. Alternatively, we can see that an object X in $\mathcal{C}$ is invertible if and only if the map "multiplication by X" $= (X \otimes -) : \mathcal{C} \to \mathcal{C}$ is an equivalence of $(\infty, 1)$-categories. Indeed, if X has an inverse X^* then the maps $(X \otimes -)$ and $(X^* \otimes -)$ are inverses since the coherences of the monoidal structure can be used to fabricate the homotopies. Conversely, if $(X \otimes -)$ is an equivalence, the essential surjectivity provides an object X^* such that $X \otimes X^* \simeq \mathbf{1}_{\mathcal{C}}$. The symmetry provides an equivalence $\mathbf{1}_{\mathcal{C}} \simeq X^* \otimes X$.

Our main goal is to produce from the data of $\mathcal{C}^{\otimes}$ and $X \in \mathcal{C}$, a new symmetric monoidal $(\infty, 1)$-category $\mathcal{C}^{\otimes}[X^{-1}]$ together with a monoidal map $\mathcal{C}^{\otimes} \to \mathcal{C}^{\otimes}[X^{-1}]$ sending X to an invertible object and universal with respect to this property. In addition, we would like this construction to hold within the world of presentable symmetric monoidal $(\infty, 1)$-categories. Our steps follow the original ideas of [103], where the authors studied the inversion of an element in a strictly commutative algebra object in a symmetric monoidal model category.

We start by analyzing the theory for a small symmetric monoidal $(\infty,1)$-category $\mathcal{C}^\otimes$. In this case, and following [63, 2.4.2.6], $\mathcal{C}^\otimes$ can be identified with an object in $CAlg(Cat_\infty)$. The objects of $Mod_{\mathcal{C}}(Cat_\infty)$ can be identified with $(\infty,1)$-categories endowed with an "action" of $\mathcal{C}$ and we will refer to them simply as $\mathcal{C}^\otimes$-*Modules*. By [63, 3.4.1.7], $CAlg(Mod_{\mathcal{C}^\otimes}(Cat_\infty))$ is equivalent to $CAlg(Cat_\infty)_{\mathcal{C}^\otimes/.}$ where the objects are small symmetric monoidal $(\infty,1)$-categories $\mathcal{D}^\otimes$ equipped with a monoidal map $\mathcal{C}^\otimes \to \mathcal{D}^\otimes$. We denote by $CAlg(Cat_\infty)^X_{\mathcal{C}^\otimes/.}$ the full subcategory of $CAlg(Cat_\infty)_{\mathcal{C}^\otimes/.}$ spanned by the algebras $\mathcal{C}^\otimes \to \mathcal{D}^\otimes$ whose structure map sends X to an invertible object. The main observation is that the objects in $CAlg(Cat_\infty)^X_{\mathcal{C}^\otimes/.}$ can be understood as local objects in $CAlg(Cat_\infty)_{\mathcal{C}^\otimes/.}$ with respect to a certain set of morphisms: there is a forgetful functor

$$CAlg(Cat_\infty)_{\mathcal{C}^\otimes/.} \simeq CAlg\big(Mod_{\mathcal{C}^\otimes}(Cat_\infty)\big) \to Mod_{\mathcal{C}^\otimes}(Cat_\infty) \tag{2.1}$$

and since $Cat_\infty^\times$ is a presentable symmetric monoidal $(\infty,1)$-category, this functor admits a left adjoint $Free_{\mathcal{C}^\otimes}(-)$ assigning to each $\mathcal{C}^\otimes$-module $\mathcal{D}$ the free commutative $\mathcal{C}^\otimes$-algebra generated by $\mathcal{D}$ (see [63, 3.1.3.5]). We will denote by $\mathcal{S}_X$ the collection of morphisms in $CAlg(Cat_\infty)_{\mathcal{C}^\otimes/.}$ consisting of the single morphism

$$Free_{\mathcal{C}^\otimes}(\mathcal{C}) \xrightarrow{Free_{\mathcal{C}^\otimes}(X\otimes -)} Free_{\mathcal{C}^\otimes}(\mathcal{C}) \tag{2.2}$$

where $\mathcal{C}$ is understood as a $\mathcal{C}^\otimes$-module in the obvious way using the monoidal structure. We prove the following

Proposition 2.1. *Let $\mathcal{C}^\otimes$ be a symmetric monoidal $(\infty,1)$-category. Then the full subcategory $CAlg(Cat_\infty)^X_{\mathcal{C}^\otimes/.}$ coincides with the full subcategory of $CAlg(Cat_\infty)_{\mathcal{C}^\otimes/.}$ spanned by the $\mathcal{S}_X$-local objects. Moreover, since $Cat_\infty^\times$ is a presentable symmetric monoidal $(\infty,1)$-category, the $(\infty,1)$-categories $CAlg(Cat_\infty)$ and $CAlg(Cat_\infty)_{\mathcal{C}^\otimes/.}$ are also presentable (see Corollary 3.2.3.5 of [63]) and the results of Proposition 5.5.4.15 in [59] follow. We deduce the existence a left adjoint $\mathcal{L}^\otimes_{(\mathcal{C}^\otimes,X)}$*

$$\begin{tikzcd} CAlg(Cat_\infty)^{\mathcal{S}_X\text{-local}}_{\mathcal{C}^\otimes/} = CAlg(Cat_\infty)^X_{\mathcal{C}^\otimes/} \arrow[r, hook] & CAlg(Cat_\infty)_{\mathcal{C}^\otimes/} \arrow[l, bend right, "\mathcal{L}^\otimes_{(\mathcal{C}^\otimes,X)}"'] \end{tikzcd} \tag{2.3}$$

In particular, the data of this adjunction provides the existence of a symmetric monoidal $(\infty,1)$-category $\mathcal{L}^\otimes_{(\mathcal{C}^\otimes,X)}(\mathcal{C}^\otimes)$ equipped with a canonical monoidal map $f : \mathcal{C}^\otimes \to \mathcal{L}^\otimes_{(\mathcal{C}^\otimes,X)}(\mathcal{C}^\otimes)$ sending X to an invertible object.

Proof. The only thing to check is that both subcategories coincide. Let $\phi : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a $\mathcal{C}$-algebra where X is sent to an invertible object. By the definition of the functor $Free_{\mathcal{C}^{\otimes}}(\mathcal{C})$ we have a commutative diagram

$$\begin{tikzcd} Map_{CAlg(Cat_{\infty})_{\mathcal{C}^{\otimes}/}}(Free_{\mathcal{C}^{\otimes}}(\mathcal{C}), \mathcal{D}^{\otimes}) \arrow[r] \arrow[d, "\sim"] & Map_{CAlg(Cat_{\infty})_{\mathcal{C}^{\otimes}/}}(Free_{\mathcal{C}^{\otimes}}(\mathcal{C}), \mathcal{D}^{\otimes}) \arrow[d, "\sim"] \\ Map_{Mod_{\mathcal{C}^{\otimes}}(Cat_{\infty})}(\mathcal{C}, \mathcal{D}) \arrow[r] & Map_{Mod_{\mathcal{C}^{\otimes}}(Cat_{\infty})}(\mathcal{C}, \mathcal{D}) \end{tikzcd} \tag{2.4}$$

where the lower horizontal map is described by the formula $\alpha \mapsto \alpha \circ (X \otimes -)$. Since ϕ is monoidal, the diagram commutes

$$\begin{tikzcd} \mathcal{C} \arrow[r, "(X\otimes -)"] \arrow[d, "\phi"] & \mathcal{C} \arrow[d, "\phi"] \\ \mathcal{D} \arrow[r, "(\phi(X)\otimes -)"'] & \mathcal{D} \end{tikzcd} \tag{2.5}$$

and the lower map is in fact homotopic to the one given by the formula $\alpha \mapsto (\phi(X) \otimes -) \circ \alpha$. Since $\phi(X)$ is invertible in $\mathcal{D}^{\otimes}$, there exists an object λ in $\mathcal{D}$ such that the maps $(\phi(X) \otimes -)$ and $(\lambda \otimes -)$ are inverses and therefore the lower map in (2.4), and as a consequence the top map, are isomorphisms of homotopy types.

Let now $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a $\mathcal{C}^{\otimes}$-algebra, local with respect to $\mathcal{S}_X$. In particular, the map

$$Map_{Mod_{\mathcal{C}^{\otimes}}(Cat_{\infty})}(\mathcal{C}, \mathcal{D}) \to Map_{Mod_{\mathcal{C}^{\otimes}}(Cat_{\infty})}(\mathcal{C}, \mathcal{D}) \tag{2.6}$$

induced by the composition with $(X \otimes -)$ is an isomorphism of homotopy types and in particular we have $\pi_0(Map_{Mod_{\mathcal{C}^{\otimes}}(Cat_{\infty})}(\mathcal{C}, \mathcal{D})) \simeq \pi_0(Map_{Mod_{\mathcal{C}^{\otimes}}(Cat_{\infty})}(\mathcal{C}, \mathcal{D}))$. We deduce the existence of a dotted arrow

$$\begin{tikzcd} \mathcal{D} \arrow[r, "X\otimes -"] \arrow[d, "\phi"] & \mathcal{D} \arrow[dl, dotted, "\alpha"] \\ \mathcal{D} & \end{tikzcd} \tag{2.7}$$

rendering the diagram of modules commutative and since α is a map of $\mathcal{C}^{\otimes}$-modules and ϕ is monoidal we find $\phi(1) \simeq \alpha(X \otimes 1) \simeq \phi(X) \otimes \alpha(1)$. Using the symmetry we find that $\alpha(1 \otimes X) \simeq \alpha(1) \otimes \phi(X) \simeq 1$ which proves that $\phi(X)$ has an inverse in $\mathcal{D}^{\otimes}$. $\square$

We will now study the properties of the base change along the morphism $\mathcal{C}^{\otimes} \to \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes}, X)}(\mathcal{C}^{\otimes})$. In order to establish some insight, let us point out that everything fits in a commutative diagram

$$\begin{tikzcd}
CAlg(Cat_\infty)_{\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})/.} \simeq CAlg(Mod_{\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})}(Cat_\infty)) \arrow[r] \arrow[d] & CAlg(Mod_{\mathcal{C}^{\otimes}}(Cat_\infty)) \simeq CAlg(Cat_\infty)_{\mathcal{C}^{\otimes}/.} \arrow[d] \\
Mod_{\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})}(Cat_\infty) \arrow[r, "f_*"] & Mod_{\mathcal{C}^{\otimes}}(Cat_\infty)
\end{tikzcd} \tag{2.8}$$

where the horizontal arrows are induced by the forgetful map given by the composition with $\mathcal{C}^{\otimes} \to \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})$ and the vertical arrows are induced by the forgetful map produced by the change of ∞-operads $\mathcal{T}riv^{\otimes} \to \mathcal{C}omm^{\otimes}$. Since Cat_∞ with the cartesian product is a presentable symmetric monoidal $(\infty,1)$-category, there is a base change functor

$$\begin{tikzcd}
Mod_{\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})}(Cat_\infty) \arrow[r] & Mod_{\mathcal{C}^{\otimes}}(Cat_\infty) \arrow[l, bend right, "\mathcal{L}_{(\mathcal{C}^{\otimes},X)}"']
\end{tikzcd} \tag{2.9}$$

and by the general theory we have an identification of $f_*(\mathcal{L}_{(\mathcal{C}^{\otimes},X)}(M)) \simeq M \otimes_{\mathcal{C}^{\otimes}} (\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}))$ given by the tensor product in $Mod_{\mathcal{C}^{\otimes}}(Cat_\infty)$. This map is monoidal and therefore induces a left adjoint

$$\begin{tikzcd}
CAlg(Cat_\infty)_{\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})/} \arrow[r] & CAlg(Cat_\infty)_{\mathcal{C}^{\otimes}/} \arrow[l, bend right, "\widetilde{\mathcal{L}}_{(\mathcal{C}^{\otimes},X)}"']
\end{tikzcd} \tag{2.10}$$

which fits in a commutative diagram

$$\begin{tikzcd}
CAlg(Cat_\infty)_{\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})/.} \simeq CAlg(Mod_{\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})}(Cat_\infty)) \arrow[d, "forget"] & CAlg(Mod_{\mathcal{C}^{\otimes}}(Cat_\infty)) \simeq CAlg(Cat_\infty)_{\mathcal{C}^{\otimes}/.} \arrow[l, "\widetilde{\mathcal{L}}_{(\mathcal{C}^{\otimes},X)}"'] \arrow[d, "forget"] \\
Mod_{\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})}(Cat_\infty) & Mod_{\mathcal{C}^{\otimes}}(Cat_\infty) \arrow[l, "\mathcal{L}_{(\mathcal{C}^{\otimes},X)}"]
\end{tikzcd} \tag{2.11}$$

where the vertical maps forget the algebra structure. We now prove the following statement, which was originally proved in [103] in the context of model categories:

Proposition 2.2. *Let $\mathcal{C}^{\otimes}$ be a small symmetric monoidal $(\infty,1)$-category and X be an object in $\mathcal{C}$. Let $f : \mathcal{C}^{\otimes} \to \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})$ be the natural map constructed above. Then*

1. *the composition map*

$$CAlg(Cat_\infty)_{\mathcal{L}^{\otimes}_{(\mathcal{C}^\otimes,X)}(\mathcal{C}^\otimes)/.} \to CAlg(Cat_\infty)_{\mathcal{C}^\otimes/.} \tag{2.12}$$

is fully faithful and its image coincides with $CAlg(Cat_\infty)^X_{\mathcal{C}^\otimes}$*;*

2. *the forgetful functor*

$$f_* : Mod_{\mathcal{L}^{\otimes}_{(\mathcal{C}^\otimes,X)}(\mathcal{C}^\otimes)}(Cat_\infty) \to Mod_{\mathcal{C}^\otimes}(Cat_\infty) \tag{2.13}$$

is fully faithful and its image coincides with the full subcategory of $Mod_{\mathcal{C}^\otimes}(Cat_\infty)$ *spanned by those* $\mathcal{C}$*-modules where* X *acts as an equivalence.*

A major consequence is that

Corollary 2.3. *The left adjoint* $\widetilde{\mathcal{L}}_{(\mathcal{C}^\otimes,X)}$ *provided by the base change is naturally equivalent to the left adjoint* $\mathcal{L}^{\otimes}_{(\mathcal{C}^\otimes,X)}$ *provided by Proposition 2.1.*

Moreover, since the diagram (2.11) commutes, we have the formula $\mathcal{L}_{(\mathcal{C}^\otimes,X)}(\mathcal{D}) \simeq \mathcal{L}^{\otimes}_{(\mathcal{C}^\otimes,X)}(\mathcal{D}^\otimes)_{\langle 1\rangle}$ for any $\mathcal{D}^\otimes \in CAlg(Cat_\infty)_{\mathcal{C}^\otimes/}$.

In order to prove Proposition 2.2, we will need some preliminary steps. We start by recalling some notation: Let $\mathcal{E}^\otimes$ be a symmetric monoidal $(\infty,1)$-category. A morphism of commutative algebras $A \to B$ in $\mathcal{E}$ is called an epimorphism (see [103]-Definition 1.2.6.1-1) if for any commutative A-algebra C, the mapping space $Map_{CAlg(\mathcal{E})}(B,C)$ is either empty or weakly contractible. In other words, the space of dotted maps of A-algebras

$$\begin{tikzcd} C & \\ A \arrow[u] \arrow[r] & B \arrow[ul, dotted] \end{tikzcd} \tag{2.14}$$

rendering the diagram commutative is either empty or consisting of a unique map, up to equivalence. We can rewrite this definition in a different way. As a result of the general theory, if $\mathcal{E}^\otimes$ is compatible with all small colimits, the ∞-category $CAlg(\mathcal{E})_{A/}$ inherits a cocartesian tensor product (see [63, 3.2.4.7]) which we denote here as $\otimes_A$. In this case it is immediate the conclusion that a map $A \to B$ is an epimorphism if and only if the canonical map $B \to B \otimes_A B$ is an equivalence. Of course, this is happens if and only if the induced colimit map $B \otimes_A B \to B$ is also an equivalence. We prove the following

Proposition 2.4. *Let* $\mathcal{E}^\otimes$ *be a symmetric monoidal* $(\infty,1)$*-category compatible with all small colimits and let* $f : A \to B$ *be a morphism of commutative algebras in* $\mathcal{E}$*. The following are equivalent:*

1. *f is an epimorphism;*
2. *The natural map $f_* : Mod_B(\mathcal{E}) \to Mod_A(\mathcal{E})$ is fully faithful;*

Moreover, if these equivalent conditions are satisfied, the forgetful map

$$CAlg(\mathcal{E})_{B/.} \to CAlg(\mathcal{E})_{A/.} \tag{2.15}$$

is also fully faithful.

Proof. With the hypothesis that the monoidal structure is compatible with colimits, the general theory gives us a base-change functor

$$(- \otimes_A B) : Mod_A(\mathcal{E}) \to Mod_B(\mathcal{E}) \tag{2.16}$$

left adjoint to the forgetful map f_*. In this case f_* will be fully faithful if and only if the counit of the adjunction is an equivalence. If the counit is an equivalence in particular we deduce that the canonical map $B \otimes_A B \to B$ is an equivalence and therefore $A \to B$ is an epimorphism. Conversely, if $A \to B$ is an epimorphism, for any B-module M we have

$$M \otimes_A B \simeq \big((M \otimes_B B) \otimes_A B\big) \simeq \big(M \otimes_B (B \otimes_A B)\big) \simeq (M \otimes_B B) \simeq M \tag{2.17}$$

It remains to prove the additional statement concerning the categories of algebras. Let us consider $u : B \to U$, $v : B \to V$ two algebras over B. We want to prove that the canonical map

$$Map_{CAlg(\mathcal{E})_{B/.}}(U, V) \to Map_{CAlg(\mathcal{E})_{A/.}}\big(f_*(U), f_*(V)\big) \tag{2.18}$$

is an isomorphism of homotopy types. The points in $Map_{CAlg(\mathcal{E})_{A/.}}(f_*(U), f_*(V))$ can be identified with commutative diagrams

$$\begin{tikzcd} & & U \arrow[dd] \\ A \arrow[r, "f"] \arrow[urr, bend left, "u\circ f"] \arrow[drr, bend right, "v\circ f"'] & B \arrow[ur, "u"] \arrow[dr, "v"'] & \\ & & V \end{tikzcd} \tag{2.19}$$

and therefore we can rewrite $Map_{CAlg(\mathcal{E})_{A/.}}(f_*(U), f_*(V))$ as a homotopy pullback diagram

$$Map_{CAlg(\mathcal{E})_{A/}}\big(B, f_*(V)\big) \times_{Map_{CAlg(\mathcal{E})_{A/.}}(A, f_*(V))} Map_{CAlg(\mathcal{E})_{B/.}}(U, V) \tag{2.20}$$

which by the fact $A \to B$ is an epimorphism and $Map_{CAlg(\mathcal{E})_{A/.}}(A, f_*(V)) \simeq *$, is the same as $Map_{CAlg(\mathcal{E})_{B/.}}(U, V)$. □

The following is the main ingredient in the proof of Proposition 2.2.

Proposition 2.5. *Let $\mathcal{C}^{\otimes}$ be a small symmetric monoidal ∞-category and let X be an object in $\mathcal{C}$. Then, the canonical map $\mathcal{C}^{\otimes} \to \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})$ is an epimorphism.*

Proof. This is a direct result of the characterization of $\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}$ as an adjoint in Proposition 2.1. Indeed, for any algebra $\phi : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$, either ϕ does not send X to an invertible object and in this case $Map_{CAlg(Cat_\infty)_{\mathcal{C}^{\otimes}/.}}(\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}), \mathcal{D}^{\otimes})$ is necessarily empty or, ϕ sends X to an invertible object and we have by the universal properties

$$Map_{CAlg(Cat_\infty)_{\mathcal{C}^{\otimes}/.}}\left(\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}), \mathcal{D}^{\otimes}\right) \simeq Map_{CAlg(Cat_\infty)_{\mathcal{C}^{\otimes}/.}}\left(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}\right) \simeq * \qquad \square \tag{2.21}$$

Proof of Proposition 2.2. By the results above we know that both maps are fully faithful. It suffices now to analyze their images.

1. If $\phi : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is in the image, $\mathcal{D}^{\otimes}$ is an algebra over $\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})$, there exists a monoidal factorization

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[r, "\phi"] \arrow[d] & \mathcal{D}^{\otimes} \\ \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}) \arrow[ur, dotted] & \end{tikzcd} \tag{2.22}$$

and therefore X is sent to an invertible object. Conversely, if $\phi : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ sends X to an invertible object, $\phi : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is local with respect to $Free_{\mathcal{C}^{\otimes}}(X \otimes -) : Free_{\mathcal{C}^{\otimes}}(\mathcal{C}) \to Free_{\mathcal{C}^{\otimes}}(\mathcal{C})$ and therefore the adjunction morphisms of Proposition 2.1 fit in a commutative diagram

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[r, "\phi"] \arrow[d] & \mathcal{D}^{\otimes} \arrow[d, "\sim"] \\ \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}) \arrow[r, "\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\phi)"] & \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{D}^{\otimes}) \end{tikzcd} \tag{2.23}$$

where the right vertical map is an equivalence and we deduce the existence of a monoidal map presenting $\mathcal{D}^{\otimes}$ as an $\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})$-algebra, therefore being in the image of f_*.

2. Again, it remains to prove the assertion about the image. If M is a $\mathcal{C}^{\otimes}$-module in the image, by definition, its module structure is obtained by the composition $\mathcal{C}^{\otimes} \times M \to \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}) \times M \to M$ and therefore the action of X on M is invertible. Conversely, let M be a $\mathcal{C}^{\otimes}$-module where X acts as an equivalence. We want to show that M is in the image of the forgetful functor. Since we know it is fully faithful, this is equivalent to showing that the unit map of the adjunction

$$M \to f_*\big(\mathcal{L}_{(\mathcal{C}^{\otimes},X)}(M)\big) \simeq M \otimes_{\mathcal{C}^{\otimes}} \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}\big(\mathcal{C}^{\otimes}\big) \tag{2.24}$$

is an equivalence. To prove this we will need a reasonable description of $\mathit{Free}_{\mathcal{C}^{\otimes}}(M)$ – the free $\mathcal{C}^{\otimes}$ algebra generated by M. Following [63, 3.1.3.9, 3.1.3.14] we know that the underlying $\mathcal{C}^{\otimes}$-module $\mathit{Free}_{\mathcal{C}^{\otimes}}(M)_{\langle 1\rangle}$ can be described as a coproduct

$$\coprod_{n\geq 0} \mathit{Sym}^n(M)_{\mathcal{C}^{\otimes}} \tag{2.25}$$

where $\mathit{Sym}^n(M)_{\mathcal{C}^{\otimes}}$ is a colimit diagram in $\mathit{Mod}_{\mathcal{C}^{\otimes}}(\mathit{Cat}_{\infty})$ which can be informally described as $M^{\otimes_{\mathcal{C}^{\otimes}}}/\Sigma_n$ where $\otimes_{\mathcal{C}^{\otimes}}$ refers to the natural symmetric monoidal structure in $\mathit{Mod}_{\mathcal{C}^{\otimes}}(\mathit{Cat}_{\infty})$. Let us proceed.

- The general machinery tells us that $\mathit{Free}_{\mathcal{C}^{\otimes}}(M)$ exists in our case and by construction it comes naturally equipped with a canonical monoidal map $\phi : \mathcal{C}^{\otimes} \to \mathit{Free}_{\mathcal{C}^{\otimes}}(M)$. We remark that the multiplication map $(\phi(X)\otimes -) : \mathit{Free}_{\mathcal{C}^{\otimes}}(M)_{\langle 1\rangle} \to \mathit{Free}_{\mathcal{C}^{\otimes}}(M)_{\langle 1\rangle}$ can be identified with the image $\mathit{Free}_{\mathcal{C}^{\otimes}}(X \otimes -)_{\langle 1\rangle}$ of the multiplication map $(X \otimes -) : M \to M$. Since this last one is an equivalence (by the assumption), we conclude that $\mathit{Free}_{\mathcal{C}^{\otimes}}(M)$ is in fact a $\mathcal{C}^{\otimes}$ algebra where X is sent to an invertible object. This means that it is in fact an $\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})$-algebra and therefore $\mathit{Free}_{\mathcal{C}^{\otimes}}(M)_{\langle 1\rangle}$ is in fact an $\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})$-module, which means that the unit map

$$\mathit{Free}_{\mathcal{C}^{\otimes}}(M)_{\langle 1\rangle} \to f_*\big(\mathcal{L}_{(\mathcal{C}^{\otimes},X)}\big(\mathit{Free}_{\mathcal{C}^{\otimes}}(M)_{\langle 1\rangle}\big)\big) \simeq \mathit{Free}_{\mathcal{C}^{\otimes}}(M)_{\langle 1\rangle} \otimes_{\mathcal{C}^{\otimes}} \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}\big(\mathcal{C}^{\otimes}\big) \tag{2.26}$$

is an equivalence.
- We observe now that we have a canonical map $M \to \mathit{Free}_{\mathcal{C}^{\otimes}}(M)_{\langle 1\rangle}$ because $\mathit{Sym}^1(M) = M$ and that this map is obviously fully faithful. The unit of the natural transformation associated to the base-change gives us a commutative diagram

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & M \otimes_{\mathcal{C}^{\otimes}} \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}) \arrow[d] \\
\mathit{Free}_{\mathcal{C}^{\otimes}}(M)_{\langle 1\rangle} \arrow[r, "\sim"] & \mathit{Free}_{\mathcal{C}^{\otimes}}(M)_{\langle 1\rangle} \otimes_{\mathcal{C}^{\otimes}} \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})
\end{tikzcd} \tag{2.27}$$

where the lower arrow is an equivalence from the discussion in the previous item. Since the monoidal structure is compatible with coproducts and using the identification $Sym^n(M)_{\mathcal{C}^\otimes} \simeq M^{\otimes^n_{\mathcal{C}^\otimes}}/\Sigma_n$, we have

$$Free_{\mathcal{C}^\otimes}(M)_{\langle 1\rangle} \otimes_{\mathcal{C}^\otimes} \mathcal{L}^\otimes_{(\mathcal{C}^\otimes,X)}(\mathcal{C}^\otimes) \simeq \coprod \big[\big(M^{\otimes^n_{\mathcal{C}^\otimes}}\big)^\otimes_{\mathcal{C}^\otimes} \mathcal{L}^\otimes_{(\mathcal{C}^\otimes,X)}(\mathcal{C}^\otimes)\big]/\Sigma_n \tag{2.28}$$

and finally, using the fact $\mathcal{C}^\otimes \to \mathcal{L}^\otimes_{(\mathcal{C}^\otimes,X)}(\mathcal{C}^\otimes)$ is an epimorphism, we have

$$\big(\mathcal{L}^\otimes_{(\mathcal{C}^\otimes,X)}(\mathcal{C}^\otimes)\big)^{\otimes^n_{\mathcal{C}^\otimes}} \simeq \mathcal{L}^\otimes_{(\mathcal{C}^\otimes,X)}(\mathcal{C}^\otimes) \tag{2.29}$$

for any $n \geq 0$. We find an equivalence

$$Free_{\mathcal{C}^\otimes}(M)_{\langle 1\rangle} \otimes_{\mathcal{C}^\otimes} \mathcal{L}^\otimes_{(\mathcal{C}^\otimes,X)}(\mathcal{C}^\otimes) \simeq Free_{\mathcal{C}^\otimes}\big(M \otimes_{\mathcal{C}^\otimes} \mathcal{L}^\otimes_{(\mathcal{C}^\otimes,X)}(\mathcal{C}^\otimes)\big)_{\langle 1\rangle} \tag{2.30}$$

The first diagram becomes

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & M \otimes_{\mathcal{C}^\otimes} \mathcal{L}^\otimes_{(\mathcal{C}^\otimes,X)}(\mathcal{C}^\otimes) \arrow[d] \\
Free_{\mathcal{C}^\otimes}(M)_{\langle 1\rangle} = \coprod_{n\geq 0} Sym^n(M)_{\mathcal{C}^\otimes} \arrow[r, "\sim"] & \coprod_{n\geq 0} Sym^n(M \otimes_{\mathcal{C}^\otimes} \mathcal{L}^\otimes_{(\mathcal{C}^\otimes,X)}(\mathcal{C}^\otimes))_{\mathcal{C}^\otimes}
\end{tikzcd} \tag{2.31}$$

where both vertical maps are now the canonical inclusions in the coproduct. Therefore, since Cat_∞ has disjoint coproducts (because coproducts can be computed as homotopy coproducts in the combinatorial model category of marked simplicial sets and here coproducts are disjoint), we conclude that the canonical map $M \to M \otimes_{\mathcal{C}^\otimes} \mathcal{L}^\otimes_{(\mathcal{C}^\otimes,X)}(\mathcal{C}^\otimes)$ is also an equivalence.

This concludes the proof. □

Our goal now is to extend our construction to the setting of presentable symmetric monoidal ∞-categories. The starting observation is that, if $\mathcal{C}^\otimes$ is a small symmetric monoidal $(\infty,1)$-category the inversion of an object X can now be rewritten by means of a pushout square in $CAlg(Cat_\infty)$: Since Cat_∞ is a symmetric monoidal $(\infty,1)$-category compatible with all colimits, the forgetful functor

$$CAlg(Cat_\infty) \to Cat_\infty \tag{2.32}$$

admits a left adjoint $free^\otimes$ which assigns to an ∞-category $\mathcal{D}$, the *free symmetric monoidal* $(\infty,1)$*-category generated by* $\mathcal{D}$. An object in $\mathcal{C}$ can be interpreted as a monoidal map $free^\otimes(\Delta[0]) \to \mathcal{C}^\otimes$ where $free^\otimes(\Delta[0])$ is the free symmetric monoidal category generated by one object $*$. By the universal property of $\mathcal{L}^\otimes_{(free^\otimes(\Delta[0]),*)}(free^\otimes(\Delta[0]))$, a monoidal map $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ sends X to an invertible object if and only if it factors as a commutative diagram

$$\begin{tikzcd} free^{\otimes}(\Delta[0]) \arrow[r] \arrow[d, "X"] & \mathcal{L}^{\otimes}_{(free^{\otimes}(\Delta[0]),*)}(free^{\otimes}(\Delta[0])) \arrow[d] \\ \mathcal{C}^{\otimes} \arrow[r] & \mathcal{D}^{\otimes} \end{tikzcd} \tag{2.33}$$

and by the combination of the universal properties, the pushout in $CAlg(Cat_{\infty})$

$$\mathcal{C}^{\otimes} \coprod_{free^{\otimes}(\Delta[0])} \mathcal{L}^{\otimes}_{(free^{\otimes}(\Delta[0]),*)}(free^{\otimes}(\Delta[0])) \tag{2.34}$$

is canonically equivalent to $\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})$. The existence of this pushout is ensured by the fact that $Cat_{\infty}^{\times}$ is compatible with all colimits (see [63, 3.2.3.2, 3.2.3.3])

We will use this pushout-version to construct the presentable theory. By [63, 4.8.1.10] if $\mathcal{C}^{\otimes}$ is a presentable symmetric monoidal $(\infty,1)$-category (not necessarily small) and X is an object in $\mathcal{C}$, the universal *monoidal* property of presheaves ensures that any diagram like (2.33) factors as

$$\begin{tikzcd} free^{\otimes}(\Delta[0]) \arrow[r] \arrow[d, "j"] & \mathcal{L}^{\otimes}_{(free^{\otimes}(\Delta[0]),*)}(free^{\otimes}(\Delta[0])) \arrow[d, "j'"] \\ \mathcal{P}(free^{\otimes}(\Delta[0]))^{\otimes} \arrow[r] \arrow[d, dashed] & \mathcal{P}(\mathcal{L}^{\otimes}_{(free^{\otimes}(\Delta[0]),*)}(free^{\otimes}(\Delta[0])))^{\otimes} \arrow[d, dashed] \\ \mathcal{C}^{\otimes} \arrow[r] & \mathcal{D}^{\otimes} \end{tikzcd} \tag{2.35}$$

where $\mathcal{P}^{\otimes}(-)$ is the natural extension of the symmetric monoidal structure to presheaves, the vertical maps j and j' are the respective Yoneda embeddings (which are monoidal maps) and the dotted arrows are given by colimit-preserving monoidal maps obtained as left Kan extensions.

Definition 2.6. Let $\mathcal{C}^{\otimes}$ be a presentable symmetric monoidal $(\infty,1)$-category and let X be an object in $\mathcal{C}$. The *formal inversion* of X in $\mathcal{C}^{\otimes}$ is the new presentable symmetric monoidal $(\infty,1)$-category $\mathcal{C}^{\otimes}[X^{-1}]$ defined by pushout

$$\mathcal{C}^{\otimes}[X^{-1}] := \mathcal{C}^{\otimes} \coprod_{\mathcal{P}(free^{\otimes}(\Delta[0]))^{\otimes}} \mathcal{P}(\mathcal{L}^{\otimes}_{(free^{\otimes}(\Delta[0]),*)}(free^{\otimes}(\Delta[0])))^{\otimes} \tag{2.36}$$

in $CAlg(\mathcal{P}r^{L})$.

Remark 2.7. Recall that $\mathcal{P}r^{L,\otimes}$ is compatible with colimits. By [63, 3.2.3.2, 3.2.3.3] the $(\infty,1)$-category $CAlg(\mathcal{P}r^L)$ has all small colimits so that the previous definition makes sense.

Remark 2.8. Let $\mathcal{C}^\otimes$ be a small symmetric monoidal $(\infty,1)$-category and let X be an object in $\mathcal{C}$. Again by [63, 4.8.1.10], the monoidal structure in $\mathcal{C}$ extends to a monoidal structure in $\mathcal{P}(\mathcal{C})$ and it makes it a presentable symmetric monoidal $(\infty,1)$-category. It is automatic by the universal properties that the inversion $\mathcal{P}(\mathcal{C})^\otimes[X^{-1}]$ in the setting of presentable $(\infty,1)$-categories is canonically equivalent to $\mathcal{P}(\mathcal{L}^\otimes_{(\mathcal{C}^\otimes,X)}(\mathcal{C}^\otimes))^\otimes$.

As in the *small* context, we analyze the base change with respect to this map. Since $(\mathcal{P}r^L)^\otimes$ is compatible with all small colimits (see [63, 4.8.1.14, 4.8.1.17]), all the machinery related to algebras and modules can be applied. The composition with the canonical map $\mathcal{C}^\otimes \to \mathcal{C}^\otimes[X^{-1}]$ produces a forgetful functor

$$Mod_{\mathcal{C}^\otimes[X^{-1}]}(\mathcal{P}r^L) \to Mod_{\mathcal{C}^\otimes}(\mathcal{P}r^L) \tag{2.37}$$

and the base-change functor $\mathcal{L}^{Pr}_{(\mathcal{C}^\otimes,X)} := (- \otimes_{\mathcal{C}^\otimes} \mathcal{C}^\otimes[X^{-1}])$ exists, is monoidal and therefore induces an adjunction

$$\begin{tikzcd} CAlg(\mathcal{P}r^L)_{\mathcal{C}^\otimes[X^{-1}]/.} \arrow[r] & CAlg(\mathcal{P}r^L)_{\mathcal{C}^\otimes/.} \arrow[l, bend right, "\mathcal{L}^{Pr,\otimes}_{(\mathcal{C}^\otimes,X)}"'] \end{tikzcd} \tag{2.38}$$

Our main result is the following:

Proposition 2.9. *Let $\mathcal{C}^\otimes$ be a presentable symmetric monoidal $(\infty,1)$-category. Then*

1. *the canonical map*

$$CAlg(\mathcal{P}r^L)_{\mathcal{C}^\otimes[X^{-1}]/.} \to CAlg(\mathcal{P}r^L)_{\mathcal{C}^\otimes/.} \tag{2.39}$$

is fully faithful and its essential image consists of full subcategory spanned by the algebras $\mathcal{C}^\otimes \to \mathcal{D}^\otimes$ sending X to an invertible object; In particular we have a canonical equivalence $\mathcal{L}^{Pr,\otimes}_{(\mathcal{C}^\otimes,X)}(\mathcal{C}^\otimes) \simeq \mathcal{C}^\otimes[X^{-1}]$.

2. *The canonical map*

$$Mod_{\mathcal{C}^\otimes[X^{-1}]}(\mathcal{P}r^L) \to Mod_{\mathcal{C}^\otimes}(\mathcal{P}r^L) \tag{2.40}$$

is fully faithful and its essential image consists of full subcategory spanned by the presentable $(\infty,1)$-categories equipped with an action of $\mathcal{C}$ where X acts as an equivalence.

Proof. Since $(\mathcal{P}r^L)^{\otimes}$ is a closed symmetric monoidal $(\infty,1)$-category (see [63, 4.8.1.14]), it is compatible with all colimits and so the results of Proposition 2.4 can be applied. We prove that $\mathcal{C}^{\otimes} \to \mathcal{C}^{\otimes}[X^{-1}]$ is an epimorphism. Indeed, if $\phi : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ does not send X to an invertible object, by the universal property of the $\mathcal{C}^{\otimes}[X^{-1}]$ as a pushout, the mapping space $Map_{CAlg(Cat_\infty)_{\mathcal{C}^{\otimes}/}}(\mathcal{C}^{\otimes}[X^{-1}], \mathcal{D}^{\otimes})$ is empty. Otherwise if ϕ sends X to an invertible object, by the universal property of the pushout we have

$$Map_{CAlg(\mathcal{P}r^L)_{\mathcal{C}^{\otimes}/}}\left(\mathcal{C}^{\otimes}\left[X^{-1}\right], \mathcal{D}^{\otimes}\right) \simeq Map_{CAlg(\mathcal{P}r^L)}\left(\mathcal{C}^{\otimes}\left[X^{-1}\right], \mathcal{D}^{\otimes}\right) \tag{2.41}$$

and the last is given by the homotopy pullback of

$$\begin{tikzcd} & Map_{CAlg(\mathcal{P}r^L)}(\mathcal{P}(\mathcal{L}^{\otimes}_{(free^{\otimes}(\Delta[0]),*)}(free^{\otimes}(\Delta[0])))^{\otimes}, \mathcal{D}^{\otimes}) \arrow[d] \\ Map_{CAlg(\mathcal{P}r^L)}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}) \arrow[r] & Map_{CAlg(\mathcal{P}r^L)}(\mathcal{P}^{\otimes}(free^{\otimes}(\Delta[0])), \mathcal{D}^{\otimes}) \end{tikzcd} \tag{2.42}$$

which, by the universal property of $\mathcal{P}^{\otimes}(-)$ is equivalent to

$$\begin{aligned} &Map_{CAlg(\mathcal{P}r^L)}\left(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}\right) \\ &\quad \times_{Map_{CAlg(Cat_\infty)}(free^{\otimes}(\Delta[0]),\mathcal{D}^{\otimes})} Map_{CAlg(Cat_\infty)}\left(\mathcal{L}^{\otimes}_{(free^{\otimes}(\Delta[0]),*)}\left(free^{\otimes}(\Delta[0])\right), \mathcal{D}^{\otimes}\right) \end{aligned} \tag{2.43}$$

and we use the fact that $free^{\otimes}(\Delta[0]) \to \mathcal{L}^{\otimes}_{(free^{\otimes}(\Delta[0]),*)}(free^{\otimes}(\Delta[0]))$ is an epimorphism to conclude the proof.

It remains now to discuss the images.

1. It is clear by the universal property of the pushout defining $\mathcal{C}^{\otimes}[X^{-1}]$;
2. If M is in the image, the action of X is clearly invertible. Let M be a $\mathcal{C}^{\otimes}$-module with an invertible action of X. By repeating exactly the same arguments as in the proof of Proposition 2.9 we get a commutative diagram in $\mathcal{P}r^L$

$$\begin{tikzcd} M \arrow[r] \arrow[d] & M \otimes_{\mathcal{C}^{\otimes}} \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}) \arrow[d] \\ Free_{\mathcal{C}^{\otimes}}(M)_{\langle 1\rangle} = \coprod_{n\geq 0} Sym^n(M)_{\mathcal{C}^{\otimes}} \arrow[r, "\sim"] & \coprod_{n\geq 0} Sym^n(M \otimes_{\mathcal{C}^{\otimes}} \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}))_{\mathcal{C}^{\otimes}} \end{tikzcd} \tag{2.44}$$

where the vertical maps are the canonical inclusions in the colimit and $Sym^n(-)_{\mathcal{C}^{\otimes}}$ is now a colimit in $Mod_{\mathcal{C}^{\otimes}}(\mathcal{P}r^L)$. We recall now that coproducts in $\mathcal{P}r^L$ are computed as products in $\mathcal{P}r^R$. Let $u : A \to B$ and $v : X \to Y$ be colimit preserving maps between presentable $(\infty, 1)$-categories and assume the coproduct map $u \coprod v : A \coprod X \to B \coprod Y$ is an equivalence. The coproduct $A \coprod X$ is canonically equivalent to the product $A \times X$ and we have commutative diagrams

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & B \arrow[d, "j"] \\ A \coprod X \arrow[r, "\sim", "u \coprod v"'] & B \coprod Y \end{tikzcd} \tag{2.45}$$

and

$$\begin{tikzcd} A & & B \arrow[ll, "\bar{u}"] \\ A \coprod X = A \times X \arrow[u, "p"'] & & B \coprod Y = B \times Y \arrow[u, "q"] \arrow[ll, "\overline{u \coprod v}"] \end{tikzcd} \tag{2.46}$$

with i and j the canonical inclusions and p and q the projections. The maps in the second diagram are right adjoints to the maps in the first, with $\overline{u \coprod v} \simeq \bar{u} \times \bar{v}$ and therefore $u \coprod v$ and $\bar{u} \times \bar{v}$ are inverses. Since the projections are essentially surjective, the inclusions i and j are fully faithful and we conclude that u has to be fully faithful and $\bar{u}$ is essentially surjective. To conclude the proof is it enough to check that u is essentially surjective or, equivalently (because u is fully faithful), that $\bar{u}$ is fully-faithful. This is the same as saying that for any diagram as in (2.76) with $\bar{u} \times \bar{v}$ fully faithful, $\bar{u}$ is necessarily fully faithful. This is true because Y is presentable and therefore has a final object e and since $\bar{v}$ commutes with limits, for any objects $b_0, b_1 \in Obj(\mathcal{A})$ we have

$$\begin{aligned} Map_B(b_0, b_1) &\simeq Map_B(b_0, b_1) \times Map_Y(e, e) \\ &\simeq Map_A\big(\bar{u}(b_0), \bar{u}(b_1)\big) \times Map_X\big(\bar{v}(e), \bar{v}(e)\big) && (2.47) \\ &\simeq Map_A\big(\bar{u}(b_0), \bar{u}(b_1)\big) \qquad \square && (2.48) \end{aligned}$$

2.2. Connection with ordinary spectra and stabilization

In the previous section we studied the formal inversion of an object X in a symmetric monoidal $(\infty, 1)$-category. Our goal for this section is to compare our formal inversion to the more familiar notion of (ordinary) spectrum-objects.

2.2.1. Stabilization

Let $\mathcal{C}$ be an $(\infty, 1)$-category and let $G : \mathcal{C} \to \mathcal{C}$ be a functor with a right adjoint $U : \mathcal{C} \to \mathcal{C}$. We define the *stabilization of* $\mathcal{C}$ *with respect to* (G, U) as the limit in Cat_∞^{big}

$$Stab_{(G,U)}(\mathcal{C}) := \mathrm{Lim} \ldots \xrightarrow{U} \mathcal{C} \xrightarrow{U} \mathcal{C} \xrightarrow{U} \mathcal{C} \tag{2.49}$$

We will refer to the objects of $Stab_{(G,U)}(\mathcal{C})$ as *spectrum objects in* $\mathcal{C}$ *with respect to* (G, U). As a limit, we have a canonical functor "evaluation at level 0" which we will denote as $\Omega^\infty_{\mathcal{C}} : Stab_{(G,U)}(\mathcal{C}) \to \mathcal{C}$.

Remark 2.10. Let $\mathcal{C}$ is a presentable $(\infty, 1)$-category together with a colimit preserving functor $G : \mathcal{C} \to \mathcal{C}$. By the Adjoint Functor Theorem we deduce the existence a right adjoint U to G. Using the equivalence $\mathcal{P}r^L \simeq (\mathcal{P}r^R)^{op}$, and the fact that both inclusions $\mathcal{P}r^L, \mathcal{P}r^R \subseteq Cat^{big}_\infty$ preserve limits, we conclude that $Stab_{(G,U)}(\mathcal{C})$ is equivalent to the colimit of

$$\mathcal{C} \xrightarrow{G} \mathcal{C} \xrightarrow{G} \mathcal{C} \xrightarrow{G} \ldots \tag{2.50}$$

Example 2.11. The construction of spectrum objects provides a method to stabilize an ∞-category: Let $\mathcal{C}$ be an $(\infty, 1)$-category with final object $*$. If $\mathcal{C}$ admits finite limits and colimits we can construct a pair of adjoint functors $\Sigma_{\mathcal{C}} : \mathcal{C}_{*/} \to \mathcal{C}_{*/}$ and $\Omega_{\mathcal{C}} : \mathcal{C}_{*/} \to \mathcal{C}_{*/}$ defined by the formula

$$\Sigma_{\mathcal{C}}(X) := * \coprod_X * \tag{2.51}$$

and

$$\Omega_{\mathcal{C}}(X) := * \times_X * \tag{2.52}$$

and by [63, Prop. 1.4.2.24] we can define the *stabilization of* $\mathcal{C}$ as the ∞-category

$$Stab(\mathcal{C}) := Stab_{(\Sigma_{\mathcal{C}}, \Omega_{\mathcal{C}})}(\mathcal{C}_{*/}). \tag{2.53}$$

By [63, Cor. 1.4.2.17], $Stab(\mathcal{C})$ is a stable ∞-category and by [63, Corollary 1.4.2.23] the functor $\Omega^\infty : Stab(\mathcal{C}) \to \mathcal{C}$ has a universal property: for any stable $(\infty, 1)$-category $\mathcal{D}$, the composition with Ω^∞ induces an equivalence

$$Fun'\big(\mathcal{D}, Stab(\mathcal{C})\big) \to Fun'(\mathcal{D}, \mathcal{C}) \tag{2.54}$$

between the full subcategories of functors preserving finite limits. Suppose now that $\mathcal{C}$ is presentable. Since $\Omega_{\mathcal{C}}$ by definition commutes with all limits and $\mathcal{P}r^R$ is closed under limits, $Stab(\mathcal{C})$ will also be presentable and Ω^∞ will also commute with all limits. Therefore, by the Adjoint Functor Theorem it will admit a left adjoint $\Sigma^\infty : \mathcal{C} \to Stab(\mathcal{C})$. Using the equivalence $\mathcal{P}r^L \simeq (\mathcal{P}r^R)^{op}$ we find (see [63, Cor. 1.4.4.5]) that Σ^∞ is characterized by the following universal property: for every stable presentable $(\infty, 1)$-category $\mathcal{D}$, the composition with Σ^∞ induces an equivalence

$$Fun^L\big(Stab(\mathcal{C}), \mathcal{D}\big) \to Fun^L(\mathcal{C}, \mathcal{D}) \tag{2.55}$$

Our goal for the rest of this section is to compare this notion of stabilization to something more familiar. Let us start with some precisions about the construction of limits in Cat_∞. By [59, Thm. 4.2.4.1], the stabilization $Stab_{(G,U)}(\mathcal{C})$ can be computed as a homotopy limit for the tower

$$... \xrightarrow{U} \mathcal{C}^\natural \xrightarrow{U} \mathcal{C}^\natural \xrightarrow{U} \mathcal{C}^\natural \tag{2.56}$$

in the simplicial model category $\hat{\Delta}_+$ of (big) marked simplicial sets of the [59, Prop. 3.1.3.7] (as a marked simplicial set, $\mathcal{C}^\natural$ is the notation for the pair $(\mathcal{C}, W)$ where W is the collection of all edges in $\mathcal{C}$ which are equivalences). By [59, Thm. 3.1.5.1], the cofibrant–fibrant objects in $\hat{\Delta}_+$ are exactly the objects of the form $\mathcal{C}^\natural$ with $\mathcal{C}$ a quasi-category and, forgetting the marked edges provides a right-Quillen equivalence from $\hat{\Delta}_+$ to $\hat{\Delta}$ with the Joyal model structure. Therefore, to obtain a model for the homotopy limit in $\hat{\Delta}_+$ we can instead compute the homotopy limit in $\hat{\Delta}$ (with the Joyal structure).

Let now us recall the following result about homotopy limits in model categories.

Lemma 2.12. *Let $\mathcal{M}$ be a simplicial model category and let $T : \mathbb{N}^{op} \to \mathcal{M}$ be tower in $\mathcal{M}$*

$$... \xrightarrow{T_3} X_2 \xrightarrow{T_2} X_1 \xrightarrow{T_1} X_0 \tag{2.57}$$

with each X_n a fibrant object of $\mathcal{M}$. In this case, the homotopy limit $holim_{(\mathbb{N}^{op})} T_n$ is weak-equivalent to the strict pullback of the diagram

$$\begin{tikzcd} & \prod_n X_n^{\Delta[1]} \arrow[d] \\ \prod_n X_n \arrow[r] & \prod_n X_n \times X_n \end{tikzcd} \tag{2.58}$$

where the vertical arrow is the fibration[9] induced by the composition with the cofibration $\partial\Delta[1] \to \Delta[1]$ and the horizontal map is the product of the compositions $\prod_n X_n \to X_n \times X_{n+1} \to X_n \times X_n$ where the last map is the product $Id_{X_n} \times T_n$. Notice that every vertice of the diagram is fibrant.

Proof. See [39]-VI-Lemma 1.12. □

Back to our situation, we conclude that the homotopy limit of

$$... \xrightarrow{U} \mathcal{C}^\natural \xrightarrow{U} \mathcal{C}^\natural \xrightarrow{U} \mathcal{C}^\natural \tag{2.59}$$

is given by the explicit strict pullback in $\hat{\Delta}_+$

[9] It is a fibration because of the simplicial assumption.

$$\begin{tikzcd} & \prod_n (\mathcal{C}^\natural)^{\Delta[1]^\sharp} \arrow[d] \\ \prod_n \mathcal{C}^\natural \arrow[r] & \prod_n \mathcal{C}^\natural \times \mathcal{C}^\natural \end{tikzcd} \tag{2.60}$$

where $\Delta[1]^\sharp$ is the notation for the simplicial set $\Delta[1]$ with all the edges marked and $(\mathcal{C}^\natural)^{\Delta[1]^\sharp}$ is the coaction of $\Delta[1]^\sharp$ on $\mathcal{C}^\natural$. In fact, it can be identified with the marked simplicial set $Fun'(\Delta[1], \mathcal{C})^\natural$ where $Fun'(\Delta[1], \mathcal{C})$ corresponds to the full-subcategory of $Fun(\Delta[1], \mathcal{C})$ spanned by the maps $\Delta[1] \to \mathcal{C}$ which are equivalences in $\mathcal{C}$.

