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Degenerations of Lie algebras and geometry of Lie groups

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Abstract

Each point of the variety of real Lie algebras is naturally identified with a left invariant Riemannian metric on a Lie group. We study the interplay between invariant-theoretic and Riemannian aspects of this variety. In particular, using the special critical point behavior of certain natural functional on the variety, we determine all the Lie groups which can be endowed with only one left invariant metric up to isometry and scaling, proving first that they correspond to Lie algebras whose only degeneration is to the abelian one. We also find all the Lie algebras which degenerate to the Lie algebra of the hyperbolic space, and all the possible degenerations for 3-dimensional real Lie algebras, by using well known descriptions of left invariant metrics satisfying some pinching curvature conditions. Finally, as another interaction, the closed $SL(n)$ -orbits on the variety are classified, and explicit curves of Einstein solvmanifolds are provided by using curves of closed orbits of the representation $\Lambda^2 SL(m) \otimes SL(n)$.

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1. Introduction

Let us consider as parameter space for the real Lie algebras of a given dimension n , the set \mathcal{L} of all Lie brackets on a fixed n -dimensional real vector space \mathfrak{g} . Since the Jacobi identity is determined by polynomial conditions, we have that \mathcal{L} is actually an algebraic subset of $\Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g} = \mathbb{R}^{(n^3-n^2)/2}$. By endowing \mathfrak{g} with a fixed inner product $\langle \cdot, \cdot \rangle$, we identify each element $\mu \in \mathcal{L}$ with the Riemannian manifold $(G_\mu, \langle \cdot, \cdot \rangle)$, that is, the corresponding simply connected Lie group G_μ endowed with the left invariant Riemannian metric determined by $\langle \cdot, \cdot \rangle$. Thus the ‘change of basis’ action of $GL(n)$ on \mathcal{L} given

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by $\varphi.\mu = \varphi\mu(\varphi^{-1}\cdot, \varphi^{-1}\cdot)$ has the following interpretation: each $\varphi \in GL(n)$ defines, by exponentiation, an isometry

$$(G_\mu, \langle \varphi\cdot, \varphi\cdot \rangle) \rightarrow (G_{\varphi.\mu}, \langle \cdot, \cdot \rangle).$$

In this way, we can say that the orbit $\mathcal{O}(\mu) = GL(n).\mu$ covers all the left invariant Riemannian metrics on G_μ . Note that the orbit by the orthogonal group $O(n).\mu$ consists of elements all isometric to μ .

This interpretation of the points of the algebraic variety \mathcal{L} , will allow us to translate invariant-theoretic properties of the orbit $\mathcal{O}(\mu)$ into Riemannian properties of left invariant metrics on G_μ , and vice-versa. The technique of fixing an inner product and varying Lie brackets isomorphic to a given one, has been used for instance in the study of curvature properties of Lie groups (see [2,19,27]), in the description of the moduli space of Einstein solvmanifolds (see [14,18]) and in spectral geometry (see [32] and the references therein).

Consider the functional on \mathcal{L} given by $F(\mu) = \text{tr} R_\mu^2$, where R_μ is a symmetric transformation of \mathfrak{g} defined via the Ricci curvature operator of μ (see (5)). The Riemannian functional F can be formally defined on the whole space $\Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$ as a homogeneous polynomial of degree 4. The gradient of F at $\mu \in \mathcal{L}$ is always tangent to the $GL(n)$ -orbit $\mathcal{O}(\mu)$. This remarkable property implies that the limits of the gradient flows of F define degenerations of the starting point, a concept introduced and studied by theoretical physicists (see [20,26,31]) and also by mathematicians (see [4,16,33,34]). Indeed, we say that μ *degenerates to* λ (denoted by $\mu \rightarrow \lambda$) if $\lambda \in \overline{\mathcal{O}(\mu)}$, the closure of $\mathcal{O}(\mu)$ with respect to the usual (metric) topology of $\Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$. As expected, the critical points of the restriction of F to the sphere S , have very nice Riemannian characterizations related with Ricci solitons and Einstein solvmanifolds, at least in the nilpotent case (see [24]).

In Section 4, we characterize the critical points of $F|_S$ in terms of R_μ , and prove that if $\mu \in \mathcal{L} \cap S$ is a critical point of $F|_S$ then any other critical point in $\mathcal{O}(\mu) \cap S$ necessarily lies in $O(n).\mu$, which is a kind of uniqueness up to isometry. After that, in Section 5, we use these results to prove:

G_μ has only one left invariant Riemannian metric up to isometry and scaling if and only if the only possible degeneration of μ is $\mu \rightarrow 0$.

We then prove that there are only two Lie algebras satisfying the last condition: the direct sum of the 3-dimensional Heisenberg Lie algebra with an abelian factor and the solvable Lie algebra $\mathfrak{s} = \mathbb{R}H \oplus \mathbb{R}^{n-1}$, where $[H, X] = X$ for any $X \in \mathbb{R}^{n-1}$ and \mathbb{R}^{n-1} is abelian. We then obtain that these Lie groups are the only ones admitting a unique left invariant metric (up to isometry and scaling).

Since the sectional curvature K_μ (and all other Riemannian curvatures) of $\mu \in \mathcal{L}$ depend continuously on μ , we realize that if $\mu \rightarrow \lambda$ and there is a $\lambda_0 \in \mathcal{O}(\lambda)$ satisfying certain strict curvature inequality, then the orbit $\mathcal{O}(\mu)$ must also contain an element satisfying the same inequality. In other words:

If $\mu \rightarrow \lambda$ and G_λ admits a left invariant Riemannian metric satisfying a pinched curvature condition, then so G_μ .

We then can use some well known algebraic characterizations of Lie groups having metrics satisfying certain pinching conditions (see [8,19]), to decide that certain Lie algebra can never degenerate to some other one. We apply this method in Section 6 to classify all the Lie algebras which degenerate to the

solvable Lie algebra \mathfrak{s} mentioned above, and to determine all the possible degenerations between 3-dimensional real Lie algebras.

In Section 7, we apply some results on the minimal vectors of representations of real reductive Lie groups given in [22,29], to the action of a reductive subgroup $G \subset GL(n)$ on $\mathcal{L} \subset \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$. We prove that the orbit $G.\mu$ is closed if and only if $p(R_\mu) = 0$, where $p: \text{sym}(\mathfrak{g}) \rightarrow \mathfrak{p}$ is the orthogonal projection and $L(G) = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of the Lie algebra $L(G)$ of G with $\mathfrak{k} \subset \mathfrak{so}(\mathfrak{g})$ and $\mathfrak{p} \subset \text{sym}(\mathfrak{g})$.

A $GL(n)$ -orbit on \mathcal{L} can never be closed, unless it is $\{0\}$. Indeed, if $\varphi_t = t^{-1}I$ then $\lim_{t \rightarrow 0} \varphi_t.\mu = 0$ and so $0 \in \overline{\mathcal{O}(\mu)}$ for every $\mu \in \mathcal{L}$. A natural question is then what happens if we do not allow multiplication by a scalar, considering for instance $SL(n)$ -orbits instead of $GL(n)$ -orbits. We apply the results obtained in Section 7 to prove in Section 8 the following interplay:

$SL(n).\mu$ is closed if and only if G_μ admits a left invariant Riemannian metric (\cdot, \cdot) such that the curvature tensor $R_{(\cdot, \cdot)}$ is a multiple of the identity.

Using this characterization, we prove that an orbit $SL(n).\mu$ is closed if and only if μ is semi-simple. Moreover, $0 \in \overline{SL(n).\mu}$ for every non-semi-simple Lie bracket $\mu \in \mathcal{L}$.

Finally, in Section 9, as another application of the results given in Section 7, we obtain the following interplay given in [18]:

For any two-step nilpotent Lie algebra $\mu \in \Lambda^2(\mathbb{R}^m)^* \otimes \mathbb{R}^n$, the orbit $SL(m) \times SL(n).\mu$ is closed if and only if a determined rank-one solvable extension of μ admit a left invariant Riemannian metric which is Einstein.

We then use closed $SL(m) \times SL(n)$ -orbits on $\Lambda^2 \mathbb{R}^m \otimes \mathbb{R}^n$ obtained in [10] by invariant-theoretic methods in the cases $(m, n) = (5, 5), (6, 3)$, to exhibit a curve and a two-parameter family of pairwise non-isometric Einstein solvmanifolds of dimension 11 and 10 respectively.

2. Variety of Lie algebras

Let \mathfrak{g} be an n -dimensional vector space over \mathbb{R} . We consider \mathfrak{g} as the underlying vector space of every n -dimensional Lie algebra over \mathbb{R} , thus identifying each Lie algebra with its Lie bracket, which is an element of $\Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$, the space of all alternating bilinear maps from $\mathfrak{g} \times \mathfrak{g}$ to \mathfrak{g} . The set \mathcal{L} of all Lie brackets is an algebraic subset of $\Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$, since the Jacobi identity is given by the following polynomial conditions:

$$\sum_{r=1}^n \mu_{ijr} \mu_{krs} + \mu_{jkr} \mu_{irs} + \mu_{kir} \mu_{jrs} = 0, \quad 1 \leq i < j < k \leq n, \quad 1 \leq s \leq n, \quad (1)$$

where $\mu(X_i, X_j) = \sum_{k=1}^n \mu_{ijk} X_k$ and $\{X_1, \dots, X_n\}$ is a fixed basis of \mathfrak{g} .

The isomorphism class of a Lie algebra $\mu \in \mathcal{L}$ is the orbit $\mathcal{O}(\mu) = GL(n).\mu$ under the ‘change of basis’ action of $GL(n) := GL(n, \mathbb{R}) = GL(\mathfrak{g})$ on \mathcal{L} given by

$$\varphi.\mu(X, Y) = \varphi\mu(\varphi^{-1}X, \varphi^{-1}Y), \quad X, Y \in \mathfrak{g}. \quad (2)$$

Definition 2.1. We say that μ *degenerates to* λ (denoted by $\mu \rightarrow \lambda$) if $\lambda \in \overline{\mathcal{O}(\mu)}$, the closure of the $GL(n)$ -orbit $\mathcal{O}(\mu)$ with respect to the usual (metric) topology of $\Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$.

