Math 19 A&B
Fall 2019
Final Exam

Solutions

Name: ____________________

PRACTICE EXAM

This exam contains 8 pages and 8 questions. Total of points is 100. For full credit you must show your work. Partial credit may be given for incorrect solutions if sufficient work is shown. Messy/unorganized answers may be penalized, even if correct.

Grade Table (for teacher use only)

Question	Points	Score
1	15	
2	15	
3	15	
4	8	
5	15	
6	8	
7	8	
8	16	
Total:	100	

HONORS PLEDGE (sign after exam is completed): I have neither given nor received aid on this exam, nor have I observed a violation of the UVM Code of Academic Integrity.

Signature: ____________________

1. (15 points) Evaluate the following indefinite integrals

(a) (5 points)

$$\int (3x^5 - 2x + 3)dx$$

$= 3\int x^5 dx - 2\int x\,dx + \int 3\,dx$

$= \boxed{3\frac{x^6}{6} - 2\frac{x^2}{2} + 3x + C}$

$= \frac{1}{2}x^6 - x^2 + 3x + C$

(b) (5 points)

$$\int (6x^2 + 10)e^{x^3+5x}dx = \int e^u \cdot 2\,du$$

$u = x^3 + 5x$

$\frac{du}{dx} = 3x^2 + 5$

$du = (3x^2+5)dx$

$2du = (6x^2+10)dx$

$= 2\int e^u du$

$= 2e^u + C$

$= \boxed{2e^{x^3+5x} + C}$

(c) (5 points)

$$\int \frac{2x}{x^2-8}dx = \int \frac{du}{u}$$

$u = x^2 - 8$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$= \int \frac{1}{u}du$

$= \int u^{-1} du$

$= \ln|u| + C$

$= \boxed{\ln|x^2-8| + C}$

2. (15 points) More integration topics

(a) (5 points) Given that

$$\frac{d}{dx}\left(\frac{1}{1+e^{-x}}\right) = \frac{e^{-x}}{(1+e^{-x})^2}.$$

Remember the antiderivative/integral is the opposite of the derivative

$$F(x) \xrightarrow{d/dx} f(x), \quad f(x) \xrightarrow{\int f(x)dx} F(x)$$

Find the particular antiderivative of

$$\frac{e^{-x}}{(1+e^{-x})^2}$$

which passes through the point $(0, 0)$.

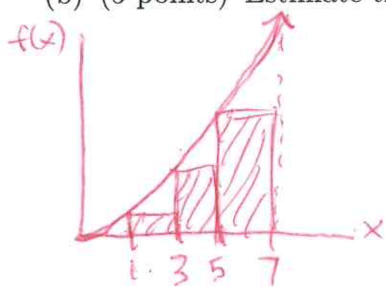
We are given $\int \frac{e^{-x}}{(1+e^{-x})^2}\,dx = \frac{1}{1+e^{-x}} + C$

Let $F(x) = \frac{1}{1+e^{-x}} + C$.

$F(0) = 0 \Rightarrow \frac{1}{1+e^{-0}} + C = 0 \Rightarrow \frac{1}{2} + C = 0 \Rightarrow C = -\frac{1}{2}$

$$\boxed{F(x) = \frac{1}{1+e^{-x}} - \frac{1}{2}}$$

(b) (5 points) Estimate the area under $f(x) = x^2$ on $[1, 7]$ using $n = 3$ left rectangles.



$\Delta x = \frac{7-1}{3} = 2$

$L_3 = 2\cdot f(1) + 2\cdot f(3) + 2\cdot f(5)$
$= 2\cdot 1 + 2\cdot 9 + 2\cdot 25$
$= 2\cdot(1+9+25)$
$= \boxed{70}$

(c) (5 points) Compute the definite integral

$$\int_3^8 (5x-6)dx = \frac{5x^2}{2} - 6x \Big|_{x=3}^{x=8}$$

$$= \left[\frac{5(8)^2}{2} - 6(8)\right] - \left[\frac{5(3)^2}{2} - 6(3)\right]$$

$$= [160 - 48] - \left[\frac{45}{2} - 18\right]$$

$$= \boxed{107.5}$$

3. (15 points) Limits – Evaluate the following limits. Make sure to briefly justify your answer. Write DNE if a limit does not exist.

(a) (5 points)

$$\lim_{x\to 1}\frac{x}{x-1} = \frac{1}{0}$$ infinite limits

Check both onesided limits

- $\lim_{x\to 1^-} \frac{x}{x-1} \approx \frac{1}{0.99-1} = \frac{1}{-.01} = -100 \to -\infty$
- $\lim_{x\to 1^+} \frac{x}{x-1} \approx \frac{1}{1.01-1} = \frac{1}{.01} = +100 \to +\infty$

these are not equal so limit DNE

$\boxed{\lim_{x\to 1} \frac{x}{x-1} = DNE}$

(b) (5 points)

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x^2-6x+8} = \frac{\sqrt{4}-2}{4^2-6\cdot 4+8} = \frac{0}{0}$$ L'H rule

$x^{1/2}$

$$= \lim_{x\to 4} \frac{\frac{1}{2}x^{-1/2}}{2x-6} = \lim_{x\to 4} \frac{1}{2}\cdot\frac{1}{\sqrt{x}(2x-6)}$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{4}(2\cdot 4-6)} = \frac{1}{2}\cdot\frac{1}{4} = \boxed{\frac{1}{8}}$$

(c) (5 points)

$$\lim_{x\to\infty}\frac{3x^8-6x^2}{-2x^6+3x^5+1} = \lim_{x\to\infty}\frac{3x^8}{-2x^6}$$

$$= \lim_{x\to\infty} -\frac{3}{2}x^2$$

$$= -\frac{3}{2}\cdot(\text{big }+)$$

$$= \text{big }-$$

$$= \boxed{-\infty}$$

Limits at infinity

Recall, for rational functions we can just look at the "leading terms."

