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THE MOVING FRAME, DIFFERENTIAL INVARIANTS AND RIGIDITY THEOREMS FOR CURVES IN HOMOGENEOUS SPACES

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Introduction

Klein conceived of a geometry as being the study for a given class of figures in a space of those properties which are left invariant by some transitive group of transformations. In modern dress, the space is a homogeneous space G/H with G a Lie group, H a closed subgroup, and with the group of transformations being the left action of G. The figures to be considered here are smooth curves $[a, b] \xrightarrow{X} G/H$; the same methods work verbatim for holomorphic curves $\Delta \xrightarrow{X} G/H$ when G and H are complex Lie groups. We hope to consider the case of higher-dimensional domains in a later paper.

The natural objects of study in this context were known classically as *differential invariants*, local expressions in X and its derivatives invariant under the action of G†. The *order of a differential invariant* is the order of the highest derivative that occurs in the local expression for it. These generalize velocity, curvature, and torsion for curves in Euclidean $\mathbb{R}^3$, which are differential invariants of orders 1, 2, and 3 respectively. E. Cartan, inspired by the methods used by Darboux in the differential geometry of surfaces, developed a beautiful approach to studying these invariants, the method of moving frames. Cartan's

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†Some classical authors include invariance under reparametrization.

idea is to associate in a natural way to each nice map to G/H a lifting of the map to G. Such a lifting procedure is called a moving frame, and one expects the pullbacks of the left-invariant (Maurer-Cartan) one-forms on G to give a complete set of differential invariants for maps to G/H. Although Cartan gave numerous examples of moving frames ([C1], [C2]), he did not give a precise proof that a moving frame exists in general, nor a procedure for constructing it.

The principal difficulty in the general theory lies in the phenomenon of degeneracy, exemplified by the undefinability of torsion of curves in $\mathbb{R}^3$ at points where the curvature vanishes. Some restriction on the class of curves to be considered is thus necessary, and the natural class turns out to be what we call curves of constant type.

Our concern in this paper is not so much with laying the theoretical foundations of the method of moving frames, but rather with developing a good way to carry out the computation of the moving frame and the differential invariants when confronted with a G/H. The author's motivation for considering this topic was the need for an explicit moving frame in a particular situation, that of maps to the period domains arising from variations of Hodge structure. Happily, it is possible to eliminate jets from the computation and use just the adjoint action of H on certain quotients of the Lie algebra of G. For H reductive, only the action of H on the tangent space of G/H need be considered.

Taking the slight liberties with the fine points of our subject to which writers of introductions believe themselves entitled, we proceed to a description of the main results. Let

$$J_m(G/H) = \left\{ m\text{-jets at } 0 \text{ of smooth curves } [-1, 1] \xrightarrow{X} G/H \right\}$$

and note that the left action of G on G/H induces one on $J_m(G/H)$. Let

$$\tau(m) = \dim(G \backslash J_m(G/H))$$

which we call the *number of differential invariants of order* $\leq m$ for non-degenerate maps. This includes the derivatives of differential invariants of order $< m$, which we can suppress by taking

$$\nu(m) = \tau(m) - \tau(m-1)$$

the *number of independent differential invariants of order* $\leq m$. Finally, let

$$i_m = \nu(m) - \nu(m-1)$$

the *number of independent differential invariants of order* m. For non-degenerate curves in Euclidean $\mathbb{R}^3$, for example,

$$\tau(3) = 6 \qquad v, v', v'', k, k', \tau$$

$$\nu(3) = 3 \qquad v, k, \tau$$

$$i_3 = 1 \qquad \tau$$

where v is velocity, k is curvature, and τ torsion.

It came as a surprise to the author that the number of differential invariants can be computed by a very simple algorithm, even without first computing the moving frame. This is the method of Cartan Polygons. Associated to the class of non-degenerate curves in G/H is a tower of subgroups of G

$$H_0 \supseteq H_1 \supseteq H_2 \supseteq H_3 \supseteq H_4 \cdots$$

where $H_0 = G$, $H_1 = H$, called the *isotropy tower* of non-degenerate curves in G/H. Given this, we obtain the number of independent differential invariants of order m by showing

$$i_m = \dim(H_{m-1}) - 2\dim(H_m) + \dim(H_{m+1}).$$

This reduces the question to calculating the isotropy tower. This may be done in three ways:

1. *Jet method*

Pick a generic map $[a, b] \xrightarrow{X} G/H$ and let $j_m \in J_m(G/H)$ be its m-jet at t. Then

$$H_{m+1} = \{g \in G | gj_m = j_m\}$$

the isotropy group of j_m under the action of G on $J_m(G/H)$. For those G/H which have what we call a non-degenerate type, this tower will be conjugate to those obtained for other values of t and for other generic maps X.

2. *Lie algebra method*

Let v be a generic element of $\mathcal{G}$, the Lie algebra of G. Let v_1 be the projection of v to $\mathcal{G}/H$. Then

$$H_2 = \{g \in H | ad_g(v_1) = v_1\}$$

the isotropy of v_1 under the adjoint action of H on $\mathcal{G}/\mathcal{H}$. Let v_2 be the projection of v on $\mathcal{G}/\mathcal{H}_2$, let H_3 be the isotropy of v_2 under the action of H_2 on $\mathcal{G}/\mathcal{H}_2$, and proceed inductively.

In case there is an H-invariant splitting

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{T}$$

as will happen if H is compact or, more generally, reductive, then picking a sequence of generic elements $v_1, v_2, v_3, \cdots$ of $\mathcal{P}$, we have

$$H_m = \{g \in H | ad_g(v_i) = v_i, i = 1, \cdots, m-1\}$$

the isotropy of $(v_1, \cdots, v_{m-1})$ under the adjoint action of H on $\mathcal{P}^{m-1}$.

3. *Geometric method*

At least in case there is an H-invariant splitting

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{T}$$

and presumably in general, if $p_1, p_2, p_3, \cdots$ is a generic sequence of points of

G/H near the identity coset, then

$$H_m = \{g \in G | gp_i = p_i,\ i = 1, \cdots, m\}$$

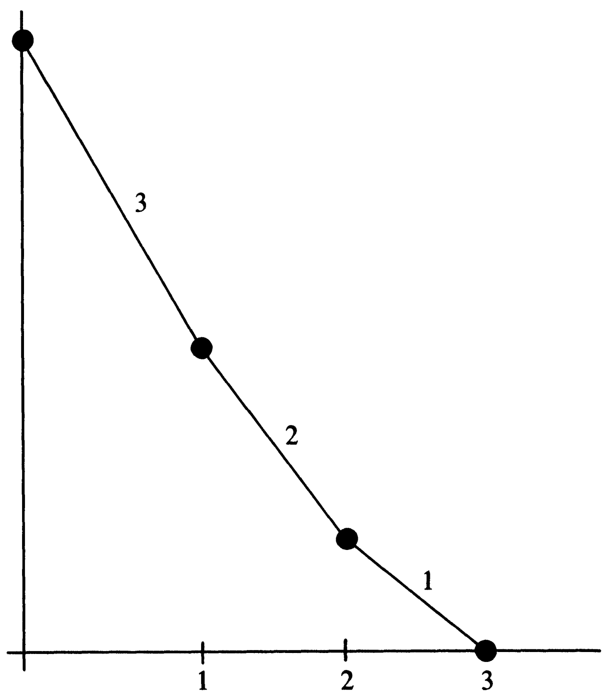
the isotropy group of a generic m-tuple of points.

Once the isotropy tower has been obtained, the Cartan Polygon is constructed by plotting the points $(m, \dim(H_m))$ for integers $m \geq 0$ and connecting them. On each line segment, the slope of that segment is written. Then i_m is the difference of the slopes of the two segments meeting above m.

As an example, letting $E(3)$ denote the Euclidean group in three variables, Euclidean $\mathbb{R}^3$ is $E(3)/SO(3)$. By the geometric method above, the isotropy tower of a non-degenerate map is seen to be

$$\begin{array}{ccccccccc} E(3) & \supset & SO(3) & \supset & SO(2) & \supset & \{e\} & = & \{e\} = \cdots \\ 6 & & 3 & & 1 & & 0 & & 0 \end{array}$$

where below each group is written its dimension. The Cartan Polygon is thus



Thus $i_m = 1$ for $m = 1, 2, 3$ and is 0 otherwise. These are velocity, curvature, and torsion.

We are able to answer some questions posed by Griffiths in his paper on moving frames [G]. Two curves

$$[a, b] \underset{X^{\#}}{\overset{X}{\rightrightarrows}} G/H$$

are said to have *contact* m at t if their m-jets at t lie in the same orbit of $J_m(G/H)$ under the action of G, and *contact* m if this holds for all t. They are said to be *congruent* if for some $g \in G$,

$$X^{\#}(t) = gX(t) \quad \text{for all} \quad t \in [a, b].$$

We answer the question of what is the minimal order of contact r which implies that two non-degenerate curves are congruent, namely

$$r = \max\{m | i_m \neq 0\} = \max\{m | \dim(H_{m-1}) \neq \dim(H_m)\}.$$

Rather surprisingly, there turn out to be a priori bounds

$$r \leq \dim(G/H) \qquad \text{if } H \text{ compact}$$

$$r \leq \dim(G/H) + 1 \qquad \text{if } H \text{ reductive,}$$

as well as a bound for r for certain types of symmetric spaces.

Section 8 is a collection of examples. Although these are placed at the end, they are first in the author's thoughts. Some interesting phenomena which can occur are:

(1) The order of contact r required for congruence of degenerate maps of constant type may exceed that for non-degenerate maps, see example 4a.

(2) Degenerate maps of constant type may have $H_k = 0$ and thus not lie in the fixed point set of any $w \in G$, see example 4a.

(3) Twisted types can occur, see example 5.

(4) The spaces $\tilde{S}_m$ with the quotient topology may fail to be Hausdorff if H is not compact, see example 5.

(5) For H not compact, there may be a continuous family of possible isotropy types and G/H may not admit a non-degenerate type, see example 8.

I gratefully acknowledge my debt to Phillip Griffiths' paper [G], which served as the point of departure for this one. I am thankful to Bill Haboush, Bob Blattner, Harsh Pittie and Joe Wolf for patiently answering various questions.

There is a large literature on moving frames and my knowledge of it is scanty. R. Hermann [H′] and John Adams [A] have written earlier papers, and the interesting results of G. Jensen [J] are coeval with mine. The book review by Hermann Weyl [W] gives an excellent discussion of Cartan's philosophy. The construction given here and the underlying notion of maps of constant type seems to be new, as do all the explicit results on the number of differential invariants of each order and on the order of contact required for congruence.

1. Definitions

This section consists of definitions to be used later on.

Congruence. Two maps $[a, b] \xrightarrow{X, X^{\#}} G/H$ are *congruent* if there exists

$w \in G$ so that $X^{\#}(t) = wX(t)$ for all t.

Contact. Let

$$J_m(G/H) = \{m\text{-jets at 0 of smooth curves } [-1,1] \xrightarrow{X} G/H\}$$

and we will denote the m-jet at t of a map

$$[a, b] \xrightarrow{X} G/H$$

by

$$j_m(X)(t).$$

Two maps

$$[a, b] \xrightarrow{X,\, X^{\#}} G/H$$

will be said to be the *same to order m at t* or *agree to order m at t* if

$$j_m(X)(t) = j_m(X^{\#})(t).$$

We say X and $X^{\#}$ have *contact m at t* if there exists $w \in G$ so that

$$j_m(wX)(t) = j_m(X^{\#})(t).$$

If for each $t \in [a, b]$ we can find such a w, possibly varying with t, we say X and $X^{\#}$ have *contact m on* $[a, b]$ or just *contact m*. This is the same as saying the maps

$$[a, b] \xrightarrow{Gj_m(X),\, Gj_m(X^{\#})} G \backslash J_m(G/H)$$

are the same.

Differential invariants. A *differential invariant of order less than or equal to m* for a class $\mathscr{S}$ of germs at 0 of smooth maps from $[-1, 1]$ to G/H is a smooth map

$$G \backslash j_m(\mathscr{S}) \xrightarrow{D} \mathbb{R}$$

where

$$j_m(\mathscr{S}) = \{m\text{-jets at 0 of maps } X \in \mathscr{S}\}.$$

If the germ of X at t, translated to 0, belongs to $\mathscr{S}$ for all $t \in [a, b]$, D induces a map

$$[a, b] \xrightarrow{D(X)} \mathbb{R}$$

by

$$D(X)(t) = D(Gj_m(X)(t)).$$

The map which associates to X the map

$$\frac{d}{dt} D(X)$$

comes from a differential invariant

$$G \backslash J_{m+1}(\mathscr{S}) \to \mathbb{R}$$

denoted D' and called the *derivative* of the differential invariant D.

The differential invariants of order less than or equal to $m - 1$ are also differential invariants of order less than or equal to m, and the complement of the former in the latter are the *differential invariants of order* m for the class $\mathscr{S}$. Setting

$$\tau(m) = \dim(G \backslash J_m(\mathscr{S}))$$

$$\nu(m) = \tau(m) - \tau(m - 1)$$

$$i_m = \nu(m) - \nu(m - 1)$$

we call $\tau(m)$ *the number of differential invariants of order* $\leq m$, $\nu(m)$ *the number of independent differential invariants of order* $\leq m$ (here independent means we delete derivatives of earlier invariants), i_m *the number of independent differential invariants of order* m.

