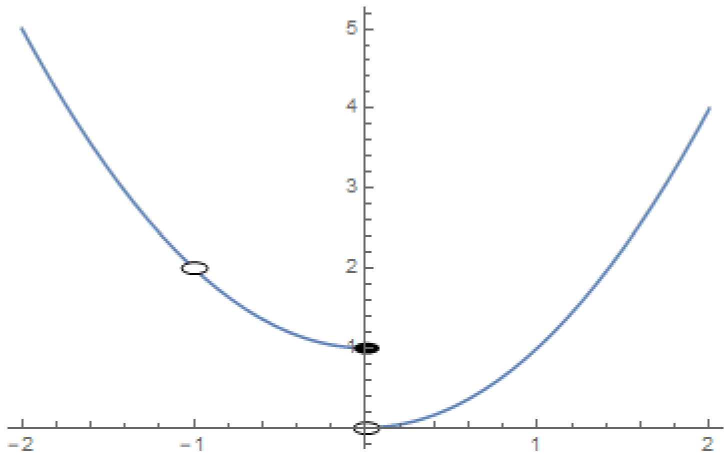
MATH 19 - Section 2.1–2.4 Solutions

Do not submit answers to these. Your assignment is found in "Assignment 1: Section 2.1-2.4".

2.1 Introduction to Limits

1. Consider the function $f(x)$ plotted below.



Determine the limits if they exist.

(a)
$$\lim_{x \to 1} f(x)$$

(b)
$$\lim_{x \to -1} f(x)$$

(c)
$$\lim_{x \to 0} f(x)$$

Solution:

(a) From **both sides** of $x = 1$ we see that the y-value is approaching 1. Thus
$$\lim_{x \to 1} f(x) = 1$$

(b) From **both sides** of $x = -1$ we see that the y-value is approaching 2. It is not important that $f(-1)$ is not defined.
$$\lim_{x \to -1} f(x) = 2$$

(c) Notice from the left $\lim_{x \to -1^-} f(x) = 1$ while $\lim_{x \to -1^+} f(x) = 2$. The one sided limits do not match so
$$\lim_{x \to 0} f(x) = \text{DNE}$$

2.2 Infinite Limits and Limits at Infinity

1. In one or two sentences, explain the following limit expressions. Sketch a graph that satisfies BOTH expressions.

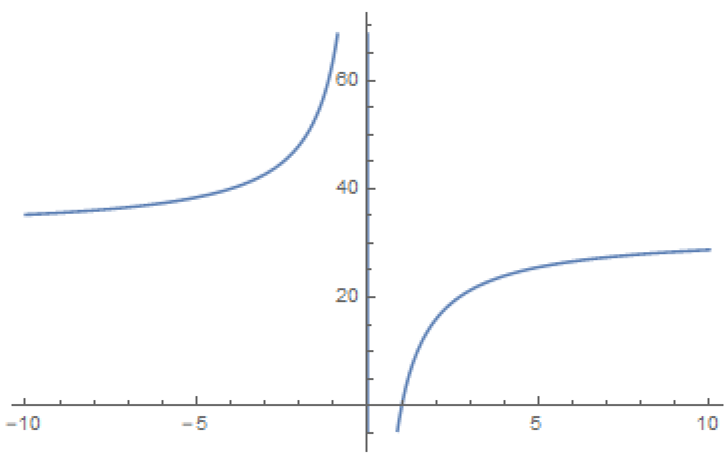
$$\text{(a)} \lim_{x\to 0^-} f(x) = \infty \quad ; \quad \text{(b)} \lim_{x\to\infty} f(x) = 32$$

Solution:

(a) The limit expression $\lim_{x\to 0^-} f(x) = \infty$ means that as x approaches zero from the left, $f(x)$ gets arbitrarily large (and positive). There is a vertical asymptote at $x = 0$.

(b) The limit expression $\lim_{x\to\infty} f(x) = 32$ means that as x gets arbitrarily large, $f(x)$ plateaus at 32. There is a horizontal asymptote at $y = 32$.

An example of a function that satisfies these limits is $f(x) = \frac{32(x-1)}{x}$, the graph of which is shown below.



