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LOCAL PLÜCKER FORMULAS FOR A SEMISIMPLE LIE GROUP

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The local Plücker formulas connect the metrics on a holomorphic curve in $\mathbb{C}P^n$ induced by the Plücker embeddings of the adjoined curves with the corresponding curvatures. The classical (global) Plücker formulas are obtained from the local ones via integration over the curve [3]. Givental' [2] remarked that the vector of curvatures is expressible through the vector of metrics by means of a Cartan matrix of type A_n. In the present paper we prove a generalization of this fact to the case of an arbitrary Cartan matrix, which was formulated by Givental' as a conjecture.

Let G be a complex simply connected semisimple Lie group of rank n, K be a compact form of G, and $\mathfrak{g}$ be the Lie algebra of G. Let F be the set of all Borel subgroups B of the group G (the flag variety of G).

We shall consider linear bundles $\mathcal{L}$ on F, to which there is a lift of the action of G on F, i.e., for $g \in G$, $B \in F$ there is given an isomorphism $g_*: \mathcal{L}_B \to \mathcal{L}_{gBg^{-1}}$. Fix a point $B_0 \in F$. The subgroup B_0 acts on the fiber $\mathcal{L}_{B_0}$ by means of some character $\omega: B_0 \to \mathbb{C}^*$. It is readily seen that the bundle $\mathcal{L}$ with the action of G is uniquely recoverable from the character ω; we denote by $\mathcal{L}(\omega)$ the result of this recovering process. In fact, any bundle $\mathcal{L}$ admits a unique, up to an automorphism, action of G, and so the linear bundles over F are in a bijective correspondence with the characters of the group B_0. Let $\omega_i: B_0 \to \mathbb{C}^*$, $i = 1, \ldots, n$, be fundamental characters, and let $\mathcal{L}_i = \mathcal{L}(\omega_i)$ be the corresponding bundles; then $V_i = \Gamma(F, \mathcal{L}_i)^*$ are fundamental representations of G (the index i runs through the vertices of the Dynkin diagram). V_i is irreducible, and therefore on $\mathbb{P}(V_i)$ there is a uniquely-defined Fubini–Study metric FS, corresponding to a unique, up to a constant, K-invariant Hermitian form on V_i. Let $f: S \to F$ be a holomorphic curve, let $\pi_i: F \to \mathbb{P}(V_i)$ be the projective morphisms associated with the sheaves $\mathcal{L}_i$, and let $\varphi_i = f^*\pi_i^*(FS)$ be the induced metrics on S. We shall assume that the metrics φ_i are not identically equal to zero, and since the problem under examination is local, we shall deal with a neighborhood of a point where all of them are different from zero. Let θ_i be the curvatures of the metrics φ_i.

Let us define a natural n-dimensional distribution $\mathcal{N}$ on the flag variety F. The tangent space T_BF is isomorphic to $\mathfrak{g}/\mathfrak{b}$, where $\mathfrak{b}$ is the Lie algebra of the group B. Set

$$\mathcal{N}(B) = \{x \in \mathfrak{g} : [x, [\mathfrak{b}, \mathfrak{b}]] \subset \mathfrak{b}\}/\mathfrak{b}.$$

<u>THEOREM 1</u>. Suppose the curve S is tangent to the distribution $\mathcal{N}$. Let φ be the vector of (1, 1)-forms composed of the metrics φ_i on S, and let θ be the vector of curvatures. Then $\theta = A\varphi$, where A is the Cartan matrix of the group G.

The connection with the classical case is as follows. Let $K = SU(n + 1)$. Then F is the flag variety in $\mathbb{C}^{n+1}$. The morphism π_i is the composition of the projection onto the Grassmannian which sends a flag into its i-plane, and the Plücker embedding; $\pi_1(S)$ is a curve in $\mathbb{C}P^n$. The condition that S belongs to the distribution $\mathcal{N}$ means that the preimage

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in S of a point $x \in \pi_1(S)$ is the moving flag of the curve $\pi_i(S)$ at the point x, i.e., $\pi_i(S)$ are Plücker-embedded adjoined curves for $\pi_i(S)$. The assertion of the theorem is identical with the local Plücker formulas for the curve $\pi_1(S)$, given in [3].