In other words, $\mu \rightarrow \lambda$ if there exists a sequence $\{\varphi_n\} \subset GL(n)$ such that $\lim_{n \rightarrow \infty} \varphi_n \cdot \mu = \lambda$. The degeneration $\mu \rightarrow \lambda$ is nontrivial when λ lies in the boundary of $\mathcal{O}(\mu)$, and in such a case, it is easy to see that the entire orbit $\mathcal{O}(\lambda)$ lies in the boundary of $\mathcal{O}(\mu)$. Every degeneration will be assumed nontrivial. If $\mu \rightarrow \lambda$ then we may say that in some sense, λ is ‘more abelian’ than μ . In fact, it is well known that $\dim \text{Der}(\lambda) > \dim \text{Der}(\mu)$, $\dim \lambda(\mathfrak{g}, \mathfrak{g}) \leq \dim \mu(\mathfrak{g}, \mathfrak{g})$, $\dim \mathfrak{z}(\lambda) \geq \dim \mathfrak{z}(\mu)$ and $\text{ab}(\lambda) \geq \text{ab}(\mu)$, where $\mathfrak{z}(\mu)$ denotes the center of μ and $\text{ab}(\mu)$ is the dimension of a maximal abelian subalgebra of μ (see [4]).

The concepts of degeneration and deformation have attracted considerable attention in the last decades (see for instance [6,16,28,35]). Seeley [33] classified all the possible degenerations of 6-dimensional complex nilpotent Lie algebras and more recently, Burde and Steinhoff [4] did the same for all the 4-dimensional complex Lie algebras. On the other hand, in [34], a very nice relationship between variation of complex structures and degenerations of Lie algebras is studied. The idea of degeneration (or contraction) was really introduced by theoretical physicists in [20].

We shall give now the basic definitions of the small degree cohomology spaces of a Lie algebra μ relative to the adjoint representation, often called *Chevalley cohomology* (see for instance [21]). Using the identifications

$$C^0 = \mathfrak{g}, \quad C^1 = \mathfrak{gl}(\mathfrak{g}), \quad C^2 = \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g},$$

the coboundary operator is given by $\delta_\mu(X) = \text{ad}_\mu X$ for any $X \in \mathfrak{g}$ and

$$\delta_\mu(A)(X, Y) = \mu(AX, Y) + \mu(X, AY) - A\mu(X, Y)$$

for all $A \in \mathfrak{gl}(\mathfrak{g})$, $X, Y \in \mathfrak{g}$. We then have that $Z^0(\mu, \mu)$ is the center of μ and $Z^1(\mu, \mu) = \text{Der}(\mu)$, the Lie algebra of derivations of μ , where $Z^k(\mu, \mu) = \text{Ker } \delta_\mu$ is the space of cocycles. There is a well known relationship between the small degree Chevalley cohomology and the tangent geometry of the algebraic variety \mathcal{L} . The orbit $\mathcal{O}(\mu)$ is the image through the action of $GL(n)$ of the point $\mu \in \mathcal{L}$, and the isotropy subgroup is precisely the automorphism group $\text{Aut}(\mu)$ of μ , which is closed in $GL(n)$. Thus $\mathcal{O}(\mu)$ can be provided with the structure of differentiable manifold of the homogeneous space $GL(n)/\text{Aut}(\mu)$. The tangent space $T_\mu \mathcal{O}(\mu)$ at the point μ on the orbit $\mathcal{O}(\mu)$ coincides with the space of coboundaries $B^2(\mu, \mu) = \delta_\mu(C^1)$, since for any $A \in \mathfrak{gl}(n)$ we have that $\frac{d}{dt}|_0 e^{tA} \cdot \mu = -\delta_\mu(A)$. It can be also proved that $T_\mu^{\text{Zar}} \mathcal{L} = Z^2(\mu, \mu)$ for any $\mu \in \mathcal{L}$ (see [15]).

Any inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} defines naturally inner products on $C^1 = \mathfrak{gl}(\mathfrak{g})$ by $\langle A, B \rangle = \text{tr } AB^t$, where the transpose is taken with respect to $\langle \cdot, \cdot \rangle$, and on $C^2 = \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$ by

$$\langle \lambda, \lambda' \rangle = \sum_{ijk} \langle \lambda(X_i, X_j), X_k \rangle \langle \lambda'(X_i, X_j), X_k \rangle, \quad (3)$$

where $\{X_1, \dots, X_n\}$ is any orthonormal basis of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$.

3. Riemannian interpretation of the variety of Lie algebras

Let G be a real Lie group and let \mathfrak{g} denote its Lie algebra with Lie bracket μ . All the Riemannian information of the left invariant metric defined by an inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} , as the sectional, Ricci or scalar curvature, are given in terms of just $\langle \cdot, \cdot \rangle$ and the Lie bracket μ of \mathfrak{g} .

We endow the vector space \mathfrak{g} with a fixed inner product $\langle \cdot, \cdot \rangle$. We then may regard each element $\mu \in \mathcal{L}$ as a Riemannian manifold in the following way: we take G_μ , the unique simply connected Lie group having Lie algebra (\mathfrak{g}, μ) , and endow G_μ with the left invariant Riemannian metric determined by $\langle \cdot, \cdot \rangle$. Throughout the paper, we shall always identify μ with this Riemannian manifold $(G_\mu, \langle \cdot, \cdot \rangle)$.

The natural $GL(n)$ -action on \mathcal{L} given in (2) has the following geometric interpretation: for any $\varphi \in GL(n)$, the map φ^{-1} determines a Riemannian isometry

$$(G_{\varphi \cdot \mu}, \langle \cdot, \cdot \rangle) \rightarrow (G_\mu, \varphi^{-1} \cdot \langle \cdot, \cdot \rangle), \quad (4)$$

where $\varphi^{-1} \cdot \langle \cdot, \cdot \rangle = \langle \varphi \cdot, \varphi \cdot \rangle$. In fact, $\varphi^{-1} : (\mathfrak{g}, \varphi \cdot \mu) \rightarrow (\mathfrak{g}, \mu)$ is a Lie algebra isomorphism and $\varphi^{-1} : (\mathfrak{g}, \langle \cdot, \cdot \rangle) \rightarrow (\mathfrak{g}, \varphi^{-1} \cdot \langle \cdot, \cdot \rangle)$ is an isometry of inner product spaces, and hence it follows from the left invariance of the metrics that φ^{-1} defines by exponentiation a Riemannian isometry between $(G_{\varphi \cdot \mu}, \langle \cdot, \cdot \rangle)$ and $(G_\mu, \varphi^{-1} \cdot \langle \cdot, \cdot \rangle)$.

We note that on the left of (4) the action of $GL(n)$ give rises to the orbit $\mathcal{O}(\mu)$, and on the right, to the set of all inner products on \mathfrak{g} . We therefore can say that the orbit $\mathcal{O}(\mu)$ contains all the left invariant Riemannian metrics on G_μ , and hence \mathcal{L} can be viewed as the space of all n -dimensional simply connected Lie groups endowed with a left invariant Riemannian metric. Recall that if φ belongs to the orthogonal group $O(n) = O(\mathfrak{g}, \langle \cdot, \cdot \rangle)$, then $\varphi \cdot \mu$ and μ are isometric.

We will use repeatedly in this paper the isometry given in (4), to translate algebraic or invariant-theoretic properties of the orbit $\mathcal{O}(\mu)$ to Riemannian properties of left invariant metrics on G_μ .

The Ricci curvature operator of μ can be viewed as a symmetric transformation $\text{Ric}_\mu : \mathfrak{g} \rightarrow \mathfrak{g}$ defined by

$$\text{Ric}_\mu = R_\mu - \frac{1}{2} B_\mu - D_\mu, \quad (5)$$

where B_μ is the Killing form of μ in terms of $\langle \cdot, \cdot \rangle$, that is,

$$\langle B_\mu X, Y \rangle = \text{tr}(\text{ad}_\mu X \text{ad}_\mu Y), \quad X, Y \in \mathfrak{g},$$

D_μ is the symmetric part of $\text{ad } Z_\mu$ and $Z_\mu \in \mathfrak{g}$ is defined by $\langle Z_\mu, X \rangle = \text{tr}(\text{ad}_\mu X)$, $X \in \mathfrak{g}$; and R_μ is given by

$$\begin{aligned} \langle R_\mu X, Y \rangle &= -\frac{1}{2} \sum_{ij} \langle \mu(X, X_i), X_j \rangle \langle \mu(Y, X_i), X_j \rangle \\ &\quad + \frac{1}{4} \sum_{ij} \langle \mu(X_i, X_j), X \rangle \langle \mu(X_i, X_j), Y \rangle, \end{aligned} \quad (6)$$

for all $X, Y \in \mathfrak{g}$, where $\{X_1, \dots, X_n\}$ is any orthonormal basis of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ (see [3, 7.39]). Note that $Z_\mu = 0$ if and only if μ is unimodular and that if μ is nilpotent then $\text{Ric}_\mu = R_\mu$. It is clear that R_μ (and also Ric_μ) can be defined, at least formally, for every element $\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$.

Lemma 3.1 [24]. *If $F : \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g} \rightarrow \mathbb{R}$ is defined by $F(\mu) = \text{tr } R_\mu^2$ (see (6)) then*

$$\text{grad}(F)_\mu = -\delta_\mu(R_\mu), \quad \mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g},$$

where $\delta_\mu : C^1 = \mathfrak{gl}(\mathfrak{g}) \rightarrow C^2 = \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$ is the cohomology coboundary operator of the Lie algebra μ .

4. Critical points of F

Lemma 3.1 gives an interesting link between the functional F and the theory of degenerations. Indeed, we have that $\pm \text{grad}(F)_\mu \in T_\mu \mathcal{O}(\mu)$ for any μ , and so it is easy to prove that the $\pm \text{grad}(F)$ flows ψ_t^+ and ψ_t^- starting from a point $\mu_0 \in \mathcal{L}$ lies in the orbit $\mathcal{O}(\mu_0)$ for every $t \in [0, \infty)$. Henceforth, if ψ_t^+ or ψ_t^- converges, as it is the case when we restrict ourselves to the sphere S of $\Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$, then the limit will determine a degeneration of μ_0 .

A precise formula for F is given by

$$F(\mu) = \text{tr} R_\mu^2 = \sum_{pr} \left(\sum_{ij} -\frac{1}{2} \mu_{pij} \mu_{rij} + \frac{1}{4} \mu_{ijp} \mu_{ijr} \right)^2, \quad (7)$$

where $\mu_{ijk} = \langle \mu(X_i, X_j), X_k \rangle$ and $\{X_1, \dots, X_n\}$ is any orthonormal basis of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$. We note that F is a homogeneous polynomial of degree 4, thus we will restrict F to the sphere S of $\Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$, and hence we have to consider sets of the form $\mathcal{O}(\mu) \cap S$, instead of the whole orbits. If one prefers to keep the action of $GL(n)$, it is equivalent to consider the functional $F: \mathbb{P}(\Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}) \rightarrow \mathbb{R}$ from the projective space $\mathbb{P}(\Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g})$ instead from the sphere S . Note that we just need to act by scalars to recover the whole orbit $\mathcal{O}(\mu)$ from the set $\mathcal{O}(\mu) \cap S$.

