在弹性动力学中, 原有的平衡方程改为

$$\nabla \cdot \vec{\vec{\sigma}} + \vec{f} = \rho \frac{\partial^2 \vec{u}}{\partial t^2} \Longleftrightarrow \frac{\partial \sigma_{ij}}{\partial x_j} + f_i = \rho \frac{\partial^2 u_i}{\partial t^2}.$$

根据 Sigmund 和 Jensen 在 2003 年的论文, 三维非均匀弹性体中的振动方程满足

$$\begin{cases} \rho \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial}{\partial x}(\lambda \nabla \cdot \vec{u}) + \nabla \cdot \left(\mu \left(\nabla u + \dfrac{\partial \vec{u}}{\partial x} \right) \right) \\ \rho \dfrac{\partial^2 v}{\partial t^2} = \dfrac{\partial}{\partial y}(\lambda \nabla \cdot \vec{u}) + \nabla \cdot \left(\mu \left(\nabla v + \dfrac{\partial \vec{u}}{\partial y} \right) \right) \\ \rho \dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial z}(\lambda \nabla \cdot \vec{u}) + \nabla \cdot \left(\mu \left(\nabla w + \dfrac{\partial \vec{u}}{\partial z} \right) \right) \end{cases}$$

我们下面进行小心的推导来说明这两个形式是一致的,

$$\frac{\partial\sigma_{ij}}{\partial x_j}=\frac{\partial}{\partial x_j}(\lambda\theta\delta_{ij}+2\mu\varepsilon_{ij})=\frac{\partial}{\partial x_i}(\lambda\nabla\cdot\vec{u})+\frac{\partial}{\partial x_j}(\mu(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}))$$

而从 Sigmund 和 Jensen 的方程出发, 第 i 个方程的最后一项中被求散度的向量是

$$\mu\left(\nabla u_i+\frac{\partial\vec{u}}{\partial x_i}\right)=\mu(\begin{pmatrix}\frac{\partial u_i}{\partial x_1}\\\frac{\partial u_i}{\partial x_2}\\\frac{\partial u_i}{\partial x_3}\end{pmatrix}+\begin{pmatrix}\frac{\partial u_1}{\partial x_i}\\\frac{\partial u_2}{\partial x_i}\\\frac{\partial u_3}{\partial x_i}\end{pmatrix})$$

所以可以看到两个形式是一致的. 但我希望将其改写成用算符表达的更简洁的形式.

$$\frac{\partial}{\partial x_j}(\mu(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}))=\mu(\frac{\partial^2u_i}{\partial x_j^2}+\frac{\partial^2u_j}{\partial x_i\partial x_j})+\frac{\partial\mu}{\partial x_j}(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i})$$

$$\frac{\partial}{\partial x_j}(\mu(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}))=\mu(\frac{\partial^2 u_i}{\partial x_j^2}+\frac{\partial^2 u_j}{\partial x_i\partial x_j})+\frac{\partial\mu}{\partial x_j}(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i})$$

前两项很容易改写, 就是 $\mu\nabla^2\vec{u}+\mu(\nabla(\nabla\cdot\vec{u}))$ 的第 i 项. 后两项是 $\nabla\mu\cdot(\nabla u_i+\frac{\partial\vec{u}}{\partial x_i})$. 这两项不是很好改写, 而 $\nabla(\lambda\nabla\cdot\vec{u})=(\nabla\cdot\vec{u})\nabla\lambda+\lambda(\nabla(\nabla\cdot\vec{u}))$, 所以有些文章中如果方程里有 $\nabla((\lambda+\mu)\nabla\cdot\vec{u})$ 这一项那是比较诡异的, 因为我看不到 $(\nabla\cdot\vec{u})\nabla\mu$ 这一项从哪来. 当介质均匀的时候也就是 λ 和 μ 是常值的时候就比较容易写成干净的样子了. 有些文章里有 $\nabla\vec{u}$ 这一项, 得到的结果应该是一个矩阵, 这是一个非常危险的记号, 它既可以诠释为对 $\vec{u}$ 的三个分量分别求梯度, 也就是矩阵的第 i 行是 ∇u_i; 又可以诠释为将 $\vec{u}$ 视作一个整体求梯度, 矩阵的第 i 行是 $\vec{u}$ 关于 x_i 求导. 前者的矩阵是

$$\begin{pmatrix}\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3}\\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3}\\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3}\end{pmatrix}$$