Let us move further. Consider now a combinatorial simplicial model category $\mathcal{M}$ and let $G : \mathcal{M} \to \mathcal{M}$ be a left simplicial Quillen functor with a right adjoint U. Using the technique described in [59, 5.2.4.6], from the adjunction data we can extract an endo-adjunction of the underlying $(\infty, 1)$-category of $\mathcal{M}$

$$\begin{tikzcd} N_\Delta(\mathcal{M}^\circ) \arrow[r, shift left, "\bar{G}"] & N_\Delta(\mathcal{M}^\circ) \arrow[l, shift left, "\bar{U}"] \end{tikzcd} \tag{2.61}$$

where $\bar{U}$ can be identified with the composition $Q \circ U$ with Q a simplicial[10] cofibrant-replacement functor in $\mathcal{M}$, which we shall fix once and for all. We can consider the stabilization $Stab_{(\bar{G},\bar{U})}(N_\Delta(\mathcal{M}^\circ))$ given by the homotopy limit

$$\ldots \xrightarrow{\bar{U}} N_\Delta\big(\mathcal{M}^\circ\big)^\natural \xrightarrow{\bar{U}} N_\Delta\big(\mathcal{M}^\circ\big)^\natural \xrightarrow{\bar{U}} N_\Delta\big(\mathcal{M}^\circ\big)^\natural \tag{2.62}$$

which we now know, is weak-equivalent to the strict pullback of

$$\begin{tikzcd} & \prod_n Fun'(\Delta[1], N_\Delta(\mathcal{M}^\circ))^\natural \arrow[d] \\ \prod_n N_\Delta(\mathcal{M}^\circ)^\natural \arrow[r] & \prod_n N_\Delta(\mathcal{M}^\circ)^\natural \times N_\Delta(\mathcal{M}^\circ)^\natural \end{tikzcd} \tag{2.63}$$

and we know that its underlying simplicial set can be computed as a pullback in $\hat{\Delta}$ by ignoring all the markings. Moreover, by [59, Prop. 4.2.4.4], we have an equivalence of $(\infty, 1)$-categories between

$$N_\Delta\big(\big(\mathcal{M}^I\big)^\circ\big) \xrightarrow{\sim} N_\Delta\big(\mathcal{M}^\circ\big)^{\Delta[1]} \tag{2.64}$$

where I is the categorical interval and $\mathcal{M}^I$ denotes the category of morphisms in $\mathcal{M}$ endowed with the projective model structure (its cofibrant–fibrant objects are the arrows

[10] (See for instance Proposition 6.3 of [75] for the existence of simplicial factorizations in a simplicial cofibrantly generated model category.)

$f : A \to B$ in $\mathcal{M}$ with both A and B cofibrant–fibrant and f a cofibration in $\mathcal{M}$). Moreover, the equivalence above restricts to a new one between the simplicial nerve of $(\mathcal{M}^I)^\circ_{triv}$ (the full simplicial subcategory of $(\mathcal{M}^I)^\circ$ spanned by the arrows $f : A \to B$ which have A and B cofibrant–fibrant and f a trivial cofibration) and $Fun'(\Delta[1], N_\Delta(\mathcal{M}^\circ))$. Using this equivalence, we find an equivalence of diagrams

$$\begin{tikzcd}
 & N_\Delta((\mathcal{M}^I)^\circ_{triv}) \arrow[rr, "\sim"] \arrow[d] & & Fun'(\Delta[1], N_\Delta(\mathcal{M}^\circ)) \arrow[d] \\
 & \prod_n N_\Delta(\mathcal{M}^\circ) \times N_\Delta(\mathcal{M}^\circ) \arrow[rr, "id"] & & \prod_n N_\Delta(\mathcal{M}^\circ) \times N_\Delta(\mathcal{M}^\circ) \\
\prod_n N_\Delta(\mathcal{M}^\circ) \arrow[ur] \arrow[rr, "id"] & & \prod_n N_\Delta(\mathcal{M}^\circ) \arrow[ur] &
\end{tikzcd} \tag{2.65}$$

The homotopy pullbacks of both diagrams are weak-equivalent but since the vertical map on the left diagram is no longer a fibration, the associated strict pullback is no longer a model for the homotopy pullback. We continue: the simplicial nerve functor N_Δ is a right-Quillen functor from the category of simplicial categories with the model structure of [10] to the category of simplicial sets with the Joyal structure. Therefore, it commutes with homotopy limits and so, the simplicial set underlying the pullback of the previous diagram is in fact given by the simplicial nerve of the homotopy pullback of

$$\begin{tikzcd}
 & \prod_n (\mathcal{M}^I)^\circ_{triv} \arrow[d] \\
\prod_n \mathcal{M}^\circ \arrow[r] & \prod_n \mathcal{M}^\circ \times \mathcal{M}^\circ
\end{tikzcd} \tag{2.66}$$

in the model category of simplicial categories.

Let us now progress in another direction. We continue with $\mathcal{M}$ a model category together with $G : \mathcal{M} \to \mathcal{M}$ a Quillen left endofunctor with a right adjoint U. We recall the construction of a category $Sp^{\mathbb{N}}(\mathcal{M}, G)$ of spectrum objects in $\mathcal{M}$ with respect to (G, U): its objects are the sequences $X = (X_0, X_1, ...)$ together with data of morphisms in $\mathcal{M}$, $\sigma_i : G(X_i) \to X_{i+1}$ (by the adjunction, this is equivalent to the data of morphisms $\bar{\sigma}_i : X_i \to U(X_{i+1})$). A morphism $X \to Y$ is a collection of morphisms in $\mathcal{M}$, $f_i : X_i \to Y_i$, compatible with the structure maps σ_i. If $\mathcal{M}$ is a cofibrantly generated model category (see Section 2.1 of [42]) we can equipped $Sp^{\mathbb{N}}(\mathcal{M}, G)$ with a *stable model structure*. First we define the projective model structure: the weak equivalences are the maps $X \to Y$ which are levelwise weak-equivalences in $\mathcal{M}$ and the fibrations are the levelwise fibrations. The cofibrations are defined by obvious left-lifting properties. By Theorem 1.13 of

[43] these form a model structure which is again cofibrantly generated and by Proposition 1.15 of [43], the cofibrant–fibrant objects are the sequences $(X_0, X_1, ...)$ where every X_i is fibrant–cofibrant in $\mathcal{M}$, and the canonical maps $G(X_i) \to G(X_{i+1})$ are cofibrations. We shall write $Sp^{\mathbb{N}}(\mathcal{M}, G)_{proj}$ to denote this model structure. The stable model structure, denoted as $Sp^{\mathbb{N}}(\mathcal{M}, G)_{stable}$, is obtained as a Bousfield localization of the projective structure so that the new fibrant–cofibrant objects are the U*-spectra*, meaning, the sequences $(X_0, X_1, ...)$ which are fibrant–cofibrant for the projective model structure and such that for every i, the adjoint of the structure map σ_i, $X_i \to U(X_{i+1})$ is a weak-equivalence. (See Theorem 3.4 of [43].)

By [43, Thm. 6.3], this construction also works if we assume $\mathcal{M}$ to be a combinatorial simplicial model category and G to be a left simplicial Quillen functor.[11] In this case, $Sp^{\mathbb{N}}(\mathcal{M}, G)$ (both with the stable and the projective structures) is again a combinatorial simplicial model category with mapping spaces given by the pullback

$$\begin{tikzcd} & \prod_n Map_{\mathcal{M}}(X_i, Y_i) \arrow[d] \\ \prod_n Map_{\mathcal{M}}(X_i, Y_i) \arrow[r] & \prod_n Map_{\mathcal{M}}(X_i, U(Y_{i+1})) \end{tikzcd} \tag{2.67}$$

where

- the horizontal map is the product of the maps

$$Map_{\mathcal{M}}(X_i, Y_i) \to Map_{\mathcal{M}}\big(X_i, U(Y_{i+1})\big) \tag{2.68}$$

  induced by the composition with the adjoint $\bar{\sigma}_i : Y_i \to U(Y_{i+1})$;
- The vertical map is the product of the compositions

$$Map_{\mathcal{M}}(X_{i+1}, Y_{i+1}) \to Map_{\mathcal{M}}\big(U(X_{i+1}), U(Y_{i+1})\big) \to Map_{\mathcal{M}}\big(X_i, U(Y_{i+1})\big) \tag{2.69}$$

  where the first map is induced by U and the second map is the composition with $X_i \to U(X_{i+1})$.

Its points correspond to the collections $f = \{f_i\}_{i\in\mathbb{N}}$ for which the diagrams

[11] The reader is left with the easy exercise of checking that the following conditions are equivalent for a Quillen adjunction (G, U) between simplicial model categories: (i) G is enriched; (ii) G is compatible with the simplicial action, meaning that for any simplicial set K and any object X we have $G(K \otimes X) \simeq K \otimes G(X)$; (iii) U is compatible with the coaction, meaning that any for any simplicial set K and object Y we have $U(Y^K) \simeq U(Y)^K$; (iv) U is enriched.

$$
\begin{tikzcd}
X_i \arrow[r] \arrow[d, "f_i"] & U(X_{i+1}) \arrow[d, "U(f_{i+1})"] \\
Y_i \arrow[r] & U(Y_{i+1})
\end{tikzcd}
\tag{2.70}
$$

commute.

As discussed in the preliminaries, the underlying $(\infty,1)$-categories of $Sp^{\mathbb{N}}(\mathcal{M},G)_{proj}$ and $Sp^{\mathbb{N}}(\mathcal{M},G)_{stable}$ are given, respectively by the simplicial nerves $N_\Delta((Sp^{\mathbb{N}}(\mathcal{M},G)_{proj})^\circ)$ and $N_\Delta((Sp^{\mathbb{N}}(\mathcal{M},G)_{stable})^\circ)$ and by construction the last appears as the full reflexive subcategory of the first, spanned by the U-spectrum objects.

Up to this point we have two different notions of spectrum-objects. Of course they are related. To understand the relation we observe first that $Sp^{\mathbb{N}}(\mathcal{M},G)$ fits in a strict pullback diagram of simplicial categories

$$
\begin{tikzcd}
Sp^{\mathbb{N}}(\mathcal{M},G) \arrow[r] \arrow[d] & \prod_n(\mathcal{M}^I) \arrow[d] \\
\prod_n \mathcal{M} \arrow[r] & \prod_n \mathcal{M}\times\mathcal{M}
\end{tikzcd}
\tag{2.71}
$$

where the top horizontal map is the product of all maps of the form $(X_i)_{i\in\mathbb{N}} \mapsto (X_i \to U(X_{i+1}))$ and the vertical-left map sends a spectrum-object to its underlying sequence of objects. The right-vertical map sends a morphism in $\mathcal{M}$ to its respective source and target and the lower-horizontal map is the product of the compositions $(X_i)_{i\in\mathbb{N}} \mapsto (X_i, X_{i+1}) \mapsto (X_1, U(X_{i+1}))$. All the maps in this diagram are compatible with the simplicial enrichment. We fabricate a new diagram which culminates in (2.66).

$$
\begin{tikzcd}
 & & & & \prod_n(\mathcal{M}^I)^\circ_{triv} \arrow[d, hook] \\
 & & & & \prod_n(\mathcal{M}^I)^\circ \arrow[d, hook] \\
Sp^{\mathbb{N}}(\mathcal{M},G)^\circ_{stable} \arrow[r, hook] \arrow[rrrruu, dashed, "f"] \arrow[dr, dashed, "a'"'] & Sp^{\mathbb{N}}(\mathcal{M},G)^\circ_{proj} \arrow[r, hook] \arrow[rrru, dashed, "e"] \arrow[d, dashed, "a"] & Sp^{\mathbb{N}}(\mathcal{M},G) \arrow[r, "x"'] \arrow[d, "y"] & \prod_n(\mathcal{M}^I) \arrow[r, "b"] \arrow[d, "z"] & \prod_n(\mathcal{M}^I) \arrow[d] \\
 & \prod_n \mathcal{M}^\circ \arrow[r, hook] \arrow[rrd, dashed, "c"] & \prod_n \mathcal{M} \arrow[r, "w"'] & \prod_n \mathcal{M}\times\mathcal{M} & \prod_n \mathcal{M}^\circ\times\mathcal{M}^\circ \\
 & & & \prod_n \mathcal{M}^\circ\times\mathcal{M}^{fib} \arrow[u, hook] \arrow[ur, "d"] &
\end{tikzcd}
\tag{2.72}
$$

where the maps

1. x, y, z, w are the maps in the diagram (2.71);
2. a is the restriction of the projection $Sp^{\mathbb{N}}(\mathcal{M}, G) \to \prod_n \mathcal{M}$ (it is well-defined because the cofibrant–fibrant objects in $Sp^{\mathbb{N}}(\mathcal{M}, G)$ are supported on sequences of cofibrant–fibrant objects in $\mathcal{M}$);
3. a' is the composition of a with the canonical inclusion;
4. b is the product of the compositions

$$\mathcal{M}^I \xrightarrow{Q} \mathcal{M}^I \times \mathcal{M}^I \longrightarrow \mathcal{M}^I \tag{2.73}$$

   where $\mathcal{Q}$ is the machine associated to our chosen simplicial functorial factorization of the form "(cofibration, trivial fibration)" (sending a morphism $f : A \to B$ in $\mathcal{M}$ to the pair $(u : A \to X, v : X \to Y)$ with u a cofibration and v a trivial fibration) and the second arrow is the projection in the first coordinate;
5. c is induced by composition of w with the canonical inclusion. Given a sequence of cofibrant–fibrant objects $(X_i)_{i\in\mathbb{N}}$, we have $w((X_i)_{i\in\mathbb{N}}) = (X_i, U(X_i + 1))_{i\in\mathbb{N}}$ with X_i fibrant–cofibrant and $U(X_{i+1})$ fibrant (because U is a right-Quillen functor). Therefore, the composition factors through $\prod_n \mathcal{M}^\circ \times \mathcal{M}^{fib}$ and c is well-defined;
6. To obtain d, we consider first the composition

$$\mathcal{M}^\circ \times \mathcal{M} \longrightarrow \mathcal{M}^\circ \times \mathcal{M}^I \xrightarrow{id\times Q} \mathcal{M}^\circ \times \left(\mathcal{M}^I \times \mathcal{M}^I\right) \longrightarrow \mathcal{M}^\circ \times \mathcal{M}^I \longrightarrow \mathcal{M}^\circ \times \mathcal{M} \tag{2.74}$$

   where the first arrow sends $(X, Y) \mapsto (X, \emptyset \to X)$, the third arrow is induced by the projection of $\mathcal{M}^I \times \mathcal{M}^I \to \mathcal{M}^I$ on the first coordinate and the last arrow is induced by taking the source. All together, this composition is sending a pair (X, Y) to the pair $(X, Q(Y))$ with Q a cofibrant-replacement of Y using the same factorization device of the item (4). In particular, if Y is already fibrant, $Q(Y)$ will be cofibrant–fibrant and we have a dotted arrow

$$\begin{tikzcd} \mathcal{M}^\circ \times \mathcal{M} \arrow[r] & \mathcal{M}^\circ \times \mathcal{M} \\ \mathcal{M}^\circ \times \mathcal{M}^{fib} \arrow[u, hook] \arrow[r, dashed] & \mathcal{M}^\circ \times \mathcal{M}^\circ \arrow[u, hook] \end{tikzcd} \tag{2.75}$$

   rendering the diagram commutative.
   By definition, d is the product of all these dotted maps;
7. e is the map induced by composing $b \circ x$ with the canonical inclusion and it is well-defined for the reasons given also in (2);
8. f is deduced from e by restricting to the U-spectra objects: If $(X_i)_{i\in\mathbb{N}}$ is a U-spectra, the canonical maps $X_i \to U(X_{i+1})$ are weak-equivalences and therefore, when we

perform the factorization encoded in the composition $b \circ x$, the first map is necessarily a trivial cofibration and therefore f factors through $\prod_n (\mathcal{M}^I)^\circ_{triv}$.

Finally, the fact that everything commutes is obvious from the definition of factorization system. All together, we found a commutative diagram

$$\begin{tikzcd} Sp^{\mathbb{N}}(\mathcal{M}, G)^\circ_{stable} \arrow[r] \arrow[d] & \prod_n (\mathcal{M}^I)^\circ_{triv} \arrow[d] \\ \prod_n \mathcal{M}^\circ \arrow[r] & \prod_n \mathcal{M}^\circ \times \mathcal{M}^\circ \end{tikzcd} \tag{2.76}$$

In summary, the upper horizontal map sends a U-spectra $X = (X_i)_{i \in \mathbb{N}}$ to the list of trivial cofibrations $(X_i \to Q(U(X_{i+1})))_{i \in \mathbb{N}}$ and the left-vertical map sends X to its underlying sequence of cofibrant–fibrant objects. By considering the simplicial nerve of the diagram above and using the equivalence of diagrams in (2.65), we obtain, using the universal property of the strict pullback, a map

$$\phi : N_\Delta\big(\big(Sp^{\mathbb{N}}(\mathcal{M}, G)_{stable}\big)^\circ\big) \to Stab_{(\bar{G},\bar{U})}\big(N_\Delta\big(\mathcal{M}^\circ\big)\big) \tag{2.77}$$

where we identify $Stab_{(\bar{G},\bar{U})}(N_\Delta(\mathcal{M}^\circ))$ with the strict pullback of the diagram (2.63).

The following result clarifies this already long story:

Proposition 2.13. *Let $\mathcal{M}$ be a combinatorial simplicial model category and let $G : \mathcal{M} \to \mathcal{M}$ be a left simplicial Quillen functor with a right adjoint U. Let $Sp^{\mathbb{N}}(\mathcal{M}, G)^{stable}$ denote the combinatorial simplicial model category of [43] equipped the stable model structure. Then, the canonical map induced by the previous commutative diagram*

$$\phi : N_\Delta\big(\big(Sp^{\mathbb{N}}(\mathcal{M}, G)_{stable}\big)^\circ\big) \to Stab_{(\bar{G},\bar{U})}\big(N_\Delta\big(\mathcal{M}^\circ\big)\big) \tag{2.78}$$

is an equivalence of $(\infty, 1)$-categories.

Proof. We will prove this by checking the map is essentially surjective and fully-faithful. We start with the essential surjectivity. For that we can restrict ourselves to study of the map induced between the maximal ∞-groupoids (Kan-complexes) on both sides.

$$N_\Delta\big(\big(Sp^{\mathbb{N}}(\mathcal{M}, G)_{stable}\big)^\circ\big)^{\simeq} \to Stab_{(\bar{G},\bar{U})}\big(N_\Delta\big(\mathcal{M}^\circ\big)\big)^{\simeq} \tag{2.79}$$

To conclude the essential surjectivity it suffices to check that the map induced between the π_0's

$$\pi_0\big(N_\Delta\big(\big(Sp^{\mathbb{N}}(\mathcal{M}, G)_{stable}\big)^\circ\big)^{\simeq}\big) \to \pi_0\big(Stab_{(\bar{G},\bar{U})}\big(N_\Delta\big(\mathcal{M}^\circ\big)\big)^{\simeq}\big) \tag{2.80}$$

is surjective. We start by analyzing the right-side. First, the operation $(-)^{\simeq}$ commutes with homotopy limits. To see this, notice that both the $(\infty,1)$-category of homotopy types $\mathcal{S}$ and the $(\infty,1)$-category of small $(\infty,1)$-categories Cat_∞ are presentable. The combinatorial simplicial model category of simplicial sets with the Quillen structure is a strict model for the first and $\hat{\Delta}_+$ models the second. By combining Theorem 3.1.5.1 and Proposition 5.2.4.6 of [59], the inclusion $\mathcal{S} \subseteq Cat_\infty$ is in fact a Bousfield (a.k.a. reflexive) localization and its the left adjoint can be understood (by its universal property) as the process of inverting all the morphisms. By combining Proposition 3.3.2.5 and Corollaries 3.3.4.3 and 3.3.4.6 of [59], we deduce that the inclusion $\mathcal{S} \subseteq Cat_\infty$ commutes with colimits. Since $\mathcal{S}$ and Cat_∞ are presentable, by the Adjoint Functor Theorem (see Corollary 5.5.2.9 of [59]), the inclusion $\mathcal{S} \subseteq Cat_\infty$ admits a right adjoint which, by its universal property can be identified with the operation $(-)^{\simeq}$. An immediate application of this fact is that $\pi_0(Stab_{(\bar{G},\bar{U})}(N_\Delta(\mathcal{M}^\circ))^{\simeq})$ is in bijection with the π_0 of the homotopy limit of the tower of Kan-complexes

$$\dots \xrightarrow{\bar{U}} N_\Delta\big(\mathcal{M}^\circ\big)^{\simeq} \xrightarrow{\bar{U}} N_\Delta\big(\mathcal{M}^\circ\big)^{\simeq} \xrightarrow{\bar{U}} N_\Delta\big(\mathcal{M}^\circ\big)^{\simeq} \tag{2.81}$$

Using the Reedy structure (on $\hat{\Delta}$ with the Quillen structure), we can find a morphism of towers

$$\begin{tikzcd} \dots \arrow[r, "\bar{U}"] & N_\Delta(\mathcal{M}^\circ)^{\simeq} \arrow[r, "\bar{U}"] \arrow[d] & N_\Delta(\mathcal{M}^\circ)^{\simeq} \arrow[r, "\bar{U}"] \arrow[d] & N_\Delta(\mathcal{M}^\circ)^{\simeq} \arrow[d] \\ \dots \arrow[r] & T_2 \arrow[r] & T_1 \arrow[r] & T_0 \end{tikzcd} \tag{2.82}$$

where the vertical maps are weak-equivalences of simplicial sets for the Quillen structure, every object is again a Kan-complex but this time the maps in the lower tower are fibrations. By the nature of the weak-equivalences, this morphism of diagrams becomes an isomorphism at the level of the π_0's

$$\begin{tikzcd} \dots \arrow[r] & \pi_0(N_\Delta(\mathcal{M}^\circ)^{\simeq}) \arrow[r, "\pi_0(\bar{U})"] \arrow[d, "\sim"] & \pi_0(N_\Delta(\mathcal{M}^\circ)^{\simeq}) \arrow[r, "\pi_0(\bar{U})"] \arrow[d, "\sim"] & \pi_0(N_\Delta(\mathcal{M}^\circ)^{\simeq}) \arrow[d, "\sim"] \\ \dots \arrow[r] & \pi_0(T_2) \arrow[r] & \pi_0(T_1) \arrow[r] & \pi_0(T_0) \end{tikzcd} \tag{2.83}$$

and therefore the limits $lim_{\mathbb{N}^{op}}\pi_0(N_\Delta(\mathcal{M}^\circ)^{\simeq})$ and $lim_{\mathbb{N}^{op}}\pi_0(T_i)$ are isomorphic. Finally, using the Milnor exact sequence associated to a tower of fibrations together with the fact that fibrations of simplicial sets are surjective (see Proposition VI-2.15 and Proposition VI-2.12-2 in [39]) we deduce an isomorphism

$$\pi_0(lim_{\mathbb{N}^{op}}T_i) \simeq lim_{\mathbb{N}^{op}}\pi_0(T_i) \tag{2.84}$$

and by combining everything we have

$$\pi_0\big(Stab_{(\bar{G},\bar{U})}\big(N_\Delta\big(\mathcal{M}^\circ\big)\big)^{\simeq}\big) \simeq lim_{\mathbb{N}^{op}}\pi_0\big(N_\Delta\big(\mathcal{M}^\circ\big)^{\simeq}\big) \tag{2.85}$$

where the right hand side can be identified with the strict limit of the tower of sets

$$\dots \longrightarrow \pi_0\big(N_\Delta\big(\mathcal{M}^\circ\big)^{\simeq}\big) \xrightarrow{\pi_0(\bar{U})} \pi_0\big(N_\Delta\big(\mathcal{M}^\circ\big)^{\simeq}\big) \xrightarrow{\pi_0(\bar{U})} \pi_0\big(N_\Delta\big(\mathcal{M}^\circ\big)^{\simeq}\big) \tag{2.86}$$

and since $\bar{U}$ can be identified with $Q \circ U$, the elements of the last can be presented as sequences $([X_i])_{i\in\mathbb{N}}$ with each $[X_i]$ an equivalence class of an object X_i in $N_\Delta(\mathcal{M}^\circ)$, satisfying $[QU(X_{i+1})] = [X_i]$, which is the same as stating the existence of an equivalence in $N_\Delta(\mathcal{M}^\circ)$ between X_i and $QU(X_{i+1})$. Since we are dealing with cofibrant–fibrant objects, we can find an actual homotopy equivalence $X_i \to QU(X_{i+1})$ and by choosing a representative for each $[X_i]$ together with composition maps $X_i \to QU(X_{i+1}) \to U(X_{i+1})$ we retrieve a U-spectra. This proves that the map is essentially surjective.

It remains to prove ϕ is fully-faithful. Given two U-spectrum objects $X = (X_i)_{i\in\mathbb{N}}$ and $Y = (Y_i)_{i\in\mathbb{N}}$, the mapping space in $N_\Delta((Sp^{\mathbb{N}}(\mathcal{M},G)_{stable})^\circ)$ between X and Y is given by the pullback[12] of the diagram

$$\begin{tikzcd} & \prod_n Map_{\mathcal{M}}(X_i,Y_i) \arrow[d] \\ \prod_n Map_{\mathcal{M}}(X_i,Y_i) \arrow[r] & \prod_n Map_{\mathcal{M}}(X_i,U(Y_{i+1})) \end{tikzcd} \tag{2.87}$$

All vertices in this diagram are given by Kan-complexes (because $\mathcal{M}$ is a simplicial model category, each Y_i and X_i is cofibrant–fibrant and U is right-Quillen) and the vertical map is a fibration. Indeed, it can be identified with product of the compositions

$$Map_{\mathcal{M}}(X_{i+1},Y_{i+1}) \to Map_{\mathcal{M}}\big(G(X_i),Y_{i+1}\big) \simeq Map_{\mathcal{M}}\big(X_i,U(Y_{i+1})\big) \tag{2.88}$$

where the last isomorphism follows from the adjunction data and the first map is the fibration induced by the composition with structure maps $G(X_i) \to X_{i+1}$ of X (which are cofibrations because X is a U-spectra). Therefore, the pullback square is a homotopy pullback.

At the same time, because of the equivalence of diagrams (2.65) the mapping spaces in $Stab_{(\bar{G},\bar{U})}(N_\Delta(\mathcal{M}^\circ))$ between the image of X and the image of Y can obtained[13] as the homotopy pullback of

[12] See the formula (2.67).

[13] The mapping spaces in the homotopy pullback are the homotopy pullback of the mapping spaces.

$$\begin{tikzcd}
& \prod_n Map_{\mathcal{M}}(X_i, Y_i) \arrow[d, "U"] \\
& \prod_n Map_{\mathcal{M}}(U(X_i), U(Y_i)) \arrow[d, "Q"] \\
& \prod_n Map_{\mathcal{M}}(QU(X_i), QU(Y_i)) \arrow[d] \\
\prod_n Map_{\mathcal{M}}(X_i, Y_i) \arrow[r] & \prod_n Map_{\mathcal{M}}(X_i, QU(Y_{i+1}))
\end{tikzcd} \tag{2.89}$$

To conclude the proof it suffices to produce a weak-equivalence between the formulas. Indeed, we produce a map from the diagram (2.89) to the diagram (2.87), using the identity maps in the outer vertices and in the corner we use the product of the maps induced by the composition with the canonical map $QU(Y_{i+1}) \to U(Y_{i+1})$.

$$Map_{\mathcal{M}}\big(X_i, QU(Y_{i+1})\big) \to Map_{\mathcal{M}}\big(X_i, U(Y_{i+1})\big) \tag{2.90}$$

Of course, this map is a trivial fibration: $\mathcal{M}$ is a simplicial model category, X_i is cofibrant and $QU(Y_{i+1}) \to U(Y_{i+1})$ is a trivial fibration. □

In the situation of Proposition 2.13, with $\mathcal{M}$ a combinatorial simplicial model category and G a left-simplicial Quillen functor, we know that $Sp^{\mathbb{N}}(\mathcal{M}, G)^{\circ}_{stable}$ is again combinatorial and simplicial and so, both the underlying $(\infty,1)$-categories $N_\Delta(\mathcal{M}^\circ)$ and $N_\Delta(Sp^{\mathbb{N}}(\mathcal{M}, G)^{\circ}_{stable})$ are presentable (see Proposition A.3.7.6 of [59]). Finally, using Remark 2.10 we deduce the existence of canonical equivalence between $N_\Delta(Sp^{\mathbb{N}}(\mathcal{M}, G)^{\circ}_{stable})$ and the colimit of the sequence

$$N_\Delta\big(\mathcal{M}^\circ\big) \xrightarrow{\bar{G}} N_\Delta\big(\mathcal{M}^\circ\big) \xrightarrow{\bar{G}} \ldots \tag{2.91}$$

2.2.2. Stabilization and symmetric monoidal structures

Let us proceed. Our goal now is to compare the construction of spectra with the formal inversion $\mathcal{C}[X]^{\otimes}$. The idea of a relation between the two comes from the following classical theorem:

Theorem 2.14. *(See Theorem 4.3 of [107].) Let $\mathcal{C}$ be a symmetric monoidal category with tensor product $\otimes$ and unit 1. Let X be an object in $\mathcal{C}$. Let $Stab_X(\mathcal{C})$ denote the colimit of the sequence*

$$\ldots \xrightarrow{X\otimes -} \mathcal{C} \xrightarrow{X\otimes -} \mathcal{C} \xrightarrow{X\otimes -} \mathcal{C} \xrightarrow{X\otimes -} \ldots \tag{2.92}$$

in Cat (up to equivalence). Then, if the action of the cyclic permutation on $X \otimes X \otimes X$ becomes an identity map in $\mathcal{C}$ after tensoring with X an appropriate amount of times *(which is the same as saying it is the identity map in $Stab_X(\mathcal{C})$) the category $Stab_X(\mathcal{C})$ admits a canonical symmetric monoidal structure and the canonical functor $\mathcal{C} \to Stab_X(\mathcal{C})$ is monoidal, sends X to an invertible object and is universal with respect to this property.*

Proof. This is well-known. See [79, Prop. 4.2.5]. □

Remark 2.15. The condition on X appearing in the previous result is trivially satisfied if the action of the cyclic permutation $(X \otimes X \otimes X)^{(1,2,3)}$ is already an identity map in $\mathcal{C}$. For instance, this particular situation holds when $\mathcal{C}$ is the pointed $\mathbb{A}^1$-homotopy category and X is $\mathbb{P}^1$ (see Theorem 4.3 and Lemma 4.4 of [107]).

Our goal now is to find an analogue for the previous theorem in the context of symmetric monoidal $(\infty, 1)$-categories.

Definition 2.16. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal $(\infty, 1)$-category and let X be an object in $\mathcal{C}$. We say that X is *symmetric* if there is a 2-equivalence in $\mathcal{C}$ between the cyclic permutation $\sigma : (X \otimes X \otimes X)^{(1,2,3)}$ and the identity map of $X \otimes X \otimes X$. In other words, we demand the existence of a 2-simplex in $\mathcal{C}$

$$\begin{tikzcd} X \otimes X \otimes X \arrow[r, "\sigma"] \arrow[d, "id"'] \arrow[dr, phantom, "\Rightarrow"] & X \otimes X \otimes X \\ X \otimes X \otimes X \arrow[ur, "id"'] & \end{tikzcd} \tag{2.93}$$

providing a homotopy between the cyclic permutation and the identity. This is equivalent to the condition that σ is the identity of $X \otimes X \otimes X$ in $h(\mathcal{C})$.

This notion of symmetry is well behaved under equivalences. Moreover, it is immediate that monoidal functors map symmetric objects to symmetric objects.

Remark 2.17. Let $\mathcal{V}$ be a symmetric monoidal model category with a cofibrant unit 1. Recall that a unit interval I is a cylinder object for the unit of the monoidal structure $I := C(1)$, together with a map $I \otimes I \to I$ such that the diagrams

$$\begin{tikzcd} 1 \otimes I \simeq I \arrow[r, "\pi"] \arrow[d, "\partial_0 \otimes Id_I"'] & 1 \arrow[d, "\partial_0"] \\ I \otimes I \arrow[r] & I \end{tikzcd} \tag{2.94}$$

$$\begin{tikzcd} I \otimes 1 \simeq I \arrow[r, "\pi"] \arrow[d, "Id_I \otimes \partial_0"'] & 1 \arrow[d, "\partial_0"] \\ I \otimes I \arrow[r] & I \end{tikzcd} \tag{2.95}$$

and

$$\begin{tikzcd} I \otimes 1 \simeq I \arrow[d, "\partial_1 \otimes Id_I"'] \arrow[dr, "Id_I"] & \\ I \otimes I \arrow[r] & I \end{tikzcd} \tag{2.96}$$

commute, where $\partial_0, \partial_1 : 1 \to I$ and $\pi : I \to 1$ are the maps providing I with a structure of cylinder object.

Recall also that two maps $f, g : A \to B$ are said to be homotopic with respect to a unit interval I if there is a map $H : A \otimes I \to B$ rendering the diagram commutative

$$\begin{tikzcd} A \simeq A \otimes 1 \arrow[dr, "Id_A \otimes \partial_0"'] \arrow[drr, bend left, "f"] & & \\ & A \otimes I \arrow[r, "H"] & B \\ A \simeq A \otimes 1 \arrow[ur, "id_A \otimes \partial_1"] \arrow[urr, bend right, "g"'] & & \end{tikzcd} \tag{2.97}$$

In [43, Defn. 10.2], the author defines an object X of $\mathcal{V}$ to be *symmetric* if it is cofibrant and if there is a *unit interval* I, together with a homotopy

$$H : X \otimes X \otimes X \otimes I \to X \otimes X \otimes X \tag{2.98}$$

between the cyclic permutation σ and the identity map. We observe that if an object X is symmetric in the sense of [43] then it is symmetric as an object in the underlying symmetric monoidal $(\infty, 1)$-category of $\mathcal{V}$ in the sense of Definition 2.16. Indeed, since $\mathcal{V}$ is a symmetric monoidal model category with a cofibrant unit, the full subcategory $\mathcal{V}^c$ of cofibrant objects is closed under the tensor product and therefore inherits a monoidal structure, which moreover preserves weak-equivalences in each variable. As explained in the preliminaries, let $N((\mathcal{V}^c)^{\otimes})[W_c^{-1}]$ denote its underlying symmetric monoidal $(\infty, 1)$-category. Its underlying $(\infty, 1)$-category is $N(\mathcal{V}^c)[W^{-1}]$ and its homotopy category is the classical localization in *Cat*. Moreover, it comes canonically equipped with a monoidal functor $L : N^{\otimes}((\mathcal{V}^c)^{\otimes}) \to N^{\otimes}((\mathcal{V}^c)^{\otimes})[W_c^{-1}]$. Now, if X is symmetric in $\mathcal{V}$ in the sense of [43], the homotopy H forces σ to become the identity in $h(N(\mathcal{V}^c)[W^{-1}])$ (because the classical localization functor is monoidal and the map $I \to 1$ is a weak-equivalence). The conclusion now follows from the commutativity of the diagram induced by the unit of the adjunction (h, N)

$$\begin{tikzcd} N(\mathcal{V}^c) \arrow[r] \arrow[d, "\sim"'] & N(\mathcal{V}^c)[W^{-1}] \arrow[d] \\ N(h(N(\mathcal{V}^c))) \arrow[r] & N(h(N(\mathcal{V}^c)[W^{-1}])) \end{tikzcd} \tag{2.99}$$

and the fact that the both horizontal arrows are monoidal and therefore send the cyclic permutation of the monoidal structure in $\mathcal{V}$ to the cyclic permutation associated to the monoidal structure in $N((\mathcal{V}^c)^{\otimes})[W_c^{-1}]$.

We now come to the generalization of Theorem 10.3 of [43]. The following results relate our formal inversion of an object to the construction of spectrum objects.

Remark 2.18. Let $\mathcal{C}^{\otimes}$ be a small monoidal $(\infty, 1)$-category and let $\overline{M}$ be an object in $Mod_{\mathcal{C}^{\otimes}}(Cat_{\infty})$ (which we will understand as a left-module). Since Cat_{∞} admits classifying objects for endomorphisms given by the categories of endofunctors, the data of $\overline{M}$ is equivalent to the data of an $(\infty, 1)$-category $M := \overline{M}(\mathbf{m})$ together with a monoidal functor $T^{\otimes} : \mathcal{C}^{\otimes} \to End(M)^{\otimes}$ where the last is endowed with the associative monoidal structure induced by the composition of maps of simplicial sets. If X is an object in $\mathcal{C}$, the endofunctor $T(X) : M \to M$ corresponds to the action of X in M by means of the operation $\mathcal{C} \times M \to M$ encoded in the module-structure. We will call it the *multiplication by* X.

Notice that if the monoidal structure $\mathcal{C}^{\otimes}$ is symmetric, the map $T(X)$ acquires the structure of a map of $\mathcal{C}$-modules. Indeed, as $T^{\otimes}$ is monoidal, it will send an object $(Y, X) \in \mathcal{C}^{\otimes}_{\langle 2\rangle}$ to $(T(Y), T(X))$ in $End(M)^{\otimes}_{\langle 2\rangle}$ and the twisting equivalence $\tau_{Y,X} : (Y, X) \simeq (X, Y)$ to an equivalence $(T(Y), T(X)) \simeq (T(X), T(Y))$. By the definition of cocartesian morphisms in $End(M)^{\otimes}$, the last equivalence provides a natural equivalence $T(Y) \circ T(X) \simeq T(X) \circ T(Y)$ that gives the coherence data making $T(X)$ a map of modules. These coherences define commutative diagrams $\Delta[1] \times \Delta[1] \to Cat_{\infty}$ that we can informally describe as

$$\begin{tikzcd} M \arrow[r, "T(Y)"] \arrow[d, "T(X)"'] & M \arrow[d, "T(X)"] \arrow[dl, Rightarrow, "T(\tau_{Y,X})"] \\ M \arrow[r, "T(Y)"'] & M \end{tikzcd} \tag{2.100}$$

More generally, the extra coherences that make $T(X)$ a map of modules are given by the higher order cyclic permutations of factors in $\mathcal{C}^{\otimes}$. The importance of this fact will become clear in the next proposition.

The following is our key result:

Proposition 2.19. *Let $\mathcal{C}^{\otimes}$ be a small symmetric monoidal $(\infty, 1)$-category and X be a symmetric object in $\mathcal{C}$. Then, for any $\mathcal{C}^{\otimes}$-module $\overline{M}$, the colimit of the diagram of $\mathcal{C}^{\otimes}$-modules*

$$\overline{Stab}_X(\overline{M}) := colimit_{Mod_{\mathcal{C}^{\otimes}}(Cat_\infty)}(\ldots \longrightarrow \overline{M} \xrightarrow{\overline{T}(X)} \overline{M} \xrightarrow{\overline{T}(X)} \overline{M} \xrightarrow{\overline{T}(X)} \ldots) \tag{2.101}$$

is a $\mathcal{C}^{\otimes}$-module where the multiplication by X is an equivalence.

Proof. Let $d : N(\mathbb{Z}) \to Mod_{\mathcal{C}}(Cat_\infty)$ be the diagram corresponding to the multiplication by X. Since $Cat_\infty^{\times}$ is compatible with all small colimits, Corollary 3.4.4.6 of [63][14] implies that d can be extended to a colimit diagram $d' : N(\mathbb{Z})^{\triangleright} \to Mod_{\mathcal{C}}(Cat_\infty)$. Moreover, this extension is a colimit diagram if and only if the composition with the forgetful functor to Cat_∞ is a colimit diagram. Let ∞ denote the new joint vertice in $N(\mathbb{Z})^{\triangleright}$ and set $\overline{Stab}_X(\overline{M}) := d'(\infty)$. Moreover, let $\phi_i := d'(i \to \infty)$. As a first step we need to understand how an object $Y \in \mathcal{C}$ acts on this new module $\overline{Stab}_X(\overline{M})$. For that purpose we observe that as the ϕ_i are, by definition, maps of modules, we have commutative diagrams

$$\begin{tikzcd}
M \arrow[r, "T(Y)"] \arrow[d, "\phi_i"] & M \arrow[d, "\phi_i"] \\
\overline{Stab}_X(\overline{M}) \arrow[r, dashed, "Y"] & \overline{Stab}_X(\overline{M})
\end{tikzcd}$$

and as the $T(X)'s$ are maps of $\mathcal{C}$-modules, this action of Y on $\overline{Stab}_X(\overline{M})$ appears as the canonical map (induced by the universal property of colimits) produced by the morphism of diagrams $D' : N(\mathbb{Z})^{\triangleright} \times \Delta[1] \to Mod_{\mathcal{C}}(Cat_\infty)$ levelwise given by $T(Y)$. This can be obtained as follows: we consider the diagram $D : N(\mathbb{Z}) \times \Delta[1] \to Mod_{\mathcal{C}}(Cat_\infty)$ obtained by composing the commutative diagrams described in Remark 2.18 side by side. This can also be written as $D : N(\mathbb{Z}) \to Fun(\Delta[1], Mod_{\mathcal{C}}(Cat_\infty))$. By [59, 5.1.2.3] this diagram admits a colimit cone $D' : N(\mathbb{Z})^{\triangleright} \to Fun(\Delta[1], Mod_{\mathcal{C}}(Cat_\infty))$ characterized by the fact that both the source and target of $D'(\infty)$ are colimit cones of the restrictions to 1 and 0. This presents the action of Y on $\overline{Stab}_X(\overline{M})$ as a colimit of the actions of Y on M. More informally, we now can picture the situation as

$$\begin{tikzcd}
\overline{Stab}_X(\overline{M}) \arrow[d, dashed, "Y"] & \ldots \arrow[r] & M \arrow[r, "T(X)"] \arrow[d, "T(Y)"'] & M \arrow[r, "T(X)"] \arrow[d, "T(Y)"] \arrow[dl, Rightarrow, "T(\tau)"] & M \arrow[r] \arrow[d, "T(Y)"] \arrow[dl, Rightarrow, "T(\tau)"] & \ldots \\
\overline{Stab}_X(\overline{M}) & \ldots \arrow[r] & M \arrow[r, "T(X)"'] & M \arrow[r, "T(X)"'] & M \arrow[r] & \ldots
\end{tikzcd} \tag{2.102}$$

[14] Since we are working the commutative setting, we could also refer to Corollary 4.2.3.5 of [63].

Our goal now is to understand this action when $Y = X$. But before that it, is important to understand the consecutive composition of two commutative diagrams

$$\begin{tikzcd}
M \arrow[r, "T(X)"] \arrow[d, "T(Y)"'] & M \arrow[r, "T(Z)"] \arrow[d, "T(Y)"] \arrow[dl, Rightarrow, "T(\tau_{X,Y})"] & M \arrow[d, "T(Y)"] \arrow[dl, Rightarrow, "T(\tau_{Z,Y})"] \\
M \arrow[r, "T(X)"'] & M \arrow[r, "T(Z)"'] & M
\end{tikzcd} \tag{2.103}$$

This can be informally describe as a new commutative square

$$\begin{tikzcd}
M \arrow[r, "T(Z)\circ T(X)"] \arrow[d, "T(Y)"'] & M \arrow[d, "T(Y)"] \arrow[dl, Rightarrow, "T(\tau_{Z,Y})\circ T(\tau_{X,Y})"] \\
M \arrow[r, "T(Z)\circ T(X)"'] & M
\end{tikzcd} \tag{2.104}$$

and our main observation is that the horizontal composition $T(\tau_{Z,Y}) \circ T(\tau_{Y,X})$ can be identified with the natural transformation $T(\sigma_{Z,X,Y})$ induced by the cyclic permutation $\sigma_{Z,X,Y} : (Z, X, Y) \to (Y, Z, X)$ in $\mathcal{C}^{\otimes}_{\langle 3\rangle}$. Indeed, $T^{\otimes}$ being monoidal, the equivalence $\sigma_{Z,X,Y}$ produces an equivalence $(T(Z), T(X), T(Y)) \simeq (T(Y), T(Z), T(X))$ which by choosing cocartesian morphisms in $End(M)^{\otimes}$ over $\langle 3\rangle \to \langle 1\rangle$, give the commutativity $T(Z) \circ T(X) \circ T(Y) \simeq T(Y) \circ T(Z) \circ T(X)$. The key point to complete the argument is that the permutation $\sigma_{Z,X,Y} : (Z, X, Y) \to (Y, Z, X)$ can be written as a composition of two consecutive twists, namely $\sigma_{Z,X,Y} \simeq (\tau_{Z,Y}, id_X) \circ (id_Z, \tau_{X,Y})$ and as $T^{\otimes}$ is functorial we have $T(\tau_{Z,Y}) \circ T(\tau_{X,Y}) \simeq T(\sigma_{Z,X,Y})$

Let us now go back to the case when Y and Z are X. In this case, since by assumption X is symmetric, there is a 2-simplex in $\mathcal{C}$ providing a homotopy between σ and the identity of $X \otimes X \otimes X$. In this case $T(\sigma)$ is equivalent to the identity and the 2-simplex rendering the commutativity of the composition (2.104) are the identity faces. By confinality, the map $\overline{Stab}_X(M) \to \overline{Stab}_X(M)$ induced by the morphism of diagrams D is equivalent to the map induced by the $\mathbb{Z}$-indexed diagram given by the composition of the commutative squares

$$\begin{tikzcd}
\dots \arrow[r, "T(X)"] & M \arrow[r, "T(X)"] \arrow[d, "T(X)"'] & M \arrow[r, "T(X)"] \arrow[d, "T(X)"'] \arrow[dl, Rightarrow, "Id"] & M \arrow[r, "T(X)"] \arrow[d, "T(X)"] \arrow[dl, Rightarrow, "Id"] & \dots \\
\dots \arrow[r, "T(X)"'] & M \arrow[r, "T(X)"'] & M \arrow[r, "T(X)"'] & M \arrow[r, "T(X)"'] & \dots
\end{tikzcd} \tag{2.105}$$

and therefore, by definition of colimit cone, it is an equivalence. $\square$

Remark 2.20. A similar argument shows that the same result holds if X is n-symmetric, meaning that, there exists $n \in \mathbb{N}, n \geq 2$ such that τ^n is equal to the identity map in $h(\mathcal{C})$.

Remark 2.18 and Proposition 2.19 apply mutatis mutandis in the presentable setting. This is true because of the following result

Proposition 2.21. *$\mathcal{P}r^L$ admits classifying objects for endomorphisms. If M is a presentable $(\infty,1)$-category, $End^L(M)$ is a classifying object for endomorphisms of M, with the associative monoidal structure given by the composition of functors.*

Proof. See [79, Prop. 3.6.3] $\square$

We can finally establish the connection between the adjoint $\mathcal{L}^{Pr}_{(\mathcal{C}^{\otimes},X)}$ and the notion of spectra.

Corollary 2.22. *Let $\mathcal{C}^{\otimes}$ be a presentable symmetric monoidal $(\infty,1)$-category and let X be a symmetric object in $\mathcal{C}$. Given a $\mathcal{C}^{\otimes}$-module M, $Stab_X(M)$ is a $\mathcal{C}^{\otimes}$-module where X acts as an equivalence and therefore the adjunction of Proposition 2.9 provides a map of $\mathcal{C}^{\otimes}$-modules*

$$\mathcal{L}^{Pr}_{(\mathcal{C}^{\otimes},X)}(M) \to Stab_X(M) \tag{2.106}$$

This map is an equivalence. In particular, the underlying ∞-category of the formal inversion $\mathcal{C}^{\otimes}[X^{-1}]$ is equivalent to the stabilization $Stab_X(\mathcal{C})$.

Proof. The map can be obtained as a composition

$$\mathcal{L}^{Pr}_{(\mathcal{C}^{\otimes},X)}(M) \to \mathcal{L}^{Pr}_{(\mathcal{C}^{\otimes},X)}\big(Stab_X(M)\big) \to Stab_X(M) \tag{2.107}$$

where the first arrow is the image of the canonical map $M \to Stab_X(M)$ by the adjunction $\mathcal{L}^{Pr}_{(\mathcal{C}^{\otimes},X)}$ and the second arrow is the counit of the adjunction. In fact, with our hypothesis and because of the previous proposition, the action of X is invertible in $Stab_X(M)$ and therefore, by Proposition 2.9 the second arrow is an equivalence It remains to prove that the first map is an equivalence. But now, since $Stab_X(M)$ is a colimit and $\mathcal{L}^{Pr}_{(\mathcal{C}^{\otimes},X)}$ is a left adjoint and therefore commutes with colimits, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{L}^{Pr}_{(\mathcal{C}^{\otimes},X)}(M) \arrow[r] \arrow[dr] & \mathcal{L}^{Pr}_{(\mathcal{C}^{\otimes},X)}(Stab_X(M)) \\
& Stab_X(\mathcal{L}^{Pr}_{(\mathcal{C}^{\otimes},X)}(M)) \arrow[u, "\sim"]
\end{tikzcd} \tag{2.108}$$

where the diagonal arrow is the colimit map induced by the stabilization of $\mathcal{L}^{Pr}_{(\mathcal{C}^{\otimes},X)}(M)$. It is enough now to observe that if M is a $\mathcal{C}^{\otimes}$-module where the action of X is already invertible, then the canonical map $M \to Stab_X(M)$ is an equivalence of modules. The *2 out of 3* argument concludes the proof. □

In particular

Corollary 2.23. *Let* $\mathcal{C}^{\otimes}$ *be a stable presentable symmetric monoidal* $(\infty,1)$*-category and let* X *be a symmetric object in* $\mathcal{C}$*. Then* $\mathcal{C}^{\otimes}[X^{-1}]$ *is again a stable presentable symmetric monoidal* $(\infty,1)$*-category.*

Proof. If $\mathcal{C}^{\otimes}$ is stable presentable, the multiplication by X is an exact functor. Moreover, since X is symmetric, the previous corollary provides an equivalence $\mathcal{C}[X^{-1}] \simeq Stab_X(\mathcal{C})$ where the last is a colimit in $\mathcal{P}r^L$. Moreover, since the whole diagram is in $\mathcal{P}r^L_{Stb}$ and the last has all colimits and the inclusion $\mathcal{P}r^L_{Stb} \subseteq \mathcal{P}r^L$ commutes with them,[15] we find that $\mathcal{C}[X^{-1}]$ is stable. Moreover, since by construction $\mathcal{C}^{\otimes}[X^{-1}]$ is a presentable symmetric monoidal $(\infty,1)$-category, we conclude it is a stable presentable symmetric monoidal $(\infty,1)$-category. □

Example 2.24. In [63] the author introduces the $(\infty,1)$-category of spectra Sp as the stabilization of the $(\infty,1)$-category of spaces. More precisely, following the notations of Example 2.11 it is given by

$$Sp := Sp^{\mathbb{N}}_{(\Sigma_{\mathcal{S}},\Omega_{\mathcal{S}})}(\mathcal{S}_{*/}) \tag{2.109}$$

where $\mathcal{S}$ denotes the $(\infty,1)$-category of spaces. By Propositions and 1.4.3.6 and 1.4.4.4 of [63] this $(\infty,1)$-category is presentable and stable and by Proposition 4.8.2.18 of [63] it admits a natural presentable stable symmetric monoidal structure $Sp^{\otimes}$ which can be described by means of a universal property: it is an initial object in $CAlg(\mathcal{P}r^L_{Stb})$. The unit of this monoidal structure is the sphere-spectrum.

