**Math 19 E** **Name:** _________________________
**Spring 2019**
**Exam 3**
**April 18**

# PRACTICE EXAM SOLUTIONS

**This exam contains 6 pages and 6 questions. Total of points is 98. For full credit you must show your work. Partial credit may be given for incorrect solutions if sufficient work is shown. Messy/unorganized answers may be penalized, even if correct.**

Grade Table (for teacher use only)

| Question | Points | Score |
|---|---|---|
| 1 | 20 | |
| 2 | 20 | |
| 3 | 16 | |
| 4 | 18 | |
| 5 | 14 | |
| 6 | 10 | |
| Total: | 98 | |

**HONORS PLEDGE** (sign after exam is completed): I have neither given nor received aid on this exam, nor have I observed a violation of the UVM Code of Academic Integrity.

Signature: ________________________________________________

1. (20 points) For the function $f(x) = x^4 + 4x^3 + 30$

   (a) (6 points) Calculate the first derivative of $f$.

   $$f'(x) = 4x^3 + 12x^2$$

   (b) (6 points) Find the partition numbers of $f'$.

   Since $f'$ is a polynomial, it is defined everywhere. The only partition numbers are when $f'(x) = 0$.

   $$f'(x) = 0 \implies 4x^3 + 12x^2 = 0 \implies 4x^2(x+3) = 0 \implies \boxed{x = 0, -3}$$

   (c) (6 points) Find the intervals where $f$ is increasing/decreasing.

   There are two partition numbers: $x = 0$ and $x = -3$. These two partition numbers divide the number line into three intervals.

   - $(-\infty, -3)$: Let's test $x = -4$. We have $f'(-4) = -64 < 0$.
   - $(-3, 0)$: Let's test $x = -1$. We have $f'(-1) = 8 > 0$.
   - $(0, \infty)$: Let's test $x = 1$. We have $f'(1) = 16 > 0$.

   The sign chart for $f'$ is

   | | | |
   |---|---|---|
   | $-$ | $+$ | $+$ |
   | $(-\infty, -3)$ | $(-3, 0)$ | $(0, \infty)$ |

   Therefore $f$ is increasing on $(-3, 0) \cup (0, \infty)$ and decreasing on $(-\infty, 3)$.

   (d) (2 points) Find any local extrema.

   $x = -3$ is a local minimum since $f$ changes from decreasing to increasing (U-shape). There is no local maximum.

2. (20 points) For the function $f(x) = x^4 + 4x^3 + 30$

(a) (6 points) Calculate the second derivative of $f$.

$$f'(x) = 4x^3 + 12x^2$$
$$f''(x) = 12x^2 + 24x$$

(b) (6 points) Determine the partition numbers of $f''$.

Since $f''$ is a polynomial, it is defined everywhere. The only partition numbers are when $f''(x) = 0$.

$$f''(x) = 0 \implies 12x^2 + 24x = 0 \implies 12x(x+2) = 0 \implies \boxed{x = 0, -2}$$

(c) (6 points) Determine the intervals where $f$ is concave up/down.

There are two partition numbers: $x = 0$ and $x = -2$. These two partition numbers divide the number line into three intervals.

- $(-\infty, -2)$: Let's test $x = -3$. We have $f''(-3) = 36 > 0$.
- $(-2, 0)$: Let's test $x = -1$. We have $f''(-1) = -12 < 0$.
- $(0, \infty)$: Let's test $x = 1$. We have $f''(1) = 36 > 0$.

The sign chart for $f'$ is

| | | |
|---|---|---|
| $+$ | $-$ | $+$ |
| $(-\infty, -2)$ | $(-2, 0)$ | $(0, \infty)$ |

Therefore $f$ is concave up on $(-\infty, -2) \cup (0, \infty)$ and concave down on $(-2, 0)$.

(d) (2 points) Determine any points of inflection.

$x = -2$ and $x = 0$ are both points of inflection since the concavity changes.

3. (16 points) For the function $f(x) = x^3 - 12x$

(a) (5 points) Calculate the first derivative.

$$f'(x) = 3x^2 - 12$$

(b) (5 points) Determine the critical numbers of $f$.

