中国科学技术大学

本科毕业论文

黎曼面上的 $\mathfrak{su}(n+1)$-Toda 方程

作者姓名：	孙天阳
学　　号：	PB19010392
专　　业：	数学与应用数学
导师姓名：	许斌　副教授
完成时间：	2023 年 6 月 10 日

摘 要

$\mathbb{C}$ 上的 $\mathfrak{su}(n+1)$-Toda 方程是指

$$\frac{\partial^2 u_i}{\partial z \partial \bar{z}} + \sum_{j=1}^{n} a_{ij} \mathrm{e}^{u_j} = 0, \quad 1 \leqslant i \leqslant n, \quad z \in \mathbb{C}$$

其中 $\left\{a_{ij}\right\}_{n\times n}$ 是 $\mathfrak{su}(n+1)$ 的 Cartan 矩阵。在本文中，我们将 $\mathbb{C}$ 上的 $\mathfrak{su}(n+1)$-Toda 方程推广到一般的黎曼面上，并建立 $\mathfrak{su}(n+1)$-Toda 方程的解与全纯曲线的局部对应和整体对应。最后，我们定义黎曼面上带奇点的 $\mathfrak{su}(n+1)$-Toda 方程，并给出解和全纯曲线在奇点附近的刻画。

关键词：$\mathfrak{su}(n+1)$-Toda、黎曼面、奇点、全纯曲线、分歧

ABSTRACT

The $\mathfrak{su}(n+1)$-Toda equation on $\mathbb{C}$ is defined as follows

$$\frac{\partial^2 u_i}{\partial z \partial \bar{z}} + \sum_{j=1}^{n} a_{ij} e^{u_j} = 0, \quad 1 \leqslant i \leqslant n, \quad z \in \mathbb{C}$$

where $\left\{a_{ij}\right\}_{n\times n}$ is the Cartan matrix of $\mathfrak{su}(n+1)$. In this paper, we generalize the $\mathfrak{su}(n+1)$ -Toda equation on $\mathbb{C}$ to general Riemann surfaces and establish the local and global correspondences between solutions of the $\mathfrak{su}(n+1)$ -Toda equation and holomorphic curves. Finally, we define the $\mathfrak{su}(n+1)$ -Toda equation with singularities on Riemann surface and characterize the behaviour of solutions and holomorphic curves around singularity.

Key Words: $\mathfrak{su}(n+1)$-Toda, Riemann surface, singularity, holomorphic curve, ramification

Contents

Chapter 1 Introduction

The $\mathfrak{su}(n+1)$-Toda equation on $\mathbb{C}$ is

$$\frac{\partial^2 u_i}{\partial z \partial \bar{z}} + \sum_{j=1}^{n} a_{ij} \mathrm{e}^{u_j} = 0, \quad 1 \leqslant i \leqslant n, \quad z \in \mathbb{C} \tag{1.1}$$

where $\{a_{ij}\}_{n\times n}$ is the Cartan matrix of $\mathfrak{su}(n+1)$ given by

$$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix}_{n\times n}.$$

When $n = 1$, (1.1) becomes the famous Liouville equation

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} + 2\mathrm{e}^{u} = 0, \quad z \in \mathbb{C}.$$

The $\mathfrak{su}(n+1)$-Toda equation (1.1) has a geometric meaning. It is actually the equation satisfied by the comformal metrics which are the pullback metric of Fubini-Study metric by a totally unramified holomorphic curve and its associated curves. Under this viewpoint, we can easily generalize (1.1) to Riemann surface

Definition 1.1 Let X be a Riemann surface and $(g_i)_{i=1}^n$ be n conformal metrics on X. We say $(g_i)_{i=1}^n$ is a solution of $\mathfrak{su}(n+1)$ -Toda equation on X if

$$\partial\bar{\partial} u_i + \sum_{j=1}^{n} a_{ij} \mathrm{e}^{u_j} \mathrm{d}z \wedge \mathrm{d}\bar{z} = 0 \tag{1.2}$$

on every coordinate chart (U, z), where $g_i = \mathrm{e}^{u_i} |\mathrm{d}z|^2$ in U.

Our first main theorem is

Theorem 1.1 Let X be a Riemann surface,

1. Suppose $f : X \to \mathbb{CP}^n$ is a totally-unramified holomorphic curve, then $\left(g_i = f_{i-1}^* g_{i-1}^{FS}\right)_{i=1}^n$ is a solution to $\mathfrak{su}(n+1)$ -Toda equation on X.
2. Suppose $(g_i)_{i=1}^n$ is a solution to $\mathfrak{su}(n+1)$ -Toda equation on X, then for any $p \in X$, there exists a neighborhood Ω of p and a totally-unramified holomorphic curve $f : \Omega \to \mathbb{CP}^n$ such that $\left(g_i|_\Omega = f_{i-1}^* g_{i-1}^{FS}\right)_{i=1}^n$. And for any two such f_1, f_2 on Ω, there exists $P \in \mathrm{PSL}(n) = \mathrm{Aut}(\mathbb{CP}^n)$ such that $f_2 = P \circ f_1$.