Moving frame. Let $\mathscr{S}$ be a subset of the germs at 0 of smooth maps from $[-1, 1]$ to G/H. Let $\mathscr{G}$ denote the Lie algebra of G. A *moving frame* for the class $\mathscr{S}$ is a smooth map

$$G \backslash \mathscr{S} \xrightarrow{F} \mathscr{G}$$

satisfying properties (1) and (2) below. If $[a, b] \xrightarrow{X} G/H$ is a smooth map so that the germ of X at t, translated to 0, lies in $\mathscr{S}$, then F induces a map

$$[a, b] \xrightarrow{F(X)} \mathscr{G}$$

by applying F to the germ of X at t, translated to 0, for each t. We require:

(1) For any X as above, $F(X)$ can be integrated to a smooth map

$$[a, b] \xrightarrow{u} G$$

lifting X, i.e., so that the diagram

$$\begin{tikzcd} & G \arrow[d, "\pi = \text{coset map}"] \\ {[a, b]} \arrow[ur, "u"] \arrow[r, "X"'] & G/H \end{tikzcd}$$

commutes. Such a u is called a *moving frame for* X.

(2) For some k, the map F factors through $G/J_k(\mathscr{S})$.

Identifying $\mathscr{G} \approx \mathbb{R}^m$, these conditions imply the coefficients of F are differential invariants of order $\leq k$.

The reason that F is not defined rather as a map $\mathscr{S} \to G$ equivariant in G is that for certain maps X of constant type, e.g., lines in Euclidean $\mathbb{R}^3$, the moving frame for X is not well-defined, while its derivative is.

In practice, one wants the k in (2) to be as low as possible for the given G/H and class $\mathcal{S}$. Our moving frame will do this.

Isotropy tower. Two towers

$$H_1 \supseteq H_2 \supseteq H_3 \supseteq \cdots$$

$$H_1^\# \supseteq H_2^\# \supseteq H_3^\# \supseteq \cdots$$

of subgroups of G will be said to be *equivalent* or *conjugate* if there exists $w \in G$ so that

$$w^{-1}H_m w = H_m^\# \quad \text{all} \quad m \in \mathbb{N}.$$

For a smooth curve

$$[a, b] \xrightarrow{X} G/H$$

the *isotropy tower of X at $t \in [a, b]$* is

$$H_1^t \supseteq H_2^t \supseteq H_3^t \supseteq \cdots$$

where

$$H_{m+1}^t = \{w \in G | j_m(wX)(t) = j_m(X)(t)\} \qquad m \in \mathbb{N}$$

i.e., H_{m+1}^t is the isotropy group of $j_m(X)(t)$ for the action of G on $J_m(G/H)$. We say X is *weakly of constant type* on $[a, b]$ if there is a fixed tower

$$H_1 \supseteq H_2 \supseteq H_3 \supseteq \cdots$$

of subgroups of G to which the isotropy tower

$$H_1^t \supseteq H_2^t \supseteq H_3^t \supseteq \cdots$$

is conjugate for all $t \in [a, b]$. This equivalence class of towers is the *weak isotropy type* or *weak type* of X. A tower

$$H_1 \supseteq H_2 \supseteq H_3 \supseteq \cdots$$

will be said to *stabilize at k* if

$$H_{k-1} \supsetneq H_k = H_{k+1} = H_{k+2} = \cdots.$$

If the weak isotropy type

$$H_1 \supseteq H_2 \supseteq H_3 \supseteq \cdots$$

of X stabilizes at k, we will say X has weak type $(H_1, H_2, H_3, \cdots, H_k)$, and this notation will always mean the tower stabilizes at the last group listed.

We may rephrase the notion of weak type as follows. Let

$$\mathcal{F}_m = \{j_m(X) \in J_m(G/H) | \text{the isotropy group of the action of } G \text{ on } j_r(X) \text{ is } H_{r+1} \text{ for } r = 0, 1, \cdots, m\}.$$

Then a map X is weakly of constant type $(H_1, H_2, \cdots)$ iff there exists a smoothly varying $w(t) \in G$ so that for all $t \in [a, b]$

$$w(t)j_m(X)(t) \in \mathscr{F}_m \qquad m = 0, 1, 2, \cdots.$$

If we choose local sections

$$\mathscr{S}_m \underset{p_m}{\overset{r_m}{\leftrightarrows}} V_m = p_m(\mathscr{F}_m) \subseteq G\backslash J_m(G/H)$$

and

$$Q_m = r_m(V_m)$$

then if we can choose $w(t)$ as above so that

$$w(t)j_m(X)(t) \in Q_m \qquad m = 0, 1, 2, \cdots$$

we say X has *constant type* $(H_1, H_2, \cdots, H_k)$.

If

$$[a, b] \xrightarrow{X} G/H$$

and we have already for all t found $w(t)$ so that

$$w(t)j_{m-1}(X) \in Q_{m-1}$$

the isotropy group of

$$w(t)j_m(X)$$

must be a subgroup H^t_{m+1} of H_m. We can modify $w(t)$ to some smoothly varying $\tilde{w}(t) \in G$ so that

$$\tilde{w}(t)j_m(X)(t) \in Q_m$$

for all t if and only if the subgroup

$$H^t_{m+1}$$

are all conjugate in H_m and for all t

$$p_m j_m(X)(t) \in V_m$$

Non-degenerate. If a generic curve $[a, b] \xrightarrow{X} G/H$ has constant type $(H_1, \cdots, H_k)$, we call curves having this type *non-degenerate*, the type $(H_1, \cdots, H_k)$ the *non-degenerate type* or *standard type*, and say G/H *admits a non-degenerate type*. Here generic means a subset $\mathscr{S}$ of the smooth curves from $[a, b]$ to G/H that is open and dense in the $\mathscr{C}^m$ topology (given locally by $\sup(\|X - Y\| + \|X' - Y'\| + \cdots + \|X^{(m)} - Y^{(m)}\|)$ for some m.

If H is compact, then the action of H_m on the preimage of Q_{m-1} has only a finite number of possible H_m-conjugacy classes for isotropy groups of points of Q_m, by a general property of compact groups (see [B]) and one of these occurs for an open dense subset of points. Thus, a G/H admits a non-degenerate type when H is compact. Non-degenerate types exist in virtually every geometrically interesting example.

2. The moving frame for curves of constant type

Let G be a Lie group, H a closed subgroup. Several conventions of notation will be used here. Script letters will denote the Lie algebras of corresponding subgroups. The adjoint action of an element $k \in G$ on a vector v in $\mathcal{G}$ or a quotient of $\mathcal{G}$ will be denoted kv. If a group K acts on a vector space V, K^v will denote the isotropy group $\{k \in K | kv = v\}$. If K is a subgroup of L, $N_L(K)$ will denote the normalizer of K in L. As a descending chain of subalgebras $\mathcal{H} = \mathcal{H}_1 \supseteq \mathcal{H}_2 \supseteq \mathcal{H}_3 \supseteq \cdots$ of $\mathcal{G}$ is defined, π_m will denote the canonical projection $\mathcal{G} \to \mathcal{G}/\mathcal{H}_m$ or $\mathcal{G}/\mathcal{H}_j \to \mathcal{G}/\mathcal{H}_m$, $j < m$, the meaning being clear from the domain.

Let $[a, b] \xrightarrow{X} G/H$ be a smooth map, and $G \xrightarrow{\pi} G/H$ the coset projection. A *lifting of X to G* is any smooth map $[a, b] \xrightarrow{u} G$ so $\pi \circ u = X$. As π is a fibering, a lifting of X always exists. We wish to choose such a lifting as canonically as possible.

To begin, choose any lifting $[a, b] \xrightarrow{u} G$ of X and let $[a, b] \xrightarrow{g} \mathcal{G}$ be its derivative. Consider the adjoint action of H on $\mathcal{G}/\mathcal{H}$ and the isotropy group $H^{\pi_1(g(t))}$, $t \in [a, b]$. We say X has *constant type at the first step* if the groups $H^{\pi_1(g(t))}$ are all conjugate in H as t varies. Assume X has this property, and let H_2 denote one subgroup in this conjugacy class. Let

$$F_1 = \{v \in \mathcal{G}/\mathcal{H} | H^v = H_2\}$$

$$T_1 = \{v \in \mathcal{G}/\mathcal{H} | H^v \text{ conjugate in } H \text{ to } H_2\}.$$

There is a natural map

$$T_1 \xrightarrow{\rho} H/N_H(H_2)$$

given by $\rho(v) = hN_H(H_2)$ if $H^v = hH_2h^{-1}$. As

$$H \to H/N_H(H_2)$$

is a fibering, we can lift the map

$$[a, b] \xrightarrow{\rho \circ \pi_1 \circ g} H/N_H(H_2)$$

to a smooth map

$$[a, b] \xrightarrow{h} H$$

so $hN_H(H_2) = \rho \circ \pi_1 \circ g$. If we replace our original lifting of X to G by

$$u^{\#} = uh$$

with derivative

$$[a, b] \xrightarrow{g^{\#}} \mathcal{G}$$

then $u^{\#}$ is a lifting of X and

$$g^{\#} = h^{-1}g + h'$$

where

$$[a, b] \xrightarrow{h'} \mathscr{H}$$

is the derivative of h. Thus

$$\pi_1(g^\#) = h^{-1}\pi_1(g)$$

and hence

$$\pi_1(g^\#) \in F_1$$

for all t. We have shown, dropping the $\#$'s, that every curve $[a, b] \xrightarrow{X} G/H$ of constant type at the first step admits a lifting u whose derivative g has $\pi_1(g) \in F_1$ for all t. If $u^\#$ is another such lifting, then

$$u^{-1}u^\# \in N_H(H_2)$$

for all t. Let

$$F_1 \xrightarrow{p_1} N_H(H_2)\diagdown F_1$$

be the quotient projection for the adjoint action of $N_H(H_2)$ on F_1. Let

$$I_1 = N_H(H_2)\diagdown F_1$$

and let

$$[a, b] \xrightarrow{\varphi_1} I_1$$

denote the map $p_1 \circ \pi_1 \circ g$. Then φ_1 is independent of which lifting of the above sort we choose. We shall see later that φ_1 contains precisely the information that comes from knowing X up to first order contact.

The next stage of the lifting process is less canonical. We must either impose a connection on the fibering

$$F_1 \xrightarrow{p_1} N_H(H_2)\diagdown F_1$$

or choose a local section

$$U_1 \xrightarrow{s_1} F_1$$

on an open subset U_1 of I_1 so $p_1 \circ s_1 = \mathrm{Id}$. The former is more elegant, but makes use of an extra derivative. Thus we adopt the latter alternative. We say s_1 is a *local section for X at t* if $\varphi_1(t) \in U_1$. Let

$$S_1 = s_1(U_1).$$

Returning to our lifting u with $\pi_1(g) \in F_1$, we have

$$N_H(H_2)s_1 \circ p_1 \circ \pi_1(g) = N_H(H_2)\pi_1(g)$$

and we can find a smooth map

$$[a, b] \xrightarrow{n} N_H(H_2)$$

so that

$$s_1 \circ p_1 \circ \pi_1(g) = n\pi_1(g) \quad \text{for all} \quad t.$$

Then

$$u^{\#} = un^{-1}$$

is a lifting of X whose derivative $g^{\#}$ has

$$\pi_1(g^{\#}) = n\pi_1(g)$$

so

$$\pi_1(g^{\#}) \in S_1$$

for all t. If $u^{\#\#}$ is another lifting of X whose derivative $g^{\#\#}$ has

$$\pi_1(g^{\#\#}) \in S_1$$

for all t, then

$$(u^{\#\#})^{-1}u^{\#} \in H_2$$

for all t. Thus, dropping the #'s, if s_1 is a local section for X at t_0, there exists a lifting u of X on $(t_0 - \epsilon, t_0 + \epsilon]$ for some $\epsilon > 0$ whose derivative g has $\pi_1(g) \in S_1$ on $[t_0 - \epsilon, t_0 + \epsilon]$. Any two such liftings differ by multiplication on the right by a smoothly varying element of H_2. For such a lifting, let

$$X_2 = uH_2$$

so

$$D_2 \xrightarrow{X_2} G/H_2$$

where

$$D_2 = \{t \mid \varphi_1(t) \in U_1\}.$$

We have the commutative diagram

$$\begin{tikzcd} & G/H_2 \arrow[d, "\pi"] \\ D_2 \arrow[ur, "X_2"] \arrow[r, "X"] & G/H. \end{tikzcd}$$

This completes the first step of the construction.

We say X has *constant type at the second step* if X_2 has constant type at the first step, and denote by H_3 the conjugacy class in H_2 of $H_2^{\pi_2(g(t))}$ where g is the derivative of a lifting u of X_2 to G. If so, the construction proceeds as before. We say X has *constant type* if, continuing inductively, we get constant type at each step. Thus we define inductively

H_{m+1} = a representative of the (constant) conjugacy class in H_m of $H_m^{\pi_m(g(t))}$ where g is the derivative of a lifting of $D_m \xrightarrow{X_m} G/H_m$ to G.

$$\tilde{S}_{m-1} = \{v \in \mathcal{G}/\mathcal{H}_m | \pi_{m-1}(v) \in S_{m-1}\}$$
$$F_m = \{v \in \tilde{S}_{m-1} | H_m^v = H_{m+1}\}$$
$$T_m = \{v \in \tilde{S}_{m-1} | H_m^v \text{ conjugate in } H_m \text{ to } H_{m+1}\}$$

$F_m \xrightarrow{p_m} N_{H_m}(H_{m+1})\backslash F_m$ the quotient by the adjoint action of $N_{H_m}(H_{m+1})$ on F_m

$$I_m = N_{H_m}(H_{m+1})\backslash F_m$$
$$\varphi_m = p_m \circ \pi_m \circ g$$
$$s_m = \text{a local section of } p_m \text{ over an open subset } U_m \text{ of } I_m$$
$$S_m = s_m(U_m)$$
$$D_{m+1} = \{t \in D_m | \varphi_m(t) \in U_m\}.$$

As H_{m+1} acts trivially on S_m, if $H_{m+1} = H_m$ we will have that H_{m+1} acts trivially on $\tilde{S}_m$ and thus $H_{m+1} = H_{m+2} = H_{m+3} = \cdots$. Thus, the process stabilizes after a finite number of steps, at most one step after $\dim(H_m) = \dim(H_{m+1})$. If $H_{k-1} \neq H_k$, but $H_k = H_{k+1} = H_{k+2} = \cdots$, we say the sequence *stabilizes at* k and that X has *type* $(H_1, \cdots, H_k)$.

