MATH 17 Lecture Notes – Aug. 27-31

Chapter 1: Mathematics of Voting

1.1) Elements of an election:

• **Candidates/alternatives:** things to choose from

• **Voters:** people who have a say in the outcome of the election. All voters assumed to have equal say (until Chapter 2).

• **Ballot**: device which voters use to express opinions of candidates.

- **Single choice ballot:** Pick only the candidate you like best. This type of ballot tells us very little about the voter's preferences.

- **Preference ballot:** Rank all candidates in order of preference (we conventionally let 1st be most preferred, 2nd be second most preferred, etc.). This tells us everything about a voter's preferences, and thus will be the focus of this chapter. In fact, it tells us who they would vote for in a single-choice ballot (simply the candidate that they ranked 1st). It tells us who they would vote for if any two candidates were pitted head-to-head (simply the candidate that appears higher in their rankings). It also tells us who they would vote for if a candidate dropped out of the race (for example, if a voter's 1st choice dropped out they would simply vote for whatever candidate they ranked 2nd).

- **Truncated preference ballot:** Rank some, but not all, candidates in order of preference (i.e. rank your top 3 candidates).

• Outcome: the result of the election

- **Winner only:** One person wins, and no distinctions is made among losers.

- **Full ranking:** All candidates are ranked from 1st to last.

- **Partial ranking:** Some, but not all, candidates are ranked (i.e. just the top 3).

• **Voting Method:** How do we convert all individuals' ballots into an outcome. In other words, how do we convert all individual preferences into one collective preference. **This will be the focus of this chapter.**

- Different voting methods may produce different results. How do we choose which one to use? It turns out this is an open question, but let's see if we can use mathematical analysis to *try* to answer it.

• **Preference Schedule:** a visual summary of all the preference ballots in an election. Think of it like stacking the ballots in piles, sorted by how each person voted. The number at the top of the column corresponds to how many people voted a given way.

Example: Suppose we have an election with three candidates: A, B, C and suppose there are six voters: Tom, Sam, Amy, Eve, Bob, and Dan.

	1st	B		1st	A
Tom's Ballot	2nd	C	Sam's Ballot	2nd	B
	3rd	A		3rd	C

	1st	C		1st	A
Amy's Ballot	2nd	A	Eve's Ballot	2nd	B
	3rd	B		3rd	C

	1st	A		1st	B
Bob's Ballot	2nd	B	Dan's Ballot	2nd	C
	3rd	C		3rd	A

There are three voters (Sam, Eve, and Bob) that voted A,B,C for 1st-2nd-3rd. This is read from the first column below. Two voters (Tom and Dan) voted B,C,A. This is read from the second column below. One voter (Amy) voted C,A,B. This is read from the third column below. The preference schedule for this election is

Number of voters	3	2	1
1st	A	B	C
2nd	B	C	A
3rd	C	A	B

This conveys the same information as listing all the individual ballots, but takes much less time to write.

1.2) The Plurality Method:

The Plurality method determines that the winner of the election is the candidate with the most 1st place votes. If we want a full ranking, the candidate with the second most 1st place votes gets 2nd place, the candidate with the third most 1st place votes gets 3rd place, etc. Note that preference ballots are not needed since we only care about each voter's first pick.

Example: Math Club Election

Number of voters	14	10	8	4	1
1st	A	C	D	B	C
2nd	B	B	C	D	D
3rd	C	D	B	C	B
4th	D	A	A	A	A

For the Plurality method we focus only on the "1st" row of the preference schedule.

A gets 14 votes.
B gets 4 votes.
C gets 10+1=11 votes.
D gets 8 votes.

The winner is A. The full ranking is A,C,D,B.

Let's start to analyze the Plurality method. Consider the following example

Example: 2010 Maine Governor Election

Candidate	Percent of Votes
Cutler	36.5
Lepage	38.2
Mitchell	19.3
Moody	5.0
Scott	1.0

• Problem 1: Close wins/sensitive to manipulation – Lepage wins over Cutler by less than 3 percent of the total voters. What if a lot of people who liked Cutler ended up voting for Lepage, just because they didn't think Cutler could win (this is called *insincere voting*)? This isn't a big concern though, as most voting methods exhibit this flaw.

• Problem 2: Don't need majority vote ($> 50\%$ of votes) to win. Lepage won with 38.2% of the votes which means the other 61.8% of voters could have HATED him. This is concerning.

There is one other problem with the Plurality method that we will cover. To see this, consider the following example

Example:

Number of voters	49	48	3
1st	R	H	F
2nd	H	S	H
3rd	F	O	S
4th	O	F	O
5th	S	R	R

By the Plurality method, R wins with 49 of the 1st place votes. However, 51 people voted R as 5th. Meanwhile, H lost by only 1 vote but ALL voters ranked H within their top two. It seems like H should win. In fact, H is what we call a **Candorcet candidate**. A Candorcet candidate is a candidate that wins all pairwise comparisons with other candidates. That is,

H vs R: 48+3=52 voters rank H above R
H vs S: 49+48+3=100 voters rank H above S
H vs F: 49+48 = 97 voters rank H above F
H vs O: 49+48+3=100 voters rank H above O.

