



Homotopy theory of simplicial sheaves in completely decomposable topologies

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ABSTRACT

There are two approaches to the homotopy theory of simplicial (pre-)sheaves. One developed by Joyal and Jardine works for all sites but produces a model structure which is not finitely generated even in the case of sheaves on a Noetherian topological space. The other one developed by Brown and Gersten gives a nice model structure for sheaves on a Noetherian space of finite dimension but does not extend to all sites. In this paper we define a class of sites for which a generalized version of the Brown–Gersten approach works.

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1. Introduction

There are two substantially different approaches to the homotopy theory of simplicial (pre)sheaves. The first one, introduced by Joyal and developed in full detail by Jardine in [6,7], provides us with a proper simplicial model structure on the category of simplicial sheaves and presheaves and allows one to give definitions of all the important objects of homotopical algebra in the context of sheaves on all sites. The drawback of this approach, which is a necessary consequence of its generality, is that some of the important classes of objects and morphisms have a very abstract definition. In particular, the existence of fibrant objects cannot be proved without the use of large cardinals and transfinite arguments even in the case when the underlying site is reasonably small. Another approach was introduced in [1] for simplicial sheaves on Noetherian topological spaces of finite dimension. It is much more explicit and gives a finitely generated model structure on the category of simplicial sheaves but does not generalize to arbitrary sites.

The goal of this paper is to define a class of sites for which a generalized analog of the Brown–Gersten approach works. We prove that for such sites the class of weak equivalences of simplicial presheaves can be generated by an explicitly given set of generating weak equivalences.

In [11] we use this result in the case of the *cdh*-topology on schemes over a field to prove a comparison theorem for the motivic homotopy categories in the Nisnevich and the *cdh*-topologies.

Let C be a small category with an initial object 0 and P be a set of commutative squares in C . One defines the *cd*-topology t_P associated with P as the topology generated by coverings of two sorts

1. the empty covering of 0 ,
2. coverings of the form $\{A \rightarrow X, Y \rightarrow X\}$ where the morphisms $A \rightarrow X$ and $Y \rightarrow X$ are sides of an element of P of the form

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow \\ A & \longrightarrow & X \end{array}$$

(1)

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A fundamental example of a cd-topology is the canonical topology on the category of open subsets of a Noetherian topological space which is associated with the set of squares of the form

$$\begin{array}{ccc} U \cap V & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & U \cup V \end{array}$$

Let us consider (1) as a square in the category $\text{PreShv}(C)$ of presheaves of sets on C via the Yoneda embedding and further as a square in the category $\Delta^{op}\text{PreShv}(C)$ of simplicial presheaves on C . Let $p_Q : K_Q \rightarrow X$ be the canonical morphism from the simplicial homotopy push-out of $B \rightarrow Y$ and $B \rightarrow A$ (given by (16)) to X . Let G_P be the union of the set of morphisms of this form with the morphism $0' \rightarrow 0$ where $0'$ is the initial object in $\text{PreShv}(C)$ and 0 is the initial object in C . Elements of G_P are called generating weak equivalences defined by P .

Denote by W_{proj} the class of projective equivalences of simplicial presheaves on C i.e. morphisms $f : X \rightarrow Y$ such that the map of simplicial sets $X(U) \rightarrow Y(U)$ defined by f is a weak equivalence for each U in C . Together with projective fibrations i.e. morphisms $p : X \rightarrow Y$ which define Kan fibrations $X(U) \rightarrow Y(U)$ for all $U \in C$, they form a finitely generated simplicial closed model structure on $\Delta^{op}\text{PreShv}(C)$ which is called the projective c.m.s. The standard localization techniques of [2] and [4] apply in the context of this closed model structure and in particular one may consider the class $cl_l(G_P)$ of G_P -local equivalences in the sense of [2, Def. 3.1.3].

One of the main results of this paper (Proposition 3.8(3)) provides an explicit list of conditions on P which imply that a morphism of simplicial presheaves on C is a local equivalence with respect to the topology defined by P if and only if it belongs to $cl_l(G_P)$.

We distinguish three types of cd-structures – complete, bounded and regular. One of the properties of complete cd-structures is that any presheaf which takes distinguished squares to pull-back squares is a sheaf in the associated topology. One of the properties of regular cd-structures is that any sheaf in the associated topology takes distinguished squares to pull-back squares. Bounded cd-structures are intuitively the ones where every object has a finite dimension. In particular any object has finite cohomological dimension with respect to the associated topology.

For any complete bounded cd-structure we prove an analog of the Brown–Gersten theorem [1, Th. 1']. Using this result we show that for a complete, regular and bounded cd-structure the class of local equivalences with respect to the associated topology is generated by G_P . We also define in this case a closed model structure on the category of simplicial t_P -sheaves which is a general version of the closed model structure introduced in [1]. This closed model structure is always finitely generated.

In Section 5 we consider cd-structures on categories with fiber products. For any morphism $f : X \rightarrow Y$ in C denote by $\check{C}(f)$ the simplicial object with terms

$$\check{C}(f)_n = X_Y^{n+1}$$

and faces and degeneracy morphisms given by partial projections and diagonals respectively. The projections $X_Y^i \rightarrow Y$ define a morphism $\check{C}(f) \rightarrow Y$ which we denote by $\eta(f)$. We show that for a regular cd-structure such that a pull-back of a distinguished square is a distinguished square the class G_P -local equivalences coincides with the class of $\eta(\text{Cov}_f(t_P))$ -local equivalences where $\text{Cov}_f(t_P)$ is the class of morphisms in $\text{PreShv}(C)$, of the form $f = \coprod f_i$ for the finite t_P -coverings $\{f_i : X_i \rightarrow X\}$. This result is important for applications where it is easier to compute a functor on morphisms of the form $\eta(f)$ than on morphisms of the form p_Q (for example if the functor commutes with fiber products but not with coproducts).

2. Topologies defined by cd-structures

Definition 2.1. Let C be a category with an initial object. A cd-structure on C is a collection P of commutative squares of the form

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ A & \xrightarrow{e} & X \end{array} \quad (2)$$

such that if $Q \in P$ and Q' is isomorphic to Q then Q' is in P .

The squares of the collection P are called distinguished squares of P . One can combine different cd-structures and/or restrict them to subcategories considering only the squares which lie in the corresponding subcategory. Note also that if C is a category with a cd-structure P and X is an object of C then P defines a cd-structure on C/X .

We define the topology t_P associated with a cd-structure P as the smallest Grothendieck topology (see [9, III.2]) such that for a distinguished square of the form (2) the sieve (p, e) generated by the morphisms

$$\{p : Y \rightarrow X, e : A \rightarrow X\} \quad (3)$$

is a covering sieve and such that the empty sieve is a covering sieve of the initial object \emptyset . The last condition implies that for any t_P -sheaf F one has $F(\emptyset) = pt$. We define simple coverings as the ones which can be obtained by iterating elementary coverings of the form (3). More precisely one has:

Definition 2.2. The class S_P of simple coverings is the smallest class of families of morphisms of the form $\{U_i \rightarrow X\}_{i \in I}$ satisfying the following two conditions:

1. any isomorphism $\{f\}$ is in S_P
2. for a distinguished square Q of the form (2) and families $\{p_i : Y_i \rightarrow Y\}_{i \in I}, \{q_j : A_j \rightarrow A\}_{j \in J}$ in S_P the family $\{p \circ p_i, e \circ q_j\}_{i \in I, j \in J}$ is in S_P .

Definition 2.3. A cd-structure is called complete if any covering sieve of an object X which is not isomorphic to \emptyset contains a sieve generated by a simple covering.

Lemma 2.4. A cd-structure P on a category C is complete if and only if the following two conditions hold:

1. any morphism with values in \emptyset is an isomorphism
2. for any distinguished square of the form (2) and any morphism $f : X' \rightarrow X$ the sieve $f^*(e, p)$ contains the sieve generated by a simple covering.