Since $f'$ is a polynomial, it is defined everywhere. The only partition numbers are when $f'(x) = 0$.

$$f'(x) = 0 \implies 3x^2 - 12 = 0 \implies 3(x^2 - 4) = 0 \implies x^2 - 4 = 0 \implies \boxed{x = \pm 2}$$

Since $f$ is defined at both $x = \pm 2$, they are both critical numbers.

(c) (6 points) Find the absolute maximum and minimum on the interval $[-3, 1]$.

The critical number $x = 2$ is outside the interval $[-3, 1]$ so we only need to test $x = -3, -2, 1$.

| $x$ | $f(x)$ |
|---|---|
| $-3$ | $9$ |
| $-2$ | $16$ |
| $1$ | $-11$ |

$x = -2$ is the absolute maximum and $x = 1$ is the absolute minimum.

4. (18 points) Evaluate the following limits. Use L'Hopital's rule if it applies – do NOT factor.

   (a) (6 points)

$$\lim_{x \to 7} \frac{x-7}{x^2 - 12x + 35} = \frac{0}{0}$$

Apply L'Hopital's rule

$$= \lim_{x \to 7} \frac{1}{2x - 12} = \frac{1}{2(7) - 12} = \boxed{\frac{1}{2}}$$

   (b) (6 points)

$$\lim_{x \to 0} \frac{e^{5x} - 1 - 5x}{x^2} = \frac{1 - 1 - 0}{0} = \frac{0}{0}$$

Apply L'Hopital's rule

$$= \lim_{x \to 0} \frac{5e^{5x} - 5}{2x} = \frac{5 - 5}{0} = \frac{0}{0}$$

Apply L'Hopital's rule again

$$= \lim_{x \to 0} \frac{25e^{5x}}{2} = \boxed{\frac{25}{2}}$$

   (c) (6 points)

$$\lim_{x \to 0^+} \frac{\ln(1 + x^4)}{x^5} = \frac{\ln(1)}{0} = \frac{0}{0}$$

Apply L'Hopital's rule

$$= \lim_{x \to 0^+} \frac{\frac{1}{1+x^4} \cdot 4x^3}{5x^4} = \lim_{x \to 0^+} \frac{4x^3}{(1 + x^4)5x^4}$$

$$= \frac{4}{5} \cdot \lim_{x \to 0^+} \frac{1}{(1 + x^4)x} = \boxed{+\infty}$$

This is because as $x$ goes to zero from the right, the denominator $(1 + x^4)x$ is a small positive number and so $1/(1 + x^4)x$ goes to positive infinity.

5. (14 points) Evaluate the following integrals.

   (a) (7 points)

$$\int (x^2 + 4x + 1)dx = \int x^2 dx + 4\int x dx + \int 1 dx$$
$$= \frac{x^3}{3} + 4 \cdot \frac{x^2}{2} + x + C$$
$$= \boxed{\frac{x^3}{3} + 2x^2 + x + C}$$

   (b) (7 points)

$$\int \left(\sqrt{x} + \frac{6}{x}\right) dx = \int x^{1/2} dx + 6\int \frac{1}{x} dx$$
$$= \frac{x^{3/2}}{3/2} + 6\ln|x| + C$$
$$= \boxed{\frac{2}{3}x^{3/2} + 6\ln|x| + C}$$

6. (10 points) The area $A$ of a healing wound changes at a rate given approximately by

$$\frac{dA}{dt} = -4t^{-3}$$

where $t$ is time in days and $A(1) = 2$. What will the area of the wound be in 10 days? ***Hint*: take the integral of dA/dt to get A(t), then use the fact that $\mathbf{A(1) = 2}$ to solve for C. Finally, compute A(10).**

$$A(t) = \int -4t^{-3} dt = -4\int t^{-3} dt = -4 \cdot \frac{t^{-2}}{-2} + C$$

which we may rewrite as

$$A(t) = \frac{2}{t^2} + C$$

Using the fact that $A(1) = 2$ we have

$$A(1) = 2 \implies \frac{2}{(1)^2} + C = 2 \implies C = 0$$

Thus

$$A(t) = \frac{2}{t^2}$$

and

$$A(10) = \frac{2}{10^2} = \boxed{0.02}$$