The version on $\mathbb{C}$ can be found in [1]. The meaning of totally-unramified and f_k is defeined in section 2.1 ; First part of the theorem 1.1 is proved in section 2.2; we introduce an important tool called Toda map in section 2.3; and construct locally a totally unramified curve from solution via Toda map in section 2.4;

Our second main theorem is

Theorem 1.2 Let X be a Riemann surface,

1. Suppose $\mathbf{f}$ defined on X is a totally-unramified unitary curve, then $\left(f_{i-1}^{*} g_{i-1}^{FS}\right)_{i=1}^{n}$ is well defined and a solution to $\mathfrak{su}(n+1)$ -Toda equation on X.
2. Suppose $(g_i)_{i=1}^{n}$ is a solution to $\mathfrak{su}(n+1)$ -Toda equation on X, then there exists a totally-unramified unitary curve $\mathbf{f}$ defined on X such that $\left(g_i|_{\Omega} = f_{i-1}^{*} g_{i-1}^{FS}\right)_{i=1}^{n}$. And for any two such $\mathbf{f}_1, \mathbf{f}_2$ on Ω, there exists $P \in \mathrm{PSU}(n+1)$ such that $\mathbf{f}_2 = P \circ \mathbf{f}_1$.

The version on any region of $\mathbb{C}$ can be found in [2].

In the last chapter, we study the $\mathfrak{su}(n+1)$-Toda equation with singularity.

Chapter 2 Local correspondence

2.1 Nondegenerate curve and its associated curve

For later use we introduce the concept of nondegenerate curve and its associated curve in this section. This part can be found in [3][Section 4,Chapter 2], but we give more details here. Let X be a Riemann surface and $f: X \to \mathbb{CP}^n$ be a holomorphic map.

Lemma 2.1 For any point $p \in X$, there exists a neighborhood Ω of p and a holomorphic map $v: \Omega \to \mathbb{C}^{n+1}\backslash\{0\}$ such that $\pi \circ v = f$, where $\pi: \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{CP}^n$. We call such v a lift of f on Ω.

Definition 2.1 Suppose v is a lift of f on a coordinate system (U, z). We define

$$\Lambda_k(v)(z): U \longrightarrow \bigwedge^{k+1} \mathbb{C}^{n+1}, \quad z \longmapsto v(z) \wedge v'(z) \wedge \cdots \wedge v^{(k)}(z), \quad 0 \leqslant k \leqslant n,$$

where $v^{(i)}(z)$ is the i-th derivative of v with respect to z. Denote

$$N_k = \binom{n+1}{k+1} - 1.$$

When $k = n$, we have

$$\Lambda_n(v)(z): U \longrightarrow \bigwedge^{n+1} \mathbb{C}^{n+1} \cong \mathbb{C},$$

where we have fixed the canonical basis $\{e_i\}_{i=0}^n$ of $\mathbb{C}^{n+1}$.

Definition 2.2

(1) We say f is nondegenerate if for any lift v on any coordinate system (U, z), $\Lambda_n(v)(z): U \to \mathbb{C}$ has at most isolated zero points in U.

(2) We say f is totally unramified if for any $p \in X$ we can find a coordinate system (U, z) around p and a lift $v: U \to \mathbb{C}^{n+1}\backslash\{0\}$ such that

$$\Lambda_n(v)(z) \equiv e_0 \wedge \cdots \wedge e_n.$$

It is obvious that totally unramified implies nondegenerate.

A more geometric definition of nondegenerate first occurs in [3][Page 173]

Proposition 2.2 The following are equivalent

(1) f is nondegenerate;

(2) $f(X)$ is not contained in any hyperplane of $\mathbb{CP}^n$.

Definition 2.3 Let f be nondegenerate. Define its k-th associated curve

$$f_k : \Sigma \longrightarrow \mathbb{P}\left(\bigwedge^{k+1} \mathbb{C}^{n+1}\right) = \mathbb{CP}^{N_k}, \quad z \longmapsto [v(z) \wedge v'(z) \wedge \cdots \wedge v^{(k)}(z)], \quad 0 \leqslant k \leqslant n.$$

Where $v(z)$ is an arbitrary lift of f on a coordinate system (U, z).

Proof of well-difinedness First, since f is nondegenerate,

$$v(z) \wedge \cdots \wedge v^{(k)}(z)$$

has at most isolated zero points for any $0 \leqslant k \leqslant n$ and any lift v,

$$[v(z) \wedge \cdots \wedge v^{(k)}(z)]$$

is well-defined. And for another lift $\tilde{v}$ of f, there exists ρ such that $\tilde{v} = \rho v$, then

$$\tilde{v}' = \rho' \cdot v + \rho \cdot v$$

so

$$\tilde{v} \wedge \tilde{v}' = \rho^2 \cdot (v \wedge v')$$

and in general

$$\tilde{v} \wedge \cdots \wedge \tilde{v}^{(k)} = \rho^{k+1} \cdot v \wedge \cdot \wedge v^{(k)}.$$

Similarly, let w be another local coordinate on S. Then

$$\frac{\partial v}{\partial w} = \frac{\partial z}{\partial w} \frac{\partial v}{\partial z}$$

and so

$$v \wedge \frac{\partial v}{\partial w} = \frac{\partial z}{\partial w} \left(v \wedge \frac{\partial v}{\partial z} \right).$$

In general, we will have

$$v \wedge \cdots \wedge \frac{\partial^k v}{\partial w^k} = \left(\frac{\partial z}{\partial w} \right)^{k(k+1)/2} v \wedge \cdots \wedge \frac{\partial^k v}{\partial z^k}.$$

∎

2.2 From curve to solution

In this section we prove the first part of theorem 1.1.

