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The space of left-invariant metrics on a Lie group up to isometry and scaling

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Abstract. We study the spaces of left-invariant Riemannian metrics on a Lie group up to isometry, and up to isometry and scaling. In this paper, we see that such spaces can be identified with the orbit spaces of certain isometric actions on noncompact symmetric spaces. We also study some Lie groups whose spaces of left-invariant metrics up to isometry and scaling are small.

1. Introduction

Lie groups with left-invariant Riemannian metrics have provided many interesting and important examples of Riemannian manifolds. In particular, left-invariant Einstein metrics and Ricci soliton metrics on Lie groups have been of interest by many mathematicians and physicists, and studied very actively. We refer to a beautiful survey [10] for informations and references.

For three-dimensional unimodular Lie groups, thanks to the famous paper by Milnor [11], the curvature properties of left-invariant metrics are very much understood. Contrary, for higher dimensional cases, the situation looks far from the complete understanding, and many questions have been remaining unsolved (for example, Alekseevskii conjecture, see [3]). A difficulty might come from the fact that the space of left-invariant metrics has higher dimension. Note that, the set of all left-invariant metrics on a Lie group G can be identified with the set of all inner products on its Lie algebra $\mathfrak{g}$,

$$\widetilde{\mathfrak{M}} := \{\langle , \rangle \ ; \ \text{an inner product on } \mathfrak{g}\}.$$

This space can be identified with the homogeneous (coset) space $\mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$, where $n := \dim \mathfrak{g}$, and has the dimension $n(n+1)/2$.

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In this paper, we consider and study the "moduli space" of the left-invariant Riemannian metrics on a Lie group. By the moduli spaces, we mean the sets of isometry classes and of the isometry and scaling classes:

$$\mathfrak{M} := \widetilde{\mathfrak{M}}/\text{isometry},$$
$$\mathfrak{PM} := \widetilde{\mathfrak{M}}/\text{isometry and scaling}.$$

It is enough to consider these spaces, since the properties we are interested in are preserved by isometry and scaling.

In Sect. 2, we will see that the spaces $\mathfrak{M}$ and $\mathfrak{PM}$ can be identified with the orbit spaces under the actions of $\mathrm{Aut}(\mathfrak{g})$ and $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$, respectively:

$$\mathfrak{M} = \mathrm{Aut}(\mathfrak{g})\backslash\widetilde{\mathfrak{M}},$$
$$\mathfrak{PM} = \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})\backslash\widetilde{\mathfrak{M}}.$$

Since $\widetilde{\mathfrak{M}}$ is a homogeneous space, these orbit spaces can also be identified with certain double coset spaces.

In Sect. 3, we consider Lie groups with $\dim \mathfrak{PM} = 0$. This condition is equivalent to $\mathfrak{PM} = \{\mathrm{pt}\}$ (since $\mathfrak{PM}$ is connected), that is, there exists only one left-invariant Riemannian metric up to isometry and scaling. In fact, such Lie groups have already been classified by Lauret [9], but, by using our expression of $\mathfrak{PM}$, we can make some arguments simpler.

In Sect. 4, we consider Lie groups with $\dim \mathfrak{PM} = 1$. We will find some examples of such Lie groups. In particular, our examples show that such Lie groups do exist in any dimensions($\geq$3).

We note that, the space $\mathfrak{PM}$ and their descriptions will be potentially useful for the problems on the existence and the non-existence of "good" (for example, Einstein or Ricci soliton) left-invariant Riemannian metrics on Lie groups. In general, it is not easy to find such good metrics, and furthermore, to show the non-existence would be very hard. To solve such problems, it is enough to work only on $\mathfrak{PM}$. Hence, for some Lie groups whose $\mathfrak{PM}$ are small, our results might be useful for studying the problems exposed above.

2. Expression of the space of left-invariant metrics

In this section, we describe the spaces $\mathfrak{M}$ and $\mathfrak{PM}$ as orbit spaces, and hence as double coset spaces.

2.1. Expression of $\widetilde{\mathfrak{M}}$

In this subsection we recall that the space

$$\widetilde{\mathfrak{M}} := \{\langle , \rangle\ ;\ \text{an inner product on } \mathfrak{g}\}$$

can be expressed as a coset space.

First of all, we recall some general theory on coset spaces. Assume that there is a transitive left-action of a group G on a set M. We denote by $g.m$ the action of $g \in G$ on $m \in M$. Let $m_0 \in M$, and

$$K := G_{m_0} := \{g \in G \mid g.m_0 = m_0\}$$

be the isotropy subgroup of G at m_0. The coset space G/K is defined as the set of cosets, that is,

$$G/K := \{gK \mid g \in G\}.$$

Thus, we can identify $M = G/K$ by the bijective map

$$f : G/K \to M : gK \mapsto g.m_0.$$

By applying this theory to the space $\widetilde{\mathfrak{M}}$, we have

Proposition 2.1. *If* $\dim \mathfrak{g} = n$*, then* $\widetilde{\mathfrak{M}} = \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$.