Proof. The first condition is necessary because if there is a morphism $U \rightarrow 0$ then the empty sieve on U is the pull-back of a covering sieve and therefore a covering sieve. Since the empty sieve does not contain the sieve generated by any simple covering this contradicts the completeness assumption. The second condition is necessary for obvious reasons. To prove that these two conditions are sufficient we have to show that they imply that the class of sieves which contain the sieves generated by simple coverings satisfy the Grothendieck topology axioms (see e.g. [9, Def. 1 p.110]). The first and the third axioms hold for the class of simple coverings in any cd-structure. An inductive argument shows that the conditions of the lemma imply that the second, stability, axiom holds. \square

Lemma 2.4 immediately implies the following three lemmas.

Lemma 2.5. Let P be a cd-structure such that

1. any morphism with values in \emptyset is an isomorphism
2. for any distinguished square Q of the form (2) and any morphism $X' \rightarrow X$ the square $Q' = Q \times_X X'$ is defined and belongs to P .

Then P is complete.

Lemma 2.6. Let C be a category and P_1, P_2 two complete cd-structures on C . Then $P_1 \cup P_2$ is complete.

Lemma 2.7. Let C be a category and P a complete cd-structure on C . Then for any object U of C the cd-structure P/U on C/U defined by P is complete.

All simple coverings are finite and therefore the topology associated with any complete cd-structure is necessarily Noetherian i.e. any t_P covering has a finite refinement. In particular, we have the following compactness result. For an object X of C denote by $\rho(X)$ the t_P -sheaf associated with the presheaf represented by X .

Lemma 2.8. Let P be a complete cd-structure. Then for any U in C the sheaf $\rho(U)$ is a compact object of $\text{Shv}(C, t_P)$ i.e. for any filtered system of sheaves F_α one has

$$\text{Hom}(\rho(U), \text{colim}(F_\alpha)) = \text{colim} \text{Hom}(\rho(U), F_\alpha). \quad (4)$$

Proof. By definition of associated sheaf and the Yoneda Lemma the right-hand side of (4) is isomorphic to $\text{colim}(F_\alpha(U))$ and the left-hand side to $(\text{colim} F_\alpha)(U)$. Therefore, to prove the lemma it is enough to show that the colimit of the family F_α in the category of presheaves coincides with its colimit in the category of sheaves i.e. that the colimit in the category of presheaves is a sheaf. Since every covering has a finite refinement this follows from the fact that finite limits in the category of sets commute with filtered colimits. \square

Lemma 2.9. Let P be a complete cd-structure and F a presheaf on C such that $F(\emptyset) = *$ and for any distinguished square Q of the form (2) the square

$$F(Q) = \begin{pmatrix} F(X) & \longrightarrow & F(Y) \\ \downarrow & & \downarrow \\ F(A) & \longrightarrow & F(B) \end{pmatrix} \quad (5)$$

is pull-back. Then F is a sheaf in the associated topology.

Proof. Let us say that a section of F on a sieve $J = \{p_U : U \rightarrow X\}$ is a collection of sections $a_U \in F(U)$ satisfying the condition $F(g)(a_U) = a_V$ for any morphism $g : V \rightarrow U$ over X . A presheaf F is a sheaf if for any object X of C , any covering sieve $J = \{p_U : U \rightarrow X\}$ of X and any section (a_U) of F on J there exists a unique section $a_X \in F(X)$ such that $a_U = F(p_U)(a_X)$ for all $p_U \in J$. Observe first that our condition on F implies that for any simple covering $\{U_i \rightarrow X\}$ the map $F(X) \rightarrow \prod F(U_i)$ is injective. Together with the completeness condition it implies that if a_X exists it is unique i.e. F is a separated presheaf. The same condition implies that it is sufficient to prove the existence for sieves of the form (p_i) where $\{p_i : U_i \rightarrow X\}$ is a simple covering. Consider the class S_F of simple coverings such that for $\{p_i\} \in S_F$ the existence condition holds for any section (a_U) . Let us show that it coincides with S_p . It clearly contains isomorphisms. Let Q, p_i and q_j be as in Definition 2.2(2) and assume that $(p_i), (q_j) \in S_F$. Our section restricted to (p_i) and (q_j) defines two section $a_Y \in F(Y)$ and $a_A \in F(A)$. If $u : W \rightarrow Y, v : W \rightarrow A$ are morphisms such that $pu = ev$ then there exists a covering sieve J on W contained in both $u^*((p_i))$ and $v^*((q_j))$. The sections $F(u)(a_Y)$ and $F(v)(a_A)$ coincide with elements from J and since F is separated they coincide with $F(W)$. We conclude that a_Y and a_A coincide with B and thus our condition on F implies that there exists a_X such that $a_A = F(e)(a_X)$ and $a_Y = F(p)(a_X)$. \square

Definition 2.10. A cd-structure P is called regular if for any $Q \in P$ of the form (2) one has

1. Q is a pull-back square
2. e is a monomorphism
3. the morphism of sheaves

$$\Delta \coprod \rho(e_B)^2 : \rho(Y) \coprod \rho(B) \times_{\rho(A)} \rho(B) \rightarrow \rho(Y) \times_{\rho(X)} \rho(Y) \quad (6)$$

is surjective.

Lemma 2.11. Let P be a cd-structure such that for any distinguished square Q of the form (2) in P one has

1. Q is a pull-back square
2. e is a monomorphism
3. the objects $Y \times_X Y$ and $B \times_A B$ exist and the derived square

$$d(Q) = \left(\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow \\ B \times_A B & \longrightarrow & Y \times_X Y \end{array} \right) \quad (7)$$

where the vertical arrows are the diagonals, is distinguished.

Then P is a regular cd-structure.

Proof. The functor ρ commutes with fiber products and therefore the third condition of Definition 2.10 can be expressed by saying that the morphism

$$\rho(Y) \coprod \rho(B \times_A B) \rightarrow \rho(Y \times_X Y)$$

is surjective. For any covering $\{f_i : U_i \rightarrow U\}$ in a topology t the morphism $\coprod_i \rho(f_i)$ is a surjection of sheaves in t . Therefore, if the squares of the form (7) are distinguished then morphisms of the form (6) are surjections. \square

The following two lemmas are straightforward.

Lemma 2.12. Let C be a category and P_1, P_2 two regular cd-structures on C . Then $P_1 \cup P_2$ is regular.

Lemma 2.13. Let C be a category and P a regular cd-structure on C . Then for any object U of C the cd-structure P/U on C/U defined by P is regular.

Example 2.14. Define the toy cd-structure as follows. The category C is the category corresponding to the diagram

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow \\ A & \xrightarrow{e} & X \end{array} \quad (8)$$

together with an additional object \emptyset which is an initial object and such that any morphism with values in \emptyset is identity. The distinguished squares are the square (8) and the squares

$$\begin{array}{ccccccc} \emptyset & \longrightarrow & \emptyset & \longrightarrow & \emptyset & \longrightarrow & \emptyset \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & A & \longrightarrow & \emptyset & \longrightarrow & \emptyset \end{array} \quad (9)$$

One verifies easily that the toy cd-structure is complete and regular.

Proposition 2.15. For any regular cd-structure P , distinguished square Q of the form (2) and a sheaf F in the associated topology the square of sets $F(Q)$ is a pull-back square.

Proof. The map $F(X) \rightarrow F(A) \times F(Y)$ is a monomorphism since $\{Y \rightarrow X, A \rightarrow X\}$ is a covering and its image coincides with the equalizer of the maps

$$\text{Hom}\left(\rho(Y) \coprod \rho(A), F\right) \rightarrow \text{Hom}\left((\rho(Y) \coprod \rho(A)) \times_{\rho(X)} (\rho(Y) \coprod \rho(A)), F\right)$$

defined by the projections. Since (6) is an epimorphism and e is a monomorphism this equalizer coincides with the equalizer of the maps $F(Y) \times F(A) \rightarrow F(B)$ defined by the morphisms from B to A and Y . \square

Proposition 2.15 has the following immediate corollaries.

Corollary 2.16. For any regular cd-structure and any distinguished square Q of the form (2) the square

$$\rho(Q) = \begin{pmatrix} \rho(B) & \longrightarrow & \rho(Y) \\ \downarrow & & \downarrow \\ \rho(A) & \longrightarrow & \rho(X) \end{pmatrix} \quad (10)$$

is a push-out square.

