

NPDE 2024 作业参考

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1 第一次书面作业

1.1 习题 1.1.1

题目： Prove Eqs. (1.1.8) and (1.1.9) for the L_2 scalar product and norm. (λ is a scalar)

$$\begin{aligned}(f, g) &= \overline{(g, f)}, & (f + g, h) &= (f, h) + (g, h), \\ (\lambda f, g) &= \overline{\lambda}(f, g), & (f, \lambda g) &= \lambda(f, g).\end{aligned}\tag{1.1.8}$$

$$\begin{aligned}\|\lambda f\| &= |\lambda| \|f\|, \\ |(f, g)| &\leq \|f\| \cdot \|g\|, \\ \|f + g\| &\leq \|f\| + \|g\|, \\ \left| \|f\| - \|g\| \right| &\leq \|f - g\|\end{aligned}\tag{1.1.9}$$

解答： 这些公式都是显然的，按照定义直接验证即可。下面提供柯西不等式的一种经典证明：不妨设 $\|g\| \neq 0$ ，则有

$$\begin{aligned}0 &\leq \|f - \lambda g\|^2 = (f - \lambda g, f - \lambda g) \\ &= (f, f) - \lambda(f, g) - \overline{\lambda}(g, f) + |\lambda|^2(g, g) \\ &= (f, f) - \frac{\overline{(f, g)}(f, g)}{(g, g)} - \frac{(f, g)(g, f)}{(g, g)} + \frac{|(f, g)|^2}{(g, g)} \quad (\lambda = \frac{\overline{(f, g)}}{(g, g)}) \\ &= (f, f) - \frac{|(f, g)|^2}{(g, g)}\end{aligned}$$

1.2 习题 1.1.2

题目： Let f be a real function with Fourier series

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{w=-\infty}^{\infty} \hat{f}(w) e^{iwx}$$

Prove that

$$S_N = \frac{1}{\sqrt{2\pi}} \sum_{w=-N}^N \hat{f}(w) e^{iwx}$$

is real for all N .

解答： 按照定义有

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} (e^{iwx}, f),$$

利用 f 是实值函数，可以得到

$$\overline{\hat{f}(w)} = \frac{1}{\sqrt{2\pi}} \overline{(e^{iwx}, f)} = \frac{1}{\sqrt{2\pi}} (e^{-iwx}, f) = \hat{f}(-w)$$

得到 $\hat{f}(0) \in \mathbb{R}$, 以及

$$\overline{\hat{f}(w)e^{iwx}} = \overline{\hat{f}(w)}e^{-iwx} = \hat{f}(-w)e^{-iwx}$$

因此

$$S_N = \frac{1}{\sqrt{2\pi}} \left(\hat{f}(0) + \sum_{w=1}^N (\hat{f}(w)e^{iwx} + \overline{\hat{f}(w)e^{iwx}}) \right) \in \mathbb{R}$$

2 第二次书面作业

2.1 习题 1.2.1

题目: Derive estimates for

$$|(D - \frac{\partial^3}{\partial x^3})e^{iwx}|$$

where $D = D_+^3, D_-D_+^2, D_-^2D_+, D_-^3, D_0D_+D_-$.

解答: 以 $D = D_+^3$ 为例,

$$\begin{aligned} h^3 D_+^3 e^{iwx} &= (e^{3iwh} - 3e^{2iwh} + 3e^{iwh} - 1)e^{iwx} \\ &= ((iwh)^3 + \frac{3}{2}(iwh)^4 + O(w^5 h^5))e^{iwx}. \end{aligned}$$

因此 $|(D_+^3 - \frac{\partial^3}{\partial x^3})e^{iwx}| = O(w^4 h)$, 同理可得:

$$\begin{aligned} |(D_-D_+^2 - \frac{\partial^3}{\partial x^3})e^{iwx}| &= O(w^4 h), \\ |(D_-^2D_+ - \frac{\partial^3}{\partial x^3})e^{iwx}| &= O(w^4 h), \\ |(D_-^3 - \frac{\partial^3}{\partial x^3})e^{iwx}| &= O(w^4 h), \\ |(D_0D_+D_- - \frac{\partial^3}{\partial x^3})e^{iwx}| &= O(w^5 h^2). \end{aligned}$$

2.2 习题 1.2.2

题目: The difference operators D_+, D_0 both approximate $\frac{\partial}{\partial x}$, but they have different norms. Explain why this is not a contradiction.

解答: 不矛盾, 因为 D_+, D_0 算子都只是求导算子的逼近, 由泛函分析知识容易知道求导算子不是一个有界算子, 当 h 趋于 0 时 D_+, D_0 算子的算子范数也是趋于无穷的, 当 h 取定时, 由于二者的近似程度不同故算子范数也不同。

2.3 习题 1.5.1

题目： Formulate and prove the generalization of Theorems 1.3.1,1.3.3 in two space dimensions.

解答： Theorem 1.3.1: The interpolation problem(1.3.1) has the unique solution

$$\tilde{u}(w) = \frac{1}{\sqrt{2\pi}}(e^{iwx}, u)_h, |w| \leq \frac{N}{2}.$$

定理 1.3.1 推广到二维情形:

Let u be a 2π -periodic gridfunction and assume that N_1, N_2 are even, there is a unique trigonometric polynomial:

$$Int_{(N_1, N_2)} u(x, y) = \frac{1}{2\pi} \sum_{w_1=-N_1/2}^{N_1/2} \sum_{w_2=-N_2/2}^{N_2/2} \tilde{u}(w_1, w_2) e^{i(w_1 x + w_2 y)},$$

where interpolates u , that is:

$$u_{jk} = Int_{(N_1, N_2)} u(x_j, y_k) = \frac{1}{2\pi} \sum_{w_1=-N_1/2}^{N_1/2} \sum_{w_2=-N_2/2}^{N_2/2} \tilde{u}(w_1, w_2) e^{i(w_1 x_j + w_2 y_k)}. \quad (*)$$

proof: The interpolation problem has a unique solution:

$$\tilde{u}(w_1, w_2) = \frac{1}{2\pi} (e^{i(w_1 x + w_2 y)}, u)_h, |w_1| \leq \frac{N_1}{2}, |w_2| \leq \frac{N_2}{2}.$$

证明: 在 (*) 式两边同时乘以 $e^{i(\mu_1 x + \mu_2 y)} h_x h_y$, 并求和可得:

$$(e^{i(\mu_1 x + \mu_2 y)}, u)_h = \frac{1}{2\pi} \sum_{w_1=-N_1/2}^{N_1/2} \sum_{w_2=-N_2/2}^{N_2/2} (e^{i(\mu_1 x + \mu_2 y)}, e^{i(w_1 x + w_2 y)})_h \tilde{u}(w_1, w_2) = 2\pi \tilde{u}(\mu_1, \mu_2).$$

上式说明了二维三角插值问题解的存在性, 特别的, 取 $u_{jk} = 0$, 由于齐次线性方程组只有零解, 故解是唯一的。

Theorem 1.3.3: Let $Int_N u$ be the interpolant of a gridfunction u . Then:

$$\|Int_N u\|^2 = \sum_{w=-N/2}^{N/2} |\tilde{u}(w)|^2 = \|u\|_h^2, \quad (1.3.4)$$

$$\|D_+^l u\|_h^2 \leq \left\| \frac{d^l}{dx^l} Int_N u \right\|^2 \leq \left(\frac{\pi}{2}\right)^{2l} \|D_+^l u\|_h^2, l = 0, 1, \dots (1.3.5)$$

定理 1.3.3 推广到二维情形:

Let $Int_N u$ be the interpolant of a gridfunction u , $N = (N_1, N_2)$. Then:

$$\|Int_N u\|^2 = \sum_{w_1=-N_1/2}^{N_1/2} \sum_{w_2=-N_2/2}^{N_2/2} |\tilde{u}(w_1, w_2)|^2 = \|u\|_h^2, \quad (**)$$

$$\|D_{+x}^{l_1} D_{+y}^{l_2} u\|_h^2 \leq \left\| \frac{\partial^{l_1+l_2}}{\partial x^{l_1} \partial y^{l_2}} \text{Int}_N u \right\|^2 \leq \left(\frac{\pi}{2}\right)^{2(l_1+l_2)} \|D_{+x}^{l_1} D_{+y}^{l_2} u\|_h^2 \quad (***) .$$

证明: (**) 式由定理 1.3.2 可以直接得到, 对于 (***) 式,

$$\begin{aligned} \|D_{+x}^{l_1} D_{+y}^{l_2} u\|_h^2 &= \|D_{+x}^{l_1} D_{+y}^{l_2} \text{Int}_N u\|_h^2 \\ &= \frac{1}{(2\pi)^2} \left\| \sum_{w_1} \sum_{w_2} \tilde{u}(w_1, w_2) \left(\frac{e^{iw_1 h_x} - 1}{h_x}\right)^{l_1} \left(\frac{e^{iw_2 h_y} - 1}{h_y}\right)^{l_2} e^{i(w_1 x + w_2 y)} \right\|_h^2 \\ &= \sum_{w_1} \sum_{w_2} |\tilde{u}(w_1, w_2)|^2 \left(\frac{e^{iw_1 h_x} - 1}{h_x}\right)^{2l_1} \left(\frac{e^{iw_2 h_y} - 1}{h_y}\right)^{2l_2} \\ &= \sum_{w_1} \sum_{w_2} |\tilde{u}(w_1, w_2)|^2 \left(\frac{2 \sin(w_1 h_x / 2)}{h_x}\right)^{2l_1} \left(\frac{2 \sin(w_2 h_y / 2)}{h_y}\right)^{2l_2} \\ &\leq \sum_{w_1} \sum_{w_2} |\tilde{u}(w_1, w_2)|^2 w_1^{l_1} w_2^{l_2} \\ &= \left\| \frac{\partial^{l_1+l_2}}{\partial x^{l_1} \partial y^{l_2}} \text{Int}_N u \right\|^2. \end{aligned}$$

再利用不等式 $\frac{2|x|}{\pi} \leq |\sin x|$, 可得:

$$\begin{aligned} \|D_{+x}^{l_1} D_{+y}^{l_2} u\|_h^2 &\geq \left(\frac{2}{\pi}\right)^{2(l_1+l_2)} \sum_{w_1} \sum_{w_2} |\tilde{u}(w_1, w_2)|^2 w_1^{2l_1} w_2^{2l_2} \\ &= \left(\frac{2}{\pi}\right)^{2(l_1+l_2)} \left\| \frac{\partial^{l_1+l_2}}{\partial x^{l_1} \partial y^{l_2}} \text{Int}_N u \right\|^2. \end{aligned}$$

故 (***) 即得证。

2.4 习题 1.5.2

题目: Compute $\|D_{+x_j}\|_h, \|D_{-x_j}\|_h, \|D_{0x_j}\|_h, j = 1, 2$, on a rectangular grid with gridsize h_j in the x_j direction, $j = 1, 2$.

解答: 对于矩形网格, $h_j = \frac{2\pi}{N_j + 1}, j = 1, 2$, 算子范数的推导过程和一维类似,

$$\|D_{+x_j} u\|_h = \frac{1}{h_j} \|(E_{x_j}^1 - E_{x_j}^0)u\|_h \leq \frac{2}{h_j} \|u\|_h$$

故有: $\|D_{+x_j}\|_h \leq \frac{2}{h_j}$. 再说明等号可以取到, 特别的, 取 $u_j = (-1)^{j_1+j_2}$, 计算可得:

$$\begin{aligned}\|u\|_h^2 &= (N_1 + 1)(N_2 + 1)h_1h_2, \\ \|D_{+x_j}u\|_h^2 &= \sum_{N_1} \sum_{N_2} ((-1)^{j_1+1+j_2} - (-1)^{j_1+j_2})^2 \frac{h_1h_2}{h_j^2} \\ &= 4(N_1 + 1)(N_2 + 1) \frac{h_1h_2}{h_j^2} \\ &= \frac{4}{h_j^2} \|u\|_h^2.\end{aligned}$$

所以 $\|D_{+x_j}\|_h = \frac{2}{h_j}$. 类似的, 分别取 $u_j = (-1)^{j_1+j_2}, i^{j_1+j_2}$, 可以得到

$$\begin{aligned}\|D_{-x_j}\|_h &= \frac{2}{h_j}, \\ \|D_{0x_j}\|_h &= \frac{1}{h_j}.\end{aligned}$$

2.5 补充作业 1

题目: 试证: 参考书 1 中 P26 的定理 1.3.3 中公式 (1.3.4); 以及当 N 为奇数时, 写出 P26 页相应的定理 1.3.2, 并证明之。

解答: 当 N 为奇数时, 定理 1.3.2 如下:

$$Int_N u^{(j)} = \frac{1}{\sqrt{2\pi}} \sum_{w=-(N+1)/2+1}^{(N+1)/2} \tilde{u}^{(j)}(w) e^{iwx}, j = 1, 2,$$

interpolate the two gridfunctions. Then:

$$(u^{(1)}, u^{(2)}) = \sum_{w=-(N+1)/2+1}^{(N+1)/2} \overline{\tilde{u}^{(1)}(w)} \tilde{u}^{(2)}(w) = (Int_N u^{(1)}, Int_N u^{(2)}).$$

证明:

$$\begin{aligned}(u^{(1)}, u^{(2)}) &= \left(\frac{1}{\sqrt{2\pi}} \sum_{w=-(N+1)/2+1}^{(N+1)/2} \tilde{u}^{(1)}(w) e^{iwx}, \frac{1}{\sqrt{2\pi}} \sum_{w=-(N+1)/2+1}^{(N+1)/2} \tilde{u}^{(2)}(w) e^{iwx} \right) \\ &= \frac{1}{2\pi} \sum_{w=-(N+1)/2+1}^{(N+1)/2} \overline{\tilde{u}^{(1)}(w)} \tilde{u}^{(2)}(w) (e^{iwx}, e^{iwx}) \\ &= \sum_{w=-(N+1)/2+1}^{(N+1)/2} \overline{\tilde{u}^{(1)}(w)} \tilde{u}^{(2)}(w) \\ &= (Int_N u^{(1)}, Int_N u^{(2)}).\end{aligned}$$

3 第一次小测

题目： 证明定理 7.1：若 $\phi(x)$ 、 $\psi(x)$ 分别满足： $\phi(x_j) = u_j$, $\psi(x_j) = v_j$, $j = 0, 1, \dots, N$ 的三角插值：

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \sum_{w=-N/2}^{N/2} \tilde{u}(w) e^{iw x}, \psi(x) = \frac{1}{\sqrt{2\pi}} \sum_{w=-N/2}^{N/2} \tilde{v}(w) e^{iw x}.$$

求证：

$$(u, v)_h = \sum_{w=-N/2}^{N/2} \overline{\tilde{u}(w)} \tilde{v}(w) = (\phi, \psi)$$

解答： 先证明第一个等号，易知

$$\begin{aligned} (u, v)_h &= \sum_{j=0}^N \overline{u_j} v_j h = \sum_{j=0}^N \overline{\phi(x_j)} \psi(x_j) h \\ &= \frac{1}{2\pi} \sum_{j=0}^N \overline{\left(\sum_{w_1} \tilde{u}(w_1) e^{i w_1 x_j} \right)} \left(\sum_{w_2} \tilde{v}(w_2) e^{i w_2 x_j} \right) h \\ &= \frac{1}{2\pi} \sum_{w_1} \sum_{w_2} \overline{\tilde{u}(w_1)} \tilde{v}(w_2) (e^{i w_1 x}, e^{i w_2 x})_h \\ &= \sum_w \overline{\tilde{u}(w)} \tilde{v}(w) \end{aligned}$$

这里省略了 w_1, w_2, w 的求和范围，并且利用了如下的性质

$$(e^{i w_1 x}, e^{i w_2 x})_h = \begin{cases} 2\pi, (w_1 = w_2) \\ 0, (0 < |w_1 - w_2| \leq N) \end{cases}$$

然后证明第二个等号，易知

$$\begin{aligned} (\phi, \psi) &= \frac{1}{2\pi} \left(\sum_w \tilde{u}(w) e^{iw x}, \sum_w \tilde{v}(w) e^{iw x} \right) \\ &= \frac{1}{2\pi} \sum_{w_1} \sum_{w_2} \overline{\tilde{u}(w_1)} \tilde{v}(w_2) (e^{i w_1 x}, e^{i w_2 x}) \\ &= \sum_w \overline{\tilde{u}(w)} \tilde{v}(w) \end{aligned}$$

这里省略了 w_1, w_2, w 的求和范围，并且利用了如下的性质

$$(e^{i w_1 x}, e^{i w_2 x}) = \begin{cases} 2\pi, (w_1 = w_2) \\ 0, (w_1 \neq w_2) \end{cases}$$

4 第三次书面作业

4.1 习题 2.1.1

题目： The convergence of the solutions in Figures 2.1.4 and 2.1.5 is rather slow. Explain why that is so and find which one of the terms I, II, or III is large for this example in the proof of Theorem 2.1.1.

