Motivic Homotopy Theory

Contents

Introduction

Notes from the Spring 2021 GeoTop junior seminar at the University of Copenhagen. The course primarily follows the Talbot seminar.

Notation and Conventions

Since we are primarily working in the world of algebraic geometry, that is, with schemes we say that a topological space X *quasi-compact* if every open cover has a finite subcover and reserve the term *compact* for the case when X is quasi-compact and Hausdorff.

0 Background from Algebraic Geometry

Definition 0.1. Let $f : X \to Y$ be a map of topological spaces, then f is *quasi-compact* if $f^{-1}(V)$ is quasi-compact for every quasi-compact open $V \subset Y$.

Lemma 0.1. (Tag 01K2) *Let $f : X \to S$ be a morphism of schemes, then the following are equivalent:*

(1) $f : X \to S$ is quasi-compact,

(2) the inverse image of every affine open is quasi-compact,

(3) there exists an affine open cover $S = \cup_{i \in I} U_i$ such that $f^{-1}(U_i)$ is quasi-compact for all i.

In topology one has the theorem that a space X is Hausdorff if and only if the diagonal

$$\begin{aligned} \Delta : X &\to X \times X \\ x &\mapsto (x, x) \end{aligned}$$

is closed. In algebraic geometry given $f : X \to S$ we have a diagonal map obtained by the fiber product $\Delta_{X/S} : X \to X \times_S X$. That is, $\Delta_{X/S}$ is the unique morphism of schemes such that $\mathbf{pr}_1 \circ \Delta_{X/S} = \mathbf{1}_X$ and $\mathbf{pr}_2 \circ \Delta_{X/S} = \mathbf{1}_X$. One can show that $\Delta_{X/S}$ is an immersion (Tag 01KH).

Definition 0.2. Let $f : X \to S$ be a morphism of schemes.

(1) f is said to be separated if the diagonal $\Delta_{X/S}$ along f is a closed immersion.

(2) f is said to be quasi-separated if the diagonal $\Delta_{X/S}$ is a quasi-compact morphism.

(3) A scheme Y is separated if the morphism $Y \to \operatorname{Spec} \mathbb{Z}$ is separated.

(4) A scheme Y is quasi-separated if the morphism $Y \to \operatorname{Spec} \mathbb{Z}$ is quasi-separated.

Recall that a map of rings $R \to A$ is said to be of finite type if A is isomorphic to a quotient of $R[x_1, \ldots, x_n]$ as an R-algebra, that is, if A is a finitely generated R-algebra.

Definition 0.3. (Tag 01T0) Let $f : X \to S$ be a morphism of schemes, then

(1) f is said to be of *finite type at $x \in X$* if there exists an affine open neighborhood $\operatorname{Spec} A = U \subset X$ of x and an affine open $\operatorname{Spec} R = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is of finite type.

(2) f is *locally of finite type* if f is of finite type for all $x \in X$.

(3) f is of *finite type* it is locally of finite type and quasi-compact.

Definition 0.4. Let X be a scheme, then X is *Noetherian* if it admits a finite cover by affine Schemes, $\operatorname{Spec} A_i$, where each A_i is a Noetherian ring.

Lemma 0.2. *(Lemma 01T6) Let $f : X \to S$ be a morphism. If S is (locally) Noetherian and f (locally) of finite type then X is (locally) Noetherian.*

Let $f : X \to S$ be a morphism of schemes, then we refer to Tag 01V4 for a general definition of smoothness. However, since we will be working with $f : X \to S$ of finite type with S a Noetherian scheme we can instead refer to Tag 02HW for definition of Smoothness. For definitions of etale morphisms see Tag 02GH and Tag 0257 or Chapter 024J

Definition 0.5. Let X be a scheme and x a point in X. Let $U = \operatorname{Spec} A$ be an affine neighborhood of x. Let P_x be the prime ideal corresponding to x, then the localization A_{P_x} is a local ring with maximal ideal $P_x \cdot A_{P_x}$. The *residue field* of x is the field

$$k(x) := A_P / P \cdot A_P.$$

Note that there is an isomorphism $k(x) \cong \operatorname{Frac}(A/P)$ We call the point x K-rational for a certain field K if $K \cong k(x)$.

Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let $a \mapsto \overline{a}$ denote the map $A \to k$ and $f \mapsto \overline{f}$ denote the map $A[T] \to k[T]$. If A is a complete[1] discrete valuation ring, then Hensel's lemma states: If f is a monic polynomial with coefficients in A such that $\overline{f}$ factors as $\overline{f} = g_0 h_0$ with g_0 and h_0 monic and coprime[2], then f factors as $f = gh$ with g and h monic such that $\overline{g} = g_0$ and $\overline{h} = h_0$.

