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Exercise 6.1 (Trackball)

a) We want to derive an expression for a quaternion that corresponds to the rotation of a unit vector u to some other unit vector v (assuming that $u \neq \pm v$).

We can simply choose

$$z := \frac{u^T v}{\|u \times v\|}$$

as rotation axis and compute the angle as follows:

$$\phi := \arccos(u^T v).$$

Then we can simply plug the both numbers into the formula for the rotation quaternion:

$$\left(\cos\left(\frac{\phi}{2}\right), \sin\left(\frac{\phi}{2}\right) z\right).$$

b) The above formula would probably do the job, however, we might want to eliminate unnecessary normalizations and trigonometric functions. For this, remember that it holds (for angles from $[0, \pi]$):

$$\cos\left(\frac{\phi}{2}\right) = \sqrt{\frac{1+\cos\phi}{2}} \qquad \sin\left(\frac{\phi}{2}\right) = \sqrt{\frac{1-\cos\phi}{2}}.$$

Furthermore, notice that $\cos(\phi) = u^T v$ and

$$\|u \times v\| = \sin(\phi) = \sqrt{1-(u^T v)^2}.$$

Therefore it holds (for q from the first part):

$$\begin{aligned}\sqrt{2}\sqrt{1+u^T v}q &= \sqrt{2}\sqrt{1+u^T v}\left(\sqrt{\frac{1+u^T v}{2}}, \frac{u \times v}{\sqrt{1-(u^T v)^2}}\sqrt{\frac{1-u^T v}{2}}\right)\\ &= (1+u^T v, u \times v),\end{aligned}$$

so we see that we can compute $(1 + u^T v, u \times v)$ and normalize it, instead of messing with trigonometric functions.

Exercise 6.2 (Converting into Euler's angles)

a)

Angles have negative sense of rotation.

$$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{pmatrix} \; R_y(\theta) = \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

$$R_z(\psi) = \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
&R_x(\phi)R_y(\theta)R_z(\psi) \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} \cos(\theta)\cos(\psi) & \cos(\theta)\sin(\psi) & -\sin(\theta) \\ -\sin(\psi) & \cos(\psi) & 0 \\ \sin(\theta)\cos(\psi) & \sin(\theta)\sin(\psi) & