CLASSIFYING SOLUTIONS OF $\mathrm{SU}(n+1)$ TODA SYSTEM AROUND A SINGULAR SOURCE VIA FUCHSIAN EQUATIONS

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ABSTRACT. Let n be a positive integer, $\gamma_1 > -1, \cdots, \gamma_n > -1$, $D = \{z \in \mathbb{C} : |z| < 1\}$, and $(a_{ij})_{n\times n}$ be the Cartan matrix of $\mathfrak{su}(n+1)$. By using the Fuchsian equation of $(n+1)$th order around a singular source of $\mathrm{SU}(n+1)$ Toda system discovered by Lin-Wei-Ye (*Invent Math*, **190**(1):169-207, 2012), we describe precisely a solution $(u_1, \cdots, u_n)$ to the $\mathrm{SU}(n+1)$ Toda system

$$\begin{cases} \frac{\partial^2 u_i}{\partial z \partial \bar{z}} + \sum_{j=1}^n a_{ij} e^{u_j} & = \pi \gamma_i \delta_0 \text{ on } D \\ \frac{\sqrt{-1}}{2} \int_{D\backslash\{0\}} e^{u_i} \mathrm{d}z \wedge \mathrm{d}\bar{z} & < \infty \end{cases} \quad \text{for all} \quad i = 1, \cdots, n$$

in terms of some $(n+1)$ holomorphic functions satisfying the normalized condition. Moreover, we show that for each $1 \le i \le n$, 0 is the cone singularity with angle $2\pi(1+\gamma_i)$ for metric $e^{u_i}|\mathrm{d}z|^2$ on $D\backslash\{0\}$, whose restriction near 0 could be characterized by some $(n-1)$ holomorphic functions non-vanishing at 0.

1. INTRODUCTION

Gervais-Matsuo [4, Section 2.2.] firstly showed that totally un-ramified holomorphic curves in $\mathbb{P}^n$ induce local solutions to $\mathrm{SU}(n+1)$ Toda systems in the sense that these systems are actually the infinitesimal Plücker formulae for these curves. A. Doliwa [3] generalized their result to Toda systems associated with non-exceptional simple Lie algebras. There have been lots of research works on the classification of solutions of Toda systems of various types which satisfy some boundary conditions on the punctured Riemann surfaces since then. We list some relevant results as follows.

Jost-Wang [7, Theorem 1.1.] classified all solutions $(u_1, \cdots, u_n)$ to the $\mathrm{SU}(n+1)$ Toda system on $\mathbb{C}$ satisfying the so-called finite energy condition:

$$\begin{cases} \frac{\partial^2 u_i}{\partial z \partial \bar{z}} + \sum_{j=1}^n a_{ij} e^{u_j} & = 0 \text{ on } \mathbb{C} \\ \frac{\sqrt{-1}}{2} \int_{\mathbb{C}} e^{u_i} \mathrm{d}z \wedge \mathrm{d}\bar{z} & < \infty \end{cases} \quad \text{for all} \quad i = 1, \cdots, n.$$

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Here we recall for the convenience of the readers

$$(a_{ij}) = \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

Equivalently, they proved that any holomorphic curve $\mathbb{C} \to \mathbb{P}^n$ associated with such a solution can be compactified to a rational normal curve $\mathbb{P}^1 \to \mathbb{P}^n$ ([7, Theorem 1.2]). Consequently, the space of such solutions is isomorphic to $\mathrm{PSL}(n+1, \mathbb{C})/\mathrm{PSU}(n+1)$ and has dimension $n(n+2)$. For the $\mathrm{SU}(n+1)$ Toda system on the twice-punctured Riemann sphere with finite energy:

$$\begin{cases} \frac{\partial^2 u_i}{\partial z \partial \bar{z}} + \sum_{j=1}^n a_{ij} e^{u_j} & = \pi \gamma_i \delta_0 \text{ on } \mathbb{C}\backslash\{0\} \quad (\gamma_i > -1) \\ \frac{\sqrt{-1}}{2} \int_{\mathbb{C}\backslash\{0\}} e^{u_i} \mathrm{d}z \wedge \mathrm{d}\bar{z} & < \infty \end{cases} \quad \text{for all} \quad i = 1, \cdots, n.$$

Lin-Wei-Ye [10, Theorem 1.1.] classified all its solutions, by which they generalized the result of Jost-Wang. The space of these solutions has dimension at most $n(n+2)$. Lin-Nie-Wei [9, Theorem 1.6] obtained the classification of all solutions to the elliptic Toda system associated with a general simple Lie algebra, where the space of all solutions is also of finite dimension. Chen-Lin [2] classified all even solutions to some SU(3) Toda systems with critical parameters on tori.

The SU(2) Toda system coincides with the classical Liouville equation

$$\frac{\partial^2 u_1}{\partial z \partial \bar{z}} + 2e^{u_1} = 0$$

whose local solutions u_1 are induced by non-degenerate meromorphic functions ([11]) and define metrics $e^{u_1}|\mathrm{d}z|^2$ with Gaussian curvature 4. R. Bryant [1, Proposition 4] show that if such a metric $e^{u_1}|\mathrm{d}z|^2$ on the punctured disk $D^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$ has finite area, i.e. $\frac{\sqrt{-1}}{2} \int_{D^*} e^{u_1} \mathrm{d}z \wedge \mathrm{d}\bar{z} < \infty$, then near 0, $e^{u_1}|\mathrm{d}z|^2$ could be expressed by $(\gamma_1+1)^2|\xi|^{2\gamma_1}|\mathrm{d}\xi|^2/(1+|\xi|^{2\gamma_1+2})^2$ for some constant $\gamma_1 > -1$, under another complex coordinate $\xi = \xi(z)$ which is defined near 0 and preserves 0, i.e. $\xi(0) = 0$.

By using the Fuchsian equation around a singular source of $\mathrm{SU}(n+1)$ Toda system discovered by Lin-Wei-Ye [10, p.201, (7.1) and (5.7)], we generalize in Theorem 1.2 (ii) the result of R. Bryant by classifying all solutions $u = (u_1, \cdots, u_n)$ to the following $\mathrm{SU}(n+1)$ Toda system

$$\begin{cases} \frac{\partial^2 u_i}{\partial z \partial \bar{z}} + \sum_{j=1}^n a_{ij} e^{u_j} & = \pi \gamma_i \delta_0 \text{ on } D \quad (\gamma_i > -1) \\ \frac{\sqrt{-1}}{2} \int_{D^*} e^{u_i} \mathrm{d}z \wedge \mathrm{d}\bar{z} & < \infty \end{cases} \quad \text{for all} \quad i = 1, \cdots, n. \tag{1.1}$$

Roughly speaking, we establish a correspondence between solutions $u = (u_1, \cdots, u_n)$ to (1.1) and $(n+1)$ holomorphic functions satisfying the normalized condition on D. Moreover, for each $1 \le i \le n$, we could characterize the germs at 0 of metric $e^{u_j}|\mathrm{d}z|^2$ with cone angle $2\pi(1+\gamma_i)$ at 0 in terms of some $(n-1)$ holomorphic functions non-vanishing at 0.

Before the statement of Theorem 1.2, we prepare some notations. Recall that the inverse matrix $(a^{ij})_{n\times n}$ of $(a_{ij})_{n\times n}$ satisfies

$$a^{ij} = \frac{i(n+2-j)}{n+2} \quad \text{for} \quad 1 \le i, j \le \langle (n+2)/2 \rangle,$$

where $\langle b \rangle$ denotes the least integer $\ge b$. Define $\alpha_i := \sum_{j=1}^n a^{ij}\gamma_j$ for $i = 1, \cdots, n$, and set

$$\begin{cases} \beta_0 & := -\alpha_1, \\ \beta_i & := \alpha_i - \alpha_{i+1} + i \quad \text{for} \quad 1 \le i \le n-1, \\ \beta_n & := \alpha_n + n\,. \end{cases} \tag{1.2}$$

Then, by the very definition of β_i's, we have $\beta_i - \beta_{i-1} = \gamma_i + 1 > 0$ for all $i = 1, \cdots, n$ and $\beta_0 + \beta_1 + \cdots + \beta_n = n(n+1)/2$. For any $(n+1)$ holomorphic functions $g_0(z), \cdots, g_k(z)$ on D with $0 \le k \le n$, we define

$$G_k\big(\beta_0, \cdots, \beta_k; g_0(z), \cdots, g_k(z)\big) := z^{k(k+1)/2 - (\beta_0 + \cdots + \beta_k)} \cdot W\big(z^{\beta_0} g_0(z), z^{\beta_1} g_1(z), \cdots, z^{\beta_k} g_k(z)\big), \tag{1.3}$$

where

$$W\big(z^{\beta_0} g_0(z), z^{\beta_1} g_1(z), \cdots, z^{\beta_k} g_k(z)\big) := \begin{vmatrix} z^{\beta_0} g_0(z) & z^{\beta_1} g_1(z) & \cdots & z^{\beta_k} g_k(z) \\ \big(z^{\beta_0} g_0(z)\big)' & \big(z^{\beta_1} g_1(z)\big)' & \cdots & \big(z^{\beta_n} g_k(z)\big)' \\ \vdots & \vdots & \cdots & \vdots \\ \big(z^{\beta_0} g_0(z)\big)^{(k)} & \big(z^{\beta_1} g_1(z)\big)^{(k)} & \cdots & \big(z^{\beta_k} g_k(z)\big)^{(k)} \end{vmatrix}.$$

Then G_k is holomorphic on D and satisfies

$$G_k|_{z=0} = \Pi_{i=0}^k\, g_i(0) \cdot \Pi_{0 \le i < j \le k} \,(\beta_i - \beta_j) \tag{1.4}$$

by Lemma 4.1. In particular, $G_n\big(\beta_0, \cdots, \beta_n; g_0(z), \cdots, g_n(z)\big)$ coincides with

$$W\,\big(z^{\beta_0} g_0(z),\; z^{\beta_1} g_1(z),\; \cdots,\; z^{\beta_n} g_n(z)\big)$$

since $\sum_{i=0}^n \beta_i = \frac{n(n+1)}{2}$.

Definition 1.1. We call that $(n+1)$ holomorphic functions $g_0(z), \cdots g_n(z)$ on D satisfy the *normalized condition* if and only if

$$G_n\big(\beta_0, \cdots, \beta_n; g_0(z), \cdots, g_n(z)\big) \equiv 1 \quad \text{on} \quad D.$$

In particular, $g_0, \cdots, g_n$ do not vanish at 0 by (1.4).

Theorem 1.2. *Let $u = (u_1, \cdots, u_n)$ be a solution to the* SU$(n+1)$ *Toda system* (1.1)*. Then we have the following two statements.*

(i) *There exists $(n+1)$ holomorphic functions $g_0, \cdots, g_n$ satisfying the normalized condition on D such that for each $1 \le k \le n$,*

$$u_k = -\sum_{j=1}^n a_{kj} \log \|\Lambda_{j-1}(\nu)\|^2, \tag{1.5}$$

where $[\nu] = [\nu_0, \cdots, \nu_n] : D^* \to \mathbb{P}^n$ is the multi-valued holomorphic curve defined by

$$z \mapsto \big[z^{\beta_0} g_0(z),\; z^{\beta_1} g_1(z),\; \cdots,\; z^{\beta_n} g_n(z)\big] \tag{1.6}$$

and the definition of $\Lambda_i(\bullet)$ will be given in Section 2. In particular, u_k equals $2\gamma_k \log|z|$ plus a bounded smooth remainder R_k near 0, where

$$\begin{aligned} R_k &= -\sum_{j=1}^{n} a_{kj} \log r_j \quad \text{with} \\ r_j &= \left|G_{j-1}\big(\beta_0,\beta_1,\cdots,\beta_{j-1}; g_0(z), g_1(z),\cdots,g_{j-1}(z)\big)\right|^2 \\ &+ \sum_{\substack{0\le i_0<i_1<\cdots<i_{j-1}\le n \\ i_{j-1}>j-1}} |z|^{2\left(\sum_{l=0}^{j-1}\beta_{i_l}-\frac{(j-1)j}{2}+\alpha_j\right)} \left|G_{j-1}\left(\beta_{i_0},\beta_{i_1},\cdots,\beta_{i_{j-1}}; g_{i_0}(z), g_{i_1}(z),\cdots,g_{i_{j-1}}(z)\right)\right|^2. \end{aligned}$$

Moreover, any curve with form (1.6) gives a solution $u=(u_1,\cdots,u_n)$ to (1.1) via (1.5) provided that the integral condition in (1.1) be weakened to

$$\frac{\sqrt{-1}}{2}\int_{0<|z|<r} e^{u_i}\mathrm{d}z\wedge \mathrm{d}\bar z<\infty \quad \text{for all} \quad 0<r<1.$$

(ii) *For all $1\le k\le n$, metrics $e^{u_k}|dz|^2$ have cone angle $2\pi(1+\gamma_k)$ at $z=0$. And there exist a complex coordinate transformation $z\mapsto \xi=\xi(z)$ near $z=0$ and preserving 0, and $(n-1)$ holomorphic functions $\tilde g_2(\xi),\cdots,\tilde g_n(\xi)$ non-vanishing at 0 such that these n metrics near 0 could be expressed in terms of these $(n-1)$ functions and $\{\beta_i\}_{i=0}^n$. In particular, as $k=1$, $e^{u_1}|\mathrm{d}z|^2$ near $z=0$ could be simplified into the form of*

$$|\xi|^{2\gamma_1}\frac{(\beta_1-\beta_0)^2+\sum\limits_{\substack{0\le i_0<i_1\le n\\ i_1>1}}|\xi|^{2(\beta_{i_0}+\beta_{i_1}-1+\alpha_2)}\left|G_1\big(\beta_{i_0},\beta_{i_1};\tilde g_{i_0}(\xi),\tilde g_{i_1}(\xi)\big)\right|^2}{\Big(1+|\xi|^{2(\beta_1-\beta_0)}+|\xi|^{2(\beta_2-\beta_0)}\,|\tilde g_2(\xi)|^2+|\xi|^{2(\beta_n-\beta_0)}\,|\tilde g_n(\xi)|^2\Big)^2}|\mathrm{d}\xi|^2.$$

Remark 1.3. Statement (i) in the theorem refines the asymptotic estimate around a singular source of solutions to SU$(n+1)$ Toda system in [7, Lemma 2.1] and [10, Theorem 1.3 (i)]. By simple computation, metric $e^{u_1}|\mathrm{d}z|^2$ in Case $n=1$ of Statement (ii) coincides with $\frac{(\gamma_1+1)^2|\xi|^{2\gamma_1}|\mathrm{d}\xi|^2}{(1+|\xi|^{2\gamma_1+2})^2}$ already given by R. Bryant [1, Proposition 4].

