**Math 19 E**
**Spring 2019**
**Final Exam**
**May 6**

**Name:** ______________________

# PRACTICE EXAM SOLUTIONS

**This exam contains 9 pages and 7 questions. Total of points is 100. For full credit you must show your work. Partial credit may be given for incorrect solutions if sufficient work is shown. Messy/unorganized answers may be penalized, even if correct.**

Grade Table (for teacher use only)

| Question | Points | Score |
|---|---|---|
| 1 | 20 | |
| 2 | 20 | |
| 3 | 20 | |
| 4 | 12 | |
| 5 | 8 | |
| 6 | 8 | |
| 7 | 12 | |
| Total: | 100 | |

**HONORS PLEDGE** (sign after exam is completed): I have neither given nor received aid on this exam, nor have I observed a violation of the UVM Code of Academic Integrity.

Signature: ______________________________________________

1. (20 points) Evaluate the following indefinite integrals

(a) (8 points)

$$\begin{aligned}\int (3x^5 - 2x + 3)dx &= 3\int x^5 dx - 2\int x dx + 3\int 1 dx \\ &= 3 \cdot \frac{x^6}{6} - 2 \cdot \frac{x^2}{2} + 3x + C \\ &= \boxed{\frac{x^6}{2} - x^2 + 3x + C}\end{aligned}$$

(b) (6 points)

$$\int (6x^2 + 10)e^{x^3+5x} dx$$

Let $u = x^3 + 5x \implies du = (3x^2 + 5)dx \implies 2du = (6x^2 + 10)dx.$

$$\begin{aligned}\int (6x^2 + 10)e^{x^3+5x} dx &= \int e^u \cdot 2du \\ &= 2\int e^u du \\ &= 2e^u + C \\ &= \boxed{2e^{x^3+5x} + C}\end{aligned}$$

(c) (6 points)

$$\int \frac{2x}{x^2 - 8} dx$$

Let $u = x^2 - 8 \implies du = 2x dx.$

$$\begin{aligned}\int \frac{2x}{x^2 - 8} dx &= \int \frac{1}{u} du \\ &= \ln|u| + C \\ &= \boxed{\ln|x^2 - 8| + C}\end{aligned}$$

2. (20 points) More integration topics

(a) (6 points) Given that

$$\frac{d}{dx}\left(\frac{1}{1+e^{-x}}\right) = \frac{e^{-x}}{(1+e^{-x})^2}.$$

Find the particular antiderivative of

$$\frac{e^{-x}}{(1+e^{-x})^2}$$

which passes through the point $(0, 0)$.

Recall that an antiderivative of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$, and we denote the family of antiderivatives of $f$ by $\int f(x)dx = F(x) + C$. Therefore,

$$\int \frac{e^{-x}}{(1+e^{-x})^2}dx = \frac{1}{1+e^{-x}} + C.$$

The particular antiderivative $F(x)$ that passes through $(0, 0)$ must satisfy

$$F(0) = \frac{1}{1+e^{-0}} + C = 0$$

which means $C = -\frac{1}{2}$. Thus

$$\boxed{F(x) = \frac{1}{1+e^{-x}} - \frac{1}{2}}$$

(b) (6 points) Given that

$$\int_0^{36} xdx = 648; \qquad \int_0^{9} \sqrt{x}dx = 18; \qquad \int_9^{36} \sqrt{x}dx = 126$$

Compute $\int_0^{36}(2x + \sqrt{x})dx$.

$$\begin{aligned}
\int_0^{36} (2x + \sqrt{x})dx &= 2\int_0^{36} xdx + \int_0^{36} \sqrt{x}dx \\
&= 2\int_0^{36} xdx + \left(\int_0^{9} \sqrt{x}dx + \int_9^{36} \sqrt{x}dx\right) \\
&= 2 \cdot 648 + 18 + 126 \\
&= \boxed{1440}
\end{aligned}$$

(c) (8 points) Compute the definite integral

$$\int_5^7 (3x+2)dx = 3\cdot\frac{x^2}{2} + 2x\Big|_{x=5}^{x=7}$$

$$= \left[3\cdot\frac{7^2}{2} + 2(7)\right] - \left[3\cdot\frac{5^2}{2} + 2(5)\right]$$

$$= \boxed{40}$$

3. (20 points) Limits – When evaluating limits make sure to briefly justify your answer. Write DNE if a limit does not exist.