Proposition 4.1. *Let F be the functional on $\Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$ given by $F(\mu) = \text{tr} R_\mu^2$ and let S denote the sphere of $\Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$. Then for $\mu \in \mathcal{L} \cap S$ the following conditions are equivalent:*

- (i) $R_\mu \in \mathbb{R}I \oplus \text{Der}(\mu)$.
- (ii) μ is a critical point of $F: S \rightarrow \mathbb{R}$.
- (iii) μ is a critical point of $F: \mathcal{O}(\mu) \cap S \rightarrow \mathbb{R}$.

Proof. The equivalence between (i) and (ii) follows from $\text{grad}(F)_\mu = -\delta_\mu(R_\mu)$ (see Lemma 3.1) and the Lagrange multiplier theorem. Indeed, $\mu \in S$ is a critical point of $F|_S$ if and only if $\delta_\mu(R_\mu) = c\mu$ for some $c \in \mathbb{R}$. Since $\delta_\mu(I) = \mu$, we have that this is equivalent to $\delta_\mu(R_\mu - cI) = 0$, or to $R_\mu - cI \in \text{Der}(\mu)$.

To see the equivalence between (ii) and (iii) we first note that we have the orthogonal decomposition

$$T_\mu \mathcal{O}(\mu) = (T_\mu \mathcal{O}(\mu) \cap T_\mu S) \oplus \mathbb{R}\mu.$$

Since $-\text{grad}(F)_\mu = \delta_\mu(R_\mu) = \frac{\partial}{\partial t} |_0 e^{-\frac{t}{2} R_\mu} \cdot \mu \in T_\mu \mathcal{O}(\mu)$ we have that μ is a critical point of $F|_{GL(n) \cdot \mu \cap S}$, that is, $\text{grad}(F)_\mu \perp T_\mu GL(n) \cdot \mu \cap S = T_\mu GL(n) \cdot \mu \cap T_\mu S$, if and only if $\delta_\mu(R_\mu) \in \mathbb{R}\mu$. \square

We now give a uniqueness result for a critical point of F .

Proposition 4.2. *There is at most one critical point of $F: S \rightarrow \mathbb{R}$ in each subset $\mathcal{O}(\mu) \cap S$ up to the action of $O(n)$.*

Proof. We first note that this is equivalent to the uniqueness of a left invariant Riemannian metric (\cdot, \cdot) on G_μ satisfying

$$R_{(\cdot, \cdot)} \in \mathbb{R}I \oplus \text{Der}(\mu) \quad (8)$$

up to the $\mathbb{R}^* \text{Aut}(\mu)$ -action on the set of inner products \mathcal{P} on \mathfrak{g} (see (2)). Indeed, if μ and $\varphi.\mu$ are both critical points of $F|_S$, then using that $(G_\mu, \varphi^{-1}.\langle \cdot, \cdot \rangle)$ and $(G_{\varphi.\mu}, \langle \cdot, \cdot \rangle)$ are isometric (see (4)) we obtain that $\langle \cdot, \cdot \rangle$ and $\varphi^{-1}.\langle \cdot, \cdot \rangle$ are two left invariant metrics satisfying (8), and so if there exists $\psi \in \mathbb{R}^* \text{Aut}(\mu)$ such that $\psi.\langle \cdot, \cdot \rangle = \varphi^{-1}.\langle \cdot, \cdot \rangle$, then $\varphi\psi \in \text{O}(n)$ and satisfies $\varphi\psi.\mu = c\varphi.\mu$ for some $c > 0$, concluding that $c = 1$ since $\varphi.\mu \in S$ and thus $\varphi.\mu \in \text{O}(n).\mu$. The converse is completely analogous.

In view of these observations, it is easy to see that the proof of the theorem is given in [18, Theorem 5.1] or [24, Theorem 3.5]. \square

Example 4.3.

- (i) In [24], nilpotent Lie brackets $\mu \in \mathcal{L}$ which are critical points of $F|_S$ are geometrically characterized in three different ways: μ is a Ricci soliton metric (see [5]); μ is a quasi-Einstein-metric (see [9]); (\mathfrak{g}, μ) admits a standard solvable extension which is Einstein (see [18]). Several examples of critical nilpotent Lie algebras are given in [24] using these characterizations.
- (ii) It is easy to see that the orbit $\mathcal{O}(\mu)$ of any semi-simple μ contains a critical point μ_0 (see Section 8).
- (iii) If $\mathfrak{g} = \mathbb{R}A \oplus \mathbb{R}^{n-1}$ and $\mu \in \mathcal{L}$ satisfies that $\mu(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}) = 0$ and $\text{ad}_\mu A|_{\mathbb{R}^{n-1}}$ is normal, then it is easy to see that μ is a critical point of $F|_S$. If $\text{ad}_\mu A|_{\mathbb{R}^{n-1}}$ is nilpotent, then it is proved in [25] that μ is a critical point of $F|_S$ as well.
- (iv) Using the examples given above, one can easily see that any 3-dimensional Lie algebra has a critical Lie bracket in its orbit, except for the defined by $\mu(X_1, X_2) = X_2, \mu(X_1, X_3) = -X_2 + 2X_3$, denoted by \mathfrak{s}_2 in Section 6.
- (v) Every nilpotent Lie algebra of dimension ≤ 5 contains a critical point of $F|_S$ in its orbit (see [25]).
- (vi) The lowest possible dimension for the existence of a curve of non-isometric (or equivalently non-isomorphic) nilpotent critical points of $F|_S$ is 7, and an example of such a curve is given in [25].
- (vii) It has been recently proved in [36] that any of the 34 6-dimensional nilpotent Lie algebras contain a critical point of $F|_S$ in its orbit.

Remark 4.4. We recently become aware that the map $R: \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ given by $R(\mu) = R_\mu$, coincides up to a scalar with the *moment map* for the action of $GL(n)$ on $\Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$ (or on its projective space), and so F is precisely the functional square norm of the moment map (see the survey [23] for further information). This implies that Proposition 4.2 would follow from the well known convexity properties of moment maps. In the nilpotent case, we then have that the Ricci operator is a moment map. We think that this relationship could give rise to many other interplays between the invariant-theoretic aspects of the representation $\Lambda^2 GL(n) \otimes GL(n)$ and the Riemannian point of view considered in this paper.

5. Lie groups with only one left invariant metric

The aim of this section is to classify all the Lie groups which admit a unique left-invariant Riemannian metric up to isometry and scaling. As an application of the results proved in Section 4 on the critical point behavior of the functional $F: S \rightarrow \mathbb{R}$, we first give the following interaction result.

Theorem 5.1. For $\mu \in \mathcal{L} \cap S$ the following conditions are equivalent:

- (i) The only possible degeneration of μ is $\mu \rightarrow 0$.
- (ii) $\mathcal{O}(\mu) \cap S$ is closed (or compact).
- (iii) $\mathcal{O}(\mu) = \mathbb{R}^* \mathcal{O}(n) \cdot \mu$ (or $\mathcal{O}(\mu) \cap S = \mathcal{O}(n) \cdot \mu$).
- (iv) $GL(n) \cdot \langle \cdot, \cdot \rangle = \mathbb{R}^* \text{Aut}(\mu) \cdot \langle \cdot, \cdot \rangle$.
- (v) There is only one left invariant metric on G_μ up to isometry and scaling.

Proof. (i) \Rightarrow (ii) We have that $\overline{\mathcal{O}(\mu)} = \mathcal{O}(\mu) \cup \{0\}$ and so $\overline{\mathcal{O}(\mu) \cap S} = \overline{\mathcal{O}(\mu)} \cap S = \mathcal{O}(\mu) \cap S$.

(ii) \Rightarrow (i) Since any degeneration $\mu \rightarrow \lambda$ with $\lambda \neq 0$ can be taken to a degeneration $\mu/\|\mu\| \rightarrow \lambda/\|\lambda\|$ such that the sequence $\{\varphi_n \cdot (\mu/\|\mu\|)\} \subset \mathcal{O}(\mu)$ defining the degeneration lies in S , we have that $\mathcal{O}(\mu) \cap S$ closed implies that the only possible degeneration of μ is $\mu \rightarrow 0$.

(ii) \Rightarrow (iii) It follows from $\mathcal{O}(\mu) \cap S$ closed that F attains their maximum and minimum values on $\mathcal{O}(\mu) \cap S$ at some points $\mu^+, \mu^- \in \mathcal{O}(\mu) \cap S$, respectively. But according to Propositions 4.1, 4.2, $F|_S$ has at most one critical point in $\mathcal{O}(\mu) \cap S$ up to the $\mathcal{O}(n)$ -action, thus $\mu^- \in \mathcal{O}(n) \cdot \mu^+$ obtaining from the $\mathcal{O}(n)$ -invariance of F that F is constant on $\mathcal{O}(\mu) \cap S$. This implies that any point of $\mathcal{O}(\mu) \cap S$ is a critical point of $F|_S$ and so $\mathcal{O}(\mu) \cap S = \mathcal{O}(n) \cdot \mu$ by Proposition 4.2, or equivalently, $\mathcal{O}(\mu) = \mathbb{R}^* \mathcal{O}(n) \cdot \mu$.

((iii) \Rightarrow (ii)) If $\mathcal{O}(\mu) = \mathbb{R}^* \mathcal{O}(n) \cdot \mu$ then $\mathcal{O}(\mu) \cap S = \mathcal{O}(n) \cdot \mu$ and so it is compact and closed.