We conclude the introduction by explaining the organization of the left three sections of this manuscript. In Section 2, for a not-necessarily simply connected domain $\Omega\subset\mathbb{C}$, we establish a correspondence between solutions to SU$(n+1)$ Toda system on Ω and totally un-ramified unitary curves $\Omega\to\mathbb{P}^n$ (Definition 2.1 and Lemma 2.3), under which the solutions are induced by the infinitesimal Plücker formulae of the curves [6, Section 2.4]. This generalizes the simply connected case used by Jost-Wang [7, Section 3]. By using the Fuchsian equation of $(n+1)$th order around $z=0$ discovered by Lin-Wei-Ye [10], we prove the former part of Statement (i) of Theorem 1.2 in Section 3 that a solution u to (1.1) is induced by the canonical unitary curve

$$D^*\to\mathbb{P}^n,\quad z\mapsto\left[z^{\beta_0}g_0(z),\, z^{\beta_1}g_1(z),\,\cdots,\, z^{\beta_n}g_n(z)\right],$$

where $g_0,\cdots,g_n$ are some $(n+1)$ holomorphic functions satisfying the normalized condition on D. We prove in the last section the left part of Theorem 1.2 by applying the infinitesimal Plücker formulae to this curve.

2. Correspondence between curves and solutions

Jost-Wang [7, Section 3] established a correspondence between solutions to SU$(n+1)$ Toda system on a simply connected domain in $\mathbb{C}$ and totally-unramified holomorphic curves from this domain to $\mathbb{P}^n$. In this section, we generalize their correspondence to a not-necessarily simply connected domain $\Omega \subset \mathbb{C}$. Before the statement of the more general correspondence, we prepare some notations as follows, where we use [6, Section 2.4] as a general reference.

Definition 2.1. We generalize the concept of associated curves in [6, pp.263-264] to the multi-valued case in the following:

(1) We call $f : \Omega \to \mathbb{P}^n$ a *projective holomorphic curve* if and only if it satisfies the following three conditions:
   (i) f is a multi-valued holomorphic map;
   (ii) f is non-degenerate, i.e. the image of a germ $\mathfrak{f}_z$ of f at any point $z \in \Omega$ is not contained in a hyperplane of $\mathbb{P}^n$; and
   (iii) the monodromy representation of f is a group homomorphism
   $$\mathcal{M}_f : \pi_1(\Omega,\, B) \to \mathrm{PSL}(n+1,\, \mathbb{C}),$$
   where $\mathrm{PSL}(n+1,\, \mathbb{C})$ is the holomorphic automorphism group of $\mathbb{P}^n$ ([6, pp.64-65]) and $B \in \Omega$ is a base point. We also say that f *has monodromy in* $\mathrm{PSL}(n+1,\, \mathbb{C})$ briefly.

(2) We call such a curve f *unitary* if and only if it has monodromy in $\mathrm{PSU}(n+1)$, which is the group of rigid motions with respect to the Fubini-Study metric
$$\omega_{\mathrm{FS}} = \frac{\sqrt{-1}}{2\pi}\,\partial\overline{\partial}\,\log\,\|Z\|^2 \quad \text{with} \quad Z \in \mathbb{C}^{n+1} - \{0\}$$
on $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$ ([6, pp.30-31]). Mimicking the definition in [6, pp.263-264], for a unitary curve $f : \Omega \to \mathbb{P}^n$, we could define its *$k$th associated curve*
$$f_k : \Omega \to G(k+1,\, n+1) \subset \mathbb{P}\big(\Lambda^{k+1}\mathbb{C}^{n+1}\big) \quad \text{for all} \quad k = 0, 1, \cdots, (n-1),$$
and they are also unitary curves.

(3) We call a unitary curve $f : \Omega \to \mathbb{P}^n$ *totally un-ramified* if and only if for each point $z \in \Omega$, each germ $\mathfrak{f}$ of f is totally un-ramified, i.e. there exists a lifting $\widehat{\mathfrak{f}} : U_z \to \mathbb{C}^{n+1}$ of $\mathfrak{f}$ such that its nth associated curve
$$\widehat{\mathfrak{f}} \wedge \widehat{\mathfrak{f}}'(z) \wedge \cdots \wedge \widehat{\mathfrak{f}}^{(n)}(z) : U_z \to \Lambda^{n+1}\big(\mathbb{C}^{n+1}\big)$$
equals $e_0 \wedge e_1 \wedge \cdots \wedge e_n$ identically, where $U_z \subset \Omega$ is some open neighborhood of z and $\{e_0, \cdots, e_n\}$ is the standard ortho-normal basis of $\mathbb{C}^{n+1}$. Hence, the nth associated curve f_n of f is also well defined.

We observe that the infinitesimal Plück formulae [6, p.269] also hold for unitary curves beside single-valued holomorphic curves, which also induce solutions to SU$(n+1)$ Toda system in the following:

Lemma 2.2. *Let $f : \Omega \to \mathbb{P}^n$ be a unitary curve and $f_0 := f,\, f_1, \cdots, f_{n-1}$ its associated curves. Let $\mathfrak{f}$ be a germ of f and $\hat{\mathfrak{f}}$ be one of its lifting. Then $\Lambda_k(\mathfrak{f},\, z) = \hat{\mathfrak{f}}(z) \wedge \hat{\mathfrak{f}}'(z) \cdots \wedge \hat{\mathfrak{f}}^{(k)}(z) \in \Lambda^{k+1}\mathbb{C}^{n+1}$ is a lifting of some germ $\mathfrak{f}_k$ of f_k. Endow $\Lambda^{k+1}\big(\mathbb{C}^{n+1}\big)$'s with induced metrics from $\big(\mathbb{C}^{n+1},\, \|\cdot\|\big)$ for $k = 0, 1, \cdots, n$, and set $\|\Lambda_{-1}\| \equiv 1$.*

(i) (Infinitesimal Plück formulae) *For* $k = 0, 1, \cdots, (n-1)$, *we have*

$$f_k^*\omega_{\mathrm{FS}} = \frac{\sqrt{-1}}{2\pi}\,\frac{\|\Lambda_{k-1}(f)\|^2 \cdot \|\Lambda_{k+1}(f)\|^2}{\|\Lambda_k(f)\|^4}\,\mathrm{d}z\wedge \mathrm{d}\bar{z}, \tag{2.1}$$

where we write the notion of $\Lambda_\bullet(f)$ *on purpose since the fraction* $\frac{\|\Lambda_{k-1}(f)\|^2\cdot\|\Lambda_{k+1}(f)\|^2}{\|\Lambda_k(f)\|^4}$ *on the right-hand side does not depend on the choice of the lifting* $\hat{\mathfrak{f}}$.