Our Corollary 2.22 provides an alternative characterization of this symmetric monoidal structure. We start with $\mathcal{S}_*$ the $(\infty,1)$-category of pointed spaces. Recall that this $(\infty,1)$-category is presentable and admits a monoidal structure given by the so-called *smash product* of pointed spaces (see Remark 4.8.2.14 of [63] and Section 2.4.2 below). We will denote it as $\mathcal{S}_*^{\wedge}$. According to Proposition 6.3.2.11 of [63], $\mathcal{S}_*^{\wedge}$ has a universal property amongst the presentable pointed symmetric monoidal $(\infty,1)$-categories: it is an initial one. The unit of this monoidal structure is the pointed space $S^0 = * \coprod *$. We will see below (Corollary 2.34 and Remark 2.35) that $\mathcal{S}_*^{\wedge}$ is the underlying symmetric monoidal $(\infty,1)$-category of the combinatorial simplicial model category of pointed simplicial sets $\hat{\Delta}_*$ equipped with the classical smash product of spaces. Since S^1 is symmetric

[15] To see this we can use the equivalence between $\mathcal{P}r^L_{Stb}$ and $Mod_{Sp}(\mathcal{P}r^L)$ [63, 4.8.2.18] and the identification of the inclusion $\mathcal{P}r^L_{Stb} \subseteq \mathcal{P}r^L$ with the forgetful functor $Mod_{Sp}(\mathcal{P}r^L) \to \mathcal{P}r^L$. Now we use the fact that $\mathcal{P}r^{L,\otimes}$ is compatible with colimits (its has internal-hom objects) and therefore colimits of modules are computed in $\mathcal{P}r^L$ using the forgetful functor [63, 3.4.4.6].

in $\hat{\Delta}_*$ with respect to this classical smash (see Lemma 6.6.2 of [42]), by Remark 2.17 it will also be symmetric in $\mathcal{S}_*^\wedge$. Our inversion $\mathcal{S}_*^\wedge[(S^1)^{-1}]$ provides a new presentable symmetric monoidal $(\infty,1)$-category and because of the symmetry of S^1, the fact that $(S^1 \wedge -)$ can be identified with $\Sigma_{\mathcal{S}}$ and Corollary 2.22, we conclude that the underlying $(\infty,1)$-category of $\mathcal{S}_*^\wedge[(S^1)^{-1}]$ is the stabilization defining Sp and therefore that $\mathcal{S}_*^\wedge[(S^1)^{-1}]$ is a presentable stable symmetric monoidal $(\infty,1)$-category. By the universal property of $Sp^\otimes$ there is a unique (up to a contractible space of choices) monoidal map

$$Sp^\otimes \to \mathcal{S}_*^\wedge\big[\big(S^1\big)^{-1}\big] \tag{2.110}$$

At the same time, since every stable presentable $(\infty,1)$-category is pointed, the universal property of $\mathcal{S}_*^\wedge$ ensures the existence of a canonical morphism

$$\mathcal{S}_*^\wedge \to Sp^\otimes \tag{2.111}$$

which is also unique up to a contractible space of choices. This morphism is just the canonical stabilization morphism and it sends S^1 to the sphere-spectrum in Sp and therefore the universal property of the localization provides a factorization

$$\mathcal{S}_*^\wedge\big[\big(S^1\big)^{-1}\big] \to Sp^\otimes \tag{2.112}$$

which is unique up to homotopy. By combining the two universal properties we find that these two maps are in fact inverses up to homotopy

Remark 2.25. The technique of inverting an object provides a way to define the monoidal stabilization of a pointed presentable symmetric monoidal $(\infty,1)$-category $\mathcal{C}^\otimes$. It follows from Proposition 6.3.2.11 of [63] that for any such $\mathcal{C}^\otimes$, there is an essentially unique (base-point preserving and colimit preserving) monoidal map $f : \mathcal{S}_*^\wedge \to \mathcal{C}^\otimes$. Let $f(S^1)$ denote the image of the topological circle through this map. The (presentable) universal property of inverting an object provides a homotopy commutative diagram of commutative algebra objects in $\mathcal{P}r^L$

$$\begin{tikzcd} Sp^\otimes \simeq \mathcal{S}_*^\wedge[(S^1)^{-1}] \arrow[d, dashed] & \mathcal{S}_*^\wedge \arrow[l] \arrow[d, "f"] \\ \mathcal{C}^\otimes[f(S^1)^{-1}] & \mathcal{C}^\otimes \arrow[l] \end{tikzcd} \tag{2.113}$$

The monoidal map $\mathcal{S}_*^\wedge \to Sp^\otimes$ produces a forgetful functor

$$CAlg\big(\mathcal{P}r^L\big)_{Sp^\otimes/} \to CAlg\big(\mathcal{P}r^L\big)_{\mathcal{S}_*^\wedge/} \tag{2.114}$$

which by Proposition 2.9 is fully faithful and admits a left adjoint given by the base-change formula $\mathcal{C}^\otimes \mapsto Sp^\otimes \otimes_{\mathcal{S}_*^\wedge} \mathcal{C}^\otimes$. The combination of the universal property of the adjunction and the universal property of inverting an object ensures the existence of an equivalence of pointed symmetric monoidal $(\infty,1)$-categories

$$\mathcal{C}^{\otimes}\left[f\left(S^1\right)^{-1}\right] \simeq Sp^{\otimes} \otimes_{\mathcal{S}_*^{\wedge}} \mathcal{C}^{\otimes} \tag{2.115}$$

Finally, combining this with Example 4.8.1.22 of [63] we deduce that the underlying $(\infty,1)$-category of $\mathcal{C}^{\otimes}[f(S^1)^{-1}]$ is the stabilization $Stab(\mathcal{C})$.

Moreover, we deduce also that if $\mathcal{C}^{\otimes}$ is a stable presentable symmetric monoidal $(\infty,1)$-category and X is any object in $\mathcal{C}$, in order to conclude that the inversion $\mathcal{C}^{\otimes}[X^{-1}]$ is stable presentable it is enough to show that $\mathcal{C}[X^{-1}]$ is pointed, thus extending the result 2.23. Indeed, by the previous discussion, $\mathcal{C}$ is stable if and only if $f(S^1)$ is invertible. Since the inversion functor $\mathcal{C}^{\otimes} \to \mathcal{C}^{\otimes}[X^{-1}]$ is monoidal, the image of $f(S^1)$ in $\mathcal{C}[X^{-1}]$ will again by invertible. Finally, if $\mathcal{C}[X^{-1}]$ is pointed, the image of $f(S^1)$ will necessarily correspond to the image of S^1 in $\mathcal{C}[X^{-1}]$, which therefore will be invertible, and so, by the previous discussion, $\mathcal{C}[X^{-1}]$ will be stable.

2.3. Connection with the symmetric spectrum objects of Hovey

We recall from [43] the construction of symmetric spectrum objects: Let $\mathcal{V}$ be a combinatorial simplicial symmetric monoidal model category and let $\mathcal{M}$ be a combinatorial simplicial $\mathcal{V}$-model category. Following Theorem 8.11 of [43], for any object X in $\mathcal{V}$ we can produce a new combinatorial simplicial $\mathcal{V}$-model category $Sp^{\Sigma}(\mathcal{M},X)$ of spectrum objects in $\mathcal{M}$ endowed with the *stable model structure* and where X acts by an equivalence. In particular, by considering $\mathcal{V}$ as a $\mathcal{V}$-model category (using the monoidal structure) the new model category $Sp^{\Sigma}(\mathcal{V},X)$ inherits the structure of a combinatorial simplicial symmetric monoidal model category and there is left simplicial Quillen monoidal map $\mathcal{V} \to Sp^{\Sigma}(\mathcal{V},X)$ sending X to an invertible object.

This general construction sends an arbitrary combinatorial simplicial $\mathcal{V}$-model category to a combinatorial simplicial $\mathcal{V}$-model category where the action of X is invertible. In fact, by Theorem 8.11 of [43] $Sp^{\Sigma}(\mathcal{M},X)$ is a combinatorial simplicial $Sp^{\Sigma}(\mathcal{V},X)$-model category. This is a first sign of the fundamental role of the construction of symmetric spectrum objects as an adjoint in the spirit of Section 2.1. We have canonical simplicial left Quillen maps

$$Sp^{\Sigma}(\mathcal{V},X) \xrightarrow{\sim} Sp^{\mathbb{N}}\left(Sp^{\Sigma}(\mathcal{V},X),X\right) \xrightarrow{\sim} Sp^{\Sigma}\left(Sp^{\mathbb{N}}(\mathcal{V},X),X\right) \longleftarrow Sp^{\mathbb{N}}(\mathcal{V},X) \tag{2.116}$$

but in general the last map is not an equivalence. By Theorem 9.1 of [43] for the last map to be an equivalence we only need $Sp^{\mathbb{N}}(\mathcal{V},X)$ to be a $\mathcal{V}$-model category where X acts as an equivalence. This is exactly the functionality of the symmetric condition on X (see Theorems 10.1 and 10.3 in [43]).

We now state our main result

Theorem 2.26. *Let $\mathcal{V}$ be a combinatorial simplicial symmetric monoidal model category whose unit is cofibrant and let X be a symmetric object in $\mathcal{V}$ in the sense of Remark 2.17. Let $Sp^{\Sigma}(\mathcal{V}, X)$ denote the combinatorial simplicial symmetric monoidal model category provided by Theorem 8.11 of [43], equipped the convolution product. Let $\mathcal{C}^{\otimes}$ and $Sp^{\Sigma}_X(\mathcal{C})^{\otimes}$ denote their underlying presentable symmetric monoidal $(\infty,1)$-categories.*[16] *The left-Quillen monoidal map $\mathcal{V} \to Sp^{\Sigma}(\mathcal{V}, X)$ induces a monoidal functor $\mathcal{C}^{\otimes} \to Sp^{\Sigma}_X(\mathcal{C})^{\otimes}$ which sends X to an invertible object, endowing $Sp^{\Sigma}_X(\mathcal{C})^{\otimes}$ with the structure of object in $CAlg(\mathcal{P}r^L)^X_{\mathcal{C}^{\otimes}/}$. In this case, the adjunction of Proposition 2.9 provides a monoidal map*

$$\mathcal{C}^{\otimes}[X^{-1}] \simeq \mathcal{L}^{Pr,\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}) \to Sp^{\Sigma}_X(\mathcal{C})^{\otimes} \tag{2.117}$$

We claim that this map is an equivalence of presentable symmetric monoidal $(\infty,1)$-categories.

Proof. By Remark 2.17 if X is symmetric in the sense of [43] then it is symmetric in $\mathcal{C}^{\otimes}$ in the sense of Definition 2.16.

By definition, the map is obtained as a composition

$$\mathcal{L}^{Pr,\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}) \longrightarrow \mathcal{L}^{Pr,\otimes}_{(\mathcal{C}^{\otimes},X)}(Sp^{\Sigma}_X(\mathcal{C})^{\otimes}) \longrightarrow Sp^{\Sigma}_X(\mathcal{C})^{\otimes} \tag{2.118}$$

where the last arrow is the counit of the adjunction of Proposition 2.9. To prove that this map is an equivalence it is enough to verify that the map between the underlying $(\infty,1)$-categories

$$\mathcal{L}^{Pr}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}) \to Sp^{\Sigma}_X(\mathcal{C}) \tag{2.119}$$

is an equivalence. But now, by the combination of Corollary 2.22 with the main result of Corollary 10.4 in [43], we find a commutative diagram of equivalences

$$\begin{tikzcd}
\mathcal{L}^{Pr}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}) \arrow[rr] \arrow[d, "\sim"] & & Sp^{\Sigma}_X(\mathcal{C}) = N_{\Delta}(Sp^{\Sigma}(\mathcal{V},X)^{\circ}) \arrow[d, "\sim"] \\
Stab_X(\mathcal{C}) \simeq N_{\Delta}(Sp^{\mathbb{N}}(\mathcal{V},X)^{\circ}) \arrow[r, "\sim"] & Stab_X(N_{\Delta}(Sp^{\Sigma}(\mathcal{V},X)^{\circ})) \simeq & N_{\Delta}(Sp^{\mathbb{N}}(Sp^{\Sigma}(\mathcal{V},X),X)^{\circ})
\end{tikzcd} \tag{2.120}$$

where the left vertical map is an equivalence because X is symmetric in $\mathcal{C}^{\otimes}$; the equivalence $Stab_X(\mathcal{C}) \simeq N_{\Delta}(Sp^{\mathbb{N}}(\mathcal{V},X)^{\circ})$ follows from Proposition 2.13 with $G = (X \otimes -)$ (it is a left Quillen functor because X is cofibrant), and the fact that $\mathcal{C}$ is presentable; the

[16] By Corollary 4.1.3.16 of [63] we have monoidal equivalences $\mathcal{C}^{\otimes} \simeq \mathcal{N}^{\otimes}_{\Delta}((\mathcal{V}^{\circ})^{\otimes})$ and $Sp^{\Sigma}_X(\mathcal{C})^{\otimes} \simeq N^{\otimes}_{\Delta}((Sp^{\Sigma}(\mathcal{V},X)^{\circ})^{\otimes})$ and therefore both $\mathcal{C}^{\otimes}$ and $Sp^{\Sigma}_X(\mathcal{C})^{\otimes}$ are presentable symmetric monoidal $(\infty,1)$-categories.

right vertical map is an equivalence because X is already invertible in $N_\Delta(Sp^\Sigma(\mathcal{V},X)^\circ)$ and because a Quillen equivalence between combinatorial model categories induces an equivalence between the underlying $(\infty,1)$-categories (see Lemma 1.3.4.21 of [63]). This same last argument, together with Corollary 10.4 of [43], justifies the fact that the lower horizontal map is an equivalence. □

Remark 2.27. In the proof of Theorem 2.26, we used the condition on X twice. The first using the result of [43] and the second with Proposition 2.13. We believe the use of this condition is not necessary. Indeed, everything comes down to prove an analogue of Proposition 2.13 for the construction of symmetric spectrum objects, replacing the natural numbers by some more complicated partially ordered set. If such a result is possible, then the construction of symmetric spectra in the combinatorial simplicial case can be presented as a colimit of a diagram of simplicial categories. In this case, Proposition 2.19 would follow immediately even without the condition on X. We will not pursue this topic here since it won't be necessary for our goals.

Example 2.28. The combination of Theorem 2.26 together with Remark 2.17 and Example 2.24 provides a canonical equivalence of presentable symmetric monoidal presentable $(\infty,1)$-categories $Sp^\otimes \simeq N_\Delta^\otimes(Sp^\Sigma(\hat{\Delta}_*,S^1))$.

2.4. A universal characterization of the motivic stable homotopy theory of schemes

Let $\mathbb{U} \in \mathbb{V} \in \mathbb{W}$ be universes. In the following sections, we shall write $Sm^{ft}(S)$ to denote the $\mathbb{V}$-small category of smooth separated $\mathbb{U}$-small schemes of finite type over a Noetherian $\mathbb{U}$-scheme S.

2.4.1. $\mathbb{A}^1$-homotopy theory of schemes

The main idea in the subject is to "do homotopy theory with schemes" in more or less the same way we do with spaces, by thinking of the affine line $\mathbb{A}^1$ as an "interval". One first difficulty is that the category of schemes does not admit all colimits. In [67], the authors constructed a *place* to realize this idea. The construction proceeds as follows: start from the category of schemes and add formally all the colimits. Then make sure that the following two principles hold:

I) the line $\mathbb{A}^1$ becomes contractible;
II) if X is a scheme and U and V are two open subschemes whose union equals X in the category of schemes then make sure that their union continues to be X in the new place;

The original construction in [67] was performed using the techniques of model category theory and this *place* is the homotopy category of a model category $\mathcal{M}_{\mathbb{A}^1}$. During the last years their methods were revisited and reformulated in many different ways. In

[31], the author presents a "universal" characterization of the original construction using the theory of Bousfield localizations for model categories[17] together with a universal characterization of the theory of simplicial presheaves, within model categories. The construction of [31] can be summarized by the expression

$$\mathcal{M}_{\mathbb{A}^1} = L_{\mathbb{A}^1} L_{HyperNis}\big(\big(SPsh\big(Sm^{ft}(S)\big)\big)\big) \tag{2.121}$$

where $SPsh(-)$ stands for simplicial presheaves with the projective model structure, $L_{HyperNis}$ corresponds the Bousfield localization with respect to the collection of the hypercovers associated to the *Nisnevich topology* (see below) and $L_{\mathbb{A}^1}$ corresponds to the Bousfield localization with respect to the collection of all projection maps $X \times \mathbb{A}^1 \to X$.

It is clear today that model categories should not be taken as fundamental objects, but rather, we should focus on their associated $(\infty, 1)$-categories. In this section, we use the insights of [31] to perform the construction of an $(\infty, 1)$-category $\mathcal{H}(S)$ directly within the setting of ∞-categories. By the construction, it will have a universal property and using the link described in Section 1.5.2 and the theory developed by J. Lurie in [59] relating Bousfield localizations to localizations of ∞-categories, we will be able to prove that $\mathcal{H}(S)$ is equivalent to the ∞-category underlying the $\mathbb{A}^1$ model category of Morel–Voevodsky.

The construction of $\mathcal{H}(S)$ proceeds as follows. We start from the category of smooth schemes of finite type over $S - Sm^{ft}(S)$ and consider it as a trivial $\mathbb{V}$-small $(\infty, 1)$-category $N(Sm^{ft}(S))$. Together with the *Nisnevich topology* [71], it acquires the structure of an ∞-*site* (see Definition 6.2.2.1 of [59]). By definition (see Def. 1.2 of [67]) the Nisnevich topology is the topology generated by the pre-topology whose covering families of an S-scheme X are the collections of étale morphisms $\{f_i : U_i \to X\}_{i \in I}$ such that for any $x \in X$ there exists an $i \in I$ and $u_i \in U_i$ such that f_i induces an isomorphism between the residual fields $k(x) \simeq k(u_i)$. Recall from [67] (Def. 1.3) that an elementary Nisnevich square is a commutative square of schemes

$$\begin{tikzcd} p^{-1}(U) \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & X \end{tikzcd} \tag{2.122}$$

such that

a) $i : U \hookrightarrow X$ is an open immersion of schemes;
b) $p : V \to X$ is an étale map;
c) the square (2.122) is a pullback. In particular $p^{-1}(U) \to V$ is also an open immersion;

[17] See [41].

d) the canonical projection $p^{-1}(X-U) \to X-U$ is an isomorphism where we consider the closed subsets $Z := X-U$ and $p^{-1}Z$ both equipped with the reduced structures of closed subschemes;

and from this we can easily deduce that

e) the square

$$\begin{tikzcd} V & p^{-1}(Z) \arrow[l] \arrow[d] \\ X \arrow[from=u, "p"'] & Z := X-U \arrow[l] \end{tikzcd} \tag{2.123}$$

is a pullback with both Z and $p^{-1}(Z)$ equipped with the reduced structures;

e) the square (2.122) is a pushout.

The crucial fact is that each family $(V \to X, U \to X)$ as above forms a Nisnevich covering and the families of this form provide a *basis* for the Nisnevich topology (see Proposition 1.4 of [67]). We consider the very big $(\infty,1)$-category $\mathcal{P}^{big}(N(Sm^{ft}(S))) := Fun(N(Sm^{ft}(S))^{op}, \widehat{\mathcal{S}})$ of presheaves of (big) homotopy types over $N(Sm^{ft}(S))$ (see Section 5.1 of [59]) which has the expected universal property (Thm. 5.1.5.6 of [59]): it is the free completion of $N(Sm^{ft}(S))$ with $\mathbb{V}$-small colimits (in the sense of ∞-categories). Using Proposition 4.2.4.4 of [59] we can immediately identify $\mathcal{P}^{big}(N(Sm^{ft}(S)))$ with the underlying ∞-category of the model category of simplicial presheaves on $Sm^{ft}(S)$ endowed with the projective model structure. The results of [59] provide an ∞-analogue for the classical Yoneda embedding, meaning that we have a fully faithful map of ∞-categories $j : N(Sm^{ft}(S)) \to \mathcal{P}^{big}(N(Sm^{ft}(S)))$ and as usual we will identify a scheme X with its image $j(X)$. We now restrict to those objects in $\mathcal{P}^{big}(N(Sm^{ft}(S)))$ which are sheaves with respect to the Nisnevich topology. Because the Nisnevich squares form a basis for the Nisnevich topology, an object $F \in \mathcal{P}^{big}(N(Sm^{ft}(S)))$ is a sheaf iff it maps Nisnevich squares to pullback squares. In particular, every representable $j(X)$ is a sheaf (because Nisnevich squares are pushouts). Following [59, 5.5.4.15], the inclusion of the full subcategory $Sh^{big}_{Nis}(Sm^{ft}(S)) \subseteq \mathcal{P}^{big}(N(Sm^{ft}(S)))$ admits a left adjoint (which is known to be exact – Lemma 6.2.2.7 of [59]) and provides a canonical example of an ∞-topos (see Definition 6.1.0.4 of [59]). More importantly to our needs, this is an example of a presentable localization of a presentable $(\infty,1)$-category.

Remark 2.29. When S is Noetherian of finite Krull dimension, the category of smooth schemes $Sm^{ft}(S)$ can be replaced by the category of *affine* smooth schemes of finite type over S, $N(AffSm^{ft})(S)$, and the resulting $(\infty,1)$-categories $Sh^{big}_{Nis}(Sm^{ft}(S))$ and $Sh^{big}_{Nis}(N(AffSm^{ft})(S))$ are equivalent. This follows because we can identify $Sm^{ft}(S)$ with a full subcategory of $\mathcal{P}^{big}(N(AffSm^{ft})(S))$ using the map sending a smooth scheme X to the representable functor $Y \in N(AffSm^{ft}(k)) \mapsto Hom_{Sm^{ft}(S)}(Y, X)$, and this identification is compatible with the Nisnevich topologies. For more details see [66]. See also [61, Section 2].

Next step, we consider the *hypercompletion* of the ∞-topos $Sh^{big}_{Nis}(Sm^{ft}(S))$ (see Section 6.5.2 of [59]). By construction, it is a presentable localization of $Sh^{big}_{Nis}(Sm^{ft}(S))$ and by Corollary 6.5.3.13 of [59] it coincides with $Sh^{big}_{Nis}(Sm^{ft}(S))^{hyp}$: the localization of $\mathcal{P}^{big}(N(Sm^{ft}(S)))$ spanned by the objects which are local with respect to the class of *Nisnevich hypercovers*.

Finally, we reach the last step: We will restrict ourselves to those sheaves in $Sh^{big}_{Nis}(Sm^{ft}(S))^{hyp}$ satisfying $\mathbb{A}^1$-invariance, meaning those sheaves F such that for any scheme X, the canonical map induced by the projection $F(X) \to F(X \times \mathbb{A}^1)$ is an equivalence. More precisely, we consider the localization of $Sh^{big}_{Nis}(Sm^{ft}(S))^{hyp}$ with respect to the class of all projection maps $\{X \times \mathbb{A}^1 \to X\}_{X \in Obj(Sm^{ft}(S))}$. We will write $\mathcal{H}(S)$ for the result of this localization and write $l_{\mathbb{A}^1} : Sh^{big}_{Nis}(Sm^{ft}(S))^{hyp} \to \mathcal{H}(S)$ for the localization functor. Notice that $\mathcal{H}(S)$ is very big, presentable with respect to the universe $\mathbb{V}$. It is also clear from the construction that $\mathcal{H}(S)$ comes naturally equipped with a universal characterization:

Theorem 2.30. *Let $Sm^{ft}(S)$ be the category of smooth schemes of finite type over a base Noetherian scheme S and let $L : N(Sm^{ft}(S)) \to \mathcal{H}(S)$ denote the composition of localizations*

$$N\big(Sm^{ft}(S)\big) \to \mathcal{P}^{big}\big(N\big(Sm^{ft}(S)\big)\big) \to Sh^{big}_{Nis}\big(Sm^{ft}(S)\big) \to Sh^{big}_{Nis}\big(Sm^{ft}(S)\big)^{hyp} \to \mathcal{H}(S) \tag{2.124}$$

Then, for any $(\infty,1)$-category $\mathcal{D}$ with all $\mathbb{V}$-small colimits, the map induced by composition with L

$$Fun^L\big(\mathcal{H}(S), \mathcal{D}\big) \to Fun\big(N\big(Sm^{ft}(S)\big), \mathcal{D}\big) \tag{2.125}$$

is fully faithful and its essential image is the full subcategory of $Fun(N(Sm^{ft}(S)), \mathcal{D})$ spanned by those functors satisfying Nisnevich descent *and* $\mathbb{A}^1$-invariance. *The left-side denotes the full subcategory of $Fun(\mathcal{H}(S), \mathcal{D})$ spanned by the colimit preserving maps.*

Proof. The proof follows from the combination of the universal property of presheaves with the universal properties of each localization in the construction and from the fact that for the Nisnevich topology in $Sm^{ft}(S)$, descent is equivalent to hyperdescent (see [106, Prop. 5.9] or [67, 3-1.16] or [61, Section 1]) and therefore the localization $Sh^{big}_{Nis}(Sm^{ft}(S)) \to Sh^{big}_{Nis}(Sm^{ft}(S))^{hyp}$ is an equivalence. □

Our goal now is to provide the evidence that $\mathcal{H}(S)$ really is the underlying $(\infty,1)$-category of the $\mathbb{A}^1$- model category of Morel–Voevodsky. In fact, we already saw that our first step coincides with the first step in the construction of $\mathcal{M}_{\mathbb{A}^1}$ – simplicial presheaves are a model for ∞-presheaves. It remains to prove that our localizations produce the same results as the Bousfield localizations. But of course, this follows from the results the appendix of [59] applied to the model category $\mathcal{M} := SPsh(Sm^{ft}(S))$.

Remark 2.31. It is important to remark that the sequence of functors in Theorem 2.30 can be promoted to a sequence of monoidal functors with respect to the cartesian monoidal structures

$$\begin{aligned} &N\big(Sm^{ft}(S)\big)^{\times} \to \mathcal{P}^{big}\big(N\big(Sm^{ft}(S)\big)\big)^{\times} \to Sh^{big}_{Nis}\big(Sm^{ft}(S)\big)^{\times} \\ &\simeq \big(Sh^{big}_{Nis}\big(Sm^{ft}(S)\big)^{hyp}\big)^{\times} \to \mathcal{H}(S)^{\times} \end{aligned} \tag{2.126}$$

The first is the Yoneda map which we know commutes with limits. The second map is the sheafification functor which we also know is left exact. The last functor is a monoidal localization because of the definition of the $\mathbb{A}^1$-equivalences. These localized monoidal structures are cartesian because of the existence of fully faithful right adjoints. Furthermore, they are presentable – this follows from the general results of [59,63]. See also our survey in [79, Remark 3.6.1]

2.4.2. The monoidal structure in $\mathcal{H}(S)_*$

Let $\mathcal{H}(S)$ be the $(\infty,1)$-category introduced in the last section. Since it is presentable it admits a final object $*$ and the $(\infty,1)$-category of pointed objects $\mathcal{H}(S)_*$ is also presentable (see [59, Prop. 5.5.2.10]). In this case, since the forgetful functor $\mathcal{H}(S)_* \to \mathcal{H}(S)$ commutes with limits, by the Adjoint Functor Theorem (see [59, Cor. 5.5.2.9]) it admits a left adjoint $()_+ : \mathcal{H}(S) \to \mathcal{H}(S)_*$ which we can identify with the formula $X \mapsto X_+ := X \coprod *$. In order to follow the stabilization methods of Morel–Voevodsky we need to explain how the cartesian product in $\mathcal{H}(S)$ extends to a symmetric monoidal structure in $\mathcal{H}(S)_*$ and how the pointing morphism becomes monoidal.

This problem fits in a more general setting. Recall that the $(\infty,1)$-category of spaces $\mathcal{S}$ is the unit for the symmetric monoidal structure $\mathcal{P}r^{L,\otimes}$. In [63, Prop. 6.3.2.11] it is proved that the pointing morphism $-\coprod * : \mathcal{S} \to \mathcal{S}_*$ endows $\mathcal{S}_*$ with the structure of an idempotent object in $\mathcal{P}r^{L,\otimes}$ and proves that its associated local objects are exactly the pointed presentable $(\infty,1)$-categories. It follows from the general theory of idempotents that the product functor $\mathcal{C} \mapsto \mathcal{C} \otimes \mathcal{S}_*$ is a left adjoint to the inclusion functor $\mathcal{P}r^L_{Pt} \subseteq \mathcal{P}r^L$. Also from the general theory, this left adjoint is monoidal. The final ingredient is that for any presentable $(\infty,1)$-category $\mathcal{C}$ there is an equivalence of $(\infty,1)$-categories $\mathcal{C}_* \simeq \mathcal{C} \otimes \mathcal{S}_*$ (see [63, Example 4.8.1.20]) and via this equivalence, the pointing map $\mathcal{C} \to \mathcal{C}_*$ is equivalent to the product map $id_{\mathcal{C}} \otimes ()_+ : \mathcal{C} \otimes \mathcal{S} \to \mathcal{C} \otimes \mathcal{S}_*$ where $()_+$ denotes the pointing map of spaces. Altogether, we have the following result

Corollary 2.32 *(Lurie). The formula $\mathcal{C} \mapsto \mathcal{C}_*$ defines a monoidal left adjoint to the inclusion $\mathcal{P}r^L_{Pt} \subseteq \mathcal{P}r^L$ and therefore induces a left adjoint to the inclusion $CAlg(\mathcal{P}r^L_{Pt}) \subseteq$*

$CAlg(\mathcal{P}r^L)$. In other words, for any presentable symmetric monoidal $(\infty,1)$-category $\mathcal{C}^{\otimes}$, there exists a pointed presentable symmetric monoidal $(\infty,1)$-category $\mathcal{C}_*^{\wedge(\otimes)}$ whose underlying $(\infty,1)$-category is $\mathcal{C}_*$, together with a monoidal functor $\mathcal{C}^{\otimes}\to\mathcal{C}_*^{\wedge(\otimes)}$ extending the pointing map $\mathcal{C}\to\mathcal{C}_*$, and satisfying the following universal property:

$(*)$ *for any pointed presentable symmetric monoidal $(\infty,1)$-category $\mathcal{D}^{\otimes}$, the composition*

$$Fun^{\otimes,L}\big(\mathcal{C}_*^{\wedge(\otimes)},\mathcal{D}^{\otimes}\big)\to Fun^{\otimes,L}\big(\mathcal{C}^{\otimes},\mathcal{D}^{\otimes}\big) \tag{2.127}$$

is an equivalence.

Remark 2.33. In the situation of the previous corollary, given a functor $F:\mathcal{C}\to\mathcal{D}$ with $\mathcal{D}$ being pointed, its canonical extension $\tilde{F}$

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{D} \\ \mathcal{C}_* \arrow[ur, dashed, "\tilde{F}"] & \end{tikzcd} \tag{2.128}$$

is naturally identified with the formula $(u:*\to X)\mapsto cofiber F(u)\in\mathcal{D}$.

The symmetric monoidal structure $\mathcal{C}_*^{\wedge(\otimes)}$ will be called the *smash product induced by* $\mathcal{C}^{\otimes}$. Of course, if $\mathcal{C}^{\otimes}$ is already pointed we have an equivalence $\mathcal{C}_*^{\wedge(\otimes)}\simeq\mathcal{C}^{\otimes}$. In the particular case when $\mathcal{C}^{\otimes}$ is cartesian, we will use the notation $\mathcal{C}_*^{\wedge}:=\mathcal{C}_*^{\wedge(\times)}$.

Let us now progress in a different direction. Let $\mathcal{M}$ be a combinatorial simplicial model category. Assume also that $\mathcal{M}$ is cartesian closed and that its final object $*$ is cofibrant. This makes $\mathcal{M}$ a symmetric monoidal model category with respect to the cartesian product and we have a monoidal equivalence

$$N_{\Delta}^{\otimes}\big((\mathcal{M}^{\circ})^{\times}\big)\simeq N_{\Delta}(\mathcal{M}^{\circ})^{\times} \tag{2.129}$$

Moreover, because the cartesian product provides a Quillen bifunctor, $N_{\Delta}(\mathcal{M}^{\circ})^{\times}$ is a presentable symmetric monoidal $(\infty,1)$-category and therefore, using the Corollary above, we can equip $N_{\Delta}(\mathcal{M}^{\circ})_*$ with a canonical presentable symmetric monoidal structure $N_{\Delta}(\mathcal{M}^{\circ})_*^{\wedge}$ for which the pointing map becomes monoidal

$$N_{\Delta}(\mathcal{M}^{\circ})^{\times}\to N_{\Delta}(\mathcal{M}^{\circ})_*^{\wedge} \tag{2.130}$$

Independently of this, we can consider the natural model structure in $\mathcal{M}_*$ (see Remark 1.1.8 in [42]). Again, it is combinatorial and simplicial and comes canonically equipped with a left-Quillen functor $(-)_+:\mathcal{M}\to\mathcal{M}_*$ defined by the formula $X\mapsto X\coprod *$. Moreover, it acquires the structure of a symmetric monoidal model category via the usual definition of the smash product, given by the formula

$$(X,x) \wedge (Y,y) := \frac{(X,x) \times (Y,y)}{(X,x) \vee (Y,y)} \tag{2.131}$$

It is well-known that this formula defines a symmetric monoidal structure with unit given by $(*)_+$ and by Proposition 4.2.9 of [42] it is compatible with the model structure in $\mathcal{M}_*$. Let $N^{\otimes}_{\Delta}(((\mathcal{M}_*)^{\circ})^{\wedge_{usual}})$ be its underlying symmetric monoidal $(\infty,1)$-category. The same result also tells us that the left-Quillen map $\mathcal{M} \to \mathcal{M}_*$ is monoidal. By functoriality (see our discussion in [79, Prop. 3.9.2]), it induces a monoidal map between the underlying symmetric monoidal $(\infty,1)$-categories.

$$f^{\otimes} : N_{\Delta}\big(\mathcal{M}^{\circ}\big)^{\times} \to N^{\otimes}_{\Delta}\big(\big((\mathcal{M}_*)^{\circ}\big)^{\wedge_{usual}}\big) \tag{2.132}$$

Of course, $N^{\otimes}_{\Delta}(((\mathcal{M}_*)^{\circ})^{\wedge_{usual}})$ is a pointed presentable symmetric monoidal $(\infty,1)$-category and by the universal property defining the smash product we obtain a monoidal map

$$N_{\Delta}\big(\mathcal{M}^{\circ}\big)^{\wedge(\otimes)}_{*} \to N^{\otimes}_{\Delta}\big(\big((\mathcal{M}_*)^{\circ}\big)^{\wedge_{usual}}\big) \tag{2.133}$$

Corollary 2.34. *The above map is an equivalence of presentable symmetric monoidal $(\infty,1)$-categories.*

Proof. It is a simple exercise with the definitions. See [79, Cor. 5.2.3]. □

Remark 2.35. If $\mathcal{M} = \hat{\Delta}$ is the model category of simplicial sets with the cartesian product, it satisfies the above conditions and we find a monoidal equivalence between $\mathcal{S}^{\wedge}_{*}$ and the underlying symmetric monoidal $(\infty,1)$-category of $\hat{\Delta}_*$ endowed with the classical smash product of pointed spaces.

Remark 2.36. If $\mathcal{C}$ is a simplicial category, the left-Quillen adjunction $\hat{\Delta} \to \hat{\Delta}_*$ extends to a left Quillen adjunction $SPsh(\mathcal{C}) \to SPsh_*(\mathcal{C})$, where $SPsh_*(\mathcal{C})$ corresponds to the category of presheaves of pointed simplicial sets over $\mathcal{C}$, endowed with the projective model structure (see [59]-Appendix). It follows that $SPsh(\mathcal{C})$ has all the good properties which intervene in the proof of Corollary 2.34 and we find a monoidal equivalence $N_{\Delta}(SPsh(\mathcal{C})^{\circ})^{\wedge}_{*} \to N^{\otimes}_{\Delta}((SPsh_*(\mathcal{C})^{\circ})^{\wedge_{usual}})$ where the last is the underlying symmetric monoidal $(\infty,1)$-category associated to the smash product in $SPsh(\mathcal{C})_*$.

Corollary 2.34 implies that

Corollary 2.37. *Let $\mathcal{H}(S)^{\times}$ be the presentable symmetric monoidal $(\infty,1)$-category underlying the model category $\mathcal{M}_{\mathbb{A}^1}$ encoding the $\mathbb{A}^1$-homotopy theory of Morel–Voevodsky together with the cartesian product. Let $(\mathcal{M}_{\mathbb{A}^1})_*$ be its pointed version with the smash product given by Lemma 2.13 of [67]. Then, the canonical map induced by the universal property of the smash product*

$$\mathcal{H}(S)_*^\wedge \to N_\Delta^\otimes\big(\big(\big((\mathcal{M}_{\mathbb{A}^1})_*\big)^\circ\big)^\wedge\big) \tag{2.134}$$

is an equivalence of presentable symmetric monoidal $(\infty, 1)$*-categories.*

In other words and as expected, $\mathcal{H}(S)_*^\wedge$ is the underlying symmetric monoidal $(\infty, 1)$-category of the classical construction.

2.4.3. The stable motivic theory

As in the original setting, we may now consider a *stabilized* version of the theory. In fact, two stabilizations are possible – one with respect to the *topological circle* $S^1 := \Delta[1]/\partial\Delta[1]$ (pointed by the image of $\partial\Delta[1]$) and another one with respect to the *algebraic circle* defined as $\mathbb{G}_m := \mathbb{A}^1 - \{0\}$. The *motivic stabilization* of the theory is by definition, the stabilization with respect to the product $\mathbb{G}_m \wedge S^1$ which we know is equivalent to $(\mathbb{P}^1, \infty)$ in $\mathcal{H}(S)_*$: consider the Nisnevich covering of $(\mathbb{P}^1, 1)$ given by two copies of $\mathbb{A}^1$ both pointed at 1, together with the maps $\mathbb{A}^1 \to \mathbb{P}^1$ sending $x \mapsto (1 : x)$, respectively, $x \mapsto (x : 1)$. Their intersection is $\mathbb{A}^1 - \{0\}$ (also pointed at 1). The result follows because this square is a pushout (as a consequence of forcing Nisnevich descent), because $\mathbb{A}^1$ is contractible in $\mathcal{H}(S)_*$ (as a consequence of forcing $\mathbb{A}^1$-invariance and Remark 2.33) and finally, because the suspension $\mathcal{H}(S)_*$ is canonically identified with the smash product with the circle (as explained by Example 2.24). The conclusion follows because $(\mathbb{P}^1, \infty)$ and $(\mathbb{P}^1, 1)$ are $\mathbb{A}^1$-homotopic via the map $x \mapsto (1 : x)$.

Definition 2.38. (See [107, Definition 5.7].) Let S be a base scheme. The *stable motivic* $\mathbb{A}^1$ ∞*-category over* S is the underlying $(\infty, 1)$-category of the presentable symmetric monoidal ∞-category $\mathcal{SH}(S)^\otimes$ defined by the formula

$$\mathcal{SH}(S)^\otimes := \mathcal{H}(S)_*^\wedge\big[(\mathbb{P}^1, \infty)^{-1}\big] \tag{2.135}$$

as in Definition 2.6.

The standard way to define the stable motivic theory is to consider the combinatorial simplicial symmetric monoidal model category $Sp^\Sigma((\mathcal{M}_{\mathbb{A}^1})_*, (\mathbb{P}^1, \infty))$ where $\mathcal{M}_*$ is equipped with the smash product. By [107, Lemma 4.4] together with Remark 2.16, we know that $(\mathbb{P}^1, \infty)$ is symmetric and consequently Theorem 2.26 ensures that $\mathcal{SH}(S)^\otimes$ recovers the classical definition. In addition, since we have an equivalence $(\mathbb{P}^1, \infty) \simeq \mathbb{G}_m \wedge S^1$, the universal properties provide canonical monoidal equivalences of presentable symmetric monoidal $(\infty, 1)$-categories

$$\begin{aligned}\mathcal{SH}(S)^\otimes &\simeq \big(\mathcal{H}(S)_*^\wedge\big)\big[\big(\mathbb{G}_m \wedge S^1\big)^{-1}\big] \simeq \big(\mathcal{H}(S)_*^\wedge\big)\big[\big((\mathbb{P}^1, \infty) \wedge S^1\big)^{-1}\big] \\ &\simeq \big(\big(\mathcal{H}(S)_*^\wedge\big)\big[\big(S^1\big)^{-1}\big]\big)\big[\big(\mathbb{P}^1, \infty\big)^{-1}\big]\end{aligned} \tag{2.136}$$

Since S^1 is symmetric in $\mathcal{S}_*^\wedge$ (see [42, Lemma 6.6.2] together with Remark 2.17) it is also symmetric in $\mathcal{H}(S)_*^\wedge$ (because it is given by the image of the unique colimit preserving monoidal map $\mathcal{S}_*^\wedge \to \mathcal{H}(S)_*^\wedge$). In this case, we can use Proposition 2.22 to deduce that the underlying ∞-category of $(\mathcal{H}(S)_*^\wedge)[(S^1)^{-1}]$ is equivalent to the stable ∞-category $Stab(\mathcal{H}(S))$. Plus, since $(\mathcal{H}(S)_*^\wedge)[(S^1)^{-1}]$ is presentable by construction, the monoidal structure is compatible with colimits, thus exact, separately on each variable. We conclude that it is a stable presentable symmetric monoidal $(\infty,1)$-category.

Finally, because $(\mathbb{P}^1,\infty)$ is symmetric, Corollary 2.23 tells us that $\mathcal{SH}(S)^\otimes$ is a *stable presentable symmetric monoidal ∞-category*. In particular its homotopy category is triangulated and inherits a canonical symmetric monoidal structure.

Corollary 2.39. *Let S be a base scheme and $Sm^{ft}(S)$ denote the category of smooth schemes of finite type over S, together with the cartesian product. The composition of monoidal functors*

$$\theta^\otimes : N\big(Sm^{ft}(S)\big)^\times \to \mathcal{P}^{big}\big(N\big(Sm^{ft}(S)\big)\big)^\times \to \mathcal{H}(S)^\times \to \mathcal{H}(S)_*^\wedge \\ \to \mathcal{H}(S)_*^\wedge\big[\big(S^1\big)^{-1}\big] \to \mathcal{SH}(S)^\otimes \tag{2.137}$$

satisfies the following universal property: for any pointed presentable symmetric monoidal $(\infty,1)$-category $\mathcal{D}^\otimes$, the composition map

$$Fun^{\otimes,L}\big(\mathcal{SH}(S)^\otimes,\mathcal{D}^\otimes\big) \to Fun^\otimes\big(N\big(Sm^{ft}(S)\big)^\times,\mathcal{D}^\otimes\big) \tag{2.138}$$

is fully faithful and its image consists of those monoidal functors $N(Sm^{ft}(S))^\times \to \mathcal{D}^\otimes$ whose underlying functor satisfy Nisnevich descent, $\mathbb{A}^1$-invariance and such that the cofiber of the image of the point at ∞, $S \xrightarrow{\infty} \mathbb{P}^1$ is an invertible object in $\mathcal{D}^\otimes$. Moreover, any pointed presentable symmetric monoidal $(\infty,1)$-category $\mathcal{D}^\otimes$ admitting such a monoidal functor is necessarily stable.

Proof. Here, $N(Sm^{ft}(S))$ denotes the standard way to interpret an ordinary 1-category as an $(\infty,1)$-category using the nerve. The Yoneda map $j : N(Sm^{ft}(S)) \to \mathcal{P}^{big}(N(Sm^{ft}(S)))$ extends to a monoidal map because of the monoidal universal property of presheaves (consult our introductory section on Higher Algebra). By Proposition 2.15 p. 74 in [67], the localization functor $\mathcal{P}^{big}(N(Sm^{ft}(S))) \to \mathcal{H}(S)$ is monoidal with respect to the cartesian structure and therefore extends to a monoidal left adjoint to the inclusion $\mathcal{H}(S)^\times \subseteq \mathcal{P}^{big}(N(Sm^{ft}(S)))^\times$. The result now follows from the discussion above, together with Corollaries 2.32 and 2.37, Corollary 2.23, Theorem 2.26 and Remark 2.33.

The last assertion follows from Remark 2.25, together with the fact that $\mathbb{P}^1$ mod out by the point at infinity is the tensor product of S^1 and $\mathbb{G}_m$, so that, since we are dealing with monoidal functors, the conditions defining the image of the composition map force the image S^1 to be tensor invertible in $\mathcal{D}^\otimes$. $\square$

To conclude this section we recall a useful description of a family of compact generators in $\mathcal{SH}(S)$.

Remark 2.40. Thanks to the results of [77] and to our discussion in Proposition 1.12, if k is a field admitting resolutions of singularities then the collection of objects generated by the image of smooth projective varieties is a family of compact generators in $\mathcal{SH}(S)$. In this case, these are also the dualizable objects in $\mathcal{SH}(k)^{\otimes}$. In particular the $(\infty,1)$-category $\mathcal{SH}(k)$ is compactly generated by dualizable objects. See our discussion in [79, Section 4.4 and Prop. 5.3.3] for a general discussion on the existence of compact generators in the stabilization.

2.4.4. Description using spectral presheaves

In this section we give an alternative description of the symmetric monoidal $(\infty,1)$-category $\mathcal{SH}(k)^{\otimes}$ using presheaves of spectra.

Remark 2.41 *(Spectral Yoneda's lemma).* Recall that any stable $(\infty,1)$-category has a natural enrichment over spectra, determined by means of a universal property [63, Chapter 1]. In this remark we recall how to use this universal property to deduce an enriched version of Yoneda's lemma for spectral presheaves. More precisely, if $\mathcal{C}$ is a small $(\infty,1)$-category, we consider the composition of the Yoneda embedding with the pointing map followed by stabilization $\Sigma^{\infty}_{+}\circ j:\mathcal{C}\hookrightarrow\mathcal{P}(\mathcal{C})\to\mathcal{P}(\mathcal{C})_{*}\to Stab(\mathcal{P}(\mathcal{C}))\simeq Fun(\mathcal{C}^{op},Sp)$ (because the stabilization is a limit). Now, given an object X in $\mathcal{C}$, we can use Yoneda's lemma for $\mathcal{P}(\mathcal{C})$ to construct a natural equivalence of functors $Map_{Fun(\mathcal{C}^{op},Sp)}(\Sigma^{\infty}_{+}\circ j(X),-)\to\Omega^{\infty}\circ ev_X$, where $ev_X: Fun(\mathcal{C}^{op},Sp)\to Sp$ is the evaluation map at X. This is possible because the delooping of presheaves is computed objectwise. To conclude, since the composition with Ω^{∞} induces an equivalence $Exc_{*}(\mathcal{C},Sp)\simeq Exc_{*}(\mathcal{C},\mathcal{S})$, we can lift the previous natural equivalence to a new one

$$Map^{Sp}_{Fun(\mathcal{C}^{op},Sp)}\big(\Sigma^{\infty}_{+}\circ j(X),-\big)\to ev_X \tag{2.139}$$

which, when evaluated at F gives us the Yoneda formula we seek. This holds for any universe: if $\mathcal{C}$ is only $\mathbb{V}$-small for some universe $\mathbb{V}$ we apply the same arguments as above to the $\mathbb{V}$-small $(\infty,1)$-category of spectra obtained from the stabilization of the $\mathbb{V}$-small $(\infty,1)$-category of spaces.

Now, we start from the $(\infty,1)$-category $N(Sm^{ft}(S))$ and consider the very big $(\infty,1)$-category $Fun(N(Sm^{ft}(S))^{op},\widehat{Sp})$[18] which is canonically equivalent to $Stab(\mathcal{P}^{big}(N(Sm^{ft}(S)))_{*})$. Using Remark 2.25 we obtain a canonical monoidal structure $Fun(N(Sm^{ft}(S))^{op},\widehat{Sp})^{\otimes}$ defined by the inversion $\mathcal{P}^{big}(N(Sm^{ft}(S)))_{*}^{\wedge(\otimes)}[(S^1)^{-1}]^{\otimes}$ where $\mathcal{P}^{big}(N(Sm^{ft}(S)))_{*}^{\wedge(\otimes)}$ is the canonical monoidal smash structure given by Proposition 2.32 extending the monoidal structure $\mathcal{P}^{big}(N(Sm^{ft}(S)))^{\otimes}$ of [63, 4.8.1.10].

We proceed as before and perform the localization with respect to the Nisnevich topology and $\mathbb{A}^1$. Extra care is needed, for the class of maps with respect to which we need to

[18] Here $\widehat{Sp}$ denotes the big $(\infty,1)$-category of spectra, obtained from the stabilization of the big $(\infty,1)$-category of spaces $\widehat{\mathcal{S}}$.

localize is not the same as for presheaves of spaces. In order to describe these two classes we recall first that $Fun(N(Sm^{ft}(S))^{op}, \widehat{Sp})$ is a stable presentable $(\infty,1)$-category and by the discussion in 1.5.3, for any $G \in Fun(N(Sm^{ft}(S))^{op}, \widehat{Sp})$ we have a mapping spectrum functor $Map^{Sp}(G,-) : Fun(N(Sm^{ft}(S))^{op}, \widehat{Sp}) \to \widehat{Sp}$ which when composed with Ω^∞ recovers the mapping space functor in $Fun(N(Sm^{ft}(S))^{op}, \widehat{Sp})$. Moreover, because of the universal property that defines it and because the composition $\Omega^\infty Map^{Sp}(G,-)$ commutes with all limits, we conclude that $Map^{Sp}(G,-)$ also commutes with all limits. In particular, by the Adjoint functor theorem [59, 5.5.2.9], it admits a left adjoint which we shall denote as $\delta_G : \widehat{Sp} \to Fun(N(Sm^{ft}(S))^{op}, \widehat{Sp})$ and for any $K \in \widehat{Sp}$ and $F \in Fun(N(Sm^{ft}(S))^{op}, \widehat{Sp})$ we have

$$Map_{Sp}\big(K, Map^{Sp}(G,F)\big) \simeq Map_{Fun(N(Sm^{ft}(S))^{op},\widehat{Sp})}\big(\delta_G(K), F\big) \tag{2.140}$$

We can now use this to define the class of maps that generate the Nisnevich localization. Namely, we localize with respect to the class of maps

$$\delta_{\Sigma^\infty_+ \circ j(U)}(K) \coprod_{\delta_{\Sigma^\infty_+ \circ j(W)}(K)} \delta_{\Sigma^\infty_+ \circ j(V)}(K) \to \delta_{\Sigma^\infty_+ \circ j(X)}(K) \tag{2.141}$$

given by the universal property of the pushout, with K in $(\widehat{Sp})^\omega$ [19] and W, V, U and X part of a Nisnevich square.

For the $\mathbb{A}^1$ localization, we localize with respect to the class of all induced maps

$$\delta_{\Sigma^\infty_+ \circ j(X\times\mathbb{A}^1)}(K) \to \delta_{\Sigma^\infty_+ \circ j(X)}(K) \tag{2.142}$$

with X in $N(Sm^{ft}(S))^{op}$ and $K \in (\widehat{Sp})^\omega$.

We observe that these localizations are monoidal.