2. Determine the following limits. Use $+\infty$ or $-\infty$ where appropriate.

(a)

$$\lim_{x\to 2^+} \frac{x^3 - 2x^2 + x + 1}{3x^2 - 1}$$

(b)

$$\lim_{x\to 1^-} \frac{2x+3}{x-1}$$

(c)

$$\lim_{x\to 5^+} \frac{x}{x-5}$$

(d)

$$\lim_{x\to 4^-} \frac{x^2+10}{(x-4)^2}$$

(e)

$$\lim_{x\to 2^+} \frac{x^2-3x+2}{x-2}$$

Solution: The strategy for evaluating limits of rational functions as they approach c is to first (try to) evaluate $f(c)$

- Case 1: If the denominator is nonzero then $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = f(c)$, due to continuity.
- Case 2: If the denominator is zero but the numerator is nonzero, then the one-sided limits may be different and they will be either $+\infty$ or $-\infty$. Think about what happens when you plug in a number really close to a. What happens when you divide by a really small positive/negative number?
- Case 3: If $0/0$, then you need to factor the numerator and denominator and simplify. Then try plugging in again.

(a) Plugging in

$$\lim_{x\to 2^+} \frac{x^3-2x^2+x+1}{3x^2-1} = \boxed{\frac{3}{11}}.$$

Since the denominator is nonzero, this is Case 1 and this is the limit.

(b) Plugging in

$$\lim_{x\to 1^-} \frac{2x+3}{x-1} = \frac{2(1)+3}{1-1} = \frac{5}{0}.$$

Since the denominator is zero and the numerator is nonzero, this is Case 2 and we have to do more work. This is a left-sided limit so we consider plugging in a number close to 1 but slightly less, i.e. $x = 0.9999$

$$\lim_{x\to 1^-} \frac{2x+3}{x-1} \approx \frac{2(0.9999)+3}{0.9999-1} \approx \frac{5}{\text{small}-} = \text{big}- = \boxed{-\infty}.$$

(c) Plugging in

$$\lim_{x\to 5^+} \frac{x}{x-5} = \frac{5}{5-5} = \frac{5}{0}.$$

This is Case 2 again. This is a right-sided limit so we consider plugging in a number close to 5 but slightly greater, i.e. $x = 5.0001$.

$$\lim_{x\to 5^+} \frac{x}{x-5} \approx \frac{5.0001}{5.0001-5} \approx \frac{5}{\text{small}+} = \text{big}+ = \boxed{+\infty}.$$

(d) Plugging in

$$\lim_{x\to 4^-} \frac{x^2+10}{(x-4)^2} = \frac{4^2+10}{(4-4)^2} = \frac{26}{0}.$$

This is Case 2 again. This is a left-sided limit so we consider plugging in a number close to 4 but slightly less, i.e. $x = 3.9999$.

$$\lim_{x\to 4^-} \frac{x^2+10}{(x-4)^2} \approx \frac{(3.9999)^2+10}{(3.9999-4)^2} \approx \frac{26}{(\text{small}-)^2} = \frac{26}{\text{small}+} = \text{big}+ = \boxed{+\infty}.$$

(e) Plugging in

$$\lim_{x\to 2^+} \frac{x^2 - 3x + 2}{x - 2} = \frac{(2)^2 - 3(2) + 2}{2 - 2} = \frac{0}{0}.$$

The numerator and denominator are both zero, so this is Case 3 and we need to factor. We can write $x^2 - 3x + 2 = (x - 2)(x - 1)$. Thus

$$\lim_{x\to 2^+} \frac{x^2 - 3x + 2}{x - 2} = \lim_{x\to 2^+} \frac{\cancel{(x - 2)}(x - 1)}{\cancel{x - 2}} = \lim_{x\to 2^+} x - 1 = 2 - 1 = \boxed{1}.$$

3. Determine the following limits. Use $+\infty$ or $-\infty$ where appropriate.

(a)

$$\lim_{x\to\infty} \frac{2x + 3}{x - 1}$$

(b)

$$\lim_{x\to -\infty} \frac{8x^2 + 1}{-x + 5}$$

(c)

$$\lim_{x\to\infty} \frac{100x}{x^2 - 1}$$

Solution: You can cite the theorem from class ($m < n, m = n, m > n$) or you can go a bit further by remembering that all we need to do is consider the leading terms in the numerator and denominator.

