Sebastian Mueller, Melanie Schepp, Andrey Tyukin

Exercise 7.1 (SLERP) Let $(X, \langle -, - \rangle)$ be an arbitrary scalar product space and $\|-\|$ the induced norm:

$$\|x\| := \sqrt{\langle x, x \rangle}.$$

In particular, X can be $\mathbb{R}^4 \cong \mathbb{H}$ with the standard scalar product.

Let $a, b \in X$ be two different unit vectors and ϕ the positive angle from $(0, 2\pi)$ such that $\cos(\phi) = \langle a, b \rangle$. Consider the following function:

$$slerp(a, b, t) := \frac{\sin\left((1-t)\phi\right)}{\sin\phi} a + \frac{\sin\left(t\phi\right)}{\sin\phi} b,$$

where $t \in [0, 1]$. We claim that this is a linear angular interpolation between a and b.
Proof: We define a unit vector orthogonal to a as follows:

$$u := \frac{b - \langle a, b \rangle a}{\|b - \langle a, b \rangle a\|},$$

this can be seen as a single step of Gram-Schmidt orthogonalization on the basis $\{a, b\}$. We want to simplify the denominator, so we compute:

$$\|b - \langle a, b \rangle a\|^2 = 1 - 2\langle a, b \rangle^2 + \langle a, b \rangle^2 = 1 - (\cos(\phi))^2 = (\sin(\phi))^2,$$

this allows us to express u as follows:

$$u = \frac{b - \cos(\phi) a}{\sin(\phi)}.$$

Now $\{a, u\}$ is an orthonormal basis, so we can express b as follows:

$$b = \langle a, b \rangle a + \langle u, b \rangle u.$$

The first coordinate is by definition just $\cos(\phi)$, the second coordinate can be computed as follows:

$$\langle u, b \rangle = \frac{1}{\sin(\phi)} \langle b - \cos(\phi) a, b \rangle = \frac{1}{\sin(\phi)} \left(\|b\|^2 - (\cos(\phi))^2 \right) = \sin(\phi).$$

This gives us a simple description of b in terms of a and u:

$$b = \cos(\phi) a + \sin(\phi) u.$$

Now all we have to do is to plug it into the $slerp$ formula:

$$\begin{aligned}
slerp(a, b, t) &= \frac{\sin((1-t)\phi)}{\sin(\phi)} a + \frac{\sin(t\phi)}{\sin(\phi)} b \\
&= \frac{\sin((1-t)\phi) + \sin(t\phi)\cos(\phi)}{\sin(\phi)} a + \sin(\phi t) u \\
&= \frac{-\cos(\phi)\sin(t\phi) + \sin(\phi)\cos(t\phi) + \sin(t\phi)\cos(\phi)}{\sin(\phi)} a + \sin(t\phi) u \\
&= \cos(t\phi) a + \sin(t\phi) u.
\end{aligned}$$

Here we used the addition theorem for $\sin$ in the third line. ■

Exercise 7.2 (SLERP + trackball) See code. Note that $slerp$ and Euler angle interpolation is activated by pressing S or E.

Exercise 7.3 (Symmetry: technical details of a proof) Let $u_i, v_i \in \mathbb{R}^3$ for all $i \in 0 \dots n$ for some $n \in \mathbb{N}$. We consider the following matrices:

$$U_i = \begin{bmatrix} 0 & -u_i^T \\ u_i & -[u_i]_\times \end{bmatrix} \quad V_i = \begin{bmatrix} 0 & -v_i^T \\ v_i & [v_i]_\times \end{bmatrix}$$

and claim that

$$\sum_{i=1}^{n} U_i^T V_i$$

is symmetric. For this it is obviously sufficient to show that each summand $U_i^T V_i$ is symmetric. Let $u \in \{u_i\}_i$ and $v \in \{v_i\}_i$ be some vectors and U, V matrices as above, but without the indices.

It holds:

$$\begin{bmatrix} 0 & -u^T \\ u & -[u]_\times \end{bmatrix}^T \begin{bmatrix} 0 & -v^T \\ v & [v]_\times \end{bmatrix} = \begin{bmatrix} 0 & u^T \\ -u & [u]_\times \end{bmatrix} \begin{bmatrix} 0 & -v^T \\ v & [v]_\times \end{bmatrix}$$
$$= \begin{bmatrix} u^T v & (u \times v)^T \\ u \times v & uv^T + vu^T + u^T v I \end{bmatrix}$$

Here we used the following identity in first row, second column:

$$u^T [v]_\times = ([v]_\times^T u)^T = (-v \times u)^T = (u \times v)^T$$

and the matrix-version of the BAC-CAB-rule in the lower right corner:

$$[a]_\times [b]_\times = ba^T - b^T a I.$$

The resulting matrix is obviously symmetric, therefore the sum of such symmetric matrices is also symmetric. ■

Exercise 7.4 (Data registration / Image registration) Code incomplete. No matrix addition or eigenvector implementation found. Please tell to use something like MATLAB next time the homework is not solvable with available tools.