MATH 19 - Exam 2 Notes

Section 3.2–3.6 & 4.1

3.2 Derivatives of Exponential and Logarithmic Functions

Recall that e is simply a number that is approximately equal to 2.718.

Derivative of e^x and $\ln(x)$

$$\frac{d}{dx}e^x = e^x$$
$$\frac{d}{dx}\ln(x) = \frac{1}{x}.$$

Derivative of b^x and $\log_b(x)$, for $b > 0$ and $b \neq 1$

$$\frac{d}{dx}b^x = b^x \ln(b)$$
$$\frac{d}{dx}\log_b(x) = \frac{1}{x} \cdot \frac{1}{\ln(b)}.$$

Notice in the case that $b = e$, we have $\ln(b) = 1$ and so these two equations reduce to the first two equations.

Other useful properties of exponents and logarithms.

$$\sqrt[n]{x} = x^{1/n}$$
$$b^{-x} = \frac{1}{b^x}$$
$$b^x b^y = b^{x+y}$$
$$(b^x)^y = b^{xy}$$
$$\ln(xy) = \ln(x) + \ln(y)$$
$$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$$
$$\ln(x^p) = p\ln(x).$$

Problems

Find the derivatives of the following functions. Use properties of logarithms to rewrite logarithmic functions instead of using the chain rule.

1.
$$f(x) = 10e^x + x^2 - x^e$$

We apply the rule for e^x to the first term. The second and third terms can both be dealt with by the power rule (recall that e is just a number so x^e is a power function).

$$f'(x) = 10e^x + 2x - ex^{e-1}$$

2.
$$f(x) = 5\ln(x) - e^x + 3x$$

We apply the rule for $\ln(x)$ to the first term.

$$f'(x) = \frac{5}{x} - e^x + 3.$$

3.
$$f(x) = \ln(6x^2) - x - 1$$

Using properties of logarithms, we rewrite $\ln(6x^2) = \ln(6) + \ln(x^2) = \ln(6) + 2\ln(x)$. Therefore the derivative of f is

$$f'(x) = \frac{d}{dx}\left(\ln(6) + 2\ln(x) - x - 1\right) = 0 + \frac{2}{x} - 1 - 0 = \frac{2}{x} - 1.$$

3.3 Derivatives of Products and Quotients

Product Rule

If $h(x) = f(x)g(x)$, then
$$h'(x) = f'(x)g(x) + g'(x)f(x).$$

Quotient Rule

If $h(x) = \frac{f(x)}{g(x)}$, then
$$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}.$$

Notice how the numerator in the quotient rule is similar to the product rule, but has a negative sign (so the order in the numerator does matter). Also remember that you divide by the dominator of the original function, squared.

Problems

Find the derivatives of the following functions. Use the product rule instead of expanding/FOILing.

1.
$$h(x) = (8x - 4)(x^2 + 3)$$
Let $f(x) = 8x - 4$ and $g(x) = x^2 + 3$, then $f'(x) = 8$ and $g'(x) = 2x$. By the product rule
$$h'(x) = 8(x^2 + 3) + 2x(8x - 4).$$

2.
$$h(x) = (x^3 - 5x)(x^4 - 6x^2)$$
Let $f(x) = x^3 - 5x$ and $g(x) = x^4 - 6x^2$, then $f'(x) = 3x^2 - 5$ and $g'(x) = 4x^3 - 12x$. By the product rule
$$h'(x) = (3x^2 - 5)(x^4 - 6x^2) + (4x^3 - 12x)(x^3 - 5x).$$

3.
$$h(x) = x^2(e^x + \ln(x))$$
Let $f(x) = x^2$ and $g(x) = e^x + \ln x$, then $f'(x) = 2x$ and $g'(x) = e^x + 1/x$. By the product rule
$$h'(x) = 2x(e^x + \ln(x)) + \left(e^x + \frac{1}{x}\right)x^2.$$

4.
$$h(x) = \frac{x^4 + 9}{x^9 + 4}$$
Let $f(x) = x^4 + 9$ and $g(x) = x^9 + 4$, then $f'(x) = 4x^3$ and $g(x) = 9x^8$. By the quotient rule
$$h'(x) = \frac{4x^3(x^9 + 4) - 9x^8(x^4 + 9)}{(x^9 + 4)^2}.$$

