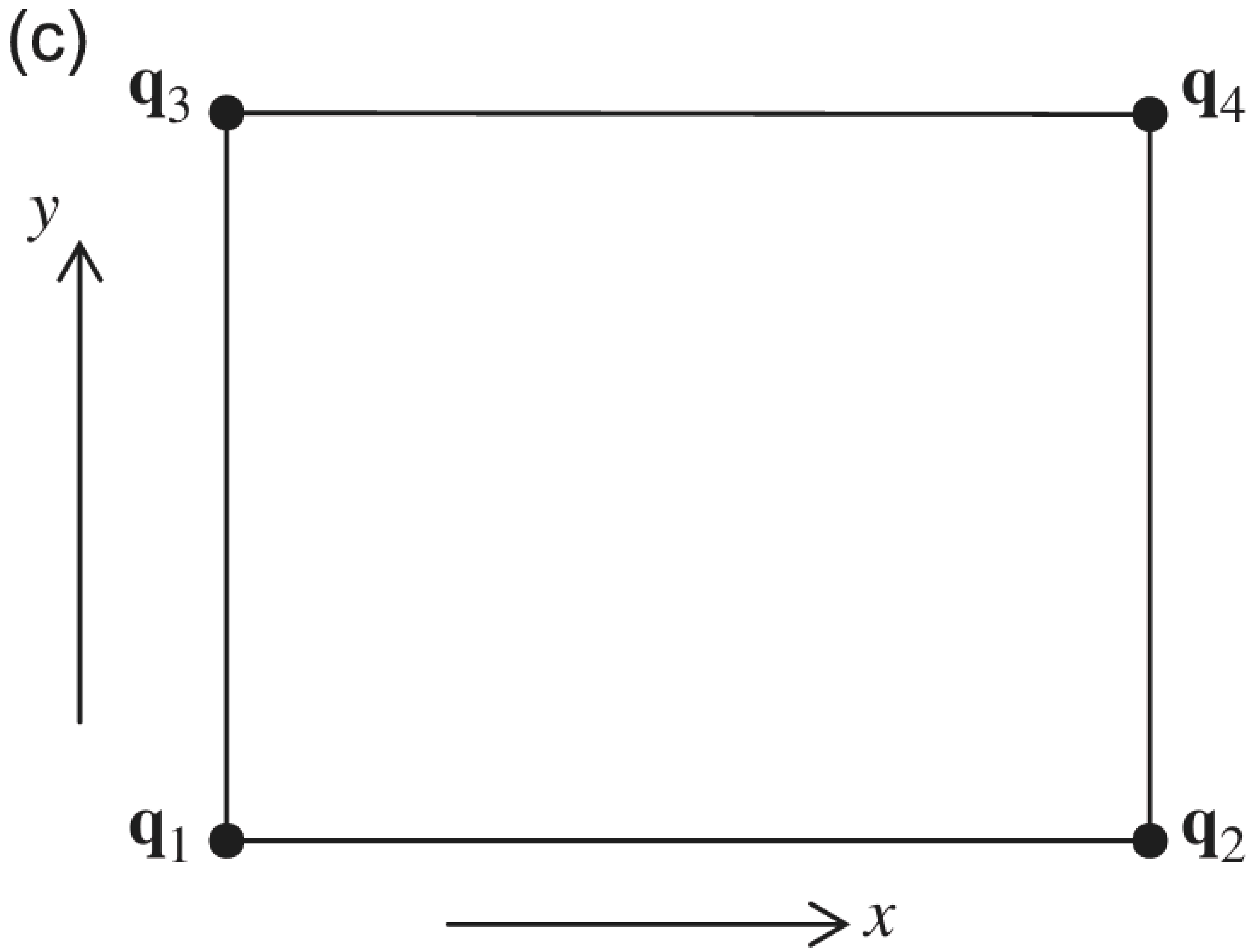
Bloch–Floquet periodic boundary condition