解答： 因为解的光滑性很差，I 占主导。

4.2 习题 2.1.2

题目： Modify the scheme (4.2.1) such that it approximates $u_t = -u_x$. Prove that the conditions (4.2.2) and (4.2.3) are also necessary for stability in this case.

$$v_j^{n+1} = (I + kD_0)v_j^n + \sigma khD_+D_-v_j^n \quad (4.2.1)$$

$$0 < \lambda \leq 2\sigma \leq 1 \quad (4.2.2)$$

$$1 \leq 2\sigma, \quad 2\sigma\lambda \leq 1 \quad (4.2.3)$$

解答： 对于 $u_t = -u_x$ 的差分格式可以写成

$$v_j^{n+1} = (I - kD_0)v_j^n + \sigma khD_+D_-v_j^n$$

将谐波解 $v_j^n = \frac{1}{\sqrt{2\pi}} e^{i\omega x_j} \hat{v}^n$ 代入化简可以得到

$$\hat{Q} = 1 - i\lambda \sin \xi - 4\sigma\lambda \sin^2 \frac{\xi}{2}, \quad \xi = \omega h, \lambda = \frac{k}{h}$$

$$|\hat{Q}|^2 = (1 - 4\sigma\lambda \sin^2 \frac{\xi}{2})^2 + \lambda^2 \sin^2 \xi$$

这与书上的 $|\hat{Q}|^2$ 相同，因此 (2.1.14) 和 (2.1.15) 都是稳定的必要条件。

4.3 习题 2.1.3

题目： Choose σ in Eq.(4.3) such that Q uses only two gridpoints. What the stability condition?

$$v_j^{n+1} = (I + kD_0)v_j^n + \sigma khD_+D_-v_j^n$$

解答： 将原式整理为

$$v_j^{n+1} = (\frac{1}{2} + \sigma)\lambda v_{j+1}^n + (1 - 2\sigma\lambda)v_j^n + (\sigma - \frac{1}{2})\lambda v_{j-1}^n, \quad \lambda = \frac{k}{h}$$

case 1: $\sigma = \frac{1}{2}$

$$\begin{aligned} v_j^{n+1} &= \lambda v_{j+1}^n + (1 - \lambda)v_j^n \\ \hat{Q} &= (1 - \lambda + \lambda \cos \xi) + i\lambda \sin \xi, \quad \xi = \omega h \\ |\hat{Q}|^2 &= 2\lambda(\lambda - 1)(1 - \cos \xi) + 1 \\ &= 1 - 4\lambda(1 - \lambda) \sin^2 \frac{\xi}{2} \end{aligned}$$

case 2: $\sigma = \frac{1}{2\lambda}$

$$\begin{aligned} v_j^{n+1} &= (\frac{1}{2} + \frac{\lambda}{2})v_{j+1}^n + (\frac{1}{2} - \frac{\lambda}{2})v_{j-1}^n \\ \hat{Q} &= \cos \xi + i\lambda \sin \xi \\ |\hat{Q}|^2 &= \cos^2 \xi + \lambda^2 \sin^2 \xi \end{aligned}$$

也可能从一般表达式算出如下非最简的结果

$$|\hat{Q}|^2 = 1 - 4(1 - \lambda^2)(\sin^2 \frac{\xi}{2} - \sin^4 \frac{\xi}{4})$$

在两种情况下都有

$$|\hat{Q}|^2 \leq 1 \iff 0 < \lambda \leq 1$$

4.4 补充题

题目： 针对方程 $u_t + u_x = 0$ ，导出其解的依赖区；其蛙跳格式的数值解的依赖区；以及 CFL 条件。

解答： 对方程 $u_t + u_x = 0$ ，特征线为 $\xi = x - t$ ，沿着特征线值不变。考虑点 $P(x_j, t_{n+1})$ 处的数值解 v_j^{n+1} ，假设其依赖区为 x_* ，则有

$$\xi = x_j - t_{n+1} = x_* - 0 \Rightarrow x_* = x_j - t_{n+1}$$

因此，方程解的依赖区为 $D_p = \{x_*\} = \{x_j - t_{n+1}\}$ 。

蛙跳格式为

$$v_j^{n+1} = v_j^{n-1} + \lambda(v_{j+1}^n - v_{j-1}^n)$$

在第一层使用 FTCS 格式

$$v_j^1 = v_j^0 + \frac{\lambda}{2}(v_{j+1}^0 - v_{j-1}^0)$$

可以推导出数值解 v_j^{n+1} 的依赖区 $N_p = \{(x_{j-n-1}, 0), (x_{j-n}, 0), \dots, (x_{j+n}, 0), (x_{j+n+1}, 0)\}$

CFL 条件:

$$\begin{aligned} D_p \subseteq N_p &\Rightarrow x_{j-n-1} \leq x_j - t_{n+1} \leq x_{j+n+1} \\ &\Rightarrow -(n+1)\Delta x \leq -(n+1)\Delta t \leq (n+1)\Delta x \\ &\Rightarrow 0 < \lambda = \frac{\Delta t}{\Delta x} \leq 1 \end{aligned}$$

注记: 在本课程范围内, 写中心差格式和更复杂格式的数值解依赖区时, 可以简单的不考虑内部缺失的点。

5 第四次书面作业

5.1 习题 2.3.1

题目: Prove that Eq.(2.3.5) is unconditionally stable for $\theta \geq \frac{1}{2}$.

$$(I - \theta k D_0) v_j^{n+1} = (I + (1 - \theta) k D_0) v_j^n, \quad j = 0, 1, \dots, N. \quad (2.3.5)$$

解答: 将谐波解 $v_j^n = \frac{1}{\sqrt{2\pi}} e^{i\omega x_j} \hat{v}^n$ 代入化简可以得到

$$\begin{aligned} \hat{Q} &= \frac{1 + i(1 - \theta)\lambda \sin \xi}{1 - i\theta\lambda \sin \xi} \\ |\hat{Q}|^2 &= \frac{1 + (1 - \theta)^2 \lambda^2 \sin^2 \xi}{1 + \theta^2 \lambda^2 \sin^2 \xi} \end{aligned}$$

其中 $\lambda = \frac{k}{h}, \xi = \omega h$, 当 $\theta \geq \frac{1}{2}$ 时, $(1 - \theta)^2 \leq \theta^2 \Rightarrow |\hat{Q}|^2 \leq 1$, 格式无条件稳定。

5.2 习题 2.4.1

题目: When deriving the order of accuracy, Taylor expansion around some point (x_*, t_*) is used. Prove that (x_*, t_*) can be chosen arbitrarily and, in particular, that it does not have to be a grid point.

解答: 不妨设在 (x_j, t_n) 展开的局部截断误差为

$$T_j^n = f(x_j, t_n)h^p + g(x_j, t_n)k^q + O(h^{p+1} + k^{q+1}) = O(h^p + k^q)$$

其中 h 和 k 分别为空间步长和时间步长, 另取附近的一点 (x_*, t_*) 满足

$$|x_j - x_*| \leq Ch, |t_n - t_*| \leq Ck$$

那么

$$\begin{aligned} f(x_j, t_n) &= f(x_*, t_n) + O(h) = f(x_*, t_*) + O(h) + O(k) \\ g(x_j, t_n) &= g(x_*, t_n) + O(h) = g(x_*, t_*) + O(h) + O(k) \end{aligned}$$

在 (x_*, t_*) 展开的局部截断误差为

$$\begin{aligned} T_{j*}^{n*} &= [f(x_*, t_*) + O(h) + O(k)] h^p + [g(x_*, t_*) + O(h) + O(k)] k^q + O(h^{p+1} + k^{q+1}) \\ &= f(x_*, t_*) h^p + g(x_*, t_*) k^q + O(h^{p+1} k + h^{p+1} + h k^q + k^{q+1}) = O(h^p + k^q) \end{aligned}$$

因此, 挑选某个具体的点 (甚至不要求是格点) 进行 Taylor 展开并不影响最终的结果。

5.3 习题 2.4.2

题目: Prove that the leap-frog scheme and the Crank-Nicholson scheme are accurate of order (2,2). Despite the same order of accuracy, one can expect that one scheme is more accurate than the other. Why is that so?

$$\begin{aligned} \text{CTCS(leap-frog)} : \quad & v_j^{n+1} = v_j^{n-1} + \lambda(v_{j+1}^n - v_{j-1}^n), \quad \lambda = \frac{k}{h} \\ \text{Crank-Nicholson} : \quad & \left(I - \frac{k}{2} D_0\right) v_j^{n+1} = \left(I + \frac{k}{2} D_0\right) v_j^n \end{aligned}$$

解答: CTCS 格式 (leap-frog) 形如:

$$\frac{v_j^{n+1} - v_j^{n-1}}{2\Delta t} = \frac{v_{j+1}^n - v_{j-1}^n}{2\Delta x}$$

在 (x_j, t^n) 展开, 局部截断误差为

$$\begin{aligned} T_j^n &= \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} - \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \\ &= \frac{2\Delta t u_t|_j^n + \frac{\Delta t^3}{3} u_{ttt}|_j^n + O(\Delta t^5)}{2\Delta t} - \frac{2\Delta x u_x|_j^n + \frac{\Delta x^3}{3} u_{xxx}|_j^n + O(\Delta x^5)}{2\Delta x} \\ &= (u_t - u_x)|_j^n + \frac{\Delta t^2}{6} u_{ttt}|_j^n - \frac{\Delta x^2}{6} u_{xxx}|_j^n + O(\Delta t^3 + \Delta x^3) \\ &= \frac{1}{6} (\Delta t^2 - \Delta x^2) u_{xxx}|_j^n + O(\Delta t^4 + \Delta x^4) \\ &= O(\Delta t^2 + \Delta x^2) \end{aligned}$$

Crank-Nicholson 格式形如:

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} = \frac{1}{2} \frac{v_{j+1}^{n+1} - v_{j-1}^{n+1}}{2\Delta x} + \frac{1}{2} \frac{v_{j+1}^n - v_{j-1}^n}{2\Delta x}$$

在 (x_j, t^n) 展开, 易得

$$\begin{aligned}\frac{u_j^{n+1} - u_j^n}{\Delta t} &= u_t|_j^n + \frac{\Delta t}{2} u_{tt}|_j^n + \frac{\Delta t^2}{6} u_{ttt}|_j^n + O(\Delta t^3) \\ \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} &= u_x|_j^n + \frac{\Delta x^2}{6} u_{xxx}|_j^n + O(\Delta x^4) \\ \frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x} &= \left[u_x|_j^n + \Delta t u_{xt}|_j^n + \frac{\Delta t^2}{2} u_{xtt}|_j^n + O(\Delta t^3) \right] \\ &\quad + \frac{\Delta x^2}{6} [u_{xxx}|_j^n + \Delta t u_{xxx t}|_j^n + O(\Delta t^2)] + O(\Delta x^4)\end{aligned}$$

局部截断误差为

$$\begin{aligned}T_j^n &= \frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{1}{2} \frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x} - \frac{1}{2} \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \\ &= \left(u_t|_j^n - \frac{1}{2} u_x|_j^n - \frac{1}{2} u_x|_j^n \right) + \left(\frac{1}{2} u_{tt}|_j^n - \frac{1}{2} u_{xt}|_j^n \right) \Delta t \\ &\quad + \left(\frac{1}{6} u_{ttt}|_j^n - \frac{1}{4} u_{xtt}|_j^n \right) \Delta t^2 - \frac{\Delta x^2}{6} u_{xxx}|_j^n + O(\Delta t^3 + \Delta t \Delta x^2 + \Delta x^3) \\ &= -\frac{1}{6} \left(\frac{1}{2} \Delta t^2 + \Delta x^2 \right) u_{xxx}|_j^n + O(\Delta t^3 + \Delta t \Delta x^2 + \Delta x^3) \\ &= O(\Delta t^2 + \Delta x^2)\end{aligned}$$

对于 $u_t = u_x$ 的求解, 固定 $\lambda = \Delta t / \Delta x$ 后两者的精度都是二阶, 但是通常 CTCS 的结果会更好: CTCS 的误差主项系数更小, 数值解最终的误差相对更小, 两者的比值约为

$$\left| \frac{\lambda^2 - 1}{1 + \frac{1}{2}\lambda^2} \right| < 1$$

在某些情况下, 如果取 $\lambda = 1$ 并采用合适的启动格式, CTCS 的数值解误差可能达到机器精度的量级, 局部截断误差的 Taylor 展开系数全部都抵消了, 但这只是从相容性的角度, 而 $\lambda < 1$ 是 CTCS 的稳定性条件。

Remark: 对于 CN 格式, 由于格式中用到了 6 个格点, 并且具有对称性, 根据上题结论可以选择在 $(x_j, t_{n+1/2})$ 处展开, 得到的局部截断误差主项是相同的, 在不同点处展开带来的影响是高阶无穷小量。

5.4 补充题一

题目: 对于方程 $u_t + au_x = 0$, 基于积分形式构造有限差分格式, 要求时间一阶空间三阶。

解答: 答案不唯一, 这里提供一种思路。任取控制体 $\Omega = [x_1, x_2] \times [t_1, t_2]$ 则

$$\int_{\Omega} u_t dxdt + a \int_{\Omega} u_x dxdt = 0$$

易得

$$\int_{x_1}^{x_2} u(x, t_2) - u(x, t_1) dx + a \int_{t_1}^{t_2} u(x_2, t) - u(x_1, t) dt = 0$$

采用合适的数值积分公式分别处理即可。

尝试一下：直接取 $\Omega = [x_{i-1}, x_{i+1}] \times [t^n, t^{n+1}]$ ，引入下列记号

$$A = \int_{x_{i-1}}^{x_{i+1}} u^{n+1} - u^n dx, B = \int_{t^n}^{t^{n+1}} u_{i+1} - u_{i-1} dt$$

对于 A 采用 simpson 数值积分

$$\int_a^b f(x) dx = \frac{b-a}{6} (f(a) + 4f(\frac{a+b}{2}) + f(b)) - \frac{(b-a)^5}{90} f^{(4)}(\xi)$$

对于 B 则采用单侧的数值积分

$$\int_a^b f(x) dx = (b-a)f(a) + \frac{(b-a)^2}{4} f'(\xi)$$

即

$$A = \frac{2\Delta x}{6} [(u_{i-1}^{n+1} - u_{i-1}^n) + 4(u_i^{n+1} - u_i^n) + (u_{i+1}^{n+1} - u_{i+1}^n)] + O(\Delta t(\Delta x)^5)$$

$$B = \Delta t(u_{i+1}^n - u_{i-1}^n) + O((\Delta t)^2 \Delta x)$$

代入 $A + aB = 0$ 得到

$$\frac{2\Delta x}{6} [(u_{i-1}^{n+1} - u_{i-1}^n) + 4(u_i^{n+1} - u_i^n) + (u_{i+1}^{n+1} - u_{i+1}^n)] + O(\Delta t(\Delta x)^5)$$

$$+ a [\Delta t(u_{i+1}^n - u_{i-1}^n) + O((\Delta t)^2 \Delta x)] = 0$$

整理得到

$$\frac{2}{6\Delta t} [(u_{i-1}^{n+1} - u_{i-1}^n) + 4(u_i^{n+1} - u_i^n) + (u_{i+1}^{n+1} - u_{i+1}^n)] + O(\Delta x^4)$$

$$+ a \left[\frac{(u_{i+1}^n - u_{i-1}^n)}{\Delta x} + O(\Delta t) \right] = 0$$

因此我们可以得到一个局部截断误差为 $O(\Delta t + (\Delta x)^4)$ 的数值格式。(不满足题目要求)