3. Motivic stable homotopy theory of noncommutative spaces over a ring

3.1. Preliminaries on dg-categories

We will assume the reader is familiar with the theory of dg-categories. There are many nice expository introductions to the subject, for instance [53] and [6]. The author has also prepared a survey in [79, Sections 6.1 and 6.2] to which we address the reader. We shall merely recall here some notations. For a fixed commutative ring k we denote by $Cat_{Ch(k)}$ the 1-category of small k-dg-categories and by $\mathcal{D}g(k)$ the $(\infty,1)$-category encoding the homotopy theory of small dg-categories up to quasi-equivalences. There is a natural functor $Alg(Ch(k)) \to Cat_{Ch(k)}$ sending a k-dg-algebra A to the dg-category with

[19] Here $(\widehat{Sp})^\omega$ denotes the full subcategory of $\widehat{Sp}$ spanned by the compact objects. Recall that $\widehat{Sp} \simeq Ind((\widehat{Sp})^\omega)$.

one object and A as its complex of endomorphisms. This functor is compatible with the two notions of weak-equivalences and therefore induces a well-defined functor between the underlying $(\infty, 1)$-categories $(-)_{dg} : Alg(\mathcal{D}(k)) \to \mathcal{D}g(k)$. We denote as $\mathcal{D}g(k)^{idem}$ the presentable $(\infty, 1)$-category of small dg-categories up to Morita equivalence. It has direct sums and a zero object. Both $\mathcal{D}g(k)^{idem}$ and $\mathcal{D}g(k)$ can be obtained as the underlying $(\infty, 1)$-categories associated to combinatorial model structures on $Ch(k)$ given in [89,88]. The $(\infty, 1)$-category $\mathcal{D}g(k)$ carries a natural symmetric monoidal structures induced by the derived tensor product of dg-categories. Moreover, this tensor structure is known to be closed and its internal-homs can be explicitly described [96]: given two small dg-categories T and T', the internal-hom $\mathbb{R}\underline{Hom}(T, T')$ is given by the full sub-dg-category spanned by the right quasi-representable cofibrant $T \otimes^{\mathbb{L}} (T')^{op}$-dg-modules. One can easily check that the functor $(-)_{dg}$ is monoidal with respect to the natural monoidal structure on dg-algebras and this derived tensor product of dg-categories. We address the reader to our discussion in [79, Remark 6.1.11]. Ultimately this follows because the product of cofibrant dg-algebras remains a dg-algebra with a cofibrant underlying complex (as proved in [83]).

It is also well-known that $\mathcal{D}g(k)^{idem}$ is a monoidal reflexive localization of $\mathcal{D}g(k)$: there is a canonical fully faithful map $\mathcal{D}g(k)^{idem} \hookrightarrow \mathcal{D}g(k)$ whose image is equivalent to the full subcategory of $\mathcal{D}g(k)$ spanned by those small dg-categories T such that the canonical map $T \to \widehat{T}_{pe}$ is an equivalence (also known as *idempotent complete dg-categories*). The monoidal left adjoint to this inclusion is given by the formula $T \mapsto \widehat{T}_{pe}$. Here, for a small dg-category T we denote by $\widehat{T}$ the big dg-category of cofibrant T^{op}-dg-modules and $\widehat{T}_{pe}$ denotes its full subcategory of perfect dg-modules, which we will understand by definition as homotopy compact objects.

Composing these two functors we obtain a monoidal functor $Perf : Alg(\mathcal{D}(k)) \to \mathcal{D}g(k)^{idem}$.

We recall also that a dg-category $T \in \mathcal{D}g(k)^{idem}$ is said to be of finite type [100] if it is a compact object in the $(\infty, 1)$-category $\mathcal{D}g(k)^{idem}$. We let $\mathcal{D}g(k)^{ft}$ denote the full subcategory of $\mathcal{D}g(k)^{idem}$ spanned by the small dg-categories of finite type. As the Morita model structure on small dg-categories is compactly generated (again, see [100]), the natural induced map $Ind(\mathcal{D}g(k)^{ft}) \to \mathcal{D}g(k)^{idem}$ is an equivalence (see our discussion in [79, Sections 2.2.2 and 6.1.4]). In particular, this formula implies that $\mathcal{D}g(k)^{ft}$ is closed under finite direct products, pushouts and contains the zero object.

We will not review here the definition of a smooth/proper dg-category but we recall that T is simultaneously smooth and proper if and only if it is a dualizable object in $\mathcal{D}g(k)^{idem}$ with respect to the derived tensor product of dg-categories. It follows that smooth and proper dg-categories are of finite type. It can also be proved that any dg-category of finite type is smooth.

We recall also that a small dg-category T is said to have a compact generator if the homotopy category of $\widehat{T}$ has a compact generator in the sense of Neeman. Using the same methods as in [84], it can be proved that T has a compact generator if and only if it is in the essential image of $Perf$. For the “only if” direction we consider the dg-algebra

B given by the opposite algebra of endomorphisms of the compact generator in $\widehat{T}$. For the "if" direction, if $T \simeq Perf(B)$ then B, seen as a dg-module over itself, is a compact generator.

Following [97] we also have the notion of *presentable dg-categories*. These are big dg-categories which are obtained as Bousfield localizations of dg-categories of the form $\widehat{T}$. Together with the colimit preserving maps these form an $(\infty, 1)$-category which we will denote as $\mathcal{D}g(k)^{lp}$. We will write $\mathcal{D}g(k)^c$ for its full subcategory spanned by the presentable dg-categories of the form $\widehat{T}$ for a small dg-category T and we write $\mathcal{D}g(k)^{cc}$ for the (non-full) subcategory of $\mathcal{D}g(k)^c$ with all objects but only those maps that preserve compact objects. An object here will be referred to as a *compactly generated dg-category*. One can easily check that both these $(\infty, 1)$-categories admit natural symmetric monoidal structures and that the assignment sending a small idempotent complete dg-category $T \in \mathcal{D}g(k)^{idem}$ to the big dg-category $\widehat{T}$ induces a monoidal equivalence of $(\infty, 1)$-categories $\mathcal{D}g(k)^{idem} \simeq \mathcal{D}g(k)^{cc}$.

To conclude this section we recall the existence of a dg-nerve functor $N_{dg} : Ch(k) \to \widehat{\Delta}$. It can be obtained by applying the Dold–Kan construction to the positive truncations of the enriching complexes followed by the simplicial nerve construction. It is a right-Quillen functor [63, 1.3.1.20] (where $\widehat{\Delta}$ is equipped with the Joyal model structure) and therefore induces an ∞-functor $\mathcal{D}g(k) \to Cat_\infty$ that commutes with limits and preserves the associated homotopy 1-categories. This functor provides a bridge between compactly generated dg-categories and stable presentable $(\infty, 1)$-categories. More precisely, there is a version of N_{dg} for big dg-categories $N_{dg} : \mathcal{D}g(k)^{big} \to Cat_\infty^{big}$ and as explained in the proof of [97, Lemma 2.3] a big dg-category is locally presentable if and only if its dg-nerve is a locally presentable $(\infty, 1)$-category. This makes the restriction a well-defined functor $N_{dg}^L : \mathcal{D}g(k)^{lp} \to \mathcal{P}r^L$. Moreover, one can also easily check that the image of this dg-nerve factors through $\mathcal{P}r_{Stb}^L \subseteq \mathcal{P}r^L$. Furthermore, new this functor is conservative and by the properties of the Dold–Kan correspondence and because of stability, reflects fully faithfulness. Moreover, it preserves the associated homotopy 1-categories [63, Remark 1.3.1.11] so that a dg-category $\widehat{T}$ has a compact generator if and only if the homotopy category $h(N_{dg}^L(\widehat{T}))$ has a compact generator.

One can restrict a bit further and obtain a well-defined map $N_{dg}^L : \mathcal{D}g(k)^{cc} \to \mathcal{P}r_{Stb,\omega}^L$ relating the notions of compactly generated categories and a homotopy commutative diagram

$$\begin{tikzcd} \mathcal{D}g(k)^{idem} \simeq \mathcal{D}g(k)^{cc} \arrow[r, hook, "non\text{-}full"] \arrow[d, "N_{dg}^L"] & \mathcal{D}g(k)^{lp} \arrow[d, "N_{dg}^L"] \\ \mathcal{P}r_{\omega,Stb}^L \arrow[r, hook, "non\text{-}full"] & \mathcal{P}r_{Stb}^L \end{tikzcd} \tag{3.1}$$

Remark 3.1. The composition $\mathcal{D}g(k)^{idem} \simeq \mathcal{D}g(k)^{cc} \to \mathcal{P}r_{\omega,Stb}^L$ has long been expected to provide an equivalence between the theory of small dg-categories up to Morita equivalence and the theory of k-linear stable presentable compactly generated $(\infty,1)$-categories. More precisely, if $\mathcal{D}(k)^{\otimes}$ denotes the derived ∞-derived category of k with its natural derived tensor product, we can understand it as an object in $CAlg(\mathcal{P}r^L_{\omega,Stb})$ and the map N^L_{dg} was expected to factor as

$$\mathcal{D}g(k)^{cc} \to Mod_{\mathcal{D}(k)}\big(\mathcal{P}r^L_{\omega,Stb}\big) \to \mathcal{P}r^L_{\omega,Stb}$$

with the last map being the forgetful functor. Moreover, the first map in this factorization was expected to be an equivalence. This result was recently established in [26].

See [79, Section 6.2] for a more detailed discussion.

3.2. From schemes to noncommutative spaces

Following [100], the notion of finite type should be understood as the correct notion of smoothness for noncommutative spaces, while the smooth dg-categories should only be understood as "formally smooth" noncommutative spaces. Finally, we are ready to introduce our smooth noncommutative geometric objects.

Definition 3.2. Let k be a ring. We define the $(\infty,1)$-category of *smooth noncommutative spaces over* k – $\mathcal{N}cS(k)$ – to be the *opposite* of $\mathcal{D}g(k)^{ft}$. It has a natural symmetric monoidal structure $\mathcal{N}cS(k)^{\otimes}$ induced from the one in $\mathcal{D}g(k)^{ft,\otimes}$, with unit object given by $L_{pe}(k)$.

Notation 3.3. We will denote our smooth noncommutative spaces using caligraphic letters $\mathcal{X}$, $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$, etc. For a smooth noncommutative space $\mathcal{X} \in \mathcal{N}cS$ we will denote by $T_{\mathcal{X}}$ its associated dg-category of finite type and by $A_{\mathcal{X}}$ a compact dg-algebra such that $T_{\mathcal{X}} \simeq Perf(A_{\mathcal{X}})$.

We will say that a smooth noncommutative space $\mathcal{X}$ is *smooth and proper* if its associated dg-category $T_{\mathcal{X}}$ is smooth and proper. We will let $\mathcal{N}cS(k)^{sp}$ denote the full subcategory of $\mathcal{N}cS(k)$ spanned by the smooth and proper noncommutative spaces. Since the smooth and proper dg-categories correspond to the dualizable objects in $\mathcal{D}g(k)^{ft}$, the subcategory $\mathcal{N}cS(k)^{sp}$ is closed under tensor products.

It follows immediately from the properties of $\mathcal{D}g(k)^{ft}$ that $\mathcal{N}cS(k)$ admits pullbacks, together with finite direct sums and a zero object. Moreover, the tensor product commutes with limits. In particular, if $\mathcal{X}$ and $\mathcal{Y}$ are two smooth noncommutative spaces, the mapping space $Map_{\mathcal{N}cS(k)}(\mathcal{X},\mathcal{Y})$ is given by the ∞-groupoid $pspe(A_{\mathcal{Y}},A_{\mathcal{X}})^{\simeq}_{\infty}$ of pseudo-perfect $A_{\mathcal{Y}} \otimes^{\mathbb{L}} A^{op}_{\mathcal{X}}$-dg-modules and equivalences between them.

We now explain how the formula $X \mapsto L_{pe}(X)$ can be properly arranged as an ∞-functor. We define it for the smooth affine schemes of finite type over k, whose 1-category we denote by $AffSm^{ft}(k)$. Recall that the full subcategory of 0-truncated objects in $CAlg(\mathcal{D}(k))^{cn}$ is equivalent to the nerve of the category of classical associative rings. In particular, we can identify the nerve of the category of commutative smooth k-algebras of finite type $N(SmCommAlg_k) \simeq N(AffSm^{ft}(k))^{op}$ with a full subcategory of $CAlg(\mathcal{D}(k))^{cn}$. Let L denote the composition

$$N(SmCommAlg_k) \hookrightarrow CAlg(\mathcal{D}(k))^{cn} \longrightarrow Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn} \hookrightarrow Alg_{\mathcal{A}ss}(\mathcal{D}(k)) \xrightarrow{Perf} \mathcal{D}g(k)^{idem} \tag{3.2}$$

where $CAlg(\mathcal{D}(k))^{cn} \to Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn}$ is the restriction of the forgetful functor to connective objects. The following is a key result:

Proposition 3.4. *Let A be a classical commutative smooth k-algebra of finite type. Then, $L(A)$ is a dg-category of finite type. In other words, L provides a well-defined functor $N(SmCommAlg_k) \to \mathcal{D}g(k)^{ft}$.*

In order to prove this result we will need the following noncommutative analogue of [103, Prop. 2.2.2.4] and [63, 7.4.3.18]:

Lemma 3.5. *Let A be an object in $Alg(\mathcal{D}(k))^{cn}$. The following are equivalent:*

1) *A is an ω-compact object in $Alg(\mathcal{D}(k))$;*
2) *$\mathbb{H}_0(A)$ is a finitely presented associative algebra over k and the cotangent complex $\mathbb{L}_A$ is a compact object in $Mod_A^{\mathcal{A}ss}(\mathcal{D}(k))$.*

Proof. We ask the reader to recall our notations and preliminaries in 1.5.6.

We follow the same methods as in [103, Prop. 2.2.2.4]. We first prove that 1) implies 2).

The fact that $\mathbb{H}_0(A)$ is finitely presented as an associative algebra follows from the fact that $\mathbb{H}_0$ commutes with colimits (it is a left adjoint), together with the fact that π_0 commutes with colimits in the $(\infty,1)$-category of spaces. The fact that $\mathbb{L}_A$ is compact follows from the universal property of the cotangent complex together with the following facts:

i) As explained before, the functor $(A \oplus -)$ of (1.17) can be identified with a delooping functor Ω^∞. Therefore it commutes with filtered colimits;
ii) by assumption, A is compact.

We now prove that 2) implies 1). To start with, we observe that since A is by assumption connective, it is enough to check that A is compact in the full subcategory $Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn}$ spanned by the connective objects. Indeed, recall that the truncation functor $\tau_{\leq 0}$ is a right adjoint to the inclusion $Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn} \subseteq Alg_{\mathcal{A}ss}(\mathcal{D}(k))$. We can easily check that $\tau_{\leq 0}$ commutes with filtered colimits (because the homology groups commute with filtered colimits) so that for any filtered system $\{\mathcal{C}_i\}_{i\in I}$ in $Alg_{\mathcal{A}ss}(\mathcal{D}(k))$ we have

$$\begin{aligned} Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, colim_I C_i) &\simeq Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn}}(A, \tau_{\leq 0} colim_I C_i) \\ &\simeq Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn}}(A, colim_I \tau_{\leq 0} C_i) \end{aligned} \tag{3.3}$$

so that A is compact in $Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn}$ if and only if it is compact in $Alg_{\mathcal{A}ss}(\mathcal{D}(k))$.

We start now by proving that A is almost compact, meaning that A is compact with respect to any filtered system in $Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn}_{\leq n}$, for every $n \geq 0$. We proceed by induction. The case $n = 0$ follows by the hypothesis. Let us suppose we know this is true for $n-1$ and prove it for n. Let $\{C_i\}_{i\in I}$ be a filtered system in $Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn}_{\leq n}$. The discussion in 1.5.5 together with Theorem 1.22 implies that for each i, C_i admits a Postnikov decomposition

$$C_i = \tau_{\leq n}(C_i) \to \tau_{\leq n-1}(C_i) \to \ldots \to \tau_{\leq 0}(C_i) \tag{3.4}$$

where each morphism is a square-zero extension providing a pullback diagram

$$\begin{tikzcd} C_i = (C_i)_{\leq n} \arrow[r] \arrow[d] & (C_i)_{\leq n-1} \arrow[d, "d_n"] \\ (C_i)_{\leq n-1} \arrow[r] & (C_i)_{\leq n-1} \oplus \mathbb{H}_n(C_i)[n+1] \end{tikzcd} \tag{3.5}$$

in $Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn}$ where the lower horizontal map is the zero map and right vertical map corresponds to the canonical derivation d_n associated to the square-zero extension $C_i \to \tau_{\leq n-1} C_i$. This diagram induces a pullback diagram of spaces

$$\begin{tikzcd} Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, C_i) \arrow[r] \arrow[d] & Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, \tau_{\leq n-1}(C_i)) \arrow[d] \\ Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, (C_i)_{\leq n-1}) \arrow[r] & Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, \tau_{\leq n-1}(C_i) \oplus \mathbb{H}_n(C_i)[n+1]) \end{tikzcd} \tag{3.6}$$

and Remark 1.21 implies that the fiber of the map

$$Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, C_i) \longrightarrow Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}\big(A, \tau_{\leq n-1}(C_i)\big) \tag{3.7}$$

over a map $u : A \to \tau_{\leq n-1}(C_i)$ is given by the space of paths in $Map_{Mod_A^{\mathcal{A}ss}}(\mathbb{L}_A, \mathbb{H}_n(C_i)[n+1])$ between the zero derivation and the point corresponding to the composition $d_n \circ u$. This reduces everything to the analysis of the diagram

$$\begin{tikzcd}
colim_I \Omega_{0,d_n \circ u} Map_{Mod_A^{\mathcal{A}ss}(\mathcal{D}(k))}(\mathbb{L}_A, \mathbb{H}_n(C_i)[n+1]) \arrow[r] \arrow[d] & \Omega_{0,d_n \circ u} Map_{Mod_A^{\mathcal{A}ss}(\mathcal{D}(k))}(\mathbb{L}_A, \mathbb{H}_n(colim_I C_i)[n+1]) \arrow[d] \\
colim_I Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, C_i) \arrow[r] \arrow[d] & Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, colim_I C_i) \arrow[d] \\
colim_I Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, \tau_{\leq n-1}(C_i)) \arrow[r] & Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, \tau_{\leq n-1}(colim_I C_i))
\end{tikzcd} \tag{3.8}$$

We observe that

a) The left column is a fiber sequence because filtered colimits are exact in the $(\infty, 1)$-category of spaces. For the same reason, there is an equivalence between the top left entry in the diagram and

$$\Omega_{0,d_n \circ u} colim_I Map_{Mod_A^{\mathcal{A}ss}(\mathcal{D}(k))}\big(\mathbb{L}_A, \mathbb{H}_n(C_i)[n+1]\big). \tag{3.9}$$

b) The right column is also a fiber sequence. This follows from the result of 1.22 and Remark 1.21 applied to the colimit algebra $colim_I C_i$.

c) The top entry on the right is equivalent to

$$\Omega_{0,d_n \circ u} Map_{Mod_A^{\mathcal{A}ss}(\mathcal{D}(k))}\big(\mathbb{L}_A, colim_I \mathbb{H}_n(C_i)[n+1]\big). \tag{3.10}$$

This is because the functor $\mathbb{H}_n$ is equivalent to the classical nth-homology functor and therefore commutes with filtered colimits.

Finally, the induction hypothesis together with the fact that $(-)_{\leq n}$ is a left adjoint (and therefore commutes with colimits), implies that the lower horizontal arrow is an equivalence. The assumption that $\mathbb{L}_A$ is compact implies that the top horizontal map is also an equivalence. It follows that the middle one is also an equivalence. This proves that A is almost compact in $Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn}$.

We now complete the proof by showing that A is compact. Since the $(\infty, 1)$-category $Mod_A^{\mathcal{A}ss}((\mathcal{D}(k)))$ is equivalent to the underlying $(\infty, 1)$-category of the model structure on strict A-bimodules in $Ch(k)$ (see 1.19) and the last is compactly generated, we know that $\mathbb{L}_A$ is a compact object in $Mod_A^{\mathcal{A}ss}((\mathcal{D}(k)))$ if and only if it is given by a finite strict cell object in the model category of bimodules. In this case, with our hypothesis that $\mathbb{L}_A$ is compact, we can find a natural number $n_0 \geq 0$ such that for any object $M \in Mod_A^{\mathcal{A}ss}(\mathcal{D}(k))$ concentrated in degrees strictly bigger than n_0 we have $\pi_0 Map_{Mod_A^{\mathcal{A}ss}(\mathcal{D}(k))}(\mathbb{L}_A, M) \simeq 0$. In particular, for any connective algebra C, the combination of the fiber sequence of Remark 1.21 and Theorem 1.22 implies that homotopy classes of maps $A \to C$ are in bijection with homotopy classes of maps $A \to \tau_{\leq n_0}(C)$, In other words, we have

$$\pi_0 Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, C) \simeq \pi_0 Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}\big(A, \tau_{\leq n_0}(C)\big) \tag{3.11}$$

We now use this to show that A is compact. Let $\{C_i\}_{i\in I}$ be a filtered system in $Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn}$. Using the fact that π_n commutes with filtered homotopy colimits of spaces and that $Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn}$ admits all limits (it is a co-reflexive localization of $Alg_{\mathcal{A}ss}(\mathcal{D}(k))$), we are reduced to showing that the natural map

$$colim_I \pi_0 Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}\big(A, \Omega^n C_i\big) \to \pi_0 Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}\big(A, colim_I \Omega^n C_i\big) \tag{3.12}$$

is an equivalence. We show that the formula is true for any filtered system of algebras $\{U_i\}_{i\in I}$, because we have a commutative diagram

$$\begin{tikzcd}
colim_I \pi_0 Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, U_i) \arrow[r] \arrow[d, "\sim"] & \pi_0 Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, colim_I U_i) \arrow[d, "\sim"] \\
colim_I \pi_0 Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, \tau_{\leq n_0}(U_i)) \arrow[r, "\sim"] & \pi_0 Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, colim_I \tau_{\leq n_0}(U_i))
\end{tikzcd} \tag{3.13}$$

where the vertical arrows are equivalences because of (3.11) together with fact that $\tau_{\leq n_0}$ is a left adjoint, and the lower horizontal map is an equivalence because A is almost compact. This concludes the proof. □

Proof of Proposition 3.4. If A is smooth as a classical commutative k-algebra it is smooth as a dg-category which by definition means it is compact as an $A \otimes_k A^{op}$-dg-module. The category of $A \otimes_k A^{op}$-dg-modules can of course be naturally identified with the category of A-bimodules $BiMod(A, A)(Ch(k))$. Using the strictification results of [63, 4.3.3.15 and 4.3.3.17] the underlying $(\infty, 1)$-category of $BiMod(A, A)(Ch(k))$ is equivalent to ${}_A BMod_A(\mathcal{D}(k)) \simeq Mod_A^{\mathcal{A}ss}(\mathcal{D}(k))$.

Of course, if A is compact in $Mod_A^{\mathcal{A}ss}(\mathcal{D}(k))$ and since $A \otimes_k A^{op}$ is also compact (it is a generator), the kernel of the multiplication map $I \to A \otimes_k A^{op} \to A$ will also be compact. Following Example 1.19 we can now identify I with the relative cotangent complex $\mathbb{L}_{A/k} \in Mod_A^{\mathcal{A}ss}(\mathcal{D}(k))$. Lemma 3.5 completes the proof. □

Using this, we define L_{pe} as the opposite of L

$$L_{pe} : N\big(AffSm^{ft}(k)\big) \to \mathcal{N}cS(k) \tag{3.14}$$

To conclude this section we observe that L_{pe} can be promoted to a monoidal functor

$$L_{pe}^{\otimes} : N\big(AffSm^{ft}(k)\big)^{\times} \to \mathcal{N}cS(k)^{\otimes} \tag{3.15}$$

where $N(\mathit{AffSm}^{ft}(k))^{\times}$ is the cartesian structure in $N(\mathit{AffSm}^{ft}(k))$ which corresponds to the coproduct of classical commutative smooth k-algebras which is, well-known, given by the classical tensor product over k.

It follows as $(-)_{dg}$ is monoidal because the tensor product in $\mathcal{D}(k)$ is compatible with the t-structure, that the composition $\mathit{CAlg}(\mathcal{D}(k))^{cn} \to \mathit{Alg}_{\mathit{Ass}}(\mathcal{D}(k))^{cn} \subseteq \mathit{Alg}_{\mathit{Ass}}(\mathcal{D}(k))$ is monoidal. Moreover, the functor *Perf* is monoidal because it is the composition of monoidal functors. We are left to check that the inclusion $N(\mathit{SmCommAlg}_k) \to \mathit{CAlg}(\mathcal{D}(k))^{cn}$ is monoidal. In other words, that for a commutative smooth k-algebra of finite type over k, the classical tensor product agrees with the derived tensor product. But this is true since smooth k-algebras are flat over k.

3.3. The motivic $\mathbb{A}^1$-homotopy theory of Kontsevich's noncommutative spaces over a ring k

We will now use our main results to fabricate a motivic $\mathbb{A}^1$-homotopy theory for smooth noncommutative spaces over a ring k. In this section we proceed in analogy with the construction of the motivic stable homotopy for schemes as described in the previous section of this work. Recall from Remark 2.29 that these constructions only depend on the category of *affine* smooth schemes of finite type over k.

Remark 3.6. There is a natural way to extend the functor L_{pe} to non-affine schemes. To do this, we observe that the classical category of schemes can be identified with a full subcategory of $\mathcal{P}^{big}(N(\mathit{AffSm}^{ft}(k)))$, by the identification of a scheme with its "functor of points". The universal property of (big) presheaves provides a colimit preserving map

$$\begin{array}{ccc} N(\mathit{AffSm}^{ft}(k)) & \xrightarrow{L_{pe}} & \mathcal{N}cS(k) \\ \downarrow & & \downarrow \\ \mathcal{P}^{big}(N(\mathit{AffSm}^{ft}(k))) & \dashrightarrow & \mathcal{P}^{big}(\mathcal{N}cS(k)) \end{array} \tag{3.16}$$

Lemma 3.27 in [100] implies that the image through this map of any smooth and proper scheme X over k is representable in $\mathcal{P}^{big}(\mathcal{N}cS(k))$. This should remain true without the properness condition.

To start with, we need to introduce an appropriate analogue for the Nisnevich topology, for the interval $\mathbb{A}^1$ and for the projective space $\mathbb{P}^1$. For the last two we have natural choices – $L_{pe}(\mathbb{A}^1)$ and $L_{pe}(\mathbb{P}^1)$: the first is a dg-category of finite type because $\mathbb{A}^1$ is smooth affine over k; the second, $L_{pe}(\mathbb{P}^1)$, is of finite type because the canonical morphism $\mathbb{P}^1 \to \mathit{Spec}(k)$ is smooth and proper (see our discussion in [79, Section 6.3.2]). The analogue of the Nisnevich topology requires a more careful discussion.

3.3.1. The noncommutative version of the Nisnevich topology

To obtain our noncommutative analogue for the Nisnevich topology we isolate the formal properties of the commutative squares in $\mathcal{N}cS(k)$

$$\begin{tikzcd} L_{pe}(p^{-1}(U)) \arrow[r] \arrow[d] & L_{pe}(V) \arrow[d] \\ L_{pe}(U) \arrow[r] & L_{pe}(X) \end{tikzcd} \tag{3.17}$$

induced by the Nisnevich squares of schemes. Following the list of properties given in Section 2.4, we start with the notion of an open embedding. For that we need some preparations. Recall that an *exact sequence* in $\mathcal{D}g(k)^{idem}$ is the data of a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "g"] \\ * \arrow[r] & C \end{tikzcd} \tag{3.18}$$

where $*$ is the zero object in $\mathcal{D}g(k)^{idem}$, such that f fully-faithful and the diagram is a pushout. Since $\mathcal{D}g(k)^{idem}$ is a reflexive localization of $\mathcal{D}g(k)$, this pushout C is canonically equivalent to the idempotent completion of the pushout B/A computed in $\mathcal{D}g(k)$. Of course, using the monoidal equivalence $\mathcal{D}g(k)^{idem} \simeq \mathcal{D}g(k)^{cc}$, the previous diagram is an exact sequence if and only if the diagram

$$\begin{tikzcd} \widehat{A} \arrow[r, "\widehat{f}"] \arrow[d] & \widehat{B} \arrow[d, "\widehat{g}"] \\ * \arrow[r] & \widehat{C} \end{tikzcd} \tag{3.19}$$

is an exact sequence in $\mathcal{D}g^{cc}(k)$ in the same sense. Thanks to the works of B. Keller in [52], we know that this notion of exact sequence extends the notion given by Verdier [104].

Proposition 3.7. *(See B. Keller [52].) The following conditions are equivalent:*

1. *a diagram as above is an exact sequence;*
2. *the functor $\widehat{f}$ induces an equivalence of $[\widehat{A}]$ with a triangulated subcategory of the triangulated category $[\widehat{B}]$ and $\widehat{g}$ exhibits the homotopy category $[\widehat{C}]$ as the Verdier quotient $[\widehat{B}]/[\widehat{A}]$;*

3. *the functor f induces an equivalence of $[A]$ with a triangulated subcategory of the triangulated category $[B]$ and the canonical map from the Verdier quotient $[B]/[A] \hookrightarrow [C]$ is cofinal (see our discussion in 1.5.4).*

Remark 3.8. We can use the functor $N_{dg}^L : \mathcal{D}g^{cc}(k) \to \mathcal{P}r^L_{\omega,Stb}$ to relate exact sequences of dg-categories in the above sense to exact sequences of stable presentable $(\infty, 1)$-categories in the sense of 1.5.4. Recall that N_{dg}^L is conservative, preserves fully-faithfulness and preserves the notion of "homotopy category" (see Remark [63, 1.3.1.11]). This result, together with Propositions 3.7 and 1.15 implies that a sequence of dg-categories $\widehat{A} \to \widehat{B} \to \widehat{C}$ in $\mathcal{D}g^{cc}(k)$ is exact in the sense discussed in this section if and only if its image $N_{dg}^L(\widehat{A}) \to N_{dg}^L(\widehat{B}) \to N_{dg}^L(\widehat{C})$ in $\mathcal{P}r^L_{\omega,Stb}$ is exact in the sense discussed in 1.5.4. It follows also that $\widehat{A}$ has a compact generator if and only if $h(N_{dg}^L(\widehat{A}))$ has a compact generator.

Remark 3.9. It is common to find in the literature the terminology of *strict exact sequence* to denote an exact sequence (3.18) in $\mathcal{D}g(k)^{idem}$ which, apart from being a pushout square, is also a pullback in $\mathcal{D}g(k)^{idem}$. It follows again from the results of [52] that in terms of the associated homotopy triangulated categories this corresponds to the additional condition that $[\widehat{A}]$ is thick in $[\widehat{B}]$. It follows however that when working in $\mathcal{D}g(k)^{idem}$ this terminology is unnecessary because every exact sequence is strict. This follows from the properties of the functor N_{dg}^L together with Corollary 1.18.

Let us now come back to the definition of open immersion. Thanks to the results of Thomason in [94, Section 5] and to the work of B. Keller in [52], we know that for a quasi-compact and quasi-separated scheme X with a quasi-compact open embedding $j : U \hookrightarrow X$, the restriction map $j^* : L_{qcoh}(X) \to L_{qcoh}(U)$ fits in a strict exact sequence in $\mathcal{D}g^{cc}(k)$

$$\begin{tikzcd} L_{qcoh}(X)_{X-U} \arrow[r] \arrow[d] & L_{qcoh}(X) \arrow[d, "j^*"] \\ * \arrow[r] & L_{qcoh}(U) \end{tikzcd} \tag{3.20}$$

where $L_{qcoh}(X)_{X-U}$ is by definition the kernel of the restriction j^*. It is also well-known that this kernel has a compact generator (see the proof of [97, Prop. 3.9]). Of course, using the equivalence $\mathcal{D}g(k)^{idem} \simeq \mathcal{D}g^{cc}(k)$, we can reformulate this in terms of an exact sequence in $\mathcal{D}g(k)^{idem}$

$$\begin{tikzcd} (L_{qcoh}(X)_{X-U})_c \arrow[r] \arrow[d] & L_{pe}(X) \arrow[d, "j^*"] \\ * \arrow[r] & L_{pe}(U) \end{tikzcd} \tag{3.21}$$

where $(L_{qcoh}(X)_{X-U})_c$ has a compact generator.

Remark 3.10. More generally, and as explained in [33, Prop. 2.9], if T is a dg-category of finite type and k is an object in T, then quotient of T by the sub-dg-category generated by k is again a dg-category of finite type.

This motivates the following definition:

Definition 3.11. Let $f : \mathcal{U} \to \mathcal{X}$ be a morphisms of smooth noncommutative spaces over k. We say that f is an *open immersion* if there exists a dg-category with a compact generator $K_{\mathcal{X}-\mathcal{U}} \in \mathcal{D}g(k)^{idem}$ together with a fully-faithful map $K_{\mathcal{X}-\mathcal{U}} \hookrightarrow T_{\mathcal{X}}$ such that the opposite of f in $\mathcal{D}g(k)^{ft}$ fits in an exact sequence in $\mathcal{D}g(k)^{idem}$:

$$\begin{tikzcd} K_{\mathcal{X}-\mathcal{U}} \arrow[r] \arrow[d] & T_{\mathcal{X}} \arrow[d] \\ * \arrow[r] & T_{\mathcal{U}} \end{tikzcd} \tag{3.22}$$

It follows from Remark 3.9 that this diagram is also a pullback square.

Definition 3.12. We will say that a commutative diagram in $\mathcal{N}cS(k)$

$$\begin{tikzcd} \mathcal{W} \arrow[r] \arrow[d] & \mathcal{V} \arrow[d] \\ \mathcal{U} \arrow[r] & \mathcal{X} \end{tikzcd} \tag{3.23}$$

is a *Nisnevich square of smooth noncommutative spaces* if the following conditions hold:

1. The maps $\mathcal{U} \to \mathcal{X}$ and $\mathcal{W} \to \mathcal{V}$ are open immersions;
2. The associated map $T_{\mathcal{X}} \to T_{\mathcal{V}}$ sends the compact generator of $K_{\mathcal{X}-\mathcal{U}} \subseteq T_{\mathcal{X}}$ to the compact generator of $K_{\mathcal{V}-\mathcal{W}} \subseteq T_{\mathcal{V}}$ and induces an equivalence $K_{\mathcal{X}-\mathcal{U}} \simeq K_{\mathcal{V}-\mathcal{W}}$;
3. The diagram is a pushout.

Convention 3.13. *We will adopt the convention that if $\mathcal{X}$ is a smooth noncommutative space whose underlying dg-category $T_{\mathcal{X}}$ is a zero object of $\mathcal{D}g(k)^{ft}$, the empty set forms a Nisnevich square of $\mathcal{X}$.*

Using the duality between smooth noncommutative spaces and dg-categories, a Nisnevich square corresponds to the data of a commutative diagram in $\mathcal{D}g(k)^{idem}$

$$\begin{tikzcd}
K_{\mathcal{X}-\mathcal{U}} \arrow[r] \arrow[d] & * \arrow[d] \\
T_{\mathcal{X}} \arrow[r] \arrow[d] & T_{\mathcal{U}} \arrow[d] \\
T_{\mathcal{V}} \arrow[r] & T_{\mathcal{W}} \\
K_{\mathcal{V}-\mathcal{W}} \arrow[r] \arrow[u] & * \arrow[u]
\end{tikzcd} \tag{3.24}$$

where:

1) all $T_{\mathcal{X}}$, $T_{\mathcal{U}}$, $T_{\mathcal{V}}$ and $T_{\mathcal{W}}$ are of finite type;
2) Both $K_{\mathcal{X}-\mathcal{U}}$ and $K_{\mathcal{V}-\mathcal{W}}$ belong to $\mathcal{D}g(k)^{idem}$, have a compact generator and the maps $K_{\mathcal{X}-\mathcal{U}} \to T_{\mathcal{X}}$ and $K_{\mathcal{V}-\mathcal{W}} \to T_{\mathcal{V}}$ are fully-faithful;
3) The associated map $T_{\mathcal{X}} \to T_{\mathcal{V}}$ sends the compact generator of $K_{\mathcal{X}-\mathcal{U}} \subseteq T_{\mathcal{X}}$ to the compact generator of $K_{\mathcal{V}-\mathcal{W}} \subseteq T_{\mathcal{V}}$ and induces an equivalence $K_{\mathcal{X}-\mathcal{U}} \simeq K_{\mathcal{V}-\mathcal{W}}$;
4) the upper and lower squares are pushouts and pullbacks in $\mathcal{D}g(k)^{idem}$ (see Remark 3.9) and the middle square is a pullback in $\mathcal{D}g(k)^{ft}$ and therefore in $\mathcal{D}g(k)^{idem}$.

These conditions also imply that the middle square is a pushout in $\mathcal{D}g(k)^{idem}$. Indeed, because the exterior diagrams are pushouts we can write $T_{\mathcal{W}} \simeq T_{\mathcal{V}} \coprod_{K_{\mathcal{V}-\mathcal{W}}} *$ and $T_{\mathcal{U}} \simeq T_{\mathcal{X}} \coprod_{K_{\mathcal{X}-\mathcal{U}}} *$. Together with the fact that $K_{\mathcal{X}-\mathcal{U}}$ and $K_{\mathcal{V}-\mathcal{W}}$ are equivalent, we have

$$T_{\mathcal{V}} \coprod_{T_{\mathcal{X}}} T_{\mathcal{U}} \simeq T_{\mathcal{V}} \coprod_{T_{\mathcal{X}}} \Big(T_{\mathcal{X}} \coprod_{K_{\mathcal{X}-\mathcal{U}}} * \Big) \simeq T_{\mathcal{V}} \coprod_{K_{\mathcal{X}-\mathcal{U}}} * \simeq T_{\mathcal{V}} \coprod_{K_{\mathcal{V}-\mathcal{W}}} * \simeq T_{\mathcal{W}} \tag{3.25}$$

Corollary 3.14. *Every Nisnevich square in $\mathcal{N}cS(k)$ is a pullback.*

Remark 3.15. Let $\mathcal{U} \to \mathcal{X}$ be an open immersion of smooth noncommutative spaces. If the associated dg-category $K_{\mathcal{X}-\mathcal{U}} \in \mathcal{D}g(k)^{idem}$ is of finite type we can then see it as the dg-category $T_{\mathcal{Z}} = K_{\mathcal{X}-\mathcal{U}}$ dual to a smooth noncommutative space $\mathcal{Z}$. Of course, since the zero map $T_{\mathcal{Z}} \to *$ is a quotient of $T_{\mathcal{Z}}$ by itself, its dual $* \to \mathcal{Z}$ is an open immersion. Moreover, since the diagram $K_{\mathcal{X}-\mathcal{U}} \hookrightarrow T_{\mathcal{X}} \to T_{\mathcal{U}}$ is also a fiber sequence (see 3.9), the square of smooth noncommutative spaces

$$\begin{tikzcd}
\mathcal{U} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
* \arrow[r] & \mathcal{Z}
\end{tikzcd} \tag{3.26}$$

is a pushout and therefore, Nisnevich.

Example 3.16. The notions of *semi-orthogonal decomposition* and *exceptional collection* for triangulated categories (see [18]) have an immediate translation to the setting of dg-categories in terms of *split short exact sequence* in $\mathcal{D}g(k)^{idem}$. Recall that an exact sequence in $\mathcal{D}g(k)^{idem}$

$$\begin{tikzcd} I \arrow[r, "f"] \arrow[d] & T \arrow[d, "g"] \\ * \arrow[r] & I' \end{tikzcd} \tag{3.27}$$

is said to *split* if the functor f (resp. g) admits a right adjoint j (resp. fully-faithful right adjoint i). Following Remark 3.15 if $\mathcal{X}$ is a smooth noncommutative space, every semi-orthogonal decomposition of the associated dg-category $T_{\mathcal{X}}$ given by dg-categories I, I' of finite type provides the data dual to a Nisnevich square

$$\begin{tikzcd} I \arrow[r] \arrow[d] & * \arrow[d] \\ T_{\mathcal{X}} \arrow[r] & I' \end{tikzcd} \tag{3.28}$$

Example 3.17. The previous example will be particularly important to us in the case $\mathcal{X} = L_{pe}(\mathbb{P}^1)$. Thanks to the results of [8] we know that $\mathbb{P}^n$ admits an exceptional collection generated by the twisting sheaves $\langle \mathcal{O}, ..., \mathcal{O}(-n)\rangle$. By the previous example, the diagram in $\mathcal{D}g(k)^{idem}$ associated to the split exact sequence

$$\begin{tikzcd} Perf(k) \arrow[r] \arrow[d] & * \arrow[d] \\ L_{pe}(\mathbb{P}^1) \arrow[r] & Perf(k) \end{tikzcd} \tag{3.29}$$

provides the data of a Nisnevich square.

We now prove that our Nisnevich squares are compatible with the monoidal product of smooth noncommutative spaces. For that we will need the following preliminary result

Lemma 3.18. *Let*

$$\begin{tikzcd} \mathcal{W} \arrow[r] \arrow[d] & \mathcal{V} \arrow[d] \\ \mathcal{U} \arrow[r] & \mathcal{X} \end{tikzcd} \tag{3.30}$$

be a Nisnevich square of smooth noncommutative spaces and let

$$\begin{tikzcd}
\widehat{T_{\mathcal{X}}} \arrow[r] \arrow[d] & \widehat{T_{\mathcal{U}}} \arrow[d] \\
\widehat{T_{\mathcal{V}}} \arrow[r] & \widehat{T_{\mathcal{W}}}
\end{tikzcd} \tag{3.31}$$

be its associated pullback diagram in $\mathcal{D}g^{cc}(k)$*. Then the image of* (3.31) *through the (non-full) inclusion* $\mathcal{D}g^{cc}(k) \to \mathcal{D}g^{c}(k)$ *remains a pullback diagram.*

Proof. The $(\infty,1)$-category $\mathcal{D}g^{lp}(k)$ has all limits and the (non-full) inclusion $\mathcal{D}g^{lp}(k) \subseteq \mathcal{D}g(k)^{big}$ preserves them. This is because the same is true for the inclusions $\mathcal{P}r^L_{Stb} \subseteq \mathcal{P}r^L \subseteq Cat_\infty^{big}$ and because of the properties of N^L_{dg}.

By definition, $\mathcal{D}g^{c}(k)$ is the full subcategory of $\mathcal{D}g^{lp}(k)$ spanned by the locally presentable dg-categories of the form $\widehat{T}$ for some small dg-category T. Therefore, we are reduced to showing that the (non-full) inclusion $\mathcal{D}g^{cc}(k) \subseteq \mathcal{D}g^{lp}(k)$ preserves the pullback diagrams (3.31) associated to Nisnevich squares. This statement is the dg-analogue of Proposition 1.16.

Following our preliminary discussion, the functor N^L_{dg} provides a commutative square

$$\begin{tikzcd}
\mathcal{D}g(k)^{idem} \simeq \mathcal{D}g(k)^{cc} \arrow[r, hook, "non\text{-}full"] \arrow[d, "N^L_{dg}"] & \mathcal{D}g(k)^{lp} \arrow[d, "N^L_{dg}"] \\
\mathcal{P}r^L_{\omega,Stb} \arrow[r, hook, "non\text{-}full"] & \mathcal{P}r^L_{Stb}
\end{tikzcd} \tag{3.32}$$

As it was also explained, N^L_{dg} preserves the notions of exact sequence and $\widehat{A}$ has a compact generator if and only if $h(N^L_{dg}(\widehat{A}))$ has a compact generator.

Consider now the pullback diagram (3.31) associated to a Nisnevich covering and let $\widehat{K} \simeq \widehat{K_{\mathcal{X}-\mathcal{U}}} \simeq \widehat{K_{\mathcal{V}-\mathcal{W}}}$ be the dg-category (with a compact generator) in $\mathcal{D}g^{cc}(k)$ associated to the open immersions. We find a diagram in $\mathcal{P}r^L_{\omega,Stb}$

$$\begin{tikzcd}
 & N^L_{dg}(\widehat{T_{\mathcal{X}}}) \arrow[r] \arrow[d] & N^L_{dg}(\widehat{T_{\mathcal{U}}}) \arrow[d] \\
N^L_{dg}(\widehat{K}) \arrow[r, hook] & N^L_{dg}(\widehat{T_{\mathcal{V}}}) \arrow[r] & N^L_{dg}(\widehat{T_{\mathcal{W}}})
\end{tikzcd} \tag{3.33}$$

Since N^L_{dg} commutes with limits, this diagram remains a pullback in $\mathcal{P}r^L_{\omega,Stb}$ and we find ourselves facing the conditions of Proposition 1.16 so that the diagram remains a pullback after the inclusion in $\mathcal{P}r^L_{Stb}$. Finally, since N^L_{dg} is conservative, the commutativity of (3.32) implies that (3.31) remains a pullback in $\mathcal{D}g(k)^{lp}$. This concludes the proof. □

We can now state the main result:

Proposition 3.19.

1) Let $\mathcal{U} \to \mathcal{X}$ be an open immersion of smooth noncommutative spaces. Then, for any smooth noncommutative space $\mathcal{Y}$, the product map

$$\mathcal{U} \otimes \mathcal{Y} \to \mathcal{X} \otimes \mathcal{Y} \tag{3.34}$$

is also an open immersion;

2) Let

$$\begin{tikzcd} \mathcal{W} \arrow[r] \arrow[d] & \mathcal{V} \arrow[d] \\ \mathcal{U} \arrow[r] & \mathcal{X} \end{tikzcd} \tag{3.35}$$

be a Nisnevich square of smooth noncommutative spaces. Then, for any smooth noncommutative space $\mathcal{Y}$, the square

$$\begin{tikzcd} \mathcal{W} \otimes \mathcal{Y} \arrow[r] \arrow[d] & \mathcal{V} \otimes \mathcal{Y} \arrow[d] \\ \mathcal{U} \otimes \mathcal{Y} \arrow[r] & \mathcal{X} \otimes \mathcal{Y} \end{tikzcd} \tag{3.36}$$

remains a Nisnevich square.

Proof. To prove 1), let

$$\begin{tikzcd} K_{\mathcal{X}-\mathcal{U}} \arrow[r] \arrow[d] & T_{\mathcal{X}} \arrow[d] \\ * \arrow[r] & T_{\mathcal{U}} \end{tikzcd} \tag{3.37}$$

be the data in $\mathcal{D}g(k)^{idem}$ corresponding to the open immersion. We are reduced to prove that by tensoring with $T_{\mathcal{Y}}$ (in $\mathcal{D}g(k)^{idem}$) the diagram

$$\begin{tikzcd} K_{\mathcal{X}-\mathcal{U}} \otimes T_{\mathcal{Y}} \arrow[r] \arrow[d] & T_{\mathcal{X}} \otimes T_{\mathcal{Y}} \arrow[d] \\ * \otimes T_{\mathcal{Y}} \arrow[r] & T_{\mathcal{U}} \otimes T_{\mathcal{Y}} \end{tikzcd} \tag{3.38}$$

remains the data of an open immersion. Observe first that since the monoidal structure in $\mathcal{D}g(k)^{idem}$ is compatible with colimits, $* \otimes T_{\mathcal{Y}}$ is again a zero object. To complete the proof it suffices to check that (i) $K_{\mathcal{X}-\mathcal{U}} \otimes T_{\mathcal{Y}}$ remains a dg-category having a compact generator; (*ii*) the map $K_{\mathcal{X}-\mathcal{U}} \otimes T_{\mathcal{Y}} \to T_{\mathcal{X}} \otimes T_{\mathcal{Y}}$ remains fully-faithful and (*iii*) the diagram (3.38) is a pushout. The first assertion follows because the tensor product of dg-categories with a compact generator has a compact generator (as Perf is monoidal). The second is obvious by the definition of fully-faithful and the construction of tensor products. The third follows from Proposition 1.6.3 in [29].

Let us now prove 2). It follows from 1) that both $\mathcal{W} \otimes \mathcal{Y} \to \mathcal{V} \otimes \mathcal{Y}$ and $\mathcal{U} \otimes \mathcal{Y} \to \mathcal{X} \otimes \mathcal{Y}$ remain open immersions, corresponding the quotients by the subcategories $K_{\mathcal{X}-\mathcal{U}} \otimes T_{\mathcal{Y}}$ and $K_{\mathcal{V}-\mathcal{W}} \otimes T_{\mathcal{Y}}$. Since the map $K_{\mathcal{X}-\mathcal{U}} \to K_{\mathcal{V}-\mathcal{W}}$ is an equivalence, the tensor product with the identity of $T_{\mathcal{Y}}$

$$K_{\mathcal{X}-\mathcal{U}} \otimes T_{\mathcal{Y}} \to K_{\mathcal{V}-\mathcal{W}} \otimes T_{\mathcal{Y}} \tag{3.39}$$

remains an equivalence. We are now left to prove that the diagram (3.36) remains a pushout. This is equivalent to prove that associated diagram of dg-categories

$$\begin{tikzcd} T_{\mathcal{X}} \otimes T_{\mathcal{Y}} \arrow[r] \arrow[d] & T_{\mathcal{U}} \otimes T_{\mathcal{Y}} \arrow[d] \\ T_{\mathcal{V}} \otimes T_{\mathcal{Y}} \arrow[r] & T_{\mathcal{W}} \otimes T_{\mathcal{Y}} \end{tikzcd} \tag{3.40}$$

remains a pullback in $\mathcal{D}g(k)^{ft}$. Since all the dg-categories in this diagram are of finite type we can find dg-algebras $T_{\mathcal{X}} = \mathit{Perf}(A_{\mathcal{X}})$, $T_{\mathcal{V}} = \mathit{Perf}(A_{\mathcal{V}})$, $T_{\mathcal{U}} = \mathit{Perf}(A_{\mathcal{U}})$, $T_{\mathcal{W}} = \mathit{Perf}(A_{\mathcal{W}})$ and $T_{\mathcal{Y}} = \mathit{Perf}(A_{\mathcal{Y}})$. It follows that the previous diagram is a pullback if and only if the diagram

$$\begin{tikzcd} \widehat{A_{\mathcal{X}} \otimes A_{\mathcal{Y}}} \arrow[r] \arrow[d] & \widehat{A_{\mathcal{U}} \otimes A_{\mathcal{Y}}} \arrow[d] \\ \widehat{A_{\mathcal{V}} \otimes A_{\mathcal{Y}}} \arrow[r] & \widehat{A_{\mathcal{W}} \otimes A_{\mathcal{Y}}} \end{tikzcd} \tag{3.41}$$

is a pullback in $\mathcal{D}g^{cc}(k)$. By the hypothesis, the diagram

$$\begin{tikzcd} \widehat{A_{\mathcal{X}}} \arrow[r] \arrow[d] & \widehat{A_{\mathcal{U}}} \arrow[d] \\ \widehat{A_{\mathcal{V}}} \arrow[r] & \widehat{A_{\mathcal{W}}} \end{tikzcd} \tag{3.42}$$

is a pullback and so, thanks to [96, Theorem 7.2-1)] and to Lemma 3.18, we have equivalences

$$\widehat{A_{\mathcal{X}} \otimes A_{\mathcal{Y}}} \simeq \mathbb{R}\underline{Hom}_c(\widehat{A_{\mathcal{Y}}}, \widehat{A_{\mathcal{X}}}) \simeq \mathbb{R}\underline{Hom}_c(\widehat{A_{\mathcal{Y}}}, \widehat{A_{\mathcal{V}}} \times_{\widehat{A_{\mathcal{W}}}} \widehat{A_{\mathcal{U}}}) \tag{3.43}$$

$$\simeq \mathbb{R}\underline{Hom}_c(\widehat{A_{\mathcal{Y}}}, \widehat{A_{\mathcal{V}}}) \times_{\mathbb{R}\underline{Hom}_c(\widehat{A_{\mathcal{Y}}}, \widehat{A_{\mathcal{W}}})} \mathbb{R}\underline{Hom}_c(\widehat{A_{\mathcal{Y}}}, \widehat{A_{\mathcal{U}}})$$

$$\simeq \widehat{A_{\mathcal{V}} \otimes A_{\mathcal{Y}}} \times_{\widehat{A_{\mathcal{W}} \otimes A_{\mathcal{Y}}}} \widehat{A_{\mathcal{U}} \otimes A_{\mathcal{Y}}} \qquad \square \tag{3.44}$$

Remark 3.20. It follows from the proof that $T_{\mathcal{Y}}$ does not need to be of finite type. It is enough the existence of a compact generator.