$$\left(\underbrace{\begin{bmatrix} \mathbf{K}_s & \mathbf{S}_p \\ 0 & \mathbf{K}_a \end{bmatrix}}_{\mathbf{K}} - \omega^2 \underbrace{\begin{bmatrix} \mathbf{M}_s & 0 \\ \mathbf{S}_u & \mathbf{M}_a \end{bmatrix}}_{\mathbf{M}} \right) \underbrace{\begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix}}_{\mathbf{q}} = \underbrace{\begin{bmatrix} \mathbf{f}_s \\ \mathbf{f}_a \end{bmatrix}}_{\mathbf{f}} + \underbrace{\begin{bmatrix} \mathbf{e}_s \\ \mathbf{e}_a \end{bmatrix}}_{\mathbf{e}},$$



$$w(x, y, z, t) = W(z)\mathrm{e}^{\mathrm{i}(\omega t - k_x x - k_y y)}$$

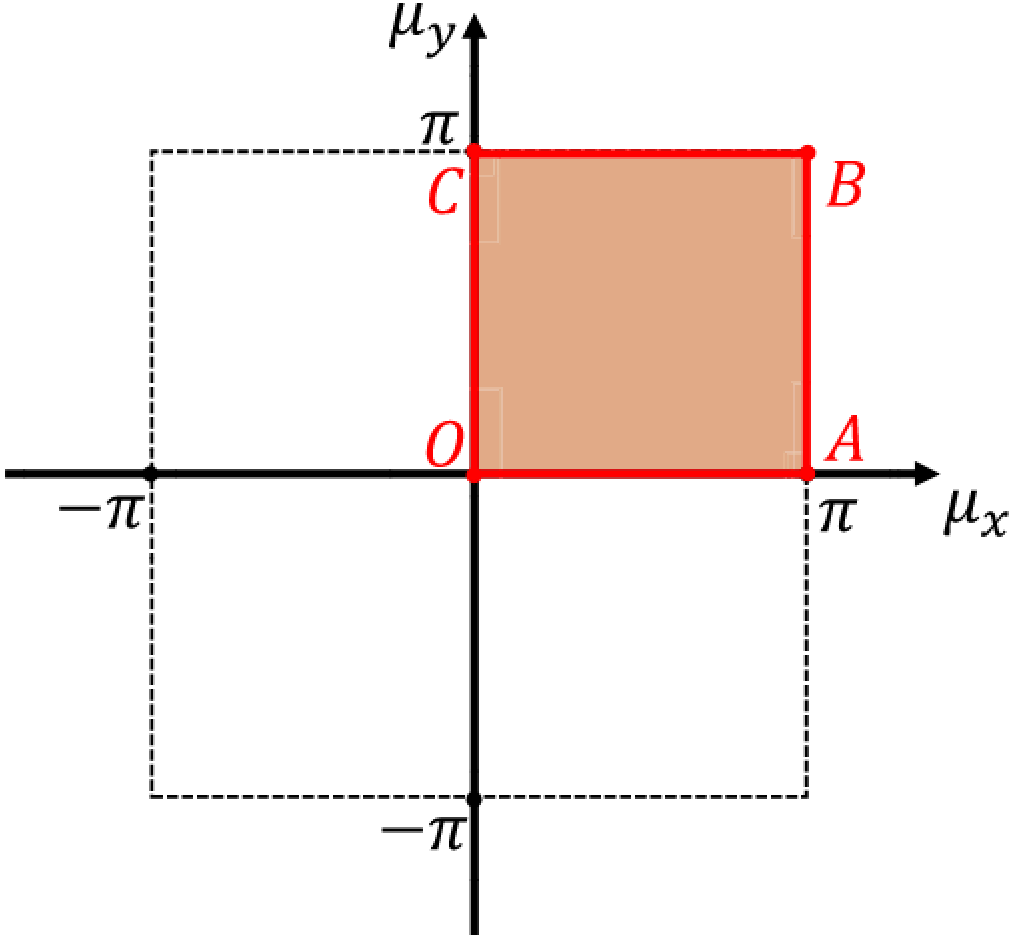
$$\mathbf{q}_2 = \lambda_x \mathbf{q}_1, \quad \mathbf{q}_3 = \lambda_y \mathbf{q}_1, \quad \mathbf{q}_4 = \lambda_x \lambda_y \mathbf{q}_1$$

$$\lambda_x = \mathrm{e}^{-\mathrm{i}\mu_x}, \quad \lambda_y = \mathrm{e}^{-\mathrm{i}\mu_y}, \quad \mu_x = k_x L_x, \quad \mu_y = k_y L_y$$