If at each step we cover I_k with open sets for which local sections of

$$F_k \xrightarrow{p_k} I_k$$

exist, so that each $t \in [a, b]$ lies in D_k for some choice of these local sections, and if X has type $(H_1, \cdots, H_k)$ in each case, we say X has *constant type* $(H_1, \cdots, H_k)$.

In a great many cases, the bundles

$$F_k \xrightarrow{p_k} I$$

have global sections. If this can be done for all k, we say the type $(H_1, \cdots, H_k)$ is *untwisted* and can take $D_m = [a, b]$ for all m.

If $H_k \neq \{e\}$, as may occur for degenerate types or if G does not act effectively on G/H, we add an extra step to lift from G/H_k to G. Choose a splitting of $\mathcal{G}$ as a vector space

$$\mathcal{G} = \mathcal{H}_k \oplus \mathcal{P}.$$

If H_k is reductive, we may take this to be H_k-invariant. We may choose a lifting u of X_k whose derivative g lies in $\mathcal{P}$ for all $t \in D_k$. Any two such liftings differ on the right by a constant element of H_k. A lifting u of X to G whose derivative g lies in $\mathcal{P}$ and has $\pi_k(g) \in S_k$ for all $t \in D_k$ is called a *moving frame for* X, relative to the choice of local sections $s_1, \cdots, s_k$ and choice of $\mathcal{P}$.

Our procedure indeed gives a moving frame in the sense of Section 1, where $\mathcal{S}$ is the subset of maps X of constant type $(H_1, \cdots, H_k)$ and so that $s_1, \cdots, s_k$

are local sections for X at all $t \in [a, b]$. Properties (1) and (3) are clear. If u is a moving frame for X, then for any $w \in G$, wu is a moving frame for wX, as u and wu have the same derivative. The coefficients of the derivative g of u will thus be differential invariants if we can show they depend pointwise on the m-jets of X for some m, and this follows from the results of Sections 3, 4, and 5—indeed, m will be the minimum order of contact sufficient for congruence of curves of this constant type.

The possible isotropy types of maps of constant type to G/H may be computed as follows: Take any $v \in \mathcal{G}$ and let

$$\begin{aligned} H_1 &= H \\ H_2 &= H_1^{\pi_1(v)} \\ H_3 &= H_2^{\pi_2(v)} \\ &\vdots \end{aligned}$$

for the adjoint actions of H_m on $\mathcal{G}/\mathcal{H}_m$, $m = 1, 2, \cdots$, continuing until the tower $H_1 \supseteq H_2 \supseteq \cdots$ stabilizes. All such types can occur because $\exp(vt)$ has the type just given—such curves generalize the helices and lines in $\mathbb{R}^3$, having all differential invariants constant. Conversely, if $[a, b] \xrightarrow{X} G/H$ has constant type $(H_1, \cdots, H_k)$, take $v = g(t)$ for some t, where g is the derivative of a moving frame u for X.

If there exists a type $(H_1, \cdots, H_k)$ so that for each m, T_m is an open dense subset of $\tilde{S}_{m-1}$, then a generic map will have constant type $(H_1, \cdots, H_k)$. This type, which is unique (up to equivalence of towers) if it exists, is called the *non-degenerate type* or *standard type* and maps of this constant type are *non-degenerate*. For H compact, a non-degenerate type is guaranteed to exist, for at each stage we take H_{m+1} to be the isotropy group of a maximal orbit for the action of H_m on $\tilde{S}_{m-1}$.

We briefly describe a method of constructing the moving frame for maps $[a, b] \xrightarrow{X} G/H$ which are only weakly of constant type (see Section 1 for the definition). Choose any lifting $[a, b] \xrightarrow{u} G$ of X with derivative $[a, b] \xrightarrow{g} \mathcal{G}$. Having weak constant type may be shown to be equivalent to the towers $H_1^t \supseteq H_2^t \supseteq H_3^t \supseteq \cdots$ defined by

$$\begin{aligned} H_1^t &= H_1 \\ H_2^t &= \{h \in H_1 | h\pi_1(g(t)) = \pi_1(g(t))\} \\ H_3^t &= \{h \in H_2 | h\pi_2(g(t)) = \pi_2(g(t))\} \\ &\vdots \end{aligned}$$

being conjugate in H for all t. We can modify the lifting so that these towers are the same for all t, say $H_1 \supseteq H_2 \supseteq H_3 \supseteq \cdots$. Two such liftings u, u^* satisfy $u^{\#}(t)^{-1}u(t) \in K$ for all t, where

$$K = N_H(H_2) \cap N_H(H_3) \cap \cdots.$$

Let

$$H_\infty = H_1 \cap H_2 \cap H_3 \cap \cdots$$

and let

$$F = \{v \in \mathcal{G}/\mathcal{H}_\infty | H^{\pi_m(v)} = H_m \quad \text{for all} \quad m\}.$$

We have the fiber bundle

$$F \xrightarrow{p} K \backslash F$$

with fiber K/H_∞. If this bundle is trivial, we say the type $(H_1, H_2, H_3, \cdots)$ is untwisted. Let $U \subseteq K\backslash F$ be an open set on which a smooth section s of p exists, and let $S = s(U)$. If $p(g(t)) \in U$ for all t, the lifting can be modified so that $g(t) \in S$ for all t. Two liftings u, $u^\#$ both having this property must have $u^\#(t)^{-1}u(t) \in H_\infty$ for all t.

Let $\mathcal{G} = \mathcal{H}_\infty \oplus \mathcal{P}$, and let $\tilde{S} = \mathcal{P} \cap \pi_\infty^{-1}(S)$. Then there exists a lifting u of X so that $g(t) \in \tilde{S}$ for all t. If u, $u^\#$ are two such liftings of X, $(u^\#)^{-1}u$ is a constant element of H_∞. Such a lifting is a moving frame for X.

3. Rigidity theorems

We say that moving frames for two smooth maps $[a, b] \xrightarrow{X, X^\#} G/H$ of the same constant type $(H_1, \cdots, H_k)$ are *compatible* if the same local sections $U_i \xrightarrow{s_i} F_i$, are used for both. As the open sets U_i were not required to be connected, for any $t_0 \in [a, b]$ we can always choose the s_i so that there exists an $\epsilon > 0$ so that X, $X^\#$ have compatible moving frames on $[t_0 - \epsilon, t_0 + \epsilon]$.

THEOREM. *Let $[a, b] \xrightarrow{X, X^\#} G/H$ be two smooth maps of the same constant type $(H_1, \cdots, H_k)$. For $t_0 \in [a, b]$, let u, $u^\#$ be compatible moving frames for X, $X^\#$ on $[t_0 - \epsilon, t_0 + \epsilon]$, with derivatives g, $g^\#$. Then*

$$(1) \begin{Bmatrix} X, X^\# \textit{ are the same} \\ \textit{to order } m \textit{ at } t_0 \end{Bmatrix} \leftrightarrow \begin{Bmatrix} u(t_0)^{-1}u^\#(t_0) \in H_{m+1} \textit{ and } \pi_{m-r}(g) = \pi_{m-r}(g^\#) \\ \textit{to order } r \textit{ at } t_0, \textit{ all } r = 0, 1, \cdots, m-1 \end{Bmatrix}$$

$$(2) \begin{Bmatrix} X, X^\# \textit{ have contact} \\ m \textit{ on } [t_0 - \epsilon, t_0 + \epsilon] \end{Bmatrix} \leftrightarrow \begin{Bmatrix} \{\pi_m(g) = \pi_m(g^\#) \textit{ on } [t_0 - \epsilon, t_0 + \epsilon].\} \end{Bmatrix}$$

Proof. Define

$$z(t) = u^{-1}(t)u^\#(t).$$

Then

$$g^\#(t) = z^{-1}(t)g(t) + z'(t)$$

where z'' is the derivative of $[a, b] \xrightarrow{z} G$. Assume

$$z(t_0) \in H_{m+1}$$

and $\pi_{m-r}(g^\#) = \pi_{m-r}(g)$ to order r, all $r = 0, 1, \cdots, m-1$. As

$$\pi_m(g) = \pi_m(g^\#) = \pi_m(z^{-1}g) + \pi_m(z') \quad \text{at} \quad t_0$$

and $z(t_0) \in H_{m+1}$, we have

$$\pi_m(z') = 0 \quad \text{at} \quad t_0.$$

So

$$z \in H_m \quad \text{to order 1 at} \quad t_0.$$

Now

$$\pi_{m-1}(g) = \pi_{m-1}(g^\#) = \pi_{m-1}(z^{-1}g) + \pi_{m-1}(z') \quad \text{to order 1 at} \quad t_0.$$

As $z \in H_m$ to order 1 at t_0,

$$\pi_{m-1}(z^{-1}g) = z^{-1}\pi_{m-k}(g) = \pi_{m-1}(g)$$

and thus

$$\pi_{m-1}(z') = 0 \quad \text{to order 1 at} \quad t_0$$

hence we have

$$z \in H_{m-1} \quad \text{to order 2 at} \quad t_0.$$

Continuing, we conclude

$$z \in H \quad \text{to order } m \text{ at} \quad t_0$$

and thus uH, $u^\#H$ agree to order m at t_0.

To prove the other direction, assume X, $X^\#$ agree to order m at t_0, so

$$z \in H \quad \text{to order } m \text{ at} \quad t_0.$$

Thus, as

$$\pi_1(g^\#) = \pi_1(z^{-1}g) + \pi_1(z')$$

we have

$$\pi_1(g^\#) = z^{-1}\pi_1(g) \quad \text{to order } m-1 \text{ at} \quad t_0.$$

As $\pi_1(g), \pi_1(g^\#) \in S_1$, we conclude

$$\pi_1(g^\#) = \pi_1(g) \quad \text{to order } m-1 \text{ at} \quad t_0$$

and also

$$z \in H_2 \quad \text{to order } m-1 \text{ at} \quad t_0$$

because the adjoint action

$$H_r/H_{r+1} \times S_r \to \mathcal{G}/\mathcal{H}_r \qquad r = 1, \cdots, k$$

is injective and has non-singular differential on $eH_{r+1} \times S_r$. Now

$$\pi_2(g^\#) = \pi_2(z^{-1}g) + \pi_2(z')$$

and thus

$$\pi_2(g^\#) = z^{-1}\pi_2(g) \quad \text{to order } m - 2 \text{ at } \quad t_0.$$

As $\pi_2(g), \pi_2(g^\#) \in S_2$, we conclude

$$\pi_2(g^\#) = \pi_2(g) \quad \text{to order } m - 2 \text{ at } \quad t_0$$

and

$$z \in H_3 \quad \text{to order } m - 2 \text{ at } \quad t_0.$$

Continuing, we obtain for $r = 0, 1, \cdots, m - 1$ that

$$\pi_{m-r}(g^\#) = \pi_{m-r}(g) \quad \text{to order } r \text{ at } \quad t_0$$

and

$$z \in H_{m-r+1} \quad \text{to order } r \text{ at } \quad t_0.$$

This proves (1), and noting that

$$\begin{Bmatrix} \pi_m(g) = \pi_m(g^\#) \text{ for} \\ \text{all } t \in [t_0 - \epsilon, t_0 + \epsilon] \end{Bmatrix} \leftrightarrow \begin{Bmatrix} \pi_{m-r}(g^\#) = \pi_{m-r}(g) \text{ to order } r \text{ at } t, \text{ for} \\ r = 0, 1, \cdots, m - 1, \text{ for all } t \in [t_0 - \epsilon, t_0 + \epsilon] \end{Bmatrix}$$

we see that (1) implies (2).

COROLLARY. *$\{w \in G | wX, X$ are the same to order m at $t_0\} = u(t_0)H_{m+1}u(t_0)^{-1}$.*

Proof. Take $X^\# = wX$. Then $X, X^\#$ agree to order m at t_0 if and only if $u(t_0)^{-1}wu(t_0) \in H_{m+1}$. This is equivalent to $w \in u(t_0)H_{m+1}u(t_0)^{-1}$.

Thus, the isotropy tower for maps of constant type defined by jets is conjugate in G to the isotropy tower defined in Section 2.

To put the main theorem of this section in the language of jets, let

$$\mathcal{S}_m \subseteq J_m(G/H)$$

be the m-jets of maps of constant type $(H_1, \cdots, H_m)$ at the m-th step, and let S_m^c be those compatible with a given choice of local sections $s_1, \cdots, s_m$. Let

$$N_m \subset J_{m-1}(S_1^c) \oplus J_{m-2}(S_2^c) \oplus \cdots \oplus J_0(S_m^c)$$

be defined to be those m-tuples $(f_{m-1}, \cdots, f_0)$ so that for all r, p such that $m - 1 \geq r \geq p \geq 0$ we have that under the natural maps

$$\begin{array}{ccc} J_{m-r}(S_r) & & J_{m-p}(S_p) \\ \downarrow & & \downarrow \\ J_{m-r}(\pi_p(S_r)) & \rightarrow & J_{m-r}(S_p) \end{array}$$

the jets f_r and f_p go to the same element of $J_{m-r}(S_p)$. Then we obtain:

COROLLARY. $S_m^c \cong G/H_{m+1} \oplus N_m$

$$G \backslash S_m^c \cong N_m.$$

We now discuss the question of congruence.

THEOREM. *Let* $[a, b] \xrightarrow{X, X^\#} G/H$ *be two smooth maps of the same constant type* $(H_1, \cdots, H_k)$.

(1) $X, X^\#$ *are congruent* $\leftrightarrow$ $X, X^\#$ *have contact* k *on* $[a, b]$.

(2) $\{w \in G | wX = X\} = u(t_0)H_k u(t_0)^{-1}$ *if* u *is a moving frame for* X *on* $[t_0 - \epsilon, t_0 + \epsilon]$.