To conclude this section we prove that our notion of Nisnevich squares of smooth noncommutative spaces is compatible with the classical notion for schemes.

Proposition 3.21. *If X is an affine smooth scheme of finite type over k and*

$$\begin{tikzcd} p^{-1}(U) \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & X \end{tikzcd} \tag{3.45}$$

is a Nisnevich square in $N(AffSm^{ft}(k))$, then the induced diagram in $\mathcal{N}cS(k)$

$$\begin{tikzcd} L_{pe}(p^{-1}(U)) \arrow[r] \arrow[d] & L_{pe}(V) \arrow[d] \\ L_{pe}(U) \arrow[r] & L_{pe}(X) \end{tikzcd} \tag{3.46}$$

is a Nisnevich square of smooth noncommutative spaces.

Proof. Indeed, it is immediate that both maps $L_{pe}(p^{-1}(U)) \to L_{pe}(V)$ and $L_{pe}(U) \to L_{pe}(X)$ are open immersions of smooth noncommutative spaces. This is exactly the example that motivated the definition. They correspond to the quotient maps in $\mathcal{D}g(k)^{idem}$

$$L_{pe}(X) \longrightarrow L_{pe}(X)/L_{pe}(X)_{X-U} \quad \text{and} \quad L_{pe}(V) \longrightarrow L_{pe}(V)/L_{pe}(V)_{V-p^{-1}(U)} \tag{3.47}$$

We are left to check that:

1) The square in $\mathcal{D}g(k)^{idem}$

$$\begin{tikzcd} L_{pe}(X) \arrow[r] \arrow[d] & L_{pe}(U) \arrow[d] \\ L_{pe}(V) \arrow[r] & L_{pe}(p^{-1}(U)) \end{tikzcd} \tag{3.48}$$

is a pullback;

2) the map $L_{pe}(X) \to L_{pe}(V)$ in $\mathcal{D}g(k)^{idem}$ induces an equivalence $L_{pe}(X)_{X-U} \simeq L_{pe}(V)_{V-p^{-1}(U)}$.

The fact that (3.48) is a pullback follows from the fact that perfect complexes satisfy descent for the étale topology (which is a refinement of the Nisnevich topology). This result was originally proven by Hirschowitz and Simpson in [87]. See also [103] for further details.

The assertion 2) follows from 1) together with the fact that both $L_{pe}(X)_{X-U}$ and $L_{pe}(V)_{V-p^{-1}(U)}$ are by definition, the kernels of the quotient maps (3.47). □

Remark 3.22. In fact, it can be proved that if a pullback diagram like (3.45) induces a Nisnevich square of smooth noncommutative spaces then it is a Nisnevich square in the classical sense. This can be deduced using the equivalence $L_{qcoh}(X)_{X-U} \simeq L_{qcoh}(V)_{p^{-1}(X-U)}$ together with the equivalences $L_{qcoh}(X)_{X-U} \simeq L_{qcoh}(\widehat{X}_{X-U})$ and $L_{qcoh}(V)_{p^{-1}(X-U)} \simeq L_{qcoh}(\widehat{V}_{p^{-1}(X-U)})$ where $\widehat{X}_{X-U}$, respectively, $\widehat{V}_{p^{-1}(X-U)}$, denotes the formal completion of X (resp. V) at the closed subset $X-U$ (resp. $p^{-1}(X-U)$) (see [37, Prop. 7.1.3 and Prop. 6.8.2]). In particular this shows that the new notion of Nisnevich square is not really a weaker form of the original notion.

Remark 3.23. This non-commutative incarnation of the Nisnevich topology is not an actual Grothendieck topology. The pushout of a Nisnevich covering of a dg-category of finite type T along a functor $T \to T'$ is not a Nisnevich covering of T' for it does not have remain a pullback.

3.3.2. The motivic stable homotopy theory of noncommutative spaces

Now that we have an analogue for the Nisnevich topology in the noncommutative setting, compatible with the classical notion for schemes, we can finally conclude our task. We apply the same formula that produces the theory of Morel–Voevodsky. We start with $\mathcal{N}cS(k)^{\otimes}$ and consider its free cocompletion $\mathcal{P}^{big}(\mathcal{N}cS(k))$ together with the natural unique monoidal product extending the monoidal operation in $\mathcal{N}cS(k)$, compatible with colimits on each variable and making the inclusion $j : \mathcal{N}cS(k) \to \mathcal{P}^{big}(\mathcal{N}cS(k))$ monoidal. In particular, $j(L_{pe}(k))$ is the unit object. Next step, consider the localization $\mathcal{P}^{big}_{Nis}(\mathcal{N}cS(k))$ of $\mathcal{P}^{big}(\mathcal{N}cS(k))$ along the set of all edges $j(\mathcal{U}) \coprod_{j(\mathcal{W})} j(\mathcal{V}) \to j(\mathcal{X})$ running over all the Nisnevich squares of smooth noncommutative space

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{V} \\ \downarrow & & \downarrow \\ \mathcal{U} & \longrightarrow & \mathcal{X} \end{array} \tag{3.49}$$

The theory of localization for presentable $(\infty,1)$-categories [59, 5.5.4.15] implies that $\mathcal{P}^{big}_{Nis}(\mathcal{N}cS(k))$ is an accessible reflexive localization of $\mathcal{P}^{big}(\mathcal{N}cS(k))$. The same result, together with the fact that the Nisnevich squares are pushouts squares, implies that every representable $j(\mathcal{X})$ is in $\mathcal{P}^{big}_{Nis}(\mathcal{N}cS(k))$. Moreover, and thanks to Proposition 3.19, we deduce that this localization is monoidal. Finally, and in analogy with the commutative case, we consider the localization

$$l^{nc}_{\mathbb{A}^1} : \mathcal{P}^{big}_{Nis}\big(\mathcal{N}cS(k)\big) \to \mathcal{H}_{nc}(k) \tag{3.50}$$

taken with respect to the set of all maps

$$j(Id_{\mathcal{X}}) \otimes j\big(L_{pe}(p)\big) : j(\mathcal{X}) \otimes j\big(L_{pe}\big(\mathbb{A}^1_k\big)\big) \longrightarrow j(\mathcal{X}) \otimes j\big(L_{pe}\big(Spec(k)\big)\big) \tag{3.51}$$

with $\mathcal{X}$ running over $\mathcal{N}cS(k)$. Here, $p : \mathbb{A}^1 \to Spec(k)$ is the canonical projection and the tensor product is computed in $\mathcal{P}^{big}_{Nis}(\mathcal{N}cS(k))$.[20] Again, this is an accessible reflective localization of $\mathcal{P}^{big}_{Nis}(\mathcal{N}cS(k))$ and it follows immediately from the definition of the localizing set that it is monoidal. With this we have a sequence of monoidal localizations

$$\mathcal{N}cS(k)^{\otimes} \xrightarrow{j} \mathcal{P}^{big}\big(\mathcal{N}cS(k)\big)^{\otimes} \longrightarrow \mathcal{P}^{big}_{Nis}\big(\mathcal{N}cS(k)\big)^{\otimes} \longrightarrow \mathcal{H}_{nc}(k)^{\otimes} \tag{3.52}$$

and by construction, $\mathcal{H}_{nc}(k)$ is a presentable symmetric monoidal $(\infty, 1)$-category and has a final object which we can identify with the image of the zero object of $\mathcal{N}cS(k)$ through the Yoneda map. Again, in analogy with the classical situation, we consider the universal pointing map

$$()^{nc}_{+} : \mathcal{H}_{nc}(k)^{\otimes} \to \mathcal{H}_{nc}(k)^{\wedge(\otimes)}_{*} \tag{3.53}$$

which is an equivalence because of our Convention 3.13: when we localize with respect to the Nisnevich topology with Convention 3.13 the $(\infty, 1)$-category $\mathcal{H}_{nc}(k)$ becomes pointed.

Finally, the compatibility between the classical and the new Nisnevich squares[21] and the respective $\mathbb{A}^1$ and $L_{pe}(\mathbb{A}^1)$-localizations, we deduce the existence of uniquely determined monoidal colimit preserving functors that make the diagram homotopy commutative

[20] Of course, since j is monoidal and the representable objects are Nisnevich local, this is the same as localizing with respect to the class of all maps $j(\mathcal{X} \otimes L_{pe}(\mathbb{A}^1_k)) \longrightarrow j(\mathcal{X} \otimes L_{pe}(Spec(k)))$.

[21] Recall that the collection of classical Nisnevich squares forms a basis for the Nisnevich topology.

$$\begin{tikzcd}
N(\mathit{AffSm}^{ft}(k))^{\times} \arrow[r, "L_{pe}"] \arrow[d, "j^{\otimes}"] & \mathcal{N}cS(k)^{\otimes} \arrow[d, "j^{\otimes}"] \\
\mathcal{P}^{big}(N(\mathit{AffSm}^{ft}(k)))^{\times} \arrow[r, dashed, "(L_{pe})_!"] \arrow[d] & \mathcal{P}^{big}(\mathcal{N}cS(k))^{\otimes} \arrow[d] \\
Sh^{big}_{Nis}(N(\mathit{AffSm}^{ft}(k)))^{\times} \arrow[r, dashed] \arrow[d, "l^{\times}_{\mathbb{A}^1}"] & \mathcal{P}^{big}_{Nis}(\mathcal{N}cS(k))^{\otimes} \arrow[d, "l^{nc,\otimes}_{\mathbb{A}^1}"] \\
\mathcal{H}(S)^{\times} \arrow[r, dashed] \arrow[d, "()_+"] & \mathcal{H}_{nc}(k)^{\otimes} \\
\mathcal{H}(k)^{\wedge}_{*} \arrow[ur, dashed, "\psi^{\otimes}"] &
\end{tikzcd} \tag{3.54}$$

If we proceed according to the classical construction, the next step would be to stabilize the theory, first with respect to S^1 (the ordinary stabilization) and then with respect to the Tate circle. It happens that the inner properties of the noncommutative world make both these steps unnecessary.

Proposition 3.24. *The presentable pointed symmetric monoidal* $(\infty, 1)$*-category* $\mathcal{H}_{nc}(k)^{\otimes}$ *is stable. Moreover, the Tate circle* $\psi(\mathbb{G}_m)$ *is already an invertible object.*

Recall that in $\mathcal{H}(k)^{\wedge}_{*}$ we have an equivalence $(\mathbb{P}^1, \infty) \simeq S^1 \wedge \mathbb{G}_m$ with $\mathbb{G}_m$ pointed at 1. Since the functor $\psi^{\otimes}$ is monoidal and commutes with colimits, we also have $\psi((\mathbb{P}^1, \infty)) \simeq S^1 \wedge \psi(\mathbb{G}_m)$. In particular, Proposition 3.24 will follow immediately from the following lemma (using Remark 2.25).

Lemma 3.25. *The object* $\psi((\mathbb{P}^1, \infty)) \in \mathcal{H}_{nc}(k)^{\otimes}$ *is invertible and equivalent to a unit of the monoidal structure.*

Proof. By definition, we have

$$\left(\mathbb{P}^1, \infty\right) := \mathit{cofiber}_{\mathcal{H}(k)}\left[l_{\mathbb{A}^1}\left(\infty : \mathit{Spec}(k) \to \mathbb{P}^1\right)\right] \tag{3.55}$$

where $\infty : \mathit{Spec}(k) \to \mathbb{P}^1$ is the point at infinity. By diagram chasing, the fact that $L_{pe}(\mathbb{P}^1)$ is a dg-category of finite type, and the fact that all the relevant maps commute with colimits we find

$$\psi\left(\left(\mathbb{P}^1, \infty\right)\right) \simeq l^{nc}_{\mathbb{A}^1}\left(\mathit{cofiber}_{\mathcal{P}_{Nis}(\mathcal{N}cS(k))}\left[j\left(L_{pe}(\infty)\right)\right]\right) \tag{3.56}$$

We claim that the last cofiber is the unit for the monoidal structure in $\mathcal{P}^{big}_{Nis}(\mathcal{N}cS(k))$, which, because the Yoneda functor is monoidal, corresponds to $j(\mathit{Perf}(k))$. To see this, we observe first that map $L_{pe}(\infty) : Perf(k) = L_{pe}(k) \to L_{pe}(\mathbb{P}^1)$ in $\mathcal{N}cS(k)$ corresponds in fact to the pullback map $L_{pe}(\mathbb{P}^1) \to Perf(k)$ along ∞ in $\mathcal{D}g(k)^{idem}$. Recall the existence of an exceptional collection in $L_{pe}(\mathbb{P}^1)$ generated by the sheaves $\mathcal{O}$ and $\mathcal{O}(-1)$. Since the pullback preserves structural sheaves, the map $Perf(k) \to L_{pe}(\mathbb{P}^1)$ in $\mathcal{N}cS(k)$ fits in the Nisnevich square of Example 3.17

$$\begin{tikzcd} Perf(k) \arrow[r] \arrow[d] & L_{pe}(\mathbb{P}^1) \arrow[d] \\ * \arrow[r] & Perf(k) \end{tikzcd} \tag{3.57}$$

dual to the split exact sequence provided by the exceptional collection. Finally, since in $\mathcal{P}^{big}_{Nis}(\mathcal{N}cS(k))$ every Nisnevich square is forced to become a pushout, we have

$$cofiber_{\mathcal{P}_{Nis}(\mathcal{N}cS(k))}\left[j\left(Perf(k) \to L_{pe}\left(\mathbb{P}^1\right)\right)\right] \simeq Perf(k) \tag{3.58}$$

which concludes the proof. □

It also follows that we have a canonical equivalence

$$\mathcal{H}_{nc}(k)^{\otimes}\left[\left(\psi\left(\mathbb{P}^1, \infty\right)^{-1}\right)\right] \simeq \mathcal{H}_{nc}(k)^{\otimes} \tag{3.59}$$

and for this reason, we reset the notations to match the classical one

$$\mathcal{SH}_{nc}(k)^{\otimes} := \mathcal{H}_{nc}(k)^{\otimes} \tag{3.60}$$

Remark 3.26. Let $\mathcal{C}$ be an $(\infty, 1)$-category with a zero object 0. Recall that a *split exact sequence* in $\mathcal{C}$ is the data of a pushout square

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[d, "p"] \\ 0 \arrow[r] & C \end{tikzcd} \tag{3.61}$$

together with maps $u : C \to B$ and $v : B \to A$ in $\mathcal{C}$ with $p \circ u \sim id_C$ and $v \circ i \sim id_A$. We easily see that if $\mathcal{C}$ is a stable $(\infty, 1)$-category, the data of a split exact sequence provides an equivalence $B \simeq A \oplus C$.

Remark 3.27. Let $\mathcal{X}$ be smooth noncommutative space whose associated dg-category $T_{\mathcal{X}}$ admits an exceptional collection generated by $n + 1$ elements. By Remark 3.16 and Example 3.17, this provides to the data of n different Nisnevich coverings. These are sent to split exact sequences in $\mathcal{SH}_{nc}(k)$ which we now know is stable. Using Remark 3.26 we find that the image of $\mathcal{X}$ in $\mathcal{SH}_{nc}(k)$ decomposes as a direct sum of $n + 1$ copies of the unit $1 = l^{nc}_{\mathbb{A}^1}(Perf(k))$. In particular the smooth noncommutative space $L_{pe}(\mathbb{P}^n)$ becomes equivalent to the direct sum $\underbrace{1 \oplus ... \oplus 1}_{n+1}$ in $\mathcal{SH}_{nc}(k)^{\otimes}$.

Finally, our universal property for inverting an object in a presentable symmetric monoidal $(\infty,1)$-category ensures the existence of a unique monoidal colimit map $\mathcal{L}^{\otimes}$ extending the diagram (3.54) to

$$\begin{tikzcd} N(AffSm^{ft}(k))^{\times} \arrow[r, "L_{pe}^{\otimes}"] \arrow[d] & \mathcal{N}cS(k)^{\otimes} \arrow[d] \\ \mathcal{SH}(k)^{\otimes} \arrow[r, dashed, "\mathcal{L}^{\otimes}"] & \mathcal{SH}_{nc}(k)^{\otimes} \end{tikzcd} \tag{3.62}$$

relating the classical stable homotopy theory of schemes with our new theory. From now we assume k is Noetherian of finite Krull dimension.

Remark 3.28. Using the same arguments of 2.4.4 we can describe the symmetric monoidal $(\infty,1)$-category $\mathcal{SH}_{nc}(k)^{\otimes}$ using presheaves of spectra. More precisely, we can start from the $(\infty,1)$-category of smooth noncommutative spaces $\mathcal{N}cS(k)$ and consider the very big $(\infty,1)$-category $Fun(\mathcal{N}cS(k)^{op}, \widehat{Sp})$. Using the equivalence $Fun(\mathcal{N}cS(k)^{op}, \widehat{Sp}) \simeq Stab(\mathcal{P}^{big}(\mathcal{N}cS(k))_*)$ together with Remark 2.25 we obtain a canonical monoidal structure $Fun(\mathcal{N}cS(k)^{op}, \widehat{Sp})^{\otimes}$ defined by the inversion $\mathcal{P}^{big}(\mathcal{N}cS(k))_*^{\wedge(\otimes)}[(S^1)^{-1}]^{\otimes}$. We proceed, and perform the localizations with respect to the noncommutative version of the Nisnevich topology and $L_{pe}(\mathbb{A}^1)$. More precisely, and using the same notations as in 2.4.4 we localize with respect to the class of all canonical maps

$$\delta_{\Sigma^{\infty}_{+}\circ j(\mathcal{U})}(K) \coprod_{\delta_{\Sigma^{\infty}_{+}\circ j(\mathcal{W})}(K)} \delta_{\Sigma^{\infty}_{+}\circ j(\mathcal{V})}(K) \to \delta_{\Sigma^{\infty}_{+}\circ j(\mathcal{X})}(K) \tag{3.63}$$

with K in $(\widehat{Sp})^{\omega}$ and $\mathcal{W}$, $\mathcal{V}$, $\mathcal{U}$ and $\mathcal{X}$ part of a Nisnevich square of noncommutative smooth spaces. For the $\mathbb{A}^1$ localization, we localize with respect to the class of all induced maps

$$\delta_{\Sigma^{\infty}_{+}\circ j(\mathcal{X}\otimes L_{pe}(\mathbb{A}^1))}(K) \to \delta_{\Sigma^{\infty}_{+}\circ j(\mathcal{X})}(K) \tag{3.64}$$

with $\mathcal{X}$ in $\mathcal{N}cS(k)$ and $K \in (\widehat{Sp})^{\omega}$. By the same argument, these are monoidal reflexive localizations. We denote the result as $Fun_{Nis, L_{pe}(\mathbb{A}^1)}(\mathcal{N}cS(k)^{op}, \widehat{Sp})^{\otimes}$. It is a stable presentable symmetric monoidal $(\infty,1)$-category and by Proposition 3.24 and the universal properties involved, it is canonically monoidal equivalent to $\mathcal{SH}_{nc}(k)^{\otimes}$.

Using this equivalence and the definition of $\mathcal{N}cS(k)$, we can identify an object $F \in \mathcal{SH}_{nc}(k)^{\otimes}$ with a functor $\mathcal{D}g(k)^{ft} \to \widehat{Sp}$ satisfying $L_{pe}(\mathbb{A}^1)$-invariance, having a descent property with respect to the Nisnevich squares and because of Convention 3.13, satisfying $F(0) = *$.

Remark 3.29 *(Strictification).* It is also important to remark that an object F in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ can always be identified up to equivalence with an actual strict functor F_s from the category of dg-categories endowed with the Morita model structure of [88] to some combinatorial model category whose underlying $(\infty,1)$-category is $\widehat{Sp}$ (for instance, the big model category of symmetric spectra Sp^{Σ} of [44]), with F_s sending Morita equivalences to weak-equivalences and commuting with filtered homotopy colimits. Indeed, as mentioned in our fast survey, $\mathcal{D}g(k)^{ft}$ generates $\mathcal{D}g(k)^{idem}$ under filtered colimits. Since Sp admits all small filtered colimits, using [59, Thm. 5.3.5.10] we find an equivalence of $(\infty,1)$-categories between $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ and $Fun_{\omega}(\mathcal{D}g(k)^{idem}, \widehat{Sp})$ – the full subcategory of $Fun(\mathcal{D}g(k)^{idem}, \widehat{Sp})$ spanned by the functors that preserve filtered colimits. Moreover, we have also seen that $\mathcal{D}g(k)^{idem}$ is the underlying $(\infty,1)$-category of the Morita model structure for small dg-categories. Finally, with the appropriate universe considerations, we can use the strictification result of [63, 1.3.4.25] and the characterization of homotopy limits and colimits in a model category as limits and colimits in its underlying $(\infty,1)$-category [63, 1.3.4.24] to deduce the existence of a canonical equivalence between $Fun_{\omega}(\mathcal{D}g(k)^{idem}, \widehat{Sp})$ and the localization along the levelwise equivalences of the category of strict functors from the category of dg-categories to the strict model for spectra Sp^{Σ}, which commute with filtered homotopy colimits and send Morita weak-equivalences to weak-equivalences in Sp^{Σ}.

4. *K*-theory and noncommutative motives

4.1. Main results

The results in the previous section establish a homotopy commutative diagram of colimit preserving monoidal functors extending the functor L_{pe}

$$
\begin{tikzcd}
N(AffSm^{ft}(k))^{\times} \arrow[r, "L_{pe}^{\otimes}"] \arrow[d, hook, "(\Sigma^{\infty}_{+}\circ j)^{\otimes}"] & \mathcal{N}cS(k)^{\otimes} \arrow[d, hook, "(\Sigma^{\infty}_{+}\circ j_{nc})^{\otimes}"] \\
Fun(N(AffSm^{ft}(k))^{op}, \widehat{Sp})^{\otimes} \arrow[r, dashed] \arrow[d, "l_{Nis}^{\otimes}"] & Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})^{\otimes} \arrow[d, "l_{Nis}^{nc,\otimes}"'] \\
Fun_{Nis}(N(AffSm^{ft}(k))^{op}, \widehat{Sp})^{\otimes} \arrow[r, dashed] \arrow[d, "l_{\mathbb{A}^1}^{\otimes}"] & Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp})^{\otimes} \arrow[d, "l_{\mathbb{A}^1}^{nc,\otimes}"'] \\
Fun_{Nis,\mathbb{A}^1}(N(AffSm^{ft}(k))^{op}, \widehat{Sp})^{\otimes} \arrow[r, dashed] \arrow[d, "\Sigma_{\mathbb{G}_m}^{\otimes}"] & Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \widehat{Sp})^{\otimes} \arrow[d, "\sim"] \\
\mathcal{SH}(k)^{\otimes} \arrow[r, dashed, "\mathcal{L}^{\otimes}"] & \mathcal{SH}_{nc}(k)^{\otimes}
\end{tikzcd}
\tag{4.1}
$$

thus providing a canonical mechanism to compare the theory of Morel–Voevodsky with our new approach.

Our goal in this section is to explore how this bridge can be used to give a canonical interpretation to the various flavors of algebraic K-theory of schemes and dg-categories. In order to state our results, we observe first that, due to the Adjoint Functor Theorem [59, Corollary 5.5.2.9], each of the dotted monoidal functors in (4.1) has a right adjoint. This is because at each level, the source and target $(\infty,1)$-categories are presentable and each dotted map is, by construction, colimit-preserving. Furthermore, since each dotted map is monoidal, these right adjoints are lax-monoidal (see [63, 7.3.2.7]). In this case, together with the lax-monoidal inclusions associated to the reflexive monoidal localizations, we have a new commutative diagram of lax-monoidal functors

$$\begin{tikzcd}
Fun(N(AffSm^{ft}(k))^{op}, \widehat{Sp})^{\otimes} & Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})^{\otimes} \arrow[l, "\mathcal{M}_1^{\otimes}"'] \\
Fun_{Nis}(N(AffSm^{ft}(k))^{op}, \widehat{Sp})^{\otimes} \arrow[u, hook] & Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp})^{\otimes} \arrow[l, "\mathcal{M}_2^{\otimes}"'] \arrow[u, hook] \\
Fun_{Nis,\mathbb{A}^1}(N(AffSm^{ft}(k))^{op}, \widehat{Sp})^{\otimes} \arrow[u, hook] & Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \widehat{Sp})^{\otimes} \arrow[l, "\mathcal{M}_3^{\otimes}"'] \arrow[u, hook] \\
\mathcal{SH}(k)^{\otimes} \arrow[u, "\Omega^{\infty,\otimes}_{\mathbb{G}_m}"] & \mathcal{SH}_{nc}(k)^{\otimes} \arrow[l, "\mathcal{M}^{\otimes}"'] \arrow[u, "\sim"]
\end{tikzcd} \tag{4.2}$$

Let us present some remarks that will be useful along this section.

Remark 4.1. The first functor $\mathcal{M}_1$ commutes with small colimits. We can deduce this either from the fact that colimits in $Fun(N(AffSm^{ft}(k))^{op}, \widehat{Sp})$ and in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ are computed objectwise (see [59, 5.1.2.3]) or from the spectral enriched version of Yoneda's lemma (2.41).

Remark 4.2. All the symmetric monoidal $(\infty,1)$-categories appearing in the previous diagram are stable and presentable. Stability follows because pushouts of local objects remain local, thanks to the fact that all colimits are computed objectwise in spectra. Therefore all these are closed monoidal. In particular, recall that if $\mathcal{C}_0 \subseteq \mathcal{C}$ is a monoidal reflexive localization and if $\mathcal{C}$ admits internal-homs $\underline{Hom}_{\mathcal{C}}$ then $\mathcal{C}_0$ admits internal-homs: given X and Y local, we can easily see that $\underline{Hom}_{\mathcal{C}}(X,Y)$ is also local and works as an internal-hom in $\mathcal{C}_0$.

We observe that each functor $\mathcal{M}_*$ is compatible with the respective internal-homs, in the sense that at each level, for every object X on the left and F on the right, we have

$$\mathcal{M}_*\big(\underline{Hom}_*(\mathcal{L}_*(X), F)\big) \simeq \underline{Hom}_*\big(X, \mathcal{M}_*(F)\big) \tag{4.3}$$

where $\mathcal{L}_*$ denotes the respective monoidal left adjoint appearing in the diagram (4.1).

Remark 4.3. Thanks to the enriched version of Yoneda's lemma for spectral presheaves (see Remark 2.41), given an object $F \in Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$, we have for each scheme X an equivalence of spectra

$$\begin{aligned} &Map^{Sp}_{Fun(N(AffSm^{ft}(k))^{op},\widehat{Sp})}\big(\Sigma^{\infty}_{+} \circ j(X), \mathcal{M}_1(F)\big) \\ &\quad \simeq Map^{Sp}_{Fun(\mathcal{D}g(k)^{ft},\widehat{Sp})}\big(\Sigma^{\infty}_{+} \circ j_{nc}\big(L_{pe}(X)\big), F\big) \simeq F\big(L_{pe}(X)\big) \end{aligned} \tag{4.4}$$

so that $\mathcal{M}_1(F)$ can be thought of as a restriction of F to the commutative world. The same is valid for $\mathcal{M}_2$ and $\mathcal{M}_3$ because the upper vertical arrows are inclusions.

This mechanism allows us to restrict noncommutative invariants to the commutative world. In this section we will be interested in the restriction of the various algebraic K-theories of dg-categories. As we shall explain below, all of them live as objects in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$. There are two of primary relevance to us, namely, K^c encoding Waldhausen's connective K-theory (Section 4.2.2) and K^S encoding the non-connective K-theory of dg-categories defined by means of Schlichting's framework in [82] (Section 4.2.3). By construction, the latest comes naturally equipped with a canonical natural transformation $K^c \to K^S$ which is an equivalence in the connective part. For the first one, it follows immediately from the spectral version of Yoneda's lemma and from the definition in [94, Section 3] that $\mathcal{M}_1(K^c)$ recovers the connective algebraic K-theory of schemes. The second one, by the comparison result [82, Theorem 7.1], recovers the non-connective K-theory of schemes of Bass–Thomason–Trobaugh of [94]. The construction of K^S in [24] using the methods of [82] is somehow ad-hoc. Our first main result explains how the non-connective version of K-theory K^S can be canonically obtained from the connective version K^c as a result of forcing our noncommutative-world version of Nisnevich descent.

Theorem 4.4. *The canonical morphism $K^c \to K^S$ presents non-connective K-theory of dg-categories as the (noncommutative) Nisnevich localization of connective K-theory.*

To prove this result we will first check that K^S is Nisnevich local. This follows from the well-known localization theorem for non-connective K-theory (see Corollary 4.18 below). The rest of the proof will require a careful discussion concerning the behavior of the noncommutative Nisnevich localization. There are two main ingredients:

Step 1) *Every Nisnevich local $F : \mathcal{D}g(k)^{ft} \to \widehat{Sp}$ is determined by its connective part by means of the Bass exact sequences.* More precisely, we show that every Nisnevich local functor $F : \mathcal{D}g(k)^{ft} \to \widehat{Sp}$ satisfies the familiar Bass exact sequences for any integer n. We will see that the proof in [94] can be easily adapted to our setting. Namely, we start by showing that every Nisnevich local F satisfies the Projective Bundle Theorem. This result is central and appears as a consequence of one of the most important features of the noncommutative world, namely, the fact that Nisnevich coverings of non-geometrical origin are allowed, in particular, those appearing from semi-orthogonal decompositions and exceptional collections. The projective bundle theorem is a direct consequence of the existence of an exceptional collection on $L_{pe}(\mathbb{P}^1)$ generated by the sheaves $\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(-1)$ (see [8]). Its existence forces the image of $L_{pe}(\mathbb{P}^1)$ in $Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ to become equivalent to the direct sum $L_{pe}(k) \oplus L_{pe}(k)$. To complete the proof we proceed as in [94, Theorem 6.1] and explain how this direct sum decomposition can be suitably adapted in order to extract the familiar Bass exact sequences out of the classical Nisnevich covering of $\mathbb{P}^1$ by two affine lines.

Step 2) *The connective truncation of the localization map $K^c \to l^{nc}_{Nis}(K^c)$ is an equivalence.*[22] In other words, the information stored in the connective part of $l_{Nis}(K^c)$ remains the information of connective K-theory. We will prove something a bit more general, namely, that this property holds not only for K^c but for the whole class of functors $F : \mathcal{D}g(k)^{ft} \to \widehat{Sp}$ satisfying the formal properties of K^c, namely, having values in connective spectra and sending Nisnevich squares of dg-categories to pullback squares of connective spectra (for K^c this follows from the fibration theorem of Waldhausen [108, 1.6.4] – see Proposition 4.16 below). These will be called *connectively-Nisnevich local*. We prove that the connective truncation functor induces a canonical equivalence between the theory of connective-Nisnevich functors and that of Nisnevich functors (see Proposition 4.26). For this we will show that if F is connectively-Nisnevich local, its noncommutative Nisnevich localization $l^{nc}_{Nis}(F)$ is equivalent to F^B – the more familiar B-construction of Thomason of [94, Def. 6.4].

Remark 4.5. Since the functor $\mathcal{M}_2$ in the diagram (4.2) sends Nisnevich local objects to Nisnevich local objects, our Theorem 4.4 provides a new proof that the spectral presheaf giving the Bass–Thomason–Trobaugh K-theory of schemes satisfies Nisnevich descent.

We can now go one step further and consider the $\mathbb{A}^1$-localization of K^S. We will prove that

Theorem 4.6. *$\mathcal{M}_3(l^{nc}_{\mathbb{A}^1}(K^S))$ is the Nisnevich local $\mathbb{A}^1$-invariant spectral presheaf giving Weibel's homotopy invariant K-theory of schemes of [109]. In particular, $\mathcal{M}(l^{nc}_{\mathbb{A}^1}K^S)$ is*

[22] Recall that $\widehat{Sp}$ has a natural t-structure $(\widehat{Sp}_{\geq 0}, \widehat{Sp}_{\leq -1})$ with $\widehat{Sp}_{\geq 0}$ the full subcategory spanned by connective spectra. As a consequence, the inclusion $\widehat{Sp}_{\geq 0} \subseteq \widehat{Sp}$ (resp. $\widehat{Sp}_{\leq -1} \subseteq \widehat{Sp}$) admits a right adjoint $\tau_{\geq 0}$ (resp. left adjoint $\tau_{\leq -1}$). In particular, we have an induced adjunction

$$\begin{tikzcd} Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \arrow[r, hook] & Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, bend left, "\tau_{\geq 0}"] \end{tikzcd}$$

with $\tau_{\geq 0}$ a right adjoint to the inclusion.

canonically equivalent to the object $\mathcal{KH}$ in $\mathcal{SH}(k)$ studied in [107] and in [22] representing homotopy invariant algebraic K-theory of schemes.

The proof of this result follows immediately from the results in [22] and from our Theorem 4.4 using a nice description of the $\mathbb{A}^1$-localization functors. This will be done in Section 4.4.

Our second main result in this section is a new representability theorem for K-theory.

Theorem 4.7. *The further localization $l^{nc}_{\mathbb{A}^1}(K^S)$ is a unit for the monoidal structure in $\mathcal{SH}_{nc}(k)^{\otimes}$.*

In [12], the author constructs an $\mathbb{A}^1$-equivalence between the split and the standard versions of Waldhausen's S-construction. In Section 4.5 we will explain how this $\mathbb{A}^1$-equivalence appears in our context and how the theorem follows as a consequence.

We deduce the following immediate corollaries

Corollary 4.8 *(Kontsevich). Let $\mathcal{X}$ and $\mathcal{Y}$ be two noncommutative spaces with $\mathcal{Y}$ smooth and proper. Then, there is a natural equivalence of spectra*

$$Map^{Sp}_{\mathcal{SH}_{nc}(k)}(\mathcal{X},\mathcal{Y}) \simeq \big(l^{nc}_{\mathbb{A}^1}K^S\big)(T_{\mathcal{X}} \otimes \check{T}_{\mathcal{Y}}) \tag{4.5}$$

where we identify $\mathcal{X}$ and $\mathcal{Y}$ with their images in $\mathcal{SH}_{nc}(k)$ and where $T_{\mathcal{X}}$ (resp. $\check{T}_{\mathcal{Y}}$) denotes the dg-category of finite type associated to $\mathcal{X}$ (resp. the dual of the dg-category associated to $\mathcal{Y}$).

Proof. This follows directly from the spectral version of the Yoneda lemma and from our Theorems 4.4 and 4.7, together with the fact that a smooth and proper noncommutative space is dualizable. □

Remark 4.9. We direct the reader to [79, Prop. 9.3.4] for an extension of Theorem 4.7 and Corollary 4.8 to non-commutative motives over a more general base scheme.

Corollary 4.10. *The object $\mathcal{KH} \in \mathcal{SH}(k)$ representing homotopy algebraic K-theory is equivalent to $\mathcal{M}(1_{nc})$. In particular, for each scheme X we have an equivalence of spectra*

$$KH(X) \simeq Map^{Sp}_{\mathcal{SH}(k)}\big(\Sigma^{\infty}_{+} \circ j(X), \mathcal{KH}\big) \simeq Map^{Sp}_{\mathcal{SH}_{nc}(k)}\big(\Sigma^{\infty}_{+} \circ j_{nc}\big(L_{pe}(X)\big), 1_{nc}\big) \tag{4.6}$$

At this point we should emphasize that a different representability result for connective K-theory is already known from the thesis of G. Tabuada [90] and for non-connective K-theory from his later works with D.C. Cisinski [24]. Our setting and proofs are independent of theirs. In the appendix we explain the relation between the two approaches. The main advantage of our theory is the existence of a canonical comparison with the original approach of Morel–Voevodsky and our new representability theorem brings some immediate consequences to the nature of this comparison. Namely, since $\mathcal{M}$ is lax-monoidal [63, 7.3.2.7], the object $\mathcal{K}H \simeq \mathcal{M}(1_{nc})$ acquires a canonical structure of commutative algebra-object in $\mathcal{SH}(k)$ induced by the trivial algebra structure on the unit object 1_{nc}. In this case, the comparison functor $\mathcal{L}^{\otimes} : \mathcal{SH}(k)^{\otimes} \to \mathcal{SH}_{nc}(k)^{\otimes}$ admits a canonical colimit preserving monoidal factorization (see our discussion in [79, Section 3.3.9]):

$$\begin{tikzcd}
\mathcal{SH}(k)^{\otimes} \arrow[r, "\mathcal{L}^{\otimes}"] \arrow[d, "-\otimes\mathcal{K}H"] & \mathcal{SH}_{nc}(k)^{\otimes} \arrow[d, "-\otimes\mathcal{L}(\mathcal{K}H)"'] \arrow[dr, "(-\otimes 1_{nc})\simeq Id"] & \\
Mod_{\mathcal{K}H}(\mathcal{SH}(k))^{\otimes} \arrow[r] & Mod_{\mathcal{L}(\mathcal{K}H)}(\mathcal{SH}_{nc}(k))^{\otimes} \arrow[r, "-\otimes_{\mathcal{L}(\mathcal{K}H)}1_{nc}"'] & Mod_{1_{nc}}(\mathcal{SH}_{nc}(k))^{\otimes}
\end{tikzcd} \tag{4.7}$$

where the first lower map is the monoidal functor induced by $\mathcal{L}$ at the level of modules and the last map is base-change with respect to the canonical morphisms of algebra objects given by the counit of the adjunction $\mathcal{L}(\mathcal{K}H) \simeq \mathcal{L} \circ \mathcal{M}(1_{nc}) \to 1_{nc}$.[23] We will write $\mathcal{L}_{\mathcal{K}H}$ for this factorization.

Warning 4.11. *We will not prove here that the commutative algebra structure in $\mathcal{K}H$ obtained from our arguments is the same as the one already appearing in the literature and deduced from different methods (for instance, see [38,69]). However, we believe that the arguments used in [69] also work in the ∞-categorical setting, so that our algebra structure should match the standard one.*

Our representability result has the following corollary showing that under the existence of resolutions of singularities the passage to the noncommutative world produces no loss of information from the K-theoretic viewpoint.

Corollary 4.12. *Let k be a field admitting resolutions of singularities. Then the canonical map*

$$\mathcal{L}_{\mathcal{K}H} : Mod_{\mathcal{K}H}\big(\mathcal{SH}(k)\big) \to \mathcal{SH}_{nc}(k) \tag{4.8}$$

is fully faithful.

Proof. Thanks to the main results of [77] the family of dualizable objects in $\mathcal{SH}(k)$ is a family of ω-compact generators for the stable $(\infty, 1)$-category $\mathcal{SH}(k)$ in the sense of Proposition 1.12. One can now easily check that the collection of all objects in the

[23] Notice that the adjunction $(\mathcal{L}, \mathcal{M})$ extends to an adjunction between the $(\infty, 1)$-categories of commutative algebra-objects, so that this counit map is a morphism of algebras. In particular, we can perform base-change with respect to it.

stable $(\infty, 1)$-category $Mod_{\mathcal{K}H}(\mathcal{SH}(k))$ of the form $X \otimes \mathcal{K}H$ with X dualizable in $\mathcal{SH}(k)$ is again a family of ω-compact generators in the sense of Proposition 1.12. See [79, Prop. 3.8.3]. Since the functor $(- \otimes \mathcal{K}H)$ is monoidal, the objects $X \otimes \mathcal{K}H$ are dualizable in $Mod_{\mathcal{K}H}(\mathcal{SH}(k))$ and as $\mathcal{L}_{\mathcal{K}H}$ is monoidal, their image in $\mathcal{SH}_{nc}(k)$ is dualizable and therefore compact (using the fact the monoidal structure is compatible with colimits in each variable). By Proposition 1.13 we are now reduced to showing that $\mathcal{L}_{\mathcal{K}H}$ is fully faithful when restricted to the full subcategory spanned by all the objects of the form $X \otimes \mathcal{K}H$ with X dualizable in $\mathcal{SH}(k)$. This follows from the canonical chain of equivalences

$$
\begin{aligned}
&Map_{Mod_{\mathcal{K}H}(\mathcal{SH}(k))}(X \otimes \mathcal{K}H, Y \otimes \mathcal{K}H) \\
&\quad \simeq Map_{\mathcal{SH}(k)}(X, Y \otimes \mathcal{K}H) && (4.9)\\
&\quad \simeq Map_{\mathcal{SH}(k)}(X \otimes \check{Y}, \mathcal{K}H) \simeq Map_{\mathcal{SH}_{nc}(k)}\big(\mathcal{L}(X \otimes \check{Y}), 1_{nc}\big) && (4.10)\\
&\quad \simeq Map_{\mathcal{SH}_{nc}(k)}\big(\mathcal{L}(X) \otimes \mathcal{L}(\check{Y}), 1_{nc}\big) \simeq Map_{\mathcal{SH}(k)}\big(\mathcal{L}(X) \otimes \mathcal{L}(\check{Y}), 1_{nc}\big) && (4.11)\\
&\quad \simeq Map_{\mathcal{SH}_{nc}(k)}\big(\mathcal{L}(X)\otimes, \mathcal{L}(Y)\big) && (4.12)
\end{aligned}
$$

where we use the adjunction properties, the fact that $\mathcal{K}H \simeq \mathcal{M}(1_{nc})$ and the fact that $\mathcal{L}$ is monoidal and therefore preserves dualizable objects. This concludes the proof. □

Although this result is new in the literature, its content has been known for a while. I think particularly of B. Toen, M. Vaquié and G. Vezzosi and also of D.-C. Cisinski and G. Tabuada.

4.2. Preliminaries on K-theory

4.2.1. Connective K-theory – a historical overview

K-theory was discovered by A. Grothendieck during his attempts to generalize the classical Riemann–Roch Theorem (see [20,35]). Given an abelian category E he was led to consider an abelian group $K_0(E)$ together with a map $\theta : Obj(E) \to K_0(E)$ universal with respect to the following property: for any exact sequence $a \to b \to c$ in E we have $\theta(b) = \theta(a) + \theta(c)$.

The essential insight leading to the introduction of higher K-theory groups is the observation by Quillen [74] that the group law on $K_0(E)$ can be understood as the π_0-reminiscent part of a grouplike homotopy commutative law on a certain space $K(E)$. Following his ideas, for any "exact category" E we are able to define such a K-theory space $K(E)$ whose homotopy groups $\pi_n(K(E))$ we interpret as level n K-theoretic information. In particular, this methodology allows us to attach a K-theory space to every scheme X using the canonical structure of exact category on $E = Vect(X)$.

An important step in this historical account is a theorem by Segal [85, 3.4] (and its later formulation in terms of model categories in [21]) establishing an equivalence between the homotopy theory of grouplike homotopy commutative algebras in spaces and the homotopy theory of connective spectra. This is the reason why connective spectra is commonly used as the natural target for K-theory and the origin of the term "connective". In the modern days this equivalence can be stated by means of an equivalence of $(\infty,1)$-categories, namely, between the $(\infty,1)$-category $CAlg^{grplike}(\mathcal{S})$ and the $(\infty,1)$-category $\mathcal{S}p_{\geq 0}$ (see [63, Theorem 5.2.6.10 and Remark 5.2.6.26]).

In [108] Waldhausen extends the domain of K-theory from exact categories to what we nowadays call "Waldhausen categories". Grosso modo, these are triples $(\mathcal{C}, W, Cof(\mathcal{C}))$ where $\mathcal{C}$ is a classical category having a zero object and both W and $Cof(\mathcal{C})$ are classes of morphisms in $\mathcal{C}$, respectively called "weak-equivalences" and "cofibrations". These triples are subject to certain conditions which we will not specify here. Waldhausen constructs a K-theory space out of this data using is the algorithm known as the "S-construction" which we review here very briefly:

Construction 4.13 *(S-construction).* Let $Ar[n]$ be the category of arrows in the linear category $[n]$. In more explicit terms it can be described as the category where objects are pairs (i,j) with $i \leq j$ and there is one morphism $(i,j) \to (l,k)$ everytime $i \leq l$ and $j \leq k$. Let now $(\mathcal{C}, W, Cof(\mathcal{C}))$ be a Waldhausen category. We let $S_n(\mathcal{C})$ denote the full subcategory of all functors $Fun(Ar[n], \mathcal{C})$ spanned by those functors A verifying:

1. $A(i,i)$ is a zero object of $\mathcal{C}$ for all $0 \leq i \leq n$;
2. for any i the maps $A(i,j) \to A(i,k)$ with $j \leq k$ are cofibrations in $\mathcal{C}$;
3. for any $i \leq j \leq k$ the induced diagram

$$\begin{tikzcd} A(i,j) \arrow[r] \arrow[d] & A(i,k) \arrow[d] \\ 0 = A(j,j) \arrow[r] & A(j,k) \end{tikzcd} \tag{4.13}$$

is a pushout $\mathcal{C}$.

In other words, the objects in $S_n(\mathcal{C})$ can be identified with sequences of cofibrations of length $n-1$ plus the data of the successive quotients. In particular, $S_0(\mathcal{C})$ is the category with a single object and $S_1(\mathcal{C})$ is equivalent to $\mathcal{C}$. Moreover, the collection of categories $\{S_n(\mathcal{C})\}_{n\in\mathbb{N}}$ assembles to form a simplicial category $S_\bullet(\mathcal{C})$ carrying at each level a canonical structure of Waldhausen category whose weak-equivalences W_n are the levelwise weak-equivalences in $\mathcal{C}$. By definition, the K-theory space of $\mathcal{C}$ is the simplicial set $K^c(\mathcal{C}) := \Omega\, colim_{\Delta^{op}} N(S_n(\mathcal{C})^{W_n})$ where $S_n(\mathcal{C})^{W_n}$ denotes the subcategory of $S_n(\mathcal{C})$ containing all the objects and only those morphisms which are weak-equivalences and N is the standard nerve functor. By iterating this procedure we can produce a connective spectrum. For the complete details see [108].

There is a natural notion of exact functor between Waldhausen categories providing a category $Wald_{Classic}$ and the K-theory assignment can be understood as a functor

$$K^c_{Wald} : N(Wald_{Classic}) \longrightarrow Sp^{\Sigma} \tag{4.14}$$

where Sp^{Σ} is a model category for the $(\infty, 1)$-category Sp.

Many Waldhausen categories used in practice appear as subcategories of a Quillen model category [73] with the cofibrations and weak-equivalences therein. We will denote by $Wald^{Model}_{Classic}$ the full subcategory of $Wald_{Classic}$ spanned by those Waldhausen categories falling into this list of examples. These Waldhausen categories have a special advantage – the factorization axioms for the model category allow us to change Construction 4.13 to consider all morphisms in $\mathcal{C}$, not only the cofibrations.

This first era of connective K-theory finishes with the works of Thomason–Trobaugh in [94] where the machinery of Waldhausen is applied to schemes and it is proven that the connective K-theory of a scheme X introduced by Quillen can be recovered from the K-theory attached to the Waldhausen structure on the category of perfect complexes on the scheme.

The current era begins with the observation that the K-theory of a Waldhausen datum $(\mathcal{C}, W, Cof(\mathcal{C}))$ is not an invariant of the classical 1-categorical localization $\mathcal{C}[W^{-1}]$: there are examples of pairs of Waldhausen categories with the same homotopy categories but with different K-theory spaces (see [81]). The crucial results of Toën–Vezzosi in [101] allow us to identify the world of $(\infty, 1)$-categories as the natural ultimate domain for K-theory. They prove that if the underlying $(\infty, 1)$-categories associated to a pair of Waldhausen categories (via the ∞-localization) are equivalent then the associated K-theory spaces are equivalent. Moreover, in the same paper, the authors remark that the classical S-construction of Waldhausen can be lifted to the setting of $(\infty, 1)$-categories. Following this insight, in [4] C. Barwick introduced the notion of a *Waldhausen* $(\infty, 1)$-*category* (which, grosso modo are pairs of $(\infty, 1)$-categories $(\mathcal{A}_0, \mathcal{A})$ with $\mathcal{A}_0$ a full subcategory of $\mathcal{A}$ containing its maximal ∞-groupoid, together with extra conditions on this pair) and develops this ∞-version of the S-construction. The collection of Waldhausen $(\infty, 1)$-categories forms itself an $(\infty, 1)$-category $\mathcal{W}ald_{\infty}$ and the result of this new ∞-version of the S-construction can be encoded as an ∞-functor $K^c_{Barwick} : \mathcal{W}ald_{\infty} \to Sp_{\geq 0}$. Moreover, there is a canonical ∞-functor linking the classical theory to this new approach

$$N\big(Wald^{Model}_{Classic}\big) \longrightarrow \mathcal{W}ald_{\infty} \tag{4.15}$$

sending a classical Waldhausen datum $(\mathcal{C}, W, Cof(\mathcal{C}))$ to the ∞-localization $N(\mathcal{C})[W^{-1}]$ together with its smallest subcategory containing the equivalences and the images of the cofibrations under the localization functor (see [4, Example 2.12]). Barwick then proves that the two S-constructions, respectively, the classical and the new ∞-version agree by means of this assignment and therefore produce the same K-theory [4, 10.6.2].

Up to our days this framework seems to be the most natural and general domain for connective K-theory. However, we should remark that a different ∞-categorical domain has been established in the paper [13] where the authors study K-theory spaces associated to pointed $(\infty,1)$-categories having all finite colimits, whose collection forms an $(\infty,1)$-category $Cat_\infty(\omega)_*$. They generalize the classical S-construction to this new domain obtaining a new ∞-functor $K^c_{BGT} : Cat_\infty(\omega)_* \to Sp_{\geq 0}$, and prove that for any Waldhausen category $\mathcal{C}$ with equivalences W (appearing as a subcategory of a model category), the K-theory space which their method assigns to the ∞-localization $N(\mathcal{C})[W^{-1}]$ is equivalent to the classical K-theory space attached to $\mathcal{C}$ through the classical methods of Waldhausen. This framework is of course related to the wider framework of [4]: following the example [4, 2.9], every pointed $(\infty,1)$-category with finite colimits has a naturally associated Waldhausen $(\infty,1)$-category. Again, this assignment can be properly understood as an ∞-functor

$$\Psi : Cat_\infty(\omega)_* \longrightarrow \mathcal{W}ald_\infty \tag{4.16}$$

We summarize this fast historical briefing with the existence of a diagram of $(\infty,1)$-categories

$$\begin{tikzcd}
 & Cat_\infty(\omega)_* \arrow[dr, "\Psi"] \arrow[dd, dotted, "K^c_{B.G.T.}"] & \\
N(Wald^{Model}_{Classic}) \arrow[ur] \arrow[rr] \arrow[dr, "K^c_{Wald}"'] & & \mathcal{W}ald_\infty \arrow[dl, "K^c_{Barwick}"] \\
 & Sp_{\geq 0} &
\end{tikzcd} \tag{4.17}$$

whose commutativity follows from the results in [4] and in [13] and from the agreement of the two ∞-categorical versions of the S-construction via Ψ. This agreement follows from the very definition of the two procedures. Consult [4, Section 5] and [13, Section 7.1] for the complete details.