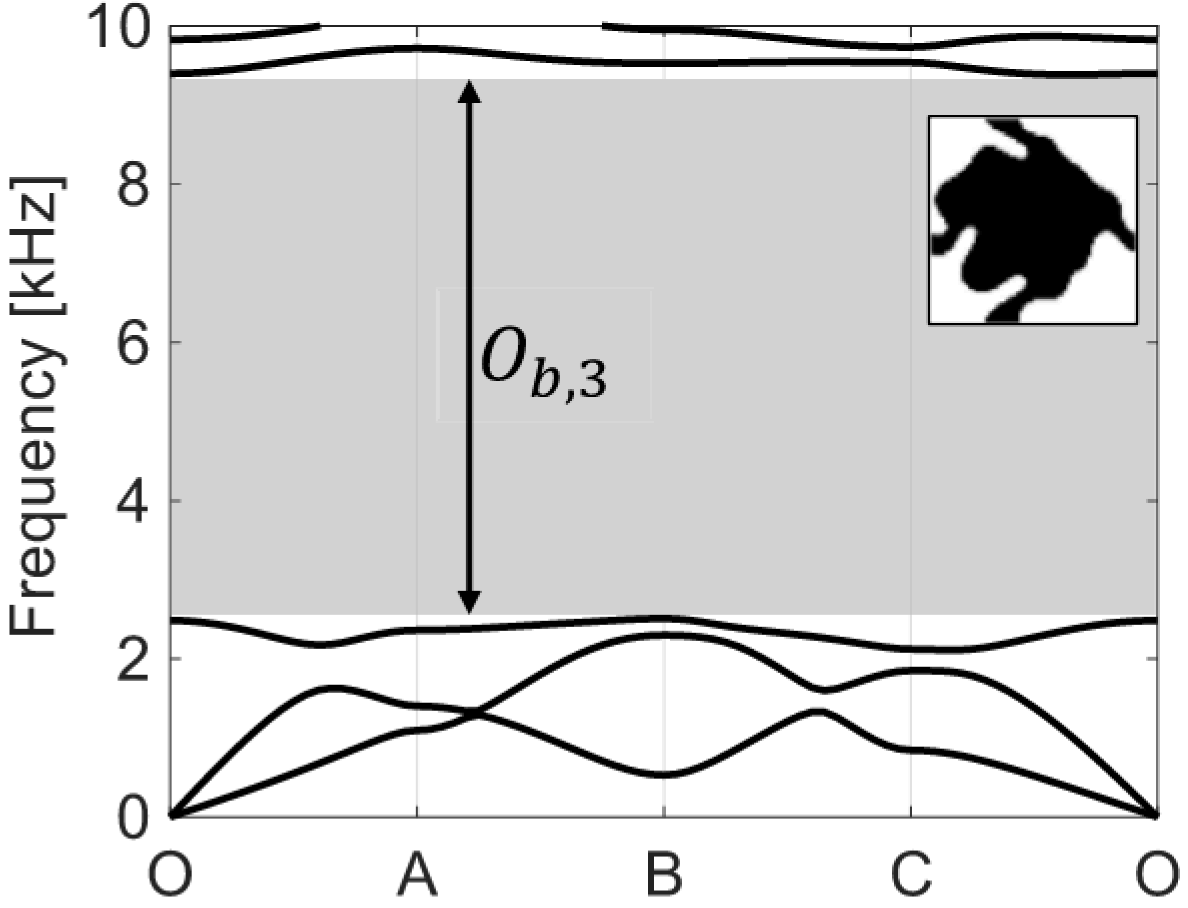
$$\boldsymbol{\Lambda}_L \begin{Bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \end{Bmatrix} = \mathbf{0}, \quad \boldsymbol{\Lambda}_L = \begin{bmatrix} \mathbf{I} & \lambda_x^{-1}\mathbf{I} & \lambda_y^{-1}\mathbf{I} & \lambda_x^{-1}\lambda_y^{-1}\mathbf{I} \end{bmatrix}$$