(a) (6 points) Evaluate

$$\lim_{x\to 1}\frac{x}{x-1}$$

Consider plugging in a number close to 1 but slightly smaller, say 0.9999. We have

$$\lim_{x\to 1^-}\frac{x}{x-1} \approx \frac{1}{-0.0001} \to -\infty.$$

Alternatively, consider plugging in a number close to 1 but slightly larger, say 1.0001. We have

$$\lim_{x\to 1^+}\frac{x}{x-1} \approx \frac{1}{+0.0001} \to +\infty.$$

The one-sided limits do not match so we conclude

$$\lim_{x\to 1}\frac{x}{x-1} = \boxed{DNE}$$

(b) (6 points) Evaluate

$$\lim_{x\to 0^+}\frac{\ln(1+x)}{x} = \frac{0}{0} \qquad \text{[apply L'Hopital's rule]}$$

$$= \lim_{x\to 0^+}\frac{1/(1+x)}{1}$$

$$= \lim_{x\to 0^+}\frac{1}{1+x}$$

$$= \boxed{1}$$

(c) (8 points) Find all horizontal and vertical asymptotes of

$$f(x) = \frac{2(x^2-9)}{x^2+5x+6}$$

The leading term (the term with the highest exponent) in the numerator is $2x^2$ and the leading term in the denominator is $x^2$. Recall that when the exponents are equal, the ratio of the coefficients on these terms gives us our horizontal asymptote. Therefore there is a horizontal asymptote at $\boxed{y=2}$.

Alternatively, if you do not remember the three cases ($m = n, m < n, m > n$), you can find horizontal asymptotes by evaluating $\lim_{x\to\infty} f(x)$. Again, we only need to look at the ratio of the leading terms.

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{2x^2-18}{x^2+5x+6} \\ &= \lim_{x\to\infty} \frac{2x^2}{x^2} \\ &= \lim_{x\to\infty} 2 \\ &= 2.\end{aligned}$$

To find vertical asymptotes we find where the denominator is zero, but at the same time the numerator has to be nonzero. We have

$$x^2+5x+6=0 \implies (x+2)(x+3)=0 \implies x=-2,\ x=-3.$$

However, $(x+3)$ is a factor of the numerator and so the numerator is zero when $x=-3$. When $x=-2$ the numerator is $-10$ which is nonzero. We conclude there is a vertical asymptote at $\boxed{x=-2}$.

4. (12 points) Compute the following derivatives

(a) (6 points)

$$\frac{d}{dx}\left(x^3 + 2\ln(x) - 4\right) = \boxed{3x^2 + \frac{2}{x}}$$

(b) (6 points)

$$\frac{d}{dx}\left(xe^{2x}\right) = \boxed{e^{2x} + 2xe^{2x}}$$

by the product and chain rules.

5. (8 points) Find the equation of the tangent line to the function

$$f(x) = \frac{e^x}{x}$$

at the point $(1, e)$.

Using the quotient rule, the derivative of $f$ is

$$f'(x) = \frac{xe^x - e^x}{x^2}.$$

The slope of the tangent line at $x = 1$ is

$$f'(1) = \frac{1e^1 - e^1}{1^2} = 0.$$

Using point-slope form, the equation of the tangent line at $(1, e)$ is

$$y - e = 0(x - 1)$$

which simplifies to the equation for a horizontal line, $\boxed{y = e}$.

6. (8 points) Find $y'$ given that

$$2y + xy - 1 = 0.$$

Recall that when we do implicit differentiation we treat $y$ as a function of $x$ and so we have to use the product rule to differentiate $xy$. Note: if we had a term like $y^4$ then we would also do the chain rule and have $\frac{d}{dx}y^4 = 4y^3y'$.

$$\begin{aligned} &2y' + y + xy' = 0 \\ &2y' + xy' = -y \\ &y'(2 + x) = -y \\ &\boxed{y' = -\frac{y}{2 + x}} \end{aligned}$$

7. (12 points) For the function $f(x) = -x^3 + 12x + 2$

(a) (8 points) Find the intervals where $f$ is increasing/decreasing

We have $f'(x) = -3x^2 + 12$. Note that $f'$ is a polynomial and therefore the only partition numbers are when $f' = 0$, i.e. when

$$-3x^2 + 12 = 0 \implies 3x^2 = 12 \implies x^2 = 4 \implies x = \pm 2.$$

There are two partition numbers: $x = -2$ and $x = 2$. These two partition numbers divide the number line into three intervals.