((iii) \Leftrightarrow (iv)) This equivalence follows from the fact that any $\varphi \in GL(n)$ defines by exponentiation an isometry between $(G_\mu, \varphi^{-1} \cdot \langle \cdot, \cdot \rangle)$ and $(G_{\varphi \cdot \mu}, \langle \cdot, \cdot \rangle)$ (see (4)). Indeed, if μ satisfies (iii), then for any $\varphi \in GL(n)$ there exists $\psi \in \mathbb{R}^* \mathcal{O}(n)$ such that $\psi \cdot \mu = \varphi \cdot \mu$ and thus $\varphi^{-1} \psi \cdot \langle \cdot, \cdot \rangle = c \varphi^{-1} \cdot \langle \cdot, \cdot \rangle$ for some $c \in \mathbb{R}$ with $\varphi^{-1} \psi \in \text{Aut}(\mu)$, obtaining that $\varphi^{-1} \cdot \langle \cdot, \cdot \rangle \in \mathbb{R}^* \text{Aut}(\mu) \cdot \langle \cdot, \cdot \rangle$ for any $\varphi \in GL(n)$. Conversely, if μ satisfies (iv), then for any $\varphi \in GL(n)$ there exists $\psi \in \mathbb{R}^* \text{Aut}(\mu)$ such that $\psi \cdot \langle \cdot, \cdot \rangle = \varphi \cdot \langle \cdot, \cdot \rangle$ and thus $\varphi \cdot \mu = c \varphi \psi^{-1} \cdot \mu$ for some $c \in \mathbb{R}$ with $\varphi \psi^{-1} \in \mathcal{O}(n)$, proving that $\varphi \cdot \mu \in \mathbb{R}^* \mathcal{O}(n) \cdot \mu$ for any $\varphi \in GL(n)$.

((iv) \Rightarrow (v)) It follows arguing as in the above paragraph.

((v) \Rightarrow (iii)) By hypothesis, we have that $(G_\mu, \langle \cdot, \cdot \rangle)$ and $(G_\mu, \varphi^{-1} \cdot \langle \cdot, \cdot \rangle)$ are isometric (up to scaling) for any $\varphi \in GL(n)$. Since the tensor $R_{\mu, \langle \cdot, \cdot \rangle}$ satisfies

$$\langle R_{\mu, \langle \cdot, \cdot \rangle} X, Y \rangle = \text{tr}(\nabla X)^2 - \frac{1}{2} \text{tr}(\nabla_X - \nabla X)^2,$$

where ∇ denotes the Levi-Civita connexion of $(G_\mu, \langle \cdot, \cdot \rangle)$ (see [7]), it follows that F is invariant under isometry. Using again the isometry between $(G_\mu, \varphi^{-1} \cdot \langle \cdot, \cdot \rangle)$ and $(G_{\varphi \cdot \mu}, \langle \cdot, \cdot \rangle)$ given in (4), we obtain that F is constant on $\mathcal{O}(\mu) \cap S$ and thus $\mathcal{O}(\mu) \cap S = \mathcal{O}(n) \cdot \mu$ by Proposition 4.2. \square

We then have to determine all the Lie algebras whose only degeneration is to the abelian Lie algebra. Let μ_{he}, μ_{hy} denote the Lie brackets defined by

$$\mu_{he}(X_1, X_2) = X_3, \quad \mu_{hy}(X_1, X_i) = X_i, \quad i = 2, \dots, n,$$

and zero otherwise. Note that μ_{he} is isomorphic to $\mathfrak{h}_3 \oplus \mathbb{R}^{n-3}$, where \mathfrak{h}_3 is the 3-dimensional Heisenberg Lie algebra and \mathbb{R}^{n-3} is an abelian factor, and μ_{hy} is the Lie algebra of the solvable Lie group for which there is a left invariant metric $\langle \cdot, \cdot \rangle$ such that $(S, \langle \cdot, \cdot \rangle)$ is isometric to the real hyperbolic symmetric space $\mathbb{R}H(n)$.

The proof of the following classification is based in techniques used by Milnor [27] in the study of curvature properties of Riemannian left invariant metrics on Lie groups.

Theorem 5.2. *Every Lie bracket which is not in $\mathcal{O}(\mu_{hy})$ degenerates to μ_{he} . Moreover, μ_{he} and μ_{hy} are the only n -dimensional Lie algebras for $n \geq 3$, for which the only degeneration is to the abelian Lie algebra.*

Proof. Let $\mu \in \mathcal{L}$ and suppose that there exists three linearly independent vectors such that $\mu(X_1, X_2) = X_3$. Complete to a basis $\{X_1, \dots, X_n\}$ and consider the diagonal $\varphi_\varepsilon \in GL(n)$ defined by

$$\varphi_\varepsilon X_1 = \varepsilon X_1, \quad \varphi_\varepsilon X_2 = \varepsilon X_2, \quad \varphi_\varepsilon X_i = \varepsilon^2 X_i, \quad i \geq 3.$$

It is easy to check that $\lim_{\varepsilon \rightarrow 0} \varphi_\varepsilon^{-1} \cdot \mu = \mu_{he}$.

Otherwise, $\mu(X, Y) \in \mathbb{R}X + \mathbb{R}Y$ for all $X, Y \in \mathfrak{g}$, and hence there exists a linear map $L: \mathfrak{g} \rightarrow \mathbb{R}$ satisfying

$$\mu(X, Y) = L(X)Y - L(Y)X, \quad \forall X, Y \in \mathfrak{g}.$$

Considering a decomposition $\mathfrak{g} = \mathbb{R}H \oplus \text{Ker}(L)$ with $L(H) = 1$, we get $\mu(H, X) = 0$ for any $X \in \text{Ker}(L)$ and $\mu(X, Y) = 0$ for all $X, Y \in \text{Ker}(L)$, that is, μ is isomorphic to μ_{hy} , proving the first part of the theorem.

For the second part, we note that μ_{hy} does not degenerates to μ_{he} since

$$\dim \text{Der}(\mu_{hy}) = n(n-1) > n^2 - 3n + 6 = \dim \text{Der}(\mu_{he}),$$

and so $\text{level}(\mu_{hy}) = 1$ (degeneration level). Furthermore, it follows from $\dim \mu_{he}(\mathfrak{g}, \mathfrak{g}) = 1 < n-1 = \dim \mu_{hy}(\mathfrak{g}, \mathfrak{g})$ that μ_{he} cannot degenerate to μ_{hy} , concluding also that $\text{level}(\mu_{he}) = 1$. \square

Remark. We have recently become aware that Gorbatsevich has given in [11] a proof of the above theorem in the complex case, via representation theory methods. Also, he studied in [11–13] a very interesting notion of degeneration: $\mu \rightarrow \lambda$ (μ, λ Lie algebras not necessarily of the same dimension) if $\lambda \oplus \mathbb{R}^k$ degenerates to $\mu \oplus \mathbb{R}^m$ in the sense considered in this paper (see Definition 2.1) for some $k, m \geq 0$, where \mathbb{R}^k and \mathbb{R}^m are abelian factors. The corresponding first three levels of degenerations are completely classified in [12]. We note that this notion of degeneration is weaker than the usual one; for instance, the level of μ_{hy} , $n = 2$, cannot be one since $\mu_{hy} \oplus \mathbb{R} \rightarrow \mu_{he}$, $n = 3$.

We then obtain from Theorems 5.1 and 5.2 the following

Corollary 5.3. *The groups $G_{\mu_{he}}$ and $G_{\mu_{hy}}$ are the only simply connected Lie groups having only one left invariant Riemannian metric up to isometry and scaling.*

It should be noted that if a Lie group admits a unique left invariant metric up to isometry and scaling, then its universal cover so. Thus the only Lie groups satisfying such a property are the finite quotients of $G_{\mu_{he}}$ and $G_{\mu_{hy}}$.

6. Curvature pinched conditions and degenerations

All the Riemannian quantities associated to a $\mu \in \mathcal{L}$, like sectional, Ricci and scalar curvatures, depend continuously on μ . Consequently, if $\mu \rightarrow \lambda$ and there is a $\lambda_0 \in \mathcal{O}(\lambda)$ satisfying certain strict curvature inequality, then the orbit $\mathcal{O}(\mu)$ must contain an element satisfying the same inequality. Indeed, we can assume that $\varphi_n \cdot \mu \rightarrow \lambda_0$ and so for some n_0 the Lie brackets $\varphi_n \cdot \mu$ will satisfy the curvature inequality for

all $n \geq n_0$. In other words, if $\mu \rightarrow \lambda$ and G_λ admits a left invariant Riemannian metric satisfying a strict pinched curvature condition, then so G_μ (see (4)).

We then can use well known algebraic characterizations of Lie groups having metrics with special curvature properties to decide in the negative, whether a Lie algebra degenerates to another one. For instance, Heintze [19] proved the following algebraic characterization of Lie groups G_μ admitting a left invariant Riemannian metric with strictly negative sectional curvature $K < 0$:

(Hn1) μ is solvable;

(Hn2) there exists $A \in \mathfrak{g}$ such that $\mathfrak{g} = \mathbb{R}A \oplus \mu(\mathfrak{g}, \mathfrak{g})$ and all the eigenvalues of $\text{ad}_\mu A|_{\mu(\mathfrak{g}, \mathfrak{g})}$ have positive real part.

More recently, Eberlein and Heber [8] showed that to have sectional curvature $-b^2 < K < -a^2 < 0$, ($a, b > 0$), besides of (Hn1) μ must satisfy the following necessary (but not sufficient) condition:

(EHb1) there exists $A \in \mathfrak{g}$ such that $\mathfrak{g} = \mathbb{R}A \oplus \mu(\mathfrak{g}, \mathfrak{g})$ and all the eigenvalues of $\text{ad}_\mu A|_{\mu(\mathfrak{g}, \mathfrak{g})}$ have real part in the open interval (a, b) (see paragraph above [8, Proposition 2.2]).

Moreover, if we want $-4 < K < -1$, then besides (Hn1) and (EHb1), μ has to satisfy necessarily that

(EHb2) the derived algebra $\mu(\mathfrak{g}, \mathfrak{g})$ is abelian (see [8, Corollary 5.2]).

The following result follows from the observations given above.

Proposition 6.1. *If $\mu \rightarrow \lambda$ and there exists $A \in \mathfrak{g}$ such that $\mathfrak{g} = \mathbb{R}A \oplus \lambda(\mathfrak{g}, \mathfrak{g})$, $\lambda(\mathfrak{g}, \mathfrak{g})$ is abelian and all the eigenvalues of $\text{ad}_\lambda A|_{\lambda(\mathfrak{g}, \mathfrak{g})}$ have real part in the open interval $(1, 2)$, then μ is solvable, $\mu(\mathfrak{g}, \mathfrak{g})$ is abelian and there exists $B \in \mathfrak{g}$ such that $\mathfrak{g} = \mathbb{R}B \oplus \mu(\mathfrak{g}, \mathfrak{g})$ and all the eigenvalues of $\text{ad}_\mu B|_{\mu(\mathfrak{g}, \mathfrak{g})}$ have real part in $(1, 2)$.*

According to [31], a problem particularly interesting from the physical point of view, is to find all the Lie algebras that degenerates to a given one. It has been showed in Section 5 that any Lie algebra other than μ_{hy} degenerates to μ_{he} . Hence a natural question takes place: which are the Lie algebras who degenerates to μ_{hy} ?

