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The associated curves $S_1 = S, S_2, \ldots, S_n$ are connected with a holomorphic curve $S \subset \mathbf{C}P^n$ in Grassmannians $\mathbf{C}G(n+1, k)$ formed by tangent lines, osculating planes, and so on. In particular, the curve $S_n \subset \mathbf{C}P^{n*}$ is projectively dual to S_1. The Plücker embeddings of the Grassmannians and the Fubini-Study metrics on projective spaces induce Hermitian metrics h_k on the associated curves and the corresponding curvature forms w_k. In both cases they are (1,1)-forms on the original curve S. The Plücker formulae [1] express the curvature vector in terms of the metric vector:

$$\omega_k = -h_{k-1} + 2h_k - h_{k+1}.$$

Below we give a conjectural explanation of the fact that in the Plücker formula $\omega = Ch$ there occurs a Cartan matrix C of type A_n.

Let G be a compact simple simply-connected Lie group, let $\mathfrak{G}$ be its complexified Lie algebra, a simple complex Lie algebra of rank n, and let F be its flag manifold. The fundamental representations $V_1, \ldots, V_n$ of G are realized in spaces of holomorphic sections of suitable linear bundles over F. The *standard projections* $F \to \mathrm{Proj}(V_k^*)$ associate with a point of F the hyperplane of those sections that vanish at this point. The G-invariant Fubini-Study metrics are uniquely defined on the projective spaces $\mathrm{Proj}(V_k^*)$. This follows from the continuity of the fundamental representations.

We associate with the holomorphic curve $S \subset F$ the vector h of Hermitian metrics on it by inducing the Fubini-Study metrics via the standard projections. Let ω be the vector of the corresponding curvatures.

We define on F the *canonical n-dimensional distribution*. A point of F is a Borel subalgebra $b \subset \mathfrak{G}$. Its nilpotents form a parabolic subalgebra $p \subset b$. The tangent space T_bF is canonically isomorphic to $\mathfrak{G}/b$. We choose in it the n-dimensional subspace N/b as follows:

$$N = \{x \in \mathfrak{G} \mid [x, p] \subset b\}.$$

Conjecture. If S is an integral curve of the canonical distribution on the flag manifold F, then $\omega = Ch$, where C is the Cartan matrix of the Lie algebra $\mathfrak{G}$.

Remark. The rows and columns of the Cartan matrix as well as the fundamental representations are numbered by rhe vertices of the Dynkin diagram.

Examples. $G = SU_{n+1}$. In this way the Plücker formulae are obtained. The canonical representation describes the infinitesimal motions of the flag

$$w^1 \subset \ldots \subset w^n \subset \mathbf{C}^{n+1},$$

under which the velocity of points in w^k lies in w^{k+1}. The moving flag of the curve in $\mathbf{C}P^n$ is an integral curve of this distribution, the fundamental representations are exterior powers $\Lambda^n\mathbf{C}^{n+1}$, so that the standard projections define the Plücker embeddings of the associated curves.

$G = \mathbf{Sp}_{2n}$. In this way one obtains the Plücker formulae for a self-dual curve in $\mathbf{Proj}(\mathbf{C}^{2n})$, that is, a curve that coincides with its projective dual. Identification of a projective space with its dual enables one to pass to the skew-orthogonal completion in the complex symplectic space $\mathbf{C}^{2n}$. The moving flag of the self-dual curve is invariant with respect to this operation and is defined by an initial segment of it consisting of isotropic subspaces. Therefore

$$S_{n-k} \simeq S_{n\circledast k}$$

and it is not difficult to verify that the curvatures of $S_1, \ldots, S_n$ are expressed in terms of their metrics by means of the Cartan matrix C of the series.

On the other hand, the isotropic flags form a symplectic group manifold F. Its fundamental representations are subspaces in

$$\Lambda^n\mathbf{C}^{2n}, \ldots, \Lambda^{2n-1}\mathbf{C}^{2n},$$

that are annihilated by the operator of exterior multiplication by the symplectic structure. The standard projections associate with the isotropic flag decomposable forms with isotropic kernel.

Therefore for an integral curve S the standard projections are the same as the Plücker embeddings of the curves $S_1, \ldots, S_n$ associated with the self-dual curve S_1. Thus our conjecture holds for symplectic groups.

Reference

[1] P. Griffiths and J. Harris, Principles of algebraic geometry, Wiley, New York 1978. MR **80b**:14001.
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