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Integrals, Solutions, and Existence Problems for Laplace Transformations of Linear Hyperbolic Systems

A. V. Zhiber and S. Ya. Startsev

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Abstract—We generalize the notions of Laplace transformations and Laplace invariants for systems of hyperbolic equations and study conditions for their existence. We prove that a hyperbolic system admits the Laplace transformation if and only if there exists a matrix of rank k mapping any vector whose components are functions of one of the independent variables into a solution of this system, where k is the defect of the corresponding Laplace invariant. We show that a chain of Laplace invariants exists only if the hyperbolic system has a entire collection of integrals and the dual system has a entire collection of solutions depending on arbitrary functions. An example is given showing that these conditions are not sufficient for the existence of a Laplace transformation.

Key words: *Laplace transform, Laplace invariant, hyperbolic system.*

1. INTRODUCTION

It is well known (see, for example, [1, 2]) that any scalar equation

$$L(u) \equiv u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0, \tag{1}$$

with nonvanishing *Laplace invariant* $h = a_x + ab - c$ admits the differential substitution $v = u_y + au$, called the *Laplace transformation*, which maps solutions of Eq. (1) into solutions of the following equation:

$$L_1(v) \equiv v_{xy} + \left(a - \frac{h_y}{h}\right)v_x + b(x,y)v_y + \left[\left(a - \frac{h_y}{h}\right)b + b_y - h\right]v = 0. \tag{2}$$

The condition $h \neq 0$ ensures the "invertibility" of this transformation: the differential substitution $u = (v_x + bv)/h$ maps solutions of Eq. (2) into solutions of Eq. (1).

The above-mentioned transformations play an important role in the so-called Laplace cascade integration methods: for certain "good" cases, a sequence of properly chosen Laplace transformations can reduce the original equation to the following form:

$$\left(\frac{\partial}{\partial x} + b\right)\left(\frac{\partial}{\partial y} + \hat{a}\right)w = 0,$$

which can readily be solved. After that, by using the inverse Laplace transformations, one can reconstruct the solution of the original equation (1) from the solution of this reduced equation. The fact that Eq. (1) can be written as

$$\left(\frac{\partial}{\partial x} + b\right)\left(\frac{\partial}{\partial y} + a\right)u - hu = 0 \tag{3}$$

ensures that the chain of Laplace transformations will not be terminated before the equation reduces to the desired form.

The reason for the increasing attention to Laplace transformations and invariants during the past decade is the discovery in a series of recent papers (see [3–10]) of deep relationships between certain important properties of the scalar nonlinear equation

$$u_{xy} = F(x, y, u, u_x, u_y), \tag{4}$$

such as an exact integrability, an existence of symmetries, differential substitutions on one hand, and properties of chains of Laplace invariants for the linearized equation

$$Z_{xy} - F_{u_x} Z_x - F_{u_y} Z_y - F_u Z = 0$$

on the other hand. Moreover, the Laplace transformation can be extended to equations of the form that differs from (4) (see [5, 11, 12]).

The next intrinsic step in this direction is a generalization of the above-mentioned results to systems of equations of the form (4). The basis for such a generalization must be the Laplace transformation of linear systems of the form (1), just as the results proved for linear equations (4) are based on the Laplace transformations of the scalar equations (1). Although Laplace invariants and transformations for scalar linear equations have been well known for over a hundred years, the Laplace transformation for systems of linear equations, apparently, were not studied at all and the first attempts were made just recently in [10, 13].

As we will show in the next section, the main problem related to generalizations of the Laplace transformations and invariants to systems of linear and nonlinear equations arises, because the Laplace transformations do not necessarily exist for any system of the form (1). Our paper is dedicated to the study of different conditions implying the existence of Laplace transformations.

In Sec. 3, under the assumption of the existence of a chain of Laplace transformations terminating by a zero Laplace invariant, we prove an explicit formula for integrals and solutions of system (1). Hence we prove that the existence of an entire collection of integrals of the original system as well as the existence of an entire collection of solutions to the system dual to Eqs. (1), which depends on arbitrary functions, is a necessary condition for the existence of such a chain. In Sec. 4, we prove an existence criterion for the Laplace transformation in terms of solutions of system (1). This criterion involves the existence of a matrix which maps any vector of functions depending on one of the independent variables into a solution of the system: the rank of this matrix must coincide with the defect of the corresponding Laplace invariant. In Sec. 5, we consider the example of a hyperbolic system which does not admit a Laplace transformation, but possesses an entire collection of integrals along each of the characteristics and a solution depending on arbitrary functions. This example show that the criterion proved in Sec. 4 cannot be strengthened by the assumption that the solution depends on derivatives of arbitrary function and the necessary conditions from Sec. 3 are not sufficient for the existence of a Laplace transformation.