(a) (This is the $m = n$ case) The leading term in the numerator is $2x$ and the leading term in the denominator is x. Therefore

$$\lim_{x\to\infty} \frac{2x + 3}{x - 1} = \lim_{x\to\infty} \frac{2\cancel{x}}{\cancel{x}} = \boxed{2}.$$

(b) (This is the $m > n$ case) The leading term in the numerator is $8x^2$ and the leading term in the denominator is $-x$. Therefore

$$\lim_{x\to -\infty} \frac{8x^2 + 1}{-x + 5} = \lim_{x\to -\infty} \frac{8x^2}{-x} = \lim_{x\to -\infty} -8x = -8(\text{big}-) = \boxed{+\infty}.$$

(c) (This is the $m < n$ case) The leading term in the numerator is $100x$ and the leading term in the denominator is x^2. Therefore

$$\lim_{x\to\infty} \frac{100x}{x^2 - 1} = \lim_{x\to\infty} \frac{100x}{x^2} = \lim_{x\to\infty} \frac{100}{x} = \boxed{0}.$$

4. Find the vertical asymptotes (if any) of the following functions

(a)

$$\frac{x + 1}{(x - 2)(x + 1)}$$

(b)

$$\frac{x^2+1}{x^2-1}$$

(c)

$$\frac{x^2-1}{x^2+1}$$

Solution: A rational function $f(x) = \frac{n(x)}{d(x)}$ has a vertical asymptote at $x = a$ if $d(a) = 0$ and $n(a) \neq 0$. That is, if $f(a) = \frac{\text{nonzero }\#}{0}$ then there is a vertical asymptote at $x = a$.

(a) The denominator is zero when $(x-2)(x+1) = 0$, i.e. when $x = 2$ or $x = -1$. When $x = 2$ the numerator is $2 + 1 = 3 \neq 0$. When $x = -1$ the numerator is $-1 + 1 = 0$. We conclude there is a vertical asymptote at $\boxed{x = 2}$.

(b) The denominator is zero when $x^2 - 1 = 0$, i.e. when $x = \pm 1$. The numerator is nonzero for both $x = \pm 1$ so we conclude there are vertical asymptotes at $\boxed{x = 1 \text{ and } x = -1}$.

(c) The denominator is never zero because $x^2 + 1 = 0$ has no solutions (technically the solution is $x = \pm\sqrt{-1}$ but this is an imaginary number). There are no vertical asymptotes.

5. Find the horizontal asymptotes (if any) of the following functions

(a)

$$\frac{x+1}{(x-2)(x+1)}$$

(b)

$$\frac{3x^2}{x+4}$$

(c)

$$\frac{x^2-2x-2}{5x^2-6x-7}$$

Solution: To finding horizontal asymptotes, take the limit as $x \to \pm\infty$. Recall the theorem from class $(m < n, m = n, m > n)$. **A rational function can have at most one horizontal asymptote.**

(a) (This is the $m < n$ case) We have

$$\lim_{x\to\infty} \frac{x+1}{(x-2)(x+1)} = \lim_{x\to\infty} \frac{1}{x-2} = 0.$$

There is a horizontal asymptote at $\boxed{y = 0}$.

(b) (This is the $m > n$ case) We have

$$\lim_{x\to\infty} \frac{3x^2}{x+4} = \lim_{x\to\infty} \frac{3x^2}{x} = +\infty.$$

and

$$\lim_{x\to-\infty} \frac{3x^2}{x+4} = \lim_{x\to-\infty} \frac{3x^2}{x} = -\infty.$$

There are **no** horizontal asymptotes.

(c) (This is the $m = n$ case) We have

$$\lim_{x\to\infty} \frac{x^2 - 2x - 2}{5x^2 - 6x - 7} = \lim_{x\to\infty} \frac{x^2}{5x^2} = \frac{1}{5}.$$

There is a horizontal asymptote at $\boxed{y = \frac{1}{5}}$.

2.3 Continuity

1. What are the conditions for continuity? Explain why the function in 2.1 # 1 was not continuous at $x = -1$ and $x = 0$.

   Solution: The conditions for continuity of $f(x)$ at $x = c$ are

   (i) $f(c)$ exists

   (ii) $\lim_{x\to c} f(x)$ exists

   (iii) $\lim_{x\to c} f(x) = f(c)$

   The function in 2.1 # 1 was not continuous at $x = -1$ because $f(-1)$ was not defined. It was not continuous at $x = 0$ because $\lim_{x\to 0} f(x)$ did not exist.

2. List the intervals where the following functions are continuous.

$$\text{(a) } 11x^{10} - x^4 \quad ; \quad \text{(b) } (x-5)^{1/4} \quad ; \quad \text{(c) } \frac{x+1}{x+2}$$

   Solution:

   (a) $f(x) = 11x^{10} - x^4$ is a polynomial and thus is continuous for all x in $(-\infty, \infty)$.