Theorem 6.2. *A Lie algebra $\mu \in \mathcal{L}$ degenerates to μ_{hy} if and only if there exists $A \in \mathfrak{g}$ such that $\mathfrak{g} = \mathbb{R}A \oplus \mu(\mathfrak{g}, \mathfrak{g})$, $\mu(\mathfrak{g}, \mathfrak{g})$ is abelian and has a basis such that $\text{ad}_\mu A|_{\mu(\mathfrak{g}, \mathfrak{g})}$ is a Jordan matrix with eigenvalue 1 and block dimensions $r_1 \geq \dots \geq r_k$ (see (9)). Moreover, two of such Lie algebras μ and μ' are isomorphic if and only if $r_i = r'_i$ for all $i = 1, \dots, k$.*

Proof. We first note that all the left invariant metrics on $G_{\mu_{hy}}$ are pairwise isometric up to scaling (see Corollary 5.3) and have constant negative sectional curvature $K \equiv -c^2$. This implies that if $\mu \rightarrow \mu_{hy}$ then μ must satisfy (Hn1), (EHb1) and (EHb2).

Now, given $\mu, \lambda \in \mathcal{L}$ both with codimension-one and abelian derived algebra, we can assume (up to isomorphism) that $\mu(\mathfrak{g}, \mathfrak{g}) = \lambda(\mathfrak{g}, \mathfrak{g}) = \mathfrak{n}$, and consider a decomposition $\mathfrak{g} = \mathbb{R}A \oplus \mathfrak{n}$. It is easy to see that μ is isomorphic to λ if and only if $\text{ad}_\mu A|_{\mathfrak{n}}$ is conjugate to $\text{ad}_\lambda A|_{\mathfrak{n}}$ up to scaling. Thus if $\mu \rightarrow \lambda \neq 0$, the

coefficients of the characteristic polynomial of $\text{ad}_{\varphi_k \cdot \mu} A|_{\mathfrak{n}}$ are multiplied by a single c_k along the degeneration $\varphi_k \cdot \mu \rightarrow \lambda$. This implies that if $\mu \rightarrow \lambda \neq 0$ then $\text{ad}_{\mu} A|_{\mathfrak{n}}$ and $\text{ad}_{\lambda} A|_{\mathfrak{n}}$ have the same set of eigenvalues (counting multiplicities) up to scaling by a single real number, not necessarily different from zero.

We deduce from the above paragraph that, if $\mu \rightarrow \mu_{hy}$ then $\text{ad}_{\mu} A|_{\mathfrak{n}}$ has only one eigenvalue and it is real and nonzero, that is, its characteristic polynomial is $f(t) = (t - c)^{n-1}$, for some $c \neq 0$. We can assume that $c = 1$ and hence there exists a basis of \mathfrak{n} such that

$$\text{ad}_{\mu} A|_{\mathfrak{n}} \begin{bmatrix} J_{r_1} & & \\ & \ddots & \\ & & J_{r_k} \end{bmatrix}, \quad J_{r_i} = \begin{bmatrix} 1 & & 0 \\ 1 & & \\ & \ddots & \ddots \\ 0 & & 1 & 1 \end{bmatrix}, \quad (9)$$

with $r_1 \geq \dots \geq r_k$, where J_{r_i} is the $r_i \times r_i$ Jordan block with eigenvalue 1.

Conversely, if $\mu \in \mathcal{L}$ satisfies that $\mu(\mathfrak{g}, \mathfrak{g})$ is abelian, of codimension-one, and $\text{ad}_{\mu} A|_{\mu(\mathfrak{g}, \mathfrak{g})}$ has characteristic polynomial $f(t) = (t - 1)^{n-1}$, then $\mu \rightarrow \mu_{hy}$. Indeed, if we define on each block

$$\varphi_{\varepsilon} = \begin{bmatrix} \varepsilon & & \\ & \varepsilon^2 & \\ & & \ddots \\ & & & \varepsilon^{r_i} \end{bmatrix},$$

and $\varphi_{\varepsilon} A = A$, then it is easy to check that $\lim_{\varepsilon \rightarrow 0} \varphi_{\varepsilon}^{-1} \text{ad}_{\mu} A|_{\mu(\mathfrak{g}, \mathfrak{g})} \varphi_{\varepsilon} = I$, the identity matrix. Thus we have that $\lim_{\varepsilon \rightarrow 0} \varphi_{\varepsilon} \cdot \mu = \mu_{hy}$. \square

With the above theorem and other previous results in place, we are now able to find all the possible degenerations for 3-dimensional real Lie algebras. A classification of such Lie algebras can be described by (see [21]):

Notation	Nonzero products	Properties
\mathbb{R}^3	—	Der = 9, unim.,
\mathfrak{h}_3	$[X_1, X_2] = X_3,$	Der = 6, unim.,
\mathfrak{r}_{α}	$[X_1, X_2] = X_2, \quad -1 \leq \alpha \leq 1$ $[X_1, X_3] = \alpha X_3,$	Der = 4, $\alpha \neq 1$; Der = 6, $\alpha = 1$, unim. iff $\alpha = -1$,
\mathfrak{s}_{β}	$[X_1, X_2] = X_3, \quad 0 \leq \beta \leq 2,$ $[X_1, X_3] = -X_2 + \beta X_3,$	Der = 4, unim. iff $\beta = 0$,
$\mathfrak{sl}(2, \mathbb{R})$	$[X_1, X_2] = 2X_2, \quad [X_1, X_3] = -2X_3,$ $[X_2, X_3] = X_1,$	Der = 3, unim.,
$\mathfrak{so}(3)$	$[X_1, X_2] = X_3, \quad [X_1, X_3] = -X_2,$ $[X_2, X_3] = X_1,$	Der = 3, unim.

Using the dimensions Der of the Lie algebras of derivations, unimodularity and Theorem 5.2 (recall that $\mu_{he} = \mathfrak{h}_3$ and $\mu_{hy} = \mathfrak{r}_1$) we get that the only possible degenerations we have to analyze are

- (i) $\mathfrak{r}_{\alpha} \rightarrow \mathfrak{r}_1, -1 < \alpha < 1$.
- (ii) $\mathfrak{s}_{\beta} \rightarrow \mathfrak{r}_1, 0 < \beta \leq 2$.
- (iii) $\mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{r}_{-1}, \mathfrak{s}_0$.
- (iv) $\mathfrak{so}(3) \rightarrow \mathfrak{r}_{-1}, \mathfrak{s}_0$.

It follows from Theorem 6.2 that the only degeneration from (i) and (ii) which is valid is $\mathfrak{s}_2 \rightarrow \mathfrak{r}_1$. The degenerations of (iii) are both true taking

$$\varphi_\varepsilon = \begin{bmatrix} 1 & & \\ & \varepsilon & \\ & & \varepsilon \end{bmatrix},$$

with respect to the basis $\{X_1, X_2, X_3\}$ for the first one and to the basis $\{\frac{1}{2}(X_2 - X_3), \frac{1}{2}(X_2 + X_3), \frac{1}{2}X_1\}$ for the second one. The degeneration $\mathfrak{so}(3) \rightarrow \mathfrak{s}_0$ in (iv) is also valid via the same φ_ε .

Finally, using some Ricci curvature estimates given in [30], we shall see that $\mathfrak{so}(3) \rightarrow \mathfrak{r}_{-1}$ is not valid. Indeed, there exist $\lambda \in \mathcal{O}(\mathfrak{r}_{-1}) \cap S$ such that the extremal values for the Ricci curvature along all the directions are $-\frac{1}{2} \leq \text{ric}_\lambda \leq 0$. If we assume that $\mathfrak{so}(3)$ degenerates to \mathfrak{r}_{-1} , then there must be a sequence $\{\mu_n\} \subset \mathcal{O}(\mathfrak{so}(3)) \cap S$ such that $\mu_n \rightarrow \lambda$ when $n \rightarrow \infty$, and thus $\text{ric}_{\mu_n} \rightarrow \text{ric}_\lambda$ uniformly. However, it follows easily from [30, Theorem 2.3] that if the maximum value of ric_{μ_n} converges to 0, then so the minimum value, obtaining that ric_{μ_n} can never converge to ric_λ , which is a contradiction.

We now give the table of all the degenerations for 3-dimensional real Lie algebras.

$\mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{s}_0, \mathfrak{r}_{-1}, \mathfrak{h}_3, \mathbb{R}^3;$	$\mathfrak{s}_\beta \rightarrow \mathfrak{h}_3, \mathbb{R}^3, \beta \neq 0, 2;$
$\mathfrak{so}(3) \rightarrow \mathfrak{s}_0, \mathfrak{h}_3, \mathbb{R}^3;$	$\mathfrak{s}_2 \rightarrow \mathfrak{r}_1, \mathfrak{h}_3, \mathbb{R}^3;$
$\mathfrak{r}_{-1} \rightarrow \mathfrak{h}_3, \mathbb{R}^3;$	$\mathfrak{r}_1 \rightarrow \mathbb{R}^3;$
$\mathfrak{s}_0 \rightarrow \mathfrak{h}_3, \mathbb{R}^3;$	$\mathfrak{h}_3 \rightarrow \mathbb{R}^3;$
$\mathfrak{r}_\alpha \rightarrow \mathfrak{h}_3, \mathbb{R}^3, \alpha \neq \pm 1;$	$\mathbb{R}^3 \rightarrow - - .$

7. Minimal vectors for actions of reductive subgroups $G \subset GL(n)$ on $V = \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$

Let \mathbf{G} be a linearly reductive complex algebraic group defined over \mathbb{R} , and let $G = \mathbf{G}(\mathbb{R})$ the group of real points of \mathbf{G} . If $\mathbf{G} \rightarrow GL(\mathbf{V})$ is a rational representation of \mathbf{G} defined over \mathbb{R} , consider $V = \mathbf{V}(\mathbb{R})$ the set of \mathbb{R} -rational points of \mathbf{V} . A vector $v \in V$ is said to be a *minimal vector* for G if $\|g.v\| \geq \|v\|$ for any $g \in G$, where the norm $\|\cdot\|$ is associated with a fixed \mathfrak{k} -invariant positive definite inner product $\langle \cdot, \cdot \rangle$ on V such that \mathfrak{p} acts on V by symmetric transformations and $L(G) = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of the Lie algebra $L(G)$ of G . Let $\mathcal{M} = \mathcal{M}(G, V)$ denote the set of minimum vectors for G .

Theorem 7.1 [22,29].