Definition 0.6. Let A be a local ring with maximal ideal $\mathfrak{m}$, then A is said to be Henselian if Hensel's lemma holds for A.

Given a ring a local ring $(A, \mathfrak{m}, k)$ there is a universal ring A^h called the Henselization of A (Tag 0BSK, Tag 07QL, Tag 03QD). That is, the assignment $A \mapsto A^h$ is functorial.

1 The Nisnevich Topology and Universal Properties of the Stable and Unstable Motivic Homotopy Categories

In this talk we introduce the Nisnevich topology for a scheme and discuss some of it's properties (including, hopefully, descent) as it is central to the definition of the Motivic homotopy category. Then we introduce the (un)stable Motivic homotopy categories and discuss/prove their universal properties following [Rob12] sections $5.1 - 5.3$. Finally, note that in many of the constructions there are the usual subtle set-theoretic issues that must be dealt with in order to avoid Russell type paradox. We will generally ignore any such subtleties in the interest of exposition as all the issues can be dealt with through the use of universes and enlarging the universe when necessary.

[1] Any discrete valuation ring may be given a metric $|x - y| = 2^{-\nu(x-y)}$ and we may take the completion with respect to this metric. Alternatively, we can consider the completion of the ring considered as a topological ring since local rings may be given a natural topology

[2] i.e. the ideal $(g_0, h_0) = (1)$ generates the whole ring

1.1 Preliminaries

We begin by recalling the notion of Grothendieck (pre)topologies as well as

Grothendieck (Pre)Topologies

We begin by discussing Grothendieck (pre)topologies. Although we attempt to work as much as possible in the setting of ∞-categories the ∞-categorical generalization of a Grothendieck topology is in some sense no generalization at all[3] so for ease of exposition we discuss Grothendieck topologies purely in the setting of 1-categories.

Definition 1.1. Let $\mathcal{C}$ be a 1-category. A *Grothendieck pretopology τ on $\mathcal{C}$* consists of for each $C \in \mathcal{C}$ a set $\mathrm{Cov}_\tau(C)$ of *covering families* where a covering is a set of morphisms $\{f_i : U_i \to C\}_{i\in I}$ in $\mathcal{C}$ such that

(1) $\{\mathbf{1}_C\}$ is in $\mathrm{Cov}_\tau(C)$ for all $C \in \mathcal{C}$;

(2) for $\{f_i : U_i \to C\} \in \mathrm{Cov}_\tau(C)$ and $g : D \to C$ a morphism in $\mathcal{C}$ the fiber product $U_i \times_C D$ exists and the natural projection $\{\mathbf{pr}_2 : U_i \times_C D \to D\}$ is in $\mathrm{Cov}_\tau(D)$;

(3) if $\{f_j : U_j \to C\}_{j\in J}$ is in $\mathrm{Cov}_\tau(C)$ and $\{g_{ij} : V_{ij} \to U_j\} \in \mathrm{Cov}_\tau(U_j)$ for all j, then $\{f_j \circ g_{ij} : V_{ij} \to C\} \in \mathrm{Cov}_\tau(C)$.

Definition 1.2. (1) Let $\mathcal{C}$ be a 1-category and $C \in \mathcal{C}$. A *sieve on C* is a subfunctor of $\mathbf{y}_C := \mathrm{Hom}_{\mathcal{C}}(-, C)$, that is, a set of objects $\mathcal{S}$ of $\mathcal{C}_{/C}$ such that for each $f : D \to C$ in $\mathcal{S}$ and each morphism $g_i : D_i \to D$ in $\mathcal{C}$ the composition $f \circ g_i : D_i \to C$ is in $\mathcal{S}$.

(2) Let $\mathcal{F} := \{f_i : D_i \to C\}$ be a collection of morphisms in $\mathcal{C}$. The *sieve generated by $\mathcal{F}$* is the collection of morphisms $h : D \to C$ which factor as $h = f_i \circ g$ for some $f_i \in \mathcal{F}$.

(3) If $\mathcal{S}$ is a sieve over C and $f_i : D_i \to C$ is a morphism in $\mathcal{C}$, then the restriction $\mathcal{S}_{D_i}$ of $\mathcal{S}$ to D_i is the sieve over D_i generated by the maps $g : D \to D_i$ such that $f_i \circ g \in \mathcal{S}$.

Definition 1.3. Let $\mathcal{C}$ be a 1-category. A *Grothendieck topology on $\mathcal{C}$* consists of for each $C \in \mathcal{C}$ a family $\mathcal{S}(C)$ of sieves over C which we call *covering sieves* such that:

(1) The sieve $\{\mathbf{1}_C\}$ is a covering sieve of C.