Proof. Let $\mathcal{G} = \mathcal{H}_k \oplus \mathcal{P}$, and for each $t_0 \in [a, b]$, choose compatible moving frames $u, u^\#$ on $[t_0 - \epsilon, t_0 + \epsilon]$ whose derivatives $g, g^\#$ lie in $\mathcal{P}$ for all $t \in [t_0 - \epsilon, t_0 + \epsilon]$. By the preceding theorem, if $X, X^\#$ have contact k on $[t_0 - \epsilon, t_0 + \epsilon]$, then

$$\pi_k(g^\#) = \pi_k(g) \quad \text{on} \quad [t_0 - \epsilon, t_0 + \epsilon].$$

As $\pi_k|_{\mathcal{P}}$ is an isomorphism,

$$g^\# = g \quad \text{on} \quad [t_0 - \epsilon, t_0 + \epsilon].$$

Integrating, X and $X^\#$ are congruent on $[t_0 - \epsilon, t_0 + \epsilon]$. This almost proves (1), for conceivably we might have $X^\# = w_1 X$ on one subinterval and $X^\# = w_2 X$ on another subinterval, but no w which works on all of $[a, b]$.

We now prove (2) and then return to complete the proof of (1). From the previous theorem, we know that $wX = X$ implies $w \in u(t_0)H_k(t_0)^{-1}$. Conversely, if $w = u(t_0)hu(t_0)^{-1}$, $h \in H_k$, we try to solve the equation

$$uz = wu \qquad [t_0 - \epsilon, t_0 + \epsilon] \xrightarrow{z} H_k.$$

Equivalently, we must solve

$$z^{-1}g + z' = g$$

with $z(t_0) = h$. As for $z \in H_k$

$$\pi_k(z^{-1}g) = \pi_k(g)$$

we can always do this. Thus $wX = X$.

Returning to (1), we see that by (2), $wX = X$ on $[t_0 - \epsilon, t_0 + \epsilon]$ if and only if $wX = X$ on some non-empty open subinterval. Thus, covering $[a, b]$ by intervals I_i for which moving frames exist for X and $X^\#$, we already have $X^\# = w_i X$ on I_i. On $I_i \cap I_j$, $w_i^{-1}w_j X = X$, but therefore $w_i^{-1}w_j X = X$ on I_i and on I_j. We may thus take $w_i = w_j$, hence X and $X^\#$ are congruent on $[a, b]$.

4. Calculations of the number of differential invariants of a given order; The Cartan polygon

Let $[a, b] \xrightarrow{X} G/H$ be a smooth map of constant type $(H_1, \cdots, H_k)$ having a moving frame $[a, b] \xrightarrow{u} G$ with derivative $[a, b] \xrightarrow{g} \mathcal{G}$. We have the commutative diagram

$$\begin{tikzcd}
 & \mathcal{G} \arrow[d, "\pi_k"] \\
 & S_k \arrow[d, "\pi_{k-1}"] \\
 & \vdots \\
 & S_2 \arrow[d, "\pi_1"] \\
{[a, b]} \arrow[uuuur, "g"] \arrow[uuur] \arrow[ur, "\pi_2(g)"'] \arrow[r, "\pi_1(g)"] & S_1
\end{tikzcd}$$

Two maps X, $X^{\#}$ of this constant type have contact m on $[a, b]$ if and only if $\pi_m(g) = \pi_m(g^{\#})$ and they are congruent if and only if $\pi_k(g) = \pi_k(g^{\#})$.

PROPOSITION. *(Existence of maps with specified invariants): For any smooth map* $[a, b] \xrightarrow{\varphi} S_k$, *there exists a smooth map* $[a, b] \xrightarrow{X} G/H$ *of constant type* $(H_1, H_2, \cdots, H_k)$ *admitting a moving frame* u *on* $[a, b]$ *with derivative* g *so that* $\pi_k(g) = \varphi$.

Proof. Choose any smooth map $[a, b] \xrightarrow{\tilde{\varphi}} \mathcal{G}$ so that $\pi_k(\tilde{\varphi}) = \varphi$. Integrate to obtain $[a, b] \xrightarrow{u} G$, and take $X = uH$.

COROLLARY. *The number* $v(m)$ *of independent differential invariants of order* $\leq m$ *for maps to* G/H *of constant type* $(H_1, \cdots, H_k)$ *is* $\dim(\pi_m(S_k))$, $m = 1, \cdots, k$.

For simplicity, we now assume G/H admits a non-degenerate type $(H_1, H_2, \cdots, H_k)$. Recall the notation

$\tau(m) = \dim(G\backslash J_m(G/H))$ = the number if functionally independent differential invariants of order $\leq m$

$v(m) = \tau(m) - \tau(m-1)$ = the number of independent differential invariants of order $\leq m$ when derivatives of previous invariants are suppressed

$i_m = v(m) - v(m-1)$ = the number of independent invariants of order precisely m.

For convenience, we note the formulas

$$\tau(m) = v(1) + v(2) + \cdots + v(m) = mi_1 + \cdots + 2i_{m-1} + i_m$$

$$v(m) = i_1 + \cdots + i_m.$$

THEOREM. *If* G/H *admits a non-degenerate type* $(H_1, \cdots, H_k)$, *for non-degenerate curves we have for* $m \geq 1$,

$$\tau(m) = (m+1)\dim(G/H) - \dim(G/H_{m+1})$$

$$v(m) = \dim(G/H) - \dim(H_m/H_{m+1})$$

$$i_m = \dim H_{m-1} - 2\dim H_m + \dim H_{m+1}.$$

Remark. Here, $H_0 = G$.

Proof. The m-jets of non-degenerate maps form an open subset U of $J_m(G/H)$, and the fibration

$$U \to G \backslash U$$

has fiber diffeomorphic to G/H_{m+1}, as H_{m+1} is conjugate to the isotropy of G on m-jets of non-degenerate maps. So

$$\tau(m) = \dim(G \backslash U) = \dim J_m(G/H) - \dim(G/H_{m+1}).$$

As

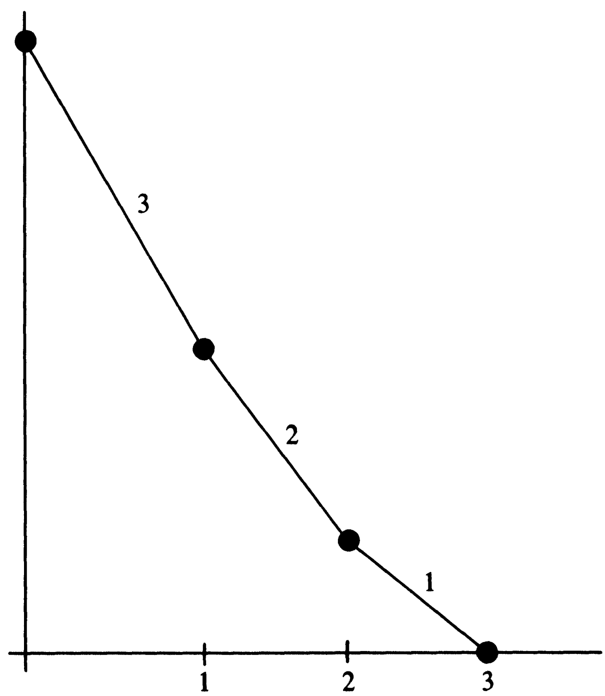
$$\dim J_m(G/H) = (m + 1) \dim(G/H)$$

the formula for $\tau(m)$ is proven. Those for $v(m)$ and i_m follow by taking successive differences.

Note in particular that the total number of independent differential invariants for non-degenerate maps is $\dim(G/H)$.

There is a convenient visual aid for computing the number of differential invariants, the *Cartan Polygon*. Graph $\dim(H_m)$ as a function of m and join the resulting points by straight lines, noting each one's slope. Then i_m is the difference of the slopes of the line segments meeting above m. The resulting polygon is always the graph of a convex function, as $i_m \geq 0$. As an example, for $G = E(3)$, the Euclidean group in these variables, $H = SO(3)$, $G/H =$ Euclidean $\mathbb{R}^3$, the type of a non-degenerate map is $(SO(3), SO(2), \{e\})$.

The Cartan polygon is thus



There is thus one invariant each of orders 1, 2 and 3. These are the velocity, curvature, and torsion respectively. More Cartan Polygons are computed in the examples in Section 8.

5. The order of contact required for congruence

THEOREM. *If G/H admits a non-degenerate type $(H_1, H_2, \cdots, H_k)$ and $r = \max\{m \mid \dim(H_m) > \dim(H_{m-1})\}$, then*

(1) *Two non-degenerate maps $[a, b] \xrightarrow{X, X^\#} G/H$ are congruent $\leftrightarrow X, X^\#$ have contact r.*

(2) *No smaller order contact will do.*

Proof. (1) As $\dim(H_m)$ stabilizes at r, we have that $\mathscr{H}_r = \mathscr{H}_k$, and hence $\pi_r = \pi_k$. In light of the results of Section 3, if u, $u^\#$ are moving frames for X, $X^\#$ with derivatives g, $g^\#$, then

$$\begin{aligned} X, X^\# \text{ have contact } r &\leftrightarrow \pi_r(g) = \pi_r(g^\#) \text{ for all } t \\ &\leftrightarrow \pi_k(g) = \pi_k(g^\#) \text{ for all } t \\ &\leftrightarrow X, X^\# \text{ congruent.} \end{aligned}$$

(2) We note that

$$r = \max\{m \mid i_m > 0\}$$

and thus by independence of invariants, as $i_r > 0$, there are non-degenerate maps X, $X^\#$ whose invariants of order $\leq r - 1$ agree, but whose invariants of order r do not. Thus they cannot be congruent.

For real-analytic maps, we have the following result:

THEOREM. *If G/H admits only finitely many types of curves of constant type, then if*

$$k_{\max} = \max\{k \mid (H_1, \cdots, H_k) \textit{ is a type for curves of constant type to } G/H\}.$$

Then two real-analytic curves $[a, b] \xrightarrow{X, X^\#} G/H$ are congruent if and only if they have contact $k_{\max}$.

Proof. A real-analytic map $[a, b] \xrightarrow{X} G/H$ has constant type on a non-empty open set. To see this, let

$$T_K = \{v \in \mathscr{G}/\mathscr{H} \mid H^v \text{ is conjugate to } K \text{ in } H\}.$$

Then T_K is an open subset of its closure, and the latter is a real-analytic subset of $\mathscr{G}/\mathscr{H}$. As there are a finite number of K's which can occur, for any lifting u of X to G with derivatives g, we have

$$H^{\pi_1(g(t))} = K$$

on a non-empty open subset of $[a, b]$ for some K. Continuing inductively, we obtain constant type on a non-empty open set.

Now if X has constant type $(H_1, \cdots, H_k)$ on (c, d), then $X^\#$ will also, and then X and $X^\#$ are congruent on (c, d), as $k_{\max} \geq k$. Hence, by the identity theorem, they are congruent on $[a, b]$.

It is interesting to note that the order of contact required for congruence of real-analytic maps may exceed that required for non-degenerate maps, see example 4a of Section 8.

6. Simplifications and further results when $\mathscr{G}$ has an H-invariant splitting $\mathscr{G} = \mathscr{H} \oplus \mathscr{T}$

It frequently happens that $\mathscr{G}$ admits an H-invariant splitting $\mathscr{G} = \mathscr{H} \oplus \mathscr{T}$, as occurs if H is compact, semisimple or, more generally, reductive. In this case the theory simplifies a great deal, and a new avenue for computing isotropy towers opens up. We can make the identification, where $J_m(G/H)_e$ denotes m-jets at 0 of curves X with $X(0) = eH$, that

$$J_m(G/H)_e \cong \mathscr{T}^m$$

by writing every m-jet at eH uniquely as

$$\exp\left(\sum_{i=1}^{m} v_i t^i\right) \qquad v_i \in \mathscr{T} \qquad i = 1, \cdots, m$$

and the action of H on $J_m(G/H)_e$ takes the simple form

$$v_i \to h v_i \qquad i = 1, \cdots, m$$

where multiplication denotes the adjoint action of H. If X denotes the curve

$$X(t) = \exp\left(\sum_{i=1}^{m} v_i t^i\right)$$

then setting

$$H_r = \{w \in G | wX, X \text{ agree to order } r - 1 \text{ at } 0\}$$

we have

$$H_r = \{h \in H | h v_i = v_i, \qquad i = 1, \cdots, r - 1\}.$$

A non-degenerate type exists if H acting on $\mathscr{T}^m$ has a principal orbit for all m and is computed by taking a sequence of generic vectors $v_1, v_2, v_3, \cdots$ of $\mathscr{T}$ and setting

$$H_m = \{h \in H | h v_i = v_i, \qquad i = 1, 2, \cdots, m - 1\}.$$

In particular,

$$H_{\dim(G/H)+1} = \text{kernel of adjoint representation of } H \text{ on } \mathscr{T}$$

and thus for the type $(H_1, \cdots, H_k)$ of a non-degenerate curve (and in fact, for any curve of constant type) we have

$$k \leq \dim(G/H) + 1$$

and thus contact of order $\dim(G/H) + 1$ suffices for congruence. This bound is achieved for affine geometry on $\mathbb{R}^n$, see example 3 of Section 8.

We have a map

$$\mathcal{T}^{m-1} \cong J_{m-1}(G/H)_e \to (G/H)^{m-1}$$

sending

$$(v_1, \cdots, v_{m-1}) \to (X(1), X(2), \cdots, X(m-1))$$

where

$$X(t) = \exp\left(\sum_{i=1}^{m-1} v_i t^i\right)$$

which is a local diffeomorphism near the origin of $\mathcal{T}^{m-1}$. It is equivariant for the action of H and induces a map

$$G\diagdown J_{m-1}(G/H) \xrightarrow{E_m} G\diagdown(G/H)^m$$

and proves that

$$\tau(m) = \dim(G\diagdown(G/H)^m)$$

and also that for the non-degenerate type

$$H_m \cong \{w \in G | wp_i = p_i, \qquad i = 1, \cdots, m\}$$

where $(p_1, \cdots, p_m)$ is a generic m-tuple of points of G/H near the diagonal in $(G/H)^m$.