通过这个尝试可知，我们希望构造的格式应当形如

$$A = \dots + O(\Delta t(\Delta x)^4), B = \dots + O((\Delta t)^2 \Delta x)$$

就可以达到时间一阶空间三阶的目的。一种想法就是既然 simpson 积分的阶数太高了，我们可以换一个差一点的数值积分公式，比如积分节点和权重不再对称等等。

例如取 $\Omega = [x_{i-2}, x_{i+2}] \times [t^n, t^{n+1}]$ ，引入下列记号

$$A = \int_{x_{i-2}}^{x_{i+2}} u^{n+1} - u^n dx, B = \int_{t^n}^{t^{n+1}} u_{i+2} - u_{i-2} dt$$

待定系数法构造一个数值积分形如

$$\int_{-2h}^{2h} f(x) dx = 4h[a_1 f(2h) + a_2 f(h) + a_3 f(0) + a_4 f(-h) + a_5 f(-2h)] + Ch^4 f^{(3)}(\xi)$$

对 $f(x) = 1, x, x^2$ 精确成立, 对 $f(x) = x^3$ 不成立。

$$\begin{aligned}\int_{-2h}^{2h} 1 dx &= 4h = 4h(a_1 + a_2 + a_3 + a_4 + a_5) \\ \int_{-2h}^{2h} x dx &= 0 = 4h(2ha_1 + ha_2 - ha_4 - 2ha_5) \\ \int_{-2h}^{2h} x^2 dx &= \frac{16}{3}h^3 = 4h(4h^2a_1 + h^2a_2 + h^2a_4 + 4h^2a_5) \\ \int_{-2h}^{2h} x^3 dx &= 0 \neq 4h(8h^3a_1 + h^3a_2 - h^3a_4 - 8h^3a_5)\end{aligned}$$

解显然不唯一, 例如取 $a_1 = \frac{1}{3}, a_2 = -\frac{1}{3}, a_3 = \frac{2}{3}, a_4 = \frac{1}{3}, a_5 = 0$ 得到如下积分公式

$$\int_{-2h}^{2h} f(x) dx = 4h \left[\frac{1}{3}f(2h) - \frac{1}{3}f(h) + \frac{2}{3}f(0) + \frac{1}{3}f(-h) \right] + Ch^4 f^{(3)}(\xi)$$

因此

$$\begin{aligned}A &= \int_{x_{i-2}}^{x_{i+2}} u^{n+1} - u^n dx \\ &= 4h \left[\frac{1}{3}(u_{i+2}^{n+1} - u_{i+2}^n) - \frac{1}{3}(u_{i+1}^{n+1} - u_{i+1}^n) + \frac{2}{3}(u_i^{n+1} - u_i^n) + \frac{1}{3}(u_{i-1}^{n+1} - u_{i-1}^n) \right] + O(\Delta t(\Delta x)^4)\end{aligned}$$

仍然有

$$B = \Delta t(u_{i+2}^n - u_{i-2}^n) + O((\Delta t)^2 \Delta x)$$

得到

$$\begin{aligned}\frac{4}{\Delta t} \left[\frac{1}{3}(u_{i+2}^{n+1} - u_{i+2}^n) - \frac{1}{3}(u_{i+1}^{n+1} - u_{i+1}^n) + \frac{2}{3}(u_i^{n+1} - u_i^n) + \frac{1}{3}(u_{i-1}^{n+1} - u_{i-1}^n) \right] + O(\Delta x^3) \\ + a \left[\frac{(u_{i+2}^n - u_{i-2}^n)}{\Delta x} + O(\Delta t) \right] = 0\end{aligned}$$

最终的数值格式为

$$\frac{4}{\Delta t} \left[\frac{1}{3}(v_{i+2}^{n+1} - v_{i+2}^n) - \frac{1}{3}(v_{i+1}^{n+1} - v_{i+1}^n) + \frac{2}{3}(v_i^{n+1} - v_i^n) + \frac{1}{3}(v_{i-1}^{n+1} - v_{i-1}^n) \right] + a \frac{(v_{i+2}^n - v_{i-2}^n)}{\Delta x} = 0$$

解答: 取 $\Omega_j^n = [t_n, t_{n+1}] \times [x_{j-1/2}, x_{j+1/2}]$ 为控制区域, 方程两边在控制区域上时空积分可得:

$$\int_{t_n}^{t_{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} u_t dx dt + \int_{t_n}^{t_{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} au_x dx dt = 0.$$

即：

$$\int_{x_{j-1/2}}^{x_{j+1/2}} (u^{n+1} - u^n) dx + \int_{t_n}^{t_{n+1}} a(u_{j+1/2} - u_{j-1/2}) dt = 0.$$

要想得到时间 1 阶, 空间 3 阶的差分格式, 事实上我们对上面两个积分项的精度要求为如下即可：

$$\int_{x_{j-1/2}}^{x_{j+1/2}} (u^{n+1} - u^n) dx = S_1 + O(\Delta x^4 \Delta t + \Delta x \Delta t^2).$$

$$\int_{t_n}^{t_{n+1}} a(u_{j+1/2} - u_{j-1/2}) dt = S_2 + O(\Delta x^4 \Delta t + \Delta x \Delta t^2).$$

S_1, S_2 为相应的数值积分, 根据已有的数值积分公式, 由于时间要求 1 阶, 空间要求 3 阶, 故一个自然的想法是对时间的积分项我们使用端点公式, 对空间的积分项使用 Simpson 公式, 于是可得到如下：

$$\int_{x_{j-1/2}}^{x_{j+1/2}} (u^{n+1} - u^n) dx = \frac{\Delta x ((u_{j+1/2}^{n+1} - u_{j+1/2}^n) + 4(u_j^{n+1} - u_j^n) + (u_{j-1/2}^{n+1} - u_{j-1/2}^n))}{6} + O(\Delta x^5 \Delta t),$$

$$\int_{t_n}^{t_{n+1}} a(u_{j+1/2} - u_{j-1/2}) dt = \Delta t a(u_{j+1/2}^n - u_{j-1/2}^n) + O(\Delta t^2 \Delta x).$$

注意: 若直接用上面的近似并不能得到相应的差分格式, 原因在于差分格式中只存在整格点值或半格点 (这取决于网格划分), 并且误差为 $O(\Delta t + \Delta x^4)$, 题目要求的是 $O(\Delta t + \Delta x^3)$ 。

因此, 我们需要将半格点的值用整格点替换, 考虑在整格点处 Taylor 展开, 可得如下：

$$u_{j+1/2}^n + u_{j-1/2}^n = 2u_j^n + \frac{\Delta x^2}{4} u_{xx}|_j^n + O(\Delta x^4),$$

$$u_{j+1/2}^n - u_{j-1/2}^n = \Delta x u_x|_j^n + \frac{\Delta x^3}{24} u_{xxx}|_j^n + O(\Delta x^5).$$

代入可得：

$$\begin{aligned} \int_{x_{j-1/2}}^{x_{j+1/2}} (u^{n+1} - u^n) dx &= \frac{\Delta x ((u_{j+1/2}^{n+1} - u_{j+1/2}^n) + 4(u_j^{n+1} - u_j^n) + (u_{j-1/2}^{n+1} - u_{j-1/2}^n))}{6} + O(\Delta x^5 \Delta t) \\ &= \frac{2}{3} \Delta x (u_j^{n+1} - u_j^n) + \frac{1}{6} \Delta x (u_{j+1/2}^{n+1} + u_{j-1/2}^{n+1}) - \frac{1}{6} \Delta x (u_{j+1/2}^n + u_{j-1/2}^n) + O(\Delta x^5 \Delta t) \\ &= \frac{2}{3} \Delta x (u_j^{n+1} - u_j^n) + \frac{1}{6} \Delta x (2u_j^{n+1} + \frac{\Delta x^2}{4} u_{xx}|_j^{n+1} + O(\Delta x^4)) \\ &\quad - \frac{1}{6} \Delta x (2u_j^n + \frac{\Delta x}{4} u_{xx}|_j^n + O(\Delta x^4)) + O(\Delta x^5 \Delta t) \\ &= \Delta x (u_j^{n+1} - u_j^n) + \frac{1}{24} \Delta x^3 (u_{xx}|_j^{n+1} - u_{xx}|_j^n) + O(\Delta x^5 \Delta t) \\ \int_{t_n}^{t_{n+1}} a(u_{j+1/2} - u_{j-1/2}) dt &= \Delta t a(u_{j+1/2}^n - u_{j-1/2}^n) + O(\Delta t^2 \Delta x) \\ &= \Delta t a(\Delta x u_x|_j^n + \frac{\Delta x^3}{24} u_{xxx}|_j^n) + O(\Delta x^5 \Delta t + \Delta t^2 \Delta x) \end{aligned}$$

由如下差分近似：

$$\Delta x u_x|_j = \frac{-u_{j+2} + 3u_{j+1} - u_j - u_{j-1}}{3} + O(\Delta x^4),$$

$$\Delta x^2 u_{xx}|_j = \Delta x^2 D_+ D_- u_j + O(\Delta x^4),$$

$$\Delta x^3 u_{xxx}|_j = \Delta x^3 D_+ D_0 D_- u_j + O(\Delta x^5).$$

故有：

$$\begin{aligned} \int_{x_{j-1/2}}^{x_{j+1/2}} (u^{n+1} - u^n) dx &= \Delta x (u_j^{n+1} - u_j^n) + \frac{1}{24} \Delta x^3 (u_{xx}|_j^{n+1} - u_{xx}|_j^n) + O(\Delta x^5 \Delta t) \\ &= \Delta x (u_j^{n+1} - u_j^n) + \frac{1}{24} \Delta x^3 (D_+ D_- u_j^{n+1} - D_+ D_- u_j^n) + O(\Delta x^5 \Delta t). \end{aligned}$$

$$\begin{aligned} \int_{t_n}^{t_{n+1}} a(u_{j+1/2} - u_{j-1/2}) dt &= \Delta t a(\Delta x u_x|_j^n + \frac{\Delta x^3}{24} u_{xxx}|_j^n) + O(\Delta x^5 \Delta t + \Delta t^2 \Delta x) \\ &= \Delta t a(\frac{-u_{j+2}^n + 3u_{j+1}^n - u_j^n - u_{j-1}^n}{3} + O(\Delta x^4) + \frac{1}{24} \Delta x^3 D_+ D_0 D_- u_j^n + O(\Delta x^5)) \\ &\quad + O(\Delta x^5 \Delta t + \Delta t^2 \Delta x) \\ &= \Delta t a(\frac{-u_{j+2}^n + 3u_{j+1}^n - u_j^n - u_{j-1}^n}{3} + \frac{1}{24} \Delta x^3 D_+ D_0 D_- u_j^n) + O(\Delta x^4 \Delta t + \Delta t^2 \Delta x) \end{aligned}$$

因此得到时间一阶，空间三阶的差分格式：

$$\Delta x (u_j^{n+1} - u_j^n) + \frac{1}{24} \Delta x^3 (D_+ D_- u_j^{n+1} - D_+ D_- u_j^n) + \Delta t a(\frac{-u_{j+2}^n + 3u_{j+1}^n - u_j^n - u_{j-1}^n}{3} + \frac{1}{24} \Delta x^3 D_+ D_0 D_- u_j^n) = 0$$

即：

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{1}{24} \Delta x^2 \frac{D_+ D_- u_j^{n+1} - D_+ D_- u_j^n}{\Delta t} + a(\frac{-u_{j+2}^n + 3u_{j+1}^n - u_j^n - u_{j-1}^n}{3 \Delta x} + \frac{1}{24} \Delta x^2 D_+ D_0 D_- u_j^n) = 0.$$

5.5 补充题二

题目： 试构造： $U_t + AU_x = 0$ 的迎风格式：其中 $U = (u, v)^T$,

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

解答： 容易知道，存在矩阵可逆矩阵 S ，使得 $S^{-1}AS = \Lambda$,

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

,

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

令 $W = SU$, 可得:

$$W_t + \Lambda W_x = 0.$$

故迎风格式为:

$$W_j^{n+1} = W_j^n - \frac{r}{2}\Lambda(W_{j+1}^n - W_{j-1}^n) + \frac{r}{2}|\Lambda|(W_{j+1}^n - 2W_j^n + W_{j-1}^n).$$

或者:

$$V_j^{n+1} = V_j^n - \frac{r}{2}A(V_{j+1}^n - V_{j-1}^n) + \frac{r}{2}|A|(V_{j+1}^n - 2V_j^n + V_{j-1}^n).$$

其中 $|A| = S^{-1}|\Lambda|S$.

6 第五次书面作业

6.1 习题 2.5.2

题目: Prove that the θ scheme is unconditionally stable for $\theta \geq \frac{1}{2}$.

$$(I - \theta k D_+ D_-) v_j^{n+1} = (I + (1 - \theta) k D_+ D_-) v_j^n, \theta \in [0, 1]$$

解答:

$$v_j^n = \frac{1}{\sqrt{2\pi}} \hat{v}^n(w) e^{iwx_j}$$

代入即可得到

$$(1 - \lambda\theta(e^{iwh} - 2 + e^{-iwh}))\hat{v}^{n+1} = (1 + \lambda(1 - \theta)(e^{iwh} - 2 + e^{-iwh}))\hat{v}^n, \lambda = \frac{k}{h^2}$$

记 $\xi = wh$ 整理得

$$(1 + 4\lambda\theta \sin^2(\frac{\xi}{2}))\hat{v}^{n+1} = (1 - 4\lambda(1 - \theta) \sin^2(\frac{\xi}{2}))\hat{v}^n$$

因此

$$\hat{Q} = \frac{1 - 4\lambda(1 - \theta) \sin^2(\frac{\xi}{2})}{1 + 4\lambda\theta \sin^2(\frac{\xi}{2})} = 1 - \frac{4\lambda \sin^2(\frac{\xi}{2})}{1 + 4\lambda\theta \sin^2(\frac{\xi}{2})}$$

要求 $-1 \leq \hat{Q} \leq 1$, 显然有 $\hat{Q} \leq 1$, 对于 $-1 \leq \hat{Q}$ 等价于

$$2 \left(1 + 4\lambda\theta \sin^2(\frac{\xi}{2}) \right) \geq 4\lambda \sin^2(\frac{\xi}{2})$$

$$2 + (2\theta - 1)4\lambda \sin^2(\frac{\xi}{2}) \geq 0$$

当 $1 \geq \theta \geq \frac{1}{2}$ 时, $|\hat{Q}| \leq 1$ 无条件稳定, 证毕。

6.2 习题 2.5.3

题目： Derive the truncation error for the backward Euler and the Crank-Nicholson methods applied to $u_t = u_{xx}$. Prove that it is $\Theta(h^2 + k)$ and $\Theta(h^2 + k^2)$, respectively. Despite this fact, at certain times the backward Euler method is more accurate for the example computed in this section. Explain this paradox.

