Math 19 E **Name:** ____________________
Spring 2019
Exam 2
March 20

PRACTICE EXAM – SOLUTIONS

This exam contains 7 pages and 6 questions. Total of points is 100. For full credit you must show your work. Partial credit may be given for incorrect solutions if sufficient work is shown. Messy/unorganized answers may be penalized, even if correct.

Grade Table (for teacher use only)

Question	Points	Score
1	40	
2	10	
3	10	
4	20	
5	10	
6	10	
Total:	100	

HONORS PLEDGE (sign after exam is completed): I have neither given nor received aid on this exam, nor have I observed a violation of the UVM Code of Academic Integrity.

Signature: ____

1. (40 points) Calculate the derivatives of the following functions.

   (a) (8 points)

$$h(x) = 3(2x^2 + 5x - 8)^4$$

Apply the chain rule (special case #1).

$$\begin{aligned} h'(x) &= 3 \cdot 4(2x^2 + 5x - 8)^3 \cdot (4x + 5) \\ &= \boxed{12(2x^2 + 5x - 8)^3(4x + 5)} \end{aligned}$$

   (b) (8 points)

$$h(x) = 10 \ln(1 - 7x)$$

Apply the chain rule (special case #3).

$$h'(x) = 10 \cdot \frac{1}{1 - 7x} \cdot (-7)$$

$$= \boxed{\frac{-70}{1 - 7x}}$$

   (c) (8 points)

$$h(x) = \frac{1 - e^x}{1 + e^x}$$

Apply the quotient rule.

$$h'(x) = \boxed{\frac{-e^x(1 + e^x) - e^x(1 - e^x)}{(1 + e^x)^2}}$$

(d) (8 points)

$$h(x) = x^2 \ln(x)$$

Apply the product rule.

$$\begin{aligned} h'(x) &= 2x\ln(x) + \frac{1}{x}x^2 \\ &= \boxed{2x\ln(x) + x} \end{aligned}$$

(e) (8 points)

$$h(x) = 5xe^{2x}$$

Apply the product rule with $f(x) = 5x$ and $g(x) = e^{2x}$. Use the chain rule (special case # 2) to differentiate e^{2x}. Then $f'(x) = 5$ and $g'(x) = 2e^{2x}$.

$$\begin{aligned} h'(x) &= 5e^{2x} + 2e^{2x}(5x) \\ &= 5e^{2x} + 10xe^{2x} \\ &= \boxed{5e^{2x}(1+2x)} \end{aligned}$$

In the last line we factored out $5e^{2x}$ from both terms, for the sake of helping with Problem 2(a).

2. (10 points) For the function $h(x) = 5xe^{2x}$ from Problem 1(e),

(a) (5 points) For what value(s) of x will the tangent line to $h(x)$ be horizontal?
We need to find when $h'(x) = 0$. Setting our result from Problem 1(e) to be equal to zero, we have

$$5e^{2x}(1+2x) = 0$$

Since $5e^{2x}$ is never zero we can divide both sides by $5e^{2x}$. We are left with

$$1 + 2x = 0$$

which has solution $\boxed{x = -1/2}$.

(b) (5 points) Find the equation of the tangent line to $h(x)$ when $x = 0$.
When $x = 0$ the corresponding y-value is $h(0) = 5(0)e^{2\cdot 0} = 0$. The slope at this point is given by the derivative

$$h'(0) = 5e^{2\cdot 0}(1 + 2\cdot 0) = 5.$$

Using point-slope form, the equation of the tangent line is

$$y - 0 = 5(x - 0)$$

which simplifies to

$$\boxed{y = 5x}$$

3. (10 points)

(a) (5 points) Suppose $h(x) = \dfrac{f(x)}{g(x)}$ and suppose we know that $f(1) = 1$, $f'(1) = 4$, $g(1) = 2$, and $g'(1) = 7$. Calculate $h'(1)$.

Apply the quotient rule.

$$h'(1) = \frac{f'(1)g(1) - g'(1)f(1)}{[g(1)]^2} = \frac{4 \cdot 2 - 7 \cdot 1}{2^2} = \frac{8-7}{4} = \boxed{\frac{1}{4}}$$

(b) (5 points) Consider the function $T(x)$ defined as

$$T(x) = \ln\left(\frac{x^3 f(x)}{g(x)}\right).$$

Find $T'(x)$. Hint 1: use properties of logarithms to simplify (otherwise you have to do product + quotient rule = yikes!!!). Hint 2: your answer will involve $f'(x)$ and $g'(x)$ due to chain rule.

Apply properties of logarithms.

$$\begin{aligned} T(x) &= \ln(x^3) + \ln(f(x)) - \ln(g(x)) \\ &= 3\ln(x) + \ln(f(x)) - \ln(g(x)). \end{aligned}$$

Now we differentiate. The second and third terms require the chain rule (special case # 3).

$$\begin{aligned} T'(x) &= 3 \cdot \frac{1}{x} + \frac{1}{f(x)} \cdot f'(x) - \frac{1}{g(x)} \cdot g'(x) \\ &= \boxed{\frac{3}{x} + \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}} \end{aligned}$$

This is an example of a problem where logarithm properties simplify things tremendously.

