IMRN International Mathematics Research Notices
2002, No. 6

Classification of Solutions of a Toda System in $\mathbb{R}^2$

Jürgen Jost and Guofang Wang

1 Introduction

Let $N > 0$ be an integer. The 2-dimensional (open) Toda system (Toda lattice) for $SU(N+1)$ is the following system:

$$-\frac{1}{2}\Delta u_i = \sum_{j=1}^{N} a_{ij} e^{u_j} \quad \text{in } \mathbb{R}^2, \tag{1.1}$$

for $i = 1, 2, \ldots, N$, where $K = (a_{ij})_{N\times N}$ is the Cartan matrix for $SU(N+1)$, given by

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}. \tag{1.2}$$

(Here the factor $1/2$ comes from $(1/2)\Delta u = u_{z\bar{z}}$.) System (1.1) is a very natural generalization of the Liouville equation

$$-\Delta u = 2e^{u}, \tag{1.3}$$

which is completely integrable, known from Liouville [26]. Roughly speaking, any solution of (1.3) in a simply connected domain arises from a holomorphic function. System (1.1) is also completely integrable. All solutions of (1.1) in a simply connected

Received 7 May 2001. Revision received 26 June 2001.

domain arise from N holomorphic functions (see [14, 21, 23, 24]). However, it is difficult to determine the precise form of these holomorphic functions, when we require additional (or boundary) conditions for (1.3) or (1.1).

Recently, Chen and Li [6] classified all solutions of (1.3) in $\mathbb{R}^2$ with

$$\int_{\mathbb{R}^2} e^u < \infty. \tag{1.4}$$

Their result is very useful for 2-dimensional problems, especially for the study of the Moser-Trudinger inequality and the mean field equation (see [12, 27]). To obtain their classification, they used an energy inequality of Ding [11] and the method of moving plane to show that any solution has a rotational symmetry. Other proofs of the classification result were given by Chou and Wan [10] using the complete integrability mentioned above and complex analysis, and by Chanillo and Kiessling [5] using an isoperimetric inequality and a global Pohozaev identity. The latter was applied to classify solutions of a class of Liouville type systems with nonnegative coefficients in [5]. See also the work of Chipot et al. [9]. The methods used by Chen, Li, Chanillo, and Kiessling rely on the maximum principle. Hence, it is difficult (or impossible) to apply their methods to study the similar problem for system (1.1). We believe that the method of Chou and Wan can be applied to study (1.1). In fact, we notice that a similar method was used by Bryant [3] in his study of pseudo-metrics. We believe that his method can be adapted to classify (1.1) by using the Nevanlinna theory for holomorphic curves into $\mathbb{C}P^N$ instead of that for holomorphic functions. For the Nevanlinna theory for holomorphic curves into $\mathbb{C}P^N$, see [16].

System (1.1) has a very close relationship with a few geometric objects, holomorphic curves into $\mathbb{C}P^N$, flat $SU(N+1)$ connections, and harmonic sequences (cf. [2, 4, 8, 15, 17, 18, 22]). To classify solutions of (1.1), it is natural to seek the help of differential geometry. Here, with such a help, we classify system (1.1) with

$$\int_{\mathbb{R}^2} e^{u_i} < \infty, \quad i = 1, 2, \ldots, N. \tag{1.5}$$

Theorem 1.1. Any C^2-solution $u = (u_1, u_2, \ldots, u_N)$ of (1.1) and (1.5) has the following form:

$$u_i(z) = \sum_{j=1}^{N} a_{ij} \log \left\|\Lambda_j(f)\right\|^2, \tag{1.6}$$

for some rational curve in $\mathbb{C}P^N$. For the definition of $\Lambda_k(f)$, see Section 3. □

Any rational curve in $\mathbb{C}P^N$ can be transformed to

$$\phi_0(z) = \left[1, z, \dots, \sqrt{\binom{N}{k}} z^k, \dots, z^N\right], \tag{1.7}$$

by a holomorphic isometry, which is an element of $PSL(N+1, \mathbb{C})$. Hence, the space of solutions of (1.1) and (1.5) is equivalent to $PSL(N+1, \mathbb{C})/PSU(N+1)$. The dimension of the solution space is $N^2 + 2N$.