If H is compact, any representation has a principal orbit, so a non-degenerate type always exists. Also, $\mathcal{T}$ admits an H-invariant metric. We obtain a representation

$$H \xrightarrow{\rho} O(n) \qquad n = \dim(G/H)$$

by the adjoint representation on $\mathcal{T}$. For generic vectors $v_1, v_2, \cdots, v_n$ of $\mathcal{T}$ we have for the non-degenerate type

$$H_m = \{h \in H | hv_i = v_i, \qquad i = 1, \cdots, m-1\}$$

and using the Gram-Schmidt process to obtain an orthogonal basis for $\mathcal{T}$ from $v_1, \cdots, v_n$ we have

$$H_m = H \cap \rho^{-1}(O(n+1-m))$$

so that $H_m \subseteq \{\pm I\}$ for $m \geq \dim(G/H)$. Thus, contact of order $\dim(G/H)$ is always sufficient for congruence of non-degenerate maps when H is compact. This upper bound is achieved for Euclidean $\mathbb{R}^n$.

As

$$\tau(m) = \dim(H\backslash\mathcal{T}^m) = \sum_{i=1}^{m} \dim(H_i\backslash\mathcal{T})$$
$$\nu(m) = \dim(H_m\backslash\mathcal{T}).$$

Now assuming H is compact,

$$H_m\backslash\mathcal{T} = (v_1, \cdots, v_m) \oplus H_m\backslash(v_1, \cdots, v_m)^\perp$$

where parentheses indicate subspace generated by. As v_m is generic in $(v_1, \cdots, v_{m-1})^\perp$, the orbit of v_m under H_{m-1} is finite if and only if the quotient H_{m-1}/K is finite, where K is the kernel of the action of H on $\mathcal{T}$. Thus

$$i_m = 0 \leftrightarrow \dim H_r = \dim H_{m-1} \quad \text{for all} \quad r \geq m-1.$$

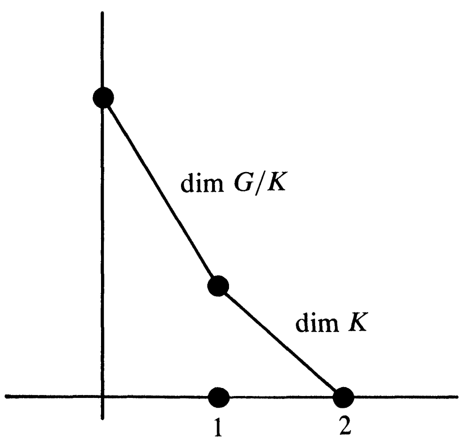
Thus for H compact, there is at least one new invariant of each order up to and including the minimum order of contact required for congruence.

7. Symmetric spaces

THEOREM. *Let G/K be a Riemannian symmetric space with no factors of Euclidean type.*

(1) *For non-degenerate maps,* $i_1 = \operatorname{rank}(G/K)$.

(2) *If* $\operatorname{rank} G = \operatorname{rank}(G/K)$, *then for non-degenerate maps* $\mathcal{H}_2 = 0$ *and second order contact implies congruence. In this case, the Cartan polygon is:*



$$i_1 = \dim G - 2\dim K = \operatorname{rank}(G/K)$$

Proof. The Cartan decomposition

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$$

is K-invariant. Let $\mathcal{A}$ be a maximal Abelian subalgebra of $\mathcal{P}$, and let $A = \exp(\mathcal{A})$. By definition,

$$\operatorname{rank}(G/K) = \dim A.$$

Now

$$G = KAK$$

(see [14]), so

$$G\backslash(G/K)^2 = K\backslash(G/K) = A.$$

By our results on m-tuples of points in Section 6, in general

$$\begin{aligned}\dim(G\backslash(G/H)^2) &= 2\dim(G/H) - \dim(G/H_2)\\ &= \dim G - 2\dim H + \dim H_2 = i_1.\end{aligned}$$

So for symmetric spaces

$$i_1 = \dim A = \operatorname{rank}(G/K).$$

If $\operatorname{rank}(G) = \operatorname{rank}(G/K)$, then $\mathscr{A}$ is a Cartan subalgebra of $\mathscr{G}$ and

$$\dim \mathscr{G} = 2\dim \mathscr{H} + \dim \mathscr{A}.$$

Thus

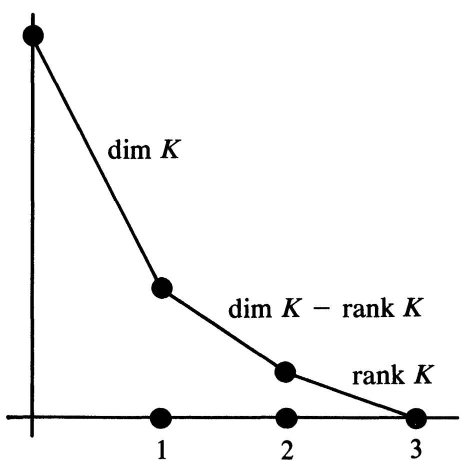
$$\dim \mathscr{H}_2 = 0$$

and the result follows.

The simply-connected irreducible non-exceptional Riemannian symmetric spaces satisfying (2) are $SL(n, \mathbb{R})/SO(n)$, $Sp(n, \mathbb{R})/U(n)$, $SO(p, q)/SO(p) \times SO(q)$ when $p = q$, and their compact duals $SU(n)/SO(n)$ and $Sp(n)/U(n)$, $SO(p+q)/SO(p) \times SO(q)$ when $p = q$, see p. 354 of [14]. It is notable that the Siegel upper half-space $\mathscr{H}_g = Sp(g, \mathbb{R})/U(g)$ is among these—the moving frame for families of curves and abelian varieties is thus especially simple.

For type IV symmetric spaces (see p. 308 of [H]), there is also a complete picture.

THEOREM. *For an irreducible Riemannian symmetric space G/K of type IV, if T is a maximal torus of K, then the non-degenerate type $(H_1, H_2, \cdots)$ has Lie algebras $\mathscr{H}_1 = \mathscr{H}$, $\mathscr{H}_2 = \mathscr{T}$, $\mathscr{H}_3 = 0$. Thus, the Cartan polygon is*



and $i_1 = \text{rank } K = \text{rank } G/K$, $i_2 = \dim K - 2 \text{ rank } K$, $i_3 = \text{rank } K$, *and* 3 *is the minimal order of contact sufficient for congruence of non-degenerate maps.*

Proof. $\mathcal{G}$ is a complex Lie algebra regarded as real, and $\mathcal{K}$ is a maximal compact subalgebra, $\dim_{\mathbb{R}}\mathcal{K} = (1/2) \dim_{\mathbb{R}}\mathcal{G}$. There is a K-invariant splitting $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$, and the adjoint action of K on $\mathcal{P}$ is equivalent to the adjoint action of K on $\mathcal{K}$. A generic vector $v \in \mathcal{K}$ belongs to the Lie algebra $\mathcal{T}$ of some maximal torus T, and we may assume the one-parameter family $\exp(vt)$ is dense in T. Thus for $k \in \mathcal{K}$, $[k, v] = 0 \leftrightarrow [k, t] = 0$ for all $t \in \mathcal{T}$. This latter contradicts the maximality of $\mathcal{T}$ unless $k \in \mathcal{T}$. As $k \in \mathcal{T}$ implies $[k, v] = 0$, we have $\mathcal{K}^v = \mathcal{T}$, or $\mathcal{K}_2 = \mathcal{T}$. There is a T-invariant splitting $\mathcal{K} = \mathcal{T} \oplus \mathcal{S}$ and the adjoint action of $\mathcal{T}$ on $\mathcal{S} = \mathcal{K}/\mathcal{T}$ is faithful, as $[t, v] = 0$ all $v \in \mathcal{S} \to [t, v] = 0$ all $v \in \mathcal{K} \to t \in \text{center}(\mathcal{K}) \to t = 0$. By standard representation theory of tori, the adjoint action of T on $\mathcal{K}/\mathcal{T}$ splits into a direct sum of trivial representations and representations of T on $\mathbb{R}^2$ by $(\theta_1, \cdots, \theta_l) \to e^{i(n_1\theta_1 + \cdots + n_l\theta_l)}$ some $n_1, \cdots, n_l \in Z$. The group T^v of a generic $v \in \mathcal{K}/\mathcal{T}$ is thus the kernel of the representation, hence zero. So $\mathcal{K}_3 = 0$. The rest follows by our general results.

8. Examples

Split examples:
1. Euclidean geometry
2. Unimodular affine geometry
3. Affine geometry
4a. The Grassmannians $G_{\mathbb{R}}(n, 2n)$
4b. The Grassmannians $G_{\mathbb{R}}(n, n + k)$, $k \neq n$
5. Minkowski space

Non-split examples:
6a. The projective line
6b. The projective plane
6c. $\mathbb{RP}_n$
7. Conformal geometry
8. An example with a continuous family of types
9. Holomorphic families of abelian varieties

Example 1. $G = E(n) = \{T_a \circ R | T_a$ translation by $a \in \mathbb{R}^n$, $R \in SO(n)\}$, $H = SO(n)$, $G/H = \mathbb{R}^n$.

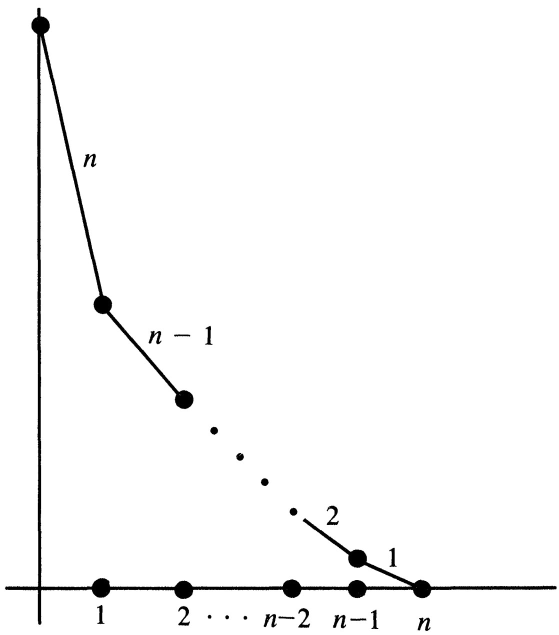
This is Euclidean geometry. There is an $SO(n)$-invariant splitting

$$\mathcal{E}(n) = \mathcal{SO}(n) \oplus \mathbb{R}^n.$$

The adjoint action of $SO(n)$ on $\mathbb{R}^n$ is just the standard representation. The possible isotropy types for maps of constant type are

$$(SO(n), SO(n - 1), \cdots, SO(n + 1 - k)) \qquad k \leq n$$

and $k = n$ for non-degenerate maps. The maps of constant degenerate type $(SO(n), SO(n-1), \cdots, SO(n+1-k))$ for $k \neq n$ lie in translates of the fixed point set of $SO(n+1-k)$, i.e., a linear subspace of codimension $n+1-k$, illustrating a result of Section 3. For non-degenerate maps, the Cartan polygon is



Thus $i_1 = i_2 = \cdots = i_n = 1$. These are the velocity and the various curvatures.

Example 2. $G = SA(n) = \{T_a \circ R | T_a$ translation by $a \in \mathbb{R}^n$, $R \in SL(n, \mathbb{R})\}$, $H = SL(n, \mathbb{R})$, $G/H = \mathbb{R}^n$.

This is unimodular affine geometry. There is an H-invariant splitting $\mathcal{G} = \mathbb{R}^n \oplus \mathcal{H}$, and the adjoint action of $SL(n, \mathbb{R})$ on $\mathbb{R}^n$ is the standard one. Thus if $v_1 \neq 0$, H^{v_1} is conjugate to

$$H_2 = \left\{ R \in SL(n, \mathbb{R}) | R \text{ has last column} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}.$$

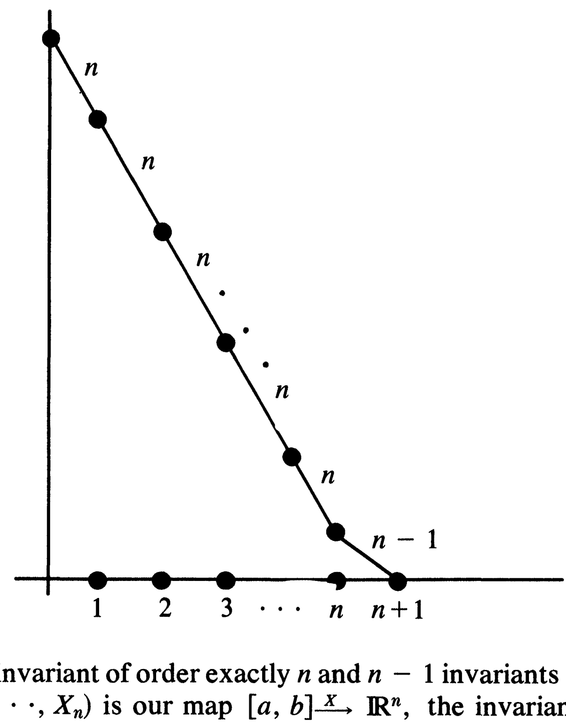
If $v_2 \in \mathbb{R}^n$ is not a multiple of

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

we get $H_2^{v_2}$ conjugate to

$$H_3 = \left\{ R \in SL(n, \mathbb{R}) | R \quad \text{has last two columns} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

and in general $H_m = \{R \in SL(n, \mathbb{R}) | R$ has the same last $m - 1$ columns as the identity matrix$\}$ for the standard type. Thus $\dim H_m = n^2 - 1 - n(m - 1)$ for $1 \leq m \leq n, H_{n+1} = \{e\}$. The Cartan polygon for non-degenerate maps is thus



so there is one invariant of order exactly n and $n - 1$ invariants of order exactly $n + 1$. If $(X_1, \cdots, X_n)$ is our map $[a, b] \xrightarrow{X} \mathbb{R}^n$, the invariants are obtained from the matrix

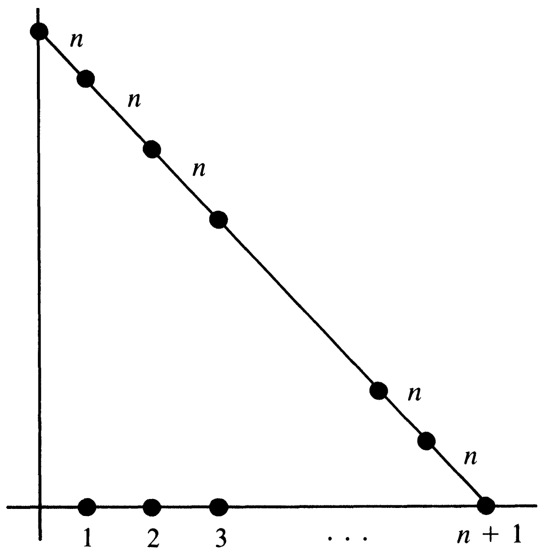
$$\begin{pmatrix} X_1' & \cdots & X_n' \\ X_1'' & & \vdots \\ \vdots & & \vdots \\ X_1^{(n+1)} & \cdots & X_n^{(n+1)} \end{pmatrix}$$

by eliminating a row and taking the determinant. If we eliminate the last row, we get an invariant of order n, if we eliminate the next-to-last row, we obtain the derivative of this invariant. Eliminating one of the remaining $n - 1$ rows gives the $n - 1$ invariants of order exactly $n + 1$.