   (b) $f(x) = (x-5)^{1/4} = \sqrt[4]{x-5}$ is an even root and thus is continuous whenever $x - 5$ is nonnegative. Clearly $x - 5 \geq 0$ for all $x \geq 5$ thus f is continuous on $(5, \infty)$.

   (c) $f(x) = \frac{x+1}{x+2}$ is a rational function and thus is continuous whenever $x + 2 \neq 0$, i.e. when $x \neq -2$. In interval notation, we say f is continuous on $(-\infty, -2) \cup (-2, \infty)$.

3. Determine the sign charts for the following functions.

   (a)

$$\frac{x-1}{x+1}$$

(b)

$$\frac{x}{x^2-9}$$

Solution:

(a) First we find the partition numbers of $f(x) = \frac{x-1}{x+1}$.

- $f(x) = 0$ when $x = 1$.
- $f(x)$ is discontinuous when $x = -1$.

The partition numbers are $x = 1, -1$. These two partition numbers divide the number line into three intervals:

- $(-\infty, -1)$: Let's test $x = -2$. We have $f(-2) = \frac{-2-1}{-2+1} = 3 > 0$.
- $(-1, 1)$: Let's test $x = 0$. We have $f(0) = \frac{0-1}{0+1} = -1 < 0$.
- $(1, \infty)$: Let's test $x = 2$. We have $f(2) = \frac{2-1}{2+1} = \frac{1}{3} > 0$.

The sign chart is

$+$	$-$	$+$
$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$

(b) First we find the partition numbers of $f(x) = \frac{x}{x^2-9}$.

- $f(x) = 0$ when $x = 0$.
- $f(x)$ is discontinuous when $x =^2 -9 = 0$. That is, when $x = \pm 3$.

The partition numbers are $x = -3, 0, 3$. These three partition numbers divide the number line into four intervals:

- $(-\infty, -3)$: Let's test $x = -4$. We have $f(-4) = \frac{-4}{(-4)^2-9} = -\frac{4}{7} < 0$.
- $(-3, 0)$: Let's test $x = -1$. We have $f(-1) = \frac{-1}{(-1)^2-9} = \frac{1}{8} > 0$.
- $(0, 3)$: Let's test $x = 1$. We have $f(1) = \frac{1}{1-9} = -\frac{1}{8} < 0$.
- $(3, \infty)$: Let's test $x = 4$. We have $f(4) = \frac{4}{4^2-9} = \frac{4}{7} > 0$.

The sign chart is

$-$	$+$	$-$	$+$
$(-\infty, -3)$	$(-3, 0)$	$(0, 3)$	$(3, \infty)$

4. Use your sign charts to solve the inequalities.

(a)

$$\frac{x-1}{x+1} > 0$$

(b)

$$\frac{x}{x^2-9} < 0$$

Solution: All we need to do is join the intervals from the previous part.

(a) $f(x) = \frac{x-1}{x+1} > 0$ for x in $(-\infty, -1) \cup (1, \infty)$.

(b) $f(x) = \frac{x}{x^2-9} < 0$ for x in $(-\infty, -3) \cup (0, 3)$.

2.4 The Derivative

1. Compute the average rate of change.

   (a)
   $$x^2 - 3x - 4 \quad \text{on} \quad [-4, 4]$$

   (b)
   $$\sqrt{x} - \frac{x^2}{3} \quad \text{on} \quad [0, 1]$$

   Solution:

   (a) The average rate of change is
   $$\text{ARC} = \frac{f(4) - f(-4)}{4 - (-4)} = \frac{[4^2 - 3(4) - 4] - [(-4)^2 - 3(-4) - 4]}{8} = \boxed{-3}.$$

   (b) The average rate of change is
   $$\text{ARC} = \frac{f(1) - f(0)}{1 - 0} = \frac{\left[\sqrt{1} - \frac{1^2}{3}\right] - \left[\sqrt{0} - \frac{0^2}{3}\right]}{1} = \boxed{\frac{2}{3}}.$$

2. Find $f'(x)$.

   (a)
   $$x^2 + 5x + 1$$

   (b)
   $$10\sqrt{x+5} - 9$$

   (c)
   $$\frac{x}{x+2}$$

   Solution:

   (a) Breaking the problem into parts:

   - $$f(x) = x^2 + 5x + 1$$
   - $$\begin{aligned} f(x+h) &= (x+h)^2 + 5(x+h) + 1 \\ &= x^2 + 2xh + h^2 + 5x + 5h + 1 \end{aligned}$$
   - $$\begin{aligned} f(x+h) - f(x) &= (\cancel{x^2} + 2xh + h^2 + \cancel{5x} + 5h + \cancel{1}) - (\cancel{x^2 + 5x + 1}) \\ &= 2xh + h^2 + 5h \end{aligned}$$