4.2.2. Connective K-theory of dg-categories

Our goal in this section is to explain how to define the connective K-theory of a dg-category and how to present this assignment as an ∞-functor $K^c : \mathcal{D}g(k)^{idem} \to \widehat{Sp}_{\geq 0}$ commuting with filtered colimits. One possible way is to use the classical theory of Waldhausen categories. As discussed in Remark 3.29, the data of an object $F \in Fun_\omega(\mathcal{D}g(k)^{idem}, \widehat{Sp})$ corresponds in an essentially unique way to the data of an actual strict functor F_s from the category of small dg-categories $Cat_{Ch(k)}$ with the Morita model structure to some combinatorial model category whose underlying $(\infty,1)$-category is $\widehat{Sp}$ such that 1) F_s sends Morita equivalences to weak-equivalences and 2) F_s preserves filtered homotopy colimits. In the case of connective K-theory such a functor can be obtained by composing the strict functor $K^c_{Wald}: Wald_{Classic} \to Sp^{\Sigma}$ of the previous section with the functor $Cat_{Ch(k)} \to Wald^{Model}_{Classic}$ defined by sending a small dg-category T to the strict category of perfect cofibrant dg-modules (obtained by forgetting the dg-enrichment), with its natural structure of Waldhausen category given by the weak-equivalences of T-dg-modules and the cofibrations of the module structure therein. This is well-defined because perfect modules are stable under homotopy pushouts and satisfy the "cube lemma" [42, 5.2.6]. The conditions 1) and 2) are also well-known to be satisfied (for instance see [12, Section 2.2]). For the most part of this work it will be enough to work with the ∞-functor $K^c : \mathcal{D}g(k)^{ft} \to \widehat{Sp}_{\geq 0}$ associated to this composition via Remark 3.29 or its canonical ω-continuous extension $\mathcal{D}g(k)^{idem} \to \widehat{Sp}$. However, some of our purposes (namely Theorem 4.7) will require an alternative approach. More precisely, and in the same spirit of [13, Section 7.1] for stable ∞-categories, we will need to have a description of the Waldhausen S-construction within the setting of dg-categories.

Construction 4.14. Let $Ar[n]_k$ be the dg-category obtained as the k-linearization of the category $Ar[n]$ described in Construction 4.13. More precisely, its objects are the objects in $Ar[n]$ and its complexes of morphisms are all given by k seen as a complex concentrated in degree zero. For each n the dg-category $Ar[n]_k$ is locally cofibrant (meaning, enriched over cofibrant complexes) so that for any locally cofibrant dg-category T we have $Ar[n]_k \otimes^{\mathbb{L}} T \simeq Ar[n]_k \otimes T$.

Recall also from [96] that the symmetric monoidal $(\infty, 1)$-category $\mathcal{D}g(k)^{\otimes}$ admits an internal-hom $\mathbb{R}\underline{Hom}(A, B)$ given by the full sub-dg-category of right-quasi-representable cofibrant $A \otimes^{\mathbb{L}} B^{op}$-dg-modules. If T is a locally cofibrant dg-category and $\widehat{T}_c$ is its idempotent-completion (which we can always assume to be locally cofibrant), we find a canonical equivalence in $\mathcal{D}g(k)$ between $\mathbb{R}\underline{Hom}(A, \widehat{T}_c)$ and $\widehat{A^{op} \otimes^{\mathbb{L}} T}_{pspe}$ – the full sub-dg-category of cofibrant pseudo-perfect $A \otimes^{\mathbb{L}} T^{op}$-dg-modules (by definition, these are cofibrant dg-modules E such that for any object $a \in A$, the T^{op}-module $E(A, -)$ is perfect). With this, we have $\mathbb{R}\underline{Hom}(Ar[n]_k, \widehat{T}_c) \simeq \widehat{Ar[n]^{op}_k \otimes^{\mathbb{L}} T}_{pspe} \simeq \widehat{Ar[n]^{op}_k \otimes T}_{pspe}$ so that the objects in this internal-hom can be identified with $Ar[n]$-indexed diagrams in the underlying strict category of perfect cofibrant T^{op}-modules (obtained by forgetting the dg-enrichment). We now set $S^{dg}_n(T)$ to be the full sub-dg-category of $\mathbb{R}\underline{Hom}(Ar[n]_k, \widehat{T}_c)$ spanned by those diagrams satisfying the conditions in Construction 4.13. These conditions make sense for the same reasons the functor $Cat_{Ch(k)} \to Wald^{Model}_{Classic}$ of the previous section also makes sense (see [12, Section 2.2]). Again, the collection of dg-categories $S^{dg}_n(T)$ for $n \geq 0$ forms a simplicial object in dg-categories and by considering each level as a category (omitting its dg-enrichment) we can recover the K-theory of T as $\Omega\, colim_{\Delta^{op}} N(S^{dg}_n(T)^{W_n})$ where W_n is the class of maps in $S^{dg}_n(T)$ given by the levelwise weak-equivalences of dg-modules and $S^{dg}_n(T)^{W_n}$ is the full subcategory of $S^{dg}_n(T)$ spanned by all the objects and only those morphisms which are in W_n.

Let now $[n]_k$ be the dg-category obtained as the k-linearization of the ordered category $[n] = \{0 \leq 1 \leq .... \leq n\}$. This dg-category is again locally cofibrant and for the same reasons as above the underlying category obtained from $\mathbb{R}\underline{Hom}([n]_k, \widehat{T}_c)$ by forgetting the dg-enrichment is the category of sequences of perfect cofibrant T^{op}-dg-modules of length $n+1$. As cofibers of maps are essentially uniquely determined up to isomorphism, we have a canonical equivalence of categories between $S_n^{dg}(T)$ and $\mathbb{R}\underline{Hom}([n-1]_k, \widehat{T}_c)$. Since the model structure on T^{op}-dg-modules satisfies the "cube lemma" [42, 5.2.6] (because $Ch(k)$ satisfies it for the projective model structure) this equivalence becomes an equivalence of pairs $(S_n^{dg}(T), W_n)$ and $(\mathbb{R}\underline{Hom}([n-1]_k, \widehat{T}_c), W'_n)$, where we consider both dg-categories as categories by forgetting the dg-enrichments and where W'_n denotes the class of maps of sequences which are levelwise given by weak-equivalences of dg-modules. Thanks to this equivalence we find a homotopy equivalence of simplicial sets between $N(S_n^{dg}(T)^{W_n})$ and $N(\mathbb{R}\underline{Hom}([n-1]_k, \widehat{T}_c)^{W'_n})$. This is a dg-version of [63, 1.2.2.4]. Finally, and thanks to the main theorem of [96] the latter is exactly the mapping space $Map_{\mathcal{D}g(k)}([n-1]_k, \widehat{T}_c)$ which by adjunction is equivalent to $Map_{\mathcal{D}g(k)^{idem}}(\widehat{([n-1]_k)}_c, \widehat{T}_c)$. Under this chain of equivalences this family of mapping spaces for $n \geq 0$ inherits the structure of a simplicial object in the $(\infty,1)$-category of spaces and the K-theory space of T can finally be rewritten as

$$\Omega\ colim_{[n]\in\Delta^{op}} Map_{\mathcal{D}g(k)^{idem}}\big(\widehat{([n-1]_k)}_c, \widehat{T}_c\big) \tag{4.18}$$

This concludes the construction.

To conclude this section we remark two important properties of K^c. The first should be well known to the reader:

Proposition 4.15. *(See [108].) The ∞-functor $K^c : \mathcal{D}g(k)^{idem} \to \widehat{Sp}$ sends exact sequences of dg-categories to fiber sequences in $\widehat{Sp}_{\geq 0}$.*

Proof. This follows from the so-called Waldhausen's Fibration Theorem [108, 1.6.4] and [94, 1.8.2], together with the dictionary between homotopy limits and homotopy colimits in the model category of spectra and limits and colimits in the $(\infty,1)$-category $\widehat{Sp}$ (see [63, 1.3.4.23 and 1.3.4.24]). $\square$

The second is a consequence of this first and will be very important to us:

Proposition 4.16. *K^c sends Nisnevich squares of noncommutative smooth spaces to pullback squares of connective spectra.*

Proof. Let

$$\begin{tikzcd} T_{\mathcal{X}} \arrow[r] \arrow[d] & T_{\mathcal{U}} \arrow[d] \\ T_{\mathcal{V}} \arrow[r] & T_{\mathcal{W}} \end{tikzcd} \tag{4.19}$$

be a Nisnevich square of dg-categories. By definition, there are dg-categories $K_{\mathcal{X}-\mathcal{U}}$ and $K_{\mathcal{V}-\mathcal{W}}$ in $\mathcal{D}g(k)^{idem}$, having compact generators, and such that the maps $T_{\mathcal{X}} \to T_{\mathcal{U}}$ and $T_{\mathcal{V}} \to T_{\mathcal{W}}$ fit into strict short exact sequences in $\mathcal{D}g(k)^{idem}$ (see Definition 3.11 and Remark 3.9)

$$\begin{tikzcd} K_{\mathcal{X}-\mathcal{U}} \arrow[r] \arrow[d] & T_{\mathcal{X}} \arrow[d] \\ 0 \arrow[r] & T_{\mathcal{U}} \end{tikzcd} \qquad \begin{tikzcd} K_{\mathcal{V}-\mathcal{W}} \arrow[r] \arrow[d] & T_{\mathcal{V}} \arrow[d] \\ 0 \arrow[r] & T_{\mathcal{W}} \end{tikzcd} \tag{4.20}$$

Again by the definition of an open immersion and because of Proposition 4.15 we have pullback squares of connective spectra

$$\begin{tikzcd} K^c(K_{\mathcal{X}-\mathcal{U}}) \arrow[r] \arrow[d] & K^c(T_{\mathcal{X}}) \arrow[d] \\ 0 \arrow[r] & K^c(T_{\mathcal{U}}) \end{tikzcd} \qquad \begin{tikzcd} K^c(K_{\mathcal{V}-\mathcal{W}}) \arrow[r] \arrow[d] & K^c(T_{\mathcal{V}}) \arrow[d] \\ 0 \arrow[r] & K^c(T_{\mathcal{W}}) \end{tikzcd} \tag{4.21}$$

With these properties in mind, we aim to show that the diagram

$$\begin{tikzcd} K^c(T_{\mathcal{X}}) \arrow[r] \arrow[d] & K^c(T_{\mathcal{U}}) \arrow[d] \\ K^c(T_{\mathcal{V}}) \arrow[r] & K^c(T_{\mathcal{W}}) \end{tikzcd} \tag{4.22}$$

is a pullback of connective spectra. For that purpose we consider the pullback squares

$$\begin{tikzcd} K^c(K_{\mathcal{X}-\mathcal{U}}) \arrow[r] \arrow[d] & K^c(K_{\mathcal{V}-\mathcal{W}}) \arrow[r] \arrow[d] & 0 \arrow[d] \\ K^c(T_{\mathcal{X}}) \arrow[r, dashed] & K^c(T_{\mathcal{V}}) \times_{K^c(T_{\mathcal{W}})} K^c(T_{\mathcal{U}}) \arrow[r] \arrow[d] & K^c(T_{\mathcal{U}}) \arrow[d] \\ & K^c(T_{\mathcal{V}}) \arrow[r] & K^c(T_{\mathcal{W}}) \end{tikzcd} \tag{4.23}$$

from which we extract a morphism of fiber sequences

$$\begin{tikzcd} K^c(K_{\mathfrak{X}-\mathfrak{U}}) \arrow[r] \arrow[d] & K^c(K_{\mathcal{V}-\mathcal{W}}) \arrow[d] \\ K^c(T_{\mathfrak{X}}) \arrow[r, dashed] \arrow[d] & K^c(T_{\mathcal{V}}) \times_{K^c(T_{\mathcal{W}})} K^c(T_{\mathfrak{U}}) \arrow[d] \\ K^c(T_{\mathfrak{U}}) \arrow[r, equal] & K^c(T_{\mathfrak{U}}) \end{tikzcd} \tag{4.24}$$

To conclude, since the square (4.19) is Nisnevich, by definition, the canonical morphism $K_{\mathfrak{X}-\mathfrak{U}} \to K_{\mathcal{V}-\mathcal{W}}$ is an equivalence in $\mathcal{D}g(k)^{idem}$ so that the top map is an equivalence $K^c(K_{\mathfrak{X}-\mathfrak{U}}) \simeq K^c(K_{\mathcal{V}-\mathcal{W}})$. Using the associated long exact sequences we conclude that the canonical morphism

$$K^c(T_{\mathfrak{X}}) \dashrightarrow K^c(T_{\mathcal{V}}) \times_{K^c(T_{\mathcal{W}})} K^c(T_{\mathfrak{U}}) \tag{4.25}$$

is also an equivalence, thus concluding the proof. □

4.2.3. Non-connective K-theory

The first attempts to define negative K-theory groups date back to the works of Bass in [5] and Karoubi in [50]. The motivation to look for these groups is very simple: the higher K-theory groups of an exact sequence of Waldhausen categories do not fit in a long exact sequence. The full solution to this problem appeared in the legendary paper of Thomason–Trobaugh [94] where the author provides a mechanism to extend the connective spectrum K^c of Waldhausen to a new non-connective spectrum K^B whose connective part recovers the classical data. His attention focuses on the K-theory of schemes and recovers the negative groups of Bass (by passing to the homotopy groups). Moreover, it satisfies the property people were waiting for [94, Thm. 7.4]: for any reasonable scheme X with an open subscheme $U \subseteq X$ with complement Z, there is a pullback–pushout sequence of spectra $K(X \text{ on } Z) \to K(X) \to K(U)$ where $K(X \text{ on } Z)$ is the K-theory spectrum associated to the category of perfect complexes on X supported on Z. Moreover, he proves that his non-connected version of K-theory satisfies descent with respect to the classical Nisnevich topology for schemes (see [94, Thm. 10.8]).

More recently, Schlichting [82] introduced a mechanism that allows us to define non-connective versions of K-theory in a wide range of situations and in [25, Sections 6 and 7] the authors applied this algorithm to the context of dg-categories. The result is a procedure that sends Morita equivalences of dg-categories to weak-equivalences of spectra and commutes with filtered homotopy colimits (for instance, see [12, 2.12]) and comes canonically equipped with a natural transformation from connective K-theory inducing an equivalence in the connective part. By applying the arguments of Remark 3.29 their construction can be encoded in a unique way in the form of an ω-continuous ∞-functor $K^S : \mathcal{D}g(k)^{idem} \to \widehat{Sp}$ together with a natural transformation $K^c \to K^S$ with $\tau_{\geq 0}K^c \simeq \tau_{\geq 0}K^S$. The *motto* of non-connective K-theory can now be stated as

Proposition 4.17. *K^S sends exact sequences in $\mathcal{D}g(k)^{idem}$ to cofiber/fiber sequences in $\widehat{Sp}$.*

Proof. This follows from [82, 12.1 Thm. 9] and from the adaptation of the Schlichting setup to dg-categories in [25, Section 6], together with the fact that our notion of exact sequences in $\mathcal{D}g(k)^{idem}$ agrees with the notion of exact sequences in [25] (see 3.7-(3)). To conclude use again the dictionary between homotopy limits and homotopy colimits in a model category and limits and colimits on the underlying $(\infty, 1)$-category. □

Using the same arguments as in Proposition 4.16, we find

Corollary 4.18. *K^S is Nisnevich local.*

The method of Thomason (the so-called B-construction) and the methods of Schlichting to create non-connective extensions of K-theory are somehow *ad hoc*. In this paper we show that these two constructions can both be understood as explicit models for the same process, namely, the Nisnevich "sheafification"[24] in the noncommutative world.

4.3. Non-connective K-theory is the Nisnevich localization of connective K-theory

In this section we give the proof of Theorem 4.4. As explained in the introduction, it goes in two steps. First, in 4.3.1, we prove that every Nisnevich local functor $F : \mathcal{D}g(k)^{ft} \to \widehat{Sp}$ satisfies the familiar Bass exact sequences for any integer n. The second step requires a more careful discussion. In 4.3.2 we introduce the notion of *connective-Nisnevich descent* for functors $F : \mathcal{D}g(k)^{ft} \to \widehat{Sp}$ with values in $\widehat{Sp}_{\geq 0}$. We will see (Proposition 4.24 below) that the full subcategory $Fun_{Nis_{\geq 0}}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$ spanned by those functors satisfying this descent condition, is an accessible reflexive localization of $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$

$$\begin{tikzcd} Fun_{Nis_{\geq 0}}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \arrow[r, hook, "\alpha"] & Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \arrow[l, bend right, "l_{nis_{\geq 0}}"'] \end{tikzcd} \tag{4.26}$$

and that as a consequence of the definition the connective truncation of a Nisnevich local is connectively-Nisnevich local, and we have a natural factorization $\overline{\tau_{\geq 0}}$

$$\begin{tikzcd} Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) & Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, "\tau_{\geq 0}"] \\ Fun_{Nis_{\geq 0}}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \arrow[u, hook, "\alpha"] & Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[u, hook, "\beta"'] \arrow[l, dashed, "\overline{\tau_{\geq 0}}"'] \end{tikzcd} \tag{4.27}$$

[24] The noncommutative Nisnevich topology is not a Grothendieck topology.

where α and β denote the inclusions. By abstract nonsense, the composition $i_! := l^{nc}_{nis} \circ i \circ \alpha$ provides a left adjoint to $\overline{\tau_{\geq 0}}$ and, because the diagram of right adjoints commutes, the diagram of left adjoints

$$\begin{tikzcd} Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \arrow[r, hook, "i"] \arrow[d, "l_{nis_{\geq 0}}"] & Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[d, "l^{nc}_{nis}"] \\ Fun_{Nis_{\geq 0}}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \arrow[r, dashed, "i_!"] & Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \end{tikzcd} \tag{4.28}$$

also commutes.

The second step in our strategy amounts to checking that the adjunction $(i_!, \overline{\tau_{\geq 0}})$ is an equivalence of $(\infty, 1)$-categories. At this point our task is greatly simplified by the first step: the fact that Nisnevich local objects satisfy the Bass exact sequences for any integer n implies that $\overline{\tau_{\geq 0}}$ is conservative. Therefore, we are reduced to prove that the counit of the adjunction $\overline{\tau_{\geq 0}} \circ i_! \to Id$ is an equivalence of functors. In other words, if F is already connectively-Nisnevich local, its Nisnevich localization preserves the connective part. In order to achieve this we will need a more explicit description of the noncommutative Nisnevich localization of a connectively-Nisnevich local F. Our main result is that the more familiar $(-)^B$ construction of Thomason–Trobaugh (which we reformulate in our setting) provides such an explicit model, namely, we prove that if F is connectively-Nisnevich local, $\tau_{\geq 0}(F^B)$ is naturally equivalent to F and F^B is Nisnevich local and naturally equivalent to $l^{nc}_{nis}(F)$.

4.3.1. Nisnevich descent forces all the Bass exact sequences

In this section we prove that every Nisnevich local $F : \mathcal{D}g(k)^{ft} \to \widehat{Sp}$ satisfies the familiar Bass exact sequences for any integer n. Our proof follows the arguments of [94, 6.1]. The first step is to show that every Nisnevich local F satisfies the Projective Bundle Theorem. As explained in the introduction, this follows from the existence of an exceptional collection in $L_{pe}(\mathbb{P}^1)$ generated by the twisting sheaves $\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(-1)$, which, following Example 3.17, provides a split short exact sequence of dg-categories

$$\begin{tikzcd} L_{pe}(k) \arrow[r, shift right, "i_{\mathcal{O}_{\mathbb{P}^1}}"'] \arrow[d] & L_{pe}(\mathbb{P}^1) \arrow[l, shift right] \arrow[d, shift right] \\ 0 \arrow[r] & L_{pe}(k) \arrow[u, shift right, "i_{\mathcal{O}_{\mathbb{P}^1(-1)}}"'] \end{tikzcd} \tag{4.29}$$

where the map $i_{\mathcal{O}_{\mathbb{P}^1}}$, resp. $i_{\mathcal{O}_{\mathbb{P}^1(-1)}}$, is the inclusion of the full triangulated subcategory generated by $\mathcal{O}_{\mathbb{P}^1}$, respectively $\mathcal{O}_{\mathbb{P}^1}(-1)$. In particular, since $\mathcal{D}g(k)^{ft}$ has direct sums, we extract canonical maps of dg-categories

$$L_{pe}(k) \oplus L_{pe}(k) \xrightarrow{\psi} L_{pe}(\mathbb{P}^1) \qquad L_{pe}(\mathbb{P}^1) \xrightarrow{\phi} L_{pe}(k) \oplus L_{pe}(k) \tag{4.30}$$

We observe now that these maps become mutually inverse once we consider them in $Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ via Yoneda's embedding. Indeed, the split exact sequence in (4.29), or more precisely, its opposite in $\mathcal{N}cS(k)$, induces a split exact sequence in $Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp})$

$$\begin{tikzcd}
l_{Nis}^{nc} \circ \Sigma_+^\infty \circ j_{nc}(L_{pe}(k)) \arrow[r, shift right] \arrow[d] & l_{Nis}^{nc} \circ \Sigma_+^\infty \circ j_{nc}(L_{pe}(\mathbb{P}^1)) \arrow[l, shift right] \arrow[d, shift right] \\
0 \arrow[r] & l_{Nis}^{nc} \circ \Sigma_+^\infty \circ j_{nc}(L_{pe}(k)) \arrow[u, shift right, bend right]
\end{tikzcd} \tag{4.31}$$

This is because $Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ is stable, together with the effects of the Nisnevich localization. Also because of stability we know that l_{Nis}^{nc} preserves direct sums. In particular, we have canonical maps

$$l_{Nis}^{nc} \circ \Sigma_+^\infty \circ j\big(L_{pe}\big(\mathbb{P}^1\big)\big) \to l_{Nis}^{nc} \circ \Sigma_+^\infty \circ j_{nc}\big(L_{pe}(k)\big) \oplus l_{Nis}^{nc} \circ \Sigma_+^\infty \circ j_{nc}\big(L_{pe}(k)\big) \tag{4.32}$$

$$l_{Nis}^{nc} \circ \Sigma_+^\infty \circ j_{nc}\big(L_{pe}(k)\big) \oplus l_{Nis}^{nc} \circ \Sigma_+^\infty \circ j_{nc}\big(L_{pe}(k)\big) \to l_{Nis}^{nc} \circ \Sigma_+^\infty \circ j\big(L_{pe}\big(\mathbb{P}^1\big)\big) \tag{4.33}$$

which can be identified with the image under $\Sigma_+^\infty \circ j_{nc}$ of the opposites of the canonical maps of dg-categories in (4.30), respectively. This is because in $\mathcal{N}cS(k)$ finite sums are the same as finite products (see the end of our discussion in [79, Section 6.1.2]), because Yoneda's embedding commutes with finite products and because the pointing map $\mathcal{S} \to \mathcal{S}_*$ and the suspension Σ^∞ commute with all colimits.

This time, and as explained in Remarks 3.26 and 3.27, because $Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ is stable, these canonical maps are inverses to each other. In other words, we have a direct sum decomposition

$$\begin{aligned}
l_{Nis}^{nc} \circ \Sigma_+^\infty \circ j\big(L_{pe}\big(\mathbb{P}^1\big)\big) &\simeq l_{Nis}^{nc} \circ \Sigma_+^\infty \circ j_{nc}\big(L_{pe}(k)\big) \oplus l_{Nis}^{nc} \circ \Sigma_+^\infty \circ j_{nc}\big(L_{pe}(k)\big) \\
&\simeq l_{Nis}^{nc} \circ \Sigma_+^\infty \circ j_{nc}\big(L_{pe}(k) \oplus L_{pe}(k)\big)
\end{aligned} \tag{4.34}$$

where the first (resp. second) component can be identified with the part of $L_{pe}(\mathbb{P}^1)$ generated by $\mathcal{O}_{\mathbb{P}^1}$ (resp. $\mathcal{O}_{\mathbb{P}^1}(-1)$).

In particular, if we denote by $\underline{Hom}$ the internal-hom in $Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ we find

Corollary 4.19. *Let F be a Nisnevich local functor $\mathcal{D}g(k)^{ft} \to \widehat{Sp}$. Then F satisfies the projective bundle theorem. In other words, we have*

$$\begin{aligned}
&\underline{Hom}\big(l_{Nis}^{nc} \circ \Sigma_+^\infty \circ j\big(L_{pe}\big(\mathbb{P}^1\big)\big), F\big) \\
&\quad\simeq \underline{Hom}\big(l_{Nis}^{nc} \circ \Sigma_+^\infty \circ j\big(L_{pe}(k)\big), F\big) \oplus \underline{Hom}\big(l_{Nis}^{nc} \circ \Sigma_+^\infty \circ j\big(L_{pe}(k)\big), F\big) \\
&\quad\simeq F \oplus F
\end{aligned}$$

As in [94, 6.1] we can now re-adapt this direct sum decomposition to a new one, suitably choosen to extract the Bass exact sequences out of the classical Zariski (therefore Nisnevich) covering of $\mathbb{P}^1$ given by

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, hook, "i"] \arrow[d, hook, "j"] & \mathbb{A}^1 \arrow[d, hook, "\alpha"] \\ \mathbb{A}^1 \arrow[r, hook, "\beta"] & \mathbb{P}^1 \end{tikzcd} \tag{4.35}$$

The basic ingredient is the induced pullback diagram of dg-categories

$$\begin{tikzcd} L_{pe}(\mathbb{P}^1) \arrow[r, "\alpha^*"] \arrow[d, "\beta^*"] & L_{pe}(\mathbb{A}^1) \arrow[d, "j^*"] \\ L_{pe}(\mathbb{A}^1) \arrow[r, "i^*"] & L_{pe}(\mathbb{G}_m) \end{tikzcd} \tag{4.36}$$

together with the composition

$$\begin{tikzcd} L_{pe}(k) \oplus L_{pe}(k) \arrow[dr, "\psi"] \arrow[drr, bend left, "\alpha^* \circ \psi"] \arrow[ddr, bend right, "\beta^* \circ \psi"'] & & \\ & L_{pe}(\mathbb{P}^1) \arrow[r, "\alpha^*"] \arrow[d, "\beta^*"] & L_{pe}(\mathbb{A}^1) \arrow[d, "j^*"] \\ & L_{pe}(\mathbb{A}^1) \arrow[r, "i^*"] & L_{pe}(\mathbb{G}_m) \end{tikzcd} \tag{4.37}$$

More precisely, we will focus on the diagram in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ induced by the opposite of the above diagram, namely,

$$\begin{tikzcd} \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{G}_m)) \arrow[r, "L_{pe}(i)"] \arrow[d, "L_{pe}(j)"] & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{A}^1)) \arrow[d, "L_{pe}(\alpha)"] \arrow[ddr, bend left] & \\ \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{A}^1)) \arrow[r, "L_{pe}(\beta)"] \arrow[drr, bend right] & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{P}^1)) \arrow[dr, "\Sigma^\infty_+ \circ j_{nc}(\psi^{op})"] & \\ & & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(k) \oplus L_{pe}(k)) \end{tikzcd} \tag{4.38}$$

Remark 4.20. It follows from Proposition 3.21, from the effects of the Nisnevich localization and from the above discussion that the exterior commutative square in (4.38) becomes a pushout–pullback square in $Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp})$.

In order to extract the Bass exact sequences, we consider a different direct sum decomposition of $l^{nc}_{Nis} \circ \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(\mathbb{P}^1))$. For that purpose, let us start by introducing a bit of notation. We let i_1, i_2 denote the canonical inclusions $L_{pe}(k) \to L_{pe}(k) \oplus L_{pe}(k)$ in $\mathcal{D}g(k)^{ft}$, and let π_1, π_2 denote the projections $L_{pe}(k) \oplus L_{pe}(k) \to L_{pe}(k)$. At the same time, let i_1^{op} and i_2^{op} denote the associated projections in $\mathcal{N}cS(k)$ and π_1^{op} and π_2^{op} the canonical inclusions. Since Yoneda's map $\Sigma^{\infty}_{+} \circ j_{nc}$ commutes with direct sums, the maps $\Sigma^{\infty}_{+} \circ j_{nc}(i_1^{op})$ and $\Sigma^{\infty}_{+} \circ j_{nc}(i_1^{op})$ can be identified with the canonical projections

$$\Sigma^{\infty}_{+} \circ j_{nc}\big(L_{pe}(k)\big) \oplus \Sigma^{\infty}_{+} \circ j_{nc}\big(L_{pe}(k)\big) \to \Sigma^{\infty}_{+} \circ j_{nc}\big(L_{pe}(k)\big) \tag{4.39}$$

and $\Sigma^{\infty}_{+} \circ j_{nc}(\pi_1^{op})$ and $\Sigma^{\infty}_{+} \circ j_{nc}(\pi_2^{op})$ with the canonical inclusions

$$\Sigma^{\infty}_{+} \circ j_{nc}\big(L_{pe}(k)\big) \to \Sigma^{\infty}_{+} \circ j_{nc}\big(L_{pe}(k)\big) \oplus \Sigma^{\infty}_{+} \circ j_{nc}\big(L_{pe}(k)\big) \tag{4.40}$$

in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$.

Let us proceed. To achieve the new decomposition, we compose the decomposition we had before with an equivalence Θ in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$

$$\Sigma^{\infty}_{+} \circ j_{nc}\big(L_{pe}(k)\big) \oplus \Sigma^{\infty}_{+} \circ j_{nc}\big(L_{pe}(k)\big) \xrightarrow{\Theta} \Sigma^{\infty}_{+} \circ j_{nc}\big(L_{pe}(k)\big) \oplus \Sigma^{\infty}_{+} \circ j_{nc}\big(L_{pe}(k)\big) \tag{4.41}$$

defined to be the map

$$\begin{tikzcd}
 & \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \\
\Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \oplus \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \arrow[ur, "\delta_1"] \arrow[dr, "\delta_2"'] \arrow[r, dashed, "\Theta"] & \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \oplus \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \arrow[u, "\Sigma^{\infty}_{+} \circ j_{nc}(i_1^{op})"'] \arrow[d, "\Sigma^{\infty}_{+} \circ j_{nc}(i_2^{op})"] \\
 & \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k))
\end{tikzcd} \tag{4.42}$$

obtained from the universal property of the direct sum, where:

- δ_1 it is the canonical dotted arrow in the diagram

$$\begin{tikzcd}
\Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \arrow[d, "\Sigma^{\infty}_{+}\circ j_{nc}(\pi_1^{op})"'] \arrow[dr, "id"] & \\
\Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \oplus \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \arrow[r, dashed] & \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \\
\Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \arrow[u, "\Sigma^{\infty}_{+}\circ j_{nc}(\pi_2^{op})"] \arrow[ur, "0"'] &
\end{tikzcd} \tag{4.43}$$

- δ_2 is the canonical map obtained from

$$\begin{tikzcd}
\Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \arrow[d, "\Sigma^{\infty}_{+}\circ j_{nc}(\pi_1^{op})"'] \arrow[dr, "id"] & \\
\Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \oplus \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \arrow[r, dashed] & \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \\
\Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \arrow[u, "\Sigma^{\infty}_{+}\circ j_{nc}(\pi_2^{op})"] \arrow[ur, "-id"'] &
\end{tikzcd} \tag{4.44}$$

Of course, it follows from this definition that Θ is an equivalence with inverse equal to itself. Finally, we consider the composition

$$\begin{tikzcd}
\Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(\mathbb{G}_m)) \arrow[r, "L_{pe}(i)"] \arrow[d, "L_{pe}(j)"] & \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(\mathbb{A}^1)) \arrow[d, "L_{pe}(\alpha)"] \arrow[ddr, bend left] & \\
\Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(\mathbb{A}^1)) \arrow[r, "L_{pe}(\beta)"] \arrow[drr, bend right] & \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(\mathbb{P}^1)) \arrow[dr, "\Theta\circ\Sigma^{\infty}_{+}\circ j_{nc}(\psi^{op})"] & \\
 & & \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \oplus \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k))
\end{tikzcd} \tag{4.45}$$

which again, as in Remark 4.20, provides a pushout–pullback square in $Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp})$. The important point of this new decomposition is the fact that both maps $\Theta \circ (\Sigma^{\infty}_{+} \circ j_{nc}(\psi^{op} \circ L_{pe}(\alpha)))$ and $\Theta \circ (\Sigma^{\infty}_{+} \circ j_{nc}(\psi^{op} \circ L_{pe}(\beta)))$ become simpler. In fact, since $\alpha^*(\mathcal{O}_{\mathbb{P}^1}) = \alpha^*(\mathcal{O}_{\mathbb{P}^1}(-1)) = \beta^*(\mathcal{O}_{\mathbb{P}^1}) = \alpha^*(\mathcal{O}_{\mathbb{P}^1}(-1)) = \mathcal{O}_{\mathbb{A}^1}$, we find that

- The composition $\Sigma^\infty_+ \circ j_{nc}(i_1^{op}) \circ \Theta \circ (\Sigma^\infty_+ \circ j_{nc}(\psi^{op} \circ L_{pe}(\alpha)))$ can be identified with the map $\Sigma^\infty_+ \circ j_{nc}(L_{pe}(p))$ induced by pullback along the canonical projection $p : \mathbb{A}^1 \to Spec(k)$. Indeed, we have

$$\begin{aligned}
&\Sigma^\infty_+ \circ j_{nc}\big(i_1^{op}\big) \circ \Theta \circ \big(\Sigma^\infty_+ \circ j_{nc}\big(\psi^{op} \circ L_{pe}(\alpha)\big)\big) && (4.46)\\
&\quad \simeq \delta_1 \circ \big(\Sigma^\infty_+ \circ j_{nc}\big(\psi^{op} \circ L_{pe}(\alpha)\big)\big) && (4.47)\\
&\quad \simeq \delta_1 \circ \big(\Sigma^\infty_+ \circ j_{nc}\big(\pi_1^{op} \circ i_1^{op} + \pi_2^{op} \circ i_2^{op}\big)\big) \circ \big(\Sigma^\infty_+ \circ j_{nc}\big(\psi^{op} \circ L_{pe}(\alpha)\big)\big) && (4.48)\\
&\quad \simeq \Sigma^\infty_+ \circ j_{nc}\big(i_1^{op} \circ \psi^{op} \circ L_{pe}(\alpha)\big) + 0 && (4.49)\\
&\quad \simeq \Sigma^\infty_+ \circ j_{nc}\big(\big(\alpha^* \circ \psi \circ i_1\big)^{op}\big) && (4.50)\\
&\quad \simeq \Sigma^\infty_+ \circ j_{nc}\big(\big(p^*\big)^{op}\big) && (4.51)\\
&\quad \simeq \Sigma^\infty_+ \circ j_{nc}\big(L_{pe}(p)\big) && (4.52)
\end{aligned}$$

  The same holds for the composition $\Sigma^\infty_+ \circ j_{nc}(i_1^{op}) \circ \Theta \circ (\Sigma^\infty_+ \circ j_{nc}(\psi^{op} \circ L_{pe}(\beta)))$;
- The maps $\Sigma^\infty_+ \circ j_{nc}(i_2^{op}) \circ \Theta \circ (\Sigma^\infty_+ \circ j_{nc}(\psi^{op} \circ L_{pe}(\alpha)))$ and $\Sigma^\infty_+ \circ j_{nc}(i_2^{op}) \circ \Theta \circ (\Sigma^\infty_+ \circ j_{nc}(\psi^{op} \circ L_{pe}(\beta)))$ are zero. Indeed, we have

$$\begin{aligned}
&\Sigma^\infty_+ \circ j_{nc}\big(i_2^{op}\big) \circ \Theta \circ \big(\Sigma^\infty_+ \circ j_{nc}\big(\psi^{op} \circ L_{pe}(\alpha)\big)\big) && (4.53)\\
&\quad \simeq \delta_2 \circ \big(\Sigma^\infty_+ \circ j_{nc}\big(\psi^{op} \circ L_{pe}(\alpha)\big)\big) && (4.54)\\
&\quad \simeq \delta_2 \circ \big(\Sigma^\infty_+ \circ j_{nc}\big(\pi_1^{op} \circ i_1^{op} + \pi_2^{op} \circ i_2^{op}\big)\big) \circ \big(\Sigma^\infty_+ \circ j_{nc}\big(\psi^{op} \circ L_{pe}(\alpha)\big)\big) && (4.55)\\
&\quad \simeq Id \circ \big(\Sigma^\infty_+ \circ j_{nc}\big(i_1^{op} \circ \psi^{op} \circ L_{pe}(\alpha)\big)\big) + (-Id) \circ \big(\Sigma^\infty_+ \circ j_{nc}\big(i_1^{op} \circ \psi^{op} \circ L_{pe}(\alpha)\big)\big) && (4.56)\\
&\quad \simeq \Sigma^\infty_+ \circ j_{nc}\big(\big(\alpha^* \circ \psi \circ i_1\big)^{op}\big) - \Sigma^\infty_+ \circ j_{nc}\big(\big(\alpha^* \circ \psi \circ i_2\big)^{op}\big) && (4.57)
\end{aligned}$$

  But since $\alpha^*(\mathcal{O}_{\mathbb{P}^1}) = \alpha^*(\mathcal{O}_{\mathbb{P}^1}(-1)) = \mathcal{O}_{\mathbb{A}^1}$, we have $\alpha^* \circ \psi \circ i_1 \simeq \alpha^* \circ \psi \circ i_2$ so that the last difference is zero. The same argument holds for β^*.

From these two facts combined we conclude that $\Theta \circ (\Sigma^\infty_+ \circ j_{nc}(\psi^{op} \circ L_{pe}(\alpha)))$ is equivalent to the sum $\Sigma^\infty_+ \circ j_{nc}(L_{pe}(p)) \oplus 0$ so that the outer commutative square of the diagram (4.45) can now be written as

$$\begin{tikzcd}
\Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{G}_m)) \arrow[r, "L_{pe}(i)"] \arrow[d, "L_{pe}(j)"] & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{A}^1)) \arrow[d, "\Sigma^\infty_+ \circ j_{nc}(L_{pe}(p)) \oplus 0"] \\
\Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{A}^1)) \arrow[r, "\Sigma^\infty_+ \circ j_{nc}(L_{pe}(p)) \oplus 0"] & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(k)) \oplus \Sigma^\infty_+ \circ j_{nc}(L_{pe}(k))
\end{tikzcd} \qquad (4.58)$$

We are almost done. To proceed, we rewrite the diagram (4.45) as

$$\begin{tikzcd}
\Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{G}_m)) \arrow[r] \arrow[d] & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{A}^1)) \oplus \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{A}^1)) \arrow[d, "(\Sigma^\infty_+ \circ j_{nc}(L_{pe}(\alpha)), -\Sigma^\infty_+ \circ j_{nc}(L_{pe}(\beta)))"'] \arrow[ddr, bend left, "(\Sigma^\infty_+ \circ j_{nc}(L_{pe}(p)), -\Sigma^\infty_+ \circ j_{nc}(L_{pe}(p)))\oplus 0"] & \\
0 \arrow[r] \arrow[drr, bend right] & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{P}^1)) \arrow[dr, "\Theta \circ \Sigma^\infty_+ \circ j_{nc}(\psi^{op})"] & \\
 & & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(k) \oplus L_{pe}(k))
\end{tikzcd} \tag{4.59}$$

where of course, since Yoneda's map $\Sigma^\infty_+ \circ j_{nc}$ commutes with direct sums, we have

$$\Sigma^\infty_+ \circ j_{nc}\big(L_{pe}\big(\mathbb{A}^1\big)\big) \oplus \Sigma^\infty_+ \circ j_{nc}\big(L_{pe}\big(\mathbb{A}^1\big)\big) \simeq \Sigma^\infty_+ \circ j_{nc}\big(L_{pe}\big(\mathbb{A}^1\big) \oplus L_{pe}\big(\mathbb{A}^1\big)\big) \tag{4.60}$$

We observe that both the inner and the outer squares become pullback–pushouts once we pass to the Nisnevich localization. Moreover, the map $\Theta \circ (\Sigma^\infty_+ \circ j_{nc}(\psi^{op}))$ becomes an equivalence.

In a different direction, we also observe that the pullback map of dg-categories p^* : $L_{pe}(k) \to L_{pe}(\mathbb{A}^1)$ admits a left inverse $s^* : L_{pe}(\mathbb{A}^1) \to L_{pe}(k)$ given by the pullback along the zero section $s : Spec(k) \to \mathbb{A}^1$.[25] In terms of noncommutative spaces, this can be rephrased by saying that $L_{pe}(p)$ has a right inverse $L_{pe}(s)$. We can use this right-inverse to construct a right inverse to the first projection of $(\Sigma^\infty_+ \circ j_{nc}(L_{pe}(p)), -\Sigma^\infty_+ \circ j_{nc}(L_{pe}(p))) \oplus 0$, namely, we consider the map $(\Sigma^\infty_+ \circ j_{nc}(L_{pe}(s)), 0)$ induced by the universal property of the direct sum in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$

$$\begin{tikzcd}
 & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{A}^1)) \\
\Sigma^\infty_+ \circ j_{nc}(L_{pe}(k)) \arrow[ur, bend left, "\Sigma^\infty_+ \circ j_{nc}(L_{pe}(s))"] \arrow[r, dashed, "(\Sigma^\infty_+ \circ j_{nc}(L_{pe}(s)),0)"] \arrow[dr, bend right, "0"'] & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{A}^1)) \oplus \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{A}^1)) \arrow[u] \arrow[d] \\
 & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{A}^1))
\end{tikzcd} \tag{4.61}$$

It is immediate to check that the composition $\Sigma^\infty_+ \circ j_{nc}(i_1^{op}) \circ ((\Sigma^\infty_+ \circ j_{nc}(L_{pe}(p)), -\Sigma^\infty_+ \circ j_{nc}(L_{pe}(p))) \oplus 0) \circ (\Sigma^\infty_+ \circ j_{nc}(L_{pe}(s)), 0)$ is the identity, so that $\Sigma^\infty_+ \circ j_{nc}(i_1^{op}) \circ ((\Sigma^\infty_+ \circ j_{nc}(L_{pe}(p)), -\Sigma^\infty_+ \circ j_{nc}(L_{pe}(p))) \oplus 0)$ has a right inverse that we can picture as a dotted arrow

[25] Which in terms of rings is given by the evaluation at zero $ev_0 : k[T] \to k$.

$$\begin{tikzcd}
\Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{G}_m)) \arrow[r] \arrow[d] & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{A}^1)) \oplus \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{A}^1)) \arrow[d, "{(L_{pe}(\alpha), -L_{pe}(\beta))}"] \arrow[dd, bend left=60, "{(L_{pe}(p), -L_{pe}(p)) \oplus 0}"] \\
0 \arrow[r] & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{P}^1)) \arrow[d, "\Theta \circ (\Sigma^\infty_+ \circ j_{nc}(\psi^{op}))"] \\
& \Sigma^\infty_+ \circ j_{nc}(L_{pe}(k) \oplus L_{pe}(k)) \arrow[d, "\Sigma^\infty_p \circ j_{nc}(i_1^{op})"] \\
& \Sigma^\infty_+ \circ j_{nc}(L_{pe}(k)) \arrow[uuu, dashed, bend right=90]
\end{tikzcd} \tag{4.62}$$

At the same time, the preceding discussion implies that the second projection

$$\begin{tikzcd}
\Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{G}_m)) \arrow[r] \arrow[d] & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{A}^1)) \oplus \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{A}^1)) \arrow[d, "{(L_{pe}(\alpha), -L_{pe}(\beta))}"] \arrow[ddd, bend left=70, "0"] \\
0 \arrow[r] & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{P}^1)) \arrow[d, "\Theta \circ (\Sigma^\infty_+ \circ j_{nc}(\psi^{op}))"] \\
& \Sigma^\infty_+ \circ j_{nc}(L_{pe}(k) \oplus L_{pe}(k)) \arrow[d, "\Sigma^\infty_+ \circ j_{nc}(i_2^{op})"] \\
& \Sigma^\infty_+ \circ j_{nc}(L_{pe}(k))
\end{tikzcd} \tag{4.63}$$

is just the zero map.

We now explain how to extract the familiar Bass exact sequence from these two diagrams. Given any object $F \in Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ and a noncommutative space $\mathcal{X}$, we set the notation $F_{\mathcal{X}} := \underline{Hom}(\Sigma^\infty_+ \circ j_{nc}(\mathcal{X}), F)$. By enriched Yoneda, this is the functor given by $F(\mathcal{X} \otimes -)$. To proceed, we consider the image of the diagram (4.62) under the functor $\underline{Hom}(-, F)$, to find a diagram

$$\begin{tikzcd}
F_{L_{pe}(k)} \simeq F \arrow[r, "i_1^F"] & F \oplus F \simeq F_{L_{pe}(k) \oplus L_{pe}(k)} \arrow[r] & F_{L_{pe}(\mathbb{P}^1)} \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{A}^1) \oplus L_{pe}(\mathbb{A}^1)} \simeq F_{L_{pe}(\mathbb{A}^1)} \oplus F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \arrow[lll, bend right=20] \\
& & 0 \arrow[r] & F_{L_{pe}(\mathbb{G}_m)}
\end{tikzcd} \tag{4.64}$$

where the first map $i_1^F : F \to F \oplus F$ can be identified with the canonical inclusion in the first coordinate and the composition $F \longrightarrow F_{L_{pe}(\mathbb{A}^1)} \oplus F_{L_{pe}(\mathbb{A}^1)} \dashrightarrow F$ is the identity.

From this we can produce a new commutative diagram by taking successive pushouts

$$\begin{tikzcd}
F \arrow[r, "i_1^F"] \arrow[d] & F \oplus F \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{P}^1)} \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{A}^1)} \oplus F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \arrow[ddr, bend left] & \\
0 \arrow[r] & F \arrow[r] & F \coprod_{F\oplus F} F_{L_{pe}(\mathbb{P}^1)} \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \arrow[dr, dashed] & \\
 & & 0 \arrow[rr] & & F_{L_{pe}(\mathbb{G}_m)}
\end{tikzcd} \tag{4.65}$$

and we notice that the vertical map $F \oplus F \to F$ can be identified with the projection in the second coordinate.

In particular, if we denote as $U(F)$ the pullback

$$\begin{tikzcd}
U(F) \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\
0 \arrow[r] & F_{L_{pe}(\mathbb{G}_m)}
\end{tikzcd} \tag{4.66}$$

we find a canonical map

$$F \coprod_{F\oplus F} F_{L_{pe}(\mathbb{P}^1)} \dashrightarrow U(F) \tag{4.67}$$

induced from the diagram (4.65) using the universal property of the pullback.

At the same time, if we apply $\underline{Hom}(-, F)$ to the diagram (4.63) we find a new commutative diagram

$$\begin{tikzcd}
F \arrow[d, "i_2^F"] \arrow[dd, bend right, "id"'] \arrow[rrr, bend left, "0"] & & & \\
F \oplus F \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{P}^1)} \arrow[rr] \arrow[d] & & F_{L_{pe}(\mathbb{A}^1)} \oplus F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\
F \arrow[r] & F \coprod_{F\oplus F} F_{L_{pe}(\mathbb{P}^1)} \arrow[rr] \arrow[dd] \arrow[dr, dashed] & & F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \arrow[dd] \\
& & U(F) \arrow[ur] \arrow[dl] & \\
& 0 \arrow[rr] & & F_{L_{pe}(\mathbb{G}_m)}
\end{tikzcd} \tag{4.68}$$

and discover that the map $F \to F \coprod_{F\oplus F} F_{L_{pe}(\mathbb{P}^1)} \to U(F)$ admits a natural factorization

$$\begin{tikzcd}
& \Omega F_{L_{pe}(\mathbb{G}_m)} \arrow[r] \arrow[d] & 0 \arrow[d] \\
F \arrow[r] \arrow[ur, dashed, "\sigma_F"] & U(F) \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\
& 0 \arrow[r] & F_{L_{pe}(\mathbb{G}_m)}
\end{tikzcd} \tag{4.69}$$

because $\Omega F_{L_{pe}(\mathbb{G}_m)}$ is the fiber of $U(F) \to F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)}$. This concludes the preliminary steps.

From now on, we suppose that F is Nisnevich local. In this case, by Corollary 4.19, the map $F \oplus F \to F_{L_{pe}(\mathbb{P}^1)}$ is an equivalence and the commutative square

$$\begin{tikzcd}
F_{L_{pe}(\mathbb{P}^1)} \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{A}^1)} \oplus F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\
0 \arrow[r] & F_{L_{pe}(\mathbb{G}_m)}
\end{tikzcd} \tag{4.70}$$

is a pushout–pullback because the image of the square (4.35) under L_{pe} is a Nisnevich square of noncommutative spaces. Using these two facts we conclude that the canonical maps constructed above, $F \to F \coprod_{F\oplus F} F_{L_{pe}(\mathbb{P}^1)}$ and $F \coprod_{F\oplus F} F_{L_{pe}(\mathbb{P}^1)} \to U(F)$ are equivalences so that the diagram

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\
0 \arrow[r] & F_{L_{pe}(\mathbb{G}_m)}
\end{tikzcd} \tag{4.71}$$

is a pullback–pushout. In particular, as in the diagram (4.69), we find the existence of a section

$$\begin{tikzcd}
& \Omega F_{L_{pe}(\mathbb{G}_m)} \arrow[r] \arrow[d] & 0 \arrow[d] \\
F \arrow[ur, dashed, "\sigma_F"] \arrow[r, "Id"] & F \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\
& 0 \arrow[r] & F_{L_{pe}(\mathbb{G}_m)}
\end{tikzcd} \tag{4.72}$$

We are almost done. To conclude, we consider the induced pullback–pushout square

$$\begin{tikzcd}
F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \arrow[r] \arrow[d] & 0 \arrow[d] \\
F_{L_{pe}(\mathbb{G}_m)} \arrow[r] & \Sigma F
\end{tikzcd} \tag{4.73}$$

where now, the suspension $\Sigma(\sigma_F)$ makes $\Sigma(F)$ a retract of $F_{L_{pe}(\mathbb{G}_m)}$. We are done now. Since the evaluation maps commute with colimits and, by definition of $F_{(-)}$, we have for each $T_{\mathcal{X}} \in \mathcal{D}g(k)^{ft}$ a pullback–pushout diagram in $\widehat{Sp}$

$$\begin{tikzcd}
F(L_{pe}(\mathbb{A}^1) \otimes T_{\mathcal{X}}) \coprod_{F(T_{\mathcal{X}})} F(L_{pe}(\mathbb{A}^1) \otimes T_{\mathcal{X}}) \arrow[r] \arrow[d] & 0 \arrow[d] \\
F(L_{pe}(\mathbb{G}_m) \otimes T_{\mathcal{X}}) \arrow[r] & \Sigma F(T_{\mathcal{X}})
\end{tikzcd} \tag{4.74}$$

and therefore a long exact sequence of abelian groups

$$\begin{aligned}
\ldots \to \pi_n\Big(F\big(L_{pe}(\mathbb{A}^1) \otimes T_{\mathcal{X}}\big) \coprod_{F(T_{\mathcal{X}})} F\big(L_{pe}(\mathbb{A}^1) \otimes T_{\mathcal{X}}\big)\Big) \to \pi_n\big(F\big(L_{pe}(\mathbb{G}_m) \otimes T_{\mathcal{X}}\big)\big) \\
\to \pi_n\big(\Sigma F(T_{\mathcal{X}})\big) = \pi_{n-1}\big(F(T_{\mathcal{X}})\big) \to \ldots
\end{aligned} \tag{4.75}$$

and because of the existence of $\Sigma(\sigma_F)$, the maps $\pi_n(F(L_{pe}(\mathbb{G}_m) \otimes T_{\mathcal{X}})) \to \pi_n(\Sigma F(T_{\mathcal{X}})) = \pi_{n-1}(F(T_{\mathcal{X}}))$ are necessarily surjective, so that the long exact sequence breaks up into short exact sequences

$$0 \to \pi_n\left(F\left(L_{pe}\left(\mathbb{A}^1\right) \otimes T_{\mathfrak{X}}\right) \coprod_{F(T_{\mathfrak{X}})} F\left(L_{pe}\left(\mathbb{A}^1\right) \otimes T_{\mathfrak{X}}\right)\right) \to \pi_n\left(F\left(L_{pe}(\mathbb{G}_m) \otimes T_{\mathfrak{X}}\right)\right)$$
$$\to \pi_n\left(\Sigma F(T_{\mathfrak{X}})\right) = \pi_{n-1}\left(F(T_{\mathfrak{X}})\right) \to 0 \tag{4.76}$$

$\forall n \in \mathbb{Z}$.