解答： Backward Euler:

$$\begin{aligned}\frac{v_j^{n+1} - v_j^n}{k} &= \frac{v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}}{h^2} \\ \frac{u_j^{n+1} - u_j^n}{k} &= (u_t)_j^{n+1} - \frac{k}{2}(u_{tt})_j^{n+1} + O(k^2) \\ \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2} &= (u_{xx})_j^{n+1} + \frac{h^2}{12}(u_{xxxx})_j^{n+1} + O(h^4)\end{aligned}$$

因此截断误差为

$$\begin{aligned}T_j^{n+1} &= \frac{u_j^{n+1} - u_j^n}{k} - \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2} \\ &= (u_t)_j^{n+1} - (u_{xx})_j^{n+1} - \frac{k}{2}(u_{tt})_j^{n+1} + \frac{h^2}{12}(u_{xxxx})_j^{n+1} + O(k^2 + h^4) \\ &= O(k + h^2)\end{aligned}$$

C-N 格式:

$$\begin{aligned}\frac{v_j^{n+1} - v_j^n}{k} &= \frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{2h^2} + \frac{v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}}{2h^2} \\ \frac{u_j^{n+1} - u_j^n}{k} &= (u_t)_j^{n+\frac{1}{2}} + \frac{k^2}{24}(u_{ttt})_j^{n+\frac{1}{2}} + O(k^4) \\ \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2h^2} &= \frac{1}{2}(u_{xx})_j^n + \frac{h^2}{24}(u_{xxxx})_j^n + \frac{h^4}{720}(u_{xxxxxx})_j^n + O(h^6) \\ \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{2h^2} &= \frac{1}{2}(u_{xx})_j^{n+1} + \frac{h^2}{24}(u_{xxxx})_j^{n+1} + \frac{h^4}{720}(u_{xxxxxx})_j^{n+1} + O(h^6)\end{aligned}$$

对于空间的两项在时间中间层 $t^{n+1/2}$ 处分析可得

$$\begin{aligned}\frac{1}{2}(u_{xx})_j^n + \frac{1}{2}(u_{xx})_j^{n+1} &= (u_{xx})_j^{n+\frac{1}{2}} + \frac{k^2}{8}(u_{ttxx})_j^{n+\frac{1}{2}} + O(k^4) \\ \frac{h^2}{24}(u_{xxxx})_j^n + \frac{h^2}{24}(u_{xxxx})_j^{n+1} &= \frac{h^2}{12}(u_{xxxx})_j^{n+\frac{1}{2}} + O(h^2k^2) \\ \frac{h^4}{720}(u_{xxxxxx})_j^n + \frac{h^4}{720}(u_{xxxxxx})_j^{n+1} &= \frac{h^4}{360}(u_{xxxxxx})_j^{n+\frac{1}{2}} + O(h^4k^2)\end{aligned}$$

因此截断误差为

$$\begin{aligned}
T_j^{n+\frac{1}{2}} &= \frac{u_j^{n+1} - u_j^n}{k} - \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2h^2} - \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{2h^2} \\
&= (u_t)_j^{n+\frac{1}{2}} - (u_{xx})_j^{n+\frac{1}{2}} + \left[\frac{k^2}{24} (u_{ttt})_j^{n+\frac{1}{2}} - \frac{k^2}{8} (u_{tttx})_j^{n+\frac{1}{2}} - \frac{h^4}{360} (u_{xxxxx})_j^{n+\frac{1}{2}} \right] \\
&\quad + \frac{h^2}{12} (u_{xxxx})_j^{n+\frac{1}{2}} + O(k^2 h^2 + h^4 k^2 + k^4) \\
&= O(k^2 + h^2)
\end{aligned}$$

C-N 格式在时间上的阶比向后欧拉高，这无法说明在特定时刻 C-N 格式的数值误差会更小，只能说明随着时间步的缩小，C-N 格式在时间上的误差的下降速率比向后欧拉更快。

6.3 补充题 1

题目：针对方程 $u_t = u_{xx} + f(x, t)$, $(x, t) \in \bar{D} = [0, 1] \times [0, T]$ 的积分形式，构造以格点处的函数为未知数的有限差分格式，并导出其局部截断误差。

常用的数值积分公式

$$\begin{aligned}
\int_a^b f(x) dx &= (b-a)f(a) + \frac{(b-a)^2}{4} f'(\xi) \\
\int_a^b f(x) dx &= \frac{(b-a)}{2} (f(a) + f(b)) - \frac{(b-a)^3}{12} f''(\xi) \\
\int_a^b f(x) dx &= (b-a)f\left(\frac{a+b}{2}\right) + \frac{(b-a)^3}{24} f''(\xi)
\end{aligned}$$

解答：（答案不唯一）取控制体 $\Omega_j^n = [x_{j-1/2}, x_{j+1/2}] \times [t_n, t_{n+1}]$ ，对方程进行积分得到

$$\int_{x_{j-1/2}}^{x_{j+1/2}} u^{n+1} - u^n dx = \int_{t_n}^{t_{n+1}} ((u_x)_{j+1/2} - (u_x)_{j-1/2}) dt + \int_{t_n}^{t_{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} f(x, t) dx dt.$$

对于第一项的处理

$$\int_{x_{j-1/2}}^{x_{j+1/2}} u^{n+1} - u^n dx = \Delta x (u_j^{n+1} - u_j^n) + O(\Delta x^3 \Delta t).$$

对于第二项的处理

$$\int_{t_n}^{t_{n+1}} ((u_x)_{j+1/2} - (u_x)_{j-1/2}) dt = \Delta t \left(u_x|_{j+1/2}^n - u_x|_{j-1/2}^n \right) + O(\Delta t^2 \Delta x) \quad (6.3.1)$$

注意到

$$\begin{aligned}
u_{j+1} - u_j &= \Delta x u_x|_{j+1/2} + \frac{\Delta x^3}{24} u_{xxx}|_{j+1/2} + O(\Delta x^5) \\
u_j - u_{j-1} &= \Delta x u_x|_{j-1/2} + \frac{\Delta x^3}{24} u_{xxx}|_{j-1/2} + O(\Delta x^5)
\end{aligned}$$

相减得到

$$u_{j+1} - 2u_j + u_{i-1} = \Delta x \left(u_x|_{j+1/2}^n - u_x|_{j-1/2}^n \right) + O(\Delta x^4)$$

代入 (6.3.1) 可得

$$\begin{aligned} \int_{t_n}^{t_{n+1}} ((u_x)_{j+1/2} - (u_x)_{j-1/2}) dt &= \Delta t \left(u_x|_{j+1/2}^n - u_x|_{j-1/2}^n \right) + O(\Delta t^2 \Delta x) \\ &= \frac{\Delta t}{\Delta x} (u_{j+1}^n - 2u_j^n + u_{i-1}^n) + O(\Delta t^2 \Delta x) + O(\Delta t \Delta x^3). \end{aligned}$$

最后，对于源项的处理是容易的

$$\begin{aligned} \int_{t_n}^{t_{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} f(x, t) dx dt &= \int_{t_n}^{t_{n+1}} \Delta x f(x_j, t) + O(\Delta x^3) dt \\ &= \Delta t \Delta x f(x_j, t_n) + O(\Delta t \Delta x^3) + O(\Delta t^3 \Delta x). \end{aligned}$$

综上可得

$$\begin{aligned} \Delta x (u_j^{n+1} - u_j^n) + O(\Delta x^3 \Delta t) &= \frac{\Delta t}{\Delta x} (u_{j+1}^n - 2u_j^n + u_{i-1}^n) + O(\Delta t^2 \Delta x) + O(\Delta t \Delta x^3) \\ &\quad + \Delta t \Delta x f(x_j, t_n) + O(\Delta t \Delta x^3) + O(\Delta t^3 \Delta x) \\ \frac{u_j^{n+1} - u_j^n}{\Delta t} + O(\Delta x^2) &= \frac{u_{j+1}^n - 2u_j^n + u_{i-1}^n}{\Delta x^2} + O(\Delta t) + O(\Delta x^2) \\ &\quad + f(x_j, t_n) + O(\Delta x^2) + O(\Delta t^2) \end{aligned}$$

因此对应的有限差分格式为

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} = \frac{v_{j+1}^n - 2v_j^n + v_{i-1}^n}{\Delta x^2} + f(x_j, t_n)$$

局部截断误差为 $O(\Delta t + \Delta x^2)$ 。

6.4 补充题 2

题目：针对方程 $u_t = u_{xx}$ ，基于其在控制体 $\Omega_j^n = [t_{n-1}, t_{n+1}] \times [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ 上的积分形式，构造以网格平均为未知数的有限差分格式，并给出精度。

解答：（答案不唯一）对方程进行积分可以得到：

$$\int_{t_{n-1}}^{t_{n+1}} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_t dx dt = \int_{t_{n-1}}^{t_{n+1}} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_{xx} dx dt$$

左端项可以化简得：

$$\begin{aligned}
\int_{t_{n-1}}^{t_{n+1}} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_t \, dx dt &= \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \int_{t_{n-1}}^{t_{n+1}} u_t \, dt dx \\
&= \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (u^{n+1} - u^{n-1}) \, dx \\
&= \Delta x (\bar{u}_j^{n+1} - \bar{u}_j^{n-1})
\end{aligned}$$

右端项可以化简得到：

$$\int_{t_{n-1}}^{t_{n+1}} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_{xx} \, dx dt = \int_{t_{n-1}}^{t_{n+1}} ((u_x)_{j+\frac{1}{2}} - (u_x)_{j-\frac{1}{2}}) \, dt \quad (6.4.1)$$

为了得到 $(u_x)_{j+\frac{1}{2}} - (u_x)_{j-\frac{1}{2}}$ 的近似，我们进行如下操作：

$$\begin{aligned}
\bar{u}_{j+1} - \bar{u}_j &= \frac{1}{\Delta x} \left(\int_{x_{j+\frac{1}{2}}}^{x_{j+\frac{3}{2}}} u(x, t) \, dx - \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t) \, dx \right) \\
&= \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (u(x + \Delta x, t) - u(x, t)) \, dx \\
&= \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \left(\Delta x u_x + \frac{\Delta x^2}{2} u_{xx} + \frac{\Delta x^3}{6} u_{xxx} + \frac{\Delta x^4}{24} u_{xxxx} + O(\Delta x^5) \right) \, dx \\
&= (u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}) + \frac{\Delta x}{2} [(u_x)_{j+\frac{1}{2}} - (u_x)_{j-\frac{1}{2}}] + \frac{\Delta x^2}{6} [(u_{xx})_{j+\frac{1}{2}} - (u_{xx})_{j-\frac{1}{2}}] \\
&\quad + \frac{\Delta x^3}{24} [(u_{xxx})_{j+\frac{1}{2}} - (u_{xxx})_{j-\frac{1}{2}}] + O(\Delta x^5)
\end{aligned}$$

同理可得：

$$\begin{aligned}
\bar{u}_j - \bar{u}_{j-1} &= (u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}) - \frac{\Delta x}{2} [(u_x)_{j+\frac{1}{2}} - (u_x)_{j-\frac{1}{2}}] + \frac{\Delta x^2}{6} [(u_{xx})_{j+\frac{1}{2}} - (u_{xx})_{j-\frac{1}{2}}] \\
&\quad - \frac{\Delta x^3}{24} [(u_{xxx})_{j+\frac{1}{2}} - (u_{xxx})_{j-\frac{1}{2}}] + O(\Delta x^5)
\end{aligned}$$

两式相减得：

$$\bar{u}_{j+1} - 2\bar{u}_j + \bar{u}_{j-1} = \Delta x [(u_x)_{j+\frac{1}{2}} - (u_x)_{j-\frac{1}{2}}] + O(\Delta x^4) \quad (6.4.2)$$

将 (6.4.2) 代入 (6.4.1)，使用中点公式得：

$$\begin{aligned}
\int_{t_{n-1}}^{t_{n+1}} ((u_x)_{j+\frac{1}{2}} - (u_x)_{j-\frac{1}{2}}) \, dt &= 2\Delta t ((u_x)_{j+\frac{1}{2}}^n - (u_x)_{j-\frac{1}{2}}^n) + O(\Delta x \Delta t^3) \\
&= \frac{2\Delta t}{\Delta x} (\bar{u}_{j+1}^n - 2\bar{u}_j^n + \bar{u}_{j-1}^n) + O(\Delta x^3 \Delta t) + O(\Delta x \Delta t^3)
\end{aligned}$$

综上所述可得

$$\frac{\bar{u}_j^{n+1} - \bar{u}_j^{n-1}}{2\Delta t} = \frac{(\bar{u}_{j+1}^n - 2\bar{u}_j^n + \bar{u}_{j-1}^n)}{\Delta x^2} + O(\Delta x^2 + \Delta t^2)$$

有限差分格式为：

$$\frac{\bar{v}_j^{n+1} - \bar{v}_j^{n-1}}{2\Delta t} = \frac{(\bar{v}_{j+1}^n - 2\bar{v}_j^n + \bar{v}_{j-1}^n)}{\Delta x^2}$$

精度为时间 2 阶，空间 2 阶。

7 第六次书面作业

7.1 习题 4.1.1

题目： Suppose that one wants to solve the problem $u_t = u_{xx} + 100u$ with initial data $u(x, 0) = e^{iwx} \hat{f}(w)$ for $0 \leq t \leq 2$, and will allow 1% relative error in the solution. Give a bound for the permissible rounding errors.

解答： 参考课本 P106 的误差分析和 P112 的适定性分析，记 $\alpha = 100$, $\varepsilon = 0.01$ 。

首先推导关于适定性的不等式，对于谐波解

$$u(x, t) = e^{iwx} \hat{u}(w, t), u(x, 0) = f(x) = e^{iwx} \hat{f}(w)$$

代入方程得到

$$\begin{cases} \hat{u}_t = -w^2 \hat{u} + \alpha \hat{u}, \\ \hat{u}(w, 0) = \hat{f}(w) \end{cases} \Rightarrow \hat{u}(w, t) = e^{(-w^2 + \alpha)t} \hat{f}(w)$$

对于一般的初值

$$u(x, 0) = \frac{1}{\sqrt{2\pi}} \sum_{w=-\infty}^{\infty} e^{iwx} \hat{f}(w)$$

也有

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \sum_{w=-\infty}^{\infty} e^{(-w^2 + \alpha)t} e^{iwx} \hat{f}(w)$$

因此

$$\|u(\cdot, t)\|^2 = \sum_{w=-\infty}^{\infty} |\hat{u}(w, t)|^2 \leq \sum_{w=-\infty}^{\infty} e^{2(-w^2 + \alpha)t} |\hat{f}(w)|^2 \leq e^{2\alpha t} \|f\|^2$$

然后进行误差分析：对初值 $u(x, 0) = f(x)$ 施加扰动变为

$$\tilde{u}(x, 0) = f(x) + \delta g(x)$$

其中 $0 < \delta \ll 1$ 以及 $\|g\| = 1$ ，以此初值可以得到另一个解 $\tilde{u}(x, t)$ ，两者的差值记作 $w = \tilde{u} - u$ ，那么 w 即为如下方程的解

$$\begin{cases} w_t = w_{xx} + \alpha w \\ w(x, 0) = \delta g(x) \end{cases}$$

由适定性分析得到

$$\|\tilde{u}(\cdot, t) - u(\cdot, t)\| = \|w(\cdot, t)\| \leq \delta e^{\alpha t} \|g\| = \delta e^{\alpha t}$$

再根据定义 $\delta = \|\tilde{u}(\cdot, 0) - u(\cdot, 0)\|$, 得到

$$\|\tilde{u}(\cdot, t) - u(\cdot, t)\| \leq e^{\alpha t} \|\tilde{u}(\cdot, 0) - u(\cdot, 0)\|$$

这个不等式表明: 完全由初值误差所引起的, 在 t 时刻的精确解误差, 可以通过因子 $e^{\alpha t}$ 控制。在 t 时刻的精确解相对误差, 满足如下估计

$$\frac{\|\tilde{u}(\cdot, t) - u(\cdot, t)\|}{\|u(\cdot, t)\|} \leq e^{\alpha t} \frac{\|\tilde{u}(\cdot, 0) - u(\cdot, 0)\|}{\|u(\cdot, 0)\|}$$

若满足下式, 则可以保证相对误差不超过 ε

$$\frac{\|\tilde{u}(\cdot, t) - u(\cdot, t)\|}{\|u(\cdot, t)\|} \leq e^{\alpha t} \frac{\|\tilde{u}(\cdot, 0) - u(\cdot, 0)\|}{\|u(\cdot, 0)\|} \leq \varepsilon$$

