Motivic Homotopy Theory

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Introduction

Notes from the Spring 2021 GeoTop junior seminar at the University of Copenhagen. The course primarily follows the Talbot seminar.

Notation and Conventions

Since we are primarily working in the world of algebraic geometry, that is, with schemes we say that a topological space X *quasi-compact* if every open cover has a finite subcover and reserve the term *compact* for the case when X is quasi-compact and Hausdorff.

0 Background from Algebraic Geometry

Definition 0.1. Let $f : X \to Y$ be a map of topological spaces, then f is *quasi-compact* if $f^{-1}(V)$ is quasi-compact for every quasi-compact open $V \subset Y$.

Lemma 0.1. (Tag 01K2) *Let $f : X \to S$ be a morphism of schemes, then the following are equivalent:*

(1) $f : X \to S$ is quasi-compact,

(2) the inverse image of every affine open is quasi-compact,

(3) there exists an affine open cover $S = \cup_{i \in I} U_i$ such that $f^{-1}(U_i)$ is quasi-compact for all i.

In topology one has the theorem that a space X is Hausdorff if and only if the diagonal

$$\Delta : X \to X \times X$$
$$x \mapsto (x, x)$$

is closed. In algebraic geometry given $f : X \to S$ we have a diagonal map obtained by the fiber product $\Delta_{X/S} : X \to X \times_S X$. That is, $\Delta_{X/S}$ is the unique morphism of schemes such that $\mathbf{pr}_1 \circ \Delta_{X/S} = \mathbf{1}_X$ and $\mathbf{pr}_2 \circ \Delta_{X/S} = \mathbf{1}_X$. One can show that $\Delta_{X/S}$ is an immersion (Tag 01KH).

Definition 0.2. Let $f : X \to S$ be a morphism of schemes.

(1) f is said to be separated if the diagonal $\Delta_{X/S}$ along f is a closed immersion.

(2) f is said to be quasi-separated if the diagonal $\Delta_{X/S}$ is a quasi-compact morphism.

(3) A scheme Y is separated if the morphism $Y \to \operatorname{Spec} \mathbb{Z}$ is separated.

(4) A scheme Y is quasi-separated if the morphism $Y \to \operatorname{Spec} \mathbb{Z}$ is quasi-separated.

Recall that a map of rings $R \to A$ is said to be of finite type if A is isomorphic to a quotient of $R[x_1, \ldots, x_n]$ as an R-algebra, that is, if A is a finitely generated R-algebra.

Definition 0.3. (Tag 01T0) Let $f : X \to S$ be a morphism of schemes, then

(1) f is said to be of *finite type at* $x \in X$ if there exists an affine open neighborhood $\operatorname{Spec} A = U \subset X$ of x and an affine open $\operatorname{Spec} R = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is of finite type.

(2) f is *locally of finite type* if f is of finite type for all $x \in X$.

(3) f is of *finite type* it is locally of finite type and quasi-compact.

Definition 0.4. Let X be a scheme, then X is *Noetherian* if it admits a finite cover by affine Schemes, $\operatorname{Spec} A_i$, where each A_i is a Noetherian ring.

Lemma 0.2. *(Lemma 01T6) Let $f : X \to S$ be a morphism. If S is (locally) Noetherian and f (locally) of finite type then X is (locally) Noetherian.*

Let $f : X \to S$ be a morphism of schemes, then we refer to Tag 01V4 for a general definition of smoothness. However, since we will be working with $f : X \to S$ of finite type with S a Noetherian scheme we can instead refer to Tag 02HW for definition of Smoothness. For definitions of etale morphisms see Tag 02GH and Tag 0257 or Chapter 024J

Definition 0.5. Let X be a scheme and x a point in X. Let $U = \operatorname{Spec} A$ be an affine neighborhood of x. Let P_x be the prime ideal corresponding to x, then the localization A_{P_x} is a local ring with maximal ideal $P_x \cdot A_{P_x}$. The *residue field* of x is the field

$$k(x) := A_P/P \cdot A_P.$$

Note that there is an isomorphism $k(x) \cong \operatorname{Frac}(A/P)$ We call the point x K-rational for a certain field K if $K \cong k(x)$.

Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let $a \mapsto \overline{a}$ denote the map $A \to k$ and $f \mapsto \overline{f}$ denote the map $A[T] \to k[T]$. If A is a complete[1] discrete valuation ring, then Hensel's lemma states: If f is a monic polynomial with coefficients in A such that $\overline{f}$ factors as $\overline{f} = g_0 h_0$ with g_0 and h_0 monic and coprime[2], then f factors as $f = gh$ with g and h monic such that $\overline{g} = g_0$ and $\overline{h} = h_0$.

Definition 0.6. Let A be a local ring with maximal ideal $\mathfrak{m}$, then A is said to be Henselian if Hensel's lemma holds for A.

Given a ring a local ring $(A, \mathfrak{m}, k)$ there is a universal ring A^h called the Henselization of A (Tag 0BSK, Tag 07QL, Tag 03QD). That is, the assignment $A \mapsto A^h$ is functorial.

1 The Nisnevich Topology and Universal Properties of the Stable and Unstable Motivic Homotopy Categories

In this talk we introduce the Nisnevich topology for a scheme and discuss some of it's properties (including, hopefully, descent) as it is central to the definition of the Motivic homotopy category. Then we introduce the (un)stable Motivic homotopy categories and discuss/prove their universal properties following [Rob12] sections 5.1 and 5.2.

1.1 Preliminaries

We begin by discussing Grothendieck (pre)topologies. Although we attempt to work as much as possible in the setting of ∞-categories the ∞-categorical generalization of a Grothendieck topol-

[1] Any discrete valuation ring may be given a metric $|x - y| = 2^{-\nu(x-y)}$ and we may take the completion with respect to this metric. Alternatively, we can consider the completion of the ring considered as a topological ring since local rings may be given a natural topology

[2] i.e. the ideal $(g_0, h_0) = (1)$ generates the whole ring

ogy is in some sense no generalization at all[3] so for ease of exposition we discuss Grothendieck topologies purely in the setting of 1-categories. Finally, note that there are occasionally set theoretic issues we can generally be overcome using the general nonsense of universes so again in the interest of exposition we ignore any of these issues.

Definition 1.1. Let $\mathcal{C}$ be a 1-category. A *Grothendieck pretopology* τ *on* $\mathcal{C}$ consists of for each $C \in \mathcal{C}$ a set $\mathrm{Cov}_\tau(C)$ of *covering families* where a covering is a set of morphisms $\{f_i : U_i \to C\}_{i \in I}$ in $\mathcal{C}$ such that

(1) $\{\mathbf{1}_C\}$ is in $\mathrm{Cov}_\tau(C)$ for all $C \in \mathcal{C}$;

(2) for $\{f_i : U_i \to C\} \in \mathrm{Cov}_\tau(C)$ and $g : D \to C$ a morphism in $\mathcal{C}$ the fiber product $U_i \times_C D$ exists and the natural projection $\{\mathbf{pr}_2 : U_i \times_C D \to D\}$ is in $\mathrm{Cov}_\tau(D)$;

(3) if $\{f_j : U_j \to C\}_{j \in J}$ is in $\mathrm{Cov}_\tau(C)$ and $\{g_{ij} : V_{ij} \to U_j\} \in \mathrm{Cov}_\tau(U_j)$ for all j, then $\{f_j \circ g_{ij} : V_{ij} \to C\} \in \mathrm{Cov}_\tau(C)$.

Definition 1.2. (1) Let $\mathcal{C}$ be a 1-category and $C \in \mathcal{C}$. A *sieve on* C is a subfunctor of $\mathbf{y}_C := \mathrm{Hom}_{\mathcal{C}}(-, C)$, that is, a set of objects of $\mathcal{C}_{/C}$ such that for each $f : D \to C$ in $\mathcal{S}$ and each morphism $g_i : D_i \to D$ in $\mathcal{C}$ the composition $f \circ g_i : D_i \to D$ is in $\mathcal{S}$.