所以很容易看出前者跟后者的矩阵相差转置的关系. 从记号 $\nabla^2\vec{u}$ 和记号 $\nabla\cdot\vec{\sigma}$ 的角度来说, 我倾向于将 $\nabla\vec{u}$ 理解为前者, 这样一来后者可以用 $(\nabla\vec{u})^T$ 来表示,

$$\rho\ddot{\vec{u}}=(\lambda+\mu)(\nabla(\nabla\cdot\vec{u}))+\mu\nabla^2\vec{u}+(\nabla\cdot\vec{u})\nabla\lambda+\nabla\mu\cdot(\nabla\vec{u}+(\nabla\vec{u})^T).$$

In the following we assume that any variation in the material parameters occurs in the (x, y)-plane only; thus we have $\lambda = \lambda(x, y)$, $\mu = \mu(x, y)$, and $\rho = \rho(x, y)$. Further, we restrict the analysis by only considering waves that propagate in the (x, y)-plane, so that $\partial \boldsymbol{u}/\partial z = 0$. Equations (2.1)–(2.3) can then be split into two coupled in-plane equations (governing the longitudinal and transverse modes) and an out-of-plane equation (the acoustic modes):

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left((2\mu + \lambda)\frac{\partial u}{\partial x} + \lambda \frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial y}\left(\mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right), \tag{2.4}$$

$$\rho \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x}\left(\mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right) + \frac{\partial}{\partial y}\left((2\mu + \lambda)\frac{\partial v}{\partial y} + \lambda \frac{\partial u}{\partial x} \right), \tag{2.5}$$

$$\rho \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(\mu \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial w}{\partial y} \right). \tag{2.6}$$