最终得到对于初值误差的要求

$$\|\tilde{u}(\cdot, 0) - u(\cdot, 0)\| \leq \varepsilon e^{-\alpha t} \|u(\cdot, t)\|$$

注记:

1. 本题所有的适定性和误差分析都仅仅是针对方程自身, 不涉及任何的数值格式;
2. 取 $\|u(\cdot, t)\| \leq e^{\alpha t} \|f\|$ 代入上式, 相当于放松了约束, 得到的只是必要条件

$$\|\tilde{u}(\cdot, 0) - u(\cdot, 0)\| \leq \varepsilon \|u(\cdot, 0)\| = \varepsilon \|f\|$$

7.2 习题 4.2.1

题目: Consider the differential equation

$$\frac{\partial u}{\partial t} = \sum_{j=0}^4 a_j \frac{\partial^j u}{\partial x^j}$$

Derive the condition for well-posedness corresponding to condition in Theorem 4.2.1. Is it true that the problem is always well posed if $\operatorname{Re} a_4 < 0$? ($a_j = \operatorname{Re} a_j + i \operatorname{Im} a_j \in \mathbb{C}$)

解答: 考虑谐波解

$$u(x, t) = \frac{1}{\sqrt{2\pi}} e^{iwx} \hat{u}(w, t), u(x, 0) = f(x) = \frac{1}{\sqrt{2\pi}} e^{iwx} \hat{f}(w)$$

代入方程得到

$$\begin{cases} \hat{u}_t = \sum_{j=0}^4 a_j (iw)^j \hat{u} = \kappa \hat{u}, \\ \hat{u}(w, 0) = \hat{f}(w) \end{cases}$$

其中记 $\kappa = \sum_{j=0}^4 a_j (iw)^j$, 得到

$$\hat{u}(w, t) = e^{\kappa t} \hat{u}(w, 0) = e^{\kappa t} \hat{f}(w)$$

因此

$$\|u(\cdot, t)\|^2 = |\hat{u}(w, t)|^2 = e^{2(\operatorname{Re} \kappa)t} |\hat{f}(w)|^2 = e^{2(\operatorname{Re} \kappa)t} \|f\|^2$$

适中等价于存在 $\alpha \in \mathbb{R}$, 使得

$$\operatorname{Re} \kappa = (\operatorname{Re} a_4)w^4 + (\operatorname{Im} a_3)w^3 - (\operatorname{Re} a_2)w^2 - (\operatorname{Im} a_1)w + \operatorname{Re} a_0 \leq \alpha, \forall w \in \mathbb{R}$$

易知, 对于实系数的一元四次多项式 $p(w)$, 若首项系数严格小于 0, 则一定存在有限上界 $p(w) \leq \alpha < +\infty$. 因此

$$\text{well-posed} \iff \operatorname{Re} \kappa \leq \alpha \iff \operatorname{Re} a_4 < 0$$

注记:

1. 这里 $\operatorname{Re} a_4 < 0$ 是充分不必要条件, 如果 $\operatorname{Re} a_4 = 0$, 还需要讨论低次项系数的影响. 对于系数而言, 充要条件为以下任一条件成立
 - (a) $\operatorname{Re} a_4 < 0$;
 - (b) $\operatorname{Re} a_4 = \operatorname{Im} a_3 = 0$, $\operatorname{Re} a_2 > 0$;
 - (c) $\operatorname{Re} a_4 = \operatorname{Im} a_3 = 0 = \operatorname{Re} a_2 = \operatorname{Im} a_1 = 0$.
2. 对于更一般的 $u(x, t) = \frac{1}{\sqrt{2\pi}} \sum_w e^{iwx} \hat{u}(w, t)$, 以及 $f(x) = \frac{1}{\sqrt{2\pi}} \sum_w e^{iwx} \hat{f}(w)$, 讨论也是类似的. 下面的几题也是如此, 主要考虑谐波解, 一般的情形略.

7.3 习题 4.3.1

题目: For which matrices A, B is the system $u_t = Au_x + Bu$ energy conserving? (i.e. $\|u(\cdot, t)\| = \|u(\cdot, 0)\|$)

解答: 考虑谐波解

$$u(x, t) = \frac{1}{\sqrt{2\pi}} e^{iwx} \hat{u}(w, t), u(x, 0) = f(x) = \frac{1}{\sqrt{2\pi}} e^{iwx} \hat{f}(w)$$

代入方程得到

$$\begin{cases} \hat{u}_t = (iwA + B)\hat{u}, \\ \hat{u}(w, 0) = \hat{f}(w) \end{cases} \Rightarrow \hat{u}(w, t) = e^{(iwA+B)t} \hat{f}(w)$$

能量守恒即

$$\|u(\cdot, t)\|^2 = \|u(\cdot, 0)\|^2 \Rightarrow |\hat{u}(w, t)|^2 = |\hat{u}(w, 0)|^2$$

对时间求导

$$\begin{aligned}\partial_t |\hat{u}(w, t)|^2 &= \langle \hat{u}, \hat{u}_t \rangle + \langle \hat{u}_t, \hat{u} \rangle \\ &= \langle \hat{u}, (iwA + B)\hat{u} \rangle + \langle (iwA + B)\hat{u}, \hat{u} \rangle \\ &= \langle \hat{u}, (iw(A - A^*) + (B + B^*))\hat{u} \rangle\end{aligned}$$

因此当 $A = A^*$, $B = -B^*$ 时, $\partial_t |\hat{u}(w, t)|^2 = 0$, 能量守恒。

典型错误: 要求双曲问题是适定的, 假设 A 特征值全为实数, 具有完备的特征向量 (可以相似对角化): 存在对角阵 Λ 全实数, S 可逆, 使得

$$A = S\Lambda S^{-1}$$

取 $v = S^{-1}u$ 换元, 得到

$$v_t = \Lambda v_x + Cv, C = S^{-1}BS$$

考虑谐波解

$$v(x, t) = \frac{1}{\sqrt{2\pi}} e^{iwx} \hat{v}(w, t), v(x, 0) = g(x) = \frac{1}{\sqrt{2\pi}} e^{iwx} \hat{g}(w)$$

代入方程得到

$$\begin{cases} \hat{v}_t = (iw\Lambda + C)\hat{v}, \\ \hat{v}(w, 0) = \hat{g}(w) \end{cases} \Rightarrow \hat{v}(w, t) = e^{(iw\Lambda + C)t} \hat{g}(w)$$

能量守恒即

$$\|v(\cdot, t)\|^2 = \|v(\cdot, 0)\|^2 \Rightarrow |\hat{v}(w, t)|^2 = |\hat{v}(w, 0)|^2$$

对时间求导

$$\begin{aligned}\partial_t |\hat{v}(w, t)|^2 &= \langle \hat{v}, \hat{v}_t \rangle + \langle \hat{v}_t, \hat{v} \rangle \\ &= \langle \hat{v}, (iw\Lambda + C)\hat{v} \rangle + \langle (iw\Lambda + C)\hat{v}, \hat{v} \rangle \\ &= \langle \hat{v}, (iw(\Lambda - \Lambda^*) + (C + C^*))\hat{v} \rangle\end{aligned}$$

注意到 Λ 为实对角阵, $\Lambda = \Lambda^*$, 因此只需要保证 $C = -C^*$ 。

注记: 错误的主要原因在于可逆线性变换前后考虑的是两种能量

$$\|v(\cdot, t)\|^2 = \langle v, v \rangle = \langle S^{-1}u, S^{-1}u \rangle = \langle u, (S^{-1})^* S^{-1}u \rangle \neq \langle u, u \rangle = \|u(\cdot, t)\|^2$$

只有加上 $S^*S = I$ 的条件 (换言之 S 为酉方阵, A 为 Hermite 方阵), 才能保证两者一致。在适定性证明中只是利用了两个能量之间的不等式

$$\|u(\cdot, t)\|^2 = \langle u, u \rangle = \langle Sv, Sv \rangle \leq |S|^2 \langle v, v \rangle \leq |S|^2 \|v(\cdot, t)\|^2$$

7.4 补充题一

题目: (均匀剖分) 用 u 在三个点: $x_{j\pm 1} = (j \pm 1)h, x_j = jh$ 处的函数值的线性组合是无法得到 u_{xx} 的 3 阶或高于 3 阶的近似。

解答： 假设存在一个线性组合能达到 3 阶或者高于 3 阶的近似，那么有：

$$\alpha u_{j-1} + \beta u_j + \gamma u_{j+1} = (u_{xx})_j + O(h^3)$$

由泰勒展开可以得到：

$$\begin{cases} \alpha + \beta + \gamma = 0 \\ h(-\alpha + \gamma) = 0 \\ \frac{h^2}{2}(\alpha + \gamma) = 1 \\ \frac{h^3}{6}(-\alpha + \gamma) = 0 \\ \frac{h^4}{24}(\alpha + \gamma) = 0 \end{cases}$$

可以看到第三个方程与第五个方程冲突了，上面方程组无解，不存在满足条件的 α, β, γ ，假设不成立。事实上，前 4 个方程有唯一解 $\alpha = \frac{1}{h^2}, \beta = -\frac{2}{h^2}, \gamma = \frac{1}{h^2}$ ，是对 u_{xx} 的二阶近似。

7.5 补充题二

题目： 针对偏微分方程 $u_t = ((0.1 + \sin^2 x)u_x)_x$ ，构造 (2,2) 阶精度的有限差分格式。

解答： 为了方便，我们记 $a(x) = 0.1 + \sin^2 x, a_{j+1/2} = a(x_{j+1/2}) = a((x_j + x_{j+1})/2)$ 。则可直接构造时间二阶、空间二阶的差分格式如下：

$$\begin{aligned} \frac{v_j^{n+1} - v_j^{n-1}}{2\Delta t} &= \frac{a_{j+1/2} v_x|_{j+1/2}^n - a_{j-1/2} v_x|_{j-1/2}^n}{\Delta x} \\ &= \frac{a_{j+1/2} \frac{v_{j+1}^n - v_j^n}{\Delta x} - a_{j-1/2} \frac{v_j^n - v_{j-1}^n}{\Delta x}}{\Delta x} \\ &= \frac{a_{j+1/2}(v_{j+1}^n - v_j^n) - a_{j-1/2}(v_j^n - v_{j-1}^n)}{\Delta x^2} \end{aligned}$$

由于：

$$\begin{aligned} \frac{v_j^{n+1} - v_j^{n-1}}{2\Delta t} &= u_t + \frac{1}{3}\Delta t^2 u_{ttt} + O(\Delta t^4)|_j^n. \\ a_{j+1/2} \frac{v_{j+1}^n - v_j^n}{\Delta x} &= (a(x) + \frac{\Delta x}{2} a_x + \frac{1}{2!}(\frac{\Delta x}{2})^2 a_{xx} + \frac{1}{3!}(\frac{\Delta x}{2})^3 a_{xxx} + O(\Delta x^4))(v_x + \frac{1}{2!}\Delta x v_{xx} + \frac{1}{3!}\Delta x^2 v_{xxx} \\ &\quad + \frac{1}{4!}\Delta x^3 v_{xxxx} + O(\Delta x^4))|_j^n \\ &= av_x + \Delta x(\frac{a_x v_x}{2} + \frac{av_{xx}}{2}) + \Delta x^2(\frac{1}{6}av_{xxx} + \frac{1}{4}a_x v_{xx} + \frac{1}{8}a_{xx} v_x) \\ &\quad + \Delta x^3(\frac{1}{24}av_{xxxx} + \frac{1}{12}a_x v_{xxx} + \frac{1}{16}a_{xx} v_{xx} + \frac{1}{48}a_{xxx} v_x) + O(\Delta x^4)|_j^n. \end{aligned}$$

$$\begin{aligned}
a_{j-1/2} \frac{v_j^n - v_{j-1}^n}{\Delta x} &= (a(x) - \frac{\Delta x}{2} a_x + \frac{1}{2!} (\frac{\Delta x}{2})^2 a_{xx} - \frac{1}{3!} (\frac{\Delta x}{2})^3 a_{xxx} + O(\Delta x^4)) (v_x - \frac{1}{2!} \Delta x v_{xx} + \frac{1}{3!} \Delta x^2 v_{xxx} \\
&\quad - \frac{1}{4!} \Delta x^3 v_{xxxx} + O(\Delta x^4))|_j^n \\
&= av_x - \Delta x (\frac{a_x v_x}{2} + \frac{av_{xx}}{2}) + \Delta x^2 (\frac{1}{6} av_{xxx} + \frac{1}{4} a_x v_{xx} + \frac{1}{8} a_{xx} v_x) \\
&\quad - \Delta x^3 (\frac{1}{24} av_{xxxx} + \frac{1}{12} a_x v_{xxx} + \frac{1}{16} a_{xx} v_{xx} + \frac{1}{48} a_{xxx} v_x) + O(\Delta x^4)|_j^n. \\
T_j^n &= \frac{v^{n+1} - v^{n-1}}{2\Delta t} - \frac{a_{j+1/2} \frac{v_{j+1}^n - v_j^n}{\Delta x} - a_{j-1/2} \frac{v_j^n - v_{j-1}^n}{\Delta x}}{\Delta x} \\
&= (v_t)_j^n - (a_x v_x + av_{xx})|_j^n + O(\Delta t^2) + O(\Delta x^2)
\end{aligned}$$

局部截断误差为时间 2 阶，空间 2 阶。

8 第二次小测

题目： 讨论 $u_t + au_x = 0$ 的 Lax 格式关于 L_∞ 模的稳定性。

解答： Lax 格式：

$$\begin{aligned}
u_j^{n+1} &= \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{a\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n) \\
&= \frac{1}{2}(1 + a\lambda)u_{j+1}^n + \frac{1}{2}(1 - a\lambda)u_{j-1}^n.
\end{aligned}$$

其中 $\lambda = \frac{\Delta t}{\Delta x}$ ，由 CFL 条件可知： $|a\frac{\Delta t}{\Delta x}| \leq 1$ ，即： $|a\lambda| \leq 1$ 。

因此，

$$|u_j^{n+1}| \leq \frac{1}{2}(1 + a\lambda)|u_{j+1}^n| + \frac{1}{2}(1 - a\lambda)|u_{j-1}^n|.$$

所以有：

$$\begin{aligned}
\|u^{n+1}\|_\infty &\leq \frac{1}{2}(1 + a\lambda)\|u^n\|_\infty + \frac{1}{2}(1 - a\lambda)\|u^n\|_\infty \\
&= \|u^n\|_\infty.
\end{aligned}$$

因此 Lax 格式关于 L_∞ 模是稳定的。

题目： 讨论 $u_t = u_{xx}$ 的 FTCS 格式关于 $\ell_{2,\Delta x}$ 模的稳定性。

解答： FTCS 格式：

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

记 $\mu = \frac{\Delta t}{\Delta x^2}$ ，则有：

$$u_j^{n+1} = \mu u_{j+1}^n + (1 - 2\mu)u_j^n + \mu u_{j-1}^n.$$

FTCS 的稳定性条件为 $\frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$ ，即： $\mu \leq \frac{1}{2}$ 。

等式两边同时平方, 乘以 Δx , 并且关于 j 求和 (假设是周期的), 则有:

$$\begin{aligned}
\|u^{n+1}\|_{\ell_2, \Delta x}^2 &= \sum_j (u_j^{n+1})^2 \Delta x = \sum_j (\mu u_{j+1}^n + (1-2\mu)u_j^n + \mu u_{j-1}^n)^2 \Delta x \\
&= \mu^2 \|u^n\|_{\ell_2, \Delta x}^2 + (1-2\mu)^2 \|u^n\|_{\ell_2, \Delta x}^2 + \mu^2 \|u^n\|_{\ell_2, \Delta x}^2 \\
&\quad + 2 \sum_j \mu(1-2\mu)u_{j+1}^n u_j^n + 2 \sum_j (1-2\mu)\mu u_j^n u_{j-1}^n + 2 \sum_j \mu^2 u_{j+1}^n u_{j-1}^n \\
&\leq \mu^2 \|u^n\|_{\ell_2, \Delta x}^2 + (1-2\mu)^2 \|u^n\|_{\ell_2, \Delta x}^2 + \mu^2 \|u^n\|_{\ell_2, \Delta x}^2 \\
&\quad + \mu(1-2\mu) \sum_j ((u_j^n)^2 + (u_{j+1}^n)^2) + (1-2\mu)\mu \sum_j ((u_j^n)^2 + (u_{j-1}^n)^2) + \mu^2 \sum_j ((u_{j+1}^n)^2 + (u_{j-1}^n)^2) \\
&= (\mu^2 + (1-2\mu)^2 + \mu^2 + 2\mu(1-2\mu) + 2(1-2\mu)\mu + 2\mu^2) \|u^n\|_{\ell_2, \Delta x}^2 \\
&= \|u^n\|_{\ell_2, \Delta x}^2.
\end{aligned}$$

因此, FTCS 格式关于 $\ell_{2, \Delta x}$ 模是稳定的。

9 第七次书面作业

9.1 习题 3.1.2*

题目: Show that the following difference schemes for approximating the solution to

$$u_t + au_x = \nu u_{xx}$$

are unconditionally stable, where $R = \frac{a\Delta t}{\Delta x}$ and $r = \frac{\nu\Delta t}{\Delta x^2}$.