- (i) *The orbit $G.v$ meets \mathcal{M} if and only if $G.v$ is closed (classical topology).*
- (ii) *For every $v \in V$, there exists a unique closed G -orbit in the closure $\overline{G.v}$.*

We note that in part (i) of the above theorem, in spite of the set \mathcal{M} depends on the invariant inner product $\langle \cdot, \cdot \rangle$ choose, the closeness of the orbit $G.v$ does not.

For each $v \in V$, we define a smooth real-valued function $F_v : G \rightarrow \mathbb{R}$ by $F_v(g) = \|g.v\|^2$. Thus $v \in \mathcal{M}$ if and only if F_v has a (necessarily minimum) critical point at $e \in G$ (see [29]). Let f_v be the differential $(dF_v)_e : L(G) \rightarrow \mathbb{R}$. Since f_v vanishes on \mathfrak{k} , it can be considered as an element of \mathfrak{p}^* , the dual space of \mathfrak{p} . We may identify \mathfrak{p} with \mathfrak{p}^* by means of the inner product $-B(X, \theta Y)$, where B is the Killing form

of $L(G)$ and θ the Cartan involution. Define the homogeneous polynomial of degree 4 on V given by $H(v) = \|f_v\|^2$. Note that $H \geq 0$ and that $v \in \mathcal{M}$ if and only if $H(v) = 0$.

In what follows, we will apply the results from invariant theory given above, to the case of G a reductive subgroup of $GL(\mathfrak{g})$ and $V = \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$. Consider a Cartan decomposition $L(G) = \mathfrak{k} \oplus \mathfrak{p}$ such that $\mathfrak{k} \subset \mathfrak{so}(\mathfrak{g})$ and $\mathfrak{p} \subset \mathfrak{sym}(\mathfrak{g})$ and endow $L(G)$ with the inner product $\langle A, B \rangle = \text{tr } AB^t$, where the transpose is taken with respect to the fixed inner product $\langle \cdot, \cdot \rangle$ we have on \mathfrak{g} . It is clear that the extended inner product $\langle \cdot, \cdot \rangle$ on V (see (3)) satisfies the required properties, that is, \mathfrak{k} and \mathfrak{p} act on V by skew-symmetric and symmetric transformations respectively. In order to get a description of the set \mathcal{M} of minimal vectors, we first need a formula for the homogeneous polynomial H .

Proposition 7.2. *If $p : \mathfrak{sym}(\mathfrak{g}) \rightarrow \mathfrak{p}$ denotes the orthogonal projection then for each $\mu \in V = \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$ we have that*

$$H(\mu) = 64 \text{tr } p(R_\mu)^2,$$

where $R_\mu \in \mathfrak{sym}(\mathfrak{g})$ is defined in (6), and so $\mathcal{M} = \{\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g} : p(R_\mu) = 0\}$.

Proof. We first calculate $f_\mu = (dF_\mu)_I \in \mathfrak{p}^*$. For every $A \in \mathfrak{p}$ we have

$$\begin{aligned} f_\mu(A) &= \frac{d}{dt} \bigg|_0 F_\mu(e^{tA}) = \frac{d}{dt} \bigg|_0 \|e^{tA} \cdot \mu\|^2 = \left\langle 2\mu, \frac{d}{dt} \bigg|_0 e^{tA} \cdot \mu \right\rangle = -2\langle \mu, \delta_\mu(A) \rangle \\ &= -2 \sum_{pij} \langle \delta_\mu(A)(X_p, X_i), X_j \rangle \langle \mu(X_p, X_i), X_j \rangle \\ &= -2 \left(\sum_{pij} \langle \mu(AX_p, X_i), X_j \rangle \langle \mu(X_p, X_i), X_j \rangle + \langle \mu(X_p, AX_i), X_j \rangle \langle \mu(X_p, X_i), X_j \rangle \right. \\ &\quad \left. - \langle A\mu(X_p, X_i), X_j \rangle \langle \mu(X_p, X_i), X_j \rangle \right) \\ &= -2 \left(\sum_{pijr} \langle \mu(X_r, X_i), X_j \rangle \langle \lambda(X_p, X_i), X_j \rangle \langle AX_p, X_r \rangle \right. \\ &\quad \left. + \sum_{pijr} \langle \mu(X_p, X_r), X_j \rangle \langle \lambda(X_p, X_i), X_j \rangle \langle AX_i, X_r \rangle \right. \\ &\quad \left. - \sum_{pijr} \langle \mu(X_p, X_i), X_r \rangle \langle \lambda(X_p, X_i), X_j \rangle \langle AX_j, X_r \rangle \right). \end{aligned}$$

By interchanging the indexes i and p in the second line of the last equality, and j and p in the third one, we get that

$$\begin{aligned} f_\mu(A) &= -4 \sum_{prij} \langle \mu(X_r, X_i), X_j \rangle \langle \mu(X_p, X_i), X_j \rangle \langle AX_p, X_r \rangle \\ &\quad + 2 \sum_{prij} \langle \mu(X_i, X_j), X_r \rangle \langle \mu(X_i, X_j), X_p \rangle \langle AX_p, X_r \rangle \\ &= \sum_{pr} 8 \langle R_\mu X_p, X_r \rangle \langle AX_p, X_r \rangle = 8 \text{tr } R_\mu A = 8 \langle R_\mu, A \rangle = 8 \langle p(R_\mu), A \rangle. \end{aligned}$$

Thus, under the identification $\mathfrak{p} \simeq \mathfrak{p}^*$, we have obtained that $f_\mu = 8p(R_\mu)$ and so $H(\mu) = \|f_\mu\|^2 = \text{tr } f_\mu^2 = 64 \text{tr } p(R_\mu)^2$, as it was to be shown. \square

We note that the transformation R_μ , introduced geometrically in Section 3 via the Ricci curvature, also appears in this context (compare with Remark 4.4). For $G = GL(n)$ we have that $H(\mu) = 64F(\mu)$ for all $\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$, where F is the functional studied in Section 4. In this case, $\mathcal{M} = \{0\}$ since the only zero of H (or F) is $\mu = 0$ (recall that $\text{tr } R_\mu = -\frac{1}{4}\|\mu\|^2$), which is in agreement to the fact that $\{0\}$ is the only closed $GL(n)$ -orbit on $\Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$.

According to Theorem 7.1, (i), the orbit $G.\mu$ is closed if and only if a minimal vector μ_0 lies in $G.\mu$. On the other hand, minimal vectors are characterized in Proposition 7.2 via certain property of the transformation R_{μ_0} . We then obtain the following interplay.

Theorem 7.3. *For $\mu \in \mathcal{L}$ and a reductive Lie group $G \subset GL(n)$, the following conditions are equivalent:*

- (i) *The orbit $G.\mu$ is closed.*
- (ii) *The corresponding Lie group G_μ admits a left invariant metric (\cdot, \cdot) such that the curvature operator $R_{(\cdot, \cdot)}$ satisfies $p(R_{(\cdot, \cdot)}) = 0$, where $p: \text{sym}(\mathfrak{g}) \rightarrow \mathfrak{p}$ denotes the orthogonal projection and $L(G) = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition for the Lie algebra $L(G)$ of G .*

Proof. If $G.\mu$ is closed, then by Theorem 7.1, (i) there exists a minimal vector $\mu_0 \in G.\mu$, say $\mu_0 = \varphi.\mu$, and thus $p(R_{\mu_0}) = 0$ (see Proposition 7.2). Since φ^{-1} defines an isometry between $(G_{\varphi.\mu}, \langle \cdot, \cdot \rangle)$ and $(G_\mu, \varphi^{-1}.\langle \cdot, \cdot \rangle)$ (see (4)), we have that

$$R_{\mu, \varphi^{-1}.\langle \cdot, \cdot \rangle} = \varphi^{-1} R_{\varphi.\mu, \langle \cdot, \cdot \rangle} \varphi = \varphi^{-1} R_{\mu_0} \varphi.$$

Thus the left invariant metric $\varphi^{-1}.\langle \cdot, \cdot \rangle$ on G_μ satisfies $p(R_{\varphi^{-1}.\langle \cdot, \cdot \rangle}) = 0$, where now p denotes the orthogonal projection $p: \text{sym}(\mathfrak{g}, \varphi^{-1}.\langle \cdot, \cdot \rangle) \rightarrow \varphi^{-1}\mathfrak{p}\varphi$ and $L(G) = \varphi^{-1}\mathfrak{k}\varphi \oplus \varphi^{-1}\mathfrak{p}\varphi$ is the appropriate Cartan decomposition. This implies that $\varphi^{-1}.\langle \cdot, \cdot \rangle$ is the required left invariant metric.

Conversely, if G_μ admits a metric (\cdot, \cdot) with the properties stated in the theorem, then we can consider \mathcal{M} and all the setup above Proposition 7.2 with respect to (\cdot, \cdot) , thus obtaining that $\mu \in \mathcal{M}$ and so $G.\mu$ is closed. \square

8. Closed $SL(n)$ -orbits on \mathcal{L}

Let \mathcal{L} be the algebraic variety of Lie brackets on a fixed n -dimensional vector space \mathfrak{g} and consider the natural action of $GL(n)$ on \mathcal{L} (see (2)). A $GL(n)$ -orbit on \mathcal{L} can never be closed, unless it is $\{0\}$. Indeed, if $\varphi_t = t^{-1}I$ then when $t \rightarrow 0$, $\lim \varphi_t.\mu = 0$ and so $0 \in \overline{G.\mu}$ for every $\mu \in \mathcal{L}$. A natural question is then what happens if we do not allow the multiplication by a scalar, considering for instance $SL(n)$ -orbits instead of $GL(n)$ -orbits. Under what conditions on μ we would have that $0 \in \overline{SL(n).\mu}$?, when $SL(n).\mu$ will be closed? We will study such a questions applying the results given in Section 7 for $G = SL(n)$ and $V = \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$. Note that $SL(n)$ is the set of real points of $\mathbf{G} = SL(n, \mathbb{C})$ and the induced representation of \mathbf{G} on $\mathbf{V} = V \otimes \mathbb{C}$ is algebraic and defined over \mathbb{R} .

According to Theorem 7.1, (i), the orbit $SL(n).\mu$ is closed if and only if a minimal vector μ_0 lies in $SL(n).\mu$. Recall that $\mathfrak{sl}(n) = \mathfrak{so}(\mathfrak{g}) \oplus \text{sym}_0(\mathfrak{g})$ is a Cartan decomposition, where $\text{sym}_0(\mathfrak{g})$ denotes

the space of traceless symmetric transformations of \mathfrak{g} . Since $\text{sym}(\mathfrak{g}) = \text{sym}_0(\mathfrak{g}) \oplus \mathbb{R}I$ is an orthogonal decomposition, we have that $\mu_0 \in \mathcal{M}$ if and only if $R_{\mu_0} \in \mathbb{R}I$. It follows from Theorem 7.3 the following Riemannian characterization.