Theorem 2.3 Let $f: X \to \mathbb{CP}^n$ be nondegenerate. Then

$$f_k^*(\omega_k) = \frac{\sqrt{-1}}{2\pi} \frac{\|\Lambda_{k-1}(v)\|^2 \|\Lambda_{k+1}(v)\|^2}{\|\Lambda_k(v)\|^4} \mathrm{d}z \wedge \mathrm{d}\bar{z}, \quad 1 \leqslant k \leqslant n-1, \tag{2.1}$$

where $v: U \to \mathbb{C}^{n+1} \backslash \{0\}$ is an arbitrary lift of f on a coordinate chart (U, z) and ω_k is the associated (1, 1) form of

This is called infinitesimal Plucker formula and its proof can be found in [3].

Besides (2.1), we have

$$f_k^*(\omega_k) = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \|\Lambda_k\|^2 = \frac{\sqrt{-1}}{2\pi} \frac{\partial^2}{\partial z \partial \bar{z}} \log \|\Lambda_k\|^2 \mathrm{d}z \wedge \mathrm{d}\bar{z}. \tag{2.2}$$

by the definition of ω_k. Compare (2.1) with (2.2) we get

$$\frac{\partial^2}{\partial z \partial \overline{z}} \log \|\Lambda_k(v)\|^2 = \frac{\|\Lambda_{k-1}(v)\|^2 \|\Lambda_{k+1}(v)\|^2}{\|\Lambda_k(v)\|^4}, \quad 1 \leqslant k \leqslant n-1.$$

Since our f is totally unramified, we can choose v such that

$$\Lambda_n(v) \equiv e_0 \wedge \cdots \wedge e_n \Longrightarrow \|\Lambda_n(v)\| \equiv 1.$$

For $k = 0$, a direct computation gives

$$\frac{\partial^2}{\partial z \partial \overline{z}} \log \|\Lambda_0\|^2 = \frac{\|\Lambda_1\|^2}{\|\Lambda_0\|^4}.$$

Let

$$u_i = \log \frac{\|\Lambda_{i-2}\|^2 \|\Lambda_i\|^2}{\|\Lambda_{i-1}\|^4}, \quad 1 \leqslant i \leqslant n.$$

It is easy to check

$$\frac{\partial^2 u_i}{\partial z \partial \bar{z}} + \sum_{j=1}^{n} a_{ij} \mathrm{e}^{u_j} = 0, \quad 1 \leqslant i \leqslant n.$$

This complete the proof of the first part of theorem 1.1.

2.3 From solution to Toda map

In this section we introduce the concept of Toda map, which is a $\mathrm{SU}(n+1)$ valued map and its first column is almost a lift of the curve.

Define

$$\begin{cases} w_0 = -\dfrac{\sum_{i=1}^{n}(n-i+1)u_i}{2(n+1)} \\ w_i = w_0 + \frac{1}{2}\sum_{j=1}^{i} u_j \qquad 1 \leqslant i \leqslant n. \end{cases}$$

Equivalently

$$\sum_{i=0}^{n} w_i = 0, \quad u_i = 2w_i - 2w_{i-1}, \quad 1 \leqslant i \leqslant n.$$

Define

$$\mathcal{U} = \begin{pmatrix} (w_0)_z & & & \\ & (w_1)_z & & \\ & & \ddots & \\ & & & (w_n)_z \end{pmatrix} + \begin{pmatrix} 0 & & & \\ \mathrm{e}^{w_1 - w_0} & 0 & & \\ & \ddots & \ddots & \\ & & \mathrm{e}^{w_n - w_{n-1}} & 0 \end{pmatrix},$$

$$\mathcal{V} = -\begin{pmatrix} (w_0)_{\bar{z}} & & & \\ & (w_1)_{\bar{z}} & & \\ & & \ddots & \\ & & & (w_n)_{\bar{z}} \end{pmatrix} - \begin{pmatrix} 0 & \mathrm{e}^{w_1 - w_0} & & \\ & 0 & \ddots & \\ & & \ddots & \mathrm{e}^{w_n - w_{n-1}} \\ & & & 0 \end{pmatrix}.$$

Then

$$\alpha = \mathcal{U}\mathrm{d}z + \mathcal{V}\mathrm{d}\bar{z}$$

is a $\mathfrak{su}(n+1)$-valued differential one-form on U. It is easy to check that

$$\frac{\partial^2 u_i}{\partial z \partial \bar{z}} + \sum_{j=1}^{n} a_{ij} \mathrm{e}^{u_j} = 0, \quad 1 \leq i \leq n$$

is equivalent to the structure equation

$$\mathrm{d}\alpha + \frac{1}{2}[\alpha, \alpha] = 0,$$

where the notation $[\alpha, \alpha]$ is defined in [4][Lemma 1.5.21].

Theorem 2.4 For any $p \in U$ there exists a neighborhood Ω of p and $\phi : \Omega \to \mathrm{SU}(n+1)$ such that $\alpha|_\Omega = \phi^* \omega$, where ω is the Maurer-Cartan form of $\mathrm{SU}(n+1)$. And for any two such ϕ_1, ϕ_2 on Ω, there exists unique $g \in \mathrm{SU}(n+1)$ such that $\phi_1 = g \cdot \phi_2$. We call such ϕ the Toda map of $(u_i)_{i=1}^n$ on Ω.

