Math 19 A&B
Fall 2019
Exam 2
November 7

Solutions

Name: ______________________

PRACTICE EXAM

This exam contains 8 pages and 6 questions. Total of points is 100. For full credit you must show your work. Partial credit may be given for incorrect solutions if sufficient work is shown. Messy/unorganized answers may be penalized, even if correct.

Grade Table (for teacher use only)

Question	Points	Score
1	30	
2	10	
3	15	
4	10	
5	25	
6	10	
Total:	100	

HONORS PLEDGE (sign after exam is completed): I have neither given nor received aid on this exam, nor have I observed a violation of the UVM Code of Academic Integrity.

Signature: ______________________________________

1. (30 points) Calculate the derivatives of the following functions.

(a) (6 points)

$$h(x) = 3(2x^2 + 5x - 8)^4$$

$$h'(x) = 12(2x^2+5x-8)^3 \cdot (4x+5)$$

(b) (6 points)

$$h(x) = 10\ln(1 - 7x)$$

$$h'(x) = \frac{10}{1-7x} \cdot (-7)$$

$$= \frac{-70}{1-7x}$$

(c) (6 points)

$$h(x) = \frac{1 - e^x}{1 + e^x}$$

$$h'(x) = \frac{-e^x(1+e^x) - e^x(1-e^x)}{(1+e^x)^2}$$

$$= \frac{-e^x(1+\cancel{e^x}+1-\cancel{e^x})}{(1+e^x)^2}$$

$$= \frac{-2e^x}{(1+e^x)^2}$$

(d) (6 points)

$$h(x) = x^2 \ln(x)$$

$$h'(x) = 2x \ln x + \frac{1}{x} \cdot x^2$$

$$= 2x \ln x + x$$

$$= x(2 \ln x + 1)$$

(e) (6 points)

$$h(x) = 5xe^{2x}$$

$$h'(x) = 5e^{2x} + 2e^{2x} \cdot 5x$$

$$= 5e^{2x} + 10xe^{2x}$$

$$= 5e^{2x}(1 + 2x)$$

2. (10 points)

   (a) (5 points) Suppose $h(x) = \frac{f(x)}{g(x)}$ and suppose we know that $f(1) = 1$, $f'(1) = 4$, $g(1) = 2$, and $g'(1) = 7$. Calculate $h'(1)$. ~~Product~~ rule Quotient

$$h'(1) = \frac{f'(1)g(1) - g'(1)f(1)}{[g(1)]^2}$$

$$= \frac{4 \cdot 2 - 7 \cdot 1}{2^2} = \frac{8-7}{4} = \boxed{\frac{1}{4}}$$

   (b) (5 points) Consider the function $T(x)$ defined as

$$T(x) = \ln\left(\frac{x^3 f(x)}{g(x)}\right).$$

Find $T'(x)$. Hint 1: use properties of logarithms to simplify (otherwise you have to do product + quotient rule = yikes!!!). Hint 2: your answer will involve $f'(x)$ and $g'(x)$ due to chain rule.

$$T(x) = \ln(x^3 f(x)) - \ln(g(x))$$

$$= \ln(x^3) + \ln(f(x)) - \ln(g(x))$$

$$= 3\ln x + \ln(f(x)) - \ln(g(x))$$

$$\boxed{T'(x) = \frac{3}{x} + \frac{1}{f(x)} \cdot f'(x) - \frac{1}{g(x)} \cdot g'(x)}$$

3. (15 points) For the implicit curve defined by the equation: $y^4 - 3y^3 + x^3 = 6$

(a) (10 points) Use implicit differentiation to find $\frac{dy}{dx}$.

$$4y^3 \frac{dy}{dx} - 9y^2 \frac{dy}{dx} + 3x^2 = 0$$

$$9y^2 \frac{dy}{dx} - 4y^3 \frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} \cdot (9y^2 - 4y^3) = 3x^2$$

$$\boxed{\frac{dy}{dx} = \frac{3x^2}{9y^2 - 4y^3}}$$

(b) (5 points) Find the equation of the tangent line at the point (2, 1).

point

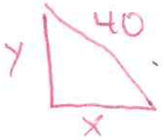
$$\text{slope} = y'\big|_{(2,1)} = \frac{3(2)^2}{9(1)^2 - 4(1)^3} = \frac{12}{9-4} = \frac{12}{5}$$

equation:

$$\boxed{y - 1 = \frac{12}{5}(x - 2)}$$

4. (10 points) **One of these two problems (but not both) will appear on the exam!**

- A 40-foot ladder is placed against a wall. If the top of the ladder is sliding down the wall at 3 feet per second, at what rate is the bottom of the ladder moving away from the wall when the bottom of the ladder is 22 feet away from the wall?



$$(x(t))^2 + (y(t))^2 = 40^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

Know $x = 22$, $\frac{dy}{dt} = -3$, want $\frac{dx}{dt}$.