(2) Let $\mathcal{F} := \{f_i : D_i \to X\}$ be a collection of morphisms in $\mathcal{C}$. The *sieve generated by* $\mathcal{F}$ is the collection of morphisms $h : D \to C$ which factor as $h = f_i \circ g$ for some $f_i \in \mathcal{F}$.

(3) If $\mathcal{S}$ is a sieve over C and $f_i : D_i \to C$ is a morphism in $\mathcal{C}$, then the restriction $\mathcal{S}_{D_i}$ of $\mathcal{S}$ to D_i is the sieve over D_i generated by the maps $g : D \to D_i$ such that $f_i \circ g \in \mathcal{S}$.

Definition 1.3. Let $\mathcal{C}$ be a 1-category. A *Grothendieck topology on* $\mathcal{C}$ consists of for each $C \in \mathcal{C}$ a family $\mathcal{S}(C)$ of sieves over C which we call *covering sieves* such that:

1. The sieve $\{\mathbf{1}_C\}$ is a covering sieve of C.
2. If $\mathcal{S}$ is a covering sieve of C and $f : D \to C$ is a morphism in $\mathcal{C}$, then $\mathcal{S}_D$ is a covering sieve of D.
3. Let $\mathcal{S}$ be a covering sieve of C. If $\mathcal{T}$ is a covering sieve of C such that for each $f : D \to C$ in $\mathcal{S}$ the restriction $\mathcal{T}_D$ is a covering sieve of D, then $\mathcal{T}$ is a covering sieve of C.

We refer to a category $\mathcal{C}$ with a Grothendieck topology τ as a site denoted $\mathcal{C}_\tau$ and where $|\mathcal{C}_\tau| = \mathcal{C}$ denotes the underlying category.

[3] This is due to the fact that there is a bijection between Grothendieck topologies on an ∞-category $\mathcal{C}$ and Grothendieck topologies on $h\mathcal{C}$. Additionally, for a 1-category one has a canonical equivalence $N(\mathcal{C}_{/C}) \simeq N(\mathcal{C})_{/C}$. Since we will primarily be interested in the nerve of a 1-category this has no effect. See HTT Remark 6.2.2.3 for details.

Observe that if for each $C \in \mathcal{C}$ we have a family $\mathcal{S}_0(C)$ of sieves over C, then there is a minimal Grothendieck topology $C \mapsto \mathcal{S}(C)$ with $S_0(C) \subset \mathcal{S}(C)$ for all C. We call this the *Grothendieck topology generated by* $\mathcal{S}_0(C)$.

Given a Grothendieck pretopology τ on $\mathcal{C}$, then we may obtain a Grothendieck topology on $\mathcal{C}$ in the following way. Since τ is a Grothendieck pretopology, then for each $C \in \mathcal{C}$ there is a covering family $\{f_i : U_i \to C\} \in \mathrm{Cov}_\tau(C)$. Thus, we may consider the sieve, $\mathcal{S}_I$, generated by the covering family $\{f_i : U_i \to C\}$. Then taking the collection of all sieves generates a Grothendieck topology on $\mathcal{C}$.

Remark 1.1. Let τ be a Grothendieck topology on a category $\mathcal{C}$. Given a full subcategory $\mathcal{C}_0$ of $\mathcal{C}$, then it will be important to consider a Grothendieck topology on $\mathcal{C}_0$ induced by τ. This is done in the following way.

Let $\mathcal{C}_0$ be a full subcategory of $\mathcal{C}$ such that if $X \in \mathcal{C}_0$, $g : Y \to X$ is a morphism in $\mathcal{C}_0$, and $\{f_i : U_i \to X\}_{i\in I}$ is in $\mathrm{Cov}_\tau(X)$, then each U_i is in $\mathcal{C}_0$ and each fiber product $Y \times_X U_i$ is in $\mathcal{C}_0$. Then we obtain a Grothendieck topology $\tau_{\mathcal{C}_0}$ on $\mathcal{C}_0$ by restricting τ to $\mathcal{C}_0$, that is, setting

$$\mathrm{Cov}_{\tau_0}(X) := \mathrm{Cov}_\tau(X)$$

for $X \in \mathcal{C}_0$.