- (a) $v_j^{n+1} + \frac{R}{2}\delta_0 v_j^{n+1} - r\delta^2 v_j^{+1} = v_j^n$;
(b) $v_j^{n+1} + \frac{R}{4}\delta_0 v_j^{n+1} - \frac{r}{2}\delta^2 v_j^{+1} = v_j^n - \frac{R}{4}\delta_0 v_j^n + \frac{r}{2}\delta^2 v_j^n$.

解答: (a) 代入 $v_j^n = \frac{1}{\sqrt{2\pi}} \hat{v}^n(w) e^{iwx_j}$ 得到

$$\left[1 + iR \sin(wh) + 4r \sin^2\left(\frac{wh}{2}\right) \right] \hat{v}^{n+1}(w) = \hat{v}^n(w)$$

因此

$$\hat{Q} = \frac{1}{1 + iR \sin(wh) + 4r \sin^2(\frac{wh}{2})}, \quad |\hat{Q}|^2 = \frac{1}{(1 + 4r \sin^2(\frac{wh}{2}))^2 + (R \sin(wh))^2} \leq 1$$

无条件稳定。

(b) 代入 $v_j^n = \frac{1}{\sqrt{2\pi}} \hat{v}^n(w) e^{iwx_j}$ 得到

$$\left[1 + i\frac{R}{2} \sin(wh) + 2r \sin^2\left(\frac{wh}{2}\right) \right] \hat{v}^{n+1}(w) = \left[1 - i\frac{R}{2} \sin(wh) - 2r \sin^2\left(\frac{wh}{2}\right) \right] \hat{v}^n(w)$$

因此

$$\hat{Q} = \frac{1 - i\frac{R}{2}\sin(wh) - 2r\sin^2(\frac{wh}{2})}{1 + i\frac{R}{2}\sin(wh) + 2r\sin^2(\frac{wh}{2})}, \quad |\hat{Q}|^2 = \frac{(1 - 2r\sin^2(\frac{wh}{2}))^2 + (\frac{R}{2}R\sin(wh))^2}{(1 + 2r\sin^2(\frac{wh}{2}))^2 + (\frac{R}{2}R\sin(wh))^2} \leq 1$$

无条件稳定。

9.2 习题 4.4.1

题目： Prove that there are positive constants $\delta, K > 0$ such that the solutions to a parabolic system $u_t = Au_{xx}$ satisfy

$$\|u(\cdot, t)\|^2 + \delta \int_0^t \|u_x(\cdot, \xi)\|^2 d\xi \leq K \|u(\cdot, 0)\|^2 \quad (9.2.1)$$

解答： 考虑谐波解

$$u(x, t) = \frac{1}{\sqrt{2\pi}} e^{iwx} \hat{u}(w, t), \quad u(x, 0) = f(x) = \frac{1}{\sqrt{2\pi}} e^{iwx} \hat{f}(w)$$

代入方程得到

$$\begin{cases} \hat{u}_t = -w^2 A \hat{u}, \\ \hat{u}(w, 0) = \hat{f}(w) \end{cases} \Rightarrow \hat{u}(w, t) = e^{-w^2 A t} \hat{f}(w)$$

易得

$$\|u(\cdot, t)\|^2 = |\hat{u}(w, t)|^2, \quad \|u_x(\cdot, t)\|^2 = w^2 |\hat{u}(w, t)|^2,$$

原式等价于

$$|\hat{u}(w, t)|^2 + \delta w^2 \int_0^t |\hat{u}(w, \xi)|^2 d\xi \leq K |\hat{u}(w, 0)|^2$$

对 $|\hat{u}(w, t)|^2$ 关于时间求导, 得到

$$\partial_t |\hat{u}(w, t)|^2 = \langle \hat{u}, \hat{u}_t \rangle + \langle \hat{u}_t, \hat{u} \rangle = \langle \hat{u}, -w^2 A \hat{u} \rangle + \langle -w^2 A \hat{u}, \hat{u} \rangle = \langle \hat{u}, -w^2 (A + A^*) \hat{u} \rangle$$

易知对于抛物方程, 存在 $\delta' > 0$ 使得 $A + A^* \geq \delta' I$, 代入得

$$\partial_t |\hat{u}(w, t)|^2 = \langle \hat{u}, -w^2 (A + A^*) \hat{u} \rangle \leq -\delta' w^2 |\hat{u}(w, t)|^2$$

因此

$$0 \geq \int_0^t (\partial_t |\hat{u}(w, \xi)|^2 + \delta' w^2 |\hat{u}(w, \xi)|^2) d\xi = |\hat{u}(w, t)|^2 - |\hat{u}(w, 0)|^2 + \delta' w^2 \int_0^t |\hat{u}(w, \xi)|^2 d\xi$$

取 $K = 1$, $\delta = \delta'$ 即可得证。

注记： 对于更一般的 $u(x, t) = \frac{1}{\sqrt{2\pi}} \sum_w e^{iwx} \hat{u}(w, t)$, 以及 $f(x) = \frac{1}{\sqrt{2\pi}} \sum_w e^{iwx} \hat{f}(w)$, 讨论也是类似的。下面一题也是如此, 因此这里主要考虑谐波解, 一般的情形略。

注记：也可以直接对 u 进行讨论，但是需要注意，这里讨论的都是周期函数，2 范数中的积分都是在 1 个周期而不是 \mathbb{R} 进行的，分部积分的边界项可以利用周期性抵消。

9.3 习题 4.4.2

题目：Is it true that the inequality in (9.2.1) holds with the same constants $\delta, K > 0$, if the system is changed to $u_t = Au_{xx} + Bu_x + Cu$, where B is Hermitian and C is skew-Hermitian?

解答：考虑谐波解

$$u(x, t) = \frac{1}{\sqrt{2\pi}} e^{iwx} \hat{u}(w, t), u(x, 0) = f(x) = \frac{1}{\sqrt{2\pi}} e^{iwx} \hat{f}(w)$$

代入方程得到

$$\begin{cases} \hat{u}_t = -w^2 A \hat{u} + iwB \hat{u} + C \hat{u}, \\ \hat{u}(w, 0) = \hat{f}(w) \end{cases} \Rightarrow \hat{u}(w, t) = e^{(-w^2 A + iwB + C)t} \hat{f}(w)$$

易得

$$\|u(\cdot, t)\|^2 = |\hat{u}(w, t)|^2, \|u_x(\cdot, t)\|^2 = w^2 |\hat{u}(w, t)|^2,$$

原式等价于

$$|\hat{u}(w, t)|^2 + \delta w^2 \int_0^t |\hat{u}(w, \xi)|^2 d\xi \leq K |\hat{u}(w, 0)|^2$$

对 $|\hat{u}(w, t)|^2$ 关于时间求导，得到

$$\begin{aligned} \partial_t |\hat{u}(w, t)|^2 &= \langle \hat{u}, \hat{u}_t \rangle + \langle \hat{u}_t, \hat{u} \rangle = \langle \hat{u}, (-w^2 A + iwB + C) \hat{u} \rangle + \langle (-w^2 A + iwB + C) \hat{u}, \hat{u} \rangle \\ &= \langle \hat{u}, (-w^2(A + A^*) + iw(B - B^*) + (C + C^*)) \hat{u} \rangle \end{aligned}$$

易知对于抛物方程，存在 $\delta' > 0$ 使得 $A + A^* \geq \delta' I$ ，再由条件得 $B = B^*, C = -C^*$ ，代入得

$$\partial_t |\hat{u}(w, t)|^2 = \langle \hat{u}, -w^2(A + A^*) \hat{u} \rangle \leq -\delta' w^2 |\hat{u}(w, t)|^2$$

因此

$$0 \geq \int_0^t (\partial_t |\hat{u}(w, \xi)|^2 + \delta' w^2 |\hat{u}(w, \xi)|^2) d\xi = |\hat{u}(w, t)|^2 - |\hat{u}(w, 0)|^2 + \delta' w^2 \int_0^t |\hat{u}(w, \xi)|^2 d\xi$$

取 $K = 1, \delta = \delta'$ 即可得证。

9.4 习题 4.5.1

题目：Consider the first order system $u_t = Au_x$. Is it possible that the Petrovskii condition

(9.4.1) is satisfied for some constant $\alpha > 0$ but not for $\alpha = 0$?

$$\operatorname{Re} \lambda \leq \alpha \quad (9.4.1)$$

解答: 代入谐波解可以得到 $\hat{P}(i\omega) = i\omega A$ 。假设存在 $\alpha > 0$, 对所有的 ω , λ 是 $\hat{P}(i\omega)$ 的特征值, 有

$$\operatorname{Re} \lambda \leq \alpha$$

利用 $\hat{P}(i\omega)$ 的表达式可得

$$\operatorname{Re} \lambda = \operatorname{Re} \lambda(\hat{P}(i\omega)) = \operatorname{Re} \lambda(i\omega A) = -\omega \operatorname{Im} \lambda(A) \leq \alpha$$

上式对任意 ω 均成立, 由 ω 的任意性可知, 必然有 $\operatorname{Im} \lambda(A) = 0$, 因此取 $\alpha = 0$ 时也成立。

9.5 习题 4.5.2

题目: Derive a matrix $\hat{H}(\omega)$ satisfying Eq.(9.5.1) and (9.5.2) for the system

$$u_t = \begin{bmatrix} 1 & 10 \\ 0 & 2 \end{bmatrix} u_x.$$

$$K^{-1}I \leq \hat{H}(\omega) \leq KI. \quad (9.5.1)$$

$$\hat{H}(\omega)\hat{P}(i\omega) + \hat{P}^*(i\omega)\hat{H}(\omega) \leq 2\alpha\hat{H}(\omega). \quad (9.5.2)$$

解答: 代入谐波解可以得到

$$\hat{P}(i\omega) = i\omega \begin{bmatrix} 1 & 10 \\ 0 & 2 \end{bmatrix}, \quad \hat{P}^*(i\omega) = -i\omega \begin{bmatrix} 1 & 0 \\ 10 & 2 \end{bmatrix}$$

我们假设 Hermite 矩阵 $\hat{H}(\omega)$ 形如

$$\hat{H}(\omega) = \begin{bmatrix} a & c \\ \bar{c} & b \end{bmatrix}$$

其中 a, b, c 是与 ω 无关的常数, 并且 $a, b \in \mathbb{R}$ 。此时 (9.5.1) 等价于要求 $\hat{H}(\omega)$ 正定, 即两个特征值严格大于 0, 具体即 $a > 0$, $b > 0$, 以及 $ab > |c|^2$ 。此时有

$$\hat{H}(\omega)\hat{P}(i\omega) + \hat{P}^*(i\omega)\hat{H}(\omega) = i\omega \begin{bmatrix} 0 & 10a + c \\ -10a - \bar{c} & 10(\bar{c} - c) \end{bmatrix}$$

代入计算 (9.5.2)

$$2\alpha\hat{H}(w) - (\hat{H}(w)\hat{P}(i\omega) + \hat{P}^*(i\omega)\hat{H}) = \begin{bmatrix} 2\alpha a & 2\alpha c - i\omega(10a + c) \\ 2\alpha\bar{c} + i\omega(10a + \bar{c}) & 2\alpha b + i\omega(c - \bar{c}) \end{bmatrix} =: A$$

要求上述矩阵半正定, 等价于要求 $2\alpha a \geq 0$ 以及

$$\det(A) = 2\alpha a(2\alpha b + 10i\omega(c - \bar{c})) - |2\alpha c - i\omega(10a + c)|^2 \geq 0, \forall \omega$$

这个式子是关于 ω 的实值二次函数, 并且二次项系数非正, 因此 $\det(A) \geq 0$ 恒成立要求二次项系数严格为 0

$$10a + c = 0 \Rightarrow c = -10a \in \mathbb{R}$$

此时条件变为 $\det(A) = 4\alpha^2(ab - c^2) \geq 0$ 。

整理上述所有条件可得

$$a > 0, c = -10a, b > 100a.$$

例如可以取 $a = 1, c = -10, b = 200$, 构造 $\hat{H}(\omega)$ 为

$$\hat{H}(\omega) = \begin{bmatrix} 1 & -10 \\ -10 & 200 \end{bmatrix}$$

取 $\alpha = 0$ 即可满足 (9.5.2)。条件(9.5.1) 要求 $K > 0$ 使得

$$\begin{aligned} \hat{H}(\omega) - K^{-1}I &= \begin{bmatrix} a - \frac{1}{K} & c \\ \bar{c} & b - \frac{1}{K} \end{bmatrix} = \begin{bmatrix} 1 - \frac{1}{K} & -10 \\ -10 & 200 - \frac{1}{K} \end{bmatrix} > 0 \\ KI - \hat{H}(\omega) &= \begin{bmatrix} K - a & -c \\ -\bar{c} & K - b \end{bmatrix} = \begin{bmatrix} K - 1 & 10 \\ 10 & K - 200 \end{bmatrix} > 0 \end{aligned}$$

取足够大的 K 即可满足, 例如 $K = 201$ 。

注记: 直接计算 $\hat{P}(i\omega) + \hat{P}^*(i\omega)$ 可得

$$\hat{P}(i\omega) + \hat{P}^*(i\omega) = i\omega \begin{bmatrix} 0 & 10 \\ -10 & 0 \end{bmatrix} = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \right)^H \begin{bmatrix} -10\omega & \\ & 10\omega \end{bmatrix} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \right)$$

找不到一个常数 α 控制这里的 10ω 和 -10ω , 使得 $\hat{P}(i\omega) + \hat{P}^*(i\omega) \leq 2\alpha I$ 成立。

注记: (对应第一本教材的 Thm4.5.2 和 Thm4.5.3) 计算可得 $\hat{P}(i\omega)$ 的特征值为 $\lambda_1 = i\omega, \lambda_2 = 2i\omega$, 始终可以相似对角化

$$\hat{P}(i\omega) = i\omega \begin{bmatrix} 1 & 10 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i\omega & \\ & 2i\omega \end{bmatrix} \begin{bmatrix} 10 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = S(\omega)\Lambda(\omega)S(\omega)^{-1}$$

由于 $S(\omega)$ 与 ω 无关, 显然存在 K 使得 $|S(\omega)||S^{-1}(\omega)| \leq K$ 。由于特征值全部为纯虚数, $\operatorname{Re} \Lambda(\omega) = 0$ 满足 The Petrovskii condition。

9.6 补充题

题目：试证：若存在常数 α ，使得对于任意 w ，下式成立

$$\hat{P}(iw) + \hat{P}^*(iw) \leq 2\alpha I.$$

则该偏微分方程的初值问题是 Well-Posed.