Can find y: $y = \sqrt{40^2 - 22^2} \approx 33.4$

$$\Rightarrow \quad 22\frac{dx}{dt} + 33.4 \cdot (-3) = 0$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{3 \cdot 33.4}{22} \approx \boxed{4.56}\ \text{ft/sec}$$

- The radius of a spherical balloon is increasing at the rate of 6 centimeters per minute. How fast is the volume changing when the radius is 24 centimers?

Note: the formula for the volume V of a sphere with radius r is given by

$$V = \frac{4}{3}\pi r^3.$$

$$\frac{dV}{dt} = \frac{4}{3}\pi\, 3r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Know $\frac{dr}{dt} = 6$, $r = 24$, want $\frac{dV}{dt}$.

$$\frac{dV}{dt} = 4\pi(24)^2 \cdot 6$$

$$= 13824\pi$$

$$\approx \boxed{43{,}429}\ \text{cm}^3/\text{min}$$

5. (25 points) Consider the function

$$f(x) = 3x^3 - 9x$$

(a) (8 points) Find the intervals on which $f(x)$ is increasing and the intervals on which $f(x)$ is decreasing.

$f'(x) = 9x^2 - 9$

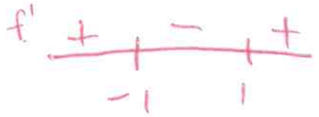
Partition #s of f'

① $f'(x) = 0 \Rightarrow 9x^2 - 9 = 0$

$\Rightarrow 9(x^2 - 1) = 0$

$\Rightarrow \boxed{x = \pm 1}$

② $f'(x)$ DNE $\Rightarrow$ none



x	$f'(x)$
−2	27
0	−9
2	27

increasing on $(-\infty, -1) \cup (1, \infty)$

decreasing on $(-1, 1)$

(b) (2 points) Find the local extrema and identify whether each is a local maximum or local minimum.

$x = -1$ is a local max

$x = 1$ is a local min

(c) (8 points) Find the intervals on which $f(x)$ is concave up and the intervals on which $f(x)$ is concave down.

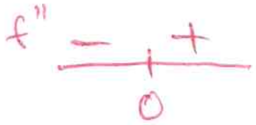
$f''(x) = 18x$

Partition #s of f''

① $f''(x) = 0 \Rightarrow 18x = 0$

$\Rightarrow \boxed{x = 0}$

② $f''(x)$ DNE $\Rightarrow$ none



x	$f''(x)$
−1	−18
1	18

concave up on $(0, \infty)$

concave down on $(-\infty, 0)$

(d) (2 points) Find the point(s) of inflections.

$x = 0$ is a point of inflection

(e) (5 points) Find the absolute maximum and absolute minimum on the interval [0, 3].

From (a), know $x = \pm 1$ are partition #s of f' (critical #s of f).

← but $x = -1$ is outside the interval $[0, 3]$ so we don't include in table!

x	$f(x)$
0	0
1	−6
3	54

$x = 1$ is ~~the~~ absolute min

$x = 3$ is absolute max

6. (10 points) Evaluate the limits

(a) (5 points)

$$\lim_{x\to\infty}\frac{x^3-1000}{\ln(x)} = \frac{\infty}{\infty} \quad \text{L'H rule}$$

$$= \lim_{x\to\infty}\frac{3x^2}{\frac{1}{x}} = \lim_{x\to\infty} 3x^3 = \boxed{+\infty}$$

(b) (5 points)

$$\lim_{x\to 0}\frac{x^2}{e^{3x}-3x-1} = \frac{0}{1-0-1} = \frac{0}{0} \quad \text{L'H}$$

$$= \lim_{x\to 0}\frac{2x}{3e^{3x}-3} = \frac{0}{3-3} = \frac{0}{0} \quad \text{L'H}$$

$$= \lim_{x\to 0}\frac{2}{9e^{3x}} = \boxed{\frac{2}{9}}$$