We prove a more general version of theorem 2.4.

Theorem 2.5 Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let α be a $\mathfrak{g}$-valued 1-form on the smooth manifold M satisfying the structural equation

$$\mathrm{d}\alpha + \frac{1}{2}[\alpha, \alpha] = 0.$$

Then, for each point $p \in M$, there exists a neighborhood Ω of p and a smooth map $\phi : \Omega \to G$ such that $\alpha|_\Omega = \phi^* \omega_G$, where ω_G is the Maurer-Cartan form of G. And for any two such ϕ_1, ϕ_2 on Ω, there exists unique $g \in G$ such that $\phi_1 = g \cdot \phi_2$.

The proof can be found in [4][Theorem 3.6.1]. For completeness, we give it here.

Proof Let $\pi_G : G \times M \to G$ and $\pi_M : G \times M \to M$ denote the canonical projections. Let $\Psi = \pi_M^* \alpha - \pi_G^* \omega_G$ and let $\mathcal{D}$ be the distribution defined by the kernel of Ψ. Actually, we don't yet know that it is a distribution. It suffices to show that it has constant rank. We are going to show

$$\pi_{M*}|\mathcal{D}_{p,g} : \mathcal{D}_{p,g} \to T_p(M)$$

is an isomorphism. In particular, this will verify that $\ker \Omega$ has constant rank $= \dim M$. If $\pi_{M*}(v, w) = 0$ for some $(v, w) \in \mathcal{D}_{(p,g)}$, we have

$$\pi_{M*}(v, w) = 0 \Longrightarrow v = 0 \Longrightarrow \alpha(v) = \omega_G(w) = 0 \Longrightarrow w = 0 \Longrightarrow (v, w) = 0.$$

So we see that $\pi_{M*}|\mathcal{D}_{(p,g)}$ is injective. Conversely, if $v \in T_p M$, then $(v, \omega_G^{-1}(\alpha(v))_p \in \mathcal{D}_{(p,g)}$, and so $\pi_{M*}|\mathcal{D}_{(p,g)}$ is surjective.

Next we are going to show that $\mathcal{D}$ is integrable. Calculating the exterior derivative of Ψ, we get

$$\begin{aligned} \mathrm{d}\Psi &= \mathrm{d}\pi_M^* \alpha - \mathrm{d}\pi_G^* \omega_G = \pi_M^*(\mathrm{d}\alpha) - \pi_G^*(\mathrm{d}\omega_G) \\ &= \pi_M^* \left(-\frac{1}{2}[\alpha, \alpha] \right) - \pi_G^* \left(-\frac{1}{2}[\omega_G, \omega_G] \right) \\ &= -\frac{1}{2}[\pi_M^* \alpha, \pi_M^* \alpha] + \frac{1}{2}[\pi_G^* \omega_G, \pi_G^* \omega_G] \end{aligned}$$

Now make the replacement $\pi_M^* \alpha = \pi_G^* \omega_G + \Psi$ so that

$$\mathrm{d}\Psi = -\frac{1}{2}[\pi_G^* \omega_G, \Psi] - \frac{1}{2}[\Psi, \pi_G^* \omega_G] - \frac{1}{2}[\Psi, \Psi].$$

Thus, $\mathrm{d}\Psi(X, Y) = 0$ whenever $\Psi(X) = \Psi(Y) = 0$. Therefore, by the Frobenius theorem, the distribution $\mathcal{D}$ is integrable. Finally, we are going to construct, for each $p \in M$, a neighborhood Ω of p in M and a smooth map $\phi : \Omega \to G$ such that $\alpha\big|_\Omega = \phi^* \omega_G$. Let $\mathcal{L}$ be a leaf through the point $(p, g) \in M \times G$. The derivative of the restriction of π_M to $\mathcal{L}$ induces isomorphism $\pi_{M*} : \mathcal{D}_{(g,p)} \to T_p M$ and so the restriction of π_M to $\mathcal{L}$ is a local diffeomorphism of a neighborhood of (p, g) to some neighborhood of (p, g) to some neighborhood Ω of $p \in M$. Let $F : \Omega \to \mathcal{L}$ be the inverse map. Since $\pi_M F = \mathrm{Id}_\Omega$, F must have the form $F(p) = (p, \phi(p))$ where $\phi : \Omega \to G$. Now $F^* \Psi = \Psi F_* = 0$, thus we have

$$0 = F^* \Psi = F^*(\pi_M^* \alpha) - F^*(\pi_G^* \omega_G) = \alpha - \phi^* \omega_G.$$

As for the uniqueness, Let $h: \Omega \to G$ given by $h(x) = \phi_2(x)\phi_1(x)^{-1}$. Then $h^*\omega_G = \mathrm{Ad}(\phi_1)(\phi_2^*\omega_G - \phi_1^*\omega_G)$, where $\mathrm{Ad}: G \to \mathrm{GL}(\mathfrak{g})$ is the adjoint representation of G. Since the later expression vanishes by assumption and $h^*\omega_G = \omega_G h_*$, we see that $h_*: TM \to TG$ induces the zero map on each tangent space. It follows that h is the constant map. In particular, $\phi_1(x) = g \cdot \phi_2(x)$ for all $x \in \Omega$ where g is some fixed element of G. ∎

When $G = \mathrm{SU}(n+1)$ and $M = U \subset \mathbb{C}$ we get theorem 2.4. The classical way of writing the Maurer-Cartan form on $\mathrm{SU}(n+1)$ is $g^{-1}\mathrm{d}g$. This has the follwing meaning. The factor g^{-1} is an abbreviation for $L_{g^{-1}*}$. For $\mathrm{d}g$, regard g as the identity map on $\mathrm{SU}(n+1)$ then $\mathrm{d}g$ is the identity map on the tangent bundle. The existence of the $\mathrm{d}g$ part is to remind you g^{-1} is not a multiplication in $\mathrm{SU}(n+1)$ but a map from $T_g(\mathrm{SU}(n+1))$ to $T_e(\mathrm{SU}(n+1))$.