Degenerate maps are those which go to a translate of a lower-dimensional linear subspace of $\mathbb{R}^n$.

Example 3. $G = A(n) = \{T_a \circ R | T_a$ translation by $a \in \mathbb{R}^n$, $R \in GL(n, \mathbb{R})\}$, $H = GL(n, \mathbb{R})$, $G/H = \mathbb{R}^n$.

This is affine geometry. Just as in the preceding example, we may obtain $H_m = \{R \in GL(n, \mathbb{R}) | R$ and the identity have the same last $m - 1$ columns$\}$ for the standard type. The Cartan polygon is now



so there are n invariants of order $n + 1$. Thus, enlarging the group slightly over that of unimodular affine geometry lost us the invariant of order n, replacing it with one of order $n + 1$. If we denote by α_m the determinant of the matrix obtained by eliminating the m'th row of

$$\begin{pmatrix} X_1 & \cdots & X_n \\ X_1' & & X_n' \\ \vdots & & \vdots \\ X_1^{(n+1)} & \cdots & X_n^{(n+1)} \end{pmatrix}$$

the invariants are $\dfrac{\alpha_1}{\alpha_{n+1}}, \cdots, \dfrac{\alpha_n}{\alpha_{n+1}}$.

Example 4a. $G_{\mathbb{R}}(n, 2n) = SO(2n)/SO(n) \times SO(n)$.
There is an H-invariant splitting

$$\mathcal{SO}(2n) = \mathcal{SO}(n) \times \mathcal{SO}(n) \oplus \mathbb{R}^{n \times n}$$

where the adjoint action of $SO(n) \times SO(n)$ on $\mathbb{R}^{n \times n}$ is

$$\begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} \text{ acts on } B \text{ by } B \to R^{-1}BS.$$

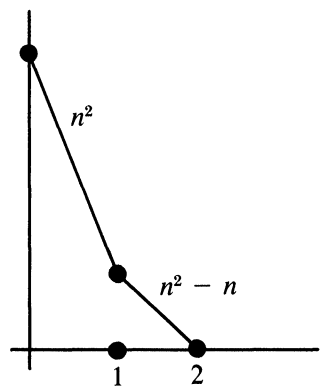
Thus

$$BB^t \to R^{-1}(BB^t)R$$
$$B^tB \to S^{-1}(B^tB)S.$$

If BB^t has distinct eigenvalues, then

$$H^B = \left\{ \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} \middle| R \in \text{center of } SO(n) \right\}$$

which is a finite group. The Cartan Polygon for non-degenerate maps is thus



Second order contact suffices for congruence. There are n first order invariants, and these are the eigenvalues of BB^t. This illustrates a result of Section 7 on symmetric spaces with $\text{rank}(G) = \text{rank}(G/H)$.

If $n = 2$, the first degenerate case is $BB^t = \lambda^2 I$, where $[a, b] \xrightarrow{\lambda} \mathbb{R}^*$. Then

$$H^B = \left\{ \begin{pmatrix} B^{-1}SB & 0 \\ 0 & S \end{pmatrix} \middle| S \in SO(2) \right\}$$

which are all conjugate in $SO(2) \times SO(2)$ to

$$H_2 = \left\{ \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} \middle| S \in SO(2) \right\}.$$

Then

$$\mathscr{H}/\mathscr{H}_2 = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \middle| A^t = -A \right\}$$

and the action of H_2 on this is $A \to S^{-1}AS$, which is trivial in the 2×2 case. The type of these degenerate maps is thus (H_1, H_2), and second order contact again suffices for rigidity. The only other constant type is that of constant maps. (Griffiths treats this case on pp. 801–805 of [G], where he defines non-degeneracy by $\det B \neq 0$ and claims second order contact implies congruence for such maps. In his proof, he tacitly assumes in the middle of p. 803 that the map has what we call constant type, and thus in a sense deserves the credit of inventing it. The result as he states it is false, for consider a curve $[a, b] \xrightarrow{X} G_{\mathbb{R}}(2, 4)$ having constant non-degenerate type (in our sense) on $[a, c)$ and $(c, b]$ but having contact ∞ with a degenerate curve at c. Now obtain a map $[a, b] \xrightarrow{X^{\#}} G_{\mathbb{R}}(2, 4)$ by twisting the $(c, b]$ half of the curve by a non-trivial element of the conjugate of H_2 which leaves the degenerate curve fixed. Thus X and $X^{\#}$ agree on $[a, c)$, are congruent on $(c, b]$, and have contact ∞ at c, hence have contact ∞ on $[a, b]$, but are not congruent.)

An interesting phenomenon occurs when $n = 3$. There are three ways a curve can degenerate at the first step—BB^t can have 2 distinct eigenvalues, can have all eigenvalues the same but non-zero, or can be the zero matrix. In the second case, $BB^t = \lambda^2 I$, $[a, b] \xrightarrow{\lambda} \mathbb{R}^*$, and we may take

$$H_2 = \left\{ \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} \middle| R \in SO(3) \right\}.$$

Now

$$\mathscr{H}/\mathscr{H}_2 = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \middle| A^t = -A \right\}$$

and H_2 acts on this by $A \to R^{-1}AR$. If $A \neq 0$,

$$H_3 = \left\{ \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} \middle| R \in SO(2) \hookrightarrow SO(3) \right\}.$$

So

$$\mathscr{H}_2/\mathscr{H}_3 \cong \mathbb{R}^2$$

and the adjoint action of H_3 is $v \to Rv$. If $v \neq 0$, then $H_4 = \{e\}$. So

$$\mathscr{H}_3/\mathscr{H}_4 \cong \mathbb{R}$$

and this latter gives us an invariant of order 4. Thus for maps to $G_{\mathbb{R}}(3, 6)$ of constant type (H_1, H_2, H_3, H_4) as above, contact of order 4 suffices for congruence and no lower order of contact will do. This provides an illustration that unlike the case of Euclidean $\mathbb{R}^n$, degenerate maps of constant type may require higher order contact for congruence than that required by non-degenerate maps.

Example 4b. $G = SO(n+k)$, $H = SO(n) \times SO(k)$, $n \neq k$, $G/H = G_{\mathbb{R}}(n, n+k)$.

Take $k < n$. We have an invariant splitting

$$\mathcal{SO}(n+k) = \mathcal{SO}(n) \times \mathcal{SO}(k) \oplus \mathbb{R}^{n \times k}$$

where the adjoint action of

$$SO(n) \times SO(k) = \left\{ \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} \middle| R \in SO(n),\ S \in SO(k) \right\}$$

on the $n \times k$ matrices is

$$B \rightarrow R^{-1}BS.$$

Again

$$BB^t \rightarrow R^{-1}BB^t R$$

$$B^tB \rightarrow S^{-1}B^tB\, S.$$

Non-degenerate maps are those for which B^tB has k distinct eigenvalues.

$$H_2/\text{finite group} = \left\{ \begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix} \middle| R \in SO(n-k) \right\}$$

considering $SO(n-k) \rightarrow SO(n)$.

Choosing a generic $C \in \mathbb{R}^{n \times k}$, we see

$$H_3/\text{finite group} = \left\{ \begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix} \middle| R \in SO(n-2k) \right\}$$

and that in general for the non-degenerate type

$$H_m/\text{finite group} = SO(n-(m-1)k).$$

The minimal order of contact sufficient for congruence of non-degenerate curves is thus the smallest integer $\geq \dfrac{n-1}{k} + 1$ whenever $k < n$. When $k = 1$, we obtain the usual result that n is the minimal order of contact sufficient for congruence of non-degenerate curves in $\mathbb{RP}_n$. Note that here $\mathbb{RP}_n$ has a metric, while in example 6 it does not.

Example 5. $G = \{T_a \circ R | T_a = \text{translation by } a \in \mathbb{R}^n,\ R \in SO_0(1, n-1)\}$, $H = SO_0(1, n-1)$, $G/H =$ Minkowski space with $n - 1$ space dimensions and 1 time dimension.

The notation $SO_0(1, n-1)$ denotes the connected component of the identity in $SO(1, n-1)$. There is an H-invariant splitting

$$\mathcal{G} = \mathcal{H} \oplus \mathbb{R}^n$$

where the adjoint action of $SO_0(1, n-1)$ on $\mathbb{R}^n$ is matrix multiplication.

For $v \in \mathbb{R}^n$, we have

$$H^v/\text{finite group} = \begin{cases} \text{an } H\text{-conjugate of } SO(1, n-2) & \text{if } \|v\| > 0 \\ \text{an } H\text{-conjugate of } SO(n-1) & \text{if } \|v\| < 0 \end{cases}$$

where $\|(v_1, \cdots, v_n)\| = -v_1^2 + v_2^2 + \cdots + v_n^2$. There is thus no non-degenerate type, as $\{v \in \mathbb{R}^n |\, \|v\| > 0\}$ and $\{v \in \mathbb{R}^n |\, \|v\| < 0\}$ are non-empty open sets. The distinction is interpreted physically

$$\|v\| > 0 \leftrightarrow v \text{ a time-like vector}$$

$$\|v\| < 0 \leftrightarrow v \text{ a space-like vector.}$$

If we consider $v_1, \cdots, v_{n-1}$ with $\|v_i\| > 0$ all i, we obtain

$$H_m = H^{v_1, \cdots, v_{m-1}} \cong SO(1, n-m) \text{ up to a finite group.}$$

The Cartan Polygon is thus the same as for Euclidean $\mathbb{R}^n$, and n is the minimal order of contact sufficient for congruence.

For $n = 2$, given a map $[a, b] \xrightarrow{X} \mathbb{R}^2$, $X = (X_1, X_2)$, with $|X_2'(t)| > |X_1'(t)|$ for all t, we choose our frame $T_x \circ R$ so that

$$R^{-1}\begin{pmatrix} X_1' \\ X_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{(X_1')^2 - (X_1')^2} \end{pmatrix}.$$

We now reparametrize so that $X_2' \equiv 1$, i.e., we use the local time of a fixed observer. Now X_1' is just the velocity v of the particle as it appears to this observer. We may see that

$$R = \begin{pmatrix} \dfrac{1}{\sqrt{1 - v^2}} & \dfrac{-v}{\sqrt{1 - v^2}} \\ \dfrac{-v}{\sqrt{1 - v^2}} & \dfrac{1}{\sqrt{1 - v^2}} \end{pmatrix}$$

which is just the usual Lorentz transformation to the particle's frame of reference. Further,

$$R^{-1}R' = \begin{pmatrix} 0 & \dfrac{dv/dt}{1 - v^2} \\ \dfrac{dv/dt}{1 - v^2} & 0 \end{pmatrix}$$

so the differential invariant of order 2 is

$$\frac{dv/dt}{1 - v^2}$$

which is the relativistic acceleration.

We note that $SO_0(1, 1)\backslash\mathbb{R}^2 - \{(0, 0)\}$ with the quotient topology is not Hausdorff, as can happen when H is non-compact. The problem is that the images of the two light rays $\{x_1 = x_2\}$, $\{x_1 = -x_2\}$ do not have disjoint neighborhoods.

Example 6a. $\mathbb{RP}_1 = SL(2, \mathbb{R})/H$, $H = \left\{\begin{pmatrix} r & 0 \\ s & \frac{1}{r} \end{pmatrix}\right\} \subseteq SL(2, \mathbb{R})$.

As an illustration of the general method, we will actually construct the moving frame and compute the invariants. Note that H is not compact, so we cannot expect nice invariant direct-sum decompositions.