Theorem 1.1 can be restated in a geometric way as follows.

Theorem 1.2. Any totally unramified holomorphic map ϕ from $\mathbb{C}$ to $\mathbb{C}P^N$ satisfying the finite energy condition (3.28) can be compactified to a rational curve. □

Theorem 1.2 is a generalization of the following well-known result: any totally unramified compact curve in $\mathbb{C}P^N$ is rational.

When $N = 1$, Theorem 1.1 is just the classification result of Chen and Li.

The Toda system is of great interest not only in geometry, but also in mathematical physics. One of our motivations to study this system is the non-abelian Chern-Simons Higgs model, in which nontopological solutions are solutions of a perturbed Toda system (see [14, 19, 28, 30, 33]).

2 Analytic aspects of the Toda system

In this section, we analyze the asymptotic behavior of solutions of (1.1), (1.3), (1.4), and (1.5) and obtain a global Rellich-Pohozaev identity. Since some results were presented in our previous work [20], we only give an outline of ideas. Similar methods were used in [5, 6, 7].

Let $I = \{1, 2, \dots, N\}$. First, we have the following lemma.

Lemma 2.1. Let u be a solution of (1.1), (1.3), (1.4), and (1.5). Then

$$u_i(z) = -\gamma_i \log|z| + a_i + O\left(|z|^{-1}\right), \quad \text{for } |z| \text{ near } \infty, \tag{2.1}$$

where $a_i \in \mathbb{R}$ are some constants and γ_i are given by

$$\gamma_i = \frac{1}{\pi} \sum_{j=1}^{N} a_{ij} \int_{\mathbb{R}^2} e^{u_j}. \tag{2.2}$$

□

Proof. First, one shows that

$$\max_{i \in I} \sup_{z \in \mathbb{R}^2} u_i(z) < \infty, \tag{2.3}$$

(see [20]). Set

$$v_i(z) = \frac{1}{\pi}\int_{\mathbb{R}^2} \big(\log|x-y| - \log(|y|+1)\big)\sum_{j=1}^{N} a_{ij}e^{u_j}(y)dy, \quad \text{for } i \in I. \tag{2.4}$$

(Note again that (1.1) has a factor 1/2.) The potential analysis implies

$$-\gamma_i \log|z| - C \leq v_i(z) \leq -\gamma_i \log|z| + C, \tag{2.5}$$

for some constant $C > 0$ (see [7]). Clearly $u_i - v_i$ is a harmonic function. Hence, (2.3) and (2.5) imply that $u_i - v_i = c_i$ for some constant c_i. That is, u has the following representation formula:

$$u_i(z) = \frac{1}{\pi}\int_{\mathbb{R}^2} \big(\log|x-y| - \log(|y|+1)\big)\sum_{j=1}^{N} a_{ij}e^{u_j}(y)dy + c_i. \tag{2.6}$$

The results (2.5), (2.6), and (1.5) imply

$$\gamma_i > 2, \quad i \in I. \tag{2.7}$$

Furthermore, we can show that (see [31])

$$u_i = \gamma_i \log|z| + a_i + O\big(|z|^{-1}\big). \tag{2.8}$$

■

Next, we have a global Rellich-Pohozaev identity for system (1.1). Such an identity was obtained in [5] for a Liouville type system with nonnegative entries. Similar arguments work for our case (see [20]). Here we give another proof that is similar to the spirit of the proof of the main theorems (Theorems 1.1 and 1.2).

Proposition 2.2. Let u be a solution of (1.1) and (1.5). Then

$$\sum_{j,k=1}^{N} a^{jk}\big(4\gamma_k - \gamma_j\gamma_k\big) = 0, \tag{2.9}$$

where the matrix (a^{ij}) is the inverse of the Cartan matrix (a_{ij}). □

Proof. Set

$$f = \sum_{j,k=1}^{N} a^{jk}\Big\{\big(u_k\big)_{zz} - \frac{1}{2}\big(u_j\big)_z \cdot \big(u_k\big)_z\Big\}. \tag{2.10}$$