2. LAPLACE TRANSFORMATION FOR SYSTEMS OF EQUATIONS

It can readily be proved that in the scalar case the following formula holds:

$$\left(\frac{\partial}{\partial y} + a_1\right) L = L_1 \left(\frac{\partial}{\partial y} + a\right), \tag{5}$$

where $a_1 = a - h_y/h$, and the differential operators L and L_1 act by the rules (1) and (2), respectively. This formula indicates that the kernel of L is mapped by the operator $\partial/\partial y + a$ into the kernel of L_1; this is the key relationship for analysis of the Laplace transformations of scalar equations.

Now let us consider the system of linear equations (1): later on, u will be an n-dimensional vector, and the coefficients a, b and c will be square matrices. If we take formula (3) as a hint for the definition of Laplace invariants and try to chose the matrix a_1 and the operator L_1 so that formula (5) holds, we obtain

$$h = a_x + ba - c, \qquad L_1 = \left(\frac{\partial}{\partial y} + a_1\right)\left(\frac{\partial}{\partial x} + b\right) - h, \qquad h_y - ha + a_1 h = 0.$$

Rewriting L_1 in the form

$$\left(\frac{\partial}{\partial x} + b\right)\left(\frac{\partial}{\partial y} + a_1\right) - h_1,$$

we obtain the following formula for the Laplace invariant h_1 of the transformed equation:

$$h_1 = \frac{\partial a_1}{\partial x} + [b, a_1] - \frac{\partial b}{\partial y} + h.$$

By repeating this procedure, we obtain the following definition.

Definition 1. The matrices h_i, $i \geq 0$, defined by the recurrence relations

$$\frac{\partial h_i}{\partial y} - h_i a_i + a_{i+1} h_i = 0, \qquad h_{i+1} = \frac{\partial a_{i+1}}{\partial x} + [b, a_{i+1}] - \frac{\partial b}{\partial y} + h_i \tag{6}$$

with the first terms

$$a_0 = a, \qquad h_0 = a_x + ba - c,$$

are called *y-Laplace invariants* of system (1).

The system of equations

$$L_i(u) \equiv \left(\frac{\partial}{\partial x} + b\right)\left(\frac{\partial}{\partial y} + a_i\right)u - h_i u = 0$$

is the *result* of the i-tuple application of y-Laplace transformations to system (1). Under such a choice of a_i, h_i, and L_i, the differential substitutions $v = u_y + a_i u$, called *Laplace transformations*, map any solution of the system $L_i(u) = 0$ into solutions of the system $L_{i+1}(v) = 0$.

Perhaps, we should point out here that Laplace transformations map solutions of the equation $L_i(u) = 0$ into solutions of the equation $L_{i+1}(v) = 0$, because a_i, h_i and L_i in Definition 1 are chosen so that the following relation holds:

$$\left(\frac{\partial}{\partial y} + a_{i+1}\right)L_i = L_{i+1}\left(\frac{\partial}{\partial y} + a_i\right). \tag{7}$$

In a similar way (by the permutation of x and y and a and b in Definition 1), we can define x-invariants and x-Laplace transformations. Further, we will discuss the results concerning only one of characteristics of system (1); the results concerning the other one can be derived by the previously described permutation.

It can readily be seen that if the matrix h_i degenerates, then the main relation (6), in the general case, cannot be resolved with respect to a_{i+1} and we cannot guarantee the existence of the Laplace invariants starting from the step at which the recurrent invariant degenerates. Meanwhile, the degenerate case is the most interesting one, because recurrent Laplace transformations simplifying an equation are terminated by an equation with zero Laplace invariant, and examples show that such a recursion includes a series of degenerate nontrivial invariants. The main difficulty of the theory for systems of equations is the fact that the existence of Laplace transformations is not guaranteed, whereas many assertions on Laplace invariants for the scalar equations can easily be generalized to systems of equations (see [13]) if one ignores this difficulty and *a priori* assumes the existence of invariants.