(ii) (From totally un-ramified unitary curves to solutions) *Assume furthermore that the unitary curve* $f : \Omega \to \mathbb{P}^n$ *is totally un-ramified. Then we could choose the lifting* $\hat{\mathfrak{f}}$ *of germ* $\mathfrak{f}$ *of* f *in* (i) *such that*

$$\Lambda_n(\mathfrak{f},\, z) = \hat{\mathfrak{f}}(z)\wedge\hat{\mathfrak{f}}'(z)\cdots\wedge\hat{\mathfrak{f}}^{(n)}(z) \equiv e_0\wedge\cdots\wedge e_n \in \Lambda^{n+1}\mathbb{C}^{n+1} \quad \text{on} \quad \Omega.$$

In particular, $\|\Lambda_n\| \equiv 1$. *Then it induces a solution* $u = (u_1, \cdots, u_n)$ *to the* $\mathrm{SU}(n+1)$ *Toda system*

$$\frac{\partial^2 u_i}{\partial z\partial\bar{z}} + \sum_{j=1}^{n} a_{ij}e^{u_j} = 0 \quad \text{on} \quad \Omega \quad \text{for all} \quad i = 1, \cdots, n. \tag{2.2}$$

in such a way that

$$u_i := -\sum_{j=1}^{n} a_{ij}\log\|\Lambda_{j-1}(f)\|^2 = \begin{cases} \log\frac{\|\Lambda_1(f)\|^2}{\|\Lambda_0(f)\|^4} & \text{for} \quad i = 1,\\ \log\frac{\|\Lambda_{i-2}(f)\|^2\cdot\|\Lambda_i(f)\|^2}{\|\Lambda_{i-1}(f)\|^4} & \text{for all} \quad i = 2,\, 3,\, \cdots, n-1,\\ \log\frac{\|\Lambda_{n-2}(f)\|^2}{\|\Lambda_{n-1}(f)\|^4} & \text{for} \quad i = n.\end{cases} \tag{2.3}$$

Proof. Since f and all its associated curves are unitary, the norm of $\Lambda_k(\mathfrak{f},\, z) = v(z)\wedge\cdots\wedge v^{(k)}(z) \in \Lambda^{k+1}\mathbb{C}^{n+1}$ does not depend on the choice of germ $\mathfrak{f}$. Hence the Plücker formulae (2.1) follows from the same argument as in [6, pp.269-270]. Statement (ii) follows from these formulae and the same argument as in [7, Section 3.4]. □

Jost-Wang [7, Section 2.1] introduced the Toda map associated with a solution to the $\mathrm{SU}(n+1)$ Toda system on a simply connected domain in $\mathbb{C}$. To obtain our correspondence, we need to introduce the notion of multi-valued Toda map on $\Omega \subset \mathbb{C}$. Let $u = (u_1, \cdots, u_n)$ be an n-tuple of real-valued smooth function on Ω and the $(n+1)$-tuple $w = (w_0, \cdots, w_n)$ of functions on Ω be defined by

$$\begin{cases} w_0 := -\frac{\sum_{i=1}^n (n-i+1)u_i}{2(n+1)} \\ w_i := w_0 + \frac{1}{2}\sum_{j=1}^i u_j, \quad 1 \le i \le n.\end{cases} \tag{2.4}$$

Then $u = (u_1, \cdots, u_n)$ solves the SU(n+1) Toda system (2.2) if and only if w satisfies the Maurer-Cartan equation $\mathcal{U}_z - \mathcal{V}_{\bar{z}} = [\mathcal{U}, \mathcal{V}]$, where

$$\mathcal{U} = \begin{pmatrix} (w_0)_z & & & \\ & (w_1)_z & & \\ & & \ddots & \\ & & & (w_n)_z \end{pmatrix} + \begin{pmatrix} 0 & & & \\ e^{w_1-w_0} & 0 & & \\ & \ddots & \ddots & \\ & & e^{w_n-w_{n-1}} & 0 \end{pmatrix}$$

and $\mathcal{V} = -\mathcal{U}^* = -\overline{\mathcal{U}}^{\mathrm{T}}$. By using the Frobenius theorem and the analytic-continuation-like argument (See [7, Section 3.1] and [12, Chapter 3]), we obtain a set of *multi-valued* Toda maps $\phi : \Omega \to \mathrm{SU}(n+1)$ associated with solution u of (2.2) such that

$$\phi^{-1}\mathrm{d}\phi = \mathcal{U}\mathrm{d}z + \mathcal{V}\mathrm{d}\bar{z} \tag{2.5}$$

and the monodromy of ϕ is a group homomorphism $\mathcal{M}_\phi : \pi_1(\Omega,\, B) \to \mathrm{SU}(n+1)$. Moreover, any two such Toda maps have the difference of a constant multiple in $\mathrm{SU}(n+1)$ from the left-hand side, and the set of all the Toda maps associated with u is isomorphic to the quotient group $\mathrm{SU}(n+1)/\mathrm{Image}\big(\mathcal{M}_\phi\big)$.

Lemma 2.3. *Suppose that $\phi : \Omega \to \mathrm{SU}(n+1)$ is a multi-valued Toda map associated to a solution $u = (u_1, \cdots, u_n)$ of* (2.2)*. Defining an $(n+1)$-tuple $(\hat{f}_0, \cdots, \hat{f}_n)$ of $\mathbb{C}^{n+1}$-multi-valued functions on Ω by*

$$(\hat{f}_0, \cdots, \hat{f}_n) = \phi \cdot \begin{pmatrix} e^{w_0} & & & \\ & e^{w_1} & & \\ & & \ddots & \\ & & & e^{w_n} \end{pmatrix},$$

we find that $f_0 := [\hat{f}_0] : \Omega \to \mathbb{P}^n$ is a totally un-ramified unitary curve on Ω which satisfies $\hat{f}_0 \wedge \hat{f}_0' \wedge \hat{f}_0^{(2)} \wedge \cdots \wedge \hat{f}_0^{(n)} = e_0 \wedge \cdots \wedge e_n$. Moreover, $(u_1, \cdots, u_n)$ coincides with the solution of (2.2) *constructed from the curve f_0 by* (2.3)*.*

Proof. We choose a germ φ of $\phi : \Omega \to \mathrm{SU}(n+1)$. Since

$$\frac{\partial \varphi}{\partial \bar{z}} = \varphi \mathcal{V} \quad \text{and} \quad \|\hat{f}_i\| = e^{w_i},$$

it follows from direct computation that the germ $(\hat{\mathfrak{f}}_0, \cdots, \hat{\mathfrak{f}}_n)$ of $(\hat{f}_0, \cdots, \hat{f}_n)$ satisfies

$$\begin{aligned} \frac{\partial}{\partial \bar{z}}\left(\hat{\mathfrak{f}}_0, \cdots, \hat{\mathfrak{f}}_n\right) &= \left(0,\, \frac{\|\hat{\mathfrak{f}}_1\|^2}{\|\hat{\mathfrak{f}}_0\|^2}\hat{\mathfrak{f}}_0,\, \frac{\|\hat{\mathfrak{f}}_2\|^2}{\|\hat{\mathfrak{f}}_1\|^2}\hat{\mathfrak{f}}_1, \cdots,\, \frac{\|\hat{\mathfrak{f}}_n\|^2}{\|\hat{\mathfrak{f}}_{n-1}\|^2}\hat{\mathfrak{f}}_{n-1}\right), \\ \frac{\partial}{\partial z}\left(\hat{\mathfrak{f}}_0, \cdots, \hat{\mathfrak{f}}_n\right) &= \left(\hat{\mathfrak{f}}_1,\, \cdots, \hat{\mathfrak{f}}_n,\, 0\right) + \left(\hat{\mathfrak{f}}_0 \frac{\partial}{\partial z}\log\|\hat{\mathfrak{f}}_0\|^2,\, \hat{\mathfrak{f}}_1 \frac{\partial}{\partial z}\log\|\hat{\mathfrak{f}}_1\|^2, \cdots,\, \hat{\mathfrak{f}}_n \frac{\partial}{\partial z}\log\|\hat{\mathfrak{f}}_n\|^2\right). \end{aligned}$$

By the first equation above, the germ $\hat{\mathfrak{f}}_0$ of $\hat{f}_0$ is holomorphic. By the second one and induction argument, we obtain that

$$\hat{\mathfrak{f}}_0 \wedge \hat{\mathfrak{f}}_0' \wedge \cdots \wedge \hat{\mathfrak{f}}_0^{(k)} = \hat{\mathfrak{f}}_0 \wedge \hat{\mathfrak{f}}_1 \wedge \cdots \wedge \hat{\mathfrak{f}}_k \tag{2.6}$$

for all $k = 0, 1, \cdots, n$. In particular, we can see

$$\hat{\mathfrak{f}}_0 \wedge \hat{\mathfrak{f}}_0' \wedge \cdots \wedge \hat{\mathfrak{f}}_0^{(n)} \equiv e_0 \wedge \cdots \wedge e_n$$

by using the definition of $(\hat{f}_0, \cdots, \hat{f}_n)$ and $w_0 + \cdots w_n = 0$. Since ϕ has monodromy in $\mathrm{SU}(n+1)$, $f_0 = [\hat{f}_0] : \Omega \to \mathbb{P}^n$ is a totally un-ramified unitary curve.