We can check that f is a holomorphic function as follows:

$$\begin{aligned} f_{\bar z} &= \frac{1}{2}\sum_{j,k=1}^{N} a^{jk}\{(\Delta u_k)_z - (u_j)_z \cdot \Delta u_k\} \\ &= -\sum_{j,k,l=1}^{N} a^{jk}\{a_{kl}e^{u_l}(u_l)_z - a_{kl}(u_j)_z e^{u_l}\} \\ &= -\sum_{j=1}^{N}\left(e^{u_j}(u_j)_z - e^{u_j}(u_j)_z\right) \\ &= 0. \end{aligned} \tag{2.11}$$

In the first equality we have used the symmetry of the matrix (a^{ij}). Using Lemma 2.1, we have the following expansion of f near infinity:

$$\frac{1}{8}\frac{1}{z^2}\sum_{j,k-1}^{N} a^{jk}(4\gamma_k - \gamma_j\gamma_k) + \frac{c_{-3}}{z^3} + \cdots . \tag{2.12}$$

Hence, f is a constant (zero, in fact) and

$$\sum_{j,k=1}^{N} a^{jk}(4\gamma_k - \gamma_j\gamma_k) = 0. \tag{2.13}$$

■

3 Geometric aspects of the Toda system

In this section, we recall some relations between the Toda system and various geometric objects, flat connections, holomorphic curves into $\mathbb{C}P^N$, and harmonic sequences. Furthermore, we relate the mild singularities of solutions of the Toda system with the holonomy of the corresponding flat connections.

3.1 From solutions of Toda systems to flat connections

Let Ω be a simply connected domain and let $u = (u_1, u_2, \dots, u_N)$ be a solution of (1.1) on Ω. Define $w_0, w_1, w_2, \dots, w_N$ by the following relations:

$$u_i = 2w_i - 2w_0, \quad \text{for } i \in I, \ \sum_{i=0}^{N} w_i = 0. \tag{3.1}$$

It is easy to check that $w_0, w_1, \ldots, w_N$ satisfy

$$\begin{aligned} -\Delta w_0 &= 2(w_0)_{z\bar{z}} = e^{w_1 - w_0}, \\ -\Delta w_1 &= 2(w_1)_{z\bar{z}} = -e^{w_1 - w_0} + e^{w_2 - w_1}, \\ &\vdots \\ -\Delta w_N &= 2(w_N)_{z\bar{z}} = -e^{w_N - w_{N-1}}. \end{aligned} \tag{3.2}$$

It is well known that (3.2) is equivalent to an integrability condition

$$\mathcal{U}_{\bar{z}} - \mathcal{V}_z = [\mathcal{U}, \mathcal{V}] \tag{3.3}$$

of the following two equations:

$$\phi^{-1} \cdot \phi_z = \mathcal{U}, \qquad \phi^{-1} \cdot \phi_{\bar{z}} = \mathcal{V}, \tag{3.4}$$

where

$$\begin{aligned} \mathcal{U} &= \begin{pmatrix} (w_0)_z & & & \\ & (w_1)_z & & \\ & & \ddots & \\ & & & (w_N)_z \end{pmatrix} + \begin{pmatrix} 0 & e^{w_1 - w_0} & & \\ & 0 & & \\ & & \ddots & e^{w_N - w_{N-1}} \\ & & & 0 \end{pmatrix}, \\ \mathcal{V} &= -\begin{pmatrix} (w_0)_{\bar{z}} & & & \\ & (w_1)_{\bar{z}} & & \\ & & \ddots & \\ & & & (w_N)_{\bar{z}} \end{pmatrix} - \begin{pmatrix} 0 & & & \\ e^{w_1 - w_0} & 0 & & \\ & & \ddots & \\ & & e^{w_N - w_{N-1}} & 0 \end{pmatrix}. \end{aligned} \tag{3.5}$$

Hence, from a solution of (1.1) (or equivalently (3.2)) we first get a one-form $\alpha = \mathcal{U}dz + \mathcal{V}d\bar{z}$. Then, with the help of the Frobenius theorem, we obtain a map $\phi : \Omega \to SU(N+1)$ such that

$$\alpha = \phi^{-1} \cdot d\phi. \tag{3.6}$$

It is clear that α (or $d + \alpha$) is a flat $SU(N+1)$ connection on the trivial bundle $\Omega \times \mathbb{C}^{N+1} \to \Omega$, that is, α satisfies the Maurer-Cartan equation

$$d\alpha + \frac{1}{2}[\alpha, \alpha] = 0, \tag{3.7}$$

which is equivalent to the integrability condition (3.3), hence (3.2).