Proof. Fix a basis $\{x_1, \ldots, x_n\}$ of $\mathfrak{g}$, and identify $\mathfrak{g} = \mathbb{R}^n$. Then,

$$g.\langle\cdot,\cdot\rangle := \langle g^{-1}\cdot, g^{-1}\cdot\rangle \quad (\text{for } g \in \mathrm{GL}_n(\mathbb{R}) \text{ and } \langle\cdot,\cdot\rangle \in \widetilde{\mathfrak{M}})$$

gives a transitive left-action of $\mathrm{GL}_n(\mathbb{R})$ on $\widetilde{\mathfrak{M}}$. Let $\langle,\rangle_0 \in \widetilde{\mathfrak{M}}$ be the inner product such that the basis $\{x_1, \ldots, x_n\}$ is orthonormal. Thus, the isotropy subgroup at $\langle,\rangle_0$ is

$$\mathrm{GL}_n(\mathbb{R})_{\langle,\rangle_0} := \{g \in GL_n(\mathbb{R}) \mid g.\langle,\rangle_0 = \langle,\rangle_0\} = \mathrm{O}(n).$$

Therefore, $\widetilde{\mathfrak{M}}$ can be identified with $\mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$. □

2.2. *Definitions of* $\mathfrak{M}$ *and* $\mathfrak{PM}$

In this subsection we define $\mathfrak{M}$ and $\mathfrak{PM}$.

Definition 2.2. Let $(\mathfrak{g}_1, \langle,\rangle_1)$ and $(\mathfrak{g}_2, \langle,\rangle_2)$ be metric Lie algebras, that is, Lie algebras with inner products.

(1) They are said to be *isometric* if there exists a Lie algebra isomorphism $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\langle\cdot,\cdot\rangle_1 = \langle\varphi(\cdot), \varphi(\cdot)\rangle_2$.
(2) They are said to be *isometric up to scaling* if there exists a Lie algebra isomorphism $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ and $c > 0$ such that $\langle\cdot,\cdot\rangle_1 = c\langle\varphi(\cdot), \varphi(\cdot)\rangle_2$.

It is easy to see that "isometry" and "isometry up to scalar" give equivalence relations.

Remark 2.3. These equivalent relations at the Lie algebra level give rise to the equivalent relations of left-invariant Riemannian metrics on Lie groups. More precisely,

(1) If $(\mathfrak{g}_1, \langle , \rangle_1)$ and $(\mathfrak{g}_2, \langle , \rangle_2)$ are isometric (resp. isometric up to scalar) in the sense of Definition 2.2, then the corresponding connected and simply connected Lie groups endowed with the induced left-invariant Riemannian metrics $(G_1, \langle , \rangle_1)$ and $(G_2, \langle , \rangle_2)$ are isometric (resp. isometric up to scalar) as Riemannian manifolds. In fact, a Lie algebra isomorphism $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ induces a Lie group isomorphism $\widetilde{\varphi} : G_1 \to G_2$ such that $\widetilde{\varphi}_* = \varphi$, since they are simply connected. One can easily see that, if φ is isometric (resp. isometric up to scalar), then so is $\widetilde{\varphi}$.
(2) The converse of (1) does not hold in general. It holds if, for example, $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are nilpotent ([16]).

By this remark, our equivalent relations at Lie algebra level preserve Riemannian geometric properties we are interested in. Hence, it is natural to consider the following quotient spaces.

Definition 2.4. For a Lie algebra $\mathfrak{g}$,

(1) $\mathfrak{M}$ denotes the quotient space of $\widetilde{\mathfrak{M}}$ by "isometry",
(2) $\mathfrak{PM}$ denotes the quotient space of $\widetilde{\mathfrak{M}}$ by "isometry up to scaling".

2.3. Expressions of $\mathfrak{M}$ *and* $\mathfrak{PM}$

In this subsection we express $\mathfrak{M}$ and $\mathfrak{PM}$ as the orbit spaces.