(No need to do L'H rule)

4. (8 points) Consider the function

$$f(x) = x^2 - 3x + 9.$$

Use the limit definition of the derivative to compute $f'(x)$. No credit will be given for using shortcuts on this problem.

(a) (2 points)

$$f(x+h) = (x+h)^2 - 3(x+h) + 9$$
$$= x^2 + 2xh + h^2 - 3x - 3h + 9$$

(b) (2 points)

$$f(x+h) - f(x) = (\cancel{x^2} + 2xh + h^2 \cancel{-3x} - 3h \cancel{+9}) - (\cancel{x^2} \cancel{-3x} \cancel{+9})$$
$$= 2xh + h^2 - 3h$$

(c) (2 points)

$$\frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2 - 3h}{h}$$
$$= 2x + h - 3$$

(d) (2 points)

$$f'(x) = \lim_{h \to 0} (2x + h - 3)$$
$$= 2x + 0 - 3$$
$$= \boxed{2x - 3}$$

DO NOT forget the derivative is a limit

5. (15 points) Compute the following derivatives. You may use shortcuts.

(a) (5 points)

$$\frac{d}{dx}\left(x^3 + 9\ln(x) - \frac{2}{x^3}\right) \quad 2x^{-3}$$

$$= \boxed{3x^2 + \frac{9}{x} - 2(-3)x^{-4}}$$

$$= 3x^2 + \frac{9}{x} + 6x^{-4}$$

(b) (5 points)

Product + Chain

$$\frac{d}{dx}\left((x^2-1)^4(3x+4)\right)$$

$$= \boxed{4(x^2-1)^3(2x)(3x+4) + 3(x^2-1)^4}$$

$$= 8x(x^2-1)^3(3x+4) + 3(x^2-1)^4$$

(c) (5 points)

No need for quotient rule.

Do need chain rule

$$\frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{2}e^x\right) + \frac{d}{dx}\left(\frac{1}{2}e^{-x}\right)$$

$$= \frac{1}{2}e^x + \frac{1}{2}e^{-x}\cdot(-1)$$

$$= \boxed{\frac{e^x - e^{-x}}{2}}$$

6. (8 points) Consider the function

Sorry this one worked out messy!

$$f(x) = \frac{e^x}{e+x}$$

(a) (4 points) Find the equation of the tangent line at $x = 1$.

Quotient rule

$$f'(x) = \frac{e^x(e+x) - e^x}{(e+x)^2}$$

Slope: $f'(1) = \frac{e(e+1) - e}{(e+1)^2} = \frac{e^2}{(e+1)^2} \approx 0.534$

point: $x=1$, $y = f(1) = \frac{e^1}{e+1} \approx 0.731$

equation:

$$\boxed{y - 0.731 = 0.534(x-1)}$$

(b) (4 points) Find where the tangent line is horizontal.

$$f'(x) = 0 \Rightarrow e^x(e+x) - e^x = 0$$
$$\Rightarrow e^{x+1} + xe^x - e^x = 0$$
$$\Rightarrow e^x(e+x-1) = 0$$
$$\Rightarrow e+x-1 = 0$$
$$\Rightarrow \boxed{x = 1-e}$$

7. (8 points) Find y' for the implicit curve defined by the equation

$$2y^2 + xy - 1 = 0.$$

$$4yy' + y + xy' = 0$$
$$\Rightarrow 4yy' + xy' = -y$$
$$\Rightarrow y'(4y+x) = -y$$
$$\Rightarrow \boxed{y' = \frac{-y}{4y+x}}$$

8. (16 points) For the function $f(x) = -x^3 + 3x^2 + 9x - 9$

(a) (5 points) Find the intervals where f is increasing/decreasing

$f'(x) = -3x^2 + 6x + 9$

$f'(x) = 0 \Rightarrow -3x^2 + 6x + 9 = 0$

$\Rightarrow -3(x^2 - 2x - 3) = 0$

$\Rightarrow -3(x-3)(x+1) = 0$

$\Rightarrow x = 3,\ x = -1$

critical #s

f': $-$ | $+$ | $-$ (at -1, 3)

x	$f'(x)$
-2	-15
0	9
4	-15

$\boxed{\text{increasing on } (-1,3) \quad \text{decreasing on } (-\infty,-1) \cup (3,\infty)}$

(b) (1 point) Find any local maxima or minima.

$x = -1$ is local min

$x = 3$ is local max

(c) (5 points) Find the intervals where f is concave up/concave down

$f''(x) = -6x + 6$

$f''(x) = 0 \Rightarrow -6x + 6 = 0$

$\Rightarrow x = 1$

f'': $+$ | $-$ (at 1)

x	$f''(x)$
0	6
2	-6

$\boxed{\text{concave up on } (-\infty, 1) \quad \text{concave down on } (1, \infty)}$

(d) (1 point) Find any points of inflection.

$x = 1$ is point of inflection

(e) (4 points) Find the absolute maximum and absolute minimum of f on $[-4, 4]$.

x	$f(x)$	
-4	67	$\leftarrow$ absolute max at $x = -4$
-1	-14	$\leftarrow$ absolute min at $x = -1$
3	18	
4	11	