Notes on equidistribution theory

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目录

1 d^c

- $\mathrm{d}^c = \mathrm{i}(\bar\partial - \partial)$
- $\mathrm{d}\mathrm{d}^c = \mathrm{i}(\partial + \bar\partial)(\bar\partial - \partial) = 2\mathrm{i}\partial\bar\partial$
- $$\begin{aligned}\mathrm{d}^c f &= \mathrm{i}\left(\frac{\partial f}{\partial \bar z}\mathrm{d}\bar z - \frac{\partial f}{\partial z}\mathrm{d}z\right)\\ &= \mathrm{i}\left(\frac{1}{2}\left(\frac{\partial f}{\partial x} + \mathrm{i}\frac{\partial f}{\partial y}\right)(\mathrm{d}x - \mathrm{i}\mathrm{d}y) - \frac{1}{2}\left(\frac{\partial f}{\partial x} - \mathrm{i}\frac{\partial f}{\partial y}\right)(\mathrm{d}x + \mathrm{i}\mathrm{d}y)\right)\\ &= \frac{\partial f}{\partial x}\mathrm{d}y - \frac{\partial f}{\partial y}\mathrm{d}x\end{aligned}$$
- $\mathrm{d}\mathrm{d}^c f = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\mathrm{d}x \wedge \mathrm{d}y$
- $\mathrm{d}f \wedge \mathrm{d}^c f = \left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial x}\right)^2\right)\mathrm{d}x \wedge \mathrm{d}y$
- 如果实函数 u, v 满足 $u + \mathrm{i}v$ 是全纯函数, 那么 $\mathrm{d}^c u = \mathrm{d}v$.

2 共轭形式

3 u_a

定义 $\mathbb{C}^2 - \{0\} - \pi^{-1}(a)$ 上的函数

$$\tilde{u}_a(z) = \log \frac{|Z|}{|\langle Z, a^{\perp} \rangle|}$$

因为对于 $\lambda \in \mathbb{C}^*$,

$$\tilde{u}_a(\lambda Z) = \log \frac{|\lambda Z|}{|\langle \lambda Z, a^{\perp} \rangle|} = \log \frac{|Z|}{|\langle Z, a^{\perp} \rangle|} = \tilde{u}_a(z)$$

所以我们可以定义 $\mathbb{CP}^1 - \{a\}$ 上的函数 u_a 使下图交换

$$\begin{tikzcd}
\mathbb{C}^2 - \{0\} - \pi^{-1}(a) \arrow[d, "\pi"'] \arrow[dr, "\tilde{u}_a"] & \\
& \mathbb{C} \\
\mathbb{CP}^1 - \{a\} \arrow[ur, "u_a"'] &
\end{tikzcd}$$

4 非积分型第一主定理

设 V 是黎曼面, $x\colon V \longrightarrow \mathbb{CP}^1$ 是非常值全纯映射, $D \subset V$ 是紧集, ∂D 光滑. 我们关心 x 在 D 中计重数意义下取到 $\mathbb{CP}^1$ 中某点 a 的次数.

定理 4.1. 固定 $a \in \mathbb{CP}^1$, 设 $x(\partial D) \cap \{a\} = \varnothing$, 那么

$$n(D,a) + \frac{1}{2\pi}\int_{\partial D} \mathrm{d}^c(x^* u_a) = \frac{1}{\pi}\int_D x^*\omega.$$

证明. 假设 x 在 D 中取不到 a, 该式退化为

$$\frac{1}{2\pi}\int_{\partial D} \mathrm{d}^c(x^* u_a) = \frac{1}{\pi}\int_D x^*\omega,$$

因为

$$\frac{1}{2}\mathrm{dd}^c u_a = \omega \quad \text{on } \mathbb{CP}^1 \setminus \{a\}\,,$$

所以该式就是 Stokes 公式. 当 $x^{-1}(a) \cap D = \{a_1, \cdots, a_n\}$ 时,

余下只需证

$$-\lim_{\varepsilon\to 0}\int_{\partial U_i} x^* \frac{1}{2\pi}\mathrm{d}^c u_a = x\text{在}a_i\text{处的重数}$$

□

5 非积分型第二主定理

6 调和穷竭

7　积分型第一主定理

引理 7.1 (Page37,Wu)**.**

$$\int_{\partial V[r_0]} \mathrm{d}^c x^* u_a = \frac{\partial}{\partial r}\bigg|_{r_0} \int_{\partial V[r]} x^* u_a \mathrm{d}^c \tau$$

证明.

$$\mathrm{d}^c x^* u_a = -\frac{\partial x^* u_a}{\partial \theta}\mathrm{d}r + \frac{\partial x^* u_a}{\partial r}\mathrm{d}\theta$$

$$\int_{\partial V[r_0]} \frac{\partial x^* u_a}{\partial r}\mathrm{d}\theta = \int_0^{\Gamma} \frac{\partial x^* u_a}{\partial r}(r_0,\theta)\mathrm{d}\theta = \int_0^{\Gamma} \frac{\partial}{\partial r}\bigg|_{r_0} x^* u_a(r,\theta)\mathrm{d}\theta = \frac{\partial}{\partial r}\bigg|_{r_0} \int_0^{\Gamma} x^* u_a \mathrm{d}\theta$$

□

引理 7.2 (Page13,Kodaira)**.**

$$\int_{\partial\Delta[s_0]} \mathrm{d}^c u_a = s_0 \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s_0} \int_0^{2\pi} u_a \mathrm{d}\theta$$

证明.