Proposition 8.1. *The orbit $SL(n).\mu$ is closed if and only if the corresponding Lie group G_μ admits a left invariant metric (\cdot, \cdot) such that its curvature operator $R_{(\cdot, \cdot)}$ is a multiple of the identity.*

As an immediate consequence, we obtain that if $\mu \in \mathcal{L}$ is nilpotent (and $\mu \neq 0$) then the orbit $SL(n).\mu$ is not closed, and moreover, $0 \in \overline{SL(n).\mu}$. Indeed, R_μ coincides with the Ricci operator when μ is nilpotent, and hence R_μ can never be a multiple of the identity since by a result due to Milnor [27], a non-abelian nilpotent Lie group can never admit an *Einstein* left invariant Riemannian metric. However, the closure $\overline{SL(n).\mu}$ must contain a closed orbit by Theorem 7.1, (ii), and so using that the set of nilpotent Lie brackets is closed, we obtain that $0 \in \overline{SL(n).\mu}$ for any nilpotent $\mu \in \mathcal{L}$.

On the other hand, if $\mu \in \mathcal{L}$ is a compact semi-simple Lie bracket then it is easy to see that $-B_\mu$ defines an Einstein left invariant Riemannian metric (\cdot, \cdot) on G_μ satisfying that $R_{(\cdot, \cdot)}$ is a multiple of the identity (see (5)). By the characterization given above, we deduce that $SL(n).\mu$ is closed. For a non-compact simple $\mu \in \mathcal{L}$, consider a Cartan decomposition $\mathfrak{g} = \mathfrak{k}_\mu \oplus \mathfrak{p}_\mu$ and the inner product (\cdot, \cdot) on \mathfrak{g} defined by $(\cdot, \cdot)|_{\mathfrak{k}_\mu \times \mathfrak{k}_\mu} = -B_\mu$, $(\cdot, \cdot)|_{\mathfrak{p}_\mu \times \mathfrak{p}_\mu} = B_\mu$ and $\langle \mathfrak{k}_\mu, \mathfrak{p}_\mu \rangle_1 = 0$. It is easy to see that the curvature operator $R_{(\cdot, \cdot)}$ of $(G_\mu, (\cdot, \cdot))$ is a multiple of the identity (see (5)), concluding that the orbit $SL(n).\mu$ is closed as well. We now give the general result.

Theorem 8.2. *Let $\mu \in \mathcal{L}$, $\mu \neq 0$.*

- (i) *The orbit $SL(n).\mu$ is closed if and only if μ is semi-simple.*
- (ii) *If μ is not semi-simple then $0 \in \overline{SL(n).\mu}$.*

Proof. (i) We first note that if $SL(n).\mu$ is closed then $\text{tr } D = 0$ for any $D \in \text{Der}(\mu)$. Otherwise, we could consider the one parameter subgroup of automorphisms $\varphi_t = \{e^{tD}\}_{t \in \mathbb{R}} = \{e^{t\frac{\text{tr } D}{n}I} e^{t(D - \frac{\text{tr } D}{n}I)}\}_{t \in \mathbb{R}}$ (assume $\text{tr } D > 0$) and thus

$$0 = \lim_{t \rightarrow -\infty} (e^{-t\frac{\text{tr } D}{n}I}.\mu) = \lim_{t \rightarrow -\infty} e^{t(D - \frac{\text{tr } D}{n}I)} \in \overline{SL(n).\mu},$$

obtaining that $SL(n).\mu$ is not closed. However, we note that such a necessary condition is not sufficient; consider for instance a characteristically nilpotent $\mu \in \mathcal{L}$, that is, $\text{Der}(\mu)$ is nilpotent (see [17]). A weaker necessary condition we have obtained is that if $SL(n).\mu$ is closed then μ is unimodular (i.e., $\text{tr ad}_\mu X = 0$ for any $X \in \mathfrak{g}$). This allows us to apply some ideas and techniques from homogeneous Riemannian geometry due to Dotti [7], used in the study of the Ricci curvature of unimodular Lie groups.

Consider a unimodular non-abelian $\mu \in \mathcal{L}$ such that the orbit $SL(n).\mu$ is closed and assume that $\mu \in \mathcal{M}$, that is, $R_\mu = cI$. Let \mathfrak{s}_μ denote the radical of μ and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}_\mu$ the orthogonal decomposition. We also decompose orthogonally $\mathfrak{s}_\mu = \mathfrak{a} \oplus \mathfrak{n}_\mu = \mathfrak{a} \oplus \mathfrak{v} \oplus \mathfrak{z}_\mu$, where $\mathfrak{n}_\mu = \mu(\mathfrak{s}_\mu, \mathfrak{s}_\mu)$ and \mathfrak{z}_μ is the center of the nilpotent Lie algebra $(\mathfrak{n}_\mu, \mu|_{\mathfrak{n}_\mu \times \mathfrak{n}_\mu})$. Suppose that $\mathfrak{z}_\mu \neq 0$ and consider the basis $\{H_i\}$, $\{A_i\}$, $\{V_i\}$ and $\{Z_i\}$ of \mathfrak{h} , \mathfrak{a} , \mathfrak{v} , \mathfrak{z} respectively. It follows from $R_\mu = cI$ that $\text{tr } R_\mu^2 = c \text{tr } R_\mu = -\frac{c}{4}\|\mu\|^2$ and thus $c < 0$

(note that if $c = 0$ then $\mu = 0$). Therefore, if $\{X_i\} = \{H_i\} \cup \{A_i\} \cup \{V_i\} \cup \{Z_i\}$ then for every $Z \in \mathfrak{z}_\mu$,

$$\begin{aligned} 0 > \langle R_\mu Z, Z \rangle &= -\frac{1}{2} \sum_{ij} \langle \mu(Z, X_i), X_j \rangle^2 + \frac{1}{4} \sum_{ij} \langle \mu(X_i, X_j), Z \rangle^2 \\ &= -\frac{1}{2} \sum_{ij} \langle \mu(Z, H_i), Z_j \rangle^2 - \frac{1}{2} \sum_{ij} \langle \mu(Z, A_i), Z_j \rangle^2 \\ &\quad + \frac{1}{2} \sum_{ij} \langle \mu(Z_i, H_j), Z \rangle^2 + \frac{1}{2} \sum_{ij} \langle \mu(Z_i, A_j), Z \rangle^2 + \beta(Z), \end{aligned}$$

where

$$\beta(Z) = \frac{1}{4} \sum_{ij} \langle \mu(X_i, X_j), Z \rangle^2 \geq 0, \quad X_i, X_j \in \{H_i\} \cup \{A_i\} \cup \{V_i\}$$

(note that both $\text{ad}_\mu \mathfrak{h}$ and $\text{ad}_\mu \mathfrak{a}$ leave invariant \mathfrak{z}_μ). This implies that

$$0 > \sum_k \langle R_\mu Z_k, Z_k \rangle = \sum_k \beta(Z_k) \geq 0,$$

which is a contradiction. Thus \mathfrak{z}_μ has to be $\{0\}$ and hence \mathfrak{s}_μ is abelian. Now, applying the same argument to a non-zero $A \in \mathfrak{s}_\mu$ we also get a contradiction, obtaining that $\mathfrak{s}_\mu = 0$ and so proving that μ is semi-simple, as it was to be shown.

Conversely, if μ is semi-simple, we define on each simple factor the metric given in the paragraph above the theorem, with a suitable scalar multiple, yielding a metric (\cdot, \cdot) on G_μ with $R_{(\cdot, \cdot)}$ being a multiple of the identity. Thus $SL(n) \cdot \mu$ is closed by Proposition 8.1.

(ii) If $\mu \in \mathcal{L}$ is a non-semi-simple Lie bracket, then by part (i) the orbit $SL(n) \cdot \mu$ is not closed. However, there must exist a unique closed orbit $SL(n) \cdot \lambda$ in the closure $\overline{SL(n) \cdot \mu}$ (see Theorem 7.1, (ii)). By part (i) we have that λ is semi-simple or $\lambda = 0$, but if λ is semi-simple then it is well known that $\mathcal{O}(\lambda)$ is open in \mathcal{L} , and so λ can never lie in $\overline{SL(n) \cdot \mu}$. This implies that $\lambda = 0$ and thus $0 \in \overline{SL(n) \cdot \mu}$. \square

9. Closed orbits and Einstein solvmanifolds

In order to vary two-step nilpotent Lie algebras, we consider the vector space $V_{m,n} = \Lambda^2 \mathfrak{v}^* \otimes \mathfrak{z}$ of alternating bilinear maps from $\mathfrak{v} \times \mathfrak{v}$ to \mathfrak{z} , where \mathfrak{v} and \mathfrak{z} are real vector spaces of dimension m and n respectively. Thus each element $\mu \in V_{m,n}$ determines a two-step nilpotent Lie bracket on $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$ by setting $\mu(\mathfrak{g}, \mathfrak{z}) = 0$. Conversely, any two-step nilpotent Lie algebra belongs to $V_{m,n}$ for some $m, n \in \mathbb{N}$. Two elements of $V_{m,n}$ define isomorphic Lie algebras if and only if they are in the same $GL(m) \times GL(n)$ -orbit, where the action of any $(\varphi, \psi) \in GL(m) \times GL(n)$ is given by

$$(\varphi, \psi) \cdot \mu(X, Y) = \psi \mu(\varphi^{-1} X, \varphi^{-1} Y), \quad X, Y \in \mathfrak{v}, \quad \mu \in V_{m,n}.$$

By fixing an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$ satisfying $\langle \mathfrak{v}, \mathfrak{z} \rangle = 0$, we proceed as in Section 3. Thus R_μ coincide with the Ricci operator of μ , and $\mu, \lambda \in V_{m,n}$ are isometric if and only if they belong to the same $O(m) \times O(n)$ -orbit (see [37]).