The case we will primarily be working with where this is important is where we have $\mathbf{Sm}^{ft}(S) := \mathbf{Sm}^{ft}_{/S}$ the full subcategory of $\mathbf{Sch}_{/S}$ consisting of smooth S-schemes of finite type. The Zariski, etale, and Nisnevich topologies will, then restrict to Grothendieck topologies on $\mathbf{Sm}^{ft}(S)$ giving sites $\mathbf{Sm}^{ft}(S)_{Zar}$, $\mathbf{Sm}^{ft}(S)_{et}$, and $\mathbf{Sm}^{ft}(S)_{Nis}$.

Example 1.1. The *indiscrete* topology on a category $\mathcal{C}$ is the topology *ind* with $\mathrm{Cov}_{ind}(X) = \{\{\mathbf{1}_X\}\}$.

Example 1.2. The canonical example is the following: let X be a topological space and $\mathcal{C} = \mathcal{U}(X)$ be the poset of open subsets of X. Then for $U \subset X$ open, $\{f_i : U_i \to U\}$ is in $\mathrm{Cov}(U)$ if $U = \cup_i U_i$.

A similar example is let **Top** be the category of topological spaces. Let $\mathrm{Cov}_{\mathbf{Top}}(U) := \mathrm{Cov}_U(U)$ gives a Grothendieck topology on **Top**. Note that for $U_i \subset U$ open and $f : V \to U$ continuous the fiber product $V \times_U U_i$ is just the open subset $f^{-1}(U_i) \subset V$. In particular, this implies for $\{f_i : U_i \to U\} \in \mathrm{Cov}(U)$, the fiber product $U_i \times_U U_j$ is just the intersection $U_i \cap U_j$.

Example 1.3. Let X be a scheme and $|X|$ denote the underlying topological space. Taking $\mathcal{U}(|X|)$ as in example 1.2 with the same covering families, that is, $\{f_i : U_i \to U\}$ is in $\mathrm{Cov}(U)$ if $U = \cup_i U_i$, then this gives the Zariski topology on X. We let X_{Zar} denote the site with underlying category $\mathcal{U}(X)$. Again as in example 1.2 we may define a Grothendieck topology on **Sch**, the category of schemes over a scheme X, denoted $\mathbf{Sch}_{Zar}$

Example 1.4. Let X be a scheme. The etale site X_{et} is the site with underlying category $\mathbf{Sch}^{ft,et}_{/X}$ consisting of $f : U \to X$ such that f is an finite type etale morphism. For a given U a covering family $\{f_i : U_i \to U\}$ is a set of finite type etale morphisms such that $\coprod_i |U_i| \to |U|$ is surjective. Again we may define a site $\mathbf{Sch}_{et}$ on the category **Sch** of schemes.

Now given a site $\mathcal{C}_\tau$ we may consider $\mathring{A}$-valued presheaves on $\mathcal{C}_\tau$ by defining $\mathrm{PSh}(\mathcal{C}_\tau) := \mathrm{Fun}(\mathcal{C}^{op}, \mathring{A})$. To define $\mathring{A}$-valued sheaves requires us to be able to state the sheaf axiom which requires $\mathring{A}$ to admit products. Recall an equalizer is the limit of a diagram

$$\bullet \rightrightarrows \bullet$$

Concretely, the equalizer of two parallel morphisms $f, g : A \to B$ is an object E and a morphism $e : E \to A$ which is universal such that $f \circ e = g \circ e$. That is, given any $z : Z \to A$ such that $f \circ z = g \circ z$, then there exists a unique morphism $u : Z \to E$ with $e \circ u = z$ as in the diagram

$$\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
Z \arrow[u, dashed, "u"] \arrow[ur, "z"'] & &
\end{tikzcd}$$

If $e : E \to A$ is the equalizer of f and g we will say that the sequence

$$\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}$$

is exact. Note that if $\mathring{A}$ is an abelian category this is equivalent to requiring that the sequence

$$0 \longrightarrow E \xrightarrow{e} A \xrightarrow{f-g} B$$

is exact in the usual sense.