2.4 From Toda map to curve

Let $\phi: \Omega \to \mathrm{SU}(n+1)$ be the Toda map of $\left(u_i\right)_{i=1}^n$ on $\Omega \subset U$. Denote the j-th column vector of

$$\phi \cdot (\mathrm{diag}(\mathrm{e}^{w_i}))$$

by

$$\hat{f}_{j-1}: \Omega \to \mathbb{C}^{n+1}, \quad 1 \leqslant j \leqslant n+1.$$

It is clear that $w_i = \log \|\hat{f}_i\|$ and $\hat{f}_i$ are orthogonal to each other.

Lemma 2.6 The following recursive relation holds

$$\begin{cases} (\hat{f}_k)_z = \hat{f}_{k+1} + (\log|\hat{f}_k|^2)_z \hat{f}_k, & 0 \leqslant k \leqslant n-1; \\ (\hat{f}_n)_z = (\log|\hat{f}_n|^2)_z \hat{f}_n; \\ (\hat{f}_k)_{\bar{z}} = -|\hat{f}_k|^2/|\hat{f}_{k-1}|^2 \cdot \hat{f}_{k-1}, & 1 \leqslant k \leqslant n; \\ (\hat{f}_0)_{\bar{z}} = 0. \end{cases}$$

Proof Differentiate

$$\left(\hat{f}_0, \cdots, \hat{f}_n\right) = \phi \cdot \begin{pmatrix} \mathrm{e}^{w_0} & & \\ & \ddots & \\ & & \mathrm{e}^{w_n} \end{pmatrix}$$

with z, we get

$$
\begin{aligned}
&\left(\left(\hat{f}_0\right)_z, \cdots, \left(\hat{f}_n\right)_z\right) \\
=&\phi_z \cdot \begin{pmatrix} e^{w_0} & & \\ & \ddots & \\ & & e^{w_n} \end{pmatrix} + \phi \cdot \begin{pmatrix} e^{w_0} & & \\ & \ddots & \\ & & e^{w_n} \end{pmatrix} \cdot \begin{pmatrix} (w_0)_z & & \\ & \ddots & \\ & & (w_n)_z \end{pmatrix} \\
=&\phi \cdot \mathcal{U} \cdot \begin{pmatrix} e^{w_0} & & \\ & \ddots & \\ & & e^{w_n} \end{pmatrix} + \left(\hat{f}_0, \cdots, \hat{f}_n\right) \cdot \begin{pmatrix} (w_0)_z & & \\ & \ddots & \\ & & (w_n)_z \end{pmatrix} \\
=&\phi \cdot \begin{pmatrix} (w_0)_z & & \\ & \ddots & \\ & & (w_n)_z \end{pmatrix} \cdot \begin{pmatrix} e^{w_0} & & \\ & \ddots & \\ & & e^{w_n} \end{pmatrix} + \phi \cdot \begin{pmatrix} 0 & & & \\ e^{w_1-w_0} & 0 & & \\ & \ddots & \ddots & \\ & & e^{w_n-w_{n-1}} & 0 \end{pmatrix} \cdot \begin{pmatrix} e^{w_0} & & \\ & \ddots & \\ & & e^{w_n} \end{pmatrix} \\
&+ \left(\hat{f}_0, \cdots, \hat{f}_n\right) \cdot \begin{pmatrix} (w_0)_z & & \\ & \ddots & \\ & & (w_n)_z \end{pmatrix} \\
=&2\left(\hat{f}_0, \cdots, \hat{f}_n\right) \cdot \begin{pmatrix} (w_0)_z & & \\ & \ddots & \\ & & (w_n)_z \end{pmatrix} \\
&+\phi \cdot \begin{pmatrix} e^{w_0} & & \\ & \ddots & \\ & & e^{w_n} \end{pmatrix} \cdot \begin{pmatrix} e^{-w_0} & & \\ & \ddots & \\ & & e^{-w_n} \end{pmatrix} \cdot \begin{pmatrix} 0 & & & \\ e^{w_1-w_0} & 0 & & \\ & \ddots & \ddots & \\ & & e^{w_n-w_{n-1}} & 0 \end{pmatrix} \cdot \begin{pmatrix} e^{w_0} & & \\ & \ddots & \\ & & e^{w_n} \end{pmatrix} \\
=&\left(\hat{f}_0, \cdots, \hat{f}_n\right) \cdot \begin{pmatrix} 2\left(\log|\hat{f}_0|\right)_z & & \\ & \ddots & \\ & & 2\left(\log|\hat{f}_n|\right)_z \end{pmatrix} + \left(\hat{f}_0, \cdots, \hat{f}_n\right) \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.
\end{aligned}
$$

In this way we get out first two recursive relations. The last two are similar. ∎

Define

$$f_0 = \pi \circ \hat{f}_0,$$

where $\pi: \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{CP}^n$. From $(\hat{f}_0)_{\bar{z}} = 0$ we know that $f_0 : \Omega \to \mathbb{CP}^n$ is holomorphic. Next we show that f_0 is totally unramified.