Lemma 3.1. The map ϕ is determined up to an element of $SU(N+1)$. That is, any two $\phi_1, \phi_2 : \Omega \to SU(N+1)$ with $\phi_1^{-1} d\phi_1 = \phi_2^{-1} d\phi_2 = \alpha$ satisfy

$$\phi_1 = g \cdot \phi_2, \tag{3.8}$$

for some element $g \in SU(N+1)$. □

We call ϕ and α a *Toda map* and *Toda form*, respectively.

3.2 Holonomy of a singular connection

Now consider an $SU(N+1)$ connection α on the punctured disk D^*. We can define its holonomy as in [29]. When $\Omega = D^*$ is not simply connected, we cannot apply the Frobenius theorem directly and we have to consider the *holonomy*. Let (r, θ) be the polar coordinates. Write α as $\alpha = \alpha_r dr + \alpha_\theta d\theta$, where α_r and α_θ are $SU(N+1)$-valued. For any given $r \in (0, 1)$, the initial value problem

$$\frac{d\phi_r}{d\theta} + \alpha_\theta \phi_r = 0, \quad \phi_r(0) = \mathrm{Id}, \tag{3.9}$$

has a unique solution $\phi_r(\theta) \in SU(N+1)$. Here Id is the identity matrix.

Lemma 3.2. If α is a flat connection on D^*, then $\phi_r(2\pi)$ is independent of r. □

Let h_α denote $\phi_r(2\pi)$, where h_α is called the *holonomy* of α.

Remark 3.3. Here, we use a slightly different definition of holonomy. The usual holonomy is defined by the conjugacy class of h_α, which is invariant under gauge transformations.

Proposition 3.4. Let $u = (u_1, u_2, \ldots, u_N)$ be a solution of (2.3) with

$$u_i(z) = -\mu_i \log|z| + O(1) \quad \text{near } 0. \tag{3.10}$$

If $\mu_i < 2$ for $i \in I$, then the corresponding flat connection α has the holonomy

$$h_\alpha = \begin{pmatrix} e^{2\pi i \beta_0} & & & \\ & e^{2\pi i \beta_1} & & \\ & & \ddots & \\ & & & e^{2\pi i \beta_N} \end{pmatrix}, \tag{3.11}$$

where $\beta_0, \beta_1, \ldots, \beta_N$ are determined by

$$\beta_i - \beta_0 = \frac{1}{2}\mu_i, \quad i \in I, \ \sum_{j=0}^{N} \beta_j = 0. \tag{3.12}$$

□

Proof. Define w_i by (3.1). From the assumption (3.10), we have

$$w_i = -\beta_i \log|z| + O(1) \quad \text{near } 0. \tag{3.13}$$

A direct computation shows that

$$\mathcal{U} = \frac{1}{2z}\begin{pmatrix} -\beta_0 & & & \\ & -\beta_1 & & \\ & & \ddots & \\ & & & -\beta_N \end{pmatrix} + o\left(\frac{1}{|z|}\right), \tag{3.14}$$

where $o(1/|z|)$ means that a matrix (b_{ij}) with entries satisfying $|z|b_{ij} \to 0$ as $|z| \to 0$. Here, we have used the condition that $\mu_i < 2$ for any $i \in I$. Similarly,

$$\mathcal{V} = \frac{1}{2\bar{z}}\begin{pmatrix} \beta_0 & & & \\ & \beta_1 & & \\ & & \ddots & \\ & & & \beta_N \end{pmatrix} + o\left(\frac{1}{|z|}\right). \tag{3.15}$$

Hence,

$$\alpha_\theta = \sqrt{-1}\begin{pmatrix} \beta_0 & & & \\ & \beta_1 & & \\ & & \ddots & \\ & & & \beta_N \end{pmatrix} + o(1). \tag{3.16}$$