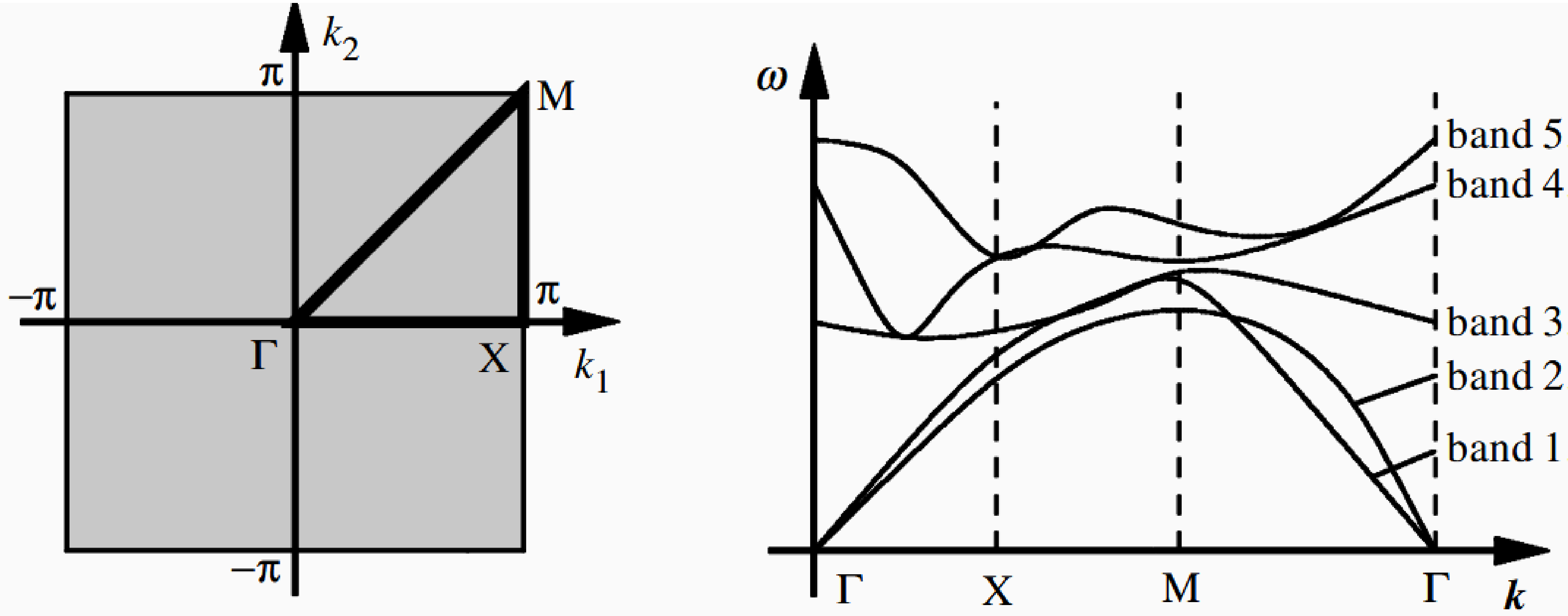
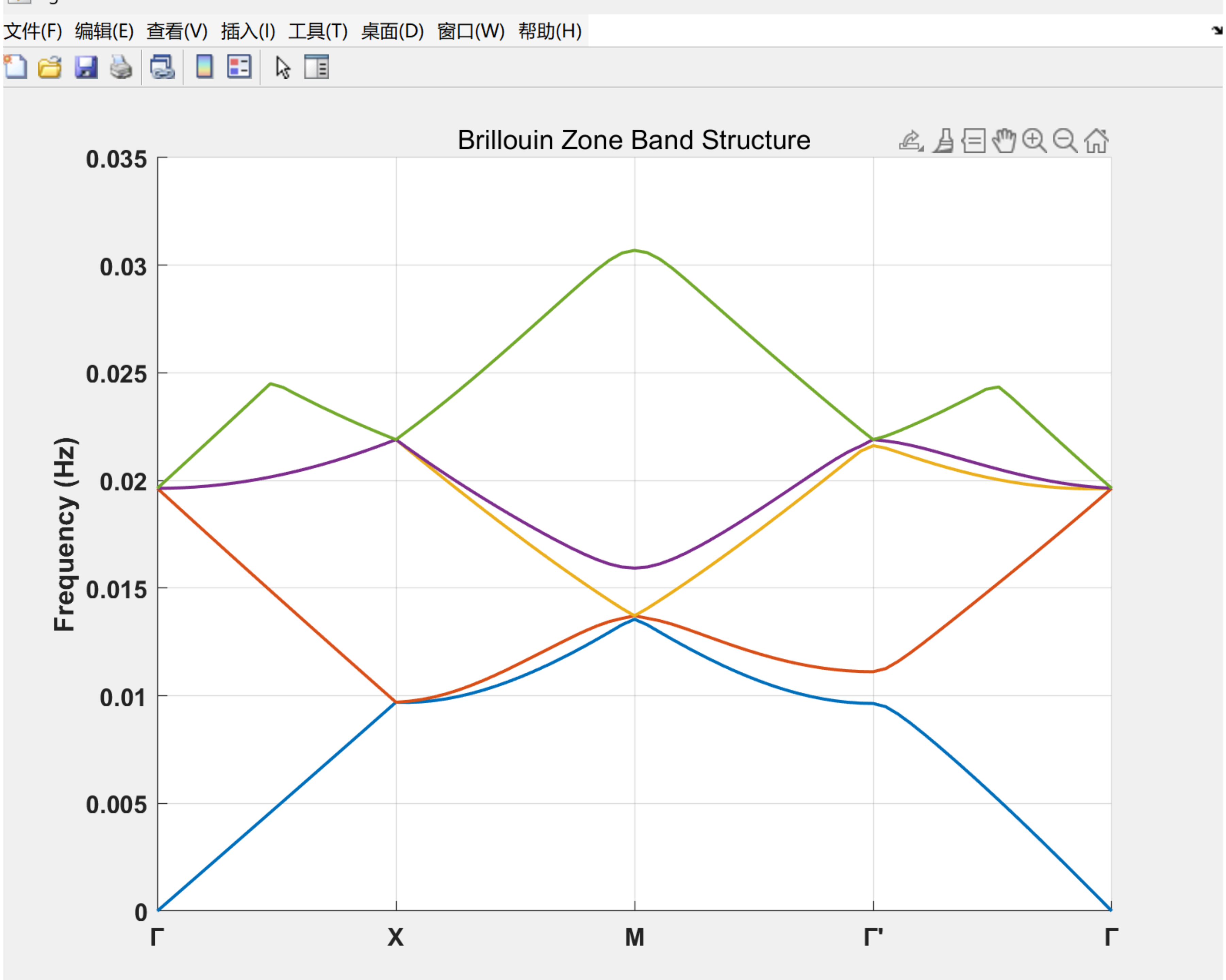


Figure 2. (a) The irreducible Brillouin zone indicating the wave vectors to be searched for the general two-dimensional case (grey area). For square symmetry, the wave equation only has to be calculated for $\boldsymbol{k}$-vector values along the curve Γ–X–M–Γ. (b) Sketch of band structure indicating lowest five eigenvalues for wave vectors along the line Γ–X–M–Γ in the irreducible Brillouin zone.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Brillouin Zone Band Structure
Frequency (Hz)
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
Γ
X
M
Γ'
Γ