We here recall some general theory of orbit spaces and double coset spaces. Assume that there is a transitive left-action of a group G on a set M. Let H be a subgroup of G, which acts naturally on M. The orbit space $H \backslash M$ of this action is defined by the set of H-orbits, that is,

$$H \backslash M := \{H.m \mid m \in M\}.$$

Let $K := G_{m_0}$ be the isotropy subgroup at $m_0 \in M$, and hence $M = G/K$. The double coset space $H \backslash G/K$ is defined by the set of double cosets,

$$H \backslash G/K := \{HgK \mid g \in G\}.$$

Thus, we can identify $H \backslash M = H \backslash G/K$ by the bijective map

$$f : H \backslash G/K \to H \backslash M : HgK \mapsto H.(g.m_0).$$

To express $\mathfrak{M}$ and $\mathfrak{PM}$ as orbit spaces, we need

$$\begin{aligned} \mathrm{Aut}(\mathfrak{g}) &:= \{h : \mathfrak{g} \to \mathfrak{g}\ ;\ \text{automorphism}\}, \\ \mathbb{R}^\times &:= \{c \cdot \mathrm{id} : \mathfrak{g} \to \mathfrak{g}\ ;\ c \in \mathbb{R} \setminus \{0\}\}. \end{aligned}$$

Both groups can naturally be regarded as subgroups of $\mathrm{GL}_n(\mathbb{R})$, if $\dim \mathfrak{g} = n$.

Theorem 2.5. *If* $\dim \mathfrak{g} = n$*, then*

(1) $\mathfrak{M} = \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}} = \mathrm{Aut}(\mathfrak{g}) \backslash \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$,
(2) $\mathfrak{PM} = \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}} = \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$.

Proof. We only prove (2), since the proof of (1) is similar. From the above mentioned general theory, one has

$$\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g})\backslash\widetilde{\mathfrak{M}} = \mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g})\backslash\mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n).$$

Hence, we have only to show that

$$\mathfrak{PM} = \mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g})\backslash\widetilde{\mathfrak{M}}.$$

This is equivalent to

$$\begin{aligned}&\langle,\rangle_1 \text{ and } \langle,\rangle_2 \text{ are isometric up to scalar}\\ &\text{if and only if they are in the same } \mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g})\text{-orbit.}\end{aligned} \tag{2.1}$$

We prove (2.1). Assume $\langle,\rangle_1$ and $\langle,\rangle_2$ are isometric up to scalar. By definition, there exist $c > 0$ and $h \in \mathrm{Aut}(\mathfrak{g})$ such that

$$\langle X, Y\rangle_1 = c\langle hX, hY\rangle_2 = \langle c^{1/2}hX, c^{1/2}hY\rangle_2 = (c^{1/2}h)^{-1}.\langle X, Y\rangle_2.$$

Since $(c^{1/2}h)^{-1} = c^{-1/2}h^{-1} \in \mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g})$, they are in the same $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g})$-orbit. Conversely, assume that $\langle,\rangle_1$ and $\langle,\rangle_2$ are in the same $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g})$-orbit. Then, there exist $c \in \mathbb{R}^{\times}$ and $h \in \mathrm{Aut}(\mathfrak{g})$ such that

$$\langle X, Y\rangle_1 = (ch).\langle X, Y\rangle_2 = \langle (ch)^{-1}X, (ch)^{-1}Y\rangle_2 = c^{-2}\langle h^{-1}X, h^{-1}Y\rangle_2.$$

Since $c^{-2} > 0$ and $h^{-1} \in \mathrm{Aut}(\mathfrak{g})$, they are isometric up to scalar. This proves (2.1). □

2.4. *An example of $\mathfrak{M}$*

As an easy example, we here see an explicit description of $\mathfrak{M}$ for the three-dimensional Heisenberg Lie algebra $\mathfrak{h}_3$. By definition, $\mathfrak{h}_3$ is the Lie algebra with a basis $\{e_1, e_2, e_3\}$ such that

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = 0.$$

Let $\langle,\rangle_0$ be the inner product of $\mathfrak{h}_3$ so that $\{e_1, e_2, e_3\}$ is orthonormal. Denote by $[\langle,\rangle]$ the isometry class of $\langle,\rangle$. For convenience, we denote by $\mathrm{diag}(a, b, c)$ the diagonal matrix whose diagonal entries are a, b, c.