4. (20 points) For the implicit curve defined by the equation: $y^4 - 3y^3 + x^3 = 6$

   (a) (10 points) Use implicit differentiation to find $\dfrac{dy}{dx}$.

   We differentiate both sides and apply the chain rule to the y terms.

$$\begin{aligned}
&\frac{d}{dx}(y^4 - 3y^3 + x^3) = \frac{d}{dx}(6) && \\
&4y^3y' - 9y^2y' + 3x^2 = 0 && \\
&4y^3y' - 9y^2y' = -3x^2 && \text{subtract } 3x^2 \\
&y'(4y^3 - 9y^2) = -3x^2 && \text{factor out } y' \\
&\boxed{y' = \frac{-3x^2}{4y^3 - 9y^2}} && \text{divide through}
\end{aligned}$$

   (b) (5 points) Find the equation of the tangent line at the point $(2, 1)$.

   At the point $(2, 1)$ the slope of the tangent line is given by the derivative

$$y'\big|_{(2,1)} = \frac{-3(2)^2}{4(1)^3 - 9(1)^2} = \frac{-12}{-5} = \frac{12}{5}.$$

   Using point-slope form, the equation of the tangent line is

$$y - 1 = \frac{12}{5}(x - 2)$$

   rearranging gives us

$$\boxed{y = \frac{12}{5}x - \frac{19}{5}}$$

   (c) (5 points) For what value(s) of x will the tangent line to the curve be horizontal.

   We need to find when $y' = 0$. Remember that a fraction is zero only if the numerator is zero. Thus $h'(x) = 0$ only if

$$-3x^2 = 0$$

   which means $\boxed{x = 0}$.

5. (10 points) **One of these two problems (but not both) will appear on the exam!**

- A 40-foot ladder is placed against a wall. If the top of the ladder is sliding down the wall at 3 feet per second, at what rate is the bottom of the ladder moving away from the wall when the bottom of the ladder is 22 feet away from the wall?

  Let $x = x(t)$ be the distance of the bottom of the ladder from the wall at time t and let $y = y(t)$ be the distance of the top of the ladder from the ground at time t. By the Pythagorean theorem we have the relation

  $$x^2 + y^2 = 40^2.$$

  Differentiating both sides of the equation above with respect to time,

  $$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

  We want to find dx/dt when $x = 22$ feet, given that $dy/dt = -3$ feet per second. Using the Pythagorean theorem to get y, we have $y = \sqrt{40^2 - 22^2} \approx 33.41$ feet. Plugging into the related-rates equation we have

  $$2(22)\frac{dx}{dt} + 2(33.41)(-3) = 0.$$

  Solving for dx/dt,

  $$\frac{dx}{dt} = \frac{-2(33.41)(-3)}{2(22)} \approx 4.6$$

  The bottom of the ladder is moving away from the wall at a rate of $\boxed{\text{4.6 feet per second}}$.

- The radius of a spherical balloon is increasing at the rate of 6 centimeters per minute. How fast is the volume changing when the radius is 24 centimers?

  Note: the formula for the volume V of a sphere with radius r is given by

  $$V = \frac{4}{3}\pi r^3.$$

  Consider $V = V(t)$ and $r = r(t)$ to be functions of time. Differentiating both sides of the volume formula with respect to time we have

  $$\begin{aligned}\frac{dV}{dt} &= \frac{4}{3}\pi \cdot 3r^2\frac{dr}{dt} \\ &= 4\pi r^2\frac{dr}{dt}.\end{aligned}$$

  We know that $dr/dt = 6$ centimeters per minute and $r = 24$ centimeters. Therefore

  $$\frac{dV}{dt} = 4\pi(24)^2(6) \approx 43,429.$$

  The volume is changing at a rate of $\boxed{\text{43,429 cubic centimeters per minute}}$.

6. (10 points) Find the intervals on which $f(x)$ is increasing and the intervals on which $f(x)$ is decreasing.

$$f(x) = 3x^3 - 9x$$

Hint: make a sign chart for the derivative of f.

The derivative of f is

$$f'(x) = 9x^2 - 9.$$

The partition numbers of f' are when $f'(x) = 0$ or f' is undefined. Since f' is polynomial, it is defined for all x. Thus the only partition numbers are when $f'(x) = 0$. Observe that

$$9x^2 - 9 = 0 \implies 9(x^2 - 1) = 0 \implies x^2 - 1 = 0 \implies x = \pm 1$$

There are two partition numbers: $x = 1$ and $x = -1$. These two partition numbers divide the number line into three intervals.

- $(-\infty, -1)$: Let's test $x = -2$. We have $f'(-2) = 9(-2)^2 - 9 = 27 > 0$.
- $(-1, 1)$: Let's test $x = 0$. We have $f'(0) = 9(0)^2 - 9 = -9 < 0$.
- $(1, \infty)$: Let's test $x = 2$. We have $f'(2) = 9(2)^2 - 9 = 27 > 0$.

The sign chart for f' is

$+$	$-$	$+$
$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$

Thus f' is greater than zero on $(-\infty, -1) \cup (1, \infty)$ and less than zero on $(-1, 1)$. By the fact we learned in class, f is increasing on $(-\infty, -1) \cup (1, \infty)$ and decreasing on $(-1, 1)$.