5.
$$h(x) = \frac{x^2 - 2x + 1}{2e^x}$$
Let $f(x) = x^2 - 2x + 1$ and $g(x) = 2e^x$, then $f'(x) = 2x - 2$ and $g'(x) = 2e^x$. By the quotient rule
$$h'(x) = \frac{(2x-2)2e^x - 2e^x(x^2 - 2x + 1)}{(2e^x)^2}.$$

6.
$$h(x) = \frac{\sqrt[3]{x} - e^x}{5x^2 - 2}$$
Let $f(x) = x^{1/3} - e^x$ and $g(x) = 5x^2 - 2$, then $f'(x) = \frac{1}{3}x^{-2/3} - e^x$ and $g'(x) = 10x$. By the quotient rule
$$h'(x) = \frac{\left(\frac{1}{3}x^{-2/3} - e^x\right)(5x^2 - 2) - 10x(x^{1/3} - e^x)}{(5x^2 - 2)^2}.$$

3.4 The Chain Rule

Review: Composite Functions

If $f(x) = x+1$ and $g(x) = x^2$ then f composed with g is denoted as $f(g(x))$ and is obtained by plugging in $g(x)$ wherever we see an x in the expression for f. Here $f(g(x)) = f(x^2) = x^2+1$. Note that $g(f(x)) = g(x+1) = (x+1)^2$ is not the same as $f(g(x))$. Other examples are
$$\begin{array}{rccc} \sqrt{x+9} & \text{with} & f(x) = \sqrt{x}, & g(x) = x + 9 \\ (2x-1)^3 & \text{with} & f(x) = x^3, & g(x) = 2x - 1 \\ e^{3x} & \text{with} & f(x) = e^x, & g(x) = 3x \\ \ln(x^2 + x + 1) & \text{with} & f(x) = \ln(x), & g(x) = x^2 + x + 1. \end{array}$$

Note: "function decomposition" (determining f and g as we did above) is not necessarily unique. For example, $e^{2x} = (e^x)^2$ and so we can let $f(x) = e^x$ and $g(x) = 2x$, or alternatively $f(x) = x^2$ and $g(x) = e^x$.

The function $e^{(x+1)^2}$ is a composition of three functions $f(g(h(x)))$ with $f(x) = e^x$, $g(x) = x^2$, and $h(x) = x + 1$.

Chain Rule

If $h(x) = f(g(x))$, then
$$h'(x) = f'(g(x))g'(x).$$
Caution: $f'(g(x))$ indicates composition of f' and g. It does NOT mean to multiply.

There are three main cases of the chain rule, which you can derive using the chain rule formula above.

1. If f is a power function so that $h(x) = (g(x))^n$, then
$$h'(x) = n(g(x))^{n-1} g'(x).$$
2. If $f(x) = e^x$ so that $h(x) = e^{g(x)}$, then
$$h'(x) = e^{g(x)} g'(x).$$
3. If $f(x) = \ln(x)$ so that $h(x) = \ln(g(x))$, then
$$h'(x) = \frac{1}{g(x)} g'(x).$$

These rules match what we have already learned. Recall that $\frac{d}{dx} x^n = nx^{n-1}$, $\frac{d}{dx} e^x = e^x$, and $\frac{d}{dx} \ln(x) = \frac{1}{x}$. The chain rule says just replace x with $g(x)$, but then multiply by the derivative of the inside function, given by $g'(x)$.

Problems

Find the derivatives of the following fucntions.

1.
$$h(x) = (5x - 2)^4$$

Method 1: Work through the general chain rule.
Let $f(x) = x^4$ and $g(x) = 5x - 2$, then $f'(x) = 4x^3$ and $g'(x) = 5$. By the chain rule
$$h'(x) = f'(g(x))g'(x) = f'(5x - 2) \cdot 5 = 4(5x - 2)^3 \cdot 5 = 20(5x - 2)^3.$$

Method 2: Recall special case #1 for power functions.
Let $g(x) = 5x - 2$ so then $g'(x) = 5$. Referring to special case #1 we have
$$h'(x) = 4(5x - 2)^3 \cdot 5 = 20(5x - 2)^3.$$