Example 2.6. For the three-dimensional Heisenberg Lie algebra $\mathfrak{h}_3$, we have

$$\mathfrak{M} = \{[\mathrm{diag}(1, 1, \lambda).\langle,\rangle_0] \mid \lambda > 0\}.$$

Proof. We have only to show that $\mathfrak{M}$ is contained in the right-hand side. Let $[\langle , \rangle] \in \mathfrak{M}$. Thus, there exists $g \in \mathrm{GL}_3(\mathbb{R})$ such that $\langle , \rangle = g.\langle , \rangle_0$. We need to show that

$$\exists \lambda > 0 \ : \ [g.\langle , \rangle_0] = [(\mathrm{diag}(1, 1, \lambda)).\langle , \rangle_0].$$

By Theorem 2.5, this is equivalent to

$$\exists \lambda > 0 \ : \ \mathrm{diag}(1, 1, \lambda) \in \mathrm{Aut}(\mathfrak{h}_3)\, g\, \mathrm{O}(3). \tag{2.2}$$

We prove (2.2). Recall that $g \in \mathrm{GL}_3(\mathbb{R})$. First of all, one knows that there exists $k \in \mathrm{O}(3)$ such that

$$gk = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Note that $0 \neq \det(gk) = a_{11}a_{22}a_{33}$. By direct calculations, one can see that

$$\mathrm{Aut}(\mathfrak{h}_3) = \left\{ \begin{bmatrix} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{11}x_{22} - x_{12}x_{21} \end{bmatrix} \mid \det \neq 0 \right\}. \tag{2.3}$$

Thus we have

$$h_1 := \frac{1}{a_{11}a_{22}} \begin{bmatrix} a_{22} & 0 & 0 \\ -a_{21} & a_{11} & 0 \\ 0 & 0 & * \end{bmatrix} \in \mathrm{Aut}(\mathfrak{h}^3).$$

This gives

$$h_1 gk = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a'_{31} & a'_{32} & a'_{33} \end{bmatrix}, \text{ where } a'_{33} \neq 0.$$

Furthermore, let us take

$$h_2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a'_{31} & -a'_{32} & 1 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{h}^3).$$

We then obtain

$$h_2 h_1 gk = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{bmatrix} \text{ for some } \lambda \neq 0.$$

One can change the sign of λ by multiplying $\mathrm{diag}(1, 1, -1) \in \mathrm{O}(3)$ from the right. This completes the proof of (2.2). □

Theoretically, one can determine $\mathfrak{M}$ and $\mathfrak{PM}$ for any Lie algebra by the same method, but in general it is very hard. We note that a similar argument can be found in [5], in which the automorphism groups of three-dimensional Lie algebras are explicitly described.

3. Lie groups with $\mathfrak{PM} = \{\mathrm{pt}\}$

In this section, we consider Lie algebras satisfying $\mathfrak{PM} = \{\mathrm{pt}\}$, that is, there exists only one inner product up to isometry and scaling. Such Lie algebras have been classified by Lauret [9]. Our description of $\mathfrak{PM}$ gives a simpler argument for a part of his theorem.

3.1. Lauret's theorem

We denote by $\mathbb{R}^n$ the n-dimensional abelian Lie algebra, and by $\mathfrak{h}_3$ the three-dimensional Heisenberg Lie algebra. We also need

Definition 3.1. A Lie algebra of dimension $n \geq 2$ is called the *Lie algebra of the real hyperbolic space* $\mathbb{R}\mathrm{H}^n$ if it has a basis $\{e_1, \ldots, e_n\}$ such that

$$[e_1, e_k] = e_k \quad (k = 2, \ldots, n)$$

and other bracket products are trivial. We denote this Lie algebra by $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$.

Note that the connected and simply connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$ acts simply transitively on the real hyperbolic space $\mathbb{R}\mathrm{H}^n$.

Theorem 3.2. (Lauret [9]) *A non-abelian Lie algebra* $\mathfrak{g}$ *satisfies* $\mathfrak{PM} = \{\mathrm{pt}\}$ *if and only if* $\mathfrak{g}$ *is isomorphic to*

(1) $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$ *($n \geq 2$), or*
(2) $\mathfrak{g} = \mathfrak{h}_3 \oplus \mathbb{R}^{n-3}$ *($n \geq 3$).*

Note that the proof by Lauret in [9] uses degenerations of bracket products and curvature pinching. Our description of $\mathfrak{PM}$ as an orbit space gives a very simple proof for the "if"-part of this theorem.

3.2. The abelian case

The proof of the following easy proposition describes the idea of our next arguments.

Proposition 3.3. *The abelian Lie algebra* $\mathfrak{g} = \mathbb{R}^n$ *satisfies* $\mathfrak{M} = \{\mathrm{pt}\}$.

Proof. Recall that

$$\mathfrak{M} = \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}}.$$

Since $\mathfrak{g} = \mathbb{R}^n$ is abelian, one has $\mathrm{Aut}(\mathbb{R}^n) = \mathrm{GL}_n(\mathbb{R})$. Therefore, the action of $\mathrm{Aut}(\mathbb{R}^n)$ is transitive. This concludes that $\mathfrak{M}$ is a point. □

Therefore, the abelian Lie algebra admits only one inner product up to isometry.