Corollary 2.17. Let P be a complete regular cd-structure. Then a presheaf F is a sheaf in the associated topology if and only if $F(\emptyset) = \text{pt}$ and for any distinguished square Q of the form (2) the square (5) is pull-back.

For a sheaf F let $\mathbf{Z}(F)$ be the sheaf of abelian groups freely generated by F .

Lemma 2.18. Let P be a regular cd-structure and Q be a distinguished square of the form (2). Then sequence of sheaves of abelian groups

$$0 \rightarrow \mathbf{Z}(\rho(B)) \rightarrow \mathbf{Z}(\rho(A)) \oplus \mathbf{Z}(\rho(Y)) \rightarrow \mathbf{Z}(\rho(X)) \rightarrow 0 \quad (11)$$

is exact.

Proof. For any site the functor $F \mapsto \mathbf{Z}(F)$ takes colimits to colimits and monomorphisms to monomorphisms. Since P is regular, the morphism $B \rightarrow Y$ is a monomorphism and therefore the sequence (11) is exact in the first term. The quotient $\mathbf{Z}(\rho(A)) \oplus \mathbf{Z}(\rho(Y))/\mathbf{Z}(\rho(B))$ is the colimit of the diagram

$$\begin{array}{ccc} \mathbf{Z}(\rho(B)) & \longrightarrow & \mathbf{Z}(\rho(Y)) \\ \downarrow & & \\ \mathbf{Z}(\rho(A)) & & \end{array}$$

and since $\mathbf{Z}(-)$ is right exact, Corollary 2.16 implies that it is $\mathbf{Z}(\rho(X))$. \square

Lemma 2.19. Let P be a regular cd-structure and F a t_P -sheaf of abelian groups on C . Then there is a function which takes any distinguished square of the form (2) to a family of homomorphisms

$$\partial_Q : H^i(B, F) \rightarrow H^{i+1}(X, F)$$

and which satisfies the following conditions

1. for a map of squares $Q' \rightarrow Q$ the square

$$\begin{array}{ccc} H^i(B, F) & \xrightarrow{\partial_Q} & H^{i+1}(X, F) \\ \downarrow & & \downarrow \\ H^i(B', F) & \xrightarrow{\partial_{Q'}} & H^{i+1}(X', F) \end{array} \quad (12)$$

commutes

2. the sequence of abelian groups

$$H^i(X, F) \rightarrow H^i(A, F) \oplus H^i(Y, F) \rightarrow H^i(B, F) \rightarrow H^{i+1}(X, F) \quad (13)$$

is exact.

Proof. Our result follows from Lemma 2.18 and the fact that for any site T , sheaf F on T and an object X of T one has

$$H^n(X, F) = \text{Ext}^n(\mathbf{Z}(\rho(X)), F). \quad \square$$

Our next goal is to define dimension for objects of a category with a cd-structure and in particular introduce a class of cd-structures of (locally) finite dimension. We start with the following auxiliary definition.

Definition 2.20. A density structure on a category C with an initial object is a function which assigns to any object X a sequence $D_0(X), D_1(X), \dots$ of families of morphisms which satisfies the following conditions:

1. X is the codomain of elements of $D_i(X)$ for all i
2. $(\emptyset \rightarrow X) \in D_0(X)$ for all X
3. isomorphisms belong to D_i for all i
4. $D_{i+1} \subset D_i$
5. if $j : U \rightarrow V$ is in $D_i(V)$ and $j' : V \rightarrow X$ is in $D_i(X)$ then $j' \circ j : U \rightarrow X$ is in $D_i(X)$.

By abuse of notation we will write $U \in D_n(X)$ instead of $(U \rightarrow X) \in D_n(X)$. A density structure is said to be locally of finite dimension if for any X there exists n such that any element of $D_{n+1}(X)$ is an isomorphism. The smallest such n is called the dimension of X with respect to $D_*(-)$.

Definition 2.21. Let C be a category with a cd-structure P and a density structure $D_*(-)$. A distinguished square Q of the form (2) is called reducing (with respect to D_*) if for any $i \geq 0$, and any $B_0 \in D_i(B), A_0 \in D_{i+1}(A), Y_0 \in D_{i+1}(Y)$ there exist $X' \in D_{i+1}(X)$, a distinguished square

$$Q' = \begin{array}{ccc} B' & \longrightarrow & Y' \\ \downarrow & & \downarrow p \\ A' & \xrightarrow{e} & X' \end{array} \quad (14)$$

and a morphism $Q' \rightarrow Q$ which coincides with the morphism $X' \rightarrow X$ on the lower right corner and whose other respective components factor through B_0, Y_0, A_0 .

We say that a distinguished square Q' is a refinement of a distinguished square Q if there is a morphism $Q' \rightarrow Q$ which is the identity on the lower right corner.

Definition 2.22. A density structure $D_*(-)$ is said to be a reducing density structure for a cd-structure P if any distinguished square in P has a refinement which is reducing with respect to $D_*(-)$. A cd-structure is called bounded if there exists a reducing density structure of locally finite dimension for it.

Example 2.23. The idea of a density structure comes from the analysis of proofs in [1]. The main example of such a structure is as follows. Let T be a topological space and $Op(T)$ be the category of open subsets of T and their inclusions. If T is a Noetherian space then there is a well-known notion of a dimension for closed subsets of T . This notion allows one to introduce a density structure on $Op(T)$ by setting $D_i(V)$ to be the set of open embeddings $U \rightarrow V$ such that the codimension of $V \setminus U$ in U is at least i . More precisely, let us call a sequence of points x_0, \dots, x_d of T an increasing sequence of length d if $x_i \neq x_{i+1}$ and $x_i \in cl(\{x_{i+1}\})$ where $cl(\{x_{i+1}\})$ is the closure of the point x_{i+1} in T . Then $j : U \rightarrow V$ is in $D_i(V)$ if for any $z \in V - U$ there exists an increasing sequence $z = x_0, x_1, \dots, x_i$ of length i .

This density structure is locally of finite dimension if T has finite dimension (as a Noetherian space) and the dimension of T with respect to this density structure is its dimension in the usual sense.

The category $Op(T)$ carries a cd-structure P with the set of squares of the form

$$\begin{array}{ccc} U \cap V & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & U \cap V \end{array}$$

and one verifies easily that the density structure described above is a reducing density structure for this cd-structure with all the distinguished squares being reducing. A closely related density structure on the category of Noetherian schemes is considered in [11].

The class of reducing squares for a given cd-structure and a density structure is again a cd-structure which we call the associated reducing cd-structure. If the density structure is reducing the reducing cd-structure generates the same topology as the original one. In this case one cd-structure is complete if and only if the other is. The following two lemmas are straightforward.

Lemma 2.24. Let C be a category and P_1, P_2 two cd-structures on C bounded by the same density structure D . Then $P_1 \cup P_2$ is bounded by D .

Lemma 2.25. Let C be a category and P a cd-structure on C bounded by a density structure D . Then for any object U of C the cd-structure P/U on C/U is bounded by the density structure D/U .

Example 2.26. Let C be the category of Example 2.14. Define the density structure on C setting

$$\begin{aligned} D_n(\emptyset) &= Id \quad \text{for } n \geq 0 \\ D_0(A) &= \{Id, \emptyset\} \quad D_n(A) = \{Id\} \quad \text{for } n > 0 \\ D_0(B) &= \{Id, \emptyset\} \quad D_n(B) = \{Id\} \quad \text{for } n > 0 \\ D_0(Y) &= \{Id, \emptyset\} \quad D_n(Y) = \{Id\} \quad \text{for } n > 0 \\ D_0(X) &= \{Id, e, \emptyset\} \quad D_1(X) = \{Id, e\} \quad D_n(X) = \{Id\} \quad \text{for } n > 1 \end{aligned}$$

Every distinguished square of the toy cd-structure is reducing with respect to this density structure and in particular the density structure is reducing. Since the dimension of all objects with respect to D is ≤ 1 the toy cd-structure is bounded.