Proof. A subgroup $P \subset G$ is called a parabolic subgroup [1, 4] if it contains a Borel subgroup. The subgroups $P_I(B)$ containing B correspond to subsets I of the vertex set of the Dynkin diagram, and the type I of a parabolic subgroup does not depend on the choice of the Borel subgroup contained in it. Let $F_I \simeq G/P_i(B)$ be the variety of parabolic subgroups of type I. Instead of the subsets $\{i\}$ and $\{1, \ldots, \hat{i}, \ldots, n\}$ we shall write the indices i and i', respectively.

I. The map π_i factors as the composition of the canonical projection $\rho_i: F \to F_{i'}$, $\rho_i(B) = P_{i'}(B)$ and an embedding $\tau_i: F_{i'} \to \mathbb{P}(V_i)$.

Let us show that the map π_i is constant on the fibers of ρ_i, i.e., the bundle $\mathscr{L}_i$ is trivial along these fibers. The fibers of ρ_i are the sets of all Borel subgroups that are contained in a fixed subgroup of the form $P_{i'}(B)$, or, equivalently, all Borel subgroups in the semisimple subgroup $G_{i'} \subset G$ generated by all simple roots except the i-th. Now $\mathscr{L}(\omega_i)|_{F(G_{i'})} = \mathscr{L}(\omega_i|_{B\cap G_{i'}})$, but $\omega_i|_{B\cap G_{i'}} = 1$ by definition. The morphisms τ_i are embeddings, since they are morphisms of G-spaces, and $P_{i'}(B)$ is a maximal proper subgroup of G.

II. Let $\mathscr{N}_i$ be the one-dimensional distribution on F defined as $\mathscr{N}_i(B) = \mathfrak{p}_i(B)/\mathfrak{b}$, where $\mathfrak{p}_i(B)$ is the Lie algebra of the group $P_i(B)$. Then, as is readily seen, $\mathscr{N}(B) = \bigoplus_{i=1}^{n} \mathscr{N}_i(B)$.

III. $d\pi_i(\mathscr{N}_j) = 0$ for $i \neq j$. Indeed, $P_j(B) \subset P_{i'}(B)$.

IV. Let $\theta(\mathscr{L})$ be the curvature of a K-invariant metric $\|\cdot\|$ in an arbitrary bundle $\mathscr{L}$ over F equipped with an action of G. Let s be a local section of $\mathscr{L}$. Then one has the following formula [3]:

$$\theta(\mathscr{L}) = -\partial\bar{\partial} \ln \| s \|,$$

from which it follows, in particular, that the mapping $\mathscr{L} \mapsto \theta(\mathscr{L})$ takes tensor products into sums.

V. $\varphi_i = f^*\theta(\mathscr{L}_i)$. Let us show that $\pi_i^*(FS) = \theta(\mathscr{L}_i)$. Let s^* be a local section of $\mathscr{L}_i^*$. Then $\theta(\mathscr{L}_i) = \partial\bar{\partial}\ln\|s^*\|$. The section s^* yields a local lift $\tilde{\pi}_i: U \to V_i$, $U \subset F$, by the rule $\tilde{\pi}_i(u)(t) = \langle s^*(u), t(u)\rangle$, $t \in \Gamma(F, \mathscr{L}_i)$, $u \in U$. Therefore, by definition, $\pi_i^*(FS) = \partial\bar{\partial}\ln\|\tilde{\pi}\| = \partial\bar{\partial}\ln\|s^*\|$.

VI. $\theta_i = f^*\theta(\mathscr{N}_i)$, where the $\mathscr{N}_i$ are regarded as abstract bundles. In fact, by III, the metric φ_i on S is lifted by means of the projection on $\mathscr{N}_i$, i.e., is induced by the isomorphism $\tau S \tilde{\to} f^*\mathscr{N}_i$.

VII. $(\mathscr{N}) = (\mathscr{L})^A$, i.e., $\mathscr{N}_i \simeq \otimes_j \mathscr{L}_j^{\otimes a_{ij}}$. The bundles $\mathscr{N}_i$ correspond, as is readily seen, to the characters effecting the Ad^{-1}-action of B on the lines $N_i \subset g/b$, i.e., to the characters α_i corresponding to the simple roots, and $(\alpha) = (\omega)^A$.

The needed assertion follows upon combining the results IV-VII.

Note. In the case $K = Sp(2n)$, i.e., the case of a Cartan matrix of type C_n, the theorem is proved in [2] by means of reduction to the classical case.

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