2.
$$h(x) = \sqrt{1 - x^2}$$

Method 1: Work through the general chain rule.
Let $f(x) = x^{1/2}$ and $g(x) = 1 - x^2$, then $f'(x) = \frac{1}{2} x^{-1/2}$ and $g'(x) = -x^2$. By the chain rule
$$h'(x) = f'(g(x))g'(x) = f'(1 - x^2) \cdot (-x^2) = \frac{1}{2} \left(1 - x^2\right)^{-1/2} \cdot (-x^2) = -\frac{x^2}{2\sqrt{1 - x^2}}.$$

Method 2: Recall special case #1 for power functions.
Let $g(x) = 1 - x^2$ so then $g'(x) = -x^2$. Referring to special case #1 we have

$$h'(x) = \frac{1}{2}\left(1 - x^2\right)^{-1/2} \cdot (-x^2) = -\frac{x^2}{2\sqrt{1 - x^2}}.$$

3.

$$h(x) = e^{x^2 + x}$$

Method 1: Work through the general chain rule.
Let $f(x) = e^x$ and $g(x) = x^2 + x$, then $f'(x) = e^x$ and $g'(x) = 2x + 1$. By the chain rule

$$h'(x) = f'(g(x))g'(x) = f'(x^2 + x) \cdot (2x + 1) = e^{x^2 + x} \cdot (2x + 1).$$

Method 2: Recall special case #2 for e^x.
Let $g(x) = x^2 + x$ so then $g'(x) = 2x + 1$. Referring to special case #2 we have

$$h'(x) = e^{x^2 + x} \cdot (2x + 1).$$

4.

$$h(x) = \ln(2x^2 - 3x + 1)$$

Method 1: Work through the general chain rule.
Let $f(x) = \ln(x)$ and $g(x) = 2x^2 - 3x + 1$, then $f'(x) = 1/x$ and $g'(x) = 4x - 3$. By the chain rule

$$h'(x) = f'(g(x))g'(x) = f'(2x^2 - 3x + 1) \cdot (4x - 3) = \frac{1}{2x^2 - 3x + 1} \cdot (4x - 3).$$

Method 2: Recall special case #3 for $\ln(x)$.
Let $g(x) = 2x^2 - 3x + 1$ so then $g'(x) = 4x - 3$. Referring to special case #2 we have

$$h'(x) = \frac{1}{2x^2 - 3x + 1} \cdot (4x - 3).$$

3.5 Implicit Differentiation

We can **explicitly** define a function $y(x)$, e.g.

$$y = x^2 + 5x + 1.$$

This is explicit because y is “solved for”, and we can immediately figure out the y-value that corresponds to any x-value. We can also **implicitly** define the same function as

$$y - x^2 - 5x = 1.$$

This is implicit because y is not solved for. In this example we can of course easily solve for y; however, this is not usually the case. For example,

$$x^2 + y^2 = 1$$

defines y implicitly. You may recall that this equation defines a circle of radius 1. We want to know how y is changing with respect to x at a given point on the circle. We use **implicit diffeentiation**. To do this, note that $y = y(x)$ is an unknown function of x and so any time there is a function of y (e.g. y^2) we must apply the chain rule and multiply by y'. Differentiating both sides we have

$$2x + 2yy' = 0$$

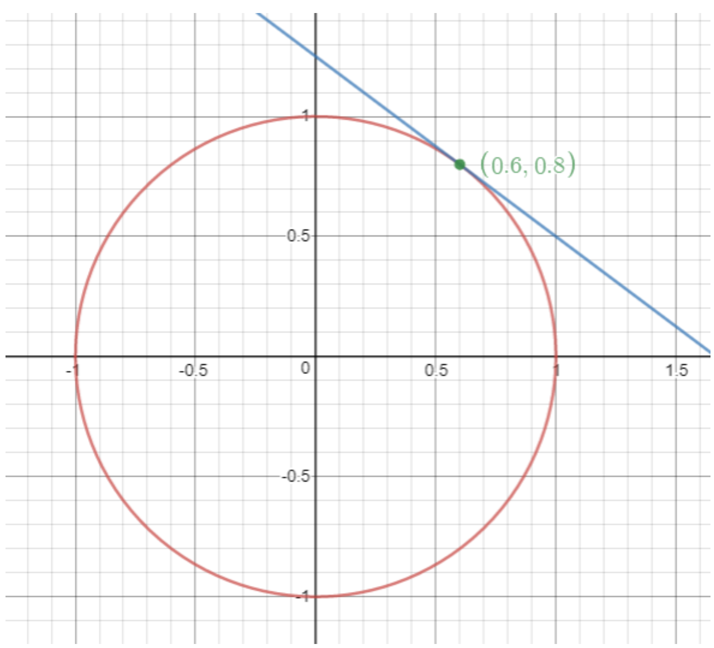
Then we subtract $2x$ from both sides and divide by $2y$ to get