$$\begin{aligned}\int_{\partial\Delta[s_0]} \mathrm{d}^c u_a &= \int_{\partial\Delta[s_0]} \frac{\partial u_a}{\partial x}\mathrm{d}y - \frac{\partial u_a}{\partial y}\mathrm{d}x = \int_{\partial\Delta[s_0]} \left(\frac{\partial u_a}{\partial x}s\cos\theta - \frac{\partial u_a}{\partial y}s\sin\theta\right)\mathrm{d}\theta \\ &= \int_{\partial\Delta[s_0]} s\frac{\partial u_a}{\partial s}\mathrm{d}\theta = \int_0^{2\pi} s_0 \frac{\partial}{\partial s}\bigg|_{s_0} u_a \mathrm{d}\theta = s_0 \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s_0} \int_0^{2\pi} u_a \mathrm{d}\theta\end{aligned}$$

□

注记.

$$r = \log s \Longrightarrow \frac{\partial}{\partial r} = s\frac{\partial}{\partial s}$$

$$\int_{r_0}^{r_n} n(t,a)\mathrm{d}t + \frac{1}{2\pi}\int_{\partial V[t]} x^* u_a \mathrm{d}^c \tau \bigg|_{r_0}^{r_n} = \int_{r_0}^{r_n} v(t)\mathrm{d}t$$

8 积分型第二主定理

引理 8.1.

$$\int_{\partial V[r]} \kappa = \frac{\partial}{\partial r}\left(\frac{1}{2}\int_{\partial V[r]}(\log h)\mathrm{d}^c\tau\right)$$

定理 8.2.

$$E(r) + N_1(r) - 2T(r) = \frac{1}{4\pi}\int_{\partial V[t]} \log h \ \mathrm{d}^c\tau \Big|_{r_0}^{r}$$

9 $N(r,a)$ 与 $T(r)$ 的关系汇总

$$T(r) = \frac{1}{\pi}\int_{\mathbb{CP}^1} N(r,a)\omega.$$

10　亏量 δ^*

定义 10.1 (Page46,Wu;Page14,Kodaira)**.** 对于 $a \in \mathbb{CP}^1$, 定义

$$\delta^*(a) = 1 - \limsup_{r\to\infty} \frac{N(r,a)}{T(r)},$$

称 $\delta^*(a)$ 为 a 的亏量.

命题 10.2.

$$0 \leqslant \delta^* \leqslant 1.$$

证明. 因为 $N(r,a)$ 和 $T(r)$ 都是非负的, 所以 $\delta^* \leqslant 1$ 是显然的. 下证 $\delta^* \geqslant 0$. 因为

$$N(r,a) < T(r) + \text{const} \Longrightarrow \frac{N(r,a)}{T(r)} < 1 + \frac{\text{const}}{T(r)},$$

而

$$T(r) = \int_{r_0}^{r} v(t)\mathrm{d}t > v(r_0)(r - r_0) \longrightarrow +\infty, \quad r \to +\infty.$$

□

定理 10.3. δ^* 在 $\mathbb{CP}^1$ 上几乎处处为零.

证明.

$$T(r) = \frac{1}{\pi}\int_{\mathbb{CP}^1} N(r,a)\omega \Longrightarrow \int_{\mathbb{CP}^1}\left(1 - \frac{N(r,a)}{T(r)}\right)\omega = 0$$

□

11 亏量关系

我们的出发点是

$$N(r,a) < T(r) + \mathrm{const}\,.$$

设 $\rho\colon \mathbb{CP}^1 \to \mathbb{R}$ 满足 ρ 是非负可积函数且 $\int_{\mathbb{CP}^1} \rho\omega = 1$. 事实上这样的 ρ 可谓一抓一大把, 我们引入 ρ 是为了给本来没有自由度的式子添加一个自由度, 让我们能够构造出一些个想要的东西.

对式子两边积分得

$$\int_{\mathbb{CP}^1} N(r,a)\rho(a)\omega < \int_{\mathbb{CP}^1} \left(T(r) + \mathrm{const}\right)\rho(a)\omega = T(r) + \mathrm{const}\,.$$

左侧

$$\int_{\mathbb{CP}^1} N(r,a)\rho(a)\omega = \int_{\mathbb{CP}^1} \left(\int_{r_0}^{r} n(t,a)\mathrm{d}t\right)\rho(a)\omega \xlongequal{\text{Fubini?}} \int_{r_0}^{r} \left(\int_{\mathbb{CP}^1} n(t,a)\rho(a)\omega\right)\mathrm{d}t$$

事实上上述计算曾出现在证明

$$\pi T(r) = \int_{\mathbb{CP}^1} N(r,a)\omega$$

的过程中. 当 $\rho \equiv 1$ 时,

$$\int_{\mathbb{CP}^1} n(t,a)\omega = \int_{V[t]} x^*\omega = \pi v(t).$$

在当下

$$\begin{aligned}
&\int_{\mathbb{CP}^1} n(t,a)\rho(a)\omega \\
=&\int_{V[t]} (x^*\rho)(x^*\omega) \\
\geqslant&\int_{V[t]-V[r_0]} (x^*\rho)(x^*\omega) && \text{意义不明的一步} \\
=&\int_{V[t]-V[r_0]} (x^*\rho)h\mathrm{d}\tau \wedge \mathrm{d}^c\tau \\
=&\int_{r_0}^{t} \left(\int_{\partial V[s]} (x^*\rho)h\mathrm{d}^c\tau\right)\mathrm{d}s
\end{aligned}$$