Theorem 2.27. Let P be a complete regular cd-structure bounded by a density structure D and X an object of C . Then for any t_p -sheaf of abelian groups F on C/X one has

$$H_{t_p}^n(X, F) = 0$$

for $n > \dim_D X$.

Proof. Replacing C by C/X we may assume that F is defined on C . By definition of dimension relative to a density structure it is sufficient to show that for any X , any n and any class $a \in H^n(X, F)$ there exists an element $j : U \rightarrow X$ of $D_n(X)$ such that $j^*(a) = 0$. We do it by induction on n . For $n = 0$ the statement follows from the fact that $\emptyset \rightarrow X$ is in $D_0(X)$ and for any t_p -sheaf F of abelian groups one has $F(\emptyset) = 0$.

Replacing P by the class of the reducing squares with respect to D we may assume that any square in P is reducing. Let a be an element in H^n and $n > 0$. Since the sheaves associated with the cohomology presheaves are zero there exists a t_p -covering $\{p_i : U_i \rightarrow X\}$ such that $p_i^*(a) = 0$ for all i . Since P is complete this covering has a simple refinement. Let S be the class of simple coverings such that if a is a class vanishing on an element of S then it vanishes on an element of D_n . It clearly contains isomorphisms. Thus to show that it coincides with the whole S_p it is enough to check that it satisfies the condition of Definition 2.2(2). Let Q be a square of the form (2) and $\{p_i : Y_i \rightarrow Y\}$, $\{q_j : A_j \rightarrow A\}$ be elements of S . Then there exist monomorphisms $Y_0 \rightarrow Y$ and $A_0 \rightarrow A$ in $D_n(Y)$ and $D_n(A)$ respectively such that a restricts to zero on A_0 and Y_0 . Setting $B_0 = B$ and using the definition of a reducing square we see that there is an element $X' \rightarrow X$ of $D_n(X)$, a distinguished square Q' based on X' and a morphism $Q' \rightarrow Q$, which coincides with the embedding $X' \rightarrow X$ on the lower right corner, such that the restriction a' of a to X' vanishes on A' and Y' . By Lemma 2.19 this implies that $a' = \partial_{Q'}(b')$ where $b' \in H^{n-1}(B')$. By induction there is an element $B_0 \rightarrow B$ in $D_{n-1}(B)$ such that b' vanishes on B_0 . Applying again the definition of a reducing square to Q' with respect to B_0 , $Y_0 = Y'$ and $A_0 = A'$ and using the naturality of homomorphisms ∂_Q we conclude that there is an element $X'' \rightarrow X'$ in $D_n(X')$ such that the restriction of a' to X'' is zero. By definition of a density structure the composition $X'' \rightarrow X$ is in $D_n(X)$ which finishes the proof. \square

3. Flasque simplicial presheaves and local equivalences

Definition 3.1. Let C be a category with a cd-structure P . A B.G.-functor on C with respect to P is a family of contravariant functors T_q , $q \geq 0$ from C to the category of pointed sets together with pointed maps $\partial_Q : T_{q+1}(B) \rightarrow T_q(X)$ given for all distinguished squares of the form (2) such that the following two conditions hold:

1. the morphisms ∂_Q are natural with respect to morphisms of distinguished squares
2. for any $q \geq 0$ the sequence of pointed sets

$$T_{q+1}(B) \rightarrow T_q(X) \rightarrow T_q(A) \times T_q(Y)$$

is exact.

The following theorem is an analog of [1, Th 1'].

Theorem 3.2. Let C be a category with a bounded complete cd-structure P . Then for any B.G.-functor (T_q, ∂_Q) on C such that the t_p -sheaves associated with T_q are trivial (i.e. isomorphic to the point sheaf pt) and $T_q(\emptyset) = pt$ for all q one has $T_q = pt$ for all q .

Proof. Replacing P with the corresponding reducing cd-structure we may assume that all distinguished squares of P are reducing. Let T_q be a B.G.-functor such that the sheaves aT_q associated with T_q 's are trivial. Let us show that for any $d \geq 0$, $q \geq 0$, X and $a \in T_q(X)$ there exists $j : U \rightarrow X$ in $D_d(X)$ such that $T_q(j)(a) = *$. We prove it by induction on d . For $d = 0$ the statement follows from the fact that $(\emptyset \rightarrow X) \in D_0(X)$ and $T_q(\emptyset) = *$. Assume that the statement is proved for d and all q and X . Let $a \in T_q(X)$ be an element. Then by our assumption there exists a covering sieve J such that for any $p : U \rightarrow X$ in J one has $T_q(p)(a) = *$. Since P is complete J contains a sieve of the form (p_i) for a simple covering $\{p_i : U_i \rightarrow X\}$. Therefore it is sufficient to show that for any simple covering (p_i) and $a \in T_q(X)$ such that $T_q(p_i)(a) = *$ there exists $(j : U \rightarrow X) \in D_{d+1}(X)$ such that $T_q(j)(a) = *$. Let S be the class of simple coverings (p_i) such that for any $a \in T_q(X)$ such that $T_q(p_i)(a) = *$ there exists $(j : U \rightarrow X) \in D_{d+1}(X)$ such that $T_q(j)(a) = *$. It contains isomorphisms since any isomorphism is in $D_{d+1}(X)$ by definition of a density structure. Let Q be a distinguished square, (p_i) and (q_j) be as in Definition 2.2(2) and (p_i) and (q_j) are in S . Let us show that $(p \circ p_i, e \circ q_j)$ is in S . Given an element $a \in T_q(X)$ such that its restrictions to Y_i and A_j are trivial we

can find $Y_0 \in D_{d+1}(Y)$ and $A_0 \in D_{d+1}(A)$ such that the restriction of a to A_0 and Y_0 is trivial. Set $B_0 = B$. Since Q is reducing we can find a map $Q' \rightarrow Q$ such that $X' \in D_{d+1}(X)$ and the restriction of a to Y' and A' is trivial. Let a' be the restriction of a to X' . Since T_q is a B.G-functor we have $a' = \partial(b')$ for some $b' \in T_{q+1}(B')$. By induction there exists $B'_0 \in D_d(B)$ such that the restriction of b' to B'_0 is trivial. Set $Y'_0 = Y', A'_0 = A'$. Since Q' is reducing we can find $Q'' \rightarrow Q'$ such that $X'' \in D_{d+1}(X')$ and $B'' \rightarrow B'$ factors through B'_0 . Since ∂ commutes with morphisms of distinguished squares we conclude that the restriction of a' to X'' is trivial. Finally observe that since $X'' \in D_{d+1}(X')$ and $X' \in D_{d+1}(X)$ one has $X'' \in D_{d+1}(X)$ by Definition 2.20(4). We conclude that $S = S_p$ which finishes the proof. \square

Definition 3.3. Let C be a category with a cd-structure P . A simplicial presheaf F on C is called flasque with respect to P if $F(\emptyset)$ is contractible and for any distinguished square of the form (2) the square of simplicial sets

$$F(Q) = \begin{pmatrix} F(X) & \longrightarrow & F(Y) \\ \downarrow & & \downarrow \\ F(A) & \longrightarrow & F(B) \end{pmatrix} \quad (15)$$

is a homotopy pull-back square.

Let $PreShv(C)$ be the category of presheaves of sets on C i.e. the category of contravariant functors from C to the category Sets of sets. The Yoneda embedding allows us to consider C as a full subcategory of $PreShv(C)$. Consider Q as a square $PreShv(C) \subset \Delta^{op}PreShv(C)$, let K_Q be the simplicial homotopy push-out of the maps $B \rightarrow Y$ and $B \rightarrow A$ given by the push-out square

$$\begin{array}{ccc} B \amalg B & \longrightarrow & B \times \Delta^1 \\ \downarrow & & \downarrow \\ A \amalg Y & \longrightarrow & K_Q \end{array} \quad (16)$$

and let $p_Q : K_Q \rightarrow X$ be the obvious morphism.

Lemma 3.4. Let F be a simplicial presheaf on C such that for any U in C the simplicial set $F(U)$ is Kan and $F(\emptyset)$ is contractible. Then F is flasque if and only if, for any $Q \in P$ of the form (2) the map of simplicial function complexes

$$F(X) = S(X, F) \rightarrow S(K_Q, F)$$

defined by p_Q is a weak equivalence.