3. CONSTRUCTIONS OF INTEGRALS AND SOLUTIONS

First of all, let us consider conditions necessary for the existence of a chain of Laplace transformations. Suppose that we can perform the Laplace transformation of system (1) several times and describe the properties of the system of equations allowing these transformations. In the scalar case, by applying the Laplace transformation we aim, as a rule, to obtain an equation with trivial Laplace invariant, and so we can assume without loss of generality that at a certain step the Laplace invariant vanishes.

It can easily be seen that the system of equations satisfying the above-formulated assumptions is characterized by the existence of entire collection of so called *y-integrals* (*x-integrals*), i.e., functions depending on x, y, and u, and on the derivatives of u with respect to y (with respect to x) whose substitutions into Eq. (1) instead of u give functions which are independent of x (of y). More precisely, the following assertion is true.

Proposition 1. *Suppose that system of equations* (1) *admits an m-tuple y-Laplace transformation and its mth y-invariant is trivial. Then this system has the collection W of n functionally independent y-integrals defined by the formula*

$$W = g(x,y)\left(\frac{\partial}{\partial y} + a_m\right)\dots\left(\frac{\partial}{\partial y} + a\right)u,$$

where the a_i are defined by formula (6) *and g is a nondegenerate matrix satisfying the relation $\partial g/\partial x - gb = 0$.*

Proof. By successively applying formula (7), we obtain

$$L_m\left(\frac{\partial}{\partial y} + a_{m-1}\right)\dots\left(\frac{\partial}{\partial y} + a\right) = \left(\frac{\partial}{\partial y} + a_m\right)\dots\left(\frac{\partial}{\partial y} + a_1\right)L.$$

Taking into account the relations

$$L_m = \left(\frac{\partial}{\partial x} + b\right)\left(\frac{\partial}{\partial y} + a_m\right) \qquad \text{and} \qquad g\left(\frac{\partial}{\partial x} + b\right) = \frac{\partial}{\partial x}\circ g,$$

we can multiply both sides of this formula by g and rewrite it in the form

$$\frac{\partial}{\partial x}\left[g(x,y)\left(\frac{\partial}{\partial y} + a_m\right)\dots\left(\frac{\partial}{\partial y} + a\right)u\right] = g(x,y)\left(\frac{\partial}{\partial y} + a_m\right)\dots\left(\frac{\partial}{\partial y} + a_1\right)L(u).$$

Then the last formula implies that the left-hand side is trivial if u is a solution of system (1). $\square$

Note that integrals are relevant mostly for the study of nonlinear systems. For linear systems, the construction of solutions is a more natural application of Laplace transformations. However, the existence of a transformation which is inverse to the Laplace transformation is necessary for the implementation of the scheme (described in the Introduction) for solving a scalar equation; on the other hand, for systems of equations, such inverse transformation may be absent. Nevertheless, we can overcome this obstacle by constructing systems of equations which can be transformed into system (1) by a direct Laplace transformation.

In order to construct such a chain of "preimages" of system (1), it is convenient to consider the formal dual system of equations $L^\top(u) = 0$. We recall that the duality between operators is established by the linear unary operation which on the set of linear operators, such as matrices, is transposition, while on the set of differential operators, it is defined by the rules

$$\left(\frac{\partial}{\partial x}\right)^\top = -\frac{\partial}{\partial x}, \qquad \left(\frac{\partial}{\partial y}\right)^\top = -\frac{\partial}{\partial y}, \qquad \text{and} \qquad (P\circ R)^\top = R^\top \circ P^\top$$

for any pair of operators P and R, where $S^\top$ denotes the operator dual to the operator S. In particular, the system dual to system (1) has the following form:

$$L^\top(u) \equiv u_{xy} - a^\top u_x - b^\top u_y + (c^\top - a_x^\top - b_y^\top)u = 0. \tag{8}$$

If this system admits an y-Laplace transformation, then the following formula holds:

$$\left(\frac{\partial}{\partial y} + \hat{a}_1\right)L^\top = (L^\top)_1\left(\frac{\partial}{\partial y} - a^\top\right).$$

Applying the formal duality map, we obtain

$$\left(\frac{\partial}{\partial y} + a\right)\Big((L^\top)_1\Big)^\top = L\left(\frac{\partial}{\partial y} - \hat{a}_1^\top\right).$$

Thus, solutions of system (1) are derived by the Laplace transformation $u = v_y - \hat{a}_1^\top v$ from solutions of the system $\big((L^\top)_1\big)^\top(v) = 0$.