3.3. Another argument for Lauret's theorem

We give another proof of the "if"-part of Lauret's theorem. The idea is same as the abelian case.

Proposition 3.4. *The following Lie algebras satisfy* $\mathfrak{PM} = \{\mathrm{pt}\}$:

(1) $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$ *($n \geq 2$), or*
(2) $\mathfrak{g} = \mathfrak{h}_3 \oplus \mathbb{R}^{n-3}$ *($n \geq 3$).*

Proof. Recall that

$$\mathfrak{PM} = \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}}.$$

Hence, we have only to show that the action of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ on $\widetilde{\mathfrak{M}}$ is transitive. To show the transitivity, we need the Lie algebra of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$. One knows (see [7]) that

$$\mathrm{Lie}(\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})) = \mathbb{R} \oplus \mathrm{Der}(\mathfrak{g}),$$

where

$$\begin{aligned} \mathrm{Der}(\mathfrak{g}) &:= \{D \in \mathfrak{gl}(\mathfrak{g}) \ ; \ D[\cdot,\cdot] = [D(\cdot),\cdot] + [\cdot, D(\cdot)]\}, \\ \mathbb{R} &:= \{c \cdot \mathrm{id} : \mathfrak{g} \to \mathfrak{g} \ ; \ c \in \mathbb{R}\}. \end{aligned}$$

One can calculate $\mathrm{Der}(\mathfrak{g})$ directly as

$$\mathrm{Der}(\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}) = \left\{ \left[\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{array} \right] \right\}$$

with respect to the canonical basis of $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$. For the Lie algebra $\mathfrak{h}_3 \oplus \mathbb{R}^{n-3}$, we use the basis $\{e_1, \ldots, e_n\}$ so that

$$[e_1, e_2] = e_n.$$

With respect to this basis, one can calculate directly that

$$\mathrm{Der}(\mathfrak{h}_3 \oplus \mathbb{R}^{n-3}) = \left\{ \left[\begin{array}{cc|ccc|c} a & * & 0 & \cdots & 0 & 0 \\ * & b & 0 & \cdots & 0 & 0 \\ \hline * & * & * & \cdots & * & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\ * & * & * & \cdots & * & 0 \\ \hline * & * & * & \cdots & * & a+b \end{array} \right] \ \middle| \ a, b \in \mathbb{R} \right\}.$$

In both cases, we have

$$\mathfrak{q} := \left\{ \left[\begin{array}{cccc} * & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ * & \cdots & \cdots & * \end{array} \right] \right\} \subset \mathbb{R} \oplus \mathrm{Der}(\mathfrak{g}).$$

Let Q be the connected Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$ with Lie algebra $\mathfrak{q}$, which consists of lower triangular matrices. Then, $Q \subset \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ holds, and one knows that Q acts transitively on $\widetilde{\mathfrak{M}}$. Therefore, the action of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ is also transitive. This completes the proof. □

Note that, our method looks hard to apply to show "only if"-part of Lauret's theorem. One of the reason is that there are so many transitive actions on $\widetilde{\mathfrak{M}} = \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$.

4. Lie groups with dim $\mathfrak{PM}$ = 1

In this section, we give some examples of Lie algebras satisfying $\dim \mathfrak{PM} = 1$. The main result of this section is

Theorem 4.1. *The following Lie algebras satisfy* $\dim \mathfrak{PM} = 1$:

(1) $(\mathbb{R}^3, [,]) : [e_1, e_2] = e_2,\ [e_1, e_3] = e_2 + e_3.$
(2) $(\mathbb{R}^3, [,]) : [e_1, e_2] = e_2,\ [e_1, e_3] = ke_3\ (-1 \leq k < 1).$
(3) $(\mathbb{R}^3, [,]) : [e_1, e_2] = ke_2 + e_3,\ [e_1, e_3] = -e_2 + ke_3\ (k \geq 0).$
(4) $(\mathbb{R}^4, [,]) : [e_1, e_2] = e_2,\ [e_1, e_3] = e_3,\ [e_1, e_4] = e_3 + ke_4.$
(5) $\mathfrak{g}_{\mathbb{R}\mathrm{H}^{n-1}} \oplus \mathbb{R}.$
(6) $\mathfrak{g}_{\mathbb{R}\mathrm{H}^2} \oplus \mathbb{R}^{n-2}.$

Note that, in (5) and (6), $\mathfrak{g}_{\mathbb{R}\mathrm{H}^k}$ is the Lie algebra of the real hyperbolic space $\mathbb{R}\mathrm{H}^k$ (see Definition 3.1), $\mathbb{R}^k$ is the abelian Lie algebra, and $\oplus$ denotes the direct sum of Lie algebras.