- Problem 3: Candorcet candidates can lose!

1.3) The Borda Count Method:

Unlike the Plurality method, the Borda Count method uses the whole preference schedule. Conventionally, the method works by giving 1 point to a candidate each time it appears in last place on a voter's ballot, 2 points each time it appears in 2nd to last place, 3 points each time it appears in 3rd to last place, etc. We sum up all the points for each candidate and determine the winner as the candidate with the most points.

Example: Math Club Election

Number of voters	14	10	8	4	1
1st (4 pts)	A	C	D	B	C
2nd (3 pts)	B	B	C	D	D
3rd (2 pts)	C	D	B	C	B
4th (1 pt)	D	A	A	A	A

A gets $14(4) + 10(1) + 8(1) + 4(1) + 1(1) = 79$ points.
B gets $14(3) + 10(3) + 8(2) + 4(4) + 1(2) = 106$ points.
C gets $14(2) + 10(4) + 8(3) + 4(2) + 1(4) = 104$ points.
D gets $14(1) + 10(2) + 8(4) + 4(3) + 1(3) = 81$ points.

The winner is B. The full ranking is B,C,D,A. Notice that this is quite different than the A,C,D,B ranking we got from the Plurality method.

Note: The Borda Count method and the next two methods also have problems, just as the Plurality method did. However, I will not cover them.

1.4) The Plurality-with-Elimination Method:

One of the problems we had with the Plurality method was that a candidate could win without having a majority ($> 50\%$) of 1st place votes. The goal of the Plurality-with-Elimination method is to eliminate candidates until someone has a majority vote. We do this by making use of preference schedules. While no candidate has a majority vote, we eliminate the candidate with the least 1st place votes and pass their votes down to the next eligible candidate.

See following pages for examples

Example: Math Club Election

# votes	14	10	8	4	1
1^{st}	A	~~C~~	D	~~B~~	~~C~~
2^{nd}	~~B~~	~~B~~	~~C~~	D	D
3^{rd}	~~C~~	D	~~B~~	~~C~~	~~B~~
4^{th}	D	A	A	A	A

Total votes $= 14+10+8+4+1 = 37$

- Need 19 votes for majority vote ($>50\%$)

	Round 0		Round 1		Round 2
A	14		14		14
B	4		~~4~~		X
C	11		11		X
D	8	$\xrightarrow[\text{Elim. B}]{+4 =}$	12	$\xrightarrow[\text{Elim. C}]{+10+1 =}$	23

D has majority vote so **D** wins.
Full ranking is D, A, C, B

Example: Problem 35 from text

# votes	8	7	5	4	3	2
1st	B	C	~~A~~	~~D~~	~~A~~	~~D~~
2nd	~~E~~	~~E~~	B	C	~~D~~	B
3rd	~~A~~	~~D~~	C	B	~~E~~	C
4th	C	~~A~~	~~D~~	~~E~~	C	~~A~~
5th	~~D~~	B	~~E~~	~~A~~	B	~~E~~

Total votes = 29 $\Rightarrow$ need 15 for majority

	Round 0	Round 1	Round 2	Round 3
A	8	8	8	X
B	8	8	+2 = 10	+5 = 15
C	7	7	+4 = 11	+3 = 14
D	6	6	X	X
E	0	X	X	X

$\xrightarrow{}$ Elim E $\qquad \xrightarrow{}$ Elim. D $\qquad \xrightarrow{}$ Elim A

B wins!

Full ranking: B, C, A, D, E

1.5) The Pairwise Comparisons Method:

A second problem we had with the Plurality method was that a Candorcet candidate could lose the election. The following method is designed to guarantee that if a Candorcet candidate exists, then they will win.

This method is basically a round-robin tournament. For each pair of candidates (say X and Y), we look at the preference schedule and determine how many people like X over Y, and how many people like Y over X. If more people prefer X to Y, then we give 1 point to X. If more people prefer Y to X, then we give 1 point to Y. If there is a tie, we give 1/2 point to both X and Y.

Example: Math Club Election

Number of voters	14	10	8	4	1
1st (4 pts)	A	C	D	B	C
2nd (3 pts)	B	B	C	D	D
3rd (2 pts)	C	D	B	C	B
4th (1 pt)	D	A	A	A	A

A vs B: A gets 14 — **B gets 10+8+4+1=23**
A vs C: A gets 14 — **C gets 10+8+4+1=23**
A vs D: A gets 14 — **D gets 10+8+4+1=23**
B vs C: B gets 14+4=18 — **C gets 10+8+1=19**
B vs D: **B gets 14+10+4=28** — C gets 8+1=9
C vs D: **C gets 14+10+1=25** — D gets 8+4=12

In the tournament: A gets 0 wins, B gets 2 wins, C gets 3 wins, and D gets 1 win. Thus, the outcome of the election is that C wins. The full ranking is C,B,D,A – which is yet again a different from the outcomes produced by the Plurality, Borda Count, and Plurality-with-Elimination methods!