We also proceed as in Section 7 for the action of $GL(m) \times GL(n)$ on $V_{m,n}$, and arguing as there we arrive at results analogous to Proposition 7.2 and Theorem 7.3. We now apply these results to the action

of the subgroup $SL(m) \times SL(n) \subset GL(m) \times GL(n)$ on $V_{m,n}$. The projection $p: \text{sym}(\mathfrak{v}) \oplus \text{sym}(\mathfrak{z}) \rightarrow \mathfrak{p}$ is the projection relative to the orthogonal decomposition

$$\text{sym}(\mathfrak{v}) \oplus \text{sym}(\mathfrak{z}) = \mathbb{R}I_{\mathfrak{v}} \oplus \text{sym}_0(\mathfrak{v}) \oplus \mathbb{R}I_{\mathfrak{z}} \oplus \text{sym}_0(\mathfrak{z}),$$

where sym_0 denotes the space of traceless symmetric transformations and $I_{\mathfrak{v}}$ and $I_{\mathfrak{z}}$ are the identity maps of \mathfrak{v} and \mathfrak{z} respectively. We then obtain the following interaction, which can also be deduced from [18, Section 6.4]:

Proposition 9.1. *For each $\mu \in V_{m,n}$ the following are equivalent:*

- (i) *The orbit $SL(m) \times SL(n) \cdot \mu$ is closed.*
- (ii) *The Lie group G_{μ} admits a left invariant metric (\cdot, \cdot) such that the curvature operator $R_{(\cdot, \cdot)}$ satisfies that the restrictions $R_{(\cdot, \cdot)}|_{\mathfrak{v}}$ and $R_{(\cdot, \cdot)}|_{\mathfrak{z}}$ are both multiples of the identity map.*

Galitski and Timashev [10] studied the invariant-theoretic aspects of the action of $SL(m) \times SL(n)$ on $V_{m,n}$ in the complex case. They proved that there is a curve μ_t in $V_{5,5}$ and a two-parameter family $\mu_{s,t}$ in $V_{6,3}$ of pairwise non-isomorphic Lie brackets whose $SL(m) \times SL(n)$ -orbits are closed. They are defined by

$$\begin{aligned} \mu_t(X_1, X_2) &= tZ_4, & \mu_t(X_2, X_3) &= tZ_5, & \mu_t(X_3, X_4) &= tZ_1, \\ \mu_t(X_1, X_3) &= Z_2, & \mu_t(X_2, X_4) &= Z_3, & \mu_t(X_3, X_5) &= Z_4, \\ \mu_t(X_1, X_4) &= -Z_5, & \mu_t(X_2, X_5) &= -Z_1, & \mu_t(X_4, X_5) &= tZ_2, \\ \mu_t(X_1, X_5) &= -tZ_3, \end{aligned} \tag{10}$$

for any $1 \leq t < \infty$; and by

$$\begin{aligned} \mu_{s,t}(X_1, X_2) &= tZ_1, & \mu_{s,t}(X_2, X_3) &= -sZ_3, & \mu_{s,t}(X_3, X_6) &= Z_1, \\ \mu_{s,t}(X_1, X_4) &= Z_3, & \mu_{s,t}(X_2, X_5) &= -Z_2, & \mu_{s,t}(X_4, X_5) &= -sZ_1, \\ \mu_{s,t}(X_1, X_6) &= sZ_2, & \mu_{s,t}(X_3, X_4) &= tZ_2, & \mu_{s,t}(X_5, X_6) &= tZ_3, \end{aligned} \tag{11}$$

for any $-\infty < t < 0$, $1 < s \leq 2$.

On the other hand, it is proved in [8] that condition (ii) in Proposition 9.1, is equivalent to the left invariant metric on the solvable Lie group S_{μ} defined as follows being Einstein: extend the inner product to $\mathfrak{s}_{\mu} = \mathbb{R}H \oplus \mathfrak{v} \oplus \mathfrak{z}$ by $\langle H, \mathfrak{v} \oplus \mathfrak{z} \rangle = 0$ and $\langle H, H \rangle = 1$, and define the Lie bracket on \mathfrak{s}_{μ} by

$$[H, X] = X, \quad [H, Z] = 2Z, \quad [\cdot, \cdot]|_{\mathfrak{v} \oplus \mathfrak{z}} = c\mu, \tag{12}$$

where $c = \frac{2}{\|\mu\|} \left(\frac{mn}{m+2n} \right)^{1/2}$. We then obtain that (10) gives rise to a curve S_{μ_t} of pairwise non-isometric 11-dimensional Einstein solvmanifolds, and (11) to a two-parameter family $S_{\mu_{s,t}}$ of pairwise non-isometric 10-dimensional Einstein solvmanifolds. We note that the family $S_{\mu_{st}}$ coincides with the family given in [14, Section 3].

Independently from how these families was found, it would be interesting to note that one can show by a very simple computation that the solvmanifolds S_{μ_t} and $S_{\mu_{s,t}}$ defined by (10), (11) and (12) are Einstein, by using for instance [18, Lemma 4.4]. Therefore, the only result which we finally need from [10] is that (10) and (11) define pairwise non-isomorphic Lie algebras, from where we obtain that the corresponding solvmanifolds are pairwise non-isometric (see [1] or [18, 2.7]). Recall that if they are non-isomorphic as complex Lie algebras, then they are also non-isomorphic as real Lie algebras.

References

- [1] D.V. Alekseevskii, Conjugacy of polar factorizations of Lie groups, *Math. Sb.* 84 (1971) 14–26. English translation: *Math. USSR Sb.* 13, 12–24.
- [2] R. Azencott, E.N. Wilson, Homogeneous manifolds with negative curvature I, *Trans. Amer. Math. Soc.* 215 (1976) 323–362.
- [3] A. Besse, Einstein manifolds, *Ergeb. Math.* 10 (1987).
- [4] D. Burde, C. Steinhoff, Classification of orbit closures of 4-dimensional complex Lie algebras, *J. Algebra* 214 (1999) 729–739.
- [5] H-D. Cao, B. Chow, Recent developments on the Ricci flow, *Bull. Amer. Math. Soc.* 36 (1) (1999) 59–74.
- [6] R. Carles, Y. Diakite, Sur le variétés d’algèbres de Lie de dimension ≤ 7 , *J. Algebra* 91 (1984) 53–63.
- [7] I. Dotti, Ricci curvature of left invariant metrics on solvable unimodular Lie groups, *Math. Z.* 180 (1982) 257–263.
- [8] P. Eberlien, J. Heber, Quarter pinched homogeneous spaces of negative curvature, *Internat. J. Math.* 7 (1996) 441–500.
- [9] D. Friedan, Nonlinear models in $2 + \varepsilon$ dimensions, *Ann. Phys.* 163 (1985) 318–419.
- [10] L. Galitski, D. Timashev, On classification of metabelian Lie algebras, *J. Lie Theory* 9 (1999) 125–156.
- [11] V. Gorbatsevich, On contractions and degeneracy of finite-dimensional algebras, *Sov. Math.* 35 (1991) 19–27.
- [12] V. Gorbatsevich, Anticommutative finite-dimensional algebras of the first three complexity levels, *St. Petersburg Math. J.* 5 (1994) 505–521.
- [13] V. Gorbatsevich, On the level of some solvable Lie algebras, *Sib. Math. J.* 39 (1998) 872–883.
- [14] C. Gordon, M. Kerr, New homogeneous Einstein metrics of negative Ricci curvature, Preprint, 2000.
- [15] M. Goze, Y. Hakimjanov, Nilpotent Lie Algebras, Kluwer Academic, Dordrecht, 1996.
- [16] F. Grunewald, J. O’Halloran, Deformations of Lie algebras, *J. Algebra* 162 (1993) 210–224.
- [17] Y. Hakimjanov, Variété des lois d’algèbres de Lie nilpotentes, *Geom. Dedicata* 40 (1991) 269–295.
- [18] J. Heber, Noncompact homogeneous Einstein spaces, *Invent. Math.* 133 (1998) 279–352.
- [19] E. Heintze, On homogeneous manifolds of negative curvature, *Math. Ann.* 211 (1974) 23–34.
- [20] E. Inonu, E.P. Wigner, *Proc. Natl. Acad. Sci. USA* 39 (510) (1953) (see also 40 (119) (1954)).
- [21] N. Jacobson, *Lie Algebras*, Interscience Publishers, New York, 1962.
- [22] G. Kempf, L. Ness, Lengths of vectors in representation spaces, in: *Lect. Notes in Math.*, Vol. 732, Springer, Berlin, 1978, pp. 233–244.
- [23] F. Kirwan, Momentum maps and reduction in algebraic geometry, *Differential Geom. Appl.* 9 (1998) 135–172.
- [24] J. Lauret, Ricci soliton homogeneous nilmanifolds, *Math. Annalen* 319 (2001) 715–733.
- [25] J. Lauret, Finding Einstein solvmanifolds by a variational method, *Math. Z.*, to appear.
- [26] M. Levy-Nahas, Deformation and contraction of Lie algebras, *J. Math. Phys.* 8 (1967) 1211–1222.
- [27] J. Milnor, Curvature of left-invariant metrics on Lie groups, *Adv. in Math.* 21 (1976) 293–329.
- [28] R.W. Richardson, Deformations of Lie subgroups and the variation of isotropy subgroups, *Acta Math.* 129 (1972) 35–73.
- [29] R.W. Richardson, P.J. Slodowy, Minimum vectors for real reductive algebraic groups, *J. London Math. Soc.* (2) 42 (1990) 409–429.
- [30] E.D. Rodionov, V.V. Slavskii, Curvatures estimations of left invariant Riemannian metrics on three dimensional Lie groups, in: *Proc. of the 7th Int. Conf. “Diff. Geom. and Appl.”*, Brno, Czech Republic, 1998, pp. 111–126.
- [31] E.J. Saletan, Contraction of Lie groups, *J. Math. Phys.* 2 (1961) 1–21.
- [32] D. Schueth, Continuous families of isospectral metrics on simply connected manifolds, *Ann. Math.* 149 (1999) 287–308.
- [33] C. Seeley, Degenerations of 6-dimensional nilpotent Lie algebras over \mathbb{C} , *Comm. Algebra* 18 (10) (1990) 3493–3505.
- [34] C. Seeley, S.S.-T. Yau, Variation of complex structures and variation of Lie algebras, *Invent. Math.* 99 (1990) 545–565.
- [35] M. Vergne, Cohomologie des algèbres de Lie nilpotentes. Application à l’étude de la variété des algèbres de Lie nilpotentes, *Bull. Soc. Math. France* 98 (1970) 81–116.
- [36] C.E. Will, Rank-one Einstein solvmanifolds of dimension 7, *Differential Geom. Appl.*, to appear.
- [37] E. Wilson, Isometry groups on homogeneous nilmanifolds, *Geom. Dedicata* 12 (1982) 337–346.