Since $\hat{\mathfrak{f}}_0, \cdots, \hat{\mathfrak{f}}_n$ are mutually orthogonal, we find by using (2.6) that

$$\|\Lambda_k([\hat{f}_0])\| = \|\hat{\mathfrak{f}}_0 \wedge \hat{\mathfrak{f}}_0' \wedge \cdots \wedge \hat{\mathfrak{f}}_0^{(k)}\| = \|\hat{\mathfrak{f}}_0 \wedge \hat{\mathfrak{f}}_1 \wedge \cdots \wedge \hat{\mathfrak{f}}_k\| = \|\hat{\mathfrak{f}}_0\| \cdot \|\hat{\mathfrak{f}}_1\| \cdots \|\hat{\mathfrak{f}}_k\|. \tag{2.7}$$

In particular, $\|\Lambda_n([\hat{f}_0])\| = e^{w_0 + \cdots + w_n} = 1$. Since for all $i = 1, \cdots, n$

$$u_i = 2w_i - 2w_{i-1} = 2\left(\log\|\hat{f}_i\| - \log\|\hat{f}_{i-1}\|\right),$$

by using (2.7) and direct computation, we obtain that $u = (u_1, \cdots, u_n)$ coincides with the one in (2.3). $\square$

Definition 2.4. We call $f_0 : \Omega \to \mathbb{P}^n$ in Lemma 2.3 a *unitary curve associated with* solution $u = (u_1, \cdots, u_n)$ of the $\mathrm{SU}(n+1)$ Toda system. The monodromy of f_0 is induced by that of the multi-valued Toda map $\phi : \Omega \to \mathrm{SU}(n+1)$. Moreover, such a unitary curve is unique up to a rigid motion in $\big(\mathbb{P}^n, \omega_{\mathrm{FS}}\big)$ ([5, (4.12)]).

3. Canonical unitary curves associated with solutions

In this section, we shall prove the former part of Statement (i) of Theorem 1.2, which is restated in the following:

Theorem 3.1. *Let u be a solution to* (1.1)*. Then there exists $(n+1)$ holomorphic functions $g_0, \cdots, g_n$ satisfying the normalized condition on D such that the following unitary curve*

$$[\nu] = [\nu_0, \cdots, \nu_n] : D^* \to \mathbb{P}^n, \quad z \mapsto \big[z^{\beta_0} g_0(z), z^{\beta_1} g_1(z), \cdots, z^{\beta_n} g_n(z)\big]$$

is associated with u. We call f_{can}'s canonical curves *associated with u.*

We cite the following lemma about the Fuchsian equation given by solution u, which was discovered by Lin-Wei-Ye [10].

Lemma 3.2. *Let u be a solution to* (1.1) *and $f_0 = [\hat{f}_0]$ the unitary curve associated with $u|_{D^*}$ which is obtained by the construction in Lemma 2.2. Then all the components of $\hat{f}_0$ form a set of fundamental solutions to the following Fuchsian equation of*

$$y^{(n+1)} + \sum_{k=0}^{n-1} Z_{k+1} y^{(k)} = 0 \tag{3.1}$$

of $(n+1)$th order on D which satisfies the following three properties:

- (i) *The coefficients Z_k are holomorphic on D^* and have poles of order $(n+2-k)$. Hence 0 is the regular singularity of* (3.1)*.*
- (ii) *$\beta_0, \beta_1, \cdots, \beta_n$ defined in* (1.2) *are the local exponents of* (3.1) *at* 0*.*

Proof. The proof of this lemma is scattered throughout the first, fifth, and seventh sections of Lin-Wei-Ye [10]. We sketch it here for completeness. By the proof of [10, Lemmas 2.1 and 5.2], where Lin-Wei-Ye used all the conditions in (1.1), we obtain that $f := e^{2w_0} = \|\hat{f}_0\|^2$ with $(\hat{f}_0)^{\mathrm{T}}(z) =: \nu(z) = (\nu_0(z), \cdots, \nu_n(z))$ satisfies the Fuchsian equation (3.1), i.e.

$$L(f) = f^{(n+1)} + \sum_{k=0}^{n-1} Z_{k+1} f^{(k)} = 0 \quad \text{on} \quad D^*,$$

whose local exponents are $\beta_0, \cdots, \beta_n$. Hence we have

$$0 = \overline{L} L(f) = \sum_{i=0}^{n} |L(\nu_i(z))|^2.$$

Therefore, $\{\nu_i\}_{i=0}^n$ is a set of fundamental solutions of (3.1). □

Proof of Theorem 3.1: It suffices to show that *there exists a matrix A in* $\mathrm{SU}(n+1)$ *such that*

$$\nu(z) A := \big(\hat{f}_0(z)\big)^{\mathrm{T}} A = \big(z^{\beta_0} g_0(z), z^{\beta_1} g_1(z), \cdots, z^{\beta_n} g_n(z)\big)$$

for some $(n+1)$ *holomorphic functions* $g_0, \cdots, g_n$*,* which satisfy the normalized condition automatically since $\Lambda_n(\nu) = \Lambda_n(\nu \cdot A) = e_0 \wedge e_1 \wedge \cdots \wedge e_n$. We divide its proof into the following two steps.

Step 1. Choose base point $B \in D^*$ and generator γ_B of $\pi_1(D^*, B)$. Since for each $A \in \mathrm{SU}(n+1)$, the unitary curve $[\nu \cdot A] : D^* \to \mathbb{P}^n$ is also associated with u and has monodromy representation conjugate to that of $[\nu]$ by A, we assume without loss of generality that the monodromy representation $\mathcal{M}_\nu$ of ν maps γ_B to the diagonal matrix

$$\begin{pmatrix} e^{2\pi\sqrt{-1}b_0} & & & \\ & e^{2\pi\sqrt{-1}b_1} & & \\ & & \ddots & \\ & & & e^{2\pi\sqrt{-1}b_n} \end{pmatrix},$$

where $b_0, \cdots b_n$ are real numbers lying in $[0, 1)$ such that $b_0 + b_1 + \cdots + b_n = 0$. Hence there exist $(n+1)$ holomorphic functions $\psi_0, \psi_1, \cdots, \psi_n$ on D^* such that

$$\nu(z) = \big(\nu_0(z), \nu_1(z), \cdots, \nu_n(z)\big) = \big(z^{b_0}\psi_0(z), z^{b_1}\psi_1(z), \cdots, z^{b_n}\psi_n(z)\big). \tag{3.2}$$

