Math 19 A&B
Fall 2019
Exam 2
November 7
Version B

Solutions

Name: ____________________

This exam contains 7 pages and 6 questions. Total of points is 100. For full credit you must show your work. Partial credit may be given for incorrect solutions if sufficient work is shown. Messy/unorganized answers may be penalized, even if correct.

Grade Table (for teacher use only)

Question	Points	Score
1	10	
2	30	
3	15	
4	10	
5	10	
6	25	
Total:	100	

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Signature: ______________________________________

1. (10 points)

   (a) (5 points) Suppose $h(x) = \dfrac{f(x)}{g(x)}$ and suppose we know that $f(6) = 3$, $f'(6) = 3$, $g(6) = -2$, and $g'(6) = 4$. Calculate $h'(6)$.

   $$h'(6) = \frac{f'(6)g(6) - g'(6)f(6)}{(g(6))^2} = \frac{3(-2) - 4(3)}{(-2)^2}$$
   $$= \frac{-6 - 12}{4} = \frac{-18}{4} = \boxed{\frac{-9}{2}}$$

   (b) (5 points) Calculate $h'(x)$ for

   $$h(x) = e^{f(x)}.$$

   $$h'(x) = e^{f(x)} \cdot f'(x)$$

2. (30 points) Calculate the derivatives of the following functions.

   (a) (10 points)

   $$h(x) = 4(x^3 - 6x + 1)^3$$

   $$h'(x) = 12(x^3 - 6x + 1)^2 \cdot (3x^2 - 6)$$

   (b) (10 points)

   $$h(x) = 8e^x(x^4 - 7x^2)$$

   $$h'(x) = 8e^x(x^4 - 7x^2) + (4x^3 - 14x) \cdot 8e^x$$

(c) (10 points)

$$h(x) = \frac{1+x^4}{1-x^4}$$

$$h'(x) = \frac{4x^3(1-x^4) - (-4x^3)(1+x^4)}{(1-x^4)^2}$$

$$= \frac{4x^3(1-x^4) + 4x^3(1+x^4)}{(1-x^4)^2}$$

$$= \frac{8x^3}{(1-x^4)^2}$$

3. (15 points) For the implicit curve defined by the equation: $\ln y = x^2 - y^2$

(a) (10 points) Use implicit differentiation to find $\frac{dy}{dx}$.

$$\frac{1}{y} y' = 2x - 2yy'$$

$$\frac{1}{y} y' + 2yy' = 2x$$

$$y'\left(\frac{1}{y} + 2y\right) = 2x$$

$$\boxed{y' = \frac{2x}{\frac{1}{y} + 2y}} = \frac{2xy}{1+2y^2}$$

(b) (5 points) Find the equation of the tangent line at the point $(1, 1)$.

Slope: $y'|_{(1,1)} = \frac{2(1)}{\frac{1}{1} + 2(1)} = \frac{2}{3}$

Point: $(1, 1)$

Equation:

$$y - 1 = \frac{2}{3}(x - 1)$$

$$\Rightarrow y = \frac{2}{3}x + \frac{1}{3}$$

4. (10 points) A 50-foot ladder is placed against a wall. If the top of the ladder is sliding down the wall at 5 feet per second, at what rate is the bottom of the ladder moving away from the wall when the bottom of the ladder is 30 feet away from the wall?

$$x^2+y^2=50^2$$

$$\cancel{2}x\frac{dx}{dt}+\cancel{2}y\frac{dy}{dt}=0$$

$$x\frac{dx}{dt}+y\frac{dy}{dt}=0$$

Know: $x=30$, $\frac{dy}{dt}=-5$, want: $\frac{dx}{dt}$

$$y=\sqrt{50^2-30^2}=40$$

$$\Rightarrow 30\frac{dx}{dt}+40(-5)=0$$

$$\Rightarrow \frac{dx}{dt}=\frac{40\cdot 5}{30}=\frac{20}{3}\approx\boxed{6.\overline{66}}\ \text{ft/sec}$$

5. (10 points) Evaluate the limit

$$\lim_{x\to 0}\frac{e^{5x}-5x-1}{x^2} = \frac{e^0-5(0)-1}{0^2}=\frac{0}{0}\quad L'H$$

$$=\lim_{x\to 0}\frac{5e^{5x}-5x}{2x}=\frac{5e^0-5(0)}{2(0)}=\frac{0}{0}\quad L'H$$

$$=\lim_{x\to 0}\frac{25e^{5x}}{2}=\boxed{\frac{25}{2}}$$

6. (25 points) Consider the function

$$f(x) = x^3 - 6x^2$$

(a) (8 points) Find the intervals on which $f(x)$ is increasing and the intervals on which $f(x)$ is decreasing.