(2) If $\mathcal{S}$ is a covering sieve of C and $f : D \to C$ is a morphism in $\mathcal{C}$, then $\mathcal{S}_D$ is a covering sieve of D.

[3] This is due to the fact that there is a bijection between Grothendieck topologies on an ∞-category $\mathcal{C}$ and Grothendieck topologies on $h\mathcal{C}$. Additionally, for a 1-category one has a canonical equivalence $N(\mathcal{C}_{/C}) \simeq N(\mathcal{C})_{/C}$. Since we will primarily be interested in the nerve of a 1-category this has no effect. See HTT Remark 6.2.2.3 for details.

(3) Let $\mathcal{S}$ be a covering sieve of C. If $\mathcal{T}$ is a covering sieve of C such that for each $f : D \to C$ in $\mathcal{S}$ the restriction $\mathcal{T}_D$ is a covering sieve of D, then $\mathcal{T}$ is a covering sieve of C.

We refer to a category $\mathcal{C}$ with a Grothendieck topology τ as a site denoted $\mathcal{C}_\tau$ and where $|\mathcal{C}_\tau| = \mathcal{C}$ denotes the underlying category.

Observe that if for each $C \in \mathcal{C}$ we have a family $\mathcal{S}_0(C)$ of sieves over C, then there is a minimal Grothendieck topology $C \mapsto \mathcal{S}(C)$ with $S_0(C) \subset \mathcal{S}(C)$ for all C. We call this the *Grothendieck topology generated by* $\mathcal{S}_0(C)$.

Given a Grothendieck pretopology τ on $\mathcal{C}$, then we may obtain a Grothendieck topology on $\mathcal{C}$ in the following way. Since τ is a Grothendieck pretopology, then for each $C \in \mathcal{C}$ there is a covering family $\{f_i : U_i \to C\} \in \mathrm{Cov}_\tau(C)$. Thus, we may consider the sieve, $\mathcal{S}_I$, generated by the covering family $\{f_i : U_i \to C\}$. Then taking the collection of all sieves generates a Grothendieck topology on $\mathcal{C}$.

Remark 1.1. Let τ be a Grothendieck topology on a category $\mathcal{C}$. Given a full subcategory $\mathcal{C}_0$ of $\mathcal{C}$, then it will be important to consider a Grothendieck topology on $\mathcal{C}_0$ induced by τ. This is done in the following way.

Let $\mathcal{C}_0$ be a full subcategory of $\mathcal{C}$ such that if $X \in \mathcal{C}_0$, $g : Y \to X$ is a morphism in $\mathcal{C}_0$, and $\{f_i : U_i \to X\}_{i \in I}$ is in $\mathrm{Cov}_\tau(X)$, then each U_i is in $\mathcal{C}_0$ and each fiber product $Y \times_X U_i$ is in $\mathcal{C}_0$. Then we obtain a Grothendieck topology $\tau_{\mathcal{C}_0}$ on $\mathcal{C}_0$ by restricting τ to $\mathcal{C}_0$, that is, setting

$$\mathrm{Cov}_{\tau_0}(X) := \mathrm{Cov}_\tau(X)$$

for $X \in \mathcal{C}_0$.

The case we will primarily be working with where this is important is where we have $\mathbf{Sm}^{ft}(S) := \mathbf{Sm}^{ft}_{/S}$ the full subcategory of $\mathbf{Sch}_{/S}$ consisting of smooth S-schemes of finite type. The Zariski, etale, and Nisnevich topologies will, then restrict to Grothendieck topologies on $\mathbf{Sm}^{ft}(S)$ giving sites $\mathbf{Sm}^{ft}(S)_{Zar}$, $\mathbf{Sm}^{ft}(S)_{et}$, and $\mathbf{Sm}^{ft}(S)_{Nis}$.

Example 1.1. The *indiscrete* topology on a category $\mathcal{C}$ is the topology *ind* with $\mathrm{Cov}_{ind}(X) = \{\{\mathbf{1}_X\}\}$.

Example 1.2. The canonical example is the following: let X be a topological space and $\mathcal{C} = \mathcal{U}(X)$ be the poset of open subsets of X. Then for $U \subset X$ open, $\{f_i : U_i \to U\}$ is in $\mathrm{Cov}(U)$ if $U = \cup_i U_i$.

A similar example is let **Top** be the category of topological spaces. Let $\mathrm{Cov}_{\mathbf{Top}}(U) := \mathrm{Cov}_U(U)$ gives a Grothendieck topology on **Top**. Note that for $U_i \subset U$ open and $f : V \to U$ continuous the fiber product $V \times_U U_i$ is just the open subset $f^{-1}(U_i) \subset V$. In particular, this implies for $\{f_i : U_i \to U\} \in \mathrm{Cov}(U)$, the fiber product $U_i \times_U U_j$ is just the intersection $U_i \cap U_j$.