Now it is easy to compute the holonomy of α. ■

3.3 From solutions of (1.1) to holomorphic curves

When we have a Toda map $\phi : \Omega \to SU(N+1)$ from a solution of the Toda system, we can get a harmonic sequence as follows. First, define $N+1$ $\mathbb{C}^{N+1}$-valued functions

$\widehat{f}_0, \widehat{f}_1, \ldots, \widehat{f}_N$ by

$$\left(\widehat{f}_0, \widehat{f}_1, \ldots, \widehat{f}_N\right) = \phi \cdot \begin{pmatrix} e^{w_0} & & & \\ & e^{w_1} & & \\ & & \ddots & \\ & & & e^{w_N} \end{pmatrix}. \tag{3.17}$$

Let f_i denote the map into $\mathbb{C}P^N$ obtained from $\widehat{f}_i$. It is easy to check that f_i is a harmonic map and satisfies

$$\left(\widehat{f}_k\right)_z = \widehat{f} + a_k \widehat{f}_k, \qquad \left(\widehat{f}_k\right)_{\bar{z}} = b_k \widehat{f}_{k-1}, \tag{3.18}$$

where

$$a_k = \left(\log|\widehat{f}_k|^2\right)_z = \left(e^{2w_k}\right)_z, \qquad b_{k-1} = -\frac{|\widehat{f}_k|^2}{|\widehat{f}_{k-1}|^2} = -w^{2(w_k - w_{k-1})}. \tag{3.19}$$

Here we assume that $\widehat{f}_{-1} = \widehat{f}_{N+2} = 0$. Hence, f_0 is a holomorphic map and f_{N+1} is an anti-holomorphic map into $\mathbb{C}P^N$. In fact, (3.18) is the Frenet frame of the holomorphic map f_0, see [18] or next subsection. Furthermore, f_0 is unramified in Ω. For the definition of the ramification index, see [18] or next subsection.

3.4 From a curve to a solution of the Toda system

From a nondegenerate (i.e., not contained in a proper projective subspace of $\mathbb{C}P^N$) holomorphic curve f_0 into $\mathbb{C}P^N$, one can get a family of associated curves into various Grassmannians as follows. Lift f_0 locally to $\mathbb{C}^{N+1}$ and denote the lift by $\nu = (\nu_0, \nu_1, \ldots, \nu_N)$. Hence, $f_0 = [\nu_0, \nu_1, \ldots, \nu_N]$. The k*th associated curve* of f_0 is defined by

$$f_k : \Omega \longrightarrow G(k+1, n+1) \subset \mathbb{C}P^{N_k}, \qquad f_k(z) = \left[\Lambda_k\right], \tag{3.20}$$

where

$$\Lambda_k = \nu(z) \wedge \nu'(z) \wedge \cdots \wedge \nu^{(k)}(z). \tag{3.21}$$

(See [18].) Here $N_k = \binom{N+1}{K+1}$.

Let ω_k be the Fubini-Study metric on $\mathbb{C}P^{N_k}$. The well-known (infinitesimal) Plücker formula is

$$f_k^*(\omega_k) = \frac{\sqrt{-1}}{2} \frac{\|\Lambda_{k-1}\|^2 \cdot \|\Lambda_{k+1}\|^2}{\|\Lambda_k\|^4} dz \wedge \bar{z}, \tag{3.22}$$

which implies

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log \|\Lambda_k\|^2 = \frac{\|\Lambda_{k-1}\|^2 \cdot \|\Lambda_{k+1}\|^2}{\|\Lambda_k\|^4}, \quad \text{for } k = 1, \ldots, N, \tag{3.23}$$

where $\|\Lambda_0\|^2 = 1$ and $\|\Lambda_N\| = \det(f, f', \ldots, f^{(N)})$. By choosing the normalization $\|\Lambda_N\| = 1$ (we can do this when we lift f), we can identify (3.23) with the Toda system (1.1) as follows. By setting

$$\nu_k = \log \|\Lambda_k\|^2, \tag{3.24}$$

system (3.23) becomes

$$-\frac{1}{2}\Delta \nu_i = \exp\left\{\sum_{j=1}^{N} a_{ij}\nu_j\right\}. \tag{3.25}$$