- $(-\infty, -2)$: Let's test $x = -3$. We have $f'(-3) = -3(-3)^2 + 12 = -15 < 0$.
- $(-2, 2)$: Let's test $x = 0$. We have $f'(0) = -3(0)^2 + 12 = 12 > 0$.
- $(2, \infty)$: Let's test $x = 3$. We have $f'(3) = -3(3)^2 + 12 = -15 < 0$.

The sign chart for $f'$ is

| $-$ | $+$ | $-$ |
|---|---|---|
| $(-\infty, -2)$ | $(-2, 2)$ | $(2, \infty)$ |

Thus $f'$ is greater than zero on $(-2, 2)$ and less than zero on $(-\infty, -2) \cup (2, \infty)$. We conclude that $f$ is increasing on $(-2, 2)$ and decreasing on $(-\infty, -2) \cup (2, \infty)$.

(b) (4 points) Find any local maxima or minima.

$\boxed{x = -2}$ is a local minimum (since $f$ changes from decreasing to increasing) and $\boxed{x = 2}$ is a local maximum (since $f$ changes from increasing to decreasing)

Bonus

1. Suppose $f$ is a continuous function and $f(x) > 0$ for all $x$. Explain why the function

$$g(t) = \int_0^t f(x)dx$$

always increases as $t$ increases.

There are several different explanations for this. By the Fundamental Theorem of Calculus $g(t) = F(t) - F(0)$. If we take the derivative with respect to $t$, note that $F(0)$ is a constant and so

$$g'(t) = F'(t) = f(t)$$

and we assumed that $f > 0$. Hence $g'(t) > 0$ for all $t$ and so $g(t)$ is increasing always.

Alternatively, simply consider what the definite integral represents. As you increase $t$ you are taking the area under the curve on a larger interval. Since the function is always positive, this will result in a greater area.

2. Compute the 100th derivative of

$$f(x) = xe^x$$

Using the product rule we take the first three derivatives and look for a pattern.

$$\frac{d}{dx}(xe^x) = e^x + xe^x$$

$$\frac{d^2}{dx^2}(xe^x) = \frac{d}{dx}(e^x + xe^x) = e^x + (e^x + xe^x) = 2e^x + xe^x$$

$$\frac{d^3}{dx^3}(xe^x) = \frac{d}{dx}(2e^x + xe^x) = 2e^x + (e^x + xe^x) = 3e^x + xe^x$$

Each time we differentiate we end up adding $e^x$. The 100th derivative will be

$$\frac{d^{100}}{dx^{100}}(xe^x) = \boxed{100e^x + xe^x}$$

3. The 3rd-degree Taylor polynomial for $f$ at 0 is defined to be the function

$$p(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{6}f'''(0)x^3$$

where $f'(0)$, $f''(0)$, and $f'''(0)$ are the 1st, 2nd, and 3rd derivatives of $f$ evaluated at zero. This gives a polynomial function $p(x)$ which, near $x = 0$, has approximately the same graph as $f(x)$.

Compute the 3rd-degree Taylor polynomial for $f(x) = e^x$.

The derivatives of $f(x)$ are easy! We have $f(x) = f'(x) = f''(x) = f'''(x) = e^x$. Therefore $f(0) = f'(0) = f''(0) = f'''(0) = 1$. The 3rd-degree Taylor polynomial for $e^x$ is

$$p(x) = \boxed{1 + x + \frac{1}{x}x^2 + \frac{1}{6}x^3}.$$

The graph of $p(x)$ (in blue) is shown below along with $e^x$ (in red). Near $x = 0$ the approximation $p(x)$ is very good.