If one want to solve the existence and non-existence problems of particular left-invariant Riemannian metrics on Lie groups, then it is enough to work only on $\mathfrak{PM}$. Theorem 4.1 implies that, although $\mathfrak{g}$ has higher dimension, $\mathfrak{PM}$ can be small. This means that, the spaces $\mathfrak{PM}$ and their descriptions as orbit spaces, have chances to be applied for the problems exposed above.

4.1. Riemannian metrics on $\widetilde{\mathfrak{M}}$

In this subsection, we mention the natural $\mathrm{GL}_n(\mathbb{R})$-invariant Riemannian metric on

$$\widetilde{\mathfrak{M}} = \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n).$$

First of all we recall some general theory (we refer to [8, Chap. X]). Let $M = G/K$ be a homogeneous manifold, where K is compact. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively, and

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

be a reductive decomposition (that is, a direct sum decomposition as a vector space such that ad_K preserves $\mathfrak{p}$). Note that $\mathfrak{p}$ can be identified with T_oM, the tangent space at the origin $o\ (= eK)$. Then, there exists one-to-one correspondence between ad_K-invariant inner products on $\mathfrak{p}$ and G-invariant Riemannian metrics on M.

For the space $\widetilde{\mathfrak{M}} = \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$, the subspace

$$\mathfrak{p} := \{X \in \mathfrak{gl}_n(\mathbb{R}) \mid {}^tX + X = 0\}$$

gives the natural reductive decomposition $\mathfrak{gl}_n(\mathbb{R}) = \mathfrak{o}(n) \oplus \mathfrak{p}$. This decomposition is reductive, since ad_g for $g \in K$ is given by

$$\mathrm{ad}_g : \mathfrak{p} \to \mathfrak{p} : X \mapsto gXg^{-1}.$$

It is easy to see that

$$\langle X, Y \rangle := \mathrm{tr}(XY) \tag{4.1}$$

is an $\mathrm{ad}_{\mathrm{O}(n)}$-invariant inner product on $\mathfrak{p}$.

Definition 4.2. The inner product $\langle , \rangle$ on $\mathfrak{p}$ defined by (4.1) is called the *natural inner product*, and the corresponding $\mathrm{GL}_n(\mathbb{R})$-invariant Riemannian metric on $\widetilde{\mathfrak{M}} = \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$ is called the *natural metric*.

The action of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ on $\widetilde{\mathfrak{M}} = \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$ is an isometric action with respect to this natural metric, since it is $\mathrm{GL}_n(\mathbb{R})$-invariant. We use this observation to prove Theorem 4.1.

Note that, a $\mathrm{GL}_n(\mathbb{R})$-invariant Riemannian metric on $\widetilde{\mathfrak{M}} = \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$ is not unique. Let

$$\mathfrak{p}_1 := \{cE_n \mid c \in \mathbb{R}\}, \quad \mathfrak{p}_0 := \{X \in \mathfrak{p} \mid \mathrm{tr}(X) = 0\}.$$

Since $\mathrm{ad}_{\mathrm{O}(n)}$ preserves the decomposition $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_0$, we have that

$$a\langle , \rangle|_{\mathfrak{p}_1 \times \mathfrak{p}_1} + b\langle , \rangle|_{\mathfrak{p}_0 \times \mathfrak{p}_0}$$

is an $\mathrm{ad}_{\mathrm{O}(n)}$-invariant inner product on $\mathfrak{p}$ for every $a, b > 0$. But, in the latter arguments, the choice of $\mathrm{GL}_n(\mathbb{R})$-invariant Riemannian metric does not matter.

We also note that $\widetilde{\mathfrak{M}} = \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$ endowed with any $\mathrm{GL}_n(\mathbb{R})$-invariant Riemannian metric is a noncompact symmetric space.

4.2. Preliminaries on orbit spaces

For the proof of Theorem 4.1, we need to recall some fundamental properties of the orbit space $H\backslash M$ of an isometric action of H on a connected Riemannian manifold M. We refer to [12, Sect. 5] and [1, Chap. 3].

Definition 4.3. The codimension of a maximal dimensional orbit is called the *cohomogeneity* of an action.

The dimension of the orbit space $\dim H\backslash M$ coincides with the cohomogeneity of the action of H on M (see [12, Remark 5.5.2]). To calculate the cohomogeneity, the slice representation is useful.

Definition 4.4. Let H_p be the isotropy subgroup of H at $p \in M$, and $\nu_p(H.p)$ be the normal space of $H.p$ at p (that is, the orthogonal complement of $T_p(H.p)$ in T_pM). Then, the action of H_p on $\nu_p(H.p)$ is called the *slice representation* of H at p.