Hence, let $\mathring{A}$ be a category with products and let F be an $\mathring{A}$-valued presheaf on $\mathcal{C}_\tau$ and $\{f_i : U_i \to X\}_{i \in I} \in \mathrm{Cov}_\tau(X)$ for some $X \in \mathcal{C}$. Then we have "restriction" morphisms

$$\begin{aligned} f_i^* &: F(X) \to F(U_i) \\ \mathbf{pr}_{1,i,j}^* &: F(U_i) \to F(U_i \times_X U_j) \\ \mathbf{pr}_{2,i,j}^* &: F(U_j) \to F(U_i \times_X U_j) \end{aligned}$$

and taking products we obtain the diagram in $\mathring{A}$

$$\begin{tikzcd}
F(X) \arrow[r, "\prod_i f_i^*"] & \prod_i F(U_i) \arrow[r, shift left, "\prod_i \mathbf{pr}_{2,i,j}^*"] \arrow[r, shift right, "\prod_j \mathbf{pr}_{1,i,j}^*"'] & \prod_{i,j} F(U_i \times_X U_j)
\end{tikzcd} \tag{1.1}$$

Definition 1.4. Let $\mathring{A}$ be a category which has arbitrary (small) products and let τ be a Grothendieck pretopology on a small category $\mathcal{C}$. An $\mathring{A}$-valued presheaf F on $\mathcal{C}$ is a sheaf for τ if for each covering $\{f_i : U_i \to X\} \in \mathrm{Cov}_\tau$ the sequence in 1.1 is exact. Let $\mathrm{Shv}_\tau^{\mathring{A}}(\mathcal{C})$ denote the category of $\mathring{A}$-valued sheaves on $\mathring{A}$ with respect to τ which is the full subcategory of $\mathrm{PSh}^{\mathring{A}}(\mathcal{C})$ with all objects sheaves. If the context is clear we write $\mathrm{Shv}(\mathcal{C})$.

1.2 The Nisnevich Topology and Descent

We now define the Nisnevich topology on a scheme X and discuss various properties.

Definition 1.5. Let $U \to X$ be a morphism of schemes and let x be a point in X. Then $f : U \to X$ is said to be *completely decomposed at x* if there exists $u \in U$ with $f(u) = x$ such that the induced map of residue fields $k(x) \to k(u)$ is an isomorphism.

One makes the observation that points u over which x is completely decomposed are in bijection with lifts in the diagram

$$\begin{tikzcd} & U \arrow[d] \\ \operatorname{Spec} k(x) \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

where the lift $\operatorname{Spec} k(x) \to U$ corresponding to u is induced by $\mathcal{O}_{U,u} \to k(u) \cong k(x)$[4]. This fact can then allows the immediate deduction of the following:

Lemma 1.1. *Given a pullback square*

$$\begin{tikzcd} V \arrow[r] \arrow[d, "\tilde{f}"'] & U \arrow[d, "f"] \\ Y \arrow[r, "g"] & X \end{tikzcd}$$

such that $f(y) = x$, then if $f : U \to X$ is completely decomposed at x, then $\tilde{f} : V \to Y$ is completely decomposed at y. That is, being completely decomposed is stable under pullback.

Definition 1.6. A collection, $\{f_i : U_i \to X\}_{i \in I}$, of morphisms of schemes is said to be a *Nisnevich covering* if

(1) I is finite

(2) each morphism f_i is etale and of finite type

(3) For each $x \in X$ there exists $i \in I$ such that f_i is completely decomposed at x.

Proposition 1.1. *$\{f_i : U_i \to X\}_{i \in I}$ is a Nisnevich covering if and only if the single induced map $\coprod_{i \in I} U_i \to X$ is a Nisnevich covering.*

[4]Using $k(u) \cong k(x)$ to get a lift is clear by considering the usual diagram relating the spectrum of the residue field of a point and the scheme however the other direction is not immediately clear to me.