If we succeed in performing the Laplace transformation m times and the mth y-Laplace invariant of system (8) vanishes, then we obtain the equation

$$\big((L^\top)_m\big)^\top(w) \equiv \left(\frac{\partial}{\partial y} - \hat{a}_m^\top\right)\left(\frac{\partial}{\partial x} + b\right)w = 0, \tag{9}$$

which can readily be integrated and transformed into system (1) by the m-tuple Laplace transformation

$$u = \left(\frac{\partial}{\partial y} - \hat{a}_1^\top\right)\dots\left(\frac{\partial}{\partial y} - \hat{a}_m^\top\right)w.$$

Taking the vector $g(x,y)f(y)$, where $f(y)$ is an arbitrary n-dimensional vector and the matrix g satisfies the relation $\partial g/\partial x + bg = 0$, as a solution of system (9), we obtain the following assertion.

Proposition 2. *Suppose that the system of equations $L^\top(u) = 0$ formally dual to system* (1) *admits an m-tuple y-Laplace transformation and its mth y-invariant vanishes. Then*

$$u = \left(\frac{\partial}{\partial y} - \hat{a}_1^\top\right)\dots\left(\frac{\partial}{\partial y} - \hat{a}_m^\top\right)(g(x,y)f(y)),$$

where the $\hat{a}_i$ denote the coefficients at $\partial/\partial x$ in $(L^\top)_i$ and the nondegenerate matrix g satisfies the relation $\partial g/\partial x + bg = 0$ and system (1) *for any n-dimensional vector $f(y)$.*

The meaning of Assertions 1 and 2 is that the condition necessary for the existence of a chain of Laplace invariants of system (1) terminating by zero is the existence of an entire collection of integrals of system (1) and an entire collection of solutions to the dual system (8) which depend on arbitrary functions. Let us explain that in the present paper we say that a collection of integrals (solutions) is *entire* if the lines (columns) of coefficients of highest derivatives of u in integrals (of highest derivatives of arbitrary function which appear in the solution) form a nondegenerate $n \times n$ matrix. (Within this paper such a definition is quite sufficient, because here we study only integrals and solutions which depend linearly on the highest derivatives.)

In the next section, we show that for the existence of a Laplace transformation, the existence of a solution to original system (1) depending on arbitrary functions is necessary.

4. SUFFICIENT CONDITIONS FOR THE EXISTENCE OF THE LAPLACE TRANSFORMATION

The following assertion was proved in the paper [10].

Proposition 3 (Sokolov). *Relation* (6) *is solvable with respect to* a_{i+1} *if and only if the differential operator* $\partial/\partial y + a_i$ *maps the kernel of* h_i *into the kernel of* h_i.

For completeness of the arguments, let us present a proof of this assertion.

Proof. By a straightforward computation, one can check that Eq. (6) is equivalent to the operator relation

$$\left(\frac{\partial}{\partial y} + a_{i+1}\right) \circ h_i = h_i \circ \left(\frac{\partial}{\partial y} + a_i\right),$$

which implies the necessity of the preservation of the kernel of h_i by the operator $\partial/\partial y + a_i$.

Now suppose that

$$\left(\frac{\partial}{\partial y} + a_i\right)(\ker h_i) \subset \ker h_i.$$

Denote by z a matrix whose columns form a basis of the kernel of h_i. Then $h_i(z_y + a_i z) = 0$. Taking into account the fact that $h_i z = 0$ and hence $h_i z_y = -(h_i)_y z$, we obtain the equality $((h_i)_y - h_i a_i)z = 0$. By the Fredholm theorem, the last relation implies the solvability of (6) with respect to a_{i+1}. This proves Proposition 3. □

Using this assertion, we can prove the following existence criterion for the Laplace transformation.