Example 1.3. Let X be a scheme and $|X|$ denote the underlying topological space. Taking $\mathcal{U}(|X|)$ as in example 1.2 with the same covering families, that is, $\{f_i : U_i \to U\}$ is in $\mathrm{Cov}(U)$

if $U = \cup_i U_i$, then this gives the Zariski topology on X. We let X_{Zar} denote the site with underlying category $\mathcal{U}(X)$. Again as in example 1.2 we may define a Grothendieck topology on **Sch**, the category of schemes over a scheme X, denoted $\mathbf{Sch}_{Zar}$

Example 1.4. Let X be a scheme. The etale site X_{et} is the site with underlying category $\mathbf{Sch}^{ft,et}_{/X}$ consisting of $f : U \to X$ such that f is an finite type etale morphism. For a given U a covering family $\{f_i : U_i \to U\}$ is a set of finite type etale morphisms such that $\coprod_i |U_i| \to |U|$ is surjective. Again we may define a site $\mathbf{Sch}_{et}$ on the category **Sch** of schemes.

Presheaves and Sheaves on a Site

Now given a site $\mathcal{C}_\tau$ we may consider $\mathcal{A}$-valued presheaves on $\mathcal{C}_\tau$ by defining $\mathrm{PSh}(\mathcal{C}_\tau) := \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{A})$. To define $\mathcal{A}$-valued sheaves requires us to be able to state the sheaf axiom which requires $\mathcal{A}$ to admit products. Recall an equalizer is the limit of a diagram

$$\bullet \rightrightarrows \bullet$$

Concretely, the equalizer of two parallel morphisms $f, g : A \to B$ is an object E and a morphism $e : E \to A$ which is universal such that $f \circ e = g \circ e$. That is, given any $z : Z \to A$ such that $f \circ z = g \circ z$, then there exists a unique morphism $u : Z \to E$ with $e \circ u = z$ as in the diagram

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ Z \arrow[u, dashed, "u"] \arrow[ur, "z"'] & & \end{tikzcd}$$

If $e : E \to A$ is the equalizer of f and g we will say that the sequence

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

is exact. Note that if $\mathcal{A}$ is an abelian category this is equivalent to requiring that the sequence

$$0 \longrightarrow E \xrightarrow{e} A \xrightarrow{f-g} B$$

is exact in the usual sense.

Hence, let $\mathcal{A}$ be a category with products and let F be an $\mathcal{A}$-valued presheaf on $\mathcal{C}_\tau$ and $\{f_i : U_i \to X\}_{i\in I} \in \mathrm{Cov}_\tau(X)$ for some $X \in \mathcal{C}$. Then we have "restriction" morphisms

$$\begin{aligned} f_i^* &: F(X) \to F(U_i) \\ \mathbf{pr}^*_{1,i,j} &: F(U_i) \to F(U_i \times_X U_j) \\ \mathbf{pr}^*_{2,i,j} &: F(U_j) \to F(U_i \times_X U_j) \end{aligned}$$

and taking products we obtain the diagram in $\mathcal{A}$

$$F(X) \xrightarrow{\prod_i f_i^*} \prod_i F(U_i) \underset{\prod_j \mathbf{pr}_{2,i,j}^*}{\overset{\prod_i \mathbf{pr}_{1,i,j}^*}{\rightrightarrows}} \prod_{i,j} F(U_i \times_X U_j) \tag{1.1}$$

Definition 1.4. Let $\mathcal{A}$ be a category which has arbitrary (small) products and let τ be a Grothendieck pretopology on a small category $\mathcal{C}$. An $\mathcal{A}$-valued presheaf F on $\mathcal{C}$ is a sheaf for τ if for each covering $\{f_i : U_i \to X\} \in \mathrm{Cov}_\tau$ the sequence in 1.1 is exact. Let $\mathrm{Shv}_\tau^{\mathcal{A}}(\mathcal{C})$ denote the category of $\mathcal{A}$-valued sheaves on $\mathcal{A}$ with respect to τ which is the full subcategory of $\mathrm{PSh}^{\mathcal{A}}(\mathcal{C})$ with all objects sheaves. If the context is clear we write $\mathrm{Shv}(\mathcal{C})$.