$$\left[\overline{\mathbf{K}}(\lambda_x, \lambda_y) - \omega^2 \overline{\mathbf{M}}(\lambda_x, \lambda_y)\right] \mathbf{q}_1 = \mathbf{0}$$

$$\overline{\mathbf{K}} = \mathbf{\Lambda}_L \mathbf{K} \mathbf{\Lambda}_R, \quad \overline{\mathbf{M}} = \mathbf{\Lambda}_L \mathbf{M} \mathbf{\Lambda}_R$$

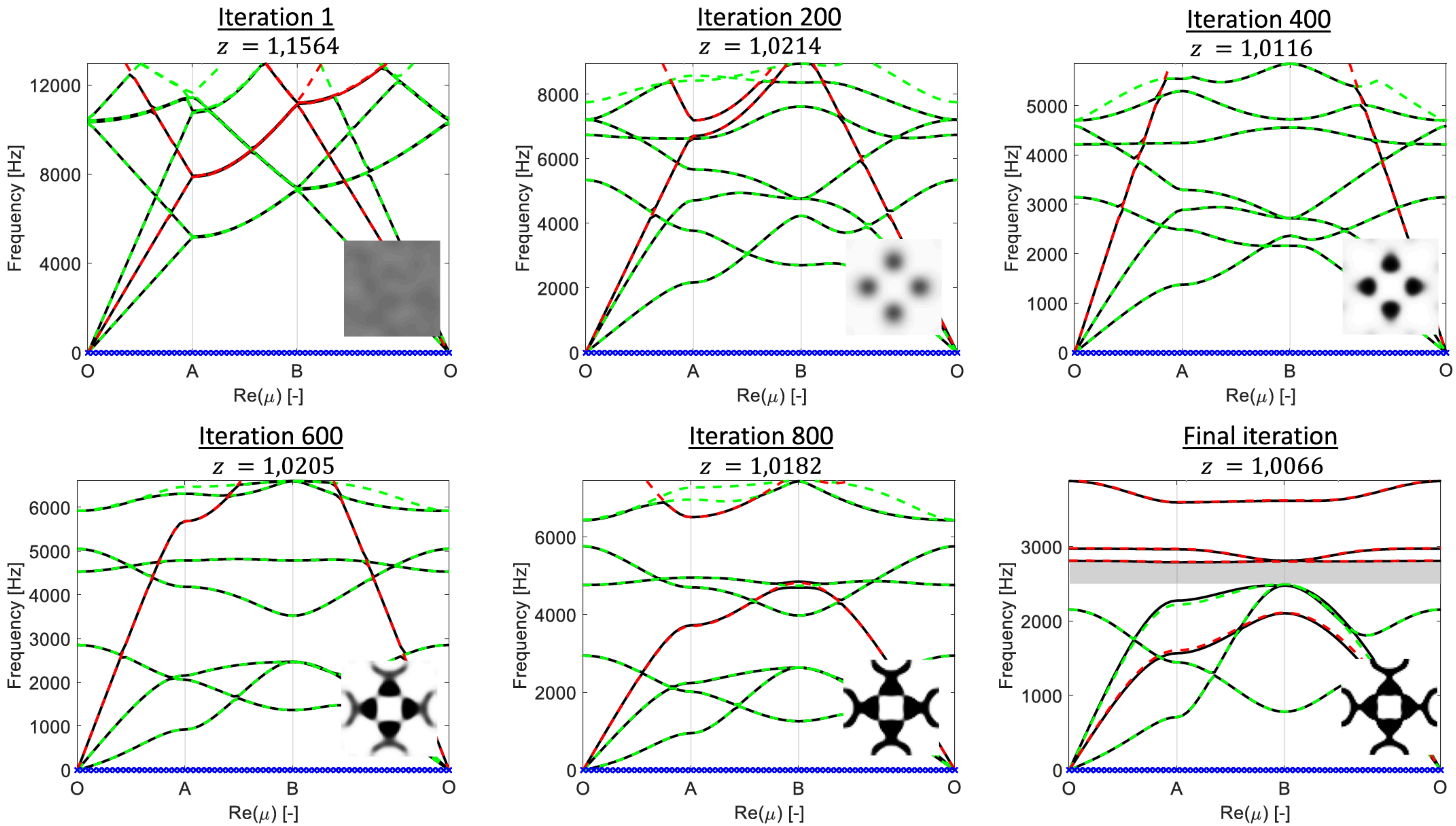


μ_y
π
C
B
O
A
$-\pi$
π
μ_x
$-\pi$

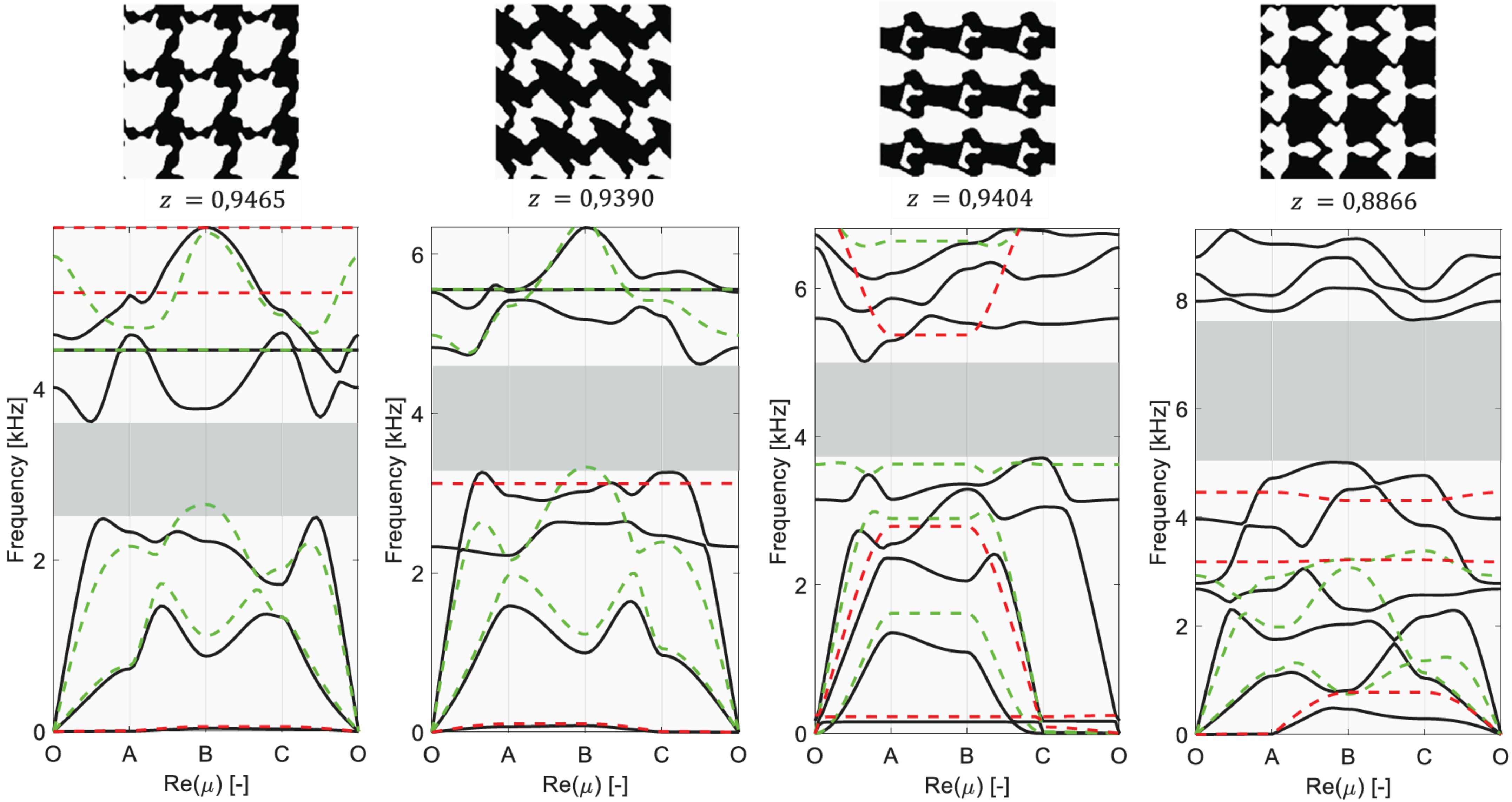


Frequency [kHz]
10
8
6
4
2
0
$O_{b,3}$
O
A
B
C
O

$$O_{i,n} = \min \omega_i(\mu_s, n+1) - \max \omega_i(\mu_s, n),$$



Iteration 1
$z = 1{,}1564$
Iteration 200
$z = 1{,}0214$
Iteration 400
$z = 1{,}0116$
Iteration 600
$z = 1{,}0205$
Iteration 800
$z = 1{,}0182$
Final iteration
$z = 1{,}0066$
Frequency [Hz]
Re(μ) [-]
O
A
B
O



$z = 0{,}9465$
$z = 0{,}9390$