The slice representation determines the cohomogeneity of the action, which we refer as the slice theorem.

Theorem 4.5. (cf. [1, Exercise 3.10.2]) *The cohomogeneity of the action of H on M coincides with the cohomogeneity of the slice representation at any point.*

4.3. Proofs of (1), (2) and (4)

In this subsection, we prove (1), (2) and (4) of Theorem 4.1. For simplicity, we write

$$H := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}), \quad \mathfrak{h} := \mathrm{Lie}(H) = \mathbb{R} \oplus \mathrm{Der}(\mathfrak{g}).$$

We will show $\dim \mathfrak{P}\mathfrak{M} = 1$, that is, the action of H on $\widetilde{\mathfrak{M}}$ is of cohomogeneity one. To show this, it is enough to find an orbit $H.p$ of codimension one (note that some of these actions can be found in [2], but we will not use this). We show

$$\mathrm{codim}\, H.o = 1,$$

where $o := \langle , \rangle_0 \in \widetilde{\mathfrak{M}}$ is the canonical inner product. Note that the dimension of the orbit $H.o = H/H_o$ can be calculated by

$$\dim H.o = \dim H - \dim H_o = \dim \mathfrak{h} - \dim \mathfrak{h} \cap \mathfrak{o}(n).$$

Proof (of (1)*).* Since $\dim \mathfrak{g} = 3$, we have

$$\dim \widetilde{\mathfrak{M}} = \dim \mathrm{GL}_3(\mathbb{R}) - \dim \mathrm{O}(3) = 6.$$

A direct calculation yields that

$$\mathrm{Der}(\mathfrak{g}) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ * & a & * \\ * & 0 & a \end{bmatrix} \mid a \in \mathbb{R} \right\}.$$

Note that $*$ denotes an arbitrary real number. Since $\mathbb{R}$ means the set of scalar matrices, we have

$$\mathfrak{h} = \mathbb{R} \oplus \mathrm{Der}(\mathfrak{g}) = \left\{ \begin{bmatrix} * & 0 & 0 \\ * & b & * \\ * & 0 & b \end{bmatrix} \mid b \in \mathbb{R} \right\}.$$

One can easily see $\mathfrak{h} \cap \mathfrak{o}(3) = \{0\}$, which concludes that

$$\dim H.o = \dim \mathfrak{h} - \dim \mathfrak{h} \cap \mathfrak{o}(n) = 5 - 0 = 5.$$

Therefore, the codimension of $H.o$ is one. $\square$

Proof (of (2)). Since $\dim \mathfrak{g} = 3$, we have

$$\dim \widetilde{\mathfrak{M}} = \dim \mathrm{GL}_3(\mathbb{R}) - \dim \mathrm{O}(3) = 6.$$

A direct calculation yields that

$$\mathrm{Der}(\mathfrak{g}) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ * & * & 0 \\ * & 0 & * \end{bmatrix} \right\}, \quad \mathfrak{h} = \mathbb{R} \oplus \mathrm{Der}(\mathfrak{g}) = \left\{ \begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ * & 0 & * \end{bmatrix} \right\}.$$

One can easily see $\mathfrak{h} \cap \mathfrak{o}(3) = \{0\}$, which concludes that

$$\dim H.o = \dim \mathfrak{h} - \dim \mathfrak{h} \cap \mathfrak{o}(3) = 5 - 0 = 5.$$

Therefore, the codimension of $H.o$ is one. $\square$

Proof (of (4)). Since $\dim \mathfrak{g} = 4$, we have

$$\dim \widetilde{\mathfrak{M}} = \dim \mathrm{GL}_4(\mathbb{R}) - \dim \mathrm{O}(4) = 16 - 6 = 10.$$

A direct calculation yields that

$$\mathrm{Der}(\mathfrak{g}) = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ * & * & 0 & * \\ * & * & a & * \\ * & 0 & 0 & a \end{bmatrix} \mid a \in \mathbb{R} \right\} \quad (\text{if } k = 1),$$

$$\mathrm{Der}(\mathfrak{g}) = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ * & * & a & (k-1)a \\ * & * & b & c \\ * & 0 & 0 & b + (k-1)c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\} \quad (\text{if } k \neq 1).$$

In both cases, one can see $\mathfrak{h} \cap \mathfrak{o}(4) = \{0\}$, which concludes that

$$\dim H.o = \dim \mathfrak{h} - \dim \mathfrak{h} \cap \mathfrak{o}(4) = 9 - 0 = 9.$$