- $$\frac{f(x+h)-f(x)}{h} = \frac{2xh+h^2+5h}{h} = 2x+h+5$$
- $$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} 2x+h+5 \\ &= 2x+0+5 \\ &= \boxed{2x+5}. \end{aligned}$$

(b) Breaking the problem into parts:

- $$f(x) = 10\sqrt{x+5} - 9$$
- $$f(x+h) = 10\sqrt{x+h+5} - 9$$
- $$\begin{aligned} f(x+h)-f(x) &= \left(10\sqrt{x+h+5} - \cancel{9}\right) - \left(10\sqrt{x+5} - \cancel{9}\right) \\ &= 10\left(\sqrt{x+h+5} - \sqrt{x+5}\right) \end{aligned}$$
- $$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{10\left(\sqrt{x+h+5} - \sqrt{x+5}\right)}{h} \\ &= 10\frac{\sqrt{x+h+5} - \sqrt{x+5}}{h} \cdot \frac{\sqrt{x+h+5} + \sqrt{x+5}}{\sqrt{x+h+5} + \sqrt{x+5}} \\ &= 10\frac{\cancel{x}+h+\cancel{5}-(\cancel{x+5})}{h\left(\sqrt{x+h+5} + \sqrt{x+5}\right)} \\ &= 10\frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h+5} + \sqrt{x+5}\right)} \\ &= \frac{10}{\sqrt{x+h+5} + \sqrt{x+5}} \end{aligned}$$
- $$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{10}{\sqrt{x+h+5} + \sqrt{x+5}} \\ &= \frac{10}{\sqrt{x+0+5} + \sqrt{x+5}} \\ &= \frac{10}{2\sqrt{x+5}} \\ &= \boxed{\frac{5}{\sqrt{x+5}}} \end{aligned}$$

(c) Breaking the problem into parts

- $$f(x) = \frac{x}{x+2}$$

- $$f(x+h) = \frac{x+h}{x+h+2}$$

- $$\begin{aligned} f(x+h) - f(x) &= \frac{x+h}{x+h+2} - \frac{x}{x+2} \\ &= \frac{(x+h)(x+2) - x(x+h+2)}{(x+h+2)(x+2)} \\ &= \frac{\cancel{x^2} + \cancel{2x} + \cancel{hx} + 2h - (\cancel{x^2 + hx + 2x})}{x^2 + hx + 2x + 2x + 2h + 4} \\ &= \frac{2h}{x^2 + hx + 4x + 2h + 4} \end{aligned}$$

- $$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{2\cancel{h}}{\cancel{h}(x^2 + hx + 4x + 2h + 4)} \\ &= \frac{2}{x^2 + hx + 4x + 2h + 4} \end{aligned}$$

- $$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{2}{x^2 + hx + 4x + 2h + 4} \\ &= \frac{2}{x^2 + 0 + 4x + 0 + 4} \\ &= \boxed{\frac{2}{x^2 + 4x + 4}} \end{aligned}$$

3. Find the equation of the tangent line of:

   (a)
   $$x^2 + 5x + 1 \quad \text{at} \quad x = 1$$

   (b)
   $$10\sqrt{x+5} - 9 \quad \text{at} \quad x = 4$$

   Solution: Recall from high school algebra that the equation of a line can be determined if you know one point on the line & the slope of the line (this is called point-slope form). Suppose (x_0, y_0) is a point on the line and the slope is m, then the equation of the line is

   $$y - y_0 = m(x - x_0)$$

(a) For the function $f(x) = x^2 + 5x + 1$, the y-value of the point with $x = 1$ is $f(1) = 7$. That is the, the point of tangency is $(1, 7)$. The slope of the tangent line at $x = 1$ is given by $f'(1) = 2(1) + 5 = 7$. The equation of the tangent line to f at $x = 1$ is

$$y - 7 = 7(x - 1)$$

which can be rearranged to

$$\boxed{y = 7x}.$$

(b) Similar to the previous problem, we have $f(4) = 21$ so $(4, 21)$ is a point on the line. The slope of the tangent line is $f'(4) = \frac{5}{\sqrt{4+5}} = \frac{5}{3}$. The equation of the tangent line to f at $x = 4$ is

$$y - 21 = \frac{5}{3}(x - 4)$$

which can be rearranged to

$$\boxed{y = \frac{5}{3}x + \frac{43}{3}}.$$