$z = 0{,}9404$
$z = 0{,}8866$
Frequency [kHz]
Re(μ) [-]
O
A
B
C
O

从三维声波方程出发，假设二维平面问题（z方向均匀）：

$$\frac{\partial^2 p}{\partial t^2} = c^2 \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \right)$$

假设简谐振动解：

$$p(x, y, t) = P(x, y) e^{i\omega t}$$

代入方程后化简为亥姆霍兹方程：

$$\nabla^2 P + \left(\frac{\omega}{c} \right)^2 P = 0$$

设解的形式为：

$$P(x,y)=X(x)\cdot Y(y)$$

代入亥姆霍兹方程，分离变量得到：

$$\frac{X''}{X}+\frac{Y''}{Y}=-\left(\frac{\omega}{c}\right)^2$$

令：

$$\frac{X''}{X}=-k_x^2,\quad \frac{Y''}{Y}=-k_y^2$$

则方程简化为两个独立的常微分方程：

$$X''+k_x^2X=0,\quad Y''+k_y^2Y=0$$

且满足关系：

$$k_x^2+k_y^2=\left(\frac{\omega}{c}\right)^2$$

对于刚性边界（法向振速为零）：

- $x = 0$ 和 $x = L_x$ 处：$\frac{\partial P}{\partial x} = 0 \Rightarrow X'(0) = X'(L_x) = 0$
- $y = 0$ 和 $y = L_y$ 处：$\frac{\partial P}{\partial y} = 0 \Rightarrow Y'(0) = Y'(L_y) = 0$

求解常微分方程：

- $X(x) = A\cos(k_x x)$，边界条件要求 $k_x L_x = n\pi$ （$n = 0, 1, 2, \ldots$）
- $Y(y) = B\cos(k_y y)$，边界条件要求 $k_y L_y = m\pi$ （$m = 0, 1, 2, \ldots$）

因此，波数被离散化为：

$$k_x = \frac{n\pi}{L_x}, \quad k_y = \frac{m\pi}{L_y}$$

将波数代入关系式 $k_x^2 + k_y^2 = (\omega/c)^2$，得到：

$$\omega_{n,m} = c\sqrt{\left(\frac{n\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2}$$

转换为频率：

$$f_{n,m} = \frac{\omega_{n,m}}{2\pi} = \frac{c}{2}\sqrt{\left(\frac{n}{L_x}\right)^2 + \left(\frac{m}{L_y}\right)^2}$$

```
>> AcousticEigenvalue
    0.0000
    1.7014
    3.4033
    3.4033
    3.8049
    4.8129
    5.1059
    6.1362
    6.8099
```

```
理论特征频率 (Hz):
         0
    1.7013
    3.4027
    3.4027
    3.8043
    4.8121
    5.1040
    6.1343
    6.8054
```