$f'(x) = 3x^2 - 12x$

~~[illegible]~~

$f'(x) = 0 \Rightarrow 3x^2 - 12x = 0$

$\Rightarrow 3x(x-4) = 0$

$\Rightarrow \boxed{x = 0,\ x = 4}$

f': $+$ | 0 | $-$ | 4 | $+$

x	$f'(x)$
-1	15
1	-9
5	15

$\boxed{f \text{ is increasing on } (-\infty, 0) \cup (4, \infty) \text{; decreasing on } (0, 4)}$

(b) (2 points) Find the local extrema and identify whether each is a local maximum or local minimum.

$x = 0$ local max

$x = 4$ local min

(c) (5 points) Find the absolute maximum and absolute minimum on the interval $[0, 10]$.

x	$f(x)$
0	0
4	-32
10	400

$x = 10$ ~~local~~ absolute max

$x = 4$ ~~local~~ absolute min

(d) (8 points) Find the intervals on which $f(x)$ is concave up and the intervals on which $f(x)$ is concave down.

$f''(x) = 6x - 12$

$f''(x) = 0 \Rightarrow 6x - 12 = 0$

$\Rightarrow \boxed{x = 2}$

f'': $-$ | $+$ at 2

x	$f''(x)$
0	-12
3	6

f is concave up on $(2, \infty)$

f is concave down on $(-\infty, 2)$

(e) (2 points) Find the point(s) of inflection.

$\boxed{x = 2}$ is a point of inflection

(Bonus: 5 points) We are skipping Section 4.6: Optimization, but here's a little peek. Follow the steps below.

Find two positive numbers whose sum is 13 and whose product is a maximum.

Let x and y be the two numbers. Then $x + y = 13$ and we want to find the absolute maximum of $f(x, y) = xy$.

Step 1. *Solve the equation $x + y = 13$ for y.*

$y = 13 - x$

Step 2. *Substitute your answer to Step 1 into $f(x, y) = xy$ to get a function which is ONLY in terms of x, call this function $g(x)$.*

$f(x,y) = x(13-x) = 13x - x^2 = g(x)$

Step 3. *Find the critical numbers of $g(x)$.*

$g'(x) = 13 - 2x$

$g'(x) = 0 \Rightarrow 13 - 2x = 0 \Rightarrow \boxed{x = \frac{13}{2} = 6.5}$

Step 4. *Find the absolute maximum of $g(x)$ on $[0, 13]$, this will give the "optimal" x.*

x	f(x)
0	0
6.5	42.25
13	0

$\underset{\text{optimal}}{x} = 6.5$

Step 5. *Again use the fact that $x + y = 13$ to find the optimal y.*

$x + y = 13 \Rightarrow y = 13 - x$

$\Rightarrow y_{optimal} = 13 - x_{optimal}$

$\Rightarrow y_{optimal} = 13 - 6.5 = 6.5$

$$\boxed{\begin{array}{l} x = 6.5 \\ y = 6.5 \end{array}}$$

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1. (10 points)

   (a) (5 points) Suppose $h(x) = \dfrac{f(x)}{g(x)}$ and suppose we know that $f(3) = 2$, $f'(3) = 2$, $g(3) = -1$, and $g'(3) = 4$. Calculate $h'(3)$.

   $$h'(3) = \frac{f'(3)g(3) - g'(3)f(3)}{(g(3))^2}$$

   $$= \frac{2(-1) - 4(2)}{(-1)^2} = \frac{-2-8}{1} = \boxed{-10}$$

   (b) (5 points) Calculate $h'(x)$ for

   $$h(x) = \ln(f(x)).$$

   $$h'(x) = \frac{1}{f(x)} \cdot f'(x)$$

2. (30 points) Calculate the derivatives of the following functions.

   (a) (10 points)

   $$h(x) = 2(x^3 - 9x + 1)^4$$

   $$h'(x) = 8(x^3-9x+1)^3 \cdot (3x^2-9)$$

   (b) (10 points)

   $$h(x) = 5e^x(x^5 - 6x^3)$$

   $$h'(x) = 5e^x(x^5-6x^3) + (5x^4-18x^2) \cdot 5e^x$$

(c) (10 points)

$$h(x) = \frac{1+x^3}{1-x^3}$$

$$h'(x) = \frac{3x^2(1-x^3) - (-3x^2)(1+x^3)}{(1-x^3)^2}$$

$$= \frac{3x^2(1-x^3) + 3x^2(1+x^3)}{(1-x^3)^2}$$

$$= \frac{6x^2}{(1-x^3)^2}$$

3. (15 points) For the implicit curve defined by the equation: $e^y = x^2 + y^2$

(a) (10 points) Use implicit differentiation to find $\frac{dy}{dx}$.

$$e^y y' = 2x + 2y y'$$

$$e^y y' - 2yy' = 2x$$

$$y'(e^y - 2y) = 2x$$

$$\boxed{y' = \frac{2x}{e^y - 2y}} = \frac{-2x}{2y - e^y}$$

(b) (5 points) Find the equation of the tangent line at the point $(1, 0)$.