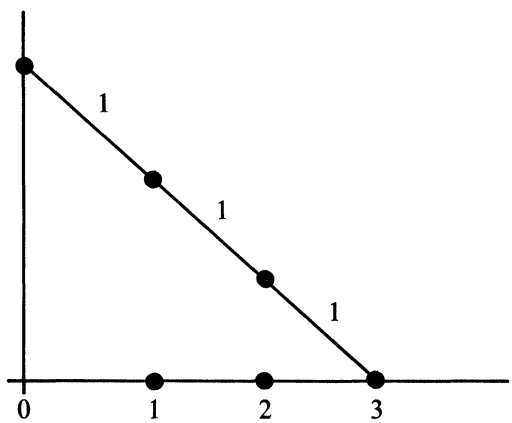
The adjoint action of H on $\mathscr{G}/\mathscr{H} \cong \mathbb{R}$ is given by $b \to \dfrac{b}{r^2}$. If $b \equiv 0$, we get a constant map, and for $b \neq 0$,

$$H_2 = \left\{\begin{pmatrix} r & 0 \\ s & r \end{pmatrix} \middle| s \in \mathbb{R}, r = \pm 1\right\}.$$

The action of H_2 on $\mathscr{G}/\mathscr{H}_2$ is given by

$$\begin{pmatrix} a & b \\ * & -a \end{pmatrix} \to \begin{pmatrix} a + b\dfrac{s}{r} & b \\ * & -a - b\dfrac{s}{r} \end{pmatrix}$$

so $H_3 = \{\pm I\}$. The Cartan polygon is thus



There is thus a unique differential invariant, which has order 3.

If $[a, b] \xrightarrow{x} \mathbb{RP}_1$, written in homogeneous coordinates $\begin{pmatrix} f \\ 1 \end{pmatrix}$, we can take $\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}$ as a preliminary lifting, as $\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} f \\ 1 \end{pmatrix}$. The derivative is $\begin{pmatrix} 0 & f' \\ 0 & 0 \end{pmatrix}$. Now in general, if $[a, b] \xrightarrow{u} SL(2, \mathbb{R})$ has derivative $g = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, then for $[a, b] \xrightarrow{h} H$, where $h = \begin{pmatrix} r & 0 \\ s & \frac{1}{r} \end{pmatrix}$, uh has derivative

$$\begin{pmatrix} a + b\dfrac{s}{r} + \dfrac{r'}{r} & \dfrac{b}{r^2} \\ cr^2 - 2asr & \\ -bs^2 + rs' - sr' & -a - b\dfrac{s}{r} - \dfrac{r'}{r} \end{pmatrix}$$

Now

$$T_1 = F_1 = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle| b \neq 0 \right\}$$

and $H \diagdown T_1 = N_H(H_2) \diagdown F_1$ is a point. We take $S_1 = \left\{ \begin{pmatrix} a & 1 \\ c & -a \end{pmatrix} \right\} / H$, so to normalize $b = 1$ we take $r = \sqrt{f'}$, $s = 0$. Our new lifting is

$$\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{f'} & 0 \\ 0 & 1/\sqrt{f'} \end{pmatrix} = \begin{pmatrix} \sqrt{f'} & f/\sqrt{f'} \\ 0 & 1/\sqrt{f'} \end{pmatrix}$$

with derivative map, using the calculations above,

$$\begin{pmatrix} \dfrac{1}{2}\dfrac{f''}{f'} & 1 \\ 0 & -\dfrac{1}{2}\dfrac{f''}{f'} \end{pmatrix}.$$

Now

$$T_2 = F_2 = \left\{ \begin{pmatrix} a & 1 \\ c & -a \end{pmatrix} \right\} / \mathscr{H}_2$$

and $H_2 \diagdown T_2$ is a point, so we take $S_2 = \left\{ \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix} \right\} / \mathscr{H}_2$. We thus want to multiply by $\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$ with $s = -\dfrac{1}{2}\dfrac{f''}{f'}$. The new lifting is thus

$$\begin{pmatrix} \sqrt{f'} & f/\sqrt{f'} \\ 0 & 1/\sqrt{f'} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\dfrac{1}{2}\dfrac{f''}{f'} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{f'} - \dfrac{1}{2}\dfrac{ff''}{(\sqrt{f'})^3} & f/\sqrt{f'} \\ -\dfrac{1}{2}\dfrac{f''}{(\sqrt{f'})^3} & 1/\sqrt{f'} \end{pmatrix}$$

and the derivative (using the formulas rather than direct calculation) is

$$\begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}$$

where

$$c = -\frac{1}{2}\left[\left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2\right].$$

The quantity in brackets in the *Schwarzian derivative* of f, and this is the differential invariant of order 3 predicted above. Minus c is classically the element of projective arc length of X. Geometrically,

$$c = 3 \lim_{h \to 0} \frac{\frac{3}{4} R(x(t), x(t+h), x(t+2h), x(t+3h)) - 1}{h^2}$$

where $R(a, b, c, d)$ is the cross-ratio $\dfrac{(a-c)(b-d)}{(a-d)(b-c)}$. (See pp. 3 and 23 of [C3].)

Example 6b. $G = SL(3, \mathbb{R})$, $H = \left\{A \in SL(3, \mathbb{R}) \middle| \text{ the last column of } A \text{ is a multiple of} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$, $G/H = \mathbb{RP}_2$.

To construct the Cartan polygon, it suffices to know $\dim(H_m) = \dim(\mathscr{H}_m)$, so we may use the adjoint action of $\mathscr{H}_m$ on $\mathscr{G}/\mathscr{H}_m$, which facilitates computations.

$$\mathscr{G} = \{(a_{ij}) | a_{11} + a_{22} + a_{33} = 0\}$$

$$\mathscr{H} = \left\{\begin{pmatrix} r & s & 0 \\ t & u & 0 \\ v & w & x \end{pmatrix} \middle| r + u + x = 0\right\}.$$

The adjoint action of $\mathscr{H}$ on $\mathscr{G}/\mathscr{H}$ is

$$\begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} \to \begin{pmatrix} (r-x)a_{13} + sa_{23} \\ ta_{13} + (u-x)a_{23} \end{pmatrix}.$$

This action of H is transitive on $\mathscr{G}/\mathscr{H} = \{0\}$, so taking

$$S_1 = \{A \in \mathscr{G} | a_{13} = 0, a_{23} = 1\}/\mathscr{H},$$

we get

$$\mathscr{H}_2 = \left\{\begin{pmatrix} r & 0 & 0 \\ t & u & 0 \\ v & x & u \end{pmatrix} \middle| r + 2u = 0\right\}.$$

The adjoint action of $\mathscr{H}_2$ on $\tilde{S}_1$ is

$$a_{12} \to (r - u)a_{12}$$

$$a_{22} - a_{33} \to ta_{12} - 2w.$$

Again by transitivity, we may choose in the non-degenerate case

$$S_2 = \{A \in \mathscr{G} | a_{12} = 2, a_{13} = 0, a_{23} = 1, a_{22} = a_{33}\}/\mathscr{H}_2$$

and then

$$\mathscr{H} = \left\{\begin{pmatrix} 0 & 0 & 0 \\ t & 0 & 0 \\ v & t & 0 \end{pmatrix}\right\}.$$

The adjoint action of $\mathscr{H}_3$ on $\tilde{S}_2$ is

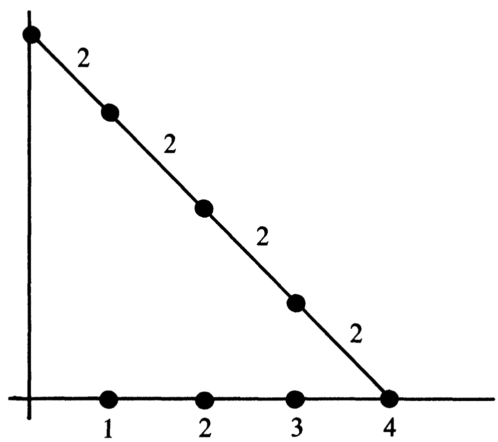
$$a_{11} - a_{22} \to -3t$$

$$a_{21} - a_{32} \to t(a_{11} - a_{22}) - 3v.$$

So

$$\mathscr{H}_3 = \{0\}$$

and the Cartan polygon for the standard type is



These are thus two invariants of order 4.

We take this opportunity to correct an error in the literature (p. 777 of [G]) to the effect that order 5 is needed for congruence of non-degenerate curves in $\mathbb{RP}_2$. Cartan notes in [C3] that if $[a, b] \xrightarrow{X} \mathbb{RP}_2$ is a non-degenerate map with homogeneous coordinates (X_1, X_2, X_3), and if

$$T = \begin{vmatrix} X_1 & X_2 & X_3 \\ X_1' & X_2' & X_3' \\ X'' & X_2'' & X_3'' \end{vmatrix}^{-1/3}$$

and $\bar{X}_i = X_i T$, $i = 1, 2, 3$, a complete set of differential invariants is given by

$$\alpha = \begin{vmatrix} \bar{X}_1 & \bar{X}_2 & \bar{X}_3 \\ \bar{X}_1'' & \bar{X}_2'' & \bar{X}_3'' \\ \bar{X}'''_1 & \bar{X}'''_2 & \bar{X}'''_3 \end{vmatrix}$$

and

$$\beta = \begin{vmatrix} \bar{X}_1' & \bar{X}_2'' & \bar{X}_3' \\ \bar{X}_1'' & \bar{X}_2'' & \bar{X}_3'' \\ \bar{X}'''_1 & \bar{X}'''_2 & \bar{X}'''_3 \end{vmatrix}.$$

One may verify, though it isn't obvious, that α has order 4 and β has order 5. However, the differential invariant $\beta + (1/3)\alpha'$ has order 4 (again by calculation), and α, $\beta + (1/3)\alpha'$ are a complete set of differential invariants for non-degenerate curves in $\mathbb{RP}_2$. One advantage of the general method of this paper is that it determines how many invariants there are of each order without such delicate calculations.

There is a nice geometric way to see that order 4 is enough. Any four points in general position in $\mathbb{P}_2$ determine a unique projective coordinate system on $\mathbb{P}_2$. Using $X(t_0)$, $X(t_0 + \Delta t)$, $X(t_0 + 2\Delta t)$, $X(t_0 + 3\Delta t)$, and taking the limit gives a lifting to $SL(3, \mathbb{R})/\{\pm I\}$, and this is determined by the order 3 jet. The derivative of this lifting gives the invariants, which thus have order ≤ 4. In fact, these invariants may also be constructed geometrically. Take the unique conic through $X(t_0), X(t_0 + \Delta t), \cdots, X(t_0 + 4\Delta t)$. On a $\mathbb{P}_1$, n points have $n - 3$ distinct cross-ratios, so 5 points on a conic have two. Taking a limit as in example 6a gives our two differential invariants of order 4.

Example 6c. $G = SL(n + 1, \mathbb{R})$,

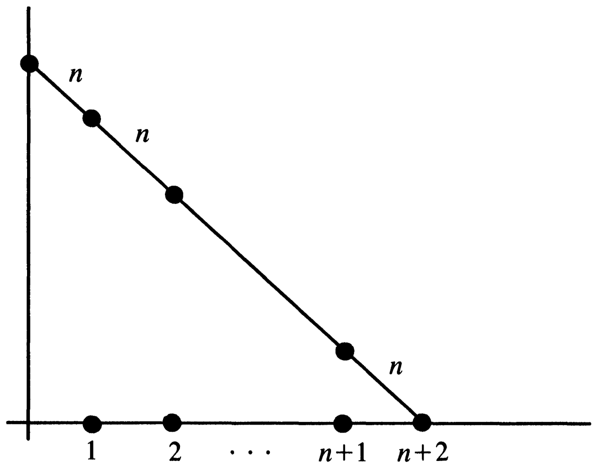
$$H = \left\{ A \in SL(n+1, \mathbb{R}) \,\middle|\, \text{the last column is a multiple of} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\},$$

$G/H = \mathbb{RP}_n$.

Here we will just indicate the result. For non-degenerate curves in $\mathbb{RP}_n$, we have

$$\mathcal{H}_m = \{(a_{ij}) \in \mathcal{SL}(n+1, \mathbb{R}) \mid a_{ij} = 0 \quad \text{for} \quad i \neq j \text{ and}$$
$$j \geq n + 2 - m,\ a_{ij} = a_{i+1,j+1} \quad \text{for} \quad i \geq j \geq n + 2 - m\}.$$

The Cartan polygon for non-degenerate curves is



and there are n invariants of order $n + 2$.

The moving frame has derivative

$$g = \begin{pmatrix} 0 & n+1 & 0 & 0 & 0 & 0 \\ \alpha_1 & 0 & & \vdots & \vdots & 0 \\ \alpha_2 & \alpha_1 & & 3 & 0 & \vdots \\ \vdots & & & 0 & 2 & 0 \\ \vdots & & & & 0 & 1 \\ \alpha_n & \cdots & \cdots & \alpha_2 & \alpha_1 & 0 \end{pmatrix}$$

and $\alpha_1, \cdots, \alpha_n$ are the n differential invariants. They may be constructed geometrically as follows—take the unique rational normal curve* through $X(t_0)$, $X(t_0 + \Delta t), \cdots, X(t_0 + (n + 2)\Delta t)$. These $n + 3$ points have n independent cross-ratios, and these, taking limits as in example 6a, are the invariants of order $n + 2$.

Example 7. Conformal geometry. Let $M(n)$ be the group acting on S^n obtained by extending from $\mathbb{R}^n$ maps of the form $T \circ D \circ R \circ I$, where T a translation, D multiplication by an element of $\mathbb{R}^*$, $R \in SO(n)$, I either inversion

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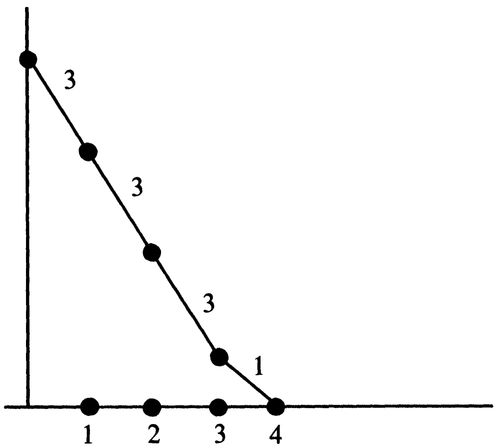
*A *rational normal curve* in $\mathbb{RP}_n$ is a smooth irreducible curve of degree n not lying in any linear subspace. These are all projectively equivalent, rational, and there is a unique rational normal curve through $n + 3$ points in general position. I am indebted to Phillip Griffiths for suggesting their relevance here.