Lemma 2.7

$$\hat{f}_0 \wedge \hat{f}_0' \wedge \cdots \wedge \hat{f}_0^{(k)} = \hat{f}_0 \wedge \hat{f}_1 \wedge \cdots \wedge \hat{f}_k.$$

Proof It is clear from lemma 2.6. ∎

As a corollary,

$$\hat{f}_0 \wedge \hat{f}_0' \wedge \cdots \wedge \hat{f}_0^{(n)} = \hat{f}_0 \wedge \hat{f}_1 \wedge \cdots \wedge \hat{f}_n.$$

Since $\hat{f}_i$ are orthogonal to each other, $\hat{f}_0 \wedge \hat{f}_0' \wedge \cdots \wedge \hat{f}_0^{(n)}$ does not vanish everywhere.

And it is easy to check that

$$\mathrm{e}^{u_k}\mathrm{d}z \wedge \mathrm{d}\bar{z} = f_k^*\omega_k.$$

The uniqueness part of theorem 1.1 can be found in [3].

Chapter 3 Global correspondence

With local correspondence in hand, our next step is to glue them together.

3.1 Unitary curve

Following the definition of multi valued holomorphic functions on $\mathbb{C}$ in [5], we can define multi valued meromorphic functions on Riemann surface. More generally, we can define multi $\mathbb{CP}^n$ valued holomorphic maps on Riemann surface. We use $\mathbf{f}$ to represent a multi $\mathbb{CP}^n$ valued holomorphic map on X and (f, Ω) to represent an element subject to $\mathbf{f}$. Denote by $\overline{(f, \Omega)}_p$ the germ at p determined by (f, Ω) .

Given $\mathbf{f}$, fix a base point $b \in X$. Denote

$$\mathcal{G}_b^{\mathbf{f}} = \{\text{germs at } b \text{ determined by } \mathbf{f}\ \} \subset \mathcal{O}_b.$$

Consider a loop $\gamma : [0, 1] \to x$ based at b. The analytic continuation along γ induces a permutation of the set $\mathcal{G}_b^{\mathbf{f}}$ and the permutation only depends on the path homotopy equivalence class of γ. In this way we get a group homomorphism

$$\Phi_b^{\mathbf{f}} : \pi_1(X, b) \longrightarrow \operatorname{Sym} \mathcal{G}_b^{\mathbf{f}}.$$

We call $\Phi_b^{\mathbf{f}}$ the monodromy of $\mathbf{f}$ based at b.

There is a special case we are most interested in. Recall that $\mathrm{PSL}(n+1, \mathbb{C})$ is the automorphism group of $\mathbb{CP}^n$. Each element P induces a permutation of $\mathcal{O}_b$

$$\overline{(f, \Omega)}_b \longmapsto \overline{(P \circ f, \Omega)}_b$$

and there is also a group homomorphism

$$\Psi : \mathrm{PSL}(n+1, \mathbb{C}) \longrightarrow \operatorname{Sym} \mathcal{O}_b.$$

Denote by $\Gamma_b^{\mathbf{f}}$ the subset of $\mathrm{PSL}(n+1, \mathbb{C})$ under which $\mathcal{G}_b^{\mathbf{f}} \subset \mathcal{O}_b$ is closed. It forms a subgroup of $\mathrm{PSL}(n+1, \mathbb{C})$ and we get

$$\Psi_b^{\mathbf{f}} : \Gamma_b^{\mathbf{f}} \longrightarrow \operatorname{Sym} \mathcal{G}_b^{\mathbf{f}}$$

If $\operatorname{Im} \Phi_b^{\mathbf{f}} \subset \operatorname{Im} \Gamma_b^{\mathbf{f}}$, we say $\mathbf{f}$ has monodromy in $\mathrm{PSL}(n+1, \mathbb{C})$. In the same way we can define what means that $\mathbf{f}$ has monodromy in $\mathrm{PSU}(n+1)$.

3.2 Proof of theorem 1.2

By the uniqueness part of theorem 1.1, we can see that **f** got from the solution has monodromy in $\mathrm{PSU}(n+1)$ and every element of **f** is totally unramified. This complete the proof of second part of theorem 1.2.

Conversely, associated curve is well defined for totally unramified unitary curve and infinitesimal Plucker formula still holds. Hence we get well-defined metric. This complete the proof of first part of theorem 1.2.

Chapter 4 Behaviour around singularity

In previous chapter we define the $\mathfrak{su}(n+1)$-Toda equation on Riemann surface X and establish the correspondence between solutions and curves. In this chapter, we investigate a particularly interesting case, i.e. $X = \tilde{X} \backslash \{p_k\}_{k=1}^m$ where $\tilde{X}$ is a compact Riemann surface. For example,

$$\frac{\partial^2 u_i}{\partial z \partial \bar{z}} + \sum_{j=1}^{n} a_{ij} \mathrm{e}^{u_j} = 0, \quad 1 \leqslant i \leqslant n, \quad z \in \mathbb{C}^*$$

can be regarded as the $\mathfrak{su}(n+1)$ -Toda equation on $\mathbb{CP}^1 \backslash \{0, \infty\}$.

