HÖLDER ESTIMATE OF SOLUTIONS TO AN ELLIPTIC SYSTEM OF TODA TYPE

1. Local Hölder estimate

Let n be a positive integer, $\gamma_i > -1$ for all $1 \leq i \leq n$ and $(a_{ij})_{1\leq i,j\leq n}$ be a real matrix which may *degenerate*. Suppose that for all $1 \leq i \leq n$, u_i are real-valued functions defined on $D = \{z \in \mathbb{C} : |z| < 1\}$ such that they are locally integrable on D, smooth on $D^* = D\backslash\{0\}$ and satisfy for all $1 \leq i \leq n$

$$\begin{aligned}
\Delta u_i + \sum_{j=1}^{n} a_{ij} e^{u_j} &= 4\pi\gamma_i\delta_0 \quad \text{on} \quad D; \\
\frac{\sqrt{-1}}{2}\int_{D^*} e^{u_i}\, \mathrm{d}z \wedge \mathrm{d}\bar{z} &< \infty.
\end{aligned}$$

Remember that $\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}}$ is the Laplacian on the complex plane $\mathbb{C}$ and the area element $\frac{\sqrt{-1}}{2}\mathrm{d}z \wedge \mathrm{d}\bar{z}$ coincides with the Lebesgue measure on $\mathbb{C}$.

Lemma 1.1. *There exist $\alpha \in (0, 1)$ only depending on $\{\gamma_i\}_{1\leq i\leq n}$ such that*

$$\big(u_i - 2\gamma_i \log|z|\big) \in C^{\alpha}_{\mathrm{loc}}(D)$$

for all $1 \leq i \leq n$.

Proof. Using the fact that $\Delta\left(\log|z|\right) = 2\pi\delta_0$ on $\mathbb{C}$ ([2, Section 2.4]), we observe that functions $V_i := u_i - 2\gamma_i \log|z|$, $1 \leq i \leq n$, satisfy

$$\begin{aligned}
\Delta V_i &= -\sum_{j=1}^{n} a_{ij}|z|^{2\gamma_j} e^{V_j} =: f_i \quad \text{on} \quad D; \\
\infty &> \frac{\sqrt{-1}}{2}\int_{D^*} |z|^{2\gamma_i} e^{V_i}\, \mathrm{d}z \wedge \mathrm{d}\bar{z}.
\end{aligned}$$

It suffices to show that there exists $\alpha \in (0, 1)$ only depending on $\{\gamma_i\}_{1\leq i\leq n}$ such that each V_i lies in $C^{\alpha}_{\mathrm{loc}}\big(\{|z| < 1/4\}\big)$. For each $1 \leq i \leq n$, we could write $V_i = V_{i,1} + V_{i,2}$ in $\{|z| \leq 1/2\}$, where $V_{i,1}$ and $V_{i,2}$ satisfy the following two boundary value problems

$$\begin{cases} \Delta\, V_{i,1} = f_i & \text{in } \{|z| < 1/2\} \\ V_{i,1} = 0 & \text{on } \{|z| = 1/2\} \end{cases} \quad \text{and} \quad \begin{cases} \Delta\, V_{i,2} = 0 & \text{in } \{|z| < 1/2\} \\ V_{i,2} = V_i & \text{on } \{|z| = 1/2\} \end{cases}, \text{ respectively.}$$

Since $V_{i,2}$ is harmonic and then smooth in $\{|z| < 1/2\}$, we're done if we could prove that each $V_{i,1}$ lies in $C^{\alpha}_{\mathrm{loc}}\big(\{|z| < 1/4\}\big)$. We divide the proof into the following three steps.

(1) Since $f_i \in L^1(D)$, by using [1, p.1227, Corollary 1.], we obtain that $e^{p|V_{i,1}|}$ lie in $L^1\big(\{|z| < 1/2\}\big)$ for all $p > 1$ and all $1 \leq i \leq n$. Since $\gamma_i > -1$, by the Hölder inequality, there exists $1 < p_0 < 2$ depending on γ_i's such that f_i lie in $L^{p_0}\big(\{|z| < 1/2\}\big)$ for all $1 \leq i \leq n$.

(2) Since all $V_{i,1}$ satisfy the very boundary value problem, by using [2, p.230, Theorem 9.9.], $V_{i,1} \in W^{2,p_0}_{\mathrm{loc}}\big(\{|z| < 1/2\}\big)$.

(3) We're able to finish the proof by using some Sobolev embedding theorems, whose details go as follows. Since we only care about the restriction of $V_{i,1}$ to $\{|z| < 1/4\}$, by using a cut-off function, we may assume $V_{i,1}$ vanishes near the circle $\{|z| = 1/2\}$. Since $\nabla V_{i,1} \in W^{1,p_0}_0\big(\{|z| < 1/2\}\big)$, by using [2, p.155, Theorem 7.10.], we obtain that $\nabla V_{i,1} \in L^{\frac{2p_0}{2-p_0}}\big(\{|z| < 1/2\}\big)$ and $V_{i,1} \in W_0^{1,\frac{2p_0}{2-p_0}}\big(\{|z| < 1/2\}\big)$. Since $\frac{2p_0}{2-p_0} > 2$, by using [2, p.163, Theorem 7.17], we have $V_{i,1} \in C^{\alpha}\big(\{|z| \leq 1/4\}\big)$ for $0 < \alpha := 2 - \frac{2}{p_0} < 1$.

$\square$

Let n be a positive integer, $\gamma_i > -1$ for all $1 \leq i \leq n$ and $(a_{ij})_{1\leq i,j\leq n}$ be a real matrix. Suppose that for all $1 \leq i \leq n$, V_i are real-valued functions defined on $D = \{z \in \mathbb{C} : |z| < 1\}$ such that they are locally integrable on D, smooth on $D^* = D\backslash\{0\}$ and satisfy

$$-4\frac{\partial^2}{\partial z \partial \bar{z}} V_i \quad = \quad |z|^{2\gamma_i} \exp\left(\sum_{j=1}^{n} a_{ij} V_j\right) \quad \text{in the sense of distribution on} \quad D;$$

$$\infty \quad > \quad \frac{\sqrt{-1}}{2} \int_{D^*} |z|^{2\gamma_i} \exp\left(\sum_{j=1}^{n} a_{ij} V_j\right) \mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

Remember that $\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}}$ is the Laplacian and the area element $\frac{\sqrt{-1}}{2}\,\mathrm{d}z \wedge \mathrm{d}\bar{z}$ coincides with the Lebesgue measure. Using the similar argument as in the proof of Lemma 1.1, we obtain that *there exist $\alpha \in (0, 1)$ such that all V_i lies in $C^{\alpha}_{\mathrm{loc}}(D)$.* Hence we provide a minor part of the details for the Brezis-Merle analysis for the $\mathrm{SU}(n+1)$ Toda system in [3, p.188].

References

[1] Haïm Brezis and Frank Merle. Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions. *Comm. Partial Differential Equations*, 16(8-9):1223–1253, 1991.

[2] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order.* Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[3] Chang-Shou Lin, Juncheng Wei, and Dong Ye. Classification and nondegeneracy of $SU(n+1)$ Toda system with singular sources. *Invent. Math.*, 190(1):169–207, 2012.