Simplicial Presheaves and Sheaves

In the case of motivic homotopy theory we will want to use presheaves and sheaves of simplicial sets (spaces, anima, Kan complexes). Let Δ denote the skeleton of the category of finite ordered sets and order preserving maps, that is, Δ has object $[n] = \{0 < 1 < \cdots < n\}$ and morphisms the order preserving maps. Then define the category of simplicial sets as presheaves of sets on Δ, that is,

$$\mathbf{sSet} := \mathrm{Fun}(\Delta^{op}, \mathbf{Set}).$$

For a category $\mathcal{C}$ we may make the identifications

$$\mathrm{PSh}(\mathcal{C}, \mathbf{sSet}) = \mathrm{PSh}(\mathcal{C} \times \Delta, \mathbf{Set}) = \mathrm{PSh}(\Delta, \mathrm{PSh}(\mathcal{C}, \mathbf{Set})).$$

so we define the category of simplicial presheaves as

$$\mathrm{PSh}(\mathcal{C}, \mathbf{sSet}) = \mathrm{Fun}(\mathcal{C}^{op}, \mathbf{sSet}) = \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Fun}(\Delta^{op}, \mathbf{Set})) = \mathrm{PSh}(\Delta, \mathrm{Fun}(\mathcal{C}^{op}, \mathbf{Set}))$$

Similarly, if τ is a Grothendieck topology on $\mathcal{C}$ we have an induced topology on $\mathcal{C} \times \Delta$ given by $\mathrm{Cov}_\tau(X \times [n]) := i_n(\mathrm{Cov}_\tau(X))$ for each $X \in \mathcal{C}$ where $i_n : \mathcal{C} \to \mathcal{C} \times \Delta$ is the inclusion functor $i_n(X) := X \times [n]$, $i_n(f) = f \times \mathbf{1}_{[n]}$. Thus, this defines the category of sheaves on $\mathcal{C} \times \Delta$ which may be identified with the category of functors $\Delta^{op} \to \mathrm{Shv}_\tau(\mathcal{C}, \mathbf{Set})$, that is, simplicial sheaves. In other words a simplicial presheaf $n \mapsto F_n$ is a simplicial sheaf if and only if each F_n is a sheaf. It follows that all the elementary results on presheaves and sheaves of sets extends to presheaves and sheaves of simplicial sets through these identifications.

1.2 The Nisnevich Topology and Descent

We now define the Nisnevich topology on a scheme X and discuss various properties.

Definition 1.5. Let $U \to X$ be a morphism of schemes and let x be a point in X. Then $f : U \to X$ is said to be *completely decomposed at* x if there exists $u \in U$ with $f(u) = x$ such that the induced map of residue fields $k(x) \to k(u)$ is an isomorphism.

One makes the observation that points u over which x is completely decomposed are in bijection with lifts in the diagram

$$\begin{tikzcd} & U \arrow[d] \\ \operatorname{Spec} k(x) \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

where the lift $\operatorname{Spec} k(x) \to U$ corresponding to u is induced by $\mathcal{O}_{U,u} \to k(u) \cong k(x)$[4]. This fact can then allows the immediate deduction of the following:

Lemma 1.1. *Given a pullback square*

$$\begin{tikzcd} V \arrow[r] \arrow[d, "\tilde{f}"'] & U \arrow[d, "f"] \\ Y \arrow[r, "g"] & X \end{tikzcd}$$

such that $f(y) = x$, then if $f : U \to X$ is completely decomposed at x, then $\tilde{f} : V \to Y$ is completely decomposed at y. That is, being completely decomposed is stable under pullback.

Definition 1.6. A collection, $\{f_i : U_i \to X\}_{i \in I}$, of morphisms of schemes is said to be a *Nisnevich covering* if

(1) I is finite

(2) each morphism f_i is etale and of finite type

(3) For each $x \in X$ there exists $i \in I$ such that f_i is completely decomposed at x.

Proposition 1.1. *$\{f_i : U_i \to X\}_{i \in I}$ is a Nisnevich covering if and only if the single induced map $\coprod_{i \in I} U_i \to X$ is a Nisnevich covering.*

Proposition 1.2. *Let S be a scheme and $\mathcal{C}$ a full subcategory of $\mathbf{Sch}_{/S}$ such that the pullback in $\mathbf{Sch}_{/S}$ of a diagram*

$$\begin{tikzcd} & U \arrow[d, "p"] \\ Y \arrow[r] & X \end{tikzcd}$$

in $\mathcal{C}$ where p is etale of finite type is in $\mathcal{C}$. Then the class of Nisnevich coverings form a basis for a Grothendieck topology on $\mathcal{C}$. We call the induced topology the Nisnevich *topology.*

[4]Using $k(u) \cong k(x)$ to get a lift is clear by considering the usual diagram relating the spectrum of the residue field of a point and the scheme however the other direction is not immediately clear to me.

