LIFETIME EFFECTS IN PHONON MEAN FIELD THEORIES

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The effect of lifetime on the amplitude of the phonon mean field is considered. The change in the mean field can be important in connection with high-order anharmonic coupling coefficients and in the calculation of soft modes.

For many properties of vibrating crystal lattices it is necessary to calculate thermodynamic expectation values of products of two phonon field operators. Usually one considers a pair of complex normal coordinates $Q(\boldsymbol{q})$, which are either introduced in the harmonic approximation [1] or, via a self-consistent type calculation, within a renormalized harmonic approximation [2]. The well-known expectation value is then given by

$$\langle Q(\boldsymbol{q})Q(-\boldsymbol{q})\rangle^{(0)} = \frac{1}{2\omega_{\boldsymbol{q}}} \coth \tfrac{1}{2}\beta\omega_{\boldsymbol{q}} \; ,$$

$$(\hbar = 1, \quad \beta = \frac{1}{kT}) \; , \tag{1}$$

where $\omega_{\boldsymbol{q}}$ is either the harmonic or the renormalized frequency of phonon mode $\boldsymbol{q}$. Expression (1) defines the phonon mean field and is in this form widely used in the literature. In this letter we want to comment on the effect of the phonon lifetime on the mean field and to point out situations where it can be important. A formal, self-consistent scheme of anharmonic lattice dynamics allowing for such effects has recently been published by Werthamer [3]. It seems, however, that no quantitative calculations including these effects have been performed.

Generally, the expectation value (1) is calculated from the retarded one-phonon Green function D_ω using the fluctuation-dissipation theorem

$$\langle Q(\boldsymbol{q})Q(-\boldsymbol{q})\rangle = -\frac{1}{\pi}\int_0^\infty \mathrm{d}\omega \coth(\tfrac{1}{2}\beta\omega)\, \mathrm{Im} D_\omega . \tag{2}$$

The finite width $2\Gamma_{\boldsymbol{q}}$ can be accounted for in the simplest way by assuming a lorentzian shape for the Green function in eq. (2):

$$D_\omega = [\omega^2 - \omega_{\boldsymbol{q}}^2 + \mathrm{i}2\omega\Gamma_{\boldsymbol{q}}]^{-1}. \tag{3}$$

Writing

$$\langle Q(\boldsymbol{q})Q(-\boldsymbol{q})\rangle = \langle Q(\boldsymbol{q})Q(-\boldsymbol{q})\rangle^{(0)} F(\gamma,t) \; , \tag{4}$$

we have introduced a reduced width $\gamma = \Gamma_{\boldsymbol{q}}/\omega_{\boldsymbol{q}}$ and a reduced temperature $t = (\beta\omega_{\boldsymbol{q}})^{-1}$. The deviations of $F(\gamma,t)$ from the value of $F = 1$ represent the effect which we would like to point out. $F(\gamma,t)$ is plotted as a function of γ for various values of t in fig. 1. The limits of low and high temperatures can be represented analytically

$$F(\gamma,0) = \frac{1}{\pi\sqrt{1-\gamma^2}} \left\{ \frac{\pi}{2} + \arctan \frac{1-2\gamma^2}{2\gamma\sqrt{1-\gamma^2}} \right\} \; , \quad (\gamma < 1)$$

$$F(\gamma,\infty) = 1 \; . \tag{5}$$

From these results we conclude that the corrections due to phonon lifetime are not important under normal conditions where low-order perturbation theory can be applied and where γ values are smaller than 0.1. They can, however, at low reduced temperatures, be important in self-consistent calculations. For example in the case of soft modes, the value of γ can be of the order of

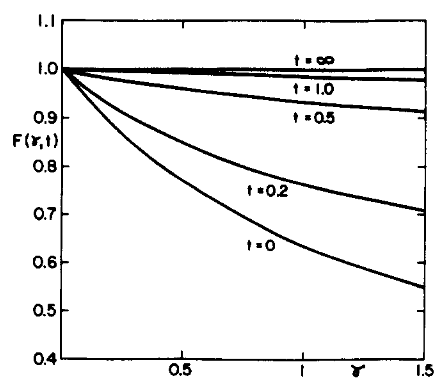

Fig. 1. Lifetime correction to phonon mean field (γ: reduced width; t : reduced temperature).

one. In the type of calculations done by Pytte and Feder [4,5] this would modify the effect of the coupling of the soft mode to its own mean field, as long as t is not too large. Another case is the self-consistent calculation of phonon frequencies with normally small values of γ, where, however, emphasis is put on summation over all higher-order derivatives of the lattice potential. Since in the mean field approximation the $2n$-th derivative of the lattice potential is connected with n-1 factors of the type (4), relatively small deviations of $F(\gamma,t)$ from unity might have considerable influence on the asymptotic contributions of the high-order derivatives of the self-energy.

References

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PERTURBATION THEORY FOR TOROIDAL PLASMA *

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The self magnetic field of an axially symmetric toroidal plasma is shown to contain positive powers of the aspect ratio and a logarithmic function of this quantity. The limit of large aspect ratio is finite.

In discussions of plasma equilibrium in toroidal geometry describing systems such as Tokamak one of the principal assumptions is that the self magnetic field of the plasma can be developed in a perturbation series. The parameter entering the perturbation expansion is the inverse aspect ratio b/R, denoted throughout this paper by ϵ. The minor radius of the torus is denoted by b while R is the major radius.

With experiments in which ϵ is in the vicinity of 1/3 theoretical discussions for non-vanishing ϵ are pertinent. Thus, the validity of a perturbative solution in ϵ of the magnetohydrodynamical equations must be considered.

For axially symmetric systems this assumption can be investigated directly by finding Green's function for the toroidal component of the vector potential due to plasma currents for arbitrary ϵ. The principal results of this calculation are presented here. In particular, the magnetic field due to an axially symmetric current loop enclosed in an ideal toroidal conductor with circular cross-section has negative powers of ϵ and a term proportional to $\epsilon \ln \epsilon$.

These results may be obtained by considering the vector potential A from which the equilibrium magnetic field can be calculated. The system can be described by toroidal coordinates x_1, x_2, x_3 which are a radial variable and cosines of the minor and major azimuthal angles respectively [1]. The scale in this coordinate system is set by the quantity $R(1-\epsilon^2)^{1/2}$, denoted throughout by a.

The variable x_3 describes rotations about the axis of symmetry. Thus, A is independent of x_3 for an axially symmetric system. Consequently, Ampere's law separates into two coupled equations for A_1 and A_2 and a third equation, independent of these two, for A_3. Green's function for the latter can be written [2] ‡.

$$\rho A_3 = \sum_{n=0}^{\infty} (1-4n^2)(Ia)\alpha_n \cos n(\zeta-\zeta_0)(x_1^2-1)^{\frac{1}{2}}[(x_1^0)^2-1]^{\frac{1}{2}}(x_1^0-x_2^0)^{-\frac{1}{2}}(x_1-x_2)^{-\frac{1}{2}} g_n\,, \tag{1}$$

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‡ The logaritmic dependence has been established by Shafranov [4].