Clearly, (3.25) is equivalent to (1.1) by setting

$$u_i = \sum_{j=1}^{N} a_{ij}\nu_j. \tag{3.26}$$

For any curve $f : \Omega \to \mathbb{C}P^n$, the *ramification index* $\beta(z_0)$ at $z_0 \in \Omega$ is defined by the unique real number such that

$$f^*\omega = \frac{\sqrt{-1}}{2}|z - z_0|^{2\beta(z_0)} \cdot h(z) \cdot dz \wedge d\bar{z} \tag{3.27}$$

with h C^∞ and nonzero at z_0, where ω is the Kähler form of the Fubini-Study metric on $\mathbb{C}P^n$. For other definitions, (see [18]). The curve f is *unramified* if for any $z_0 \in \Omega$ the ramification index $\beta(z_0)$ vanishes. Hence, a solution of the Toda system (1.1) corresponds to an unramified holomorphic curve.

Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a holomorphic curve and f_k its kth associated curves. The finite energy condition is defined by

$$\int_{\mathbb{R}^2} f^*(\omega_k) < \infty, \quad \text{for any } k \in I. \tag{3.28}$$

Inequality (3.28) means that the area of the kth associated curve is bounded.

4 Proof of the main theorems

Now we begin to prove our main theorems.

Proof of Theorem 1.1. Let

$$v_i(z) = u_i\left(\frac{\bar{z}}{|z|^2}\right) - 4\log|z|, \quad i \in I. \tag{4.1}$$

On $\mathbb{R}^2 \setminus \{0\}$, $v = (v_1, v_2, \dots, v_N)$ satisfies (1.1). Applying Lemma 2.1, we have

$$v_i(z) = (\gamma_i - 4)\log|z| + O(1) \quad \text{near } 0. \tag{4.2}$$

Hence, using (2.7), Proposition 3.4 implies that the holonomy of the corresponding Toda form of v is

$$h_\alpha = \begin{pmatrix} e^{2\pi i\beta_0} & & & \\ & e^{2\pi i\beta_1} & & \\ & & \ddots & \\ & & & e^{2\pi i\beta_N} \end{pmatrix}, \tag{4.3}$$

where $\beta_0, \beta_1, \dots, \beta_N$ are determined by

$$\beta_i - \beta_0 = \frac{1}{2}(\gamma_i - 4), \qquad \sum_{j=0}^{N} \beta_j = 0. \tag{4.4}$$

Now we know that the holonomy is trivial, that is, h_α is the identity matrix, which clearly implies

$$\beta_i = 1 \mod \mathbb{Z}, \quad \text{for } i = 0, 1, \dots, N. \tag{4.5}$$

Hence, we have

$$\gamma_i = 2 \mod \mathbb{Z}, \quad \text{for any } i \in I, \tag{4.6}$$

which, together with (2.7), implies that $\gamma_i \geq 4$ for any $i \in I$. Thus, $4\gamma_k - \gamma_j\gamma_k \leq 0$ for any $j, k \in I$. On the other hand, one can check that the matrix (a^{ij}) admits only positive entries. In fact, a direct computation shows that

$$a^{ij} = \frac{i(N+2-j)}{N+2}, \quad \text{for } i, j \leq \left\{\frac{N+2}{2}\right\}, \tag{4.7}$$

where $\{b\}$ is the least integer greater than or equal to b. Other entries are determined by an obvious symmetry. Altogether, we obtain

$$\sum_{j,k=1}^{N} a^{jk}(4\gamma_k - \gamma_j\gamma_k) \leq 0, \tag{4.8}$$

and the equality holds if and only if $\gamma_i = 4$ for any $i \in I$. Applying the global Rellich-Pohozaev identity (2.9), we have

$$\gamma_i = 4, \quad \text{for any } i \in I. \tag{4.9}$$

Hence, v_i is bounded near 0. The elliptic theory implies that v_i is smooth. From the discussions presented in Section 3, it follows that the corresponding holomorphic curve f can be viewed as an unramified map from $\mathbb{S}^2$ to $\mathbb{C}P^N$, hence this curve is a rational curve, namely

