Applications of the Laplacian on Surfaces

Max Gray, Ty Trusty

1 Introduction

We discussed the Laplace operator on functions $f \in C^2(0,1)$ in class as the negative gradient ∇E of the Dirichlet energy $E(f) = \int_0^1 (f'(x))^2 dx$, finding it be the negative second derivative of the parameter, $\Delta f = -\nabla E(f) = f''$. A similar process can be used to derive a Laplacian on functions over an n-manifold other than $\mathbb{R}$, and we use a geometric argument from some lecture notes via Stanford[1]. We provide the derivation for functions $S \to \mathbb{R}$ for a surface S, but the argument is similar for others (such as immersions $S \to \mathbb{R}^3$, which we will use to do mean curvature flow). The approach is the following: let S be a surface in $\mathbb{R}^2$, and fix a function $f : S \to \mathbb{R}$ over our surface S, a function g for which we have $\forall \phi : S \to \mathbb{R}, \langle \phi, g \rangle_{L^2(S)} = \langle \phi, \Delta f \rangle_{L^2(S)}$, where $\langle u, v \rangle_{L^2(S)} = \int_S uv \ dA$. To discretize the surface S, we use a mesh $M = (V, F)$, where V is the vertex set and F is the set of triangular faces. To discretize the function space $L^2(S)$, we use the basis functions $h_i : S \to \mathbb{R}$, associated to each vertex $v_i \in V$ where $h_i(v_i) = 1, h_i(v_j) = 0, j \neq i$, which are piecewise-linear on the faces of M adjacent to the associated vertices. Thus we write for our functions f, g that

$$f(x) = \sum_i a_i h_i(x), \ g(x) = \sum_i b_i h_i(x).$$

Since the h_i are a basis for the discretized function space and Δ is a linear operator, we need only show that this equivalence under inner products holds on each h_i - we have for any $\phi(x) = \sum_i c_i h_i(x)$ that $\langle \phi, g \rangle = \langle \sum_i c_i h_i, g \rangle = \sum_i c_i \langle h_i, g \rangle$. Now we need that $\int_M h_i \Delta f \ dA = \int_M h_i g \ dA$ for all $i \in [|V|]$. Dealing first with the left hand side, we have via integration by parts and enforcing that $h_i = 0$ on the boundary of M that

$$\begin{aligned}
\int_M h_i \Delta f \ dA &= -\int_M \nabla h_i \cdot \nabla f dA && (1)\\
&= -\int_M \nabla h_i \cdot \nabla (\sum_j a_i h_j) dA && (2)\\
&= -\sum_j a_j \int_M \nabla h_i \cdot \nabla h_j dA, && (3)
\end{aligned}$$

and for the right, we have

$$\begin{aligned}
\int_M h_i g \ dA &= \int_M h_i \cdot \sum_j b_j h_j \ dA && (4)\\
&= \sum_j b_j \int_M h_i \cdot h_j \ dA. && (5)
\end{aligned}$$

Defining matrices A, L and vectors a, b such that

$$A_{ij} = \int_M h_i \cdot h_j \ dA, \ L_{ij} = \int_M \nabla h_i \cdot \nabla h_j \ dA$$

[1] https://graphics.stanford.edu/courses/cs468-13-spring/assets/lecture12-lu.pdf

, and a, b contain the coefficients in the h_i basis $\mathcal{H}$ of f, g respectively, we have that $Ab = La$, or more clearly, $[g]_{\mathcal{H}} = A^{-1}L[f]_{\mathcal{H}}$. This gives that $A^{-1}L = [\Delta]_{\mathcal{H}}$. It can be shown by further geometric arguments that L can be written in terms of the cotangents of angles in M, but we are out of space!

2 Conformal Mean Curvature Flow

Mean curvature flow is the flow of a surface toward a local minimum of mean curvature. The flow can be written as a differential equation in terms of the immersion $\phi : S \to \mathbb{R}^3$ of a 2-manifold S, or the corresponding explicit discretization in the notation of the above section, where δt is the time step size:

$$\frac{\partial \phi_t}{\partial t} = \Delta_t \phi_t,\ \phi^{i+1} = \phi^i + \delta t A_i^{-1} L_i \phi^i,$$

where Δ_t is with respect to the metric induced by the immersion ϕ_t, and in the discrete case ϕ is represented by the $|V| \times 3$ matrix of vertex positions. However, even when converted to a Verlet integrator, the discretized flow develops singularities quickly. This motivates a different discretization due to Kazhdan et al.,[2] who perform an involved derivation and conclude that fixing the matrix $L_i = L_0$ for all time stabilizes the flow by making it conformal, i.e. angle-preserving. The discrete flow is then

$$\phi^{i+1} = \phi^i + \delta t A_i^{-1} L_0 \phi^i,$$

which we implement as the semi-implicit system

$$(A_t - \delta t L_0)\phi^{i+1} = A_t \phi^i.$$

3 Approximating Geodesic Distances

It was known to Varadhan that heat flow could be related to geodesic distance by the equation $\phi(x, y) = \lim_{t\to 0} \sqrt{-4t \log k_{t,x}(y)}$, where $k_{t,x}(y)$ is the heat that has flowed from point x to point y on a Riemannian manifold at time t, and ϕ gives geodesic distance between points. However, attempts to use this formula directly have suffered from issues with numerical stability. Crane et al.[3] propose to use the idea of this relationship without requiring a precise reconstruction of the heat kernel, which Varadhan's formula demands. The idea is that since heat diffuses outward from a source point, the spatial derivative of the heat should align with a distance field on the manifold. Thus we can integrate the heat equation

$$\frac{\partial u}{\partial t} = \Delta u$$

for a fixed time, normalize this field to get $X = -\nabla u/|\nabla u|$ (since a distance field ϕ must have $|\nabla \phi| = 1$ everywhere), and then find the scalar field ϕ over the manifold which minimizes $\int_M |\nabla \phi - X|^2 dA$. The corresponding Euler-Lagrange equations can be written $\nabla \phi = \nabla \cdot X$, which can be solved with a standard Poisson solver and is thus an efficient method.

[2] http://www.cs.jhu.edu/ misha/MyPapers/SGP12.pdf

[3] https://arxiv.org/pdf/1204.6216.pdf