Proposition 1.2. *Let S be a scheme and $\mathcal{C}$ a full subcategory of $\mathbf{Sch}_{/S}$ such that the pullback in $\mathbf{Sch}_{/S}$ of a diagram*

$$\begin{array}{ccc} & & U \\ & & \downarrow{\scriptstyle p} \\ Y & \longrightarrow & X \end{array}$$

in $\mathcal{C}$ where p is etale of finite type is in $\mathcal{C}$. Then the class of Nisnevich coverings form a basis for a Grothendieck topology on $\mathcal{C}$. We call the induced topology the Nisnevich *topology.*

From the definition once can see that any Zariski cover is a Nisnevich cover and any Nisnevich cover will be an etale cover. Hence, it follows that the Zariski topology is coarser than the Nisenvich topology is coarser then the etale topology. Recall that for a scheme X, then the representable presheaf $X := \mathrm{Hom}(-, X)$ is a sheaf in the etale topology. Hence, it follows that every representable presheaf in the Nisnevich topology is a sheaf, that is, for an appropriate site $\mathcal{C}_\tau$ with $\tau = Zar, Nis, et$ we have the sequence of fully faithful embeddings

$$\mathcal{C} \subset \mathrm{Shv}(\mathcal{C}_{et}) \subset \mathrm{Shv}(\mathcal{C}_{Nis}) \subset \mathrm{Shv}(\mathcal{C}_{Zar}) \subset \mathrm{PSh}(\mathcal{C}).$$

One additionally has the following proposition which provides an alternative characterization of the Nisnevich topology.

Proposition 1.3. *Let $U \to X$ be an etale morphism and $x \in X$. Then the following are equivalent:*

(1) $U \to X$ is competely decomposed at x;

(2) The morphism $U \times_X \operatorname{Spec} \mathcal{O}^h_{X,x} \to \operatorname{Spec} \mathcal{O}^h_{X,x}$ has a section.

In some sense this says that the "points" in the Nisnevich topology are determined by Hensel local schemes. To prove the proposition one uses the following theorem from [Mil80].

Theorem 1.1. ([Mil80]) Theorem I.4.2(a), (b), (d) *Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let x be the closed point of $X = \operatorname{Spec} A$, then the following are equivalent:*

(a) A is Henselian

(b) any finite A-algebra B is a direct product of local rings $B = \prod_{i\in I} B_i$ where the B_i are necessarily isomorphic to the rings $B_{\mathfrak{m}_i}$ with the $\mathfrak{m}_i$ the maximal ideals of B.

(c) If $f : Y \to X$ is etale and there is a point $y \in Y$ such that $f(y) = x$ and $k(y) \cong k(x)$, then f has a section $s : X \to Y$.

Proof. (of Proposition 1.3) Assume (1) holds, then the morphism $U \times_X \operatorname{Spec} \mathcal{O}^h_{X,x} \to \operatorname{Spec} \mathcal{O}^h_{X,x}$ is etale by properties of Henselization and pullbacks and completely decomposed at the closed points of $\operatorname{Spec} \mathcal{O}^h_{X,x}$ (Why?). Thus, it follows by [Mil80] Theorem I.4.2(d) that there is a section so (2) holds.

For (2) $\implies$ (1) consider the extension of scalars using $\mathcal{O}^h_{X,x} \to k(x)$, then one see that $U \times_X \operatorname{Spec} k(x) \to \operatorname{Spec} k(x)$ has a section which is equivalent to a lift in (1.2). □

Definition 1.7. A commutative square

$$\begin{tikzcd} p^{-1}(U) = U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & X \end{tikzcd}$$

is said to be an *elementary distinguished square* if

(1) the square is a pullback;

(2) $i : U \overset{X}{\hookrightarrow}$ is an open immersion;

(3) $p : V \to X$ is etale;

(4) the canonical projection $p^{-1}(Z) \to Z$ is an isomorphism where $Z := X \setminus U$ and $p^{-1}(Z)$ have the reduced structure of closed subschemes.

Observe that a distinguished square determines a Nisnevich covering by taking $V \to X$ and $U \to X$.

1.3 The Motivic Homotopy Category

1.4 The Stable Motivic Homotopy Category

2 Algebraic K-theory and Representability in Motivic Homotopy Theory

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