Theorem 1. *The system of equations* (1) *admits an* y*-Laplace transformation if and only if there exists a matrix* $S(x,y)$ *such that* $u = S(x,y)f(x)$ *is a solution of system* (1) *for any* n*-dimensional vector* $f(x)$ *and the sum of the ranks of matrices* S *and* h *equals* n, *where* n *is the number of dependent variables of system* (1) *and* h *is a trivial* y*-Laplace invariant of this system* ($h = a_x + ba - c$).

Proof. One can easily prove that the existence of a solution of the previously described form is equivalent to the following relations:

$$\left(\frac{\partial}{\partial y} + a\right)S = 0, \qquad hS = 0. \tag{10}$$

The last equality implies that we can choose the basis of h consisting of columns of the matrix S and to represent any element from $\ker h$ in the form Sz, where z is a vector. Taking into account the first relation in Eqs. (10), we see that the operator $\partial/\partial y + a$ maps the vector Sz into the vector Sz_y, which also belongs to the kernel of h. With regard to Proposition 3, this proves that the existence of the Laplace transformation is sufficient for the existence of the above solution.

Before we pass to the proof of the necessary conditions, note that the existence a Laplace transformation for system (1) is invariant with respect to changes of variables of the form $u = T(x,y)\hat{u}$. Indeed, substituting $h = T\hat{h}T^{-1}$ and $a = T\hat{a}T^{-1} - T_y T^{-1}$ into relation (6) for $i = 0$, we see that the solvability of this relation is preserved under the above change of variables.

Now let us choose a nondegenerate matrix $G(x,y)$ so that $\partial G/\partial y + aG = 0$. The change of variables $u = G\hat{u}$ transforms system (1) into the equation

$$\left(\frac{\partial}{\partial x} + \hat{b}\right)\frac{\partial \hat{u}}{\partial y} - \hat{h}\hat{u} = 0, \qquad \hat{h} = G^{-1}hG.$$

Suppose that the rank of $\hat{h}$ equals m. After renumbering the components of the vector $\hat{u}$ in accordance to the corresponding change of variables, without loss of generality we can suppose that the first m columns of the matrix $\hat{h}$ are linearly independent. Since the remaining columns can be represented as linear combinations of the first m columns, there exists a matrix

$$T = \begin{pmatrix} E_m & Z \\ 0 & E_{n-m} \end{pmatrix},$$

where E_i is the unit $i \times i$ matrix such that all the columns of the matrix $\hat{h}T$ starting from the $(m+1)$th are null vectors. Changing the variable $\hat{u} = T\tilde{u}$, we obtain the system of equations

$$\left(\frac{\partial}{\partial x} + \tilde{b}\right)\left(\frac{\partial}{\partial y} + T^{-1}T_y\right)\tilde{u} - \tilde{h}\tilde{u} = 0, \qquad \tilde{h} = T^{-1}\hat{h}T.$$

For the basis of the kernel of $\tilde{h}$, we can choose the vectors $e_{m+1}, \dots, e_n$, where e_i is the vector whose ith component equals 1 and the remaining components consist of zeros. Since

$$T^{-1} = \begin{pmatrix} E_m & -Z \\ 0 & E_{n-m} \end{pmatrix}, \qquad T^{-1}T_y = \begin{pmatrix} 0 & Z_y \\ 0 & 0 \end{pmatrix},$$

we see that the operator $\partial/\partial y + T^{-1}T_y$ can map all the vectors e_i, $i = m+1, \dots, n$, into the kernel of $\tilde{h}$ only if $Z_y = 0$. Then the matrix

$$S = G\begin{pmatrix} Z \\ E_{n-m} \end{pmatrix}$$

satisfy relations (10). This completes the proof. □

5. INSUFFICIENCY OF INTEGRALS AND SOLUTIONS OF HIGHER ORDER FOR THE EXISTENCE OF LAPLACE TRANSFORMATIONS

The disadvantage of the criterion given by Theorem 1 is that it allows one to perform the Laplace transformation just once. For applications, as a rule, several successive Laplace transformations are required. On the other hand, in the scalar case, Laplace transformations and invariants are applied in order to find equations possessing integrals and/or solutions of the form $\sum_{i=0}^{m} \alpha_i(x,y) f^{(i)}$, where f is an arbitrary function of one variable from the set of independent variables. Another natural aim is to ensure the applicability of Laplace transformations to systems of equations with similar properties by deriving the existence of a chain of Laplace transformations of necessary length from the existence of entire collections of integrals and solutions of the previously described form for system (1). It seems that Theorem 1 and Propositions 1 and 2 provide arguments in favor of such a relationship between the integrals, solutions, and the existence of Laplace transformations. However, in fact, the next example shows that such a relationship does not exist.