From the definition once can see that any Zariski cover is a Nisnevich cover and any Nisnevich cover will be an etale cover. Hence, it follows that the Zariski topology is coarser than the Nisenvich topology is coarser then the etale topology. Recall that for a scheme X, then the representable presheaf $X := \mathrm{Hom}(-, X)$ is a sheaf in the etale topology. Hence, it follows that every representable presheaf in the Nisnevich topology is a sheaf, that is, for an appropriate site $\mathcal{C}_\tau$ with $\tau = Zar, Nis, et$ we have the sequence of fully faithful embeddings

$$\mathcal{C} \subset \mathrm{Shv}(\mathcal{C}_{et}) \subset \mathrm{Shv}(\mathcal{C}_{Nis}) \subset \mathrm{Shv}(\mathcal{C}_{Zar}) \subset \mathrm{PSh}(\mathcal{C}).$$

One additionally has the following proposition which provides an alternative characterization of the Nisnevich topology.

Proposition 1.3. *Let $U \to X$ be an etale morphism and $x \in X$. Then the following are equivalent:*

(1) $U \to X$ is competely decomposed at x;

(2) The morphism $U \times_X \operatorname{Spec} \mathcal{O}^h_{X,x} \to \operatorname{Spec} \mathcal{O}^h_{X,x}$ has a section.

In some sense this says that the "points" in the Nisnevich topology are determined by Hensel local schemes. To prove the proposition one uses the following theorem from [Mil80].

Theorem 1.1. ([Mil80]) Theorem I.4.2(a), (b), (d) *Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let x be the closed point of $X = \operatorname{Spec} A$, then the following are equivalent:*

(a) A is Henselian

(b) any finite A-algebra B is a direct product of local rings $B = \prod_{i \in I} B_i$ where the B_i are necessarily isomorphic to the rings $B_{\mathfrak{m}_i}$ with the $\mathfrak{m}_i$ the maximal ideals of B.

(c) If $f : Y \to X$ is etale and there is a point $y \in Y$ such that $f(y) = x$ and $k(y) \cong k(x)$, then f has a section $s : X \to Y$.

Proof. (of Proposition 1.3) Assume (1) holds, then the morphism $U \times_X \operatorname{Spec} \mathcal{O}^h_{X,x} \to \operatorname{Spec} \mathcal{O}^h_{X,x}$ is etale by properties of Henselization and pullbacks and completely decomposed at the closed points of $\operatorname{Spec} \mathcal{O}^h_{X,x}$ (Why?). Thus, it follows by [Mil80] Theorem I.4.2(d) that there is a section so (2) holds.

For (2) $\implies$ (1) consider the extension of scalars using $\mathcal{O}^h_{X,x} \to k(x)$, then one see that $U \times_X \operatorname{Spec} k(x) \to \operatorname{Spec} k(x)$ has a section which is equivalent to a lift in (1.2). $\square$

Definition 1.7. A commutative square

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & X \end{tikzcd}$$

is said to be an *elementary distinguished square* if

(1) the square is a pullback;

(2) $i : U \overset{i}{\hookrightarrow} X$ is an open immersion;

(3) $p : V \to X$ is etale;

(4) the canonical projection $p^{-1}(Z) \to Z$ is an isomorphism where $Z := X \setminus U$ and $p^{-1}(Z)$ have the reduced structure of closed subschemes.

Moreover, one can see that an elementary distinguished square is in fact a pushout.

An important fact is that $\{i : U \hookrightarrow X, p : V \to X\}$ is a Nisenvich covering and we will see later (Theorem 1.2) that they in fact for a basis for the Nisnevich topology.

Lemma 1.2. *Suppose $p : V \to X$ is etale, $Z \subseteq X$ is a closed subset, and Z and $p^{-1}(Z)$ are given the reduced structures. Then the square*

$$\begin{tikzcd} p^{-1}(Z) \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ Z \arrow[r, hook] & X \end{tikzcd}$$

is a pullback.

Proof. Indeed the pullback $V \times_X Z \to V$ is a closed immersion with underlying closed subset $p^{-1}(Z)$. Additionally, $V \times_X Z$ is reduced since an etale scheme over a reduced scheme is reduced (ref?). Hence, the uniqueness of reduced subschemes on a given closed subset (ref?) ensures that the map $p^{-1}(Z) \to V \times_X Z$ induced by the usual pullback square is an isomorphism. □

Proposition 1.4. *Let S be a scheme and $\mathcal{C}$ a full subcategory of $\mathbf{Sch}_{/S}$, then the elementary distinguished squares (1.7) are pushouts in $\mathrm{Shv}(\mathcal{C}_{Nis})$.*

Proof. □

Corollary 1.1. *Let $\mathcal{C}$ be a full subcategory of $\mathbf{Sch}_{/S}$. Given an elementary distinguished square in $\mathcal{C}$ and if F is a sheaf of abelian groups on $\mathcal{C}_{Nis}$, then there is a natural associated long exact Mayer-Vietoris sequence*