Step 2. We divide the $(n+1)$ local exponents $\beta_0 < \beta_1 < \cdots < \beta_n$ into the following k groups

$$\beta^{(1)}_1, \beta^{(1)}_2, ...\beta^{(1)}_{i_1}; \quad \beta^{(2)}_1, \beta^{(2)}_2, ...\beta^{(2)}_{i_2}; \quad \cdots; \quad \beta^{(k)}_1, \beta^{(k)}_2, ...\beta^{(k)}_{i_k}$$

such that in one of these k groups, each local exponent differs from the other by integers and the local exponents are in strictly ascending order; and any two local exponents lying in different groups are mutually distinct modulo integers. Using both the unitary monodromy property of (3.2) and the Frobenius method [8, Section 3.4.1] solving the Fuchsian equation (3.1), we could rule out the possible logarithmic singularities of solutions to (3.1). Hence, there exists M in $\mathrm{GL}(n+1, \mathbb{C})$ such that

$$\begin{aligned} \nu(z) &= \big(\nu_0(z), \nu_1(z), \cdots, \nu_n(z)\big) = \big(z^{b_0}\psi_0(z), z^{b_1}\psi_(z), \cdots, z^{b_n}\psi_n(z)\big) \\ &= \big(y^{(1)}_1, y^{(1)}_2, ..., y^{(1)}_{i_1}; y^{(2)}_1, y^{(2)}_1, ..., y^{(2)}_{i_2}; \cdots; y^{(k)}_1, y^{(k)}_2, ..., y^{(k)}_{i_k}\big) \cdot M, \end{aligned}$$

where for all $j = 1, 2, \cdots, k$ and $m = 1, 2, \cdots, i_j$,

$$y^{(j)}_m(z) = z^{\beta^{(j)}_m}\phi^{(j)}_m(z)$$

such that $\phi^{(j)}_m(z)$ are holomorphic functions on D such that $\phi^{(j)}_m(0) \neq 0$. In particular, for all $m = 1, 2, \cdots, i_j$, $y^{(j)}_m(z)$ has the same monodromy generated by multiplying $e^{2\pi\sqrt{-1}\beta^{(j)}_m}$. Since the monodromy of the set $(\nu_0, \nu_1, \cdots, \nu_n)$ of fundamental solutions to (3.1) is generated by the preceding diagonal matrix,

$$M = \mathrm{diag}(C_1, \cdots, C_k), \quad \text{where} \quad C_j \in \mathrm{GL}(i_j, \mathbb{C}).$$

For all $j = 1, \cdots, k$, we rewrite C_j as the product $C_j = B_jA_j$ for all $j = 1, \cdots, k$, where B_j is a lower triangular matrix and A_j is a unitary matrix. Recalling $\beta^{(j)}_1 < \beta^{(j)}_2 < \cdots < \beta^{(j)}_{i_j}$, we may assume that the lower triangular matrix B_j is the identity one and $C_j = A_j$ since

$$\Big(z^{\beta^{(j)}_1}\phi^{(j)}_1(z), z^{\beta^{(j)}_2}\phi^{(j)}_2(z), \cdots, z^{\beta^{(j)}_{i_j}}\phi^{(j)}_{i_j}(z)\Big) \cdot B_j = \Big(z^{\beta^{(j)}_1}g^{(j)}_1(z), z^{\beta^{(j)}_2}g^{(j)}_2(z), \cdots, z^{\beta^{(j)}_{i_j}}g^{(j)}_{i_j}(z)\Big),$$

where $g^{(j)}_1(z), \cdots, g^{(j)}_{i_j}(z)$ are holomorphic functions on D and do not vanish at 0. We are done by taking $A = \mathrm{diag}(A_1, \cdots, A_k)$. $\square$

4. Completion of the proof for Theorem 1.2.

In the preceding section, we proved an important part of Theorem 1.2., i.e. there exists a canonical unitary curve $[\nu(z)] = [\nu_0(z), \cdots, \nu_n(z)]$ associated with each solution $u = (u_1, \cdots, u_n)$ to (1.1). We shall complete the proof of the theorem in this section by applying both the infinitesimal Plücker formulae and the D^*-case of (2.3) to $\nu(z)$ and its associated curves $\nu(z) \wedge \nu'(z) \wedge \cdots \wedge \nu^{(k)}(z)$ for all $k = 1, \cdots, n$. Here we recall that

$$\nu(z) \wedge \nu'(z) \wedge \cdots \wedge \nu^{(n)}(z) \equiv e_0 \wedge \cdots \wedge e_n.$$

To this end, we prepare a lemma relevant to linear algebra in the following:

Lemma 4.1. *Let $g_0(z), g_1(z), \cdots, g_k(z)$ be holomorphic functions on D where $0 \le k \le n$. Then there exists another holomorphic function*

$$G_k(z) = G_k\big(\beta_0, \beta_1, \cdots, \beta_k; g_0(z), g_1(z), \cdots, g_k(z)\big)$$

in D such that

$$\begin{vmatrix} z^{\beta_0} g_0(z) & z^{\beta_1} g_1(z) & \cdots & z^{\beta_k} g_n(z) \\ \big(z^{\beta_0} g_0(z)\big)' & \big(z^{\beta_1} g_1(z)\big)' & \cdots & \big(z^{\beta_n} g_k(z)\big)' \\ \vdots & \vdots & \cdots & \vdots \\ \big(z^{\beta_0} g_0(z)\big)^{(k)} & \big(z^{\beta_1} g_1(z)\big)^{(k)} & \cdots & \big(z^{\beta_k} g_k(z)\big)^{(k)} \end{vmatrix} \tag{4.1}$$
$$= z^{\sum_{i=0}^{k} \beta_i - \frac{k(k+1)}{2}} \cdot G_k\big(\beta_0, \beta_1, \cdots, \beta_k; g_0(z), g_1(z), \cdots, g_k(z)\big).$$

In particular, $G_k(0) = \prod_{i=0}^{k} g_i(0) \cdot \prod_{0 \le i < j \le k} (\beta_i - \beta_j)$.

Proof. By the Leibnitz rule, for all $0 \le \ell \le k$, we have $\big(z^{\beta_i} g_i(z)\big)^{(\ell)} = z^{\beta_i - \ell} \cdot g_{i\ell}(z)$, where

$$g_{i\ell}(z) = \beta_i(\beta_i - 1)\cdots(\beta_i - \ell + 1) g_i(z) + C_\ell^1 \beta_i(\beta_i - 1)\cdots(\beta_i - \ell + 2) z g_i'(z) + \cdots + z^\ell g_i^{(\ell)}(z) \tag{4.2}$$