Example. Consider the system of equations

$$\left[\frac{\partial^2}{\partial x\,\partial y} - \frac{3}{x+y}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\frac{\partial}{\partial x} - \begin{pmatrix} \frac{1}{x+y} & \frac{1}{x+y} - y \\ 0 & -x \end{pmatrix}\frac{\partial}{\partial y} + \begin{pmatrix} \frac{4}{(x+y)^2} & \frac{4}{(x+y)^2} - 2 \\ 0 & 0 \end{pmatrix}\right]\begin{pmatrix} u \\ v \end{pmatrix} = 0. \tag{11}$$

This system possesses the two functionally independent x-integrals

$$w_1 = \frac{u_x + u_x}{(x+y)^3} - \frac{u+v}{(x+y)^4} + \frac{v}{(x+y)^2},$$
$$w_2 = v_x + xv$$

and the functionally independent y-integrals

$$\overline{w}_1 = -\frac{u_{yyy}}{x+y} + 3\frac{u_{yy}+v_{yy}}{(x+y)^2} - 6\frac{u_y+v_y}{(x+y)^3} + 6\frac{u+v}{(x+y)^4} - v_{yyy}e^{x^2/2}\int\left(\frac{ye^{-x^2/2}}{x+y} - \frac{e^{-x^2/2}}{(x+y)^2}\right)dx,$$
$$\overline{w}_2 = e^{x^2/2}v_{yyy}.$$

Moreover, system (11) has an entire collection of solutions of the form

$$\begin{pmatrix} u \\ v \end{pmatrix} = \left[\begin{pmatrix} (x+y)^3 & -1 \\ 0 & 1 \end{pmatrix}\frac{\partial}{\partial x} - \begin{pmatrix} 2(x+y)^2+2 & x+y \\ -2 & 0 \end{pmatrix}\right]\begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix}$$
$$+ \begin{pmatrix} x+y & (x+y)\int\left(\frac{1}{(x+y)^2} - \frac{y}{x+y}\right)e^{-x^2/2}\,dx \\ 0 & e^{-x^2/2} \end{pmatrix}\begin{pmatrix} f_1(y) \\ f_2(y) \end{pmatrix}$$

for any choice of the functions $f_1(y)$, $f_2(y)$, $g_1(x)$, and $g_2(x)$.

Proposition 2 brings to mind the idea that probably a sufficient condition for the existence of Laplace transformations is the existence of entire collections of integrals and solutions of the dual system depending on arbitrary functions. In the previous example, this assumption is satisfied: the system dual to (11) possesses both the integrals and the solutions of the required form. This follows from the general fact that system (1) admits the collection of integrals of the form

$$\sum_{i=0}^{k}\alpha_i(x,y)\frac{\partial^i u}{\partial y^i}$$

if and only if the system dual to (1) possesses solutions of the form

$$\sum_{i=0}^{k-1}\beta_i(x,y)f^{(i)}(y),$$

where the α_i and β_i are certain matrices and $\beta_{k-1} = \alpha_k^\top$, and $f(y)$ is an arbitrary vector of functions (for the proof of Proposition 3, see the paper [13, p. 150]).

Along each characteristic curve, system (11) and its dual possess an entire collection of integrals and an entire collection of solutions depending on arbitrary functions, i.e., all conditions which hypothetically must justify the existence of the Laplace transformation. However, by writing relation (6) for (11), we can easily observe that this relation is not solvable even for $i = 0$, and system (11) does not admit an y-Laplace transformation.

Note that there exists an entire family of equations possessing similar properties, and we choose one representative from this set only for simplicity.

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Institute of Mathematics and Computer Center,
Russian Academy of Sciences, Ufa
E-mail: (A. V. Zhiber) zhiber@imat.rb.ru
(S. Ya. Startsev) starts@imat.rb.ru