Proof. The simplicial set $S(K_Q, F)$ is given by the pull-back square

$$\begin{array}{ccc} S(K_Q, F) & \longrightarrow & F(B)^{\Delta^1} \\ \downarrow & & \downarrow \\ F(Y) \times F(A) & \longrightarrow & F(B) \times F(B) \end{array}$$

Therefore, if $F(B)$ is a Kan simplicial set it is a model for the homotopy limit of the diagram $(F(A) \rightarrow F(B), F(Y) \rightarrow F(B))$ which implies the statement of the lemma. \square

For a presheaf F denote by aF the associated sheaf in the t_p -topology. Recall that a morphism of simplicial presheaves $f : F \rightarrow G$ is called a local equivalence with respect to the topology t_p if one has:

1. the morphism $a\pi_0(F) \rightarrow a\pi_0(G)$ defined by f is an isomorphism
2. for any object X of C , any $x \in F(X)$ and any $n \geq 1$ the morphism of associated sheaves $a\pi_n(F, x) \rightarrow a\pi_n(G, f(x))$ on C/X defined by f is an isomorphism.

Local equivalences with respect to the trivial topology are morphisms $f : X \rightarrow Y$ such that for any U in C the map of simplicial sets $X(U) \rightarrow Y(U)$ defined by f is a weak equivalence. We call these morphisms projective equivalences.

Lemma 3.5. Let C be a category with a complete bounded cd-structure P . A morphism $f : F \rightarrow G$ of flasque simplicial presheaves is a t_p -local equivalence if and only if it is a projective equivalence.

Proof. Most of the proof is copied from [10, Lemma 3.1.18]. The if part is obvious. Assume that f is a t_p -local equivalence. Using the closed model structure on the category of simplicial presheaves or an appropriate explicit construction we can find a commutative diagram of simplicial presheaves

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ F' & \longrightarrow & G' \end{array} \quad (17)$$

such that for any X in C , the maps $F(X) \rightarrow F'(X)$ and $G(X) \rightarrow G'(X)$ are weak equivalences of simplicial sets and the map $F'(X) \rightarrow G'(X)$ is a Kan fibration of Kan simplicial sets. Replacing F, G by F', G' we may assume that the maps $F(X) \rightarrow G(X)$ are Kan fibrations between Kan simplicial sets.

It is sufficient to prove that for any X in C and $y \in G(X)$ the fiber $K(X)$ of the map $F(X) \rightarrow G(X)$ over y is contractible (i.e. weakly equivalent to point and in particular nonempty). The simplicial presheaf

$$(p : U \rightarrow X) \mapsto \text{fiber}_{G(p)(y)}(F(U) \rightarrow G(U))$$

on C/X is clearly flasque with respect to the induced cd-structure on C/X . The cd-structure on C/X induced by a complete (resp. bounded) cd-structure is complete (resp. bounded). Therefore, it is sufficient to prove the lemma for $G = pt$ in which case we have to show that for any X the (Kan) simplicial set $F(X)$ is contractible. In addition we may assume that C has a final object pt .

Assume first that $F(pt) \neq \emptyset$ and let $a \in F_0(pt)$ be an element. Consider the family of functors T_q on C of the form

$$X \mapsto \pi_q(F(X), a|_X).$$

It is a B.G.-functor and the associated t_p -sheaves are trivial since $F \rightarrow pt$ is a local equivalence. We also have $T_q(\emptyset) = *$. Thus $T_q(X) = pt$ for all X by [Theorem 3.2](#).

It remains to prove that $F(pt)$ is not empty. We already know that for any X such that $F(X)$ is not empty it is contractible. Since P is complete the sheaf associated with $\pi_0(F)$ is pt there exists a simple covering $\{U_i \rightarrow pt\}$ such that $F(U_i) \neq \emptyset$. Consider the class S of simple coverings $\{U_i \rightarrow X\}$ such that if $F(U_i) \neq \emptyset$ for all i then $F(X) \neq \emptyset$. Let us show that it coincides with the class of all simple coverings. It clearly contains isomorphisms. Let Q, p_i, q_j be as in [Definition 2.2\(2\)](#) and $(p_i), (q_j)$ be in S . Then if $F(U_i), F(V_j)$ are nonempty for all i, j then by assumption $F(Y)$ and $F(A)$ are nonempty. Thus $F(B)$ is nonempty and therefore all these simplicial sets are contractible. It remains to note that if in a homotopy pull-back square of the form (15) $F(A), F(Y)$ and $F(B)$ are contractible then $F(X)$ is nonempty. \square

Lemma 3.6. Consider a commutative square Q of the form (2) in the category of presheaves on a site and assume that e is a monomorphism. Then the square of sheaves associated with Q is a push-out square if and only if the morphism of simplicial presheaves $K_Q \rightarrow X$ is a local equivalence.

Proof. For each U in C the simplicial set $K_Q(U) = K_{Q(U)}$ has the property that

$$\pi_0(K_Q(U)) = Y(U) \coprod_{B(U)} A(U)$$

This implies that $\pi_0(K_Q) = Y \coprod_B A$. Therefore, we have

$$a\pi_0(K_Q) = a\left(Y \coprod_B A\right)$$

and we conclude that if $K_Q \rightarrow X$ is a local equivalence then Q is a push-out square. Assume now that e is a monomorphism. Then, for each U in C the simplicial set $K_{Q(U)}$ is weakly equivalent to the (simplicial) set $Y(U) \coprod_{B(U)} A(U)$ i.e. the morphism $K_Q \rightarrow Y \coprod_B A$ is a projective equivalence. Since the associated sheaf functor takes projective equivalences to local equivalences we conclude that for a push-out square Q the morphism $K_Q \rightarrow X$ is a local equivalence. \square

Let G_P^0 be the collection of morphisms (in $\Delta^{op}PreShv(C)$) of the form $K_Q \rightarrow X$ where $Q \in P$. Denote by G_P the union of G_P^0 with the morphism $\emptyset' \rightarrow \emptyset$ where \emptyset' is the initial object of $PreShv(C)$ and \emptyset is the presheaf represented by the initial object of C . The elements of G_P are called the generating weak equivalences defined by P . Combining [Lemma 3.6](#) with [Corollary 2.16](#) we get the following result.

Lemma 3.7. Let P be a regular cd-structure. Then all elements of G_P are local equivalences for the associated topology.

For any small category C , the class of projective equivalences together with the class of projective fibrations i.e. morphisms $F \rightarrow G$ such that for all $U \in C$ the map of simplicial sets $F(U) \rightarrow G(U)$ is a Kan fibration, form a simplicial closed model structure on $\Delta^{op}PreShv(C)$ which is called the projective c.m.s. Therefore, for any set of maps E in $\Delta^{op}PreShv(C)$ it makes sense to consider the class $cl_l(E)$ of E -local equivalences in the sense of [2, Def. 3.1.3].

The following result shows that for a complete regular and bounded cd-structure the class $cl_l(G_P)$ coincides with the class of t_p -local equivalences.

Proposition 3.8. Let C be a category with an initial object. Then:

1. for any cd-structure P , a simplicial presheaf F is G_P -local if and only if it is projectively fibrant and flasque,
2. if P is regular, then a G_P -local equivalence is a t_p -local equivalence,
3. if P is regular, complete and bounded then a t_p -local equivalence is a G_P -local equivalence.

Proof. One verifies easily that sources and targets of elements of G_p are cofibrant. Therefore, the first assertion is equivalent to [Lemma 3.4](#). By the standard factorization techniques we can construct for any $f : F \rightarrow G$ in $\Delta^{op}PreShv(C)$ a commutative square

$$\begin{array}{ccc} F & \xrightarrow{f} & G \\ e_F \downarrow & & \downarrow e_G \\ Ex(F) & \xrightarrow{Ex(f)} & Ex(G) \end{array}$$

where $Ex(F)$ and $Ex(G)$ are G_p -local and the morphisms e_F and e_G are transfinite compositions of push-outs of elements of G'_p where G'_p is a set of projective cofibrations projectively equivalent to elements of G_p (see e.g. [3, Th. 2.1, 14, p.32]).