are holomorphic functions on D. Therefore, there holds

$$\begin{vmatrix} z^{\beta_0} g_0(z) & z^{\beta_1} g_1(z) & \cdots & z^{\beta_k} g_k(z) \\ \big(z^{\beta_0} g_0(z)\big)' & \big(z^{\beta_1} g_1(z)\big)' & \cdots & \big(z^{\beta_k} g_k(z)\big)' \\ \vdots & \vdots & \cdots & \vdots \\ \big(z^{\beta_0} g_0(z)\big)^{(k)} & \big(z^{\beta_1} g_1(z)\big)^{(k)} & \cdots & \big(z^{\beta_k} g_k(z)\big)^{(k)} \end{vmatrix}$$
$$= \begin{vmatrix} z^{\beta_0} g_0(z) & z^{\beta_1} g_1(z) & \cdots & z^{\beta_k} g_k(z) \\ z^{\beta_0 - 1} g_{01}(z) & z^{\beta_1 - 1} g_{11}(z) & \cdots & z^{\beta_k - 1} g_{k1}(z) \\ \vdots & \vdots & \cdots & \vdots \\ z^{\beta_0 - k} g_{0k}(z) & z^{\beta_1 - k} g_{1k}(z) & \cdots & z^{\beta_k - k} g_{kk}(z) \end{vmatrix}$$
$$= z^{\sum_{i=0}^{k} \beta_i - \frac{k(k+1)}{2}} \cdot \begin{vmatrix} g_0(z) & g_1(z) & \cdots & g_k(z) \\ g_{01}(z) & g_{11}(z) & \cdots & g_{k1}(z) \\ \vdots & \vdots & \cdots & \vdots \\ g_{0k}(z) & g_{1k}(z) & \cdots & g_{kk}(z) \end{vmatrix} =: z^{\sum_{i=0}^{k} \beta_i - \frac{k(k+1)}{2}} G_k(z),$$

where

$$G_k(z) = G_k\big(\beta_0, \beta_1, \cdots, \beta_k; g_0(z), g_1(z), \cdots, g_k(z)\big) := \begin{vmatrix} g_0(z) & g_1(z) & \cdots & g_k(z) \\ g_{01}(z) & g_{11}(z) & \cdots & g_{k1}(z) \\ \vdots & \vdots & \cdots & \vdots \\ g_{0k}(z) & g_{1k}(z) & \cdots & g_{kk}(z) \end{vmatrix}.$$

Finally, we find by (4.2) that $G_k(0)$ equals

$$\begin{vmatrix} g_0(0) & g_1(0) & \cdots & g_n(0) \\ \beta_0 g_0(0) & \beta_1 g_1(0) & \cdots & \beta_n g_n(0) \\ \vdots & \vdots & \cdots & \vdots \\ \beta_0(\beta_0-1)\cdots(\beta_0-k+1)g_0(0) & \beta_1\cdots(\beta_1-k+1)g_1(0) & \cdots & \beta_n\cdots(\beta_n-k+1)g_n(0) \end{vmatrix}$$

$$= \quad g_0(0)g_1(0)\cdots g_n(0)\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \beta_0 & \beta_1 & \cdots & \beta_n \\ \vdots & \vdots & \cdots & \cdots \\ \beta_0^n & \beta_1^n & \cdots & \beta_n^n \end{vmatrix} = \prod_{i=0}^n g_i(0)\cdot \prod_{0\le i<j\le n}(\beta_i - \beta_j).$$

□

Using this lemma, we obtain the following three formulae relevant to the lifting $\nu(z) = \big(z^{\beta_0}g_0(z), z^{\beta_1}g_1(z), \cdots, z^{\beta_n}g_n(z)\big)$ of the canonical curve $[\nu(z)]$ associated with solution u to (1.1).

Formula 1. *For all $k = 0, 1, \cdots .n$, we have*

$$\begin{aligned} \Lambda_k(z) &= \nu(z)\wedge\nu'(z)\wedge\cdots\wedge\nu^{(k)}(z) \\ &= \sum_{0\le i_0<i_1<\cdots<i_k\le n}\begin{vmatrix} z^{\beta_{i_0}}g_{i_0}(z) & z^{\beta_{i_1}}g_{i_1}(z) & \cdots & z^{\beta_{i_k}}g_{i_k}(z) \\ \big(z^{\beta_{i_0}}g_{i_0}(z)\big)' & \big(z^{\beta_{i_1}}g_{i_1}(z)\big)' & \cdots & \big(z^{\beta_{i_k}}g_{i_k}(z)\big)' \\ \vdots & \vdots & \cdots & \vdots \\ \big(z^{\beta_{i_0}}g_{i_0}(z)\big)^{(k)} & \big(z^{\beta_{i_1}}g_{i_1}(z)\big)^{(k)} & \cdots & \big(z^{\beta_{i_k}}g_{i_k}(z)\big)^{(k)} \end{vmatrix} e_{i_0}\wedge e_{i_1}\wedge\cdots\wedge e_{i_k} \\ &= \sum_{0\le i_0<i_1<\cdots<i_k\le n} z^{\sum_{j=0}^k \beta_{i_j} - \frac{k(k+1)}{2}}\cdot G_k\big(\beta_{i_0}, \beta_{i_1}, \cdots, \beta_{i_k}; g_{i_0}(z), g_{i_1}(z), \cdots g_{i_k}(z)\big)e_{i_0}\wedge e_{i_1}\wedge\cdots\wedge e_{i_k} \end{aligned}$$

Recall that $g_0, \cdots, g_n$ satisfy the normalized condition (1.1) *and $\beta_0 + \cdots + \beta_n = n(n+1)/2$, which implies that $\Lambda_n(z) = e_0\wedge\cdots\wedge e_n$.*

Proof. It follows from a straightforward computation by using the very expression $\big(z^{\beta_0}g_0(z), z^{\beta_1}g_1(z), \cdots, z^{\beta_n}g_n(z)\big)$ of $\nu(z)$. □

By the definition of $\beta_0 < \beta_1 < \cdots \beta_n$, the summand with the lowest degree with respect to z on the left hand side of Formula 1 has form

$$\begin{aligned} & z^{\sum_{j=0}^k \beta_j - \frac{k(k+1)}{2}}\cdot G_k\big(\beta_0, \beta_1, ...\beta_k; g_0(z), g_1(z), \cdots, g_k(z)\big)e_0\wedge e_1\wedge\cdots\wedge e_k \\ = \quad & z^{-\alpha_{k+1}}G_k\big(\beta_0, \beta_1, ...\beta_k; g_0(z), g_1(z), \cdots, g_k(z)\big)e_0\wedge e_1\wedge\cdots\wedge e_k. \end{aligned}$$

Therefore, $\Lambda_k(z)$ equals $z^{-\alpha_{k+1}}\,G_k\big(\beta_0,\beta_1,\cdots,\beta_k;g_0(z),g_1(z),\cdots,g_k(z)\big)e_0\wedge\cdots\wedge e_k$ plus

$$\sum_{\substack{0\le i_0<i_1<\cdots<i_k\le n\\ i_k>k}} z^{\sum_{j=0}^{k}\beta_{i_j}-\frac{k(k+1)}{2}}\cdot G_k\big(\beta_{i_0},\beta_{i_1},\cdots,\beta_{i_k};g_{i_0}(z),g_{i_1}(z),\cdots g_{i_k}(z)\big)e_{i_0}\wedge e_{i_1}\wedge\cdots\wedge e_{i_k}.$$

Hence, we reach the last two formulae in the following:

Formula 2. $\|\Lambda_k\|^2=|z|^{-2\alpha_{k+1}}\Bigg(\Big|G_k\big(\beta_0,\beta_1,\cdots,\beta_k;g_0(z),g_1(z),\cdots,g_k(z)\big)\Big|^2$

$$+\sum_{\substack{0\le i_0<i_1<\cdots<i_k\le n\\ i_k>k}}|z|^{2(\sum_{j=0}^{k}\beta_{i_j}-\frac{k(k+1)}{2}+\alpha_{k+1})}\Big|G_k\big(\beta_{i_0},\beta_{i_1},\cdots,\beta_{i_k};g_{i_0}(z),g_{i_1}(z),\cdots,g_{i_k}(z)\big)\Big|^2\Bigg).$$

In particular, Therefore, $\log\|\Lambda_k\|^2$ equals $-2\alpha_{k+1}\log|z|$ plus a bounded smooth function near 0 by Lemma 4.1.