At the same time, since the square

$$\begin{tikzcd} F \arrow[r, "i_1^F"] \arrow[d] & F \oplus F \arrow[r] & F_{L_{pe}(\mathbb{A}^1)} \oplus F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\ 0 \arrow[rr] & & F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \end{tikzcd} \tag{4.77}$$

is also a pullback–pushout and the top map $F \to F_{L_{pe}(\mathbb{A}^1)} \oplus F_{L_{pe}(\mathbb{A}^1)}$ admits a left inverse, the associated long exact sequence

$$\ldots \to \pi_n\left(F\left(L_{pe}\left(\mathbb{A}^1\right) \otimes T_{\mathfrak{X}}\right) \coprod_{F(T_{\mathfrak{X}})} F\left(L_{pe}\left(\mathbb{A}^1\right) \otimes T_{\mathfrak{X}}\right)\right) \to \pi_n\left(F(T_{\mathfrak{X}})\right)$$
$$\to \pi_n\left(F\left(L_{pe}\left(\mathbb{A}^1\right) \otimes T_{\mathfrak{X}}\right) \oplus F\left(L_{pe}\left(\mathbb{A}^1\right) \otimes T_{\mathfrak{X}}\right)\right) \to \ldots \tag{4.78}$$

breaks up into short exact sequences

$$0 \to \pi_n\left(F(T_{\mathfrak{X}})\right) \to \pi_n\left(F\left(L_{pe}\left(\mathbb{A}^1\right) \otimes T_{\mathfrak{X}}\right) \oplus F\left(L_{pe}\left(\mathbb{A}^1\right) \otimes T_{\mathfrak{X}}\right)\right)$$
$$\to \pi_n\left(F\left(L_{pe}\left(\mathbb{A}^1\right) \otimes T_{\mathfrak{X}}\right) \coprod_{F(\mathfrak{X})} F\left(L_{pe}\left(\mathbb{A}^1\right) \otimes T_{\mathfrak{X}}\right)\right) \to 0 \tag{4.79}$$

Combining the two short exact sequences (4.76) and (4.79) we find the familiar exact sequences of Bass–Thomason–Trobaugh

$$0 \to \pi_n\left(F(T_{\mathfrak{X}})\right) \to \pi_n\left(F\left(L_{pe}\left(\mathbb{A}^1\right) \otimes T_{\mathfrak{X}}\right) \oplus F\left(L_{pe}\left(\mathbb{A}^1\right) \otimes T_{\mathfrak{X}}\right)\right) \tag{4.80}$$
$$\to \pi_n\left(F\left(L_{pe}(\mathbb{G}_m) \otimes T_{\mathfrak{X}}\right)\right) \to \pi_{n-1}\left(F(T_{\mathfrak{X}})\right) \to 0 \tag{4.81}$$

This concludes this section.

4.3.2. Nisnevich vs connective-Nisnevich descent and the Thomason–Trobaugh $(-)^B$-construction

In this section we study the class of functors sharing the same formal properties of K^c, namely, the one of sending Nisnevich squares to pullback squares of connective spectra. This will take us through a small digression aiming to understand how the truncation functor $\tau_{\geq 0}$ interacts with the Nisnevich localization.

Definition 4.21. Let $F \in \mathit{Fun}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$. We say that F is *connectively-Nisnevich local* if for any Nisnevich square of dg-categories

$$\begin{tikzcd} T_{\mathcal{X}} \arrow[r] \arrow[d] & T_{\mathcal{U}} \arrow[d] \\ T_{\mathcal{V}} \arrow[r] & T_{\mathcal{W}} \end{tikzcd} \tag{4.82}$$

the induced square

$$\begin{tikzcd} F(T_{\mathcal{X}}) \arrow[r] \arrow[d] & F(T_{\mathcal{U}}) \arrow[d] \\ F(T_{\mathcal{V}}) \arrow[r] & F(T_{\mathcal{W}}) \end{tikzcd} \tag{4.83}$$

is a pullback of connective spectra.

Remark 4.22. It follows that if F belongs to $\mathit{Fun}_{Nis}(Dg^{ft}, \widehat{Sp})$, its connective truncation $\tau_{\geq 0}(F)$ is connectively-Nisnevich local. This is because $\tau_{\geq 0}$ acts objectwise and is a right adjoint to the inclusion of connective spectra into all spectra, thus preserving pullbacks.

It is also convenient to isolate the following small technical remark:

Remark 4.23. Let $\mathcal{C}$ be a stable $(\infty, 1)$-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a subcategory such that the inclusion preserves direct sums. Then, if

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[d, "p"] \\ 0 \arrow[r] & C \end{tikzcd} \tag{4.84}$$

is a pullback square in $\mathcal{C}_0$ such that

- the map i admits a left inverse v;
- the map p admits a right inverse u;
- the sum $i \circ v + u \circ p$ is homotopic to the identity,

we conclude, by the same arguments given in Remark 3.26, that $B \simeq A \oplus C$. Moreover, under the hypothesis that the inclusion preserves direct sums, the square remains a pullback after the inclusion $\mathcal{C}_0 \subseteq \mathcal{C}$ and therefore a pushout. In particular, it becomes a split exact sequence in $\mathcal{C}$. This holds for any universe.

In particular, for any pullback square of dg-categories associated to a Nisnevich square of noncommutative spaces (4.82) such that $T_{\mathcal{U}}$ is zero and the sequence splits, the induced diagram of connective spectra (4.83) makes $F(T_{\mathcal{V}})$ canonically equivalent to the direct sum $F(T_{\mathcal{X}}) \oplus F(T_{\mathcal{W}})$ in $\widehat{Sp}$.

We let $Fun_{Nis_{\geq 0}}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$ denote the full subcategory of $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$ spanned by the connectively-Nisnevich local functors. For technical reasons it is convenient to observe that the inclusion $Fun_{Nis_{\geq 0}}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \subseteq Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$ admits a left adjoint $l_{nis_{\geq 0}}$. More precisely

Proposition 4.24. *$Fun_{Nis_{\geq 0}}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$ is an accessible reflexive localization of $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$.*

Proof. We evoke Proposition 5.5.4.15 of [59] so that we are reduced to showing the existence of a small class of maps S in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$ such that an object F is connectively-Nisnevich local if and only if it is local with respect to the maps in S.

To define S, we ask the reader to bring back to recollect our discussion and notations in 2.4.4 and in Remark 3.28. Using the same notations, we define S to be the collection of all maps

$$\delta_{\Sigma^{\infty}_{+}\circ j_{nc}(\mathcal{U})}(K) \coprod_{\delta_{\Sigma^{\infty}_{+}\circ j_{nc}(\mathcal{W})}(K)} \delta_{\Sigma^{\infty}_{+}\circ j_{nc}(\mathcal{V})}(K) \to \delta_{\Sigma^{\infty}_{+}\circ j_{nc}(\mathcal{X})}(K) \tag{4.85}$$

given by the universal property of the pushout, this time with K in $\widehat{Sp}_{\geq 0} \cap (\widehat{Sp})^{\omega}$[26] and $\mathcal{W}, \mathcal{V}, \mathcal{U}$ and $\mathcal{X}$ part of a Nisnevich square of noncommutative smooth spaces. As before, the fact that S satisfies the required property follows directly from the definition of the functors $\delta_{\Sigma^{\infty}_{+}\circ j_{nc}(-)}$ as left adjoints to Map^{Sp} and from the enriched version of Yoneda's lemma. □

It follows directly from the definition of the class S in the previous proof and from the description of the class of maps that generate the Nisnevich localization in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ (see Remark 3.28) that the inclusion

$$i : Fun\big(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}\big) \hookrightarrow Fun\big(\mathcal{D}g(k)^{ft}, \widehat{Sp}\big) \tag{4.86}$$

sends connective-Nisnevich local equivalences to Nisnevich local equivalences. In particular, the universal property of the localization provides us with a canonical colimit preserving map

[26] Here $(\widehat{Sp})^{\omega}$ denotes the full subcategory of $\widehat{Sp}$ spanned by the compact objects. Recall that $\widehat{Sp} \simeq Ind((\widehat{Sp})^{\omega})$.

$$\begin{tikzcd} Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \arrow[r, hook, "i"] \arrow[d, "l_{nis_{\geq 0}}"] & Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[d, "l^{nc}_{nis}"] \\ Fun_{Nis_{\geq 0}}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \arrow[r, dashed] & Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \end{tikzcd} \tag{4.87}$$

rendering the diagram commutative. Moreover, since the localizations are presentable, the Adjoint Functor Theorem implies the existence of a right adjoint which makes the associated diagram of right adjoints

$$\begin{tikzcd} Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) & Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, "\tau_{\geq 0}"'] \\ Fun_{Nis_{\geq 0}}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \arrow[u, hook, "\alpha"] & Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, dashed] \arrow[u, hook, "\beta"'] \end{tikzcd} \tag{4.88}$$

commute. At the same time, Remark 4.22 implies the existence of the two commutative diagrams (4.27) and (4.28). By comparison with the new diagrams, we find that the canonical colimit preserving map $Fun_{Nis_{\geq 0}}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \dashrightarrow Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ can be identified with the composition $i_! := l^{nc}_{Nis} \circ i \circ \alpha$ and that its right adjoint can be identified with $\overline{\tau_{\geq 0}}$, the restriction of the truncation functor $\tau_{\geq 0}$ to the Nisnevich local functors.

Our goal is to prove that this adjunction

$$\begin{tikzcd} Fun_{Nis_{\geq 0}}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \arrow[r, shift left, "i_!"] & Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, bend left, "\overline{\tau_{\geq 0}}"] \end{tikzcd} \tag{4.89}$$

is an equivalence. Our results from 4.3.1 already provide one step towards this:

Proposition 4.25. *The functor $\overline{\tau_{\geq 0}}$ is conservative.*

Proof. Recall from 4.3.1 that for any Nisnevich local F we can construct a pullback–pushout square

$$\begin{tikzcd} F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \arrow[r] \arrow[d] & 0 \arrow[d] \\ F_{L_{pe}(\mathbb{G}_m)} \arrow[r] & \Sigma F \end{tikzcd} \tag{4.90}$$

such that for any $T_{\mathcal{X}} \in \mathcal{D}g(k)^{ft}$, the associated long exact sequence of homotopy groups breaks up into short exact sequences for any $n \in \mathbb{N}$

$$0 \to \pi_n\Big(F\big(L_{pe}(\mathbb{A}^1)\otimes T_{\mathcal{X}}\big) \coprod_{F(T_{\mathcal{X}})} F\big(L_{pe}(\mathbb{A}^1)\otimes T_{\mathcal{X}}\big)\Big) \to \pi_n\big(F\big(L_{pe}(\mathbb{G}_m)\otimes T_{\mathcal{X}}\big)\big)$$
$$\to \pi_{n-1}\big(F(T_{\mathcal{X}})\big) \to 0 \tag{4.91}$$

Therefore, given a morphism $f : F \to G$ in $Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp})$, we have an induced diagram

$$\begin{tikzcd}
 & G_{L_{pe}(\mathbb{A}^1)} \coprod_G G_{L_{pe}(\mathbb{A}^1)} \arrow[rr] \arrow[dd] & & 0 \arrow[dd] \\
F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \arrow[ur] \arrow[rr] \arrow[dd] & & 0 \arrow[ur] \arrow[dd] & \\
 & G_{L_{pe}(\mathbb{G}_m)} \arrow[rr] & & \Sigma G \\
F_{L_{pe}(\mathbb{G}_m)} \arrow[ur] \arrow[rr] & & \Sigma F \arrow[ur] &
\end{tikzcd} \tag{4.92}$$

which induces natural maps of short exact sequences

$$\begin{tikzcd}
\pi_n(F(L_{pe}(\mathbb{A}^1)\otimes T_{\mathcal{X}}) \coprod_{F(T_{\mathcal{X}})} F(L_{pe}(\mathbb{A}^1)\otimes T_{\mathcal{X}})) \arrow[r] \arrow[d] & \pi_n(F(L_{pe}(\mathbb{G}_m)\otimes T_{\mathcal{X}})) \arrow[r] \arrow[d] & \pi_{n-1}(F(T_{\mathcal{X}})) \arrow[d] \\
\pi_n(G(L_{pe}(\mathbb{A}^1)\otimes T_{\mathcal{X}}) \coprod_{G(T_{\mathcal{X}})} G(L_{pe}(\mathbb{A}^1)\otimes T_{\mathcal{X}})) \arrow[r] & \pi_n(G(L_{pe}(\mathbb{G}_m)\otimes T_{\mathcal{X}})) \arrow[r] & \pi_{n-1}(G(T_{\mathcal{X}}))
\end{tikzcd} \tag{4.93}$$

In particular, if f is an equivalence in the connective part, by induction on $n = 0, -1, -2, ...$, we conclude that f is an equivalence. □

With this result, in order to prove that $i_!$ is an equivalence we are reduced to showing that the counit of the adjunction $\overline{\tau_{\geq 0}} \circ i_! \to Id$ is a natural equivalence of functors. Notice that since α and i are fully-faithful, this amounts to show that for any F connectively-Nisnevich local, the canonical map $i \circ \tau_{\geq 0} \circ l^{nc}_{Nis} \circ i \circ \alpha(F) \to i \circ \alpha(F)$ is an equivalence. Of course, to achieve this we will need a more explicit description of the noncommutative Nisnevich localization functor l^{nc}_{Nis} restricted to connectively-Nisnevich local objects. There is a naive candidate, namely, the familiar $(-)^B$ construction of Thomason–Trobaugh [94, 6.4]. Our goal to the end of this section is to prove the following proposition confirming that this guess is correct:

Proposition 4.26. *There is an accessible localization functor* $(-)^B : Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \to Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ *encoding the* B*-construction of [94, 6.4] such that for any* $F \in Fun_{Nis_{\geq 0}}(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ *we have:*

- $\overline{\tau_{\geq 0}}(i \circ \alpha(F)^B) \simeq F$*.*
- *the object* $(i \circ \alpha(F))^B$ *is Nisnevich local;*
- *there is a canonical equivalence* $(i \circ \alpha(F))^B \simeq l^{nc}_{Nis}(i \circ \alpha(F))$*;*

In particular, the natural transformation $\overline{\tau_{\geq 0}} \circ i_! \to Id$ *is an equivalence. Together with Proposition 4.25 we have an equivalence of* $(\infty, 1)$*-categories between the theory of connectively-Nisnevich local functors and the theory of Nisnevich local functors.*

With these results available we can already uncover the proof of our first main theorem:

Proof of Theorem 4.4. Thanks to Corollary 4.18 we already know that K^S is Nisnevich local. In this case, and by the universal property of the localization, the canonical map $K^c \to K^S$ admits a canonical uniquely determined factorization

$$\begin{tikzcd} K^c \arrow[d] \arrow[dr] & \\ l^{nc}_{Nis}(K^c) \arrow[r, dashed] & K^S \end{tikzcd} \tag{4.94}$$

so that we are reduced to showing that this canonical morphism $l^{nc}_{Nis}(K^c) \to K^S$ is an equivalence. But since these are Nisnevich local objects and since we now know by Proposition 4.26 that the truncation functor $\tau_{\geq 0}$ is an equivalence when restricted to Nisnevich locals, it suffices to check that the induced map $\tau_{\geq 0} l^{nc}_{Nis}(K^c) \to \tau_{\geq 0} K^S$ is an equivalence. But this follows because all the morphisms in the image of the commutative diagram (4.94) become equivalences after applying $\tau_{\geq 0}$. This follows from the construction of K^S and again by the results in Proposition 4.26. □

We now start our small journey towards the proof of Proposition 4.26. To start with we need to specify how the B-construction of [94, 6.4] can be formulated in our setting:

Construction 4.27 *(Thomason–Trobaugh* $(-)^B$*-construction).* We begin by asking the reader to recall the diagrams constructed in 4.3.1, or more precisely, that for any $F \in Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$, we found a commutative diagram

$$\begin{tikzcd} & \Omega F_{L_{pe}(\mathbb{G}_m)} \arrow[r] \arrow[d] & 0 \arrow[d] \\ F \arrow[ur, dashed, "\sigma_F"] \arrow[r, "\alpha_F"] & U(F) \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\ & 0 \arrow[r] & F_{L_{pe}(\mathbb{G}_m)} \end{tikzcd} \tag{4.95}$$

where both squares are pushout–pullbacks. Iterating this construction, we find a sequence of canonical maps

$$F \xrightarrow{\alpha_F} U(F) \xrightarrow{\alpha_{U(F)}} U\big(U(F)\big) \xrightarrow{\alpha_{U^2(F)}} \dots. \tag{4.96}$$

and we define F^B to be the colimit for sequence (which is of course unique up to canonical equivalence). The assignment $F \mapsto F^B$ provides an endofunctor $(-)^B$ of the $(\infty, 1)$-category $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$. To see this we can use the fact the monoidal structure in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ admits internal-homs $\underline{Hom}$. More precisely, we consider the diagram of natural transformations induced by the image of the diagram (4.62) under the first entry of $Hom(-,-)$. With this, and keeping the notations we have been using, we define f_1 to be the functor cofiber of $Id = (-)_{L_{pe}(k)} \to (-)_{L_{pe}(\mathbb{A}^1)} \oplus (-)_{L_{pe}(\mathbb{A}^1)}$. The universal property of the cofiber gives us a canonical natural transformation $f_1 \to (-)_{L_{pe}(\mathbb{G}_m)}$ and define a new functor U as the fiber of this map (recall that colimits and limits in the category of functors are determined objectwise). Finally, we consider $(-)^B$ as the colimit of the natural transformations

$$\begin{tikzcd} Id \arrow[r, "\alpha"] \arrow[dr] & U = Id \circ U \arrow[r] \arrow[d] & U^2 = Id \circ U^2 \arrow[r] \arrow[dl] & \dots \\ & (-)^B & & \end{tikzcd} \tag{4.97}$$

We prove that for any F the object F^B satisfies the exact sequences of Bass–Thomason–Trobaugh for any $n \in \mathbb{Z}$. The proof requires some technical steps which we summarize in the following lemma:

Lemma 4.28. *We have:*

1. *The functor U commutes with small colimits;*
2. *The two maps $U = Id \circ U \to U^2$ and $U = U \circ Id \to U^2$ induced by the natural transformation $Id \to U$, are homotopic;*
3. *The natural transformation $(-)^B \circ Id \to (-)^B \circ U$ is an equivalence;*
4. *The natural map $(-)^B \circ U \to U \circ (-)^B$ is an equivalence.*

Proof. These are routine exercises. See [79, Props. 7.2.10, 7.2.11, 7.2.12, 7.2.13]. □

We can now put these results together and show that

Corollary 4.29. *For any object $F \in Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ the object F^B satisfies the Bass–Thomason–Trobaugh exact sequences for any $n \in \mathbb{Z}$.*

Proof. By combining Lemma 4.28-(3) and (4), we deduce that the canonical map $F^B \to U(F^B)$ is an equivalence. Therefore, we have a pullback–pushout square

$$\begin{tikzcd}
(F^B)_{L_{pe}(\mathbb{A}^1)} \coprod_{F^B} (F^B)_{L_{pe}(\mathbb{A}^1)} \arrow[r] \arrow[d] & 0 \arrow[d] \\
(F^B)_{L_{pe}(\mathbb{G}_m)} \arrow[r] & \Sigma F^B
\end{tikzcd} \tag{4.98}$$

and using exactly the same arguments as in 4.3.1 we find that for any $T_{\mathfrak{X}} \in \mathcal{D}g(k)^{ft}$, the associated long exact sequence breaks up into short exact sequences

$$\begin{aligned}
0 &\to \pi_n\left(F^B\big(L_{pe}(\mathbb{A}^1) \otimes T_{\mathfrak{X}}\big) \coprod_{F^B(T_{\mathfrak{X}})} F^B\big(L_{pe}(\mathbb{A}^1) \otimes T_{\mathfrak{X}}\big)\right) \to \pi_n\big(F^B\big(L_{pe}(\mathbb{G}_m) \otimes T_{\mathfrak{X}}\big)\big) \\
&\to \pi_{n-1}\big(F^B(T_{\mathfrak{X}})\big) \to 0
\end{aligned} \tag{4.99}$$

and again by the same arguments we are able to extract the familiar exact sequences of Bass–Thomason–Trobaugh, for all $n \in \mathbb{Z}$. □

Remark 4.30. As the canonical map $F^B \to U(F^B)$ is an equivalence it follows from Construction 4.27 that when we construct the diagram (4.95) with F^B

$$\begin{tikzcd}
 & \Omega(F^B_{L_{pe}(\mathbb{G}_m)}) \arrow[r] \arrow[d] & 0 \arrow[d] \\
F^B \arrow[r] \arrow[ur, dashed, "\sigma_{F^B}"] & U(F^B) \arrow[r] \arrow[d] & (F^B)_{L_{pe}(\mathbb{A}^1)} \coprod_{F^B} (F^B)_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\
 & 0 \arrow[r] & (F^B)_{L_{pe}(\mathbb{G}_m)}
\end{tikzcd} \tag{4.100}$$

the section σ_{F^B} makes F^B a retract of $\Omega(F^B_{L_{pe}(\mathbb{G}_m)})$. In particular, by iteratively applying the construction $\Omega(-)_{L_{pe}(\mathbb{G}_m)}$ we find (because $\Sigma^\infty_+ \circ j_{nc}$ is monoidal) that for any $n \geq 1$, the composition

$$F^B \to \Omega\big(F^B_{L_{pe}(\mathbb{G}_m)}\big) \to \ldots \to \Omega^n\big(F^B_{L_{pe}(\mathbb{G}_m)^{\otimes n}}\big) \to \ldots \to \Omega\big(F^B_{L_{pe}(\mathbb{G}_m)}\big) \to F^B \tag{4.101}$$

is the identity map so that, for any $n \geq 1$, F^B is a retract of $\Omega^n(F^B_{L_{pe}(\mathbb{G}_m)^{\otimes n}})$. Equivalently, for any $n \geq 1$, the suspension $\Sigma^n F^B$ is a retract of $(F^B)_{L_{pe}(\mathbb{G}_m)^{\otimes n}}$.

We will now show that the construction $(-)^B$ defines a localization:

Proposition 4.31. *The functor $(-)^B : Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \to Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ of Construction 4.27 is an accessible localization functor.*

This result follows from Lemma 4.28-(1) and (2) together with the following general result:

Lemma 4.32. *Let $\mathcal{C}$ be a presentable $(\infty, 1)$-category and let $U : \mathcal{C} \to \mathcal{C}$ be a colimit preserving endofunctor of $\mathcal{C}$, together with a natural transformation $f : Id_{\mathcal{C}} \to U$ such that the two obvious maps $U \circ Id_{\mathcal{C}} \to U^2$ and $Id_{\mathcal{C}} \circ U \to U^2$ are equivalent. Let*

$$\begin{tikzcd} Id \arrow[r, "f"] \arrow[drr, "i_0"'] & U = Id_{\mathcal{C}} \circ U \arrow[r] \arrow[dr, "i_1"] & U^2 = Id_{\mathcal{C}} \circ U^2 \arrow[r] \arrow[d, "i_2"] & \ldots. \\ & & T & \end{tikzcd} \tag{4.102}$$

be a colimit cone for the horizontal sequence (indexed by $\mathbb{N}$), necessarily in $Fun^L(\mathcal{C}, \mathcal{C})$. Then, the functor $T : \mathcal{C} \to \mathcal{C}$ provides a reflexive localization of $\mathcal{C}$. Moreover, since T commutes with small colimits the localization is accessible.

Proof. We will omit the proof here. The reader can find it in [79, Lemma 7.2.17]. □

Remark 4.33. It follows from Proposition 4.31 and from Construction 4.27 that an object $F \in Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ is local with respect to the localization $(-)^B$ if and only if the diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\ 0 \arrow[r] & F_{L_{pe}(\mathbb{G}_m)} \end{tikzcd} \tag{4.103}$$

is a pullback–pushout square. In particular, the discussion in 4.3.1 implies that any Nisnevich local object F is $(-)^B$-local.

We now come to a series of technical steps in order prove each of the items in Proposition 4.26. First thing, we give a precise sense to what it means for a functor F with connective values to satisfy all the Bass exact sequences for $n \geq 1$.

Definition 4.34. Let $F \in Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$ and consider its associated diagram (4.95) (constructed in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$, where we identify F with its inclusion). We say that F satisfies all Bass exact sequences for $n \geq 1$ if the canonical induced map of connective functors $F \to \tau_{\geq 0} U(F)$ is an equivalence, or, in other words, since $\tau_{\geq 0}$ commutes with limits and because of the definition of $U(F)$, if the diagram (4.103) is a pullback in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$.

Remark 4.35. Let $F \in Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$ and consider the pullback–pushout diagram in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$

$$\begin{tikzcd}
\Omega(F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)}) \arrow[r] \arrow[d] & \Omega(F_{L_{pe}(\mathbb{G}_m)}) \arrow[d] \\
0 \arrow[r] & U(F)
\end{tikzcd} \tag{4.104}$$

Since, $\tau_{\geq 0}$ preserves pullbacks, we obtain a pullback diagram in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$

$$\begin{tikzcd}
\tau_{\geq 0}\Omega(F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)}) \arrow[r] \arrow[d] & \tau_{\geq 0}\Omega(F_{L_{pe}(\mathbb{G}_m)}) \arrow[d] \\
0 \arrow[r] & \tau_{\geq 0}U(F)
\end{tikzcd} \tag{4.105}$$

If F satisfies the condition in the previous definition, then the zero truncation of the composition

$$F \overset{\sigma_F}{\dashrightarrow} \Omega(F_{L_{pe}(\mathbb{G}_m)}) \longrightarrow U(F) \tag{4.106}$$

makes F a retract of $\tau_{\geq 0}\Omega(F_{L_{pe}(\mathbb{G}_m)})$. With this, and as before, once evaluated at $T_{\mathcal{X}} \in \mathcal{D}g(k)^{ft}$, the long exact sequence associated to the pullback (4.105) splits up into short exact sequences

$$\begin{aligned}
0 \to \pi_n \Big(F\big(L_{pe}(\mathbb{A}^1) \otimes T_{\mathcal{X}}\big) \coprod_{F(T_{\mathcal{X}})} F\big(L_{pe}(\mathbb{A}^1) \otimes T_{\mathcal{X}}\big) \Big) \\
\to \pi_n\big(F\big(L_{pe}(\mathbb{G}_m) \otimes T_{\mathcal{X}}\big)\big) \to \pi_{n-1}\big(F(T_{\mathcal{X}})\big) \to 0
\end{aligned} \tag{4.107}$$

$\forall n \geq 1$, and again by the same arguments, we can extract the exact sequences of Bass–Thomason.

Lemma 4.36. *If $F \in Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ has connective values and satisfies all the Bass exact sequences for $n \geq 1$ (in the sense of Definition 4.34), then the canonical map $F \simeq \tau_{\geq 0}F \to \tau_{\geq 0}F^B$ is an equivalence.*

Proof. Assuming that F satisfies the condition in Definition 4.34, meaning the canonical map $F \to \tau_{\geq 0}U(F)$ is an equivalence, we will show that for any $k \geq 2$, the canonical map $F \to \tau_{\geq 0}U^k(F)$ is an equivalence. Once we have this, the conclusion of the lemma will follow from the fact $\tau_{\geq 0}$ commutes with filtered colimits (because the t-structure in $\widehat{Sp}$ is determined by the stable homotopy groups and these commute with filtered colimits), so that

$$\tau_{\geq 0}\big(F^B\big) \simeq \tau_{\geq 0}\big(colim_{i\in\mathbb{N}}U^i(F)\big) \simeq colim_{i\in\mathbb{N}}\tau_{\geq 0}\big(U^i(F)\big) \simeq colim_{i\in\mathbb{N}}F \simeq F \quad (4.108)$$

So, let us prove the assertion for $k = 2$. By definition, we have a pullback–pushout square in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$

$$\begin{tikzcd} U^2(F) \arrow[r] \arrow[d] & U(F)_{L_{pe}(\mathbb{A}^1)} \coprod_{U(F)} U(F)_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\ 0 \arrow[r] & U(F)_{L_{pe}(\mathbb{G}_m)} \end{tikzcd} \quad (4.109)$$

and since $\tau_{\geq 0}$ preserves pullbacks, we find

$$\tau_{\geq 0}U^2(F) \simeq \tau_{\geq 0}\left(U(F)_{L_{pe}(\mathbb{A}^1)} \coprod_{U(F)} U(F)_{L_{pe}(\mathbb{A}^1)}\right) \times_{\tau_{\geq 0}(U(F)_{L_{pe}(\mathbb{G}_m)})} 0 \qquad (4.110)$$

We observe that

(i) $\tau_{\geq 0}(U(F)_{L_{pe}(\mathbb{G}_m)}) \simeq F_{L_{pe}(\mathbb{G}_m)}$.
(ii) $\tau_{\geq 0}(U(F)_{L_{pe}(\mathbb{A}^1)} \coprod_{U(F)} U(F)_{L_{pe}(\mathbb{A}^1)}) \simeq F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)}$

To deduce the first equivalence, we use the equivalence $\tau_{\geq 0}U(F) \simeq F$ together with the fact that $(-)_{L_{pe}(\mathbb{G}_m)}$ commutes with $\tau_{\geq 0}$. The second equivalence requires a more sophisticated discussion. Recall from Section 4.3.1 that for any $G \in Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ we are able to construct a pushout square in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$

$$\begin{tikzcd} G \arrow[r, "i_1^G"] \arrow[d] & G \oplus G \arrow[r] & G_{L_{pe}(\mathbb{P}^1)} \arrow[r] & G_{L_{pe}(\mathbb{A}^1)} \oplus G_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\ 0 \arrow[rrr] & & & G_{L_{pe}(\mathbb{A}^1)} \coprod_G G_{L_{pe}(\mathbb{A}^1)} \end{tikzcd} \quad (4.111)$$

such that the top horizontal composition admits a left inverse. Applying this construction to $G = F$ and to $G = U(F)$, we construct a map between the associated pullback–pushout squares

$$\begin{tikzcd}[row sep=small, column sep=small]
 & U(F) \arrow[rr] \arrow[dd] & & U(F)_{L_{pe}(\mathbb{A}^1)} \oplus U(F)_{L_{pe}(\mathbb{A}^1)} \arrow[dd] \\
F \arrow[ur] \arrow[rr] \arrow[dd] & & F_{L_{pe}(\mathbb{A}^1)} \oplus F_{L_{pe}(\mathbb{A}^1)} \arrow[ur] \arrow[dd] & \\
 & 0 \arrow[rr] & & U(F)_{L_{pe}(\mathbb{A}^1)} \coprod_{U(F)} U(F)_{L_{pe}(\mathbb{A}^1)} \\
0 \arrow[ur] \arrow[rr] & & F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \arrow[ur] &
\end{tikzcd} \tag{4.112}$$

(obtained using the natural transformation $\underline{Hom}(-,F) \to \underline{Hom}(-,U(F))$ induced by canonical morphism $F \to U(F)$).

Both the front and back faces are pullback–pushouts and both the top horizontal maps admit left-inverses.

Finally, since $\tau_{\geq 0}U(F) \simeq F$ and because the top horizontal maps admit left-inverses, the long exact sequences associated to each square breaks up into short exact sequences, and for each $n \geq 0$ and each $T_{\mathcal{X}} \in \mathcal{D}g(k)^{ft}$ we find natural maps of short exact sequences

$$\begin{tikzcd}
\pi_n(U(F)(T_{\mathcal{X}})) \arrow[r] & \pi_n((U(F)_{L_{pe}(\mathbb{A}^1)} \oplus U(F)_{L_{pe}(\mathbb{A}^1)})(T_{\mathcal{X}})) \arrow[r] & \pi_n((U(F)_{L_{pe}(\mathbb{A}^1)} \coprod_{U(F)} U(F)_{L_{pe}(\mathbb{A}^1)})(T_{\mathcal{X}})) \\
\pi_n(F(T_{\mathcal{X}})) \arrow[r] \arrow[u, "\sim"] & \pi_n((F_{L_{pe}(\mathbb{A}^1)} \oplus F_{L_{pe}(\mathbb{A}^1)})(T_{\mathcal{X}})) \arrow[r] \arrow[u, "\sim"] & \pi_n((F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)})(T_{\mathcal{X}})) \arrow[u]
\end{tikzcd} \tag{4.113}$$

implying the equivalence in (ii).

Finally, we deal with the case $k > 2$. Applying the same strategy for $G = F$ and $G = U^k(F)$, we consider the analogue of the diagram (4.112) induced by the canonical morphism $F \to U^k(F)$. By induction, we deduce that $\tau_{\geq 0}U^{k+1}(F) \simeq F$. This concludes the proof. □

Proposition 4.37. *Let F be a connectively-Nisnevich local object. Then, it satisfies the Projective Bundle Theorem and all the Bass exact sequences for $n \geq 1$. In particular, by Lemma 4.36 we have $F \simeq \tau_{\geq 0}F \simeq \tau_{\geq 0}F^B$.*

Proof. To start with, we prove that if is F connectively-Nisnevich local then it satisfies the Projective Bundle Theorem. Indeed, we can use the arguments used in 4.3.1 together with the definition of being connectively-Nisnevich local to construct a pullback diagram in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$

$$\begin{tikzcd} F \simeq F_{L_{pe}(k)} \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{P}^1)} \arrow[d] \\ 0 \arrow[r] & F \simeq F_{L_{pe}(k)} \end{tikzcd} \tag{4.114}$$

with splittings, which, as explained in Remark 4.23, provide a canonical equivalence $F_{L_{pe}(\mathbb{P}^1)} \simeq F \oplus F$ in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$. Secondly, and again by the definition of connectively-Nisnevich local, we can easily deduce that the canonical diagram

$$\begin{tikzcd} F_{L_{pe}(\mathbb{P}^1)} \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{A}^1)} \oplus F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\ 0 \arrow[r] & F_{L_{pe}(\mathbb{G}_m)} \end{tikzcd} \tag{4.115}$$

associated to the covering of $\mathbb{P}^1$ by two affine lines (4.35) is a pullback in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$.

With these two ingredients we prove that if is F connectively-Nisnevich local then it satisfies all the Bass exact sequences for $n \geq 1$ in the sense of Definition 4.34, namely, we show that the canonical map $F \simeq \tau_{\geq 0}F \to \tau_{\geq 0}U(F)$ is an equivalence, or, in other words, that the diagram (4.103) is a pullback within connective functors.

Consider the pushout squares in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ described in (4.65). More precisely, since F satisfies the Projective Bundle Theorem, we are interested in the pullback–pushout square

$$\begin{tikzcd} F \oplus F \simeq F_{L_{pe}(\mathbb{P}^1)} \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{A}^1)} \oplus F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\ F \simeq F \coprod_{F \oplus F} F_{L_{pe}(\mathbb{P}^1)} \arrow[r] & F_{L_{pe}(\mathbb{A}^1)} \coprod_{F} F_{L_{pe}(\mathbb{A}^1)} \end{tikzcd} \tag{4.116}$$

which, in particular, is a pullback square in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$ once truncated at level zero. Combining with the pullback square (4.115) we find a series of pullback squares in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$.

$$\begin{tikzcd}
\Omega F \arrow[r] \arrow[d] & 0 \arrow[d] & \\
\Omega(F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)}) \arrow[r] \arrow[dd] & \Omega(F_{L_{pe}(\mathbb{G}_m)}) \arrow[r] \arrow[d] & 0 \arrow[d] \\
& F_{L_{pe}(\mathbb{P}^1)} \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{A}^1)} \oplus F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\
0 \arrow[r] & F \arrow[r] & F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)}
\end{tikzcd} \tag{4.117}$$

Now comes the important ingredient: since the diagram (4.115) is a pullback, we can still deduce (as before) the existence of a canonical map σ_F such that the composition

$$F \overset{\sigma_F}{\dashrightarrow} \Omega F_{L_{pe}(\mathbb{G}_m)} \longrightarrow F_{L_{pe}(\mathbb{P}^1)} \longrightarrow F \tag{4.118}$$

is the identity. We now explain how the existence of this section allows us to prove that the diagram (4.103) is a pullback. More precisely, by using σ_F at each copy of F in (4.103) and applying the construction $\Omega(-)_{L_{pe}(\mathbb{G}_m)}$ we find the square (4.103) as a retract of the square

$$\begin{tikzcd}
\Omega F_{L_{pe}(\mathbb{G}_m)} \arrow[r] \arrow[d] & (\Omega F_{L_{pe}(\mathbb{G}_m)})_{L_{pe}(\mathbb{A}^1)} \coprod_{\Omega(F)_{L_{pe}(\mathbb{G}_m)}} (\Omega F_{L_{pe}(\mathbb{G}_m)})_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\
0 \arrow[r] & \Omega(F_{L_{pe}(\mathbb{G}_m)})_{L_{pe}(\mathbb{G}_m)}
\end{tikzcd} \tag{4.119}$$

but since both Ω and $\underline{Hom}(\Sigma_+^\infty \circ j_{nc}(L_{pe}(\mathbb{G}_m)), -)$ commute with colimits, we can easily identify this last square with the image of the top left pullback square in (4.117) under $\underline{Hom}(\Sigma_+^\infty \circ j_{nc}(L_{pe}(\mathbb{G}_m)), -)$ and conclude that this is also a pullback square. We conclude the proof using the fact that the retract of a pullback square is a pullback. $\square$

We now address the second item of Proposition 4.26, namely,

Proposition 4.38. *Let $F \in Fun_{Nis_{\geq 0}}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$. Then, the object $(i \circ \alpha(F))^B$ is Nisnevich local.*

The proof of this proposition is based on a very helpful criterium to decide if a given F is Nisnevich local by studying its truncations $\tau_{\geq 0}\Sigma^n F$, namely:

Lemma 4.39. *Let F be any object in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$. Then, if for any $n \geq 0$ the truncations $\tau_{\geq 0}\Sigma^n F$ are connectively-Nisnevich local, the object F itself is Nisnevich local.*

This lemma follows from a somewhat more general situation, which we isolate in the following remark:

Remark 4.40. Let $\mathcal{C}$ be a stable $(\infty, 1)$-category with a right-complete t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ and let $\tau_{\geq n}$ and $\tau_{\leq n}$ denote the associated truncation functors (see [63, Section 1.2.1]). We observe that a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} \tag{4.120}$$

in $\mathcal{C}$ is a pullback (therefore pushout) if and only if for any $n \geq 0$ the truncated squares

$$\begin{tikzcd} \tau_{\geq 0}\Sigma^n A \arrow[r] \arrow[d] & \tau_{\geq 0}\Sigma^n B \arrow[d] \\ \tau_{\geq 0}\Sigma^n C \arrow[r] & \tau_{\geq 0}\Sigma^n D \end{tikzcd} \tag{4.121}$$

are pullbacks in $\mathcal{C}_{\geq n}$. This is a simple exercise. See [79, Remark 7.5.25] for the details.

Proof of Lemma 4.39. Just apply Remark 4.40 to the commutative squares of spectra

$$\begin{tikzcd} F(T_{\mathcal{X}}) \arrow[r] \arrow[d] & F(T_{\mathcal{U}}) \arrow[d] \\ F(T_{\mathcal{V}}) \arrow[r] & F(T_{\mathcal{W}}) \end{tikzcd} \tag{4.122}$$

induced by the Nisnevich squares of noncommutative spaces. The discussion therein works because the t-structure in $\widehat{Sp}$ is known to be right-complete (see [63, 1.4.3.6]). □

Proof of Proposition 4.38. As explained in Remark 4.30, for any $n \geq 1$, the suspension $\Sigma^n F^B$ is a retract of $(F^B)_{L_{pe}(\mathbb{G}_m)^{\otimes n}}$. In particular, $\tau_{\geq 0}\Sigma^n F^B$ is a retract of $\tau_{\geq 0}((F^B)_{L_{pe}(\mathbb{G}_m)^{\otimes n}})$ which is a mere notation for $\tau_{\geq 0}\underline{Hom}(\Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{G}_m)^{\otimes n}), F^B)$ so that

$$\begin{aligned} \tau_{\geq 0}\big((F^B)_{L_{pe}(\mathbb{G}_m)^{\otimes n}}\big) &\simeq \underline{Hom}\big(\Sigma^\infty_+ \circ j_{nc}\big(L_{pe}(\mathbb{G}_m)^{\otimes n}\big), \tau_{\geq 0}F^B\big) \\ &\simeq \underline{Hom}\big(\Sigma^\infty_+ \circ j_{nc}\big(L_{pe}(\mathbb{G}_m)^{\otimes n}\big), F\big) \end{aligned} \tag{4.123}$$

where the first equivalence follows because the t-structure in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ is determined objectwise by the t-structure in Sp and the second follows from Proposition 4.37. In particular, since F is connectively-Nisnevich local, $\underline{Hom}(\Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{G}_m)^{\otimes n}), F)$ is also connectively-Nisnevich local so that $\tau_{\geq 0}\Sigma^n F^B$ is the retract of a connectively-Nisnevich local and therefore, it is itself local.[27] We conclude using Lemma 4.39, observing that for $n = 0$ the condition follows by the hypothesis that F is connectively-Nisnevich local. □

Finally,

Corollary 4.41. *Let F be any object in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$. Then, there is a canonical equivalence*

$$\big(i \circ \alpha(F)\big)^B \simeq l^{nc}_{Nis}\big(i \circ \alpha(F)\big) \tag{4.124}$$

Proof. This follows from Proposition 4.31, Remark 4.33 and Proposition 4.38, using the universal properties of the two localizations. □

Proof of Proposition 4.26. The three items correspond, respectively to Propositions 4.37, 4.38 and to Corollary 4.41. The conclusion now follows from the universal property of the two localizations. □

4.4. Comparing the commutative and the noncommutative $\mathbb{A}^1$-localizations

In this section we prove Theorem 4.6. We start by asking the reader to recall the diagrams (4.1) and (4.2) and to recall that after Theorem 4.4, together with Yoneda's lemma, $\mathcal{M}_2(l^{nc}_{Nis}(K^c))$ is the Bass–Thomason–Trobaugh K-theory of schemes. Recall also that, by definition,[28] Weibel's homotopy invariant K-theory of [109] is the "commutative" localization $l_{\mathbb{A}^1}(\mathcal{M}_2(l^{nc}_{Nis}(K^c)))$. With these ingredients the conclusion of Theorem 4.6 will follow if we prove that the commutative and noncommutative versions of the $\mathbb{A}^1$-localizations make the diagram

$$\begin{tikzcd} Fun_{Nis}(N(AffSm^{ft}(k))^{op}, \widehat{Sp}) \arrow[d, "l_{\mathbb{A}^1}"] & Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, "\mathcal{M}_2"'] \arrow[d, "l^{nc}_{\mathbb{A}^1}"] \\ Fun_{Nis,\mathbb{A}^1}(N(AffSm^{ft}(k))^{op}, \widehat{Sp}) & Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, "\mathcal{M}_3"'] \end{tikzcd} \tag{4.125}$$

commute. In fact, we will be able to prove something slightly more general. We begin by recalling a well-known explicit formula for the $\mathbb{A}^1$-localization of presheaves of spectra. Let $\Delta_{\mathbb{A}^1}$ be the cosimplicial affine scheme given by

$$\Delta^n_{\mathbb{A}^1} := Spec\big(k[t_0, ..., t_n]/(t_0 + ... + t_n - 1)\big) \tag{4.126}$$

[27] In general, the retract of a local object in a reflexive localization is local. This is, ultimately, because the retract of an equivalence is an equivalence.

[28] Either we take it as a definition or as a consequence of the explicit formula given in this section.

Notice that at each level we have (non-canonical) isomorphisms $\Delta^n_{\mathbb{A}^1} \simeq (\mathbb{A}^1_k)^n$. After [22], the endofunctor of $\mathcal{C} = \mathit{Fun}(N(\mathit{AffSm}^{ft}(k))^{op}, \widehat{Sp})$ defined by the formula

$$F \mapsto \mathit{colim}_{n\in\Delta^{op}} \underline{\mathit{Hom}}\big(\Delta^n_{\mathbb{A}^1}, F\big) \tag{4.127}$$

with $\underline{\mathit{Hom}}$ the internal-hom for presheaves of spectra, is an explicit model for the $\mathbb{A}^1$-localization in the commutative world. To see that this indeed gives something $\mathbb{A}^1$-local we use the $\mathbb{A}^1$-homotopy $m : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ between the identity of $\mathbb{A}^1$ and the constant map at zero. The map m is given by the usual multiplication. It follows from this explicit description that the $\mathbb{A}^1$-localization preserves Nisnevich local objects (this is because in a stable context, sifted colimits commute with pullbacks and the Nisnevich local condition is determined by certain squares being pullbacks).

The important point now is that this mechanism applies mutatis mutandis in the noncommutative world. Indeed, by taking the composition

$$\Delta^{nc}_{\mathbb{A}^1} : \Delta \xrightarrow{\Delta_{\mathbb{A}^1}} N\big(\mathit{AffSm}^{ft}(k)\big) \xrightarrow{L_{pe}} \mathcal{N}cS(k) \tag{4.128}$$

we obtain a cosimplicial noncommutative space and as L_{pe} is monoidal we get $(\Delta^{nc,n}_{\mathbb{A}^1}) \simeq L_{pe}(\mathbb{A}^1)^{\otimes_n}$. Moreover, we can use exactly the same arguments to prove that the endofunctor of $\mathcal{C} = \mathit{Fun}(\mathcal{N}cS(k)^{op}, \widehat{Sp})$ defined by the formula

$$F \mapsto \mathit{colim}_{n\in\Delta^{op}} \underline{\mathit{Hom}}\big(\Delta^{nc,n}_{\mathbb{A}^1}, F\big) \tag{4.129}$$

is an explicit model for the noncommutative $\mathbb{A}^1$-localization functor on spectral presheaves and also by the same arguments, we conclude that Nisnevich local objects are preserved under this localization.

With this we can now reduce the proof that the diagram 4.125 commutes to the proof that the following diagram commutes

$$\begin{array}{ccc} \mathit{Fun}(N(\mathit{AffSm}^{ft}(k))^{op}, \widehat{Sp}) & \xleftarrow{\mathcal{M}_1} & \mathit{Fun}(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \\ \downarrow{\scriptstyle l_{\mathbb{A}^1}} & & \downarrow{\scriptstyle l^{nc}_{\mathbb{A}^1}} \\ \mathit{Fun}_{\mathbb{A}^1}(N(\mathit{AffSm}^{ft}(k))^{op}, \widehat{Sp}) & \xleftarrow{\mathcal{M}'} & \mathit{Fun}_{L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \end{array} \tag{4.130}$$

where the lower part corresponds to the reflexive $\mathbb{A}^1$-localizations and $\mathcal{M}'$ is the right adjoint of this context obtained by the same formal arguments as $\mathcal{M}_2$ and $\mathcal{M}_3$. The commutativity of this diagram is measured by the existence of a canonical natural transformation of functors $l_{\mathbb{A}^1} \circ \mathcal{M}_1 \to \mathcal{M}' \circ l^{nc}_{\mathbb{A}^1}$ induced by the fact that $\mathcal{M}'$ sends $L_{pe}(\mathbb{A}^1)$-local objects to $\mathbb{A}^1$-local objects, together with the universal property of $l_{\mathbb{A}^1}$. The diagram commutes if and only if this natural transformation is an equivalence of functors. In particular, since the diagram of right adjoints commutes

$$\begin{tikzcd} Fun(N(AffSm^{ft}(k))^{op}, \widehat{Sp}) & Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, "\mathcal{M}_1"'] \\ Fun_{\mathbb{A}^1}(N(AffSm^{ft}(k))^{op}, \widehat{Sp}) \arrow[u, hook, "\alpha"] & Fun_{L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, "\mathcal{M}'"'] \arrow[u, hook, "\beta"'] \end{tikzcd} \tag{4.131}$$

and the vertical maps are fully-faithful, it will be enough to show that the induced natural transformation $\alpha \circ l_{\mathbb{A}^1} \circ \mathcal{M}_1 \to \alpha \circ \mathcal{M}' \circ l^{nc}_{\mathbb{A}^1}$ is an equivalence. But now, using our explicit descriptions for the $\mathbb{A}^1$-localization functors we know that for each $F \in Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ we have

$$\alpha \circ l_{\mathbb{A}^1}\big(\mathcal{M}_1(F)\big) \simeq colim_{n\in\Delta^{op}}\, \underline{Hom}\big(\Sigma^\infty_+ \circ j\big(\mathbb{A}^1\big)^{\otimes^n}, \mathcal{M}_1(F)\big) \tag{4.132}$$

$$\simeq colim_{n\in\Delta^{op}}\, \mathcal{M}_1\big(\underline{Hom}\big(\Sigma^\infty_+ \circ j_{nc}\big(L_{pe}\big(\mathbb{A}^1\big)\big)^{\otimes^n}, F\big)\big) \tag{4.133}$$

$$\simeq \mathcal{M}_1\big(colim_{n\in\Delta^{op}}\, \underline{Hom}\big(\Sigma^\infty_+ \circ j_{nc}\big(L_{pe}\big(\mathbb{A}^1\big)\big)^{\otimes^n}, F\big)\big) \tag{4.134}$$

$$\simeq \mathcal{M}_1\big(\beta \circ l^{nc}_{\mathbb{A}^1}(F)\big) \simeq \alpha \circ \mathcal{M}' \circ l^{nc}_{\mathbb{A}^1}(F) \tag{4.135}$$

where the first and penultimate equivalences follow from the explicit formulas for the $\mathbb{A}^1$-localizations, the middle equivalences follow, respectively, from Remarks 4.2 and 4.1 and the last equivalence follows from the commutativity of the diagram (4.131).

In particular, when applied to $F = l_{Nis}(K^c)$ we conclude the proof of Theorem 4.6.

