Math 17: Exam 1 Solutions Name:

There are 4 pages and 6 questions in total. There are 50 possible points and the point values for each page are 13, 12, 13, and 12 respectively. For full credit you must show your work. Partial credit may be given for incorrect solutions if sufficient work is shown.

1. Shown below is the preference schedule for an election with four candidates (A, B, C, and D).

Number of voters	27	15	11	9	8	1
1st	C	A	~~B~~	~~D~~	~~B~~	~~B~~
2nd	~~D~~	~~B~~	~~D~~	A	A	A
3rd	~~B~~	~~D~~	A	~~B~~	C	~~D~~
4th	A	C	C	C	~~D~~	C

(a) Determine the full ranking using the Plurality method. (3 pt)

# of 1st votes – A : 15, B : 20, C : 27, D : 9. **The full ranking is: C,B,A,D.**

(b) Determine the full ranking using Plurality-with-Elimination. (8 pt)

There are 71 votes total. Therefore 36 votes are needed for a majority vote.

	Round 0	Round 1	Round 2
A	15	24	$\boxed{44}$
B	20	~~20~~	×
C	27	27	~~27~~
D	~~9~~	×	×

The full ranking is: A,C,B,D.

(c) What issue does the Plurality method have, but which the Plurality-with-Elimination method resolves? (2 pt)

The Plurality method violates the Majority fairness criterion. That is, the winning candidate may have less than 50% of the votes. In part (a) we see that this is the case – the winner only has $27/71 = 38\%$ of the total votes. The Plurality-with-Elimination method guarantees the winner will have a majority vote.

2. Consider the weighted voting system $[7 : 5, 2, 1]$.

   (a) Which player(s), if any, have veto power? Justify your answer. (3 pt)

   A player P_i is said to have veto power if $V - w_i < q$. In words, a player has veto power if their vote is necessary to pass a vote (i.e. if they are a critical player in the grand coaliation). P_1 has veto power because $V - w_1 = 8 - 5 = 3$ which is less than $q = 7$. Similarly, P_2 has veto power because $V - w_2 = 8 - 2 = 6$ which is less than 7. P_3 does not have veto power because $V - w_3 = 8 - 1 = 7$ which is still enough to pass a vote.

   Thus, P_1 and P_2 both have veto power, P_3 does not.

   (b) Determine the Banzhaf power distribution of this voting system. (7 pt)

Winning coalition	Weight of coalition	Critical players
$\{P_1, P_2\}$	7	P_1, P_2
$\{P_1, P_2, P_3\}$	8	P_1, P_2

   $b_1 =$ 2, $b_2 =$ 2, $b_3 =$ 0

   $T =$ 4

   $\beta_1 = \frac{2}{4}$, $\beta_2 = \frac{2}{4}$, $\beta_3 = \frac{0}{4}$

3. What is the conceptual difference between the Banzhaf measurement of power and the Shapley-Shubik measurement of power? (2 pt)

   Shapley-Shubik power takes into account the order in which the votes are cast (via sequential coalitions) whereas Banzhaf power does not.

4. Suppose we are dividing three shares s_1, s_2, s_3 among three players Alex, Benson, and Christine. The value system for each player is given in the table below.

	s_1	s_2	s_3
Alex	25%	40%	35%
Benson	28%	35%	37%
Christine	$33\frac{1}{3}$%	$33\frac{1}{3}$%	$33\frac{1}{3}$%

(a) Determine the fair shares for each player. (2 pt)

Alex: s_2, s_3, Benson: s_2, s_3, Christine: s_1, s_2, s_3.

(b) Determine all possible fair divisions. *Hint*: there are only two. (2 pt)

(i) Alex: s_2, Benson: s_3, Christine: s_1
(ii) Alex: s_3, Benson: s_2, Christine: s_1

(c) Describe each possible fair division using the terms: efficient, inefficient, envy-free. Justify your answer. *Hint*: you may need to use more than one term. (2 pt)

Value of Fair Division

	Alex	Benson	Christine
(i)	40%	37%	$33.\bar{3}$%
(ii)	35%	35%	$33.\bar{3}$%

Fair division (i) is envy-free, and therefore also efficient, because everyone gets their highest valued share. Fair division (ii) is inefficient because everyone is worse off when compared to (i).

5. Suppose we want to divide three items (a car, a laptop, and a guitar) among two siblings (Drake & Josh). From the bids shown below, use the Method of Sealed bids to determine the final settlement. (7 pt)

	Car	Laptop	Guitar	Total	Fair Share Value
Drake	\$7000	\$500	\$500	\$8000	\$4000
Josh	\$5000	\$900	\$100	\$6000	\$3000

Drake must pay $\$7500 - \$4000 = \$3500$ to money pot. Josh gets $\$3000 - \$900 = \$2100$ from money pot. Thus, there is a surplus of $\$3500 - \$2100 = \$1400$ and so the surplus division is $\$1400/2 = \$700$ each. The final settlment is:

Drake: Car, Guitar, pays \$3500**, gets** \$700**.**
Josh: Laptop, gets \$2100**, gets** \$700**.**

6. A university consists of five colleges: Arts and Sciences (A & S), Business, Education, Nursing, and Engineering. The university wants to hire 250 faculty in total. The amount of faculty that are hired for each college should be in proportion to the student enrollment for the college. Below is the student enrollment for each college.

College	A & S	Business	Education	Nursing	Engineering	Total
Enrollment	1,646	762	2,081	1,066	6,945	12,500
Standard quota	32.92	15.24	41.62	21.32	138.90	250
Lower quota	32	15	41	21	138	247
Residue	0.92	0.24	0.62	0.32	0.90	3
Apportionment	33	15	42	21	139	250

(a) Calculate the standard divisor and interpret it in the context of the problem. (2 + 2 pt)

$SD = \frac{12{,}500 \text{ students}}{250 \text{ faculty}} = 50$ (students/faculty)

50 students per 1 faculty hire

(b) Find the apportionment of faculty using Hamilton's Method. (8 pt)
Show your work for this problem by adding to the table above.