Formula 3. For all $k=1,2,\cdots,n$, we have

$$\begin{aligned}u_k&=-\sum_{j=1}^{n}a_{kj}\log\|\Lambda_{j-1}\|^2\\&=-\sum_{j=1}^{n}a_{kj}\big(-2\alpha_j\log|z|+O(1)\big)\\&=2\gamma_k\log|z|+O(1),\end{aligned}\tag{4.3}$$

where

$$\begin{aligned}O(1)\quad=\quad&-\sum_{j=1}^{n}a_{kj}\log\Bigg(\Big|G_{j-1}\big(\beta_0,\beta_1,\cdots,\beta_{j-1};g_0(z),g_1(z),\cdots,g_{j-1}(z)\big)\Big|^2\\&+\sum_{\substack{0\le i_0<i_1<\cdots<i_{j-1}\le n\\ i_{j-1}>j-1}}|z|^{2(\sum_{l=0}^{j-1}\beta_{i_l}-\frac{(j-1)j}{2}+\alpha_j)}\Big|G_{j-1}\big(\beta_{i_0},\beta_{i_1},\cdots,\beta_{i_{j-1}};g_{i_0}(z),g_{i_1}(z),\cdots,g_{i_{j-1}}(z)\big)\Big|^2\Bigg)\end{aligned}$$

is a bounded smooth function near 0 by Lemma 4.1.

Proof of Theorem 1.2 (i) Formula 3 coincides with the second sentence of Theorem 1.2. (i). As long as the last sentence is concerned, any unitary curve $D^*\to\mathbb{P}^n$ with form (1.6) induces a solution $u=(u_1,\cdots,u_n)$ to the $\mathrm{SU}(n+1)$ system in D^*. By Formula 3, u_k equals $2\gamma_k\log|z|$ plus a bounded smooth function near 0 for all $k=1,\cdots,n$, which implies $u=(u_1,\cdots,u_n)$ satisfies the system of PDEs in (1.1). It is clear that e^{u_k} is locally integrable in D by Formula 3. QED

Proof of Theorem 1.2 (ii) The idea of the proof goes as follows: by using some complex coordinate transformation $z\mapsto\xi(z)$ near 0 and preserving 0, we could simply further the expression of the canonical curve $\nu(z)$ under the new coordinate ξ. Then we obtain the desired form for the Kähler metric

$$\frac{\sqrt{-1}}{2\pi}\,e^{u_1}\,\mathrm{d}z\wedge\mathrm{d}\bar z\quad\text{on}\quad D^*,$$

which coincides with the pull-back metric $[\nu]^*\big(\omega_{\rm FS}\big)$ by (2.1) and (2.3). The details consist of the following three steps.

Step 1. Recall that $\nu(z)=\big(z^{\beta_0}g_0(z), z^{\beta_1}g_1(z),\cdots, z^{\beta_n}g_n(z)\big)$, $g_0,\cdots,g_n$ satisfy the normalized condition so that $g_0(0)g_1(0)\cdots g_n(0)\neq 0$. Then we choose the new complex coordinate

$$\xi = z\cdot\left(\frac{g_1(z)}{g_0(z)}\right)^{\frac{1}{\beta_1-\beta_0}} \tag{4.4}$$

near $z=0$ and preserving 0. Then, under this new coordinate ξ, there exist $(n-1)$ holomorphic functions $\tilde g_2(\xi),\cdots,\tilde g_n(\xi)$ near 0 and non-vanishing at 0 such that ν has the simpler form of

$$\tilde\nu(\xi):=\nu\big(z(\xi)\big)=\Big(\xi^{\beta_0},\,\xi^{\beta_1},\,\xi^{\beta_2}\tilde g_2(\xi),\,\cdots,\,\xi^{\beta_n}\tilde g_n(\xi)\Big).$$

Step 2. The preceding curve $\tilde\nu(\xi)$ does not satisfy the normalized condition with respect to ξ near 0 in general, which will not bring us trouble since the pull-back metric $\frac{\sqrt{-1}}{2\pi}\,e^{u_1}\,{\rm d}z\wedge{\rm d}\bar z=[\nu]^*(\omega_{\rm FS})$ is invariant under the coordinate transformation. On one hand, by using Formula 3, we have

$$\begin{aligned}[\nu]^*\big(\omega_{\rm FS}\big) \quad &= \quad \frac{\Big|G_1\big(\beta_0,\beta_1;g_0(z),g_1(z)\big)\Big|^2+\sum\limits_{\substack{0\le i_0<i_1\le n\\ i_1>1}}|z|^{2(\beta_{i_0}+\beta_{i_1}-1+\alpha_2)}\Big|G_1\big(\beta_{i_0},\beta_{i_1};g_{i_0}(z),g_{i_1}(z)\big)\Big|^2}{\Big(|g_0(z)|^2+|z|^{2(\beta_1-\beta_0)}|g_1(z)|^2+\cdots+|z|^{2(\beta_n-\beta_0)}|g_n(z)|^2\Big)^2}\\ &\quad\cdot\frac{\sqrt{-1}}{2\pi}\,|z|^{2\gamma_1}\,dz\wedge d\bar z.\end{aligned}$$

On the other hand, substituting the simpler form (4.4) of $\nu(z)$ to the preceding equality, we could simplify the pull-back metric $[\nu]^*\big(\omega_{\rm FS}\big)$ to the form of

$$|\xi|^{2\gamma_1}\,\frac{(\beta_1-\beta_0)^2+\sum\limits_{\substack{0\le i_0<i_1\le n\\ i_1>1}}|\xi|^{2(\beta_{i_0}+\beta_{i_1}-1+\alpha_2)}\Big|G_1\big(\beta_{i_0},\beta_{i_1};\tilde g_{i_0}(\xi),\tilde g_{i_1}(\xi)\big)\Big|^2}{\Big(1+|\xi|^{2(\beta_1-\beta_0)}+|\xi|^{2(\beta_2-\beta_0)}|\tilde g_2(\xi)|^2+|\xi|^{2(\beta_n-\beta_0)}|\tilde g_n(\xi)|^2\Big)^2}\,\frac{\sqrt{-1}}{2\pi}{\rm d}\xi\wedge{\rm d}\bar\xi.$$

In particular, the pull-back metric $[\nu]^*\big(\omega_{\rm FS}\big)$ has cone singularity at 0 with angle $2\pi(1+\gamma_1)$.

Step 3. Since the Kähler metric $\frac{\sqrt{-1}}{2\pi}\,e^{u_k}\,{\rm d}z\wedge{\rm d}\bar z$ on D^* coincides with the pull-back metric

$$[\nu\wedge\nu'\wedge\cdots\wedge\nu^{(k-1)}]^*\big(\omega_{\rm FS}\big)=\frac{\sqrt{-1}}{2\pi}\,\frac{\|\Lambda_{k-2}(\nu)\|^2\cdot\|\Lambda_k(\nu)\|^2}{\|\Lambda_{k-1}(\nu)\|^4}\,{\rm d}z\wedge{\rm d}\bar z$$

by (2.1) and (2.3) for all $k=2,3,\cdots,n$, this metric has cone singularity at 0 of angle $2\pi(1+\gamma_k)$ and could be simplified correspondingly by using both Formula 3. and (4.4). $\square$

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