Projective cofibrations are monomorphisms and the push-outs of t_p -local equivalences which are monomorphisms are again t_p -local equivalences (see [6, Prop. 2.2, p.60]). Transfinite compositions of t_p -local equivalences are t_p -local equivalences. Therefore, if P is regular then [Lemma 3.7](#) implies that the morphisms e_F and e_G are t_p -local equivalences.

If f is a G_p -local equivalence, then $Ex(f)$ is a projective equivalence as a G_p -local equivalence between G_p -local objects. Together with the previous paragraph this proves the second assertion of the proposition.

The third assertion follows by the same diagram from the first two and [Lemma 3.5](#). \square

Topologies defined by cd-structures also exhibit a somewhat better than average functoriality behavior. Recall, from [10], that a continuous map of sites $f : (T', t') \rightarrow (T, t)$ is called reasonable if for any Jardine–Joyal fibrant simplicial sheaf X' on (T', t') the morphism $f_*(X') \rightarrow Ex_{JJ}(f_*(X'))$ from $f_*(X')$ to its Jardine–Joyal fibrant replacement is a projective equivalence.

Proposition 3.9. *Let (T', t') be a site and (T, t_p) the site defined by a complete regular and bounded cd-structure P on T . Let further $f : (T', t') \rightarrow (T, t_p)$ be a continuous map of sites such that for any distinguished Q square in P of the form (2), the morphism $f^*(\rho(e))$ is a monomorphism and the square $f^*(\rho(Q))$ is a pull-back square. Then f is reasonable.*

Proof. Let X' be a Jardine–Joyal fibrant simplicial sheaf on (T', t') . For any regular cd-structure, Jardine–Joyal fibrant simplicial sheaves are flasque. Therefore, in order to show that $f_*(X') \rightarrow Ex_{JJ}(f_*(X'))$ is a projective equivalence it is sufficient, by [Lemma 3.5](#) to show that $f_*(X')$ is flasque. The first condition of [Definition 3.3](#) holds since $f_*(X')$ is a t_p -sheaf. By adjunction $f_*(X')(Q) = Hom(f^*(\rho(Q)), X')$ and therefore the second condition follows from our assumptions. \square

4. The Brown–Gersten closed model structure

Let C be a category with a cd-structure P and $Shv(C, t_p)$ the category of sheaves of sets on C with respect to the associated topology. We say that a morphism in $\Delta^{op}Shv(C, t_p)$ is a local (resp. projective) equivalence if it is a local (resp. projective) equivalence as a morphism of simplicial presheaves. If we define weak equivalences in $\Delta^{op}Shv(C, t_p)$ as the local equivalences with respect to t_p , cofibrations as all monomorphisms and fibrations by the right lifting property we get a closed model structure on $\Delta^{op}Shv(C, t_p)$ which we call the Joyal–Jardine closed model structure (see [8,5,6]). In this section we show that if P is complete regular and bounded then there is another closed model structure on $\Delta^{op}Shv(C, t_p)$ with the same class of weak equivalences which we call the Brown–Gersten closed model structure. When C is the category of open subsets of a topological space and P is the standard cd-structure we recover the closed model structure constructed in [1]. The Brown–Gersten closed model structure is often more convenient than the Joyal–Jardine one since it is finitely generated. Our definitions and proofs follow closely the ones given in [1].

For a simplicial set we denote by the same letter the corresponding constant simplicial sheaf on C . Recall that $\Delta^{n,k}$ is the simplicial subset of the boundary $\partial\Delta^n$ of the standard simplex Δ^n which is the complement to the k th face of dimension $n - 1$. Denote by I_p the class of morphisms of the following two types:

1. for any object X of C the morphisms $\Delta^{n,k} \times \rho(X) \rightarrow \Delta^n \times \rho(X)$
2. for any distinguished square of the form (2) the morphisms

$$\Delta^n \times \rho(A) \coprod_{\Delta^{n,k} \times \rho(A)} \Delta^{n,k} \times \rho(X) \rightarrow \Delta^n \times \rho(X)$$

and by I_p the class of morphisms of the following two types:

1. for any object X of C the morphisms $\partial\Delta^n \times \rho(X) \rightarrow \Delta^n \times \rho(X)$
2. for any distinguished square of the form (2) the morphisms

$$\Delta^n \times \rho(A) \coprod_{\partial\Delta^n \times \rho(A)} \partial\Delta^n \times \rho(X) \rightarrow \Delta^n \times \rho(X).$$

The following lemma is straightforward.

Lemma 4.1. A morphism f has the right lifting property with respect to J_P if and only if the maps $E(X) \rightarrow B(X)$ are Kan fibrations for all X in C and the maps

$$E(X) \rightarrow B(X) \times_{B(A)} E(A)$$

are Kan fibrations for all distinguished squares Q of the form (2).

A map f has the right lifting property with respect to I_P if the maps $E(X) \rightarrow B(X)$ are trivial Kan fibrations for all X in C and the maps

$$E(X) \rightarrow B(X) \times_{B(A)} E(A)$$

are Kan fibrations for all distinguished squares Q of the form (2).

Definition 4.2. Let P be a regular cd-structure. A morphism $f : E \rightarrow B$ in $\Delta^{op}Shv(C, t_P)$ is called a Brown–Gersten fibration if it has the right lifting property with respect to elements of J_P .

Lemma 4.3. Let P be a regular cd-structure, then any Brown–Gersten fibrant simplicial sheaf is flasque.

Proof. Let F be a fibrant simplicial sheaf. Since F is a sheaf we have $F(0) = pt$ and therefore the first condition of Definition 3.3 is satisfied. Lemma 4.1 implies that for any distinguished square of the form (2) the map of simplicial sets $F(X) \rightarrow F(A)$ is a Kan fibration. On the other hand since F is a sheaf and P is regular Proposition 2.15 shows that the square $F(Q)$ is a pull-back square. This implies that it is a homotopy pull-back square. \square

A morphism is called a Brown–Gersten cofibration if it has the left lifting property with respect to all trivial Brown–Gersten fibrations that is morphisms which are Brown–Gersten fibrations and (local) equivalences. The proof of the following lemma is parallel to the proof of [1, Lemma, p.274].

Lemma 4.4. Let P be a complete, bounded and regular cd-structure. Then a morphism $f : E \rightarrow B$ is a trivial Brown–Gersten fibration if and only if it has the right lifting property with respect to elements of I_P .

Proof. Lemma 4.1 implies that any map which has the right lifting property with respect to I_P is a projective equivalence and has the right lifting property with respect to J_P . This proves the “if” part of the lemma. To prove the “only if” part we have to check that a morphism f which has the right lifting property for elements of J_P and which is at t_P -local equivalence has the right lifting property with respect to I_P . Using again Lemma 4.1 one concludes that it is sufficient to show that under our assumption f is a projective equivalence i.e. that for any X in C the map of simplicial sets $E(X) \rightarrow B(X)$ defined by f is a weak equivalence. Since this map is a fibration it is sufficient to check that its fiber over any point $x_0 \in B(X)$ is contractible. The point x_0 defines a map $\rho(X) \rightarrow B$ and the contractibility of the fiber is equivalent to the condition that the pull-back $f_{x_0} : \rho(X) \times_B E \rightarrow \rho(X)$ of f with respect to this map is a projective equivalence. This map is a local equivalence in the t_P -topology (if our site has enough points it is obvious; for the general case see [6, Th.1.12]) and has the right lifting property with respect to J_P . Observe now that the morphism $\rho(U) \rightarrow pt$ has the right lifting property with respect to J_P and therefore so does the morphism $\rho(X) \times_B E \rightarrow pt$. We conclude that f_{x_0} is a local equivalence between two fibrant objects and since, by Lemma 4.3, Brown–Gersten fibrant objects are flasque we conclude that f_{x_0} is a projective equivalence by Lemma 3.5. \square

Theorem 4.5. Let P be a complete, bounded and regular cd-structure. Then the classes of local equivalences, Brown–Gersten fibrations and Brown–Gersten cofibrations form a closed model structure on $\Delta^{op}Shv(C, t_P)$.