In order to understand the behaviour of the solution around every p_k, we investigate

$$\frac{\partial^2 u_i}{\partial z \partial \bar{z}} + \sum_{j=1}^{n} a_{ij} \mathrm{e}^{u_j} = 0, \quad 1 \leqslant i \leqslant n, \quad z \in D^*.$$

To get meaningful results, we often assume additionally

$$\frac{\sqrt{-1}}{2} \int_{D^*} \mathrm{e}^{u_i} \mathrm{d}z \wedge \mathrm{d}\bar{z} < +\infty, \quad 1 \leqslant i \leqslant n. \tag{4.1}$$

A result that has not yet undergone peer review is

Theorem 4.1 Suppose $\{u_i\}_{i=1}^n$ is a solution to

$$\begin{cases} \dfrac{\partial^2 u_i}{\partial z \partial \bar{z}} + \displaystyle\sum_{j=1}^{n} a_{ij} \mathrm{e}^{u_j} = 0, \quad 1 \leqslant i \leqslant n, \quad z \in D^*, \\ \dfrac{\sqrt{-1}}{2} \displaystyle\int_{D^*} \mathrm{e}^{u_i} \mathrm{d}z \wedge \mathrm{d}\bar{z} < +\infty, \quad 1 \leqslant i \leqslant n. \end{cases}$$

Then

(1) The curve associated to $\{u_i\}_{i=1}^n$ can be written as

$$z \longmapsto \left[z^{\beta_0} g_0(z), \cdots, z^{\beta_n} g_n(z) \right],$$

where $\beta_0 < \beta_1 < \cdots < \beta_n$ and g_i is holomorphic on the whole D.

(2) There exists a lift F of curve such that the components of F satisfy the $(n+1)$-Fuchs equation around 0 with β_i as local exponents.

(3) $u_k = 2\gamma_k \log|z| + O(1)$, where $\gamma_i = \beta_i - \beta_{i-1} - 1$.

(4)

$$\frac{\partial^2 u_i}{\partial z \partial \bar{z}} + \sum_{j=1}^{n} a_{ij} \mathrm{e}^{u_j} = \pi \gamma_i \delta_0, \quad 1 \leqslant i \leqslant n, \quad z \in D \tag{4.2}$$

in the sense of distribution.

Convesely, we can not derive (4.1) from (4.2) in general. But I cannot provide explicit counterexamples here. In [2], we study

$$\begin{cases} \dfrac{\partial^2 u_i}{\partial z \partial \bar{z}} + \displaystyle\sum_{j=1}^{n} a_{ij} \mathrm{e}^{u_j} = \pi \gamma_i \delta_0, \quad 1 \leqslant i \leqslant n, \quad z \in D. \\ \dfrac{\sqrt{-1}}{2} \displaystyle\int_{D^*} \mathrm{e}^{u_i} \mathrm{d}z \wedge \mathrm{d}\bar{z} < +\infty, \quad 1 \leqslant i \leqslant n. \end{cases}$$

where $\gamma_i > -1$. And the main result of [2] is

Theorem 4.2 The curve associated to $\{u_i\}_{i=1}^n$ can be written as

$$z \longmapsto \left[z^{\beta_0} g_0(z), \cdots, z^{\beta_n} g_n(z) \right],$$

where g_i is holomorphic on the whole D and

$$\begin{cases} \beta_0 = -\alpha_1, \\ \beta_n = \alpha_n + n, \\ \beta_i = \alpha_i - \alpha_{i+1} + i, \quad 1 \leqslant i \leqslant n-1 \\ \alpha_i = \sum_{j=1}^n a^{ij} \gamma_j, \quad 1 \leqslant i \leqslant n \end{cases}$$

Besides, it is easy to check that $\beta_0 < \beta_1 < \cdots < \beta_n$ and $\gamma_i = \beta_i - \beta_{i-1} - 1$.

When we consider equations on punctured compact Riemann surfaces, we encounter a similar dilemma, so we still need the condition of finite energy.

Theorem 4.3 Let X be a compact Riemann surface and $\{p_k\}_{1 \leqslant k \leqslant m} \subset X$. Let $(g_i)_{i=1}^n$ is a solution of $\mathfrak{su}(n+1)$ -Toda equation on $X \backslash \{p_k\}_{1 \leqslant k \leqslant m}$ such that

$$\int_{X \backslash \{p_k\}_{1 \leqslant k \leqslant m}} \omega_i < +\infty, \quad 1 \leqslant i \leqslant n,$$

where ω_i is the volume form of g_i. Let $\mathbf{f}$ be the curve associated to $\{g_i\}_{i=1}^n$, then

(1) For any p_k, there exists $A_k \in \mathrm{PSU(n+1)}$ such that $A_k \circ \mathbf{f}$ can be written as

$$z \longmapsto \left[z^{\beta_{0k}} g_{0k}(z), \cdots, z^{\beta_{nk}} g_{nk}(z) \right]$$

around p_k,where $\beta_{0k} < \beta_{1k} < \cdots < \beta_{nk}$ and g_{ik} is holomorphic around p_k.

(2) There exists a lift F of a component of $A_k \circ \mathbf{f}$ around p_k such that the components of F satisfy the $(n+1)$-Fuchs equation around p_k with β_{ik} as local exponents.