$$f = \left[1, z, \dots, \sqrt{\binom{N}{k}} z^k, \dots, z^N\right], \tag{4.10}$$

up to a holomorphic isometry, an element in $PSL(N+1, \mathbb{C})$. Now we can investigate any solution of (1.1) and (1.5) as in Section 3.4. From a holomorphic curve f, we get the kth associated curves f_k and Λ_k. The solution of (1.1), $u = (u_1, u_2, \dots, u_N)$, is given by

$$u_i = \sum_{j=1}^{N} a_{ij} \log \|\Lambda_j\|^2. \tag{4.11}$$

This completes the proof of Theorem 1.1. ■

Proof of Theorem 1.2. From such a holomorphic curve $f : \mathbb{C} \to \mathbb{C}P^N$, we get a solution u of (1.1). It is clear that condition (3.28) implies that u satisfies (1.5). As in the proof of Theorem 1.1, f can be extended to a curve from $\mathbb{S}^2$ to $\mathbb{C}P^N$, which is totally unramified. Hence, it is a rational curve. ■

Corollary 4.1. The space of solutions of (1.1) and (1.5) is $PSL(N+1, \mathbb{C})/PSU(N+1)$. □

Proof. The proof follows from Theorem 1.1 and Lemma 3.1. ■

Corollary 4.2. Any solution $u = (u_1, u_2, \dots, u_N)$ of (1.1) and (1.5) satisfies

$$\frac{1}{\pi} \sum_{j=1}^{N} a_{ij} \int_{\mathbb{R}^2} e^{u_j} = 4. \tag{4.12}$$

In particular, if $N = 2$, then

$$2\int_{\mathbb{R}^2} e^{u_1} = 2\int_{\mathbb{R}^2} e^{u_2} = 8\pi. \tag{4.13}$$

□

References

[1] J. Bolton, G. R. Jensen, M. Rigoli, and L. M. Woodward, *On conformal minimal immersions of* S^2 *into* $\mathbf{C}P^n$, Math. Ann. **279** (1988), no. 4, 599–620.

[2] J. Bolton and L. M. Woodward, *Some geometrical aspects of the* 2*-dimensional Toda equations*, Geometry, Topology and Physics (Campinas, 1996), de Gruyter, Berlin, 1997, pp. 69–81.

[3] R. L. Bryant, *Surfaces of mean curvature one in hyperbolic space*, Astérisque (1987), no. 154-155, 12–12, 321–347.

[4] E. Calabi, *Isometric imbedding of complex manifolds*, Ann. of Math. (2) **58** (1953), 1–23.

[5] S. Chanillo and M. K.-H. Kiessling, *Conformally invariant systems of nonlinear PDE of Liouville type*, Geom. Funct. Anal. **5** (1995), no. 6, 924–947.

[6] W. X. Chen and C. Li, *Classification of solutions of some nonlinear elliptic equations*, Duke Math. J. **63** (1991), no. 3, 615–622.

[7] ———, *What kinds of singular surfaces can admit constant curvature?*, Duke Math. J. **78** (1995), no. 2, 437–451.

[8] S. S. Chern and J. G. Wolfson, *Harmonic maps of the two-sphere into a complex Grassmann manifold. II*, Ann. of Math. (2) **125** (1987), no. 2, 301–335.

[9] M. Chipot, I. Shafrir, and G. Wolansky, *On the solutions of Liouville systems*, J. Differential Equations **140** (1997), no. 1, 59–105.

[10] K. S. Chou and T. Y.-H. Wan, *Asymptotic radial symmetry for solutions of* $\Delta u + e^u = 0$ *in a punctured disc*, Pacific J. Math. **163** (1994), no. 2, 269–276.

[11] W. Ding, *On the best constant in a Sobolev inequality on compact* 2*-manifolds and application*, unpublished manuscript, 1984.

[12] W. Ding, J. Jost, J. Li, and G. Wang, *The differential equation* $\Delta u = 8\pi - 8\pi h e^u$ *on a compact Riemann surface*, Asian J. Math. **1** (1997), no. 2, 230–248.