Therefore, the codimension of $H.o$ is one. $\square$

4.4. Proofs of (3), (5) and (6)

To prove these cases, we need the slice representation of $H := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ at $o := \langle , \rangle_0$. To determine the slice representation, we use the identifications

$$\begin{aligned} T_o\widetilde{\mathfrak{M}} = \mathfrak{p} &= \{X \in \mathfrak{gl}_n(\mathbb{R}) \mid {}^tX + X = 0\}, \\ \nu_o(H.o) &= \{X \in \mathfrak{p} \mid \langle X, \pi(\mathfrak{h}) \rangle = 0\}. \end{aligned}$$

Note that $\pi : \mathfrak{gl}_n(\mathbb{R}) \to \mathfrak{p}$ is the natural projection, and $\langle , \rangle$ is the natural inner product on $\mathfrak{p}$ (see Definition 4.2). The slice representation coincides with the adjoint action of H_p on $\nu_o(H.o)$.

Proof (of (3)*).* We show that the slice representation at o is of cohomogeneity one. A direct calculation yields that

$$\mathrm{Der}(\mathfrak{g}) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ * & a & -b \\ * & b & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\},$$

and thus

$$\mathfrak{h} = \mathbb{R} \oplus \mathrm{Der}(\mathfrak{g}) = \left\{ \begin{bmatrix} * & 0 & 0 \\ * & a & -b \\ * & b & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}.$$

Hence, the Lie algebra $\mathfrak{h}_o$ of the isotropy subgroup H_o is

$$\mathfrak{h}_o = \mathfrak{h} \cap \mathfrak{o}(3) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -x \\ 0 & x & 0 \end{bmatrix} \mid x \in \mathbb{R} \right\} \cong \mathfrak{o}(2).$$

The normal space $\nu_o(H.o)$ of $H.o$ at o is

$$\nu_o(H.o) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & c & d \\ 0 & d & -c \end{bmatrix} \mid c, d \in \mathbb{R} \right\}.$$

Thus, on the Lie algebra level, the slice representation is equivalent to the natural action of $\mathfrak{o}(2)$ on $\mathbb{R}^2$. This is obviously of cohomogeneity one, hence we completes the proof by the slice theorem. □

The proofs of (5) and (6) follow from the next proposition.

Proposition 4.6. *For $m_1, m_2 \geq 1$, the Lie algebra $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}\mathrm{H}^{m_1+1}} \oplus \mathbb{R}^{m_2}$ satisfies* $\dim \mathfrak{P}\mathfrak{M} = \min\{m_1, m_2\}$.

Proof. Let $\{e_1, \ldots, e_{m_1+m_2+1}\}$ be a basis of $\mathfrak{g}$ so that the bracket product satisfies

$$[e_1, e_i] = e_i \quad (\text{for } i = 2, \ldots, m_1 + 1).$$

The Lie algebra $\mathfrak{h} := \mathbb{R} \oplus \mathrm{Der}(\mathfrak{g})$ can be calculated directly as

$$\mathrm{Der}(\mathfrak{g}) = \left\{ \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline * & * & 0 \\ \hline * & 0 & * \end{array} \right] \right\}, \quad \mathfrak{h} = \left\{ \left[\begin{array}{c|c|c} * & 0 & 0 \\ \hline * & * & 0 \\ \hline * & 0 & * \end{array} \right] \right\},$$

where the sizes of the block decompositions are $(1, m_1, m_2)$. The Lie algebra $\mathfrak{h}_o$ of the isotropy subgroup H_o is

$$\mathfrak{h}_o = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & Y \end{bmatrix} \mid X \in \mathfrak{o}(m_1),\ Y \in \mathfrak{o}(m_2) \right\} \cong \mathfrak{o}(m_1) \oplus \mathfrak{o}(m_2),$$

and the normal space is

$$\nu_o(H.o) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & A \\ 0 & {}^tA & 0 \end{bmatrix} \mid A \in M(m_1, m_2; \mathbb{R}) \right\} \cong M(m_1, m_2; \mathbb{R}).$$

Therefore, on the Lie algebra level, the slice representation is equivalent to the natural (tensor product) action of $\mathfrak{o}(m_1) \oplus \mathfrak{o}(m_2)$ on $M(m_1, m_2; \mathbb{R}) = \mathbb{R}^{m_1} \otimes \mathbb{R}^{m_2}$. This coincides with the isotropy representation of the Grassmannian manifold

$$G_{m_1}(\mathbb{R}^{m_1+m_2}) = \mathrm{O}(m_1 + m_2)/\mathrm{O}(m_1) \times \mathrm{O}(m_2).$$

From a standard theory of symmetric spaces, the cohomogeneity of this representation coincides with rank $(G_{m_1}(\mathbb{R}^{m_1+m_2})) = \min\{m_1, m_2\}$ (refer to [6] for detail). □

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