(3) $u_i = 2\gamma_{ik} \log |z| + O(1)$, where z a coordinate around p_k and $\gamma_{ik} = \beta_{ik} - \beta_{i-1,k} - 1$.

(3)
$$\frac{\sqrt{-1}}{2}\left(\partial\bar{\partial}u_i+\sum_{j=1}^{n}a_{ij}\mathrm{e}^{u_j}\mathrm{d}z\wedge\mathrm{d}\bar{z}\right)=\pi\sum_{k=1}^{m}\gamma_{ik}\delta_{p_k},\quad 1\leqslant i\leqslant n$$
in the sense of current.

Proof Only (1) and (2) need to be explained. Around every p_k, we can use theorem 4.1 directly. When consider the global curve $\mathbf{f}$, we can only guarantee it has simplest form around one p_k at one time. So in general $\mathbf{f}$ differ by a $A_k\in \mathrm{PSU(n+1)}$ around p_k compared with the simplest form.

∎

We can also specify the information at the singularities in advance.

Definition 4.1 Let X be a compact Riemann surface and $\left\{p_k\right\}_{1\leqslant k\leqslant m}\subset X$. Let
$$(\gamma_{ik})_{1\leqslant i\leqslant n,1\leqslant k\leqslant m}\in\mathbb{R}^{n\times m},\quad \gamma_{ik}>-1$$
and for any $1\leqslant k\leqslant m$, there exists $1\leqslant i\leqslant n$ such that $\gamma_{ik}\neq 0$. Let
$$D_i=\sum_{k=1}^{m}\gamma_{ik}[p_k],\quad 1\leqslant i\leqslant n.$$
Let $(g_i)_{i=1}^n$ be n conformal metrics on $X\backslash\left\{p_k\right\}_{1\leqslant k\leqslant m}$ such that
$$\int_{X\backslash\left\{p_k\right\}_{1\leqslant k\leqslant m}}\omega_i<+\infty.$$
We say $(g_i)_{i=1}^n$ is a solution to the $\mathfrak{su}(n+1)$-Toda equation on $X\backslash\left\{p_k\right\}_{1\leqslant k\leqslant m}$ with singularity $(D_i)_{i=1}^n$, if
$$\frac{\sqrt{-1}}{2}\left(\partial\bar{\partial}u_i+\sum_{j=1}^{n}a_{ij}\mathrm{e}^{u_j}\mathrm{d}z\wedge\mathrm{d}\bar{z}\right)=\pi\sum_{k=1}^{m}\gamma_{ik}\delta_{p_k},\quad 1\leqslant i\leqslant n. \tag{4.3}$$
in the sense of current.

Theorem 4.4 Let $(g_i)_{i=1}^n$ be a solution to (4.3) and $\mathbf{f}$ associated to $(g_i)_{i=1}^n$, then for any p_k, there exists $A_k\in\mathrm{PSU}(n+1)$ such that $A_k\circ\mathbf{f}$ can be written as
$$z\longmapsto\left[z^{\beta_{0k}}g_{0k}(z),\cdots,z^{\beta_{nk}}g_{nk}(z)\right]$$

around p_k,where g_{ik} is holomorphic around p_k and

$$\begin{cases} \beta_{0k} = -\alpha_{1k}, \\ \beta_{nk} = \alpha_{nk} + n, \\ \beta_{ik} = \alpha_{ik} - \alpha_{i+1,k} + i, & 1 \leqslant i \leqslant n-1 \\ \alpha_{ik} = \sum_{j=1}^{n} a^{ij}\gamma_{jk}, & 1 \leqslant i \leqslant n \end{cases}$$

Besides, it is easy to check that $\beta_{0k} < \beta_{1k} < \cdots < \beta_{nk}$ and $\gamma_{ik} = \beta_{ik} - \beta_{i-1,k} - 1$.

Proof The proof is similar to theorem 4.3 and can be reduced to theorem 4.2. ∎

We must point out that the premise of above two theorems is the existence of solutions, about which we know little. One open problem is to find sufficient or necessary conditions that make the equation (4.3) have a solution. If (4.3) has a solution, another open problem is how to characterize the moduli space of the solutions.

Bibliography

[1] Jürgen Jost and Guofang Wang. Classification of solutions of a toda system in $\mathbb{R}$ 2. *International Mathematics Research Notices*, 2002(6):277–290, 2002.

[2] Jingyu Mu, Yiqian Shi, Tianyang Sun, and Bin Xu. Classifying solutions of su(n+1) toda system around a singular source via fuchsian equations, 2023.

[3] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. John Wiley & Sons, 2014.

[4] Richard W Sharpe. *Differential geometry: Cartan's generalization of Klein's Erlangen program*, volume 166. Springer Science & Business Media, 2000.

[5] Lars V Ahlfors. Complex analysis: an introduction to the theory of analytic functions of one complex variable. *New York, London*, 177, 1953.

Acknowledgements

我要感谢我的父母，感谢他们四年来对我的支持。

我要感谢我的班主任兼导师许斌，感谢他在我拖延时鞭策我，在我迷茫时点拨我，在我焦虑时宽慰我，在我忐忑时支持我。

我要感谢我的师兄弟牟景宇、魏春晖、王陈曦和杨笛龙，毕设的完成离不开他们的帮助。

我要感谢我的同学王子达、鲁子伟、李轩宇、鲍越恒和周可树，四年来与你们的交流使我受益良多。

2023 年 6 月