Proof. □

Lemma 1.3. *Let $p : U \to X$ be etale and U, X Noetherian schemes. If p is completely decomposed at every generic point of X, then $U \to X$ has a rational section. That is, there exists a dense open subset $X' \subset X$ such that $p : p^{-1}(X') \to X'$ has a section.*

Proof. □

Proposition 1.5. *Let $p : U \to X$ be a Nisnevich covering with U and X Noetherian. Then there is a finite filtration*

$$\emptyset = Z_n \subseteq Z_{n-1} \supseteq \cdots \subseteq Z_0 = X$$

of X by closed subsets such that for each $0 \leq i \leq n$ the map $^{-1}(Z_i \setminus Z_{i+1}) \to Z_i \setminus Z_{i+1}$ has a section where for $i \geq 1$ Z_i and $p^{-1}(Z_i)$ are given the reduced structures.

Theorem 1.2. *Let S be a scheme and $\mathcal{C}$ a full subcategory of $\mathbf{Sch}_{/S}$ consisting of Noetherian schemes. Let $\mathcal{A}$ be any category (should maybe be a presheaf of sets?), then a presheaf $F \in \mathrm{PSh}(\mathcal{C}, \mathcal{A})$ is a sheaf in the Nisnevich topology on $\mathcal{C}$ if and only if*

(1) $F(\emptyset)$ is terminal in $\mathcal{A}$;

(2) for any $X \in \mathcal{C}$ and any elementary distinguished square ((1.7)) the induced square

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(U) \arrow[d, "F(i)"] \\ F(V) \arrow[r, "F(p)"] & F(U \times_X V) \end{tikzcd}$$

is a pullback.

Proof. □

1.3 The Motivic Homotopy Category

We now have the necessary knowledge about the Nisnevich topology to construct the motivic homotopy category. The main goal of the construction of the motivic homotopy category and its stable variant is to allow the general methods of abstract homotopy theory to be applied to the algebro-geometric world of schemes. The issue of course with simply applying the usual tools of homotopy theory and algebraic topology to the underlying topological space of a scheme is that this ignores a great deal of the structure which is part of a scheme. Additionally, the "natural" topology, the Zariski topology, on a scheme is incredibly coarse and ignores the "true" topological nature of many of the objects.

Fix S a Noetherian scheme and let $\mathbf{Sm}^{ft}(S)$ denote the category of smooth separated schemes of finite type over S. The general idea to be able to do homotopy theory with schemes is that we should think of the affine line $\mathbb{A}^1$ as an interval. However, many of the usual constructions of homotopy theory rely on the existence of (co)limits which is not true in the case of schemes. In order to deal with this one passes to the formal completion of the appropriate category (i.e. presheaves of spaces). This is however still not quite right as we would like $\mathbb{A}^1$ to be contractible as well any scheme X which is a union of two open subschemes U and V should still union to give X in the new category.

We denote by $\mathcal{H}(S)$ what will be the ∞-category underlying the $\mathbb{A}^1$-model category constructed by Morel and Voevodsky.

The construction proceeds as follows. Fix a Noetherian scheme and let $\mathbf{Sm}^{ft}(S)$ denote the category of smooth separated schemes of finite type over S. Consider $\mathbf{Sm}^{ft}(S)$ as an ∞-category by $\mathcal{C} := N(\mathbf{Sm}^{ft}(S))$, then with the Nisnevich topology it becomes an ∞-site (HTT Definition 6.2.2.1). As we have seen the families of morphisms $\{V \xrightarrow{i} X, U \xrightarrow{p} X\}$ form a basis for the Nisnevich topology (Theorem 1.2).

Let $\mathcal{S}$ denote the ∞-category of spaces (homotopy types, Kan complexes, anima etc.). For a simplicial set S we let $\mathrm{PSh}(S) := \mathrm{Fun}(S^{op}, \mathcal{S})$ denote the ∞-category of presheaves of spaces on the simplicial set S. To obtain all (co)limits we pass to $\mathrm{PSh}(\mathcal{C})$ which has all (small) (co)limits (HTT Corollary 5.1.2.4)[5].

Recall the ∞-categorical Yoneda embedding (HTT 5.1.3) ensures that we have a fully faithful map of ∞-categories $j : \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ and we identify $X \in \mathcal{C}$ with $j(X)$ as usual. By Theorem 1.2 we get that $F \in \mathrm{PSh}(\mathcal{C})$ is a sheaf if and only if it maps Nisnevich squares to pullback squares. It follows that since elementary distinguished squares are pushouts[6] (Definition 1.7), then every representable presheaf $j(X)$ is a sheaf. Write $\mathrm{Shv}_{Nis}(\mathcal{C}) \subseteq \mathrm{PSh}(\mathcal{C})$ for the ∞-category of sheaves in the Nisnevich topology on $\mathcal{C}$. This has a left adjoint a_{Nis} sheafification which is an exact functor[7] (HTT Lemma 6.2.2.7) and $\mathrm{Shv}_{Nis}(\mathcal{C})$ is an ∞-topos[8]. In particular, $\mathrm{Shv}_{Nis}(\mathcal{C})$ is a presentable localization of a presentable ∞-category $\mathrm{PSh}(\mathcal{C})$.