$$y' = -\frac{x}{y}.$$

If we want to know what that rate of change is at the point $(0.6, 0.8)$ then we plug into the above derivative

$$y'\Big|_{(0.6,0.8)} = -\frac{0.6}{0.8} = -\frac{3}{4}.$$

In other words, the tangent line at the point $(0.6, 0.8)$ has slope $-3/4$. This is depicted below.



Problems

Use implicit differentiation to find y'.

1.
$$xy + y^3 = 5$$

We differentiate both sides. Note: for the xy term we have to use the product rule, and for the y^3 term we have to use the chain rule (case 1).

$$\frac{d}{dx}xy + \frac{d}{dx}y^3 = \frac{d}{dx}5$$
$$(y + y'x) + 3y^2y' = 0$$
$$y'x + 3y^2y' = -y$$
$$y'(x + 3y^2) = -y$$
$$\boxed{y' = -\frac{y}{x + 3y^2}}$$

2. Also find the derivative at the point $(1, 0)$.
$$e^y = x^2 + y^2$$

Note: for both the y^2 and e^y terms we have to use the chain rule (case 1 and 2).

$$\frac{d}{dx}e^y = \frac{d}{dx}x^2 + \frac{d}{dx}y^2$$
$$e^y y' = 2x + 2yy'$$
$$e^y y' - 2yy' = 2x$$
$$y'(e^y - 2y) = 2x$$
$$\boxed{y' = \frac{2x}{e^y - 2y}}$$

The derivative at the point $(1, 0)$ is found by plugging the point into the equation above.
$$y'\Big|_{(1,0)} = \frac{2(1)}{e^0 - 2(0)} = 2.$$

3. Also find the derivative at the point $(2, 1)$.

$$\ln y = 2y^2 - x$$

Note: for both the y^2 and $\ln y$ terms we have to use the chain rule (case 1 and 3).

$$\frac{d}{dx}\ln y = \frac{d}{dx}2y^2 - \frac{d}{dx}x$$

$$\frac{1}{y}y' = 4yy' - 1$$

$$\frac{1}{y}y' - 4yy' = -1$$

$$y'\left(\frac{1}{y} - 4y\right) = -1$$

$$\boxed{y' = -\frac{1}{\frac{1}{y} - 4y}}$$

The derivative at the point $(2, 1)$ is found by plugging the point into the equation above.

$$y'|_{(2,1)} = -\frac{1}{\frac{1}{1} - 4(1)} = -\frac{1}{-3} = \frac{1}{3}.$$

4. Also find the equation of the tangent line at the point $(2, -2)$.

$$y^2 + 3xy + x^3 = 0$$

Note: for the y^2 term we have to use the chain rule (case 1) and for the $3xy$ term we need to use the product rule.

$$\frac{d}{dx}y^2 + \frac{d}{dx}3xy + \frac{d}{dx}x^3 = \frac{d}{dx}0$$

$$2yy' + (3y + 3y'x) + 3x^2 = 0$$

$$2yy' + 3y'x = -3y - 3x^2$$

$$y'(2y + 3x) = -3y - 3x^2$$

$$\boxed{y' = \frac{-3y - 3x^2}{2y + 3x}}$$

To find the equation of the tangent line at $(2, -2)$ we need to find the slope (derivative) at the point. Plugging into y' above we have

$$y'\Big|_{(2,-2)} = \frac{-3(-2) - 3(2)^2}{2(-2) + 3(2)} = \frac{6 - 12}{-4 + 6} = \frac{-6}{2} = -3.$$