Proof. Parallel to the proof of [1, Th.2]. \square

Lemma 4.6. If P is a regular cd-structure then any Brown–Gersten cofibration is a monomorphism.

Proof. Let $f : A \rightarrow B$ be a cofibration. Using the standard technique we can find a decomposition $e \circ p$ of the morphism $A \rightarrow pt$ such that p is a trivial fibration and e is obtained from elements of I_P by push-outs and infinite compositions. Since P is regular elements of I_P are monomorphisms and therefore e is a monomorphism. The definition of cofibrations implies that there is a morphism g such that $g \circ f = e$ which forces f to be a monomorphism. \square

Proposition 4.7. The Brown–Gersten closed model structure is finitely generated (in the sense of [4, Def. 4.1]) and cellular (in the sense of [2]).

Proof. Lemma 2.8 implies easily that domains and codomains of elements of I_P and J_P are finite. Together with our definition of fibrations and Lemma 4.4 it implies that the Brown–Gersten closed model structure is finitely generated. A finitely generated closed model structure is cellular if any cofibration is an effective monomorphism. Therefore, the second statement follows from Lemma 4.6 and the fact that any monomorphism in the category of sheaves is effective. \square

Proposition 4.8. The Brown–Gersten closed model structure is both left and right proper.

Proof. Consider a commutative square of the form

$$\begin{array}{ccc} B & \xrightarrow{e'} & Y \\ p' \downarrow & & \downarrow p \\ A & \xrightarrow{e} & X \end{array} \quad (18)$$

Assume first that it is a push-out square, e' is a cofibration and p' is a local equivalence. By Lemma 4.6 e' is a monomorphism and therefore p is a local equivalence by [7, Prop. 1.4].

Assume that (18) is a pull-back square, p is a fibration and e is a local equivalence. The definition of Brown–Gersten fibrations implies in particular that for any object U of C the morphism of simplicial sets $Y(U) \rightarrow X(U)$ is a Kan fibration. In particular it is a local fibration in the sense of [6]. The same reasoning as in the proof of [7, Prop. 1.4] shows now that p' is a local equivalence. \square

5. Cech morphisms

Let C be a category with fiber products. For any morphism $f : X \rightarrow Y$ in C denote by $\check{C}(f)$ the simplicial object with the terms

$$\check{C}(f)_n = X_Y^{n+1}$$

and faces and degeneracy morphisms given by partial projections and diagonals respectively. The projections $X_Y^i \rightarrow Y$ define a morphism $\check{C}(f) \rightarrow Y$ which we denote by $\eta(f)$. For a class of morphisms A in C we denote by $\eta(A)$ the class of morphisms of the form $\eta(f)$ for $f \in A$.

In this section we work in the category $C^{\sqcup < \infty}$ which is defined as the full subcategory of $PreShv(C)$ which consists of finite coproducts of representable presheaves. One verifies easily that this category has fiber products compatible with the fiber products in C .

For a cd-structure P on a category C denote by $Cov_f(t_P)$ the class of morphisms in $C^{\sqcup < \infty}$ of the form $f = \coprod f_i$ for all finite t_P -coverings $\{f_i : X_i \rightarrow X\}$ in C . Since the empty family is a covering of the initial object the morphism $\emptyset' \rightarrow \emptyset$ belongs to $Cov_f(t_P)$.

In this section we consider the relationship between the classes $\eta(Cov_f(t_P))$ and G_P . To connect these two classes we use the operation of $(\Delta, \sqcup < \infty)$ -closure (cf. [12]). For a class of morphisms E in $\Delta^{op} C^{\sqcup < \infty}$ one defines $cl_{\Delta, \sqcup < \infty}(E)$ as the smallest class of morphisms which contains E and satisfies the following conditions:

1. simplicial homotopy equivalences are in $cl_{\Delta, \sqcup < \infty}(E)$,
2. if for a composable pair of morphisms f, g two out of the three elements of the set $\{f, g, g \circ f\}$ are in E then so is the third,
3. if $f : B \rightarrow B'$ is a morphism of bisimplicial objects over $C^{\sqcup < \infty}$ whose rows or columns are in $cl_{\Delta, \sqcup < \infty}(E)$ then the diagonal $\Delta(f) : \Delta(B) \rightarrow \Delta(B')$ is in $cl_{\Delta, \sqcup < \infty}(E)$,
4. $cl_{\Delta, \sqcup < \infty}(E)$ is closed under finite coproducts.

The goal of this section is to show that, for a good enough cd-structure P on a category C with fiber products, the class $cl_{\Delta, \sqcup < \infty}(G_P)$ coincides with the class $cl_{\Delta, \sqcup < \infty}(\eta(Cov_f(t_P)))$.

Lemma 5.1. *Let $f : X \rightarrow Y$ be a morphism in C which has a section. Then $\eta(f)$ is a simplicial homotopy equivalence.*

Proof. If f has a section $Y \rightarrow X$ it defined a map $Y \rightarrow \check{C}(f)$ in the obvious way. The composition $Y \rightarrow \check{C}(f) \rightarrow Y$ is the identity. On the other hand, for any simplicial object K over Y one has

$$Hom_Y(K, \check{C}(f)) = Hom_Y(K_0, X)$$

where K_0 is the object of 0-simplexes of K . In particular any two morphisms with values in $\check{C}(f)$ over Y are homotopic. This implies that the composition $\check{C}(f) \rightarrow Y \rightarrow \check{C}(f)$ is homotopic to identity. \square

Lemma 5.2. *Let C be a category with fiber products and Q a fiber square of the form (2) such that both morphisms $A \rightarrow X$ and $Y \rightarrow X$ are monomorphisms. Denote by π_Q the morphism $Y \coprod A \rightarrow X$ in $C^{\sqcup < \infty}$. Then one has*

$$p_Q \in cl_{\Delta, \sqcup < \infty}(\eta(\pi_Q))$$

Proof. Consider the fiber square

$$\begin{array}{ccc} K_Q \times_X \check{C}(\pi_Q) & \longrightarrow & \check{C}(\pi_Q) \\ \downarrow & & \downarrow \\ K_Q & \xrightarrow{p_Q} & X \end{array} \quad (19)$$

Let us show that the upper horizontal and the left vertical arrows belong to $cl_{\Delta, \sqcup_{<\infty}}(\emptyset)$. Thinking of the fiber product as of the diagonal of the corresponding bisimplicial object we see that the left vertical arrow is contained in the Δ -closure of the family of morphisms of the form

$$(Y \coprod A)^n \times \check{C}(\pi_Q) \rightarrow (Y \coprod A)^n$$

where the products are taken over X . Since

$$(Y \coprod A)_X^n = \coprod_{i+j=n} Y^i \times_X A^j$$

it is contained in the $(\Delta, \sqcup_{<\infty})$ -closure of morphisms of the form

$$Y^i \times_X A^j \times_X \check{C}(\pi_Q) \rightarrow Y^i \times_X A^j$$

which are isomorphic to morphisms of the form $\eta(\pi_Q \times_X Id_{Y^i \times_X A^j})$. Since $i + j = n > 0$ the morphism $\pi_Q \times_X Id_{Y^i \times_X A^j}$ has a section. Lemma 5.1 implies now that $\eta(\pi_Q \times_X Id_{Y^i \times_X A^j})$ are homotopy equivalences and therefore the left vertical arrow is in $cl_{\Delta, \sqcup_{<\infty}}(\emptyset)$.