[13] A. Doliwa, *Holomorphic curves and Toda systems*, Lett. Math. Phys. **39** (1997), no. 1, 21–32.

[14] G. Dunne, *Self-Dual Chern-Simons Theories*, Lecture Notes in Physics. New Series m: Monographs, vol. m36, Springer-Verlag, Berlin, 1995.

[15] J. Eells and J. C. Wood, *Harmonic maps from surfaces to complex projective spaces*, Adv. in Math. **49** (1983), no. 3, 217–263.

[16] H. Fujimoto, *Value Distribution Theory of the Gauss Map of Minimal Surfaces in* $\mathbf{R}^m$, Aspects of Mathematics, vol. E29, Friedr. Vieweg & Sohn, Braunschweig, 1993.

[17] J.-L. Gervais and Y. Matsuo, *Classical* A_n*-W-geometry*, Comm. Math. Phys. **152** (1993), no. 2, 317–368.

[18] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, New York, 1978.

[19] M. A. Guest, *Harmonic Maps, Loop Groups, and Integrable Systems*, London Mathematical Society Student Texts, vol. 38, Cambridge University Press, Cambridge, 1997.

[20] J. Jost and G. Wang, *Analytic aspects of the Toda system: I. A Moser-Trudinger inequality*, Comm. Pure Appl. Math. **54** (2001), no. 11, 1289–1319.

[21] B. Kostant, *The solution to a generalized Toda lattice and representation theory*, Adv. in Math. **34** (1979), no. 3, 195–338.

[22] H. B. Lawson Jr., *The Riemannian geometry of holomorphic curves*, Bol. Soc. Brasil. Mat. **2** (1971), no. 1, 45–62.

[23] A. N. Leznov and M. V. Saveliev, *Representation of zero curvature for the system of nonlinear partial differential equations* $x_{\alpha,z\bar{z}} = \exp(kx)_\alpha$ *and its integrability*, Lett. Math. Phys. **3** (1979), no. 6, 489–494.

[24] ———, *Representation theory and integration of nonlinear spherically symmetric equations to gauge theories*, Comm. Math. Phys. **74** (1980), no. 2, 111–118.

[25] Y. Li and M. Zhu, *Uniqueness theorems through the method of moving spheres*, Duke Math. J. **80** (1995), no. 2, 383–417.

[26] J. Liouville, *Sur l'équation aux différences partielles* $\frac{d^2}{du\,dv}\log\lambda \pm \frac{\lambda}{2a^2} = 0$, Math. Pures Appl. **18** (1853), 71–71.

[27] M. Nolasco and G. Tarantello, *On a sharp Sobolev-type inequality on two-dimensional compact manifolds*, Arch. Ration. Mech. Anal. **145** (1998), no. 2, 161–195.

[28] ———, *Vortex condensates for the* $\mathrm{SU}(3)$ *Chern-Simons theory*, Comm. Math. Phys. **213** (2000), no. 3, 599–639.

[29] L. M. Sibner and R. J. Sibner, *Classification of singular Sobolev connections by their holonomy*, Comm. Math. Phys. **144** (1992), no. 2, 337–350.

[30] G. Wang and L. Zhang, *Non-topological solutions of the relativistic* $\mathrm{SU}(3)$ *Chern-Simons Higgs model*, Comm. Math. Phys. **202** (1999), no. 3, 501–515.

[31] G. Wang and X. Zhu, *Extremal Hermitian metrics on Riemann surfaces with singularities*, Duke Math. J. **104** (2000), no. 2, 181–210.

[32] L. M. Woodward, private communication.

[33] Y. Yang, *The relativistic non-abelian Chern-Simons equations*, Comm. Math. Phys. **186** (1997), no. 1, 199–218.

Jürgen Jost: Max-Planck-Institute for Mathematics in Sciences, Inselstraße 22-26 D-04103 Leipzig, Germany
E-mail address: juergen.jost@mis.mpg.de

Guofang Wang: Max-Planck-Institute for Mathematics in Sciences, Inselstraße 22-26 D-04103 Leipzig, Germany
E-mail address: gwang@mis.mpg.de