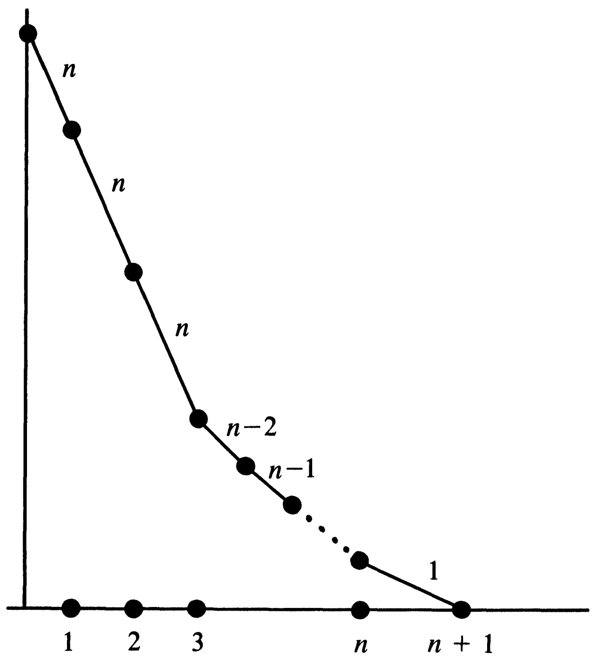
in a unit sphere or the identity. $\text{Dim}\,(M(n)) = \frac{(n+1)(n+2)}{2}$. Let $M_0(n) = \{A \in M(n) \mid A(0) = 0\}$. In low dimensions, $M(1) = SL(2, \mathbb{R})$ and $n(2) = SL(2, \mathbb{C})$ acting as fractional linear transformations, and we obtain the projective geometries $\mathbb{RP}_1 = M(1)/M_0(1)$, $\mathbb{CP}_1 = M(2)/M_0(2)$. For $n \geq 3$, $M(n)$ is the group of conformal self-maps of $\mathbb{R}^n$.

For $n = 3$, the Cartan polygon for non-degenerate curves is



so $i_1 = 0$, $i_2 = 0$, $i_3 = 2$, $i_4 = 1$. There are two global invariants of 4-tuples of non-circular points, obtained by taking the sphere through these points, which is $\cong \mathbb{CP}_1$, and take the complex cross-ratio. All invariants of m-tuples of points come from taking cross-ratios for sub 4-tuples. The differential invariants of order 3 of X are obtained by taking the projection of X on the osculating sphere at t from its center and taking the two third order invariants of this curve in projective $\mathbb{CP}_1$, i.e., the real and imaginary parts of the complex Schwarzian derivative. The fourth order invariant is obtained by taking $\lim_{\Delta t \to 0} \frac{\theta}{\Delta t}$, where θ is the angle of intersection of the osculating spheres to X at t and $t + \Delta t$.

For $n \geq 3$, we use the relation that dim H_m = dim of the stabilizer of a generic m tuple of points. For $m \geq 3$, elements of H_m are conformal transformations leaving the $(m-2)$-sphere through the m points pointwise fixed. This is seen to be $SO(n + 2 - m)$. As there are no invariants of 3 tuples, the Cartan polygon for non-degenerate curves in conformal $\mathbb{R}^n$ is



Thus $i_1 = i_2 = 0$, $i_3 = 2$, and $i_4 = i_5 = \cdots = i_{n+1} = 1$.

Example 8. An example with a continuous family of types.

$$G = \left\{ \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & 0 \\ 0 & d & 1 & 0 \\ 0 & e & 0 & 1 \end{pmatrix} \middle| \; a, b, c, d, e \in \mathbb{R} \right\}$$

$$H = \left\{ \begin{pmatrix} 1 & 0 & \beta & \gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \; \beta, \gamma \in \mathbb{R} \right\}.$$

This gives an example where there are an infinite number of possible types, in fact a continuous family. H acts on $\mathcal{G}/\mathcal{H}$ by

$$\begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & e & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & a - \beta d - \gamma e & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & e & 0 & 0 \end{pmatrix}.$$

So

$$H^{(a,d,e)} = \left\{ \begin{pmatrix} 1 & 0 & \beta & \gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big| \beta d + \gamma e = 0 \right\}.$$

Since H is abelian, in fact $\cong \mathbb{R}^2$, these lines give a family of non-conjugate isotropy types parametrized by $\mathbb{P}_1$.

It is nevertheless possible to find a moving frame in this situation. If $[a, b] \xrightarrow{u} G$ is a smooth map with derivative

$$g = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & e & 0 & 0 \end{pmatrix}$$

and if $[a, b] \xrightarrow{h} H$ is a smooth map given by

$$h = \begin{pmatrix} 1 & 0 & \beta & \gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then $u^\# = uh$ has derivative

$$g^\# = \begin{pmatrix} 0 & a^\# & b^\# & c^\# \\ 0 & 0 & 0 & 0 \\ 0 & d^\# & 0 & 0 \\ 0 & e^\# & 0 & 0 \end{pmatrix}$$

where

$$a^\# = a - \beta d - \gamma e \qquad b^\# = b + \beta' \qquad c^\# = c + \gamma'$$
$$d^\# = d \qquad e^\# = e.$$

Thus d and e are first order differential invariants. We can choose β, γ to arrange

$$a^\# = 0 \quad \text{and} \quad b^\# d^\# + c^\# e^\# = 0$$

by choosing β, γ to satisfy

$$\beta d + \gamma e = a$$

$$\beta d' + \gamma e' = a' + bd + ce$$

which uniquely determines β and γ under the non-degeneracy condition $\begin{vmatrix} d & e \\ d' & e' \end{vmatrix} \neq 0$. The lifting thus obtained is of order two, and thus we expect contact 3 to be required. We get three invariants, d and e of order 1, and one of order 3, which we may take to be

$$b + \left(\frac{\begin{vmatrix} a & e \\ a' + bd + ce & e' \end{vmatrix}}{\begin{vmatrix} d & e \\ d' & e' \end{vmatrix}} \right).$$

This example owes much to a suggestion of W. Haboush.

Example 9. A holomorphic moving frame for one-parameter families of abelian varieties.

In order to have complex groups, we consider $\mathscr{H}_g$ as a subset of $Sp(g, \mathbb{C})/H$

$$Sp(g, \mathbb{C}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid AB^t = BA^t,\ CD^t = DC^t,\ AD^t - BC^t = I \right\}$$

$$H = \left\{ \begin{pmatrix} A & 0 \\ C & (A^t)^{-1} \end{pmatrix} \mid CA^{-1} \text{ symmetric, } A, B, C, D \in \mathbb{C}^{g \times g} \right\}$$

with Lie algebras

$$Sp(g, \mathbb{C}) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mid B, C \text{ symmetric} \right\}$$

$$\mathscr{H} = \left\{ \begin{pmatrix} A & 0 \\ C & -A^t \end{pmatrix} \mid C \text{ symmetric} \right\}.$$

We consider only the non-degenerate case. The adjoint action of H on

$$\mathscr{G}/\mathscr{H} = \left\{ \begin{pmatrix} * & B \\ * & * \end{pmatrix} \mid B \text{ symmetric} \right\}$$

sends

$$B \to ABA^t.$$

If $\det B \neq 0$, we can choose A so $ABA^t = I$. The isotropy group is

$$H_2 = \left\{ \begin{pmatrix} A & 0 \\ C & A \end{pmatrix} \mid A^{-1}C \text{ symmetric, } AA^t = I \right\}.$$

The adjoint action of H_2 on normalized elements of $\mathscr{G}/\mathscr{H}_2$, i.e., those of the form

$$\begin{pmatrix} E & I \\ * & -E^t \end{pmatrix} \bmod H_2, \qquad E \text{ symmetric,}$$

is

$$E \to AEA^{-1} - CA^{-1}.$$

We can always pick $A = I$, $C = E$ to move E to 0. The isotropy is then

$$H_3 = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \mid AA^t =, I \right\}.$$

The adjoint action of H_3 on normalized elements

$$\begin{pmatrix} 0 & I \\ S & 0 \end{pmatrix}$$

mod H_3, S symmetric, is

$$S \to ASA^{-1}.$$

For a generic S, there is a unique A, modulo a finite group, so that

$$ASA^{-1} = D, \qquad D \text{ diagonal.}$$

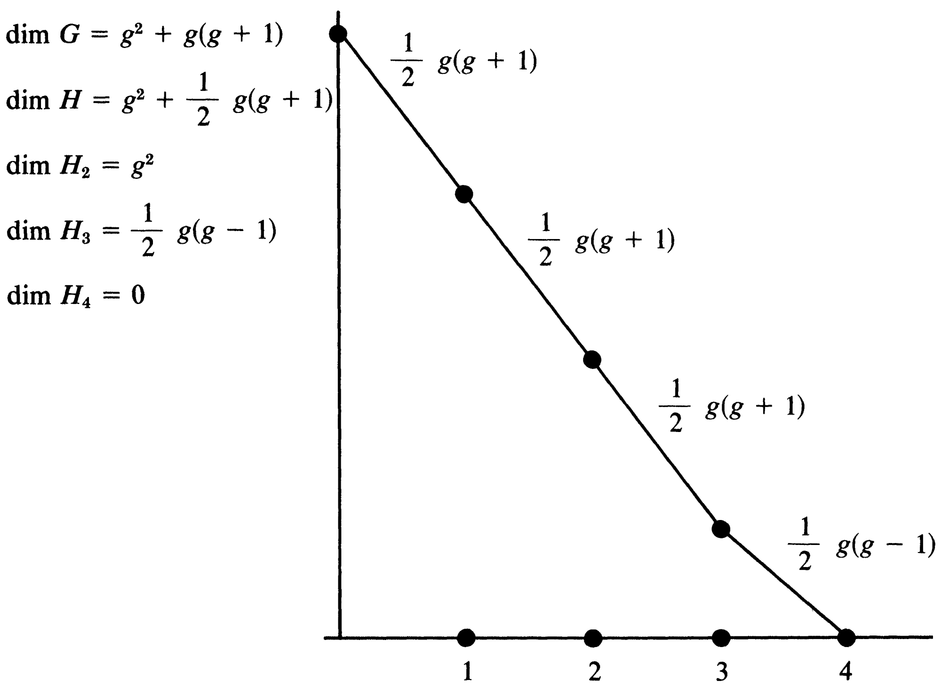
The isotropy is

$$H_4 = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \middle| A = \begin{pmatrix} \pm 1 & & 0 \\ & \pm 1 & \\ 0 & & \pm 1 \end{pmatrix} \right\} \text{ a finite group}$$

and H_4 acts trivially on normalized elements of $\mathscr{G}/\mathscr{H}_4$. The derivative of our lifting will thus have the form

$$\begin{pmatrix} U & I \\ D & U \end{pmatrix} U \text{ anti-symmetric,} \qquad D \text{ diagonal.}$$

The Cartan polygon, in terms of complex dimensions, is



There are thus g holomorphic invariants of order 3 (the entries of D) and $(1/2)g(g-1)$ holomorphic invariants of order 4 (the entries of U).

For $g = 1$, the invariants of order 4 don't appear, as $(1/2)g(g-1) = 0$. The one invariant of order 3 is the Schwarzian derivative.

The moving frame lifting $\Delta \xrightarrow[\text{holo}]{F} \mathscr{H}_g \subset Sp(g, \mathbb{C})/H$ may be constructed more explicitly as follows. Viewing F as a symmetric $g \times g$ complex matrix, the preliminary lifting $\begin{pmatrix} I & F \\ 0 & I \end{pmatrix}$ has derivative $\begin{pmatrix} 0 & F' \\ 0 & 0 \end{pmatrix}$. Choose A so that $A^tF'A = I$, possible if $\det(F') \neq 0$. The modified lifting

$$\begin{pmatrix} I & F \\ 0 & I \end{pmatrix}\begin{pmatrix} (A^t)^{-1} & 0 \\ 0 & A \end{pmatrix} \text{ has derivative } \begin{pmatrix} -(A^{-1}A')^t & I \\ 0 & A^{-1}A' \end{pmatrix}.$$

Let $(A^{-1}A')_s$ and $(A^{-1}A')_a$ be the symmetric and anti-symmetric parts of $A^{-1}A'$. The lifting

$$\begin{pmatrix} I & F \\ 0 & I \end{pmatrix}\begin{pmatrix} (A^t)^{-1} & 0 \\ 0 & A \end{pmatrix}\begin{pmatrix} I & 0 \\ (A^{-1}A')_s & I \end{pmatrix} \text{ has derivative } \begin{pmatrix} (A^{-1}A')_a & I \\ G & (A^{-1}A')_a \end{pmatrix}$$

where $G = [(A^{-1}A')_a, (A^{-1}A')_s] + (A^{-1}A')_s^2 - (A^{-1}A')_s'$. If G is a generic symmetric matrix and thus diagonalizable by an element of $O(n, \mathbb{C})$, so $B^tGB = D$ where $BB^t = I$, D diagonal, then the lifting

$$\begin{pmatrix} I & F \\ 0 & I \end{pmatrix}\begin{pmatrix} (A^t)^{-1} & 0 \\ 0 & A \end{pmatrix}\begin{pmatrix} I & 0 \\ (A^{-1}A')_s & I \end{pmatrix}\begin{pmatrix} B & 0 \\ 0 & B^t \end{pmatrix} \text{ has derivative}$$

$$\begin{pmatrix} U & I \\ D & U \end{pmatrix} \text{ where } \begin{array}{l} D = B^tGB \text{ diagonal} \\ U = B^t(A^{-1}A')_aB + B^{-1}B' \text{ anti-symmetric.} \end{array}$$

The entries of D and U are differential invariants of orders 3 and 4 respectively.

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