解答：初值问题的解满足

$$\hat{u}(w, t) = e^{\hat{P}(iw)t} \hat{f}(w)$$

利用方程可得

$$\frac{d}{dt} \langle \hat{u}, \hat{u} \rangle = \langle \hat{P}\hat{u}, \hat{u} \rangle + \langle \hat{u}, \hat{P}\hat{u} \rangle = \langle \hat{u}, (\hat{P} + \hat{P}^*)\hat{u} \rangle \leq 2\alpha \langle \hat{u}, \hat{u} \rangle$$

因此

$$|\hat{u}(w, t)|^2 \leq e^{2\alpha t} |\hat{u}(w, 0)|^2 = e^{2\alpha t} |\hat{f}(w)|^2.$$

进而可得原问题是 Well-Posed. (教材 P129 定理 4.5.4 有证明, 相关的记号和概念也在那一节中有定义)

10 第八次书面作业

10.1 习题 5.8.3

题目：Analyze the consistency and stability of difference scheme(5.8.21)-(5.8.22).

$$u_{jk}^{n+1/2} = u_{jk}^n - \frac{R_x}{2} \delta_{x0} u_{jk}^n + \frac{R_x^2}{2} \delta_x^2 u_{jk}^n, \quad (5.8.21)$$

$$u_{jk}^{n+1} = u_{jk}^{n+1/2} - \frac{R_y}{2} \delta_{y0} u_{jk}^{n+1/2} + \frac{R_y^2}{2} \delta_y^2 u_{jk}^{n+1/2}. \quad (5.8.22)$$

解答：见第二本参考书 P246 页。

10.2 补充题 1

题目：分析偏微分方程 $u_t + u_x - \nu_2 u_{xx} + \mu_3 u_{xxx} = 0$ 的耗散性、色散性，其中 ν_2, μ_3 分别为常实数。

解答：取简谐波

$$u(x, t) = e^{i(wx+kt)}$$

代入 PDE 得到

$$\begin{aligned} (ik) + (iw) - \nu_2(iw)^2 + \mu_3(iw)^3 &= 0 \\ \Rightarrow k &= -w + i\nu_2 w^2 + \mu_3 w^3 \end{aligned}$$

因此色散关系 $k = k(w) = \alpha(w) + i\beta(w)$ 对应为

$$\alpha(w) = -w + \mu_3 w^3, \beta(w) = \nu_2 w^2$$

还有放大因子 $\lambda_e = e^{ik\Delta t}$ 对应为

$$|\lambda_e| = e^{-\beta(w)\Delta t} = e^{-\nu_2 w^2 \Delta t}, \varphi_e = \alpha(w)\Delta t = (-w + \mu_3 w^3)\Delta t$$

因此对于 $u_t + u_x - \nu_2 u_{xx} + \mu_3 u_{xxx} = 0$:

1. 耗散性

- (a) 若 $\nu_2 > 0$, $\beta(w) > 0$, PDE 具有耗散性, 解是稳定的;
- (b) 若 $\nu_2 = 0$, $\beta(w) = 0$, PDE 没有耗散性;
- (c) 若 $\nu_2 < 0$, $\beta(w) < 0$, PDE 是逆耗散的。

2. 色散性

- (a) 若 $\mu_3 = 0$, $-\frac{\alpha(w)}{w} = 1$, PDE 无色散
- (b) 若 $\mu_3 \neq 0$, $-\frac{\alpha(w)}{w} = 1 - \mu_3 w^2$, PDE 有色散。

10.3 补充题 2

题目: 分析偏微分方程 $u_t = u_{xx}$ 的耗散性和色散性, 以及用两种方法分别分析其 FTCS 格式的耗散性和色散性。

解答: 用上一题的结论可知 PDE 具有耗散性, 但是没有色散性。接下来分析 FTCS 格式的性质。
方法一:

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} = \frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{\Delta x^2},$$

$$v_j^{n+1} = (1 - 2r)v_j^n + r(v_{j+1}^n + v_{j-1}^n), r = \frac{\Delta t}{\Delta x^2}$$

放大因子为

$$\lambda = 1 + (e^{iw\Delta x} - 2 + e^{-iw\Delta x})r = 1 - 4\sin^2\left(\frac{w\Delta x}{2}\right)r = |\lambda|e^{i\varphi}$$

其中 $\varphi = 0$, FTCS 格式无色散。当 $0 < r \leq \frac{1}{2}$ 时方程的 FTCS 格式是耗散。我们对 λ 和 λ_e 做 Taylor 展开可以得到:

$$\lambda = 1 - 4\sin^2\left(\frac{w\Delta x}{2}\right)r = 1 - \omega^2 \Delta t + \frac{\omega^4 \Delta t^2}{12r} + \dots$$

$$\lambda_e = e^{-\omega^2 \Delta t} = 1 - \omega^2 \Delta t + \frac{\omega^4 \Delta t^2}{2} + \dots$$

当 $0 < r < \frac{1}{6}$ 时, $\lambda > \lambda_e$, 数值逆耗散。当 $\frac{1}{6} < r \leq \frac{1}{2}$ 时, $\lambda < \lambda_e$, 数值正耗散。方法无数值色散。
方法二 (MPDE 方法):

假设 $u(x, t)$ 是与差分格式等价的 PDE 的精确解，则有

$$0 = \frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} = \left\{ u_t + \frac{\Delta t}{2} u_{tt} + \frac{\Delta t^3}{6} u_{ttt} + \cdots - u_{xx} - \frac{\Delta x^2}{12} u_{xxx} - \frac{\Delta x^4}{360} u_{xxxx} + \cdots \right\}_j^n$$

对上式关于 x, t 求导可以得到：

$$\begin{cases} u_t - u_{xx} + \frac{\Delta t}{2} u_{tt} - \frac{\Delta x^2}{12} u_{xxx} + \frac{\Delta t^2}{6} u_{ttt} - \frac{\Delta x^4}{360} u_{xxxx} + \cdots = 0 \\ u_{tt} - u_{xxt} + \frac{\Delta t}{2} u_{ttt} - \frac{\Delta x^2}{12} u_{xxx} + \cdots = 0 \\ u_{xt} - u_{xxx} + \frac{\Delta t}{2} u_{xtt} - \frac{\Delta x^2}{12} u_{xxx} \cdots = 0 \end{cases}$$

消去 u_{tt}, u_{xxt} 可得

$$u_t - u_{xx} + \left(\frac{\Delta t}{2} - \frac{\Delta x^2}{12}\right) u_{xxx} - \frac{\Delta t^2}{12} u_{ttt} - \frac{\Delta t^2}{4} u_{xtt} + \frac{\Delta x^2 \Delta t}{24} u_{xxx} + \left(\frac{\Delta x^2 \Delta t}{24} - \frac{\Delta x^4}{360}\right) u_{xxxx} + \cdots = 0$$

将方程的时间 t 的导数转化为空间 x 的导数的过程中，不会出现关于 x 的奇数阶项，因此 FTCS 格式无色散。由于 PDE 也是无色散的，因此是无数值色散。对于耗散性，我们需要考虑上式的前三项。代入谐波解

$$u(x, t) = e^{i(\omega x + kt)}$$

解得

$$k = i(\omega^2 + \omega^4 \left(\frac{\Delta t}{2} - \frac{\Delta x^2}{12}\right)) = i(\omega^2 + \omega^4 \Delta x^2 \left(\frac{r}{2} - \frac{1}{12}\right))$$

当 $0 < r \leq \frac{1}{2}$ 时，FTCS 格式是耗散的。数值耗散需要与 PDE 比较，由 ω^4 的系数决定，当 $0 < r < \frac{1}{6}$ 时，数值逆耗散。当 $\frac{1}{6} < r \leq \frac{1}{2}$ 时，数值正耗散。

11 第九次书面作业

11.1 习题 5.8.7(a)(b)*

题目：

(a) Find the numerical domains of the dependence at the point $(j\Delta x, k\Delta y, (n+1)\Delta t)$ for the following differences schemes.

(b) Determinate the CFL condition for the difference schemes given in part (a).

1. $v_{jk}^{n+1} = a_2 v_{j-1, k}^n + a_3 v_{jk}^n + a_4 v_{j+1, k}^n$
2. $v_{jk}^{n+1} = a_1 v_{j-1, k}^n + a_3 v_{jk}^n + a_4 v_{j+1, k}^n + a_5 v_{j, k+1}^n$
3. $v_{jk}^{n+1} = a_1 v_{j-1, k}^n + a_3 v_{jk}^n + a_5 v_{j, k+1}^n$
4. $v_{jk}^{n+1} = a_6 v_{j+1, k-1}^n + a_7 v_{j+1, k-1}^n + a_8 v_{j-1, k-1}^n + a_9 v_{j-1, k+1}^n$

对于二维方程 $u_t + au_x + bu_y = 0$, 精确解 $u(x, y, t)$ 的依赖区为

$$D_p = \{(x - at, y - bt)\}$$

数值解依赖区简单地视作每一个分量对应依赖区间的笛卡尔积

$$N_p = [x_1, x_2] \times [y_1, y_2]$$

解答: 以第一个格式为例, 数值解依赖区为

$$[j\Delta x, (j + n + 1)\Delta x] \times [(k - n - 1)\Delta y, k\Delta y]$$

要求精确解依赖区 $\{(j\Delta x - a(n + 1)\Delta t, k\Delta y - b(n + 1)\Delta t)\}$ 落在数值解依赖区

$$\begin{aligned} j\Delta x &\leq j\Delta x - a(n + 1)\Delta t \leq (j + n + 1)\Delta x \\ (k - n - 1)\Delta y &\leq k\Delta y - b(n + 1)\Delta t \leq k\Delta y \end{aligned}$$

得到

$$\begin{aligned} -1 &\leq R_x = \frac{a\Delta t}{\Delta x} \leq 0 \\ 0 &\leq R_y = \frac{b\Delta t}{\Delta y} \leq 1 \end{aligned}$$

其它三个类似可得。

四个格式的数值依赖区依次为:

$$\begin{aligned} &[j\Delta x, (j + n + 1)\Delta x] \times [(k - n - 1)\Delta y, k\Delta y] \\ &[(j - n - 1)\Delta x, (j + n + 1)\Delta x] \times [k\Delta y, (k + n + 1)\Delta y] \\ &[(j - n - 1)\Delta x, \Delta x] \times [k\Delta y, (k + n + 1)\Delta y] \\ &[(j - n - 1)\Delta x, (j + n + 1)\Delta x] \times [(k - n - 1)\Delta y, (k + n + 1)\Delta y] \end{aligned}$$

CFL 条件依次为:

$$\begin{aligned} R_x &\in [-1, 0], R_y \in [0, 1] \\ R_x &\in [-1, 1], R_y \in [-1, 0] \\ R_x &\in [0, 1], R_y \in [-1, 0] \\ R_x &\in [-1, 1], R_y \in [-1, 1] \end{aligned}$$

11.2 习题 4.4.11*

题目： Show that the three dimensional analog of the Peaceman-Rachford scheme,

$$\begin{aligned}\left(1 - \frac{r_x}{3}\delta_x^2\right)v_{jk\ell}^{n+\frac{1}{3}} &= \left(1 + \frac{r_y}{3}\delta_y^2 + \frac{r_z}{3}\delta_z^2\right)v_{jk\ell}^n \\ \left(1 - \frac{r_y}{3}\delta_y^2\right)v_{jk\ell}^{n+\frac{2}{3}} &= \left(1 + \frac{r_x}{3}\delta_x^2 + \frac{r_z}{3}\delta_z^2\right)v_{jk\ell}^{n+\frac{1}{3}} \\ \left(1 - \frac{r_z}{3}\delta_z^2\right)v_{jk\ell}^{n+1} &= \left(1 + \frac{r_x}{3}\delta_x^2 + \frac{r_y}{3}\delta_y^2\right)v_{jk\ell}^{n+\frac{2}{3}}\end{aligned}$$

is conditionally stable and order $\mathcal{O}(\Delta t + \Delta x^2 + \Delta y^2 + \Delta z^2)$.

解答： 取 $v_{jk}^n = \hat{v}^n e^{i(w_x x_j + w_y y_k + w_z z_\ell)}$ 代入计算每一步的放大因子，使用记号 $\xi_x = w_x \Delta x$, $\xi_y = w_y \Delta y$, $\xi_z = w_z \Delta z$

$$\begin{aligned}\hat{Q}_1 &= \frac{1 - \frac{4}{3}r_y \sin^2(\frac{\xi_y}{2}) - \frac{4}{3}r_z \sin^2(\frac{\xi_z}{2})}{1 + \frac{4}{3}r_x \sin^2(\frac{\xi_x}{2})} \\ \hat{Q}_2 &= \frac{1 - \frac{4}{3}r_x \sin^2(\frac{\xi_x}{2}) - \frac{4}{3}r_z \sin^2(\frac{\xi_z}{2})}{1 + \frac{4}{3}r_y \sin^2(\frac{\xi_y}{2})} \\ \hat{Q}_3 &= \frac{1 - \frac{4}{3}r_x \sin^2(\frac{\xi_x}{2}) - \frac{4}{3}r_y \sin^2(\frac{\xi_y}{2})}{1 + \frac{4}{3}r_z \sin^2(\frac{\xi_z}{2})}\end{aligned}$$

那么

$$\hat{Q} = \frac{1 - \frac{4}{3}r_y \sin^2(\frac{\xi_y}{2}) - \frac{4}{3}r_z \sin^2(\frac{\xi_z}{2})}{1 + \frac{4}{3}r_x \sin^2(\frac{\xi_x}{2})} \frac{1 - \frac{4}{3}r_x \sin^2(\frac{\xi_x}{2}) - \frac{4}{3}r_z \sin^2(\frac{\xi_z}{2})}{1 + \frac{4}{3}r_y \sin^2(\frac{\xi_y}{2})} \frac{1 - \frac{4}{3}r_x \sin^2(\frac{\xi_x}{2}) - \frac{4}{3}r_y \sin^2(\frac{\xi_y}{2})}{1 + \frac{4}{3}r_z \sin^2(\frac{\xi_z}{2})}$$

若满足 $0 < r_x + r_y + r_z \leq \frac{3}{2}$, 则有

$$|\hat{Q}_1| \leq 1, |\hat{Q}_2| \leq 1, |\hat{Q}_3| \leq 1 \Rightarrow |\hat{Q}| \leq 1$$

但是无法得到无条件的稳定性。

将格式整理为如下形式

$$\left(1 - \frac{r_x}{3}\delta_x^2 - \frac{r_y}{3}\delta_y^2 - \frac{r_z}{3}\delta_z^2\right)v_{ij\ell}^{n+1} + Fv_{ij\ell}^{n+1} = \left(1 + \frac{2r_x}{3}\delta_x^2 + \frac{2r_y}{3}\delta_y^2 + \frac{2r_z}{3}\delta_z^2\right)v_{ij\ell}^n + Gv_{ij\ell}^n$$

这里的 F 和 G 是两个离散算子，定义为

$$\begin{aligned}F &= \left(1 - \frac{r_x}{3}\delta_x^2\right)\left(1 - \frac{r_y}{3}\delta_y^2\right)\left(1 - \frac{r_z}{3}\delta_z^2\right) - \left(1 - \frac{r_x}{3}\delta_x^2 - \frac{r_y}{3}\delta_y^2 - \frac{r_z}{3}\delta_z^2\right) \\ &= \frac{r_x r_y}{9}\delta_x^2\delta_y^2 + \frac{r_y r_z}{9}\delta_y^2\delta_z^2 + \frac{r_x r_z}{9}\delta_x^2\delta_z^2 - \frac{r_x r_y r_z}{27}\delta_x^2\delta_y^2\delta_z^2 \\ G &= \left(1 + \frac{r_y}{3}\delta_y^2 + \frac{r_z}{3}\delta_z^2\right)\left(1 + \frac{r_x}{3}\delta_x^2 + \frac{r_z}{3}\delta_z^2\right)\left(1 + \frac{r_x}{3}\delta_x^2 + \frac{r_y}{3}\delta_y^2\right) \\ &\quad - \left(1 + \frac{2r_x}{3}\delta_x^2 + \frac{2r_y}{3}\delta_y^2 + \frac{2r_z}{3}\delta_z^2\right)\end{aligned}$$