For the upper horizontal arrow the same argument shows that it belongs to the $(\Delta, \sqcup_{<\infty})$ -closure of morphisms of the form $p_{Q'}$ where $Q' = Q \times_X Y^i \times_X A^j$ and $i + j > 0$. Since both $A \rightarrow X$ and $Y \rightarrow X$ are monomorphisms each of these squares has the property that its vertical or horizontal sides are isomorphism. For such squares the morphisms p_Q are homotopy equivalences by [12, Lemma 2.9]. \square

Lemma 5.3. *Let C be a category with fiber products and P a regular cd-structure on C . Then for any distinguished square Q one has*

$$p_Q \in cl_{\Delta, \sqcup_{<\infty}}(\eta(Cov_f(t_P))).$$

Proof. Let $\pi_Q : Y \coprod A \rightarrow X$ be the element of $Cov_f(t_P)$ defined by Q . Consider again a square of the form (19). The same argument as in the proof of Lemma 5.2 shows that the left vertical arrow is in $cl_{\Delta, \sqcup_{<\infty}}(\emptyset)$ and the upper horizontal arrow belongs to the $(\Delta, \sqcup_{<\infty})$ -closure of morphisms of the form $p_{Q'}$ where $Q' = Q \times_X Y^i \times_X A^j$ and $i + j > 0$. For since $A \rightarrow X$ is a monomorphism by the definition of a regular cd-structure for $j > 0$ the horizontal arrows of Q' are isomorphisms and the corresponding morphism is a homotopy equivalence by [12, Lemma 2.9]. Consider the squares $Q_i = Q \times_X Y^i$ for $i > 0$. Since $A \rightarrow X$ is a monomorphism we have $B \times_X Y = B \times_X B$ and therefore the square Q_1 is isomorphic to the lower square of the following diagram

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow \\ B \times_X B & \longrightarrow & Y \times_X Y \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array} \quad (20)$$

where the upper vertical arrows are the diagonals. Denote the upper square of this diagram by Q' . By [12, Lemma 2.14] we conclude that it is sufficient to show that one has

$$p_{Q' \times_X Y^i} \in cl_{\Delta, \sqcup_{<\infty}}(\eta(Cov_f(t_P))) \quad (21)$$

for $i \geq 0$. Both morphisms in Q' are monomorphisms and $B \times_X B \coprod Y \rightarrow Y \times_X Y$ is in $Cov_f(t_P)$ by definition of a regular cd-structure. Therefore the same holds for $Q' \times_X Y^i$ and (21) follows from Lemma 5.2. \square

Proposition 5.4. *Let A be a class of morphisms in C which is closed under pull-backs. Then one has:*

1. if f, g is a pair of morphisms such that gf is defined and $\eta(gf) \in cl_{\Delta}(\eta(A))$ then $\eta(g) \in cl_{\Delta}(\eta(A))$
2. if f, g is a pair of morphisms such that gf is defined and $\eta(f), \eta(g) \in cl_{\Delta}(\eta(A))$ then $\eta(gf) \in cl_{\Delta}(\eta(A))$.

Proof. The first statement follows from Lemma 5.6. The second one from Lemma 5.7. \square

Lemma 5.5. *Let $p : A \rightarrow X, q : B \rightarrow X$ be two morphisms. Then one has*

$$\eta(p) \in cl_{\Delta}(\{\eta(p \times_X Id_{B^m}), \eta(q \times_X Id_{A^n})\}_{m>0, n \geq 0}) \quad (22)$$

Proof. Consider the bisimplicial object $\check{C}(p, q)$ build on p and q . We have a commutative diagram

$$\begin{array}{ccc} \Delta \check{C}(p, q) & \longrightarrow & \check{C}(q) \\ \downarrow & & \downarrow \\ \check{C}(p) & \longrightarrow & X \end{array} \quad (23)$$

The upper horizontal arrow belongs to $cl_{\Delta}(\{\eta(p \times_X Id_{B^m})\}_{m>0})$ and the left vertical one to $cl_{\Delta}(\{\eta(q \times_X Id_{A^n})\}_{n>0})$. This implies (22) by the “2 out of 3” property of the Δ -closed classes. \square

Lemma 5.6. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a composable pair of morphisms. Then one has

$$\eta(g) \in cl_{\Delta}(\{\eta(gf \times_Z Id_{Y^n})\}_{n \geq 0}) \quad (24)$$

Proof. Applying Lemma 5.5 to the pair $p = g, q = gf$ we get

$$\eta(g) \in cl_{\Delta}(\{\eta(g \times_Z Id_{X^m}), \eta((gf) \times_Z Id_{Y^n})\}_{m > 0, n \geq 0})$$

Since morphisms $g \times_Z Id_{X^m}$ for $m > 0$ have sections Lemma 5.1 implies (24). \square

Lemma 5.7. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a composable pair of morphisms. Then one has

$$\eta(gf) \in cl_{\Delta}(\{\eta(g \times Id_{X^n}), \eta(f \times Id_{X^n} \times Id_{Y^l})\}_{n, l \geq 0}) \quad (25)$$

where all the products are taken over Z .

Proof. All the products in the proof are over Z unless the opposite is specified. Let us show first that for $l \geq 0$ one has

$$\eta((gf) \times Id_{Y^{l+1}}) \in cl_{\Delta}(\eta(f \times Id_{X^n} \times Id_{Y^l}))_{n, l \geq 0} \quad (26)$$

For that we apply Lemma 5.5 to $p = (gf) \times Id_{Y^{l+1}}$ and $q = (Y^l \times X \xrightarrow{Id \times f} Y^l \times Y)$. We have

$$((gf) \times Id_{Y^{l+1}}) \times_{Y^{l+1}} Id_{(Y^l \times X)^m} = (gf) \times Id_{(Y^l \times X)^m} \quad (27)$$

and

$$q \times_{Y^{l+1}} Id_{(X \times_{Y^{l+1}})^n} = q \times Id_{X^n} = f \times Id_{X^n} \times Id_{Y^l}$$

and therefore

$$\eta((gf) \times Id_{Y^{l+1}}) \in cl_{\Delta}(\{\eta((gf) \times Id_{(Y^l \times X)^m}), \eta(f \times Id_{X^n} \times Id_{Y^l})\}_{m > 0, n \geq 0}).$$

For $m > 0$ the morphisms (27) have sections and therefore Lemma 5.1 implies (26). Apply now Lemma 5.5 to morphisms $p = gf$ and $q = g$. We get

$$\eta(gf) \in cl_{\Delta}(\{\eta(gf \times Id_{Y^m}), \eta(g \times Id_{X^n})\}_{m > 0, n \geq 0})$$

which together with (26) implies (25). \square

Proposition 5.8. Let P be a cd-structure on a category C with fiber products such that for any distinguished square Q and a morphism $X' \rightarrow X$ the square $Q \times_X X'$ is distinguished. Then one has

$$\eta(Cov_f(t_P)) \subset cl_{\Delta, \sqcup < \infty}(G_P).$$

Proof. Our assumption on P implies in particular that it is complete. The class $Cov_f(t_P)$ is then the smallest class which contains the morphism $\emptyset' \rightarrow \emptyset$, morphisms of the form $\pi_Q : Y \coprod A \rightarrow X$ for distinguished squares Q of the form (2) and is closed under finite coproducts and under operations described in parts (1) and (2) of Proposition 5.4. This implies that it is sufficient to prove that for any Q in P one has

$$\eta(\pi_Q) \in cl_{\Delta, \sqcup < \infty}(\{p_{Q'}\}_{Q' \in P}).$$

This follows from the diagram (19) in the same way as in the proof of Lemma 5.2. \square

Combining Lemma 5.3 and Proposition 5.8 we get the following result.

Proposition 5.9. Let C be a category with fiber products and P be a regular cd-structure such that for any distinguished square Q and a morphism $X' \rightarrow X$ the square $Q \times_X X'$ is distinguished. Then one has

$$cl_{\Delta, \sqcup < \infty}(\eta(Cov_f(t_P))) = cl_{\Delta, \sqcup < \infty}(G_P). \quad (28)$$

Corollary 5.10. Under the assumptions of the proposition one has

$$cl_l(\eta(Cov_f(t_P))) = cl_l(G_P). \quad (29)$$

Proof. The category of presheaves on C is equivalent to the category of radditive functors on $C^{\sqcup < \infty}$. Therefore we may apply the results of [12]. In particular, by [12, Th. 3.49] combined with an obvious argument for coproducts we conclude that for any E in $\Delta^{op}PreShv(C)$ the class $cl_l(E)$ is $(\Delta, \sqcup < \infty)$ -closed. We conclude that (28) implies (29). \square

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