4.5. The $\mathbb{A}^1$-localization of non-connective K-theory is the unit non-commutative motive

In this section we prove Theorem 4.7. We start by gathering some necessary preliminary remarks. To start with, and as explained in Remark 3.28 we have two different equivalent ways to construct $\mathcal{SH}_{nc}(k)$: one by using presheaves of spaces, forcing Nisnevich descent and $\mathbb{A}^1$-invariance and a second one by using presheaves of spectra and forcing again the Nisnevich and $\mathbb{A}^1$-localizations. These two approaches are related by means of a commutative diagram of monoidal functors

$$
\begin{tikzcd}
 & \mathcal{N}cS(k) \arrow[dl] \arrow[dr] & \\
Fun(\mathcal{D}g(k)^{ft}, \widehat{\mathcal{S}}) \arrow[rr, "\Sigma^{\infty}_{+}"] \arrow[d, "l^{nc}_{0,Nis}"] & & Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[d, "l^{nc}_{Nis}"] \\
Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{\mathcal{S}}) \arrow[rr, "\Sigma^{\infty}_{+,Nis}"] \arrow[d, "l^{nc}_{0,\mathbb{A}^1}"] & & Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[d, "l^{nc}_{\mathbb{A}^1}"] \\
\mathcal{SH}_{nc}(k) := Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \widehat{\mathcal{S}}) \arrow[rr, "\Sigma^{\infty}_{+,Nis,\mathbb{A}^1}", "\sim"'] & & Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \widehat{Sp})
\end{tikzcd}
\tag{4.136}
$$

induced by the universal properties involved and the last induced $\Sigma^{\infty}_{+,Nis,\mathbb{A}^1}$ is an equivalence because of the results in Proposition 3.24. To be completely precise we have to check that the class of maps with respect to which we localize the theory of presheaves of spaces is sent to the class of maps with respect to which we localize spectral presheaves. Following the description of the last given in Remark 3.28 it is enough to see that for any representable object $j(\mathcal{X})$ we have $\Sigma^{\infty}_{+} j(\mathcal{X}) \simeq \delta_{j(\mathcal{X})}(S)$ where the S is the sphere spectrum. This is because $Map^{Sp}(-)$ is an internal-hom in $\widehat{Sp}$ and the sphere spectrum is a unit for the monoidal structure.

In this section we will be considering the associated commutative diagram of right adjoints

$$
\begin{tikzcd}
Fun(\mathcal{D}g(k)^{ft}, \widehat{\mathcal{S}}) & Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, "\Omega^{\infty}"'] \\
Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{\mathcal{S}}) \arrow[u, hook] & Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, "\Omega^{\infty}_{Nis}"'] \arrow[u, hook] \\
Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \widehat{\mathcal{S}}) \arrow[u, hook] & Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, "\sim", "\Omega^{\infty}_{Nis,\mathbb{A}^1}"'] \arrow[u, hook]
\end{tikzcd}
\tag{4.137}
$$

where again the last map is an equivalence. We will now explain how to use this diagram to reduce the proof that $l^{nc}_{\mathbb{A}^1}(K^S)$ is unit for the monoidal structure in $Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ to the proof that $l^{nc}_{0,\mathbb{A}^1}(\Omega^{\infty}(K^c))$ is a unit for the monoidal structure in $Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \widehat{\mathcal{S}})$. This will require some preliminaries. First we recall that thanks to Proposition 4.26 we have an equivalence

$$
Fun_{Nis\geq 0}\big(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}\big) \xleftarrow[\tau_{\geq 0}]{\sim} Fun_{Nis}\big(\mathcal{D}g(k)^{ft}, \widehat{Sp}\big)
\tag{4.138}
$$

This equivalence provides a compatibility for the $\mathbb{A}^1$-localizations, in the sense that the diagram

$$\begin{tikzcd} Fun_{Nis\geq 0}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \arrow[d, "l^{nc}_{\geq 0,\mathbb{A}^1}"] & Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, "\sim", "\tau_{\geq 0}"'] \arrow[d, "l^{nc}_{\mathbb{A}^1}"] \\ Fun_{Nis\geq 0, L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) & Fun_{Nis, L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, "\sim", "\tau_{\geq 0}"'] \end{tikzcd} \quad (4.139)$$

commutes. Here $l^{nc}_{\geq 0,\mathbb{A}^1}$ is the (noncommutative) $\mathbb{A}^1$-localization functor for connectively-Nisnevich local presheaves.

The second preliminary result is a consequence of the equivalence between $\widehat{Sp}_{\geq 0}$ and the $(\infty,1)$-category of grouplike commutative algebra objects $CAlg^{grplike}(\widehat{\mathcal{S}})$ (see [63, 5.1.3.17]) and the equivalence of this last one with $Fun^{Segal\text{-}grplike}(N(Fin_*), \widehat{\mathcal{S}})$ – the full subcategory of the $(\infty,1)$-category $Fun(N(Fin_*), \widehat{\mathcal{S}})$ spanned by those functors satisfying the standard Segal condition and which are grouplike (see [63, 2.4.2.5]). See the final discussion in this section where this notion is discussed.

We can easily check that this equivalence induces equivalences

$$Fun^{Segal\text{-}grplike}\big(N(Fin_*), Fun_{Nis}\big(\mathcal{D}g(k)^{ft}, \widehat{\mathcal{S}}\big)\big) \simeq Fun_{Nis\geq 0}\big(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}\big) \quad (4.140)$$

and

$$Fun^{Segal\text{-}grplike}\big(N(Fin_*), Fun_{Nis, L_{pe}(\mathbb{A}^1)}\big(\mathcal{D}g(k)^{ft}, \widehat{\mathcal{S}}\big)\big) \simeq Fun_{Nis\geq 0, L_{pe}(\mathbb{A}^1)}\big(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}\big) \quad (4.141)$$

and we claim that the $\mathbb{A}^1$-localization functor $l^{nc}_{\geq 0,\mathbb{A}^1}$ can be identified along this equivalence with the functor induced by the levelwise application of the $\mathbb{A}^1$-localization functor for spaces $l^{nc}_{0,\mathbb{A}^1}$. To confirm that this is indeed the case we observe first that the composition with $l^{nc}_{0,\mathbb{A}^1}$ produces a left-adjoint to the inclusion

$$Fun\big(N(Fin_*), Fun_{Nis, L_{pe}(\mathbb{A}^1)}\big(\mathcal{D}g(k)^{ft}, \widehat{\mathcal{S}}\big)\big) \subseteq Fun\big(N(Fin_*), Fun_{Nis}\big(\mathcal{D}g(k)^{ft}, \widehat{\mathcal{S}}\big)\big) \quad (4.142)$$

so that it suffices to check that this left-adjoint preserves Segal-grouplike objects. To prove this we will need an explicit description of the $\mathbb{A}^1$-localization functor of Nisnevich local objects $\mathcal{D}g(k)^{ft} \to \widehat{\mathcal{S}}$. Unfortunately, the explicit formula (4.129) will not work directly in the unstable case because when we apply the formula to a Nisnevich object the result might not be Nisnevich local. In any case the formula defines a reflexive localization of $Fun(\mathcal{D}g(k)^{ft}, \widehat{\mathcal{S}})$ so that we have the following situation: a presentable $(\infty,1)$-category $\mathcal{C} := Fun(\mathcal{D}g(k)^{ft}, \widehat{\mathcal{S}})$ together with two reflexive accessible localizations:

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, hook, "i_1"'] & \mathcal{C} \arrow[l, bend right, "L_1"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}_2 \arrow[r, hook, "i_2"'] & \mathcal{C} \arrow[l, bend right, "L_2"'] \end{tikzcd}$$

with (L_1, i_1) corresponding to the localization produced by the formula (4.129) and (L_2, i_2) corresponding to the Nisnevich localization. Let $f_1 := i_1 \circ L_1$ and $f_2 := i_2 \circ L_2$. Our goal is to describe $\mathcal{C}_1 \cap \mathcal{C}_2$ as an accessible reflexive localization of $\mathcal{C}_2$ and to understand how the left adjoint L_1 has to be modified in order to produce a left adjoint to the inclusion $\mathcal{C}_1 \cap \mathcal{C}_2 \subseteq \mathcal{C}_2$. The idea is that the intersection localization functor can be obtained by an infinite iteration of the composition $f_2 \circ f_1$. We observe that:

a) f_1 commutes with colimits (his is because $L_{pe}(\mathbb{A})$ is completely compact as an object in $\mathcal{C}$ and because sifted colimits commute with colimits);
b) f_2 commutes with filtered colimits (this is because Nisnevich coverings are defined via a pullback condition and filtered colimits preserve pullbacks);
c) Let us denote by S_{12} the class of maps $F \to E$ in $\mathcal{C}_2$ such for any object $X \in \mathcal{C}_1 \cap \mathcal{C}_2$ the composition $Map(E, X) \to Map(F, X)$ is an equivalence. As the generating $\mathbb{A}^1$-equivalences, by definition, live in the Nisnevich local category, S_{12} corresponds to the strongly saturated closured of this class (see [59, 55.4.15]). In particular, L_2 sends f_1-equivalences to maps in S_{12}.

We will now follow [67, Lemma 1-3.20, Lemma 2.2.6] and produce a new localization of $\mathcal{C}$ that will give the right answer. We start by considering the endofunctor $\mathcal{C}$ defined by the formula

$$F \mapsto V(F) := colim_{n \in \mathbb{N}} \big(\underbrace{(f_2 \circ f_1) \circ ... \circ (f_2 \circ f_1)}_{n} (F) \big) \tag{4.143}$$

and we consider its restriction to $\mathcal{C}_2$ given by the composition

$$\tilde{V} : \mathcal{C}_2 \xhookrightarrow{i_1} \mathcal{C} \xrightarrow{V} \mathcal{C} \xrightarrow{L_2} \mathcal{C}_2$$

We have the following lemma:

Lemma 4.42. *The endofunctor $\tilde{V} : \mathcal{C}_2 \to \mathcal{C}_2$ is a localization functor of $\mathcal{C}_2$ with local objects corresponding to the intersection $\mathcal{C}_1 \cap \mathcal{C}_2$.*

Proof. This lemma was proved in [67, Lemma 1-3.20, Lemma 2.2.6] in the case where

$$\mathcal{C} = Fun\big(N\big(AffSm^{ft}(k)\big)^{op}, \widehat{\mathcal{S}}\big)$$

with $f_1(F) := colim_{n\in\Delta^{op}}\underline{Hom}(\Delta^n_{\mathbb{A}^1}, F)$[29] and f_2 is the endofunctor corresponding to the Nisnevich localization. As we shall now explain the same proof works also in our context.

The key ingredients to prove the lemma are the properties a)–c) above, together with the explicit description of f_1. We will only sketch the main steps. We leave it to as an exercise to the reader to check that the formula (4.143) indeed defines a localization functor. We now need to prove that this localization indeed provides a left adjoint to the inclusion $\mathcal{C}_1 \cap \mathcal{C}_2 \subseteq \mathcal{C}_2$. For this purpose we observe that the canonical map $F \to \tilde{V}(F)$ is in S_{12} and that $\tilde{V}(F)$ is $\mathbb{A}^1$-Nisnevich-local. The first follows by the definition of $\mathbb{A}^1$-equivalence in $\mathcal{C}_2$: take X an object in $\mathcal{C}_1 \cap \mathcal{C}_2$ and it is immediate from the definitions to see that the composition map $Map_2(\tilde{V}(F), X) \to Map_2(F, X)$ is an equivalence in $\mathcal{C}_2$. The second requires us to use the explicit description of f_1: $L_{pe}(\mathbb{A}^1)$ is an interval-object and each of the inclusions $i_0, i_1 : L_{pe}(k) \to L_{pe}(\mathbb{A}^1)$ admits a left inverse given by the projection $p : L_{pe}(\mathbb{A}^1) \to L_{pe}(k)$. In particular, for every non-commutative space $\mathcal{X}$ we have $Map_{Nis}(\mathcal{X}, \tilde{V}(F))$ as a retract of $Map_{Nis}(\mathcal{X} \otimes L_{pe}(\mathbb{A}^1), \tilde{V}(F))$. It suffices then to show that the composition

$$Map_{Nis}\big(\mathcal{X} \otimes L_{pe}(\mathbb{A}^1), \tilde{V}(F)\big) \to Map_{Nis}\big(\mathcal{X}, \tilde{V}(F)\big) \to Map_{Nis}\big(\mathcal{X} \otimes L_{pe}(\mathbb{A}^1), \tilde{V}(F)\big)$$

is homotopic to the identity. As both $\mathcal{X}$ and $L_{pe}(\mathbb{A}^1)$ are compact, this composition can be obtained as

$$\begin{aligned} &colim_n\big(Map_{\mathcal{C}}\big(\mathcal{X} \otimes L_{pe}(\mathbb{A}^1), (f_2 \circ f_1)^n(i_2F)\big) \\ &\quad \to Map_{\mathcal{C}}\big(\mathcal{X}, (f_2 \circ f_1)^n(i_2F)\big) \to Map_{\mathcal{C}}\big(\mathcal{X} \otimes L_{pe}(\mathbb{A}^1), (f_2 \circ f_1)^n(i_2F)\big)\big) \end{aligned}$$

To conclude we use the fact that the composition $i \circ p$ is strongly $\mathbb{A}^1$-homotopic to the identity so that $f_1(i \circ p) \simeq f_1(id_{\mathcal{X}\otimes L_{pe}(\mathbb{A}^1)})$ and in particular, $f_2 \circ f_1(i \circ p) \simeq f_2 \circ f_1(id_{\mathcal{X}\otimes L_{pe}(\mathbb{A}^1)})$. Using this we see that the composition

$$\begin{aligned} &Map_{\mathcal{C}}\big(\mathcal{X} \otimes L_{pe}(\mathbb{A}^1), (f_2 \circ f_1)^n(i_2F)\big) \\ &\quad \to Map_{\mathcal{C}}\big(\mathcal{X}, (f_2 \circ f_1)^n(i_2F)\big) \to Map_{\mathcal{C}}\big(\mathcal{X} \otimes L_{pe}(\mathbb{A}^1), (f_2 \circ f_1)^n(i_2F)\big) \\ &\quad \to Map_{\mathcal{C}}\big(\mathcal{X} \otimes L_{pe}(\mathbb{A}^1), (f_2 \circ f_1)^{n+1}(i_2F)\big) \end{aligned}$$

becomes the identity map when we take the colimit. □

This description can now be used to prove that the composition with $l^{nc}_{0,\mathbb{A}^1}$ preserves the Segal-grouplike condition. Indeed, this follows immediately from 1) this explicit description together with 2) the fact that products in $Fun_{Nis}(\mathcal{D}g(k)^{ft}, \mathcal{S})$ are computed objectwise in spaces; 3) the fact that in spaces both sifted and filtered colimits commute

[29] In the original formulation of this result the authors use a different description of $f_1(F)$ that follows from the fact that the geometric realization of a simplicial space is homotopy equivalent to the diagonal of the underlying bisimplicial set.

with finite products (see [59, 5.5.8.11, 5.5.8.12] for the sifted case) and finally 4), the fact that the Nisnevich localization commutes with finite products – this is a consequence of [59, 5.5.4.15] together with the fact that in $\mathcal{D}g(k)^{idem}$ finite products are the same as finite coproducts so that the product of a Nisnevich square of dg-categories of finite type with a dg-category of finite type remains of finite type.

Finally, the grouplike condition follows also from this, together with the functoriality of $l^{nc}_{0,\mathbb{A}^1}$. As a summary of this discussion, we concluded the existence of a commutative diagram

$$\begin{tikzcd} Fun^{Segal\text{-}grplike}(N(Fin_*), Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{\mathcal{S}})) \arrow[d, "(l^{nc}_{0,\mathbb{A}^1}\circ -)"] & Fun_{Nis\geq 0}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \arrow[l, "\sim"'] \arrow[d, "l^{nc}_{\geq 0,\mathbb{A}^1}"] \\ Fun^{Segal\text{-}grplike}(N(Fin_*), Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \widehat{\mathcal{S}})) & Fun_{Nis\geq 0,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \arrow[l, "\sim"'] \end{tikzcd} \tag{4.144}$$

Finally, combining the commutativity of this diagram with the diagram (4.139) we obtain the commutativity of the diagram

$$\begin{tikzcd} Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{\mathcal{S}}) \arrow[d, "l^{nc}_{0,\mathbb{A}^1}"] & Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, "\Omega^\infty_{Nis}"] \arrow[d, "l^{nc}_{\mathbb{A}^1}"] \\ Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \widehat{\mathcal{S}}) & Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, "\sim", "\Omega^\infty_{Nis,\mathbb{A}^1}"'] \end{tikzcd} \tag{4.145}$$

This follows because Ω^∞_{Nis} can now be identified with the evaluation at $\langle 1\rangle \in N(Fin_*)$ by means of the commutativity and form of the diagrams (4.137), (4.139) and (4.144).

The following lemma is the last step in our preliminaries:

Lemma 4.43. *Let F be a connectively-Nisnevich local object in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$. Then, $\Omega^\infty(F)$ is Nisnevich local and the canonical map $\Omega^\infty(F) \simeq l^{nc}_{0,Nis}(\Omega^\infty(F)) \to \Omega_{Nis}(l^{nc}_{Nis}(F))$ is an equivalence in $Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{\mathcal{S}})$.*

Proof. The proof depends on two observations. The first is that if F is connectively-Nisnevich local, the looping $\Omega^\infty(F)$ is Nisnevich local as a functor $\mathcal{D}g(k)^{ft} \to \widehat{\mathcal{S}}$. This is because the composition $\widehat{Sp}_{\geq 0} \hookrightarrow \widehat{Sp} \xrightarrow{\Omega^\infty} \widehat{\mathcal{S}}$ preserves limits (one possible way to see this is to use the equivalence between connective spectra and grouplike commutative algebras in $\widehat{\mathcal{S}}$ for the cartesian product [63, Theorem 5.1.3.16 and Remark 5.1.3.17] and the fact that this equivalence identifies the looping functor Ω^∞ with the forgetful functor which we know as a left adjoint and therefore commutes with limits). The conclusion now follows from the definition of connectively-Nisnevich local. The second observation is that the looping functor Ω^∞ only captures the connective part of a spectrum. This follows from the very definition of the canonical t-structure in $\widehat{Sp}$ (see [63, 1.4.3.4]). In particular, since F is connectively-Nisnevich local, our Proposition 4.26 implies that the canonical morphism $F \to l^{nc}_{Nis}(F)$ is an equivalence in the connective part so that its image under Ω^∞ is an equivalence. Putting together these two observations we have equivalences fitting in a commutative diagram

$$\begin{tikzcd} \Omega^\infty(F) \arrow[d, "\sim"] \arrow[dr, "\sim"] & \\ l^{nc}_{0,Nis}(\Omega^\infty(F)) \arrow[r, "\delta"] & \Omega^\infty(l^{nc}_{Nis}(F)) \end{tikzcd} \tag{4.146}$$

so that the canonical map δ induced by the universal property of the localization is also an equivalence. □

Finally, we uncover the formulas

$$\Omega^\infty_{Nis,\mathbb{A}^1}\big(l^{nc}_{\mathbb{A}^1}(K^S)\big) \simeq l^{nc}_{0,\mathbb{A}^1}\big(\Omega^\infty_{Nis}(K^S)\big) \simeq l_{0,\mathbb{A}^1}\big(\Omega^\infty(K^c)\big) \tag{4.147}$$

where the first equivalence follows from the preceding discussion and the last one follows from the previous lemma.

The first task is done. Now we explain the equivalence between $l_{0,\mathbb{A}^1}(\Omega^\infty(K^c))$ and the unit for the monoidal structure in $Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \widehat{\mathcal{S}})$.

Our starting point is the formula (4.18) describing the K-theory space of an idempotent complete dg-category T by means of a colimit of mapping spaces. Since colimits and limits of functors are determined objectwise, the functor $\Omega^\infty K^c$ can itself be written as $\Omega\, colim_{[n]\in\Delta^{op}}\, Seq$ where Seq is the object in the $(\infty,1)$-category $Fun(\Delta^{op}, Fun(\mathcal{D}g(k)^{idem}, \widehat{\mathcal{S}}))$ resulting from the last stage of Construction 4.14.

Remark 4.44. More precisely, at the end Construction 4.14 we obtained a functor

$$N(Cat_{Ch(k)}) \to Fun\big(N(\Delta^{op}), N(\widehat{\Delta}_{big})\big) \to Fun\big(N(\Delta^{op}), \widehat{\mathcal{S}}\big) \tag{4.148}$$

where the second map is induced by the localization functor $N(\widehat{\Delta}_{big}) \to \widehat{\mathcal{S}}$ with $\widehat{\Delta}_{big}$ the very big category big of simplicial sets equipped with the standard model structure. By the description of each space at level n as a mapping space we conclude that this composition sends Morita equivalences of dg-categories to equivalences and therefore by the universal property the localization extends to a unique functor $\mathcal{D}g(k)^{idem} \to Fun(N(\Delta)^{op}, \mathcal{S})$ which, using the equivalence between $Fun(\mathcal{D}g(k)^{idem}, Fun(N(\Delta^{op}), \widehat{\mathcal{S}}))$ and $Fun(N(\Delta^{op}), Fun(\mathcal{D}g(k)^{idem}, \widehat{\mathcal{S}}))$ gives what we call Seq.

The value of Seq at zero is the constant functor with value $*$ and its value at $n \geq 1$ is $Map_{\mathcal{D}g(k)^{idem}}((\widehat{[n-1]_k})_c, -)$. The boundary and degeneracy maps are obtained from the

S-construction as explained in Construction 4.14. We observe now that the dg-categories $\widehat{([n-1]_k)}_c$, for any $n \geq 0$, are of finite type so that each level of the simplicial object Seq is in the full subcategory of ω-continuous functors. Moreover, we can think of the dg-categories $\widehat{([n]_k)}_c$ as non-commutative spaces I_n so that by means of the Yoneda map $j_{nc} : \mathcal{N}cS(k) \hookrightarrow Fun(\mathcal{D}g(k)^{ft}, \mathcal{S})$ we can identify Seq_n with the representable $Map_{\mathcal{N}cS(k)}(-, I_{n-1})$. In particular, since the Yoneda map is fully-faithful, the simplicial object Seq is the image through j_{nc} of a uniquely determined simplicial object $Seq_{nc} \in Fun(\Delta^{op}, \mathcal{N}cS(k))$ whose value at level n is the noncommutative space I_{n-1}. Finally, with these notations we can write $\Omega^\infty K^c$ as $\Omega\, colim_{[n]\in\Delta^{op}}\, j_{nc} \circ Seq_{nc}$ so that our main goal is to understand the localization $l^{nc}_{0,\mathbb{A}^1}(\Omega\, colim_{[n]\in\Delta^{op}}\, j_{nc} \circ Seq_{nc})$. As the zero level of the simplicial object $j_{nc} \circ Seq_{nc}$ is contractible, the realization $colim_{[n]\in\Delta^{op}}\, j_{nc} \circ Seq_{nc}$ is 1-connective.[30]

We have the following lemma:

Lemma 4.45. *The canonical map*

$$l^{nc}_{0,\mathbb{A}^1}(\Omega\, colim_{[n]\in\Delta^{op}}\, j_{nc} \circ Seq_{nc}) \to \Omega\, l^{nc}_{0,\mathbb{A}^1}(colim_{[n]\in\Delta^{op}}\, j_{nc} \circ Seq_{nc}) \tag{4.149}$$

is an equivalence.

Proof. The key observation is that the presheaf of spaces $colim_{[n]\in\Delta^{op}}\, j_{nc} \circ Seq_{nc}$ is the zero level of a presheaf of connective spectra (for instance, as constructed by Waldhausen in [108]). The important point is that this spectral presheaf satisfies Nisnevich descent as a result of the Waldhausen localization theorem [108, 1.6.4]. In particular, it is Nisnevich local. The result now follows from the commutativity of the diagram (4.145). □

Our main goal now is to understand the simplicial object Seq. Following Waldhausen [108] we recall the existence of a weaker version of the S-construction that considers only those sequence of cofibrations that split. More precisely, and using the same terminology as in Construction 4.14 we denote by $\mathbb{R}\underline{Hom}^{split}(Ar[n]_k, \widehat{T}_c)$ the full sub dg-category of $\mathbb{R}\underline{Hom}(Ar[n]_k, \widehat{T}_c)$ spanned by those $Ar[n]$-indexed diagrams satisfying the conditions given in Construction 4.13 and where the top sequence is given by the canonical inclusions $E_1 \to E_1 \oplus E_2 \to E_1 \oplus E_2 \oplus E_3 \to \ldots \to E_1 \oplus \ldots \oplus E_n$ for some list of perfect modules $(E_1, \ldots, E_n)$. These are called *split cofibrations*. As in the standard S-construction, the categories subjacent to $\mathbb{R}\underline{Hom}^{split}(Ar[n]_k, \widehat{T}_c)$ carries a notion of weak-equivalences W^{Split}_n and assemble to form a simplicial space $[n] \to N(\mathbb{R}\underline{Hom}^{split}(Ar[n]_k, \widehat{T}_c)^{W^{Split}_n})$.

As in Construction 4.14 we can now describe these spaces in a somewhat more simple form. As the dg-categories 1_k are cofibrant (see [89]) they are also locally-cofibrant and for any $n \geq 0$ the coproduct $\coprod_{i=1}^n 1_k$ is a homotopy coproduct. Moreover, for any locally-cofibrant dg-category T we have equivalences $\mathbb{R}\underline{Hom}(\coprod_{i=1}^n 1_k, \widehat{T}_c) \simeq$

[30] Recall that a space is said to be n-connective if it is non-empty and all its homotopy groups for $i < n$ are zero.

$\prod_{i=1}^{n} (\widehat{1_k \otimes^{\mathbb{L}} T})_{pspe} \simeq \prod_{i=1}^{n} (\widehat{1_k \otimes T})_{pspe} \simeq \prod_{i=1}^{n} \widehat{T}_c$ In this case, for every $n \geq 0$ and for every dg-category T there is an equivalence between the category subjacent to $\mathbb{R}\underline{Hom}^{split}(Ar[n]_k, \widehat{T}_c)$ and the category subjacent to $\mathbb{R}\underline{Hom}(\coprod_{i=1}^{n} 1_k, \widehat{T}_c)$, defined by sending a sequence $E_1 \to E_1 \oplus E_2 \to E_1 \oplus E_2 \oplus E_3 \to ... \to E_1 \oplus ... \oplus E_n$ to the successive quotients $(E_1, ..., E_n)$. This correspondence is functorial and defines an equivalence because of the universal property of direct sums. Moreover, and again thanks to the cube lemma, this equivalence preserves the natural notions of weak-equivalences. Finally, and again due to the main theorem of [96] we found the spaces $N(\mathbb{R}\underline{Hom}^{split}(Ar[n]_k, \widehat{T}_c)^{W_n^{Split}})$ and $Map_{\mathcal{D}g(k)^{idem}}(\bigoplus_{i=1}^{n} \widehat{(1_k)}_c, \widehat{T}_c)$ to be equivalent so that by the same arguments as in Remark 4.44 we obtain a simplicial object $Split \in Fun(N(\Delta)^{op}, Fun(\mathcal{D}g(k)^{idem}, \widehat{\mathcal{S}}))$, which, because the dg-categories $\bigoplus_{i=1}^{n} \widehat{(1_k)}_c$ are of finite type, lives in the full subcategory of ω-continuous functors, therefore being an object in $Fun(N(\Delta^{op}), \mathcal{P}(\mathcal{N}cS(k)))$. Moreover, for each $n \geq 0$ $Split_n$ is representable by the noncommutative space associated to the dg-category $\bigoplus_{i=1}^{n} \widehat{(1_k)}_c$ so that by Yoneda the whole simplicial object $Split$ is of the form $j_{nc} \circ \Theta$ for a simplicial object $\Theta \in Fun(N(\Delta^{op}), \mathcal{N}cS(k))$ with level n given by $\bigoplus_{i=1}^{n} \widehat{(1_k)}_c$.

Finally, the inclusion of split cofibrations into all sequences of morphisms provides a strict map of simplicial objects in the model category $\widehat{\Delta}$ between $[n] \to N(\mathbb{R}\underline{Hom}^{split}(Ar[n]_k, \widehat{T}_c)^{W_n})$ and $[n] \to N(S_n^{dg}(T)^{W_n})$ and we define λ

$$\lambda : j_{nc}(\Theta) \simeq Split \to Seq \tag{4.150}$$

to be the image of this map under the composition in (4.148). This is where the result of [12] becomes crucial:

Proposition 4.46. *(See [12, Prop. 4.6].) The map λ is a levelwise noncommutative $\mathbb{A}^1$-equivalence in $Fun(\mathcal{D}g(k)^{ft}, \mathcal{S})$.*

Proof. In [12, Prop. 4.6] the author uses an inductive argument to prove that for any $n \geq 0$ the map λ_n is an $\mathbb{A}^1$-equivalence.

For $n = 1$, λ_1 is an equivalence. For $n = 2$ we need some further adaptation to our case. Namely, we are required to construct a noncommutative $\mathbb{A}^1$-homotopy between the identity of the noncommutative space I_{2-1} and the zero map. Such a homotopy corresponds to a co-homotopy in $\mathcal{D}g(k)^{idem}$, namely, a map $H : \widehat{([1]_k)}_c \to \widehat{([1]_k)}_c \otimes^{\mathbb{L}} L_{pe}(\mathbb{A}^1)$ in $\mathcal{D}g(k)^{idem}$ fitting in a commutative diagram

$$\begin{tikzcd} \widehat{([1]_k)}_c \arrow[r, "H"] \arrow[rr, bend left, "Id"] \arrow[rr, bend right, "0"'] & \widehat{([1]_k)}_c \otimes^{\mathbb{L}} L_{pe}(\mathbb{A}^1) \arrow[r, "ev_1"] \arrow[r, "ev_0"'] & \widehat{([1]_k)}_c \end{tikzcd} \tag{4.151}$$

Recall that $L_{pe}(\mathbb{A}^1)$ is canonically equivalent to $\widehat{k[X]}_c$ – the idempotent completion of the dg-category with one object and $k[X]$ concentrated in degree zero as endomorphisms. In this case the term in the middle is equivalent to $(\widehat{([1]_k) \otimes k}[X])_c$. We define H to be the map induced by the universal property of the idempotent completion $\widehat{(-)}_c : \mathcal{D}g(k) \to \mathcal{D}g(k)^{idem}$ by means of the composition

$$\big([1]_k\big) \to \big([1]_k\big) \otimes k[X] \subseteq \big(\widehat{([1]_k) \otimes k}[X]\big)_c \tag{4.152}$$

where the first map is obtained from the strict dg-functor defined by the identity on the objects, by the inclusion $k \subseteq k[X]$ on the endomorphisms of 0 and by the composition $k \subseteq k[X] \to k[X]$ on the complex of maps between 0 and 1 and on the endomorphisms of 1, where the last map is the multiplication by the variable X. This makes the diagram above commute and provides the required homotopy. We conclude as in [12, Prop. 4.6] to find that λ_2 is an $\mathbb{A}^1$-equivalence given by a strong $\mathbb{A}^1$-homotopy.

We now conclude with the induction step: it follows from the observation that the canonical map $Seq_n \to Seq_{n-1} \times_{Seq_n} Seq_2$ is an equivalence of presheaves. The conclusion now follows because the fiber product of strong $\mathbb{A}^1$-homotopy equivalences remains a strong $\mathbb{A}^1$-homotopy equivalence (it is easy to write down the homotopies for the fiber product). □

Finally, the fact that any colimit of $\mathbb{A}^1$-equivalences is an $\mathbb{A}^1$-equivalence gives us the following corollary:

Corollary 4.47. *The map induced by λ between the colimits $colim_{\Delta^{op}} j_{nc} \circ \Theta \to colim_{\Delta^{op}} j_{nc} \circ Seq_{nc}$ is an $\mathbb{A}^1$-equivalence. Moreover, and since $l^{nc}_{0,\mathbb{A}^1}$ commutes with colimits and representable objects are Nisnevich local, we have equivalences*

$$\begin{aligned} colim_{\Delta^{op}} l^{nc}_{0,\mathbb{A}^1} \circ j_{nc} \circ \Theta &\simeq l^{nc}_{0,\mathbb{A}^1}(colim_{\Delta^{op}} j_{nc} \circ \Theta) \simeq l^{nc}_{0,\mathbb{A}^1}(colim_{\Delta^{op}} j_{nc} \circ Seq_{nc}) \\ &\simeq colim_{\Delta^{op}} l^{nc}_{0,\mathbb{A}^1} \circ j_{nc} \circ Seq_{nc} \end{aligned} \tag{4.153}$$

in $\mathcal{SH}_{nc}(k)$.

Our next move requires a small preliminary digression. To start with, recall that any $(\infty, 1)$-category endowed with finite sums and an initial object or finite products and a final object, can be considered as a symmetric monoidal $(\infty, 1)$-category with respect to these two operations, respectively denoted as $\mathcal{C}^{\coprod}$ and $\mathcal{C}^{\times}$ (see [63, Sections 2.4.1 and 2.4.3]). Monoidal structures appearing from this mechanism are called, respectively, *cartesian* and *cocartesian*. In particular, if $\mathcal{C}$ has direct sums and a zero object, these monoidal structures coincide $\mathcal{C}^{\oplus}$ (this follows from Proposition [63, 2.4.3.19]). In this particular situation the theory of algebras over a given ∞-operad $\mathcal{O}^{\otimes}$ gets simplified: the $(\infty, 1)$-category of $\mathcal{O}$-algebras on $\mathcal{C}^{\oplus}$ is equivalent to a full subcategory of $Fun(\mathcal{O}^{\otimes}, \mathcal{C})$, spanned by a class of functors satisfying the standard Segal conditions (see [63, 2.4.2.1, 2.4.2.5]). In the particular case of associative algebras, and since the category Δ^{op} is a “model” for the associative operad (see [63, 4.1.2.6, 4.1.2.10, 4.1.2.14] for the precise statement) an associative algebra in $\mathcal{C}^{\oplus}$ is just a simplicial object in $\mathcal{C}$ satisfying the Segal condition.

$$Alg_{\mathcal{A}ss}(\mathcal{C}) \simeq Fun^{Segal}\big(N\big(\Delta^{op}\big), \mathcal{C}\big) \tag{4.154}$$

We shall now come back to our situation and observe that

Lemma 4.48. *The simplicial object Θ satisfies the Segal conditions.*

Proof. As the Yoneda embedding preserves limits and is fully-faithful it suffices to check that *Split* satisfies the Segal conditions. But this is obvious from the definition of the simplicial structure given by the S-construction. At each level the map appearing in the Segal condition is the map sending a sequence of dg-modules $E_0 \to E_0 \oplus E_1 \to \ldots .E_0 \oplus \ldots \oplus E_{n-1}$ to the quotients $(E_0, \ldots, E_{n-1})$. □

We now characterize the simplicial object Θ in a somewhat more canonical fashion. An important aspect of a cocartesian symmetric monoidal structure $\mathcal{C}^{\coprod}$ is that any object X in $\mathcal{C}$ admits a unique algebra structure, determined by the codiagonal map $X \coprod X \to X$. More precisely (see [63, 2.4.3.16] for the general result), the forgetful map $Alg_{\mathcal{A}ss}(\mathcal{C}) \to \mathcal{C}$ is an equivalence of $(\infty, 1)$-categories.[31] By choosing an inverse to this equivalence and composing with the equivalence (4.154) we obtain an ∞-functor

$$\mathcal{C} \to Alg_{\mathcal{A}ss}(\mathcal{C}) \simeq Fun^{Segal}\big(N(\Delta)^{op}, \mathcal{C}\big) \tag{4.155}$$

providing for any object in $\mathcal{C}$ a uniquely determined simplicial object, encoding the algebra structure induced by the codiagonal.[32] Because of the Segal condition this simplicial object is a zero object of $\mathcal{C}$ in degree zero, X in degree one and more generally is $X^{\oplus_n}$

[31] Recall that the associative operad is unital.

[32] The fact that the multiplication can be identified with the codiagonal map follows from the simplicial identities and from the universal property defining the codiagonal.

in degree n. We now apply this discussion to $\mathcal{C} = \mathcal{N}cS(k)$ (it has direct sums and a zero object because $\mathcal{D}g(k)^{idem}$ has and the inclusion $\mathcal{D}g(k)^{ft} \subseteq \mathcal{D}g(k)^{idem}$ preserves them) and to $X = \widehat{(1_k)}_c$. Since the simplicial object Θ satisfies the Segal condition and its first level is equivalent to X, the equivalence (4.155) tells us that it is necessarily the simplicial object codifying the unique associative algebra structure on X given by the codiagonal.

With Corollary 4.47 we are now reduced to study the colimit of the simplicial object $l^{nc}_{\mathbb{A}^1} \circ j_{nc} \circ \Theta$ in $\mathcal{SH}_{nc}(k)$. As the last is a stable $(\infty, 1)$-category it has direct sums and therefore can be understood as the underlying $(\infty, 1)$-category of a symmetric monoidal structure $\mathcal{SH}_{nc}(k)^{\oplus}$ which is simultaneously cartesian and cocartesian. As the canonical composition $\mathcal{N}cS(k) \to \mathcal{SH}_{nc}(k)$ preserves direct sums (this follows from 1) the fact the Yoneda functor preserves limits; 2) the fact representables are Nisnevich local; 3) the fact the $\mathbb{A}^1$-localization preserves finite products (as explained when confirming that it preserves the Segal conditions) and finally 4) the fact that $\mathcal{SH}_{nc}(k)$ is stable) it can be lifted in an essentially unique way to a monoidal functor $\mathcal{N}cS(k)^{\oplus} \to \mathcal{SH}_{nc}(k)^{\oplus}$ [63, Cor. 2.4.1.8]. This monoidal map allows us to transport algebras and provides a commutative diagram

$$\begin{tikzcd}
Fun^{Segal}(N(\Delta)^{op}, \mathcal{N}cS(k)) \arrow[r] \arrow[d, "\sim"] \arrow[dd, bend right=60, "ev_{[1]}"'] & Fun^{Segal}(N(\Delta)^{op}, \mathcal{SH}_{nc}(k)) \arrow[d, "\sim"] \arrow[dd, bend left=60, "ev_{[1]}"] \\
Alg_{\mathcal{A}ss}(\mathcal{N}cS(k)) \arrow[r] \arrow[d, "\sim"] & Alg_{\mathcal{A}ss}(\mathcal{SH}_{nc}(k)) \arrow[d, "\sim"] \\
\mathcal{N}cS(k) \arrow[r] & \mathcal{SH}_{nc}(k)
\end{tikzcd} \tag{4.156}$$

where the upper map is the composition with $\mathcal{N}cS(k) \to \mathcal{SH}_{nc}(k)$. It follows from the description of Θ above and from the commutativity of this diagram that the simplicial object $l^{nc}_{\mathbb{A}^1} \circ j_{nc} \circ \Theta$ in $\mathcal{SH}_{nc}(k)$ corresponds to the unique commutative algebra structure on $1_{nc} := l^{nc}_{\mathbb{A}^1} \circ j_{nc}(L_{pe}(k))$ created by the codiagonal.

Our next task is to study the theory of associative algebras on a stable $(\infty, 1)$-category equipped with its natural simultaneously cartesian and cocartesian monoidal structure induced by the existence of direct sums. We recall some terminology. If $\mathcal{C}^{\otimes}$ is a cartesian symmetric monoidal structure, an associative algebra on $\mathcal{C}$ is said to be *grouplike* if the simplicial object which codifies it $A \in Fun^{Segal}(N(\Delta^{op}), \mathcal{C})$ is a groupoid object in $\mathcal{C}$ in the sense of the definition [59, 6.1.2.7]. We let $Alg^{grplike}_{\mathcal{A}ss}(\mathcal{C})$ denote the full subcategory of $Alg_{\mathcal{A}ss}(\mathcal{C})$ spanned by the grouplike associative algebras.

Let now Δ^{op}_{+} be the standard augmentation of the category Δ^{op}. Following [59, 6.1.2.11], an object $U_{+} \in Fun(\Delta^{op}_{+}, \mathcal{C})$ is said to be a *Cech nerve of the morphism* $U_0 \to U_{-1}$ if the restriction $U_{+}|_{N(\Delta^{op})}$ is a groupoid object and the commutative diagram

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d] & U_0 \arrow[d] \\ U_0 \arrow[r] & U_{-1} \end{tikzcd} \tag{4.157}$$

is a pullback diagram in $\mathcal{C}$. Again by [59, 6.1.2.11], a Cech nerve U_+ is determined by the map $U_0 \to U_{-1}$ in an essentially unique way as the right-Kan extension along the inclusion $N(\Delta^{op}_{+,\leq 0}) \subseteq N(\Delta^{op}_{+})$.

We have the following lemma:

Lemma 4.49. *Let $\mathcal{C}^{\otimes}$ be a cartesian symmetric monoidal $(\infty,1)$-category whose underlying $(\infty,1)$-category is stable. Then*

1. *The inclusion $Alg^{grplike}(\mathcal{C}) \subseteq Alg(\mathcal{C})$ is an equivalence;*
2. *For any object X in $\mathcal{C}$ the simplicial object associated to X by means of the composition (4.155) is a Cech nerve of the canonical morphism $0 \to \Sigma X$.*

Proof. The first assertion is true because in any stable $(\infty,1)$-category every morphism $f : X \to Y$ has an inverse $-f$ with respect to the additive structure.[33] In particular, for any object $X \in \mathcal{C}$ there is map $-Id_X$ providing an inverse for the algebra structure given by the codiagonal map $X \oplus X \to X$. More precisely, let X be an object in $\mathcal{C}$ and let U_X be the simplicial object associated to X by means of the mechanism (4.155). By construction this simplicial object satisfies the Segal condition and in particular we have $(U_X)_0 \simeq 0$ and $(U_X)_1 \simeq X$. We aim to prove that this simplicial object is a groupoid object. For that we observe that for a simplicial object A to be a groupoid object it is equivalent to ask for A to satisfy the Segal conditions and to ask for the induced map

$$A([2]) \xrightarrow{A(\partial_1)\times A(\partial_0)} A([1]) \times A([1]) \tag{4.158}$$

to be an equivalence. Indeed, if A is a groupoid object, by the description in [59, 6.1.2.6-4")] it satisfies these two requirements automatically. The converse follows by applying the same arguments as in the proof of [59, 6.1.2.6-4)' implies 3)], together with the observation that for the induction step to work we don't need the full condition in $4'$) but only the Segal condition. The induction basis is equivalent to the Segal conditions for $n = 2$ together with the condition that (4.158) is an equivalence.

In our case (4.158) is the map $\nabla \times id_X : X \oplus X \to X \oplus X$ where ∇ is the codiagonal map $X \oplus X \to X$. Of course, since the identity of X admits an inverse $(-Id_X)$ the map $(\nabla \circ (Id_X \times (-Id_X))) \times Id_X$ is an explicit inverse for $\nabla \times id_X$.

Let us now prove 2). Again by construction, we know that the colimit of the truncation $(U_X)_{|N(\Delta_{\leq 1})^{op}}$ is canonically equivalent to the suspension ΣX. Therefore U_X admits a

33 More precisely $\pi_0 Map(X, Y)$ has a canonical structure of abelian group.

canonical augmentation $(U_X)^+ : N(\Delta_+^{op}) \to \mathcal{C}$ with $(U_X)^+_{-1} \simeq \Sigma X$. It follows from 1) that U is a groupoid object and since $\mathcal{C}$ is stable, the diagram

$$\begin{tikzcd} (U_X)_1 \simeq X \arrow[r] \arrow[d] & (U_X)_0 \simeq 0 \arrow[d] \\ (U_X)_0 \simeq 0 \arrow[r] & (U_X)_{-1} \simeq \Sigma X \end{tikzcd} \tag{4.159}$$

is a pullback so that $(U_X)^+$ is the Cech nerve of the canonical map $0 \to \Sigma X$. $\square$

In particular, we find that the simplicial object $l^{nc}_{0,\mathbb{A}^1} \circ j_{nc} \circ \Theta$ is a Cech nerve of the canonical map $0 \to \Sigma 1_{nc}$. Finally, recall that a morphism $A \to B$ is said to be an effective epimorphism if the colimit of its Cech nerve is B. The following lemma holds the final step

Lemma 4.50. *Let $\mathcal{C}$ be a stable $(\infty,1)$-category. Then, for any object X in $\mathcal{C}$, the canonical morphism $0 \to X$ is an effective epimorphism.*

Proof. Let $U : N(\Delta^{op}) \to \mathcal{C}$ be a simplicial object in $\mathcal{C}$. Then the colimit of U can be computed as the sequential colimit of the successive colimits of its truncations $U_{|_{N(\Delta^{op}_{\leq n})}}$. Using the descriptions of Cech nerves as right-Kan extensions (see above) we know that if U^+ is the Cech nerve of the map $0 \to X$, its level n is given by the n-fold fiber product of 0 over X. As $\mathcal{C}$ is stable this n-dimensional limit cube will also be a colimit n-cube so that the colimit of the truncation at level n will necessarily be X (see Proposition [63, 1.2.4.13]). Since this holds for every $n \geq 0$ the colimit of the Cech nerve is necessarily canonically equivalent to X. $\square$

We are done. Since $\mathcal{SH}_{nc}(k)$ is stable we have $colim_{\Delta^{op}}\, l^{nc}_{0,\mathbb{A}^1} \circ j_{nc} \circ \Theta \simeq \Sigma 1_{nc}$ so that, by Lemma 4.45 we have $l^{nc}_{0,\mathbb{A}^1}(\Omega^\infty(K^c))$ is equivalent to $\Omega\Sigma 1_{nc} \simeq 1_{nc}$.

Appendix A. Comparison with the approach of Cisinski–Tabuada

In this appendix we explain the relation between our approach to noncommutative motives and the approach studied by G. Tabuada in [90,92] and Cisinski–Tabuada in [25, 24]. Both theories have the $(\infty,1)$-category $Fun_\omega(\mathcal{D}g(k)^{idem}, \widehat{Sp})$ as a common ground. To start with we observe that our version $\mathcal{SH}_{nc}(k)$ can be identified with the full subcategory spanned by those functors F sending Nisnevich squares of dg-categories to pullback–pushout squares in spectra and satisfying $\mathbb{A}^1$-invariance. Indeed, our original definition of $\mathcal{SH}_{nc}(k)$ as a localization of $Fun(\mathcal{D}g(k)^{ft}.\widehat{Sp})$ can be transported along the equivalence

$$Fun_\omega\big(\mathcal{D}g(k)^{idem}, \widehat{Sp}\big) \simeq Fun\big(\mathcal{D}g(k)^{ft}, \widehat{Sp}\big) \tag{A.1}$$

Tabuada's approach focuses on the full subcategory $Fun_{\omega,Loc}(\mathcal{D}g(k)^{idem}, \widehat{Sp})$ spanned by those functors sending exact sequences of dg-categories to fiber/cofiber sequences in spectra. His main theorem is the existence of a stable presentable $(\infty,1)$-category which we denote here as $\mathcal{M}^{Tab}_{Loc}$, together with a functor $\mathcal{D}g(k)^{idem} \to \mathcal{M}^{Tab}_{Loc}$ preserving filtered colimits, sending exact sequences to fiber/cofiber sequences and universal in this sense. We can also easily see that $\mathcal{M}^{Tab}_{Loc}$ is a stable presentable symmetric monoidal $(\infty,1)$-category with the monoidal structure extending the monoidal structure in $\mathcal{D}g(k)^{idem}$. This result was originally formulated using the language of derivators (see [64] for an introduction) but we can easily extend it to the setting of $(\infty,1)$-categories by applying the same construction and the general machinery developed by J. Lurie in [63,59]. In particular we have an equivalence of $(\infty,1)$-categories

$$Fun_{\omega,Loc}\big(\mathcal{D}g(k)^{idem}, \widehat{Sp}\big) \simeq Fun^{L}\big(\mathcal{M}^{Tab}_{Loc}, \widehat{Sp}\big) \tag{A.2}$$

As we can see this is a theorem about a specific class of objects inside $Fun_{\omega}(\mathcal{D}g(k)^{idem}, \widehat{Sp})$, namely, those that satisfy localization. The comparison with our approach starts with the observation that any object F satisfying localization satisfies also our condition of Nisnevich descent so that we have an inclusion of full subcategories $Fun_{\omega,Loc}(\mathcal{D}g(k)^{idem}, \widehat{Sp}) \subseteq Fun_{\omega,Nis}(\mathcal{D}g(k)^{idem}, \widehat{Sp})$. In particular, we can identify

$$\mathcal{SH}^{Loc}_{nc}(k) := Fun_{\omega,Loc,\mathbb{A}^1}\big(\mathcal{D}g(k)^{idem}, \widehat{Sp}\big) \tag{A.3}$$

with a full subcategory of $\mathcal{SH}_{nc}(S)$. We summarize this in the following diagram

$$\begin{tikzcd}
 & Fun_{\omega}(\mathcal{D}g(k)^{idem}, \widehat{Sp}) & \\
Fun_{\omega,Nis}(\mathcal{D}g(k)^{idem}, \widehat{Sp}) \arrow[ur, hook] & & Fun_{\omega,Loc}(\mathcal{D}g(k)^{idem}, \widehat{Sp}) \arrow[ul, hook] \arrow[ll, hook'] \\
Fun_{\omega,Nis,\mathbb{A}^1}(\mathcal{D}g(k)^{idem}, \widehat{Sp}) =: \mathcal{SH}_{nc}(k) \arrow[u, hook] & & \mathcal{SH}^{Loc}_{nc}(k) \arrow[u, hook] \arrow[ll, hook']
\end{tikzcd} \tag{A.4}$$

The second observation is that the construction $\mathcal{M}^{Tab}_{Loc}$ of Tabuada and the formula (A.2) admits analogues adapted to each of the full subcategories in this diagram. More precisely one can easily show the existence of new stable presentable symmetric monoidal $(\infty,1)$-categories $\mathcal{M}^{Tab}_{Nis}$, $\mathcal{M}^{Tab}_{Nis,\mathbb{A}^1}$, $\mathcal{M}^{Tab}_{Loc,\mathbb{A}^1}$ all equipped with ω-continuous monoidal functors from $\mathcal{D}g(k)^{idem}$, universal with respect to each of the obvious respective properties. In particular we find an equivalence

$$Fun^{L}\big(\mathcal{M}^{Tab}_{Nis,\mathbb{A}^1}, \widehat{Sp}\big) \simeq \mathcal{SH}_{nc}(k) \tag{A.5}$$

exhibiting the duality between our approach and the corresponding Nisnevich-$\mathbb{A}^1$-version of Tabuada's construction (recall that the very big $(\infty,1)$-category of big stable presentable $(\infty,1)$-categories has a natural symmetric monoidal structure [63, 4.8.2.10, 4.8.2.18 and 4.8.1.17] where the big $(\infty,1)$-category of spectra $\widehat{Sp}$ is a unit and $Fun^L(-,-)$ is the internal-hom). We redirect the reader to our discussion in [79, Chapter 8] for further details about this comparison.

As emphasized before, the main advantage of our approach is the easy comparison with the motivic stable homotopy theory of schemes. The duality presented in this appendix explains why the original approach of Cisinski–Tabuada can't have a direct comparison. A second main advantage of our methods is the interpretation of non-commutative K-theory as a Nisnevich sheafification of connective K-theory.

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