Using point-slope form, the equation of the tangent line is given by

$$y - (-2) = -3(x - 2)$$

rearranging then gives

$$\boxed{y = -3x + 4}$$

5.

$$y^4 + \frac{x}{y} + e^{2y} = 0$$

Note: for the y^4 term we have to use the chain rule (case 1), for the x/y term we have to use the quotient rule, and for e^{2y} we have to use the chain rule (case 2).

$$\begin{aligned}
\frac{d}{dx}y^4 + \frac{d}{dx}\frac{x}{y} + \frac{d}{dx}e^{2y} &= \frac{d}{dx}0 \\
4y^3y' + \frac{y - y'x}{y^2} + 2y'e^{2y} &= 0 \\
4y^3y' + \frac{1}{y} - \frac{y'x}{y^2} + 2y'e^{2y} &= 0 \\
4y^3y' - \frac{y'x}{y^2} + 2y'e^{2y} &= -\frac{1}{y} \\
y'\left(4y^3 - \frac{x}{y^2} + 2e^{2y}\right) &= -\frac{1}{y}
\end{aligned}$$

$$\boxed{y' = -\frac{1}{y(4y^3 - \frac{x}{y^2} + 2e^{2y})}}$$

6. Also find the equation of the tangent line at $(0.5, 0.5)$.

$$(x - y)^4 = x + y - 1$$

Note: for the $(x - y)^4$ term we have to use the chain rule (case 1). The derivative of

$x - y$ is $1 - y'$.

$$\begin{aligned}
\frac{d}{dx}(x-y)^4 &= \frac{d}{dx}x + \frac{d}{dx}y - \frac{d}{dx}1 \\
4(x-y)^3(1-y') &= 1 + y' \\
4(x-y)^3 - 4(x-y)^3 y' &= 1 + y' \\
4(x-y)^3 - 1 &= 4(x-y)^3 y' + y' \\
4(x-y)^3 - 1 &= y'(1 + 4(x-y)^3) \\
\boxed{y' = \frac{4(x-y)^3 - 1}{4(x-y)^3 + 1}}
\end{aligned}$$

To find the equation of the tangent line at $(0.5, 0.5)$ we need to find the slope (derivative) at the point. Plugging into y' above we have

$$y'\big|_{(0.5,0.5)} = \frac{4(0.5-0.5)^3 - 1}{4(0.5-0.5)^3 + 1} = \frac{-1}{1} = -1.$$

Using point-slope form, the equation of the tangent line is given by

$$y - 0.5 = -(x - 0.5)$$

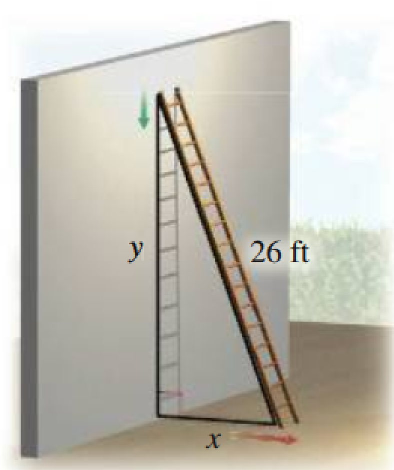
rearranging gives us

$$\boxed{y = -x + 1}$$

3.6 Related Rates

The objective of this chapter is to use implicit differentiation to solve application problems.

Example 1: A 26-foot ladder is placed against a wall. If the top of the ladder is sliding down the wall at 2 feet per second, at what rate is the bottom of the ladder moving away from the wall when the bottom of the ladder is 10 feet away from the wall?



Let x be the distance of the bottom of the ladder from the wall and let y be the distance of the top of the ladder from the ground. Both x and y are changing with respect to time and can be thought of as functions of time, that is, $x = x(t)$ and $y = y(t)$. By the Pythagorean theorem,

$$x^2 + y^2 = 26^2.$$

Differentiating both sides of the equation above with respect to time,

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

The rates dx/dt and dy/dt are related by the equation above. This is a **related-rates** problem.