局部截断误差满足

$$\begin{aligned}\Delta t T_{ij\ell}^n &= \left(1 - \frac{r_x}{3}\delta_x^2 - \frac{r_y}{3}\delta_y^2 - \frac{r_z}{3}\delta_z^2\right) u_{ij\ell}^{n+1} - \left(1 + \frac{2r_x}{3}\delta_x^2 + \frac{2r_y}{3}\delta_y^2 + \frac{2r_z}{3}\delta_z^2\right) u_{ij\ell}^n \\ &\quad + (Fu_{ij\ell}^{n+1} - Gu_{ij\ell}^n)\end{aligned}$$

计算可得

$$T_{ij\ell}^n = \frac{1}{\Delta t} (Fu_{ij\ell}^{n+1} - Gu_{ij\ell}^n) + O(\Delta t + \Delta x^2 + \Delta y^2 + \Delta z^2)$$

只需要保证 $\frac{1}{\Delta t} (Fu_{ij\ell}^{n+1} - Gu_{ij\ell}^n)$ 不影响阶数即可，例如

$$\frac{1}{\Delta t} r_x r_y \delta_x^2 \delta_y^2 u_{ij\ell}^{n+1} = \Delta t (u_{xxyy} + O(\Delta x^2) + O(\Delta y^2))$$

因此最终的局部截断误差为

$$T_{ij\ell}^n = O(\Delta t + \Delta x^2 + \Delta y^2 + \Delta z^2)$$

11.3 补充题 1

题目： 对于二维常系数扩散方程 $u_t = au_{xx} + bu_{yy}$ ，证明 Crank-Nicolson 格式的截断误差、分析其稳定性。

解答： Crank-Nicolson 格式具体为

$$\frac{v_{jk}^{n+1} - v_{jk}^n}{\Delta t} = \frac{a}{2\Delta x^2} \delta_x^2 (v_{jk}^n + v_{jk}^{n+1}) + \frac{b}{2\Delta y^2} \delta_y^2 (v_{jk}^n + v_{jk}^{n+1})$$

计算 $(x_j, y_k, t^{n+1/2})$ 处的局部截断误差

$$T_{jk}^{n+1/2} = \frac{u_{jk}^{n+1} - u_{jk}^n}{\Delta t} - \frac{a}{2\Delta x^2} \delta_x^2 (u_{jk}^n + u_{jk}^{n+1}) - \frac{b}{2\Delta y^2} \delta_y^2 (u_{jk}^n + u_{jk}^{n+1})$$

对于时间离散部分

$$\frac{u_{jk}^{n+1} - u_{jk}^n}{\Delta t} = \left(u_t + \frac{\Delta t^2}{24} u_{ttt} + O(\Delta t^4) \right) \Big|_{jk}^{n+1/2}$$

对于空间离散部分，注意到

$$\begin{aligned}\frac{1}{\Delta x^2} \delta_x^2 u_{jk}^n &= \left(u_{xx} + \frac{\Delta x^2}{12} u_{xxxx} + O(\Delta x^4) \right) \Big|_{jk}^n \\ \frac{1}{\Delta x^2} \delta_x^2 u_{jk}^{n+1} &= \left(u_{xx} + \frac{\Delta x^2}{12} u_{xxxx} + O(\Delta x^4) \right) \Big|_{jk}^{n+1}\end{aligned}$$

因此在两个时间层取平均得到

$$\begin{aligned}
& \frac{1}{2\Delta x^2} \delta_x^2 (u_{jk}^n + u_{jk}^{n+1}) \\
&= \left(u_{xx} + \frac{\Delta t^2}{8} u_{xxtt} + O(\Delta t^4) + \frac{\Delta x^2}{12} u_{xxx} + \frac{\Delta x^2 \Delta t^2}{96} u_{xxxxtt} + O(\Delta x^2 \Delta t^4) + O(\Delta x^4) \right) \Big|_{jk}^{n+1/2} \\
&= (u_{xx} + O(\Delta t^2) + O(\Delta x^2)) \Big|_{jk}^{n+1/2}
\end{aligned}$$

同理可得

$$\frac{1}{2\Delta y^2} \delta_y^2 (u_{jk}^n + u_{jk}^{n+1}) = (u_{yy} + O(\Delta t^2) + O(\Delta y^2)) \Big|_{jk}^{n+1/2}$$

因此

$$\begin{aligned}
T_{jk}^{n+1/2} &= (u_t - au_{xx} - bu_{yy}) \Big|_{jk}^{n+1/2} + O(\Delta t^2) + O(\Delta x^2) + O(\Delta y^2) \\
&= O(\Delta t^2) + O(\Delta x^2) + O(\Delta y^2)
\end{aligned}$$

稳定性：首先将 CN 格式整理为如下形式

$$v_{jk}^{n+1} - \frac{a\Delta t}{2\Delta x^2} \delta_x^2 v_{jk}^{n+1} - \frac{b\Delta t}{2\Delta y^2} \delta_y^2 v_{jk}^{n+1} = v_{jk}^n + \frac{a\Delta t}{2\Delta x^2} \delta_x^2 v_{jk}^n + \frac{b\Delta t}{2\Delta y^2} \delta_y^2 v_{jk}^n$$

取 $v_{jk}^n = \hat{v}^n e^{i(w_x x_j + w_y y_k)}$ 进行代入，注意到

$$\delta_x^2 v_{jk}^n = (e^{iw_x \Delta x} - 2 + e^{-iw_x \Delta x}) \hat{v}^n e^{i(w_x x_j + w_y y_k)} = -4 \sin^2\left(\frac{w_x \Delta x}{2}\right) \hat{v}^n e^{i(w_x x_j + w_y y_k)}$$

代入可得

$$\left(1 + \frac{a\Delta t}{2\Delta x^2} 4 \sin^2\left(\frac{w_x \Delta x}{2}\right) + \frac{b\Delta t}{2\Delta y^2} 4 \sin^2\left(\frac{w_y \Delta y}{2}\right) \right) \hat{v}^{n+1} = \left(1 - \frac{a\Delta t}{2\Delta x^2} 4 \sin^2\left(\frac{w_x \Delta x}{2}\right) - \frac{b\Delta t}{2\Delta y^2} 4 \sin^2\left(\frac{w_y \Delta y}{2}\right) \right) \hat{v}^n$$

因此放大因子为

$$\hat{Q} = \frac{1 - \frac{a\Delta t}{2\Delta x^2} 4 \sin^2\left(\frac{w_x \Delta x}{2}\right) - \frac{b\Delta t}{2\Delta y^2} 4 \sin^2\left(\frac{w_y \Delta y}{2}\right)}{1 + \frac{a\Delta t}{2\Delta x^2} 4 \sin^2\left(\frac{w_x \Delta x}{2}\right) + \frac{b\Delta t}{2\Delta y^2} 4 \sin^2\left(\frac{w_y \Delta y}{2}\right)}$$

需要证明 $|\hat{Q}| \leq 1$ 始终成立，由于 $a, b > 0$ ，显然有 $\hat{Q} < 1$ ；此外

$$\hat{Q} + 1 = \frac{2}{1 + \frac{a\Delta t}{2\Delta x^2} 4 \sin^2\left(\frac{w_x \Delta x}{2}\right) + \frac{b\Delta t}{2\Delta y^2} 4 \sin^2\left(\frac{w_y \Delta y}{2}\right)} > 0$$

因此 $\hat{Q} > -1$ ，得证无条件稳定性。

11.4 补充题 2

题目：构造 $u_t + u_x + u_y = 0$ 的 ADI 格式，并推导截断误差、分析其稳定性。

解答： ADI 格式如下

$$\begin{cases} \left(1 + \frac{\Delta t}{2\Delta x}\delta_x\right) v_{jk}^{n+1/2} = \left(1 - \frac{\Delta t}{2\Delta y}\delta_y\right) v_{jk}^n \\ \left(1 + \frac{\Delta t}{2\Delta y}\delta_y\right) v_{jk}^{n+1} = \left(1 - \frac{\Delta t}{2\Delta x}\delta_x\right) v_{jk}^{n+1/2} \end{cases}$$

或者写作

$$\left(1 + \frac{\Delta t}{2\Delta x}\delta_x\right) \left(1 + \frac{\Delta t}{2\Delta y}\delta_y\right) v_{jk}^{n+1} = \left(1 - \frac{\Delta t}{2\Delta x}\delta_x\right) \left(1 - \frac{\Delta t}{2\Delta y}\delta_y\right) v_{jk}^n$$

这里使用的记号 δ_x 和 δ_y 代表对一阶导的离散，可以使用中心差 ($\delta_x = \frac{1}{2}(E^1 - E^{-1})$)，也可以使用迎风差 ($\delta_x = E^1 - E^0$)。

对于 ADI 格式的稳定性：取 $v_{jk}^n = \hat{v}^n e^{i(w_x x_j + w_y y_k)}$ 进行代入，注意到

$$\delta_x v_{jk}^n = \frac{1}{2}(e^{iw_x \Delta x} - e^{-iw_x \Delta x}) \hat{v}^n e^{i(w_x x_j + w_y y_k)} = i \sin(w_x \Delta x) \hat{v}^n e^{i(w_x x_j + w_y y_k)}. \quad (\text{central})$$

$$\delta_x v_{jk}^n = (1 - e^{-iw_x \Delta x}) \hat{v}^n e^{i(w_x x_j + w_y y_k)}, \quad (\text{upwind})$$

因此放大因子为

$$\hat{Q} = \frac{\left(1 - \frac{\Delta t}{2\Delta y}(i \sin(w_y \Delta y))\right) \left(1 - \frac{\Delta t}{2\Delta x}(i \sin(w_x \Delta x))\right)}{\left(1 + \frac{\Delta t}{2\Delta y}(i \sin(w_x \Delta x))\right) \left(1 + \frac{\Delta t}{2\Delta y}(i \sin(w_y \Delta y))\right)}, \quad (\text{central})$$

$$\hat{Q} = \frac{\left(1 - \frac{\Delta t}{2\Delta y}(1 - e^{-iw_y \Delta y})\right) \left(1 - \frac{\Delta t}{2\Delta x}(1 - e^{-iw_x \Delta x})\right)}{\left(1 + \frac{\Delta t}{2\Delta x}(1 - e^{-iw_x \Delta x})\right) \left(1 + \frac{\Delta t}{2\Delta y}(1 - e^{-iw_y \Delta y})\right)}, \quad (\text{upwind})$$

对于基于中心差的 ADI 格式，显然有 $|\hat{Q}| = 1$ ，对于迎风的 ADI 格式，计算可得

$$\begin{aligned} \left|1 - \frac{\Delta t}{2\Delta y}(1 - e^{-iw_y \Delta y})\right|^2 &= \left(1 - \frac{\Delta t}{2\Delta y}2 \sin^2\left(\frac{w_y \Delta y}{2}\right)\right)^2 + \left(\frac{\Delta t}{2\Delta y} \sin(w_y \Delta y)\right)^2 \\ \left|1 + \frac{\Delta t}{2\Delta y}(1 - e^{-iw_y \Delta y})\right|^2 &= \left(1 + \frac{\Delta t}{2\Delta y}2 \sin^2\left(\frac{w_y \Delta y}{2}\right)\right)^2 + \left(\frac{\Delta t}{2\Delta y} \sin(w_y \Delta y)\right)^2 \end{aligned}$$

因此 $|\hat{Q}| \leq 1$ 。

对于截断误差的计算比较繁琐，考虑与 ADI 格式类似的 CN 格式

$$\begin{aligned} \tilde{v}_{jk}^{n+1} + \frac{a\Delta t}{2\Delta x}\delta_x \tilde{v}_{jk}^{n+1} + \frac{b\Delta t}{2\Delta y}\delta_y \tilde{v}_{jk}^{n+1} &= v_{jk}^n - \frac{a\Delta t}{2\Delta x}\delta_x v_{jk}^n - \frac{b\Delta t}{2\Delta y}\delta_y v_{jk}^n \\ \left(1 + \frac{a\Delta t}{2\Delta x}\delta_x + \frac{b\Delta t}{2\Delta y}\delta_y\right) \tilde{v}_{jk}^{n+1} &= \left(1 - \frac{a\Delta t}{2\Delta x}\delta_x - \frac{b\Delta t}{2\Delta y}\delta_y\right) v_{jk}^n \end{aligned}$$

CN 格式的局部截断误差（记作 $\tilde{T}_{jk}^{n+1/2}$ ）满足

$$\Delta t \tilde{T}_{jk}^{n+1} = \left(1 + \frac{a\Delta t}{2\Delta x} \delta_x + \frac{b\Delta t}{2\Delta y} \delta_y\right) u_{jk}^{n+1} - \left(1 - \frac{a\Delta t}{2\Delta x} \delta_x - \frac{b\Delta t}{2\Delta y} \delta_y\right) u_{jk}^n$$

ADI 格式的局部截断误差（记作 $T_{jk}^{n+1/2}$ ）满足

$$\Delta t T_{jk}^{n+1} = \left(1 + \frac{\Delta t}{2\Delta x} \delta_x\right) \left(1 + \frac{\Delta t}{2\Delta y} \delta_y\right) u_{jk}^{n+1} - \left(1 - \frac{\Delta t}{2\Delta x} \delta_x\right) \left(1 - \frac{\Delta t}{2\Delta y} \delta_y\right) u_{jk}^n$$

相减可得

$$\Delta t T_{jk}^{n+1} - \Delta t \tilde{T}_{jk}^{n+1} = \frac{\Delta t^2}{4\Delta x \Delta y} \delta_x \delta_y (u_{jk}^{n+1} - u_{jk}^n)$$

只需要证明这一项是高阶量，不影响局部截断误差的阶数即可。

如果 δ_x 和 δ_y 使用中心差，易知对于 CN 格式的局部截断误差为

$$\tilde{T}_{jk}^{n+1} = O(\Delta t^2) + O(\Delta x^2) + O(\Delta y^2)$$

此时

$$\begin{aligned} \frac{\Delta t^2}{4\Delta x \Delta y} \delta_x \delta_y u_{jk}^n &= \Delta t^2 (u_{xy} + O(\Delta x^2) + O(\Delta y^2))_{jk}^n \\ \frac{\Delta t^2}{4\Delta x \Delta y} \delta_x \delta_y (u_{jk}^{n+1} - u_{jk}^n) &= \Delta t^2 (u_{xyt} \Delta t + O(\Delta t^2) + O(\Delta x^2) + O(\Delta y^2))_{jk}^{n+1/2} \end{aligned}$$

因此对于 ADI 格式仍然有

$$T_{jk}^{n+1} = O(\Delta t^2) + O(\Delta x^2) + O(\Delta y^2)$$

如果 δ_x 和 δ_y 使用迎风差，则对于 CN 格式的局部截断误差为

$$\tilde{T}_{jk}^{n+1} = O(\Delta t^2) + O(\Delta x) + O(\Delta y)$$

此时

$$\begin{aligned} \frac{\Delta t^2}{4\Delta x \Delta y} \delta_x \delta_y u_{jk}^n &= \Delta t^2 (u_{xy} + O(\Delta x) + O(\Delta y))_{jk}^n \\ \frac{\Delta t^2}{4\Delta x \Delta y} \delta_x \delta_y (u_{jk}^{n+1} - u_{jk}^n) &= \Delta t^2 (u_{xyt} \Delta t + O(\Delta t^2) + O(\Delta x) + O(\Delta y))_{jk}^{n+1/2} \end{aligned}$$

因此对于 ADI 格式仍然有

$$T_{jk}^{n+1} = O(\Delta t^2) + O(\Delta x) + O(\Delta y)$$