Slope: $y'|_{(1,0)} = \frac{2(1)}{e^0 - 2(0)} = \frac{2}{1} = 2$

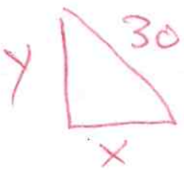
Point: $(1, 0)$

Equation: $y - 0 = 2(x-1)$

$$y = 2(x-1)$$

$$y = 2x - 2$$

4. (10 points) A 30-foot ladder is placed against a wall. If the top of the ladder is sliding down the wall at 2 feet per second, at what rate is the bottom of the ladder moving away from the wall when the bottom of the ladder is 18 feet away from the wall?



$$x^2 + y^2 = 30^2$$

$$\cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt} = 0$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

Know $x = 18$, $\frac{dy}{dt} = -2$, want $\frac{dx}{dt}$.

$$y = \sqrt{30^2 - 18^2} = 24$$

$$\Rightarrow 18\frac{dx}{dt} + 24(-2) = 0$$

$$\Rightarrow \frac{dx}{dt} = \frac{24(2)}{18} = \frac{48}{18} = \frac{8}{3} = \boxed{2.66} \text{ ft/sec}$$

5. (10 points) Evaluate the limit

$$\lim_{x\to 0} \frac{e^{6x} - 6x - 1}{x^2} = \frac{e^0 - 6(0) - 1}{0^2} = \frac{0}{0} \text{ L'H}$$

$$= \lim_{x\to 0} \frac{6e^{6x} - 6}{2x} = \frac{6e^0 - 6}{2(0)} = \frac{0}{0} \text{ L'H}$$

$$= \lim_{x\to 0} \frac{36e^{6x}}{2} = \boxed{\frac{36}{2}}$$

$$= 18$$

6. (25 points) Consider the function

$$f(x) = x^3 - 9x^2$$

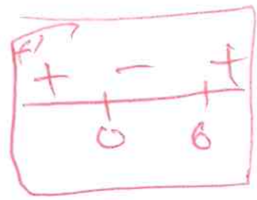
(a) (8 points) Find the intervals on which $f(x)$ is increasing and the intervals on which $f(x)$ is decreasing.

$f'(x) = 3x^2 - 18x$

$f'(x) = 0 \Rightarrow 3x^2 - 18x = 0$

$\Rightarrow 3x(x-6) = 0$

$\Rightarrow \boxed{x = 0, x = 6}$



x	f'(x)
-1	21
1	-15
7	21

$\boxed{\begin{array}{l} f \text{ is increasing on } (-\infty, 0) \cup (6, \infty) \\ \text{decreasing on } (0, 6) \end{array}}$

(b) (2 points) Find the local extrema and identify whether each is a local maximum or local minimum.

$x = 0$ is a local max

$x = 6$ is a local min

(c) (5 points) Find the absolute maximum and absolute minimum on the interval $[0, 10]$.

x	f(x)
0	0
6	-108
10	100

$\boxed{\begin{array}{l} \text{absolute max: } x = 10 \\ \text{absolute min: } x = 6 \end{array}}$

(d) (8 points) Find the intervals on which $f(x)$ is concave up and the intervals on which $f(x)$ is concave down.

$f''(x) = 6x - 18$

$f''(x) = 0 \Rightarrow 6x - 18 = 0$

$\Rightarrow x = 3$.

f'': $-$ | $+$ at 3

x	$f''(x)$
2	-6
4	6

$\boxed{\begin{array}{l} f \text{ is concave up on } (3, \infty) \\ \text{concave down on } (-\infty, 3) \end{array}}$

(e) (2 points) Find the point(s) of inflection.

$\boxed{x = 3}$ is a point of inflection

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$g'(x) = 13 - 2x$

$g'(x) = 0 \Rightarrow x = \frac{13}{2} = 6.5$

Step 4. *Find the absolute maximum of $g(x)$ on $[0, 13]$, this will give the "optimal" x.*

x	$g(x)$
0	0
6.5	42.25
13	0

$x_{optimal} = 6.5$ is absolute max

Step 5. *Again use the fact that $x + y = 13$ to find the optimal y.*

$x + y = 13 \Rightarrow y = 13 - x$

$\Rightarrow y_{optimal} = 13 - x_{optimal}$

$\Rightarrow y_{optimal} = 13 - 6.5 = 6.5$

$$\boxed{\begin{array}{l} x = 6.5 \\ y = 6.5 \end{array}}$$