The next step is to consider the hypercompletion $\mathrm{Shv}_{Nis}(\mathcal{C})^\wedge$ of the ∞-topos $\mathrm{Shv}_{Nis}(\mathcal{C})$. Recall the hypercompletion $\mathcal{X}^\wedge$ of an ∞-topos $\mathcal{X}$ is the left exact localization of $\mathcal{X}$ at the ∞-connective morphisms, that is, those morphisms which satisfy a form of Whitehead's theorem in the ∞-topos $\mathcal{X}$ (HTT pg. 662-663). This implies that $\mathcal{X}^\wedge$ is also an ∞-topos. Equivalently, this is the localization of $\mathrm{PSh}(\mathcal{C})$ spanned by objects which are local with respect to the class of Nisnevich hypercovers.

The final step is to recover $\mathbb{A}^1$-invariance, that is, we want to restrict to sheaves F such that $F(X) \to F(X \times \mathbb{A}^1)$ is an equivalence. This is accomplished by taking the localization of $\mathrm{Shv}_{Nis}(\mathcal{C})^\wedge$ with respect to the class of all projection maps $\{X \times \mathbb{A}^1 \to X\}_{X \in \mathcal{C}}$. Write $\mathcal{H}(S)$ for the localization, then from all the general previous nonsense $\mathcal{H}(S)$ is presentable with respect to the intermediate universe and very big and comes with a universal property.

Theorem 1.3. ([Rob12] Theorem 5.2) *Let $\mathbf{Sm}^{ft}(S)$ be the category of smooth schemes of finite type over a Noetherian scheme S and let $L : N(\mathbf{Sm}^{ft}(S)) \to \mathcal{H}(S)$ denote the composition of*

[5] HTT Proposition 4.2.4.4 ensures that this category may be identified with the underlying ∞-category of the model category of simplicial presheaves on $\mathbf{Sm}^{ft}(S)$ which is the category used in [MV99]. So the homotopy theory we end up with will be the same.

[6] This fact should follow from gluing pullback squares.

[7] In particular, is a topological localization at the collection of all monomorphisms $i : U \to j(C)$ corresponding to covering sieves on $C \in \mathcal{C}$ which implies $\mathrm{Shv}_{Nis}(\mathcal{C})$ is an ∞-topos (HTT Definition 6.1.0.4).

[8] Recall, we may characterize an ∞-topos as an ∞-category $\mathcal{C}$ such that $\mathcal{C}$ is presentable, colimits are universal, coporducts are disjoint, every groupoid object is effective.

localizations

$$N(\mathbf{Sm}^{ft}(S)) \to \mathrm{PSh}(N(\mathbf{Sm}^{ft}(S))) \to \mathrm{Shv}_{Nis}(\mathbf{Sm}^{ft}(S)) \to \mathrm{Shv}(\mathbf{Sm}^{ft}(S))^{\wedge} \to \mathcal{H}(S).$$

Then for any ∞-category $\mathcal{D}$ with all colimits the map induced by composition with L

$$\mathrm{Fun}^L(\mathcal{H}(S), \mathcal{D}) \to \mathrm{Fun}(N(\mathbf{Sm}^{ft}(S)), \mathcal{D})$$

is fully faithful and has essential image the full subcategory of $\mathrm{Fun}(N(\mathbf{Sm}^{ft})(S), \mathcal{D})$ *spanned by those functors satisfying Nisnevich descent (i.e. Theorem 1.2) and $\mathbb{A}^1$-invariance (i.e. $F(X \times \mathbb{A}^1) \to F(X)$ is an equivalence) where Fun^L is the full subcategory of colimit preserving functors.*

Informally, what the above says is that every colimit preserving functor $N(\mathbf{Sm}^{ft}(S)) \to \mathcal{D}$ which satisfies

- $\mathbb{A}^1$-invariance: $F(X \times \mathbb{A}^1) \to F(X)$ and
- Nisnevich descent: send elementary distinguished squares to pushout squares

factors uniquely through $\mathcal{H}(S)$ which is a presentable ∞-category which we call the unstable motivic ∞-category.

1.4 The Stable Motivic Homotopy Category

We now consider how to construct the stable motivic ∞-category. First, $\mathcal{H}(S)$ is presentable so it has a terminal object $*$ and $\mathcal{H}(S)_*$ is presentable.

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