We want to find dx/dt when $x = 10$ feet, given that $dy/dt = -2$ feet per second. Using the Pythagorean theorem to get y, we have $y = \sqrt{26^2 - 10^2} = 24$ feet. Plugging into the related-rates equation we have

$$2(10)\frac{dx}{dt} + 2(24)(-2) = 0.$$

Solving for dx/dt,

$$\frac{dx}{dt} = \frac{-2(24)(-2)}{2(10)} = 4.8.$$

The bottom of the ladder is moving away from the wall at a rate of 4.8 feet per second.

Example 2: The radius of a spherical balloon is increasing at the rate of 3 centimeters per minute. How fast is the volume changing when the radius is 10 centimers?

Note: the formula for the volume V of a sphere with radius r is given by

$$V = \frac{4}{3}\pi r^3.$$

Consider $V = V(t)$ and $r = r(t)$ to both be functions of time. Differentiating both sides we have

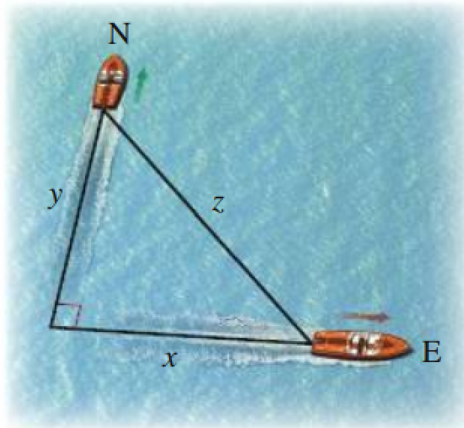
$$\begin{aligned}\frac{dV}{dt} &= \frac{4}{3}\pi \cdot 3r^2\frac{dr}{dt} \\ &= 4\pi r^2\frac{dr}{dt}.\end{aligned}$$

We know that $dr/dt = 3$ centimeters per minute and $r = 10$ centimeters. Therefore

$$\frac{dV}{dt} = 4\pi(10)^2(3) = 1200\pi \approx 3770.$$

The volume is changing at a rate of 3770 cubic centimeters per minute.

Example 3: Suppose that two motorboats leave from the same point at the same time. If one travels north at 15 miles per hour and the other travels east at 20 miles per hour, how fast will the distance be changing after 2 hours?



Let $x = x(t)$ be the distance that the boat traveling east has covered at time t and let $y = y(t)$ be the distance that the boat traveling north has covered at time t. The distance between the two boats is $z = z(t)$ which is given by the Pythagorean theorem

$$x^2 + y^2 = z^2.$$

Differentiating both sides of the equation above with respect to time,

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}.$$

Let's divide both sides by 2 to make things nicer.

$$x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}.$$

Solving for dz/dt we get

$$\frac{dz}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z}.$$

We know $dx/dt = 20$ mph and $dy/dy = 15$ mph. After 2 hours we have $x = 2(20) = 40$ miles and similarly $y = 2(15) = 30$ miles. By the Pythagorean theorem we have that $z = \sqrt{40^2 + 30^2} = 50$ miles. Plugging this into our related-rates equation,

$$\frac{dz}{dt} = \frac{40(20) + 30(15)}{50} = 25.$$

The distance is changing at a rate of 25 miles per hour.

4.1 (beginning only) First Derivative and Graphs

Note: the first derivative is what we have be doing, and is denoted $f'(x)$. In another chapter, we will use the second derivative (i.e. the derivative of $f'(x)$, which is usually denoted $f''(x)$).

Fact
If $f'(x) > 0$ on the interval (a, b) then f is increasing on (a, b). If $f'(x) < 0$ on the interval (a, b) then f is decreasing on (a, b).

Examples

1. Determine when the function $f(x) = 8x - x^2$ is increasing and when it is decreasing.

   We need to make a sign chart for f'. First off, we compute $f'(x) = 8 - 2x$. Recall that the partition numbers of f are the x such that $f'(x) = 0$ or $f'(x)$ is undefined. We have

   $$f'(x) = 0 \implies 8 - 2x = 0 \implies x = 4$$

   and

   $f'(x)$ is undefined **nowhere** (since f' is a polynomial).

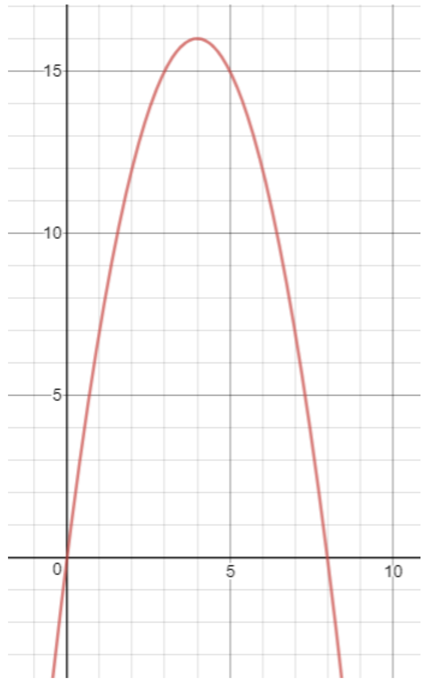
   Thus the only partition number is $x = 4$. This partition number divides the number line into two intervals

   - $(-\infty, 4)$: Let's test $x = 0$. We have $f'(0) = 8 - 2(0) = 8 > 0$.
   - $(4, \infty)$: Let's test $x = 5$ we have $f'(5) = 8 - 2(5) = -2 < 0$.

   The sign chart for f' is

$+$	$-$
$(-\infty, 4)$	$(4, \infty)$

   Thus f' is greater than zero on $(-\infty, 4)$ and less than zero on $(4, \infty)$. By the fact stated earlier, f is increasing on $(-\infty, 4)$ and decreasing on $(4, \infty)$. The function $f(x)$ is graphed below to confirm this.



2. Determine when the function $f(x) = x^3 - 27x + 30$ is increasing and when it is decreasing.

   We have $f'(x) = 3x^2 - 27$. Again, f' is a polynomial and thus always defined. The only partition numbers are when $f'(x) = 0$. We see

   $$f'(x) = 0 \implies 3x^2 - 27 = 0 \implies x^2 = 9 \implies x = \pm 3.$$

   We have two partition numbers $x = \pm 3$, which divide the number line into three intervals

   - $(-\infty, -3)$: Let's test $x = -4$. We have $f'(-4) = 3(-4)^2 - 27 = 21 > 0$.
   - $(-3, 3)$: Let's test $x = 0$. We have $f'(0) = 3(0)^2 - 27 = -27 < 0$.
   - $(3, \infty)$: Let's test $x = 4$ we have $f'(4) = 3(4)^2 - 27 = 21 > 0$.

   The sign chart for f' is

$+$	$-$	$+$
$(-\infty, -3)$	$(-3, 3)$	$(3, \infty)$

   Therefore f is increasing on $(-\infty, -3) \cup (3, \infty)$ and decreasing on $(-3, 3)$.

Miscellaneous

Up to this point, you should be comfortable with all the derivative rules and techniques (including implicit differentiation and related-rates problems). A few other things you should know,

1. Know how to find the slope (or equation) of the tangent line at a given point.

2. Know how to find where the tangent line is horizontal (set the derivative equal to zero and solve for x).

3. Prove the quotient rule using the product rule (you also need the chain rule). Hint:

$$\frac{f(x)}{g(x)} = f(x)[g(x)]^{-1}$$

Let $F(x) = f(x)$ and $G(x) = [g(x)]^{-1}$. Then the derivatives are $F'(x) = f'(x)$ and by the chain rule $G'(x) = -[g(x)]^{-2}g'(x)$. By the product rule

$$\begin{aligned}
\frac{d}{dx}\frac{f(x)}{g(x)} &= \frac{d}{dx}f(x)[g(x)]^{-1} \\
&= \frac{d}{dx}F(x)G(x) \\
&= F'(x)G(x) + G'(x)F(x) \\
&= f'(x)[g(x)]^{-1} - [g(x)]^{-2}g'(x)f(x) \\
&= \frac{f'(x)}{g(x)} - \frac{g'(x)f(x)}{[g(x)]^2}
\end{aligned}$$

Now multiply the fraction on the left by $g(x)/g(x)$ and we get a common denominator.

$$\begin{aligned}
\frac{d}{dx}\frac{f(x)}{g(x)} &= \frac{f'(x)g(x)}{[g(x)]^2} - \frac{g'(x)f(x)}{[g(x)]^2} \\
&= \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}
\end{aligned}$$