Math 19 E
Spring 2019
Exam 2
March 20
Version A

Name: ________________________

SOLUTIONS

This exam contains 7 pages and 6 questions. Total of points is 100. For full credit you must show your work. Partial credit may be given for incorrect solutions if sufficient work is shown. Messy/unorganized answers may be penalized, even if correct.

Grade Table (for teacher use only)

Question	Points	Score
1	40	
2	10	
3	10	
4	20	
5	10	
6	10	
Total:	100	

HONORS PLEDGE (sign after exam is completed): I have neither given nor received aid on this exam, nor have I observed a violation of the UVM Code of Academic Integrity.

Signature: ____

1. (40 points) Calculate the derivatives of the following functions.

   (a) (8 points)

$$h(x) = 7x^4 - 9x + 10$$

$$h'(x) = 28x^3 - 9$$

   (b) (8 points)

$$h(x) = 10\ln(x^2 + 2x + 1)$$

$$h'(x) = \frac{10}{x^2 + 2x + 1} \cdot (2x + 2)$$

   (c) (8 points)

$$h(x) = \frac{1 - e^x}{1 + x^2}$$

$$h'(x) = \frac{-e^x(1 + x^2) - 2x(1 - e^x)}{(1 + x^2)^2}$$

(d) (8 points)

$$h(x) = (x+1)^4 \ln(x)$$

$$h'(x) = 4(x+1)^3 \ln(x) + \frac{(x+1)^4}{x}$$

(e) (8 points)

$$h(x) = e^{x^2+12x}$$

$$h'(x) = e^{x^2+12x}(2x+12)$$

2. (10 points) For the function $h(x) = e^{x^2+12x}$ from Problem 1(e), find the equation of the tangent line to $h(x)$ when $x = 0$.

   The y-value when $x = 0$ is $h(0) = e^{0^2+12\cdot 0} = 1$. The slope is $h'(0) = e^{0^2+12\cdot 0} \cdot (2\cdot 0+12) = 12$. Using point-slope form, the equation of the tangent line to h at $x = 0$ is

$$y - 1 = 12(x - 0)$$

   which can be rearranged to

$$y = 12x + 1$$

3. (10 points)

(a) (5 points) Suppose $h(x) = \dfrac{f(x)}{g(x)}$ and suppose we know that $f(3) = 1$, $f'(3) = 3$, $g(3) = 4$, and $g'(3) = 3$. Calculate $h'(3)$.

$$h'(3) = \frac{3 \cdot 4 - 3 \cdot 1}{4^2} = \frac{12 - 3}{16} = \frac{9}{16}$$

(b) (5 points) Consider the function $T(x)$ defined as

$$T(x) = \ln\left(\frac{f(x)}{x^2 g(x)}\right).$$

Find $T'(x)$. Hint: use properties of logarithms to simplify.

Rewrite $T(x)$ as

$$\begin{aligned} T(x) &= \ln(f(x)) - \ln(x^2) - \ln(g(x)) \\ &= \ln(f(x)) - 2\ln(x) - \ln(g(x)) \end{aligned}$$

Using the chain rule

$$\boxed{T'(x) = \frac{f'(x)}{f(x)} - \frac{2}{x} - \frac{g'(x)}{g(x)}}$$

Alternatively you could use the product, quotient, and chain rules. Note that the derivative of $x^2 g(x)$ is $2xg(x) + g'(x)x^2$.

$$\begin{aligned} T'(x) &= \frac{1}{\frac{f(x)}{x^2 g(x)}} \cdot \frac{d}{dx}\left(\frac{f(x)}{x^2 g(x)}\right) \\ &= \frac{x^2 g(x)}{f(x)} \cdot \frac{f'(x)[x^2 g(x)] - [2xg(x) + g'(x)x^2]f(x)}{[x^2 g(x)]^2} \\ &= \frac{f'(x)[x^2 g(x)] - [2xg(x) + g'(x)x^2]f(x)}{x^2 f(x) g(x)} \\ &= \frac{x^2 f'(x) g(x)}{x^2 f(x) g(x)} - \frac{2x f(x) g(x)}{x^2 f(x) g(x)} - \frac{x^2 f(x) g'(x)}{x^2 f(x) g(x)} \\ &= \frac{f'(x)}{f(x)} - \frac{2}{x} - \frac{g'(x)}{g(x)} \end{aligned}$$

4. (20 points) For the implicit curve defined by the equation: $x^2 + y^2 = e^y$

(a) (10 points) Use implicit differentiation to find $\frac{dy}{dx}$.

$$x^2 + y^2 = e^y$$
$$2x + 2yy' = e^y y'$$
$$2x = e^y y' - 2yy'$$
$$2x = y'(e^y - 2y)$$
$$\boxed{y' = \frac{2x}{e^y - 2y}}$$

(b) (5 points) Find the equation of the tangent line at the point $(1, 0)$.

The slope is

$$y'\Big|_{(1,0)} = \frac{2(1)}{e^0 - 2(0)} = \frac{2}{1} = 2.$$

Using point-slope form, the equation of the tangent line at $(1, 0)$ is

$$y - 0 = 2(x - 1)$$

which can be rearranged to

$$y = 2x - 2$$

(c) (5 points) For what value(s) of x will the tangent line to the curve be horizontal.

The tangent line will be horizontal when $y' = 0$, i.e. when

$$\frac{2x}{e^y - 2y} = 0.$$

A fraction is zero if and only if the numerator is zero which happens when $2x = 0$, which implies $\boxed{x = 0}$.

5. (10 points) A 40-foot ladder is placed against a wall. If the top of the ladder is sliding down the wall at 4 feet per second, at what rate is the bottom of the ladder moving away from the wall when the bottom of the ladder is 20 feet away from the wall?

   Let $x = x(t)$ be the distance of the bottom of the ladder from the wall at time t and let $y = y(t)$ be the distance of the top of the ladder from the ground at time t. By the Pythagorean theorem we have the relation

$$x^2 + y^2 = 40^2.$$

   Implicitly differentiating both sides with respect to time we have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

   which can be solved for $\dfrac{dx}{dt}$

$$\frac{dx}{dt} = \frac{-2y\frac{dy}{dt}}{2x}.$$

   We know $x = 20$ and $\frac{dy}{dt} = -4$. By the Pythagorean theorem we can find y.

$$y = \sqrt{40^2 - 20^2} = \sqrt{1200} \approx 34.64.$$

   Plugging into the related-rates equation

$$\frac{dx}{dt} = \frac{-2(-4)(34.64)}{2(20)} \approx 6.928.$$

6. (10 points) Find the intervals on which $f(x)$ is increasing and the intervals on which $f(x)$ is decreasing.

$$f(x) = x^3 - 48x + 3$$

Hint: make a sign chart for the derivative of f.

We have $f'(x) = 3x^2 - 48$. Note that f' is a polynomial and therefore the only partition numbers are when $f' = 0$, i.e. when

$$3x^2 - 48 = 0 \implies 3x^2 = 48 \implies x^2 = 16 \implies x = \pm 4.$$

There are two partition numbers: $x = -4$ and $x = 4$. These two partition numbers divide the number line into three intervals.

- $(-\infty, -4)$: Let's test $x = -5$. We have $f'(-5) = 3(-5)^2 - 48 = 27 > 0$.
- $(-4, 4)$: Let's test $x = 0$. We have $f'(0) = 3(0)^2 - 48 = -48 < 0$.
- $(4, \infty)$: Let's test $x = 5$. We have $f'(5) = 3(5)^2 - 48 = 27 > 0$.

The sign chart for f' is

$+$	$-$	$+$
$(-\infty, -4)$	$(-4, 4)$	$(4, \infty)$

Thus f' is greater than zero on $(-\infty, -4) \cup (4, \infty)$ and less than zero on $(-4, 4)$. By the fact we learned in class, f is increasing on $(-\infty, -4) \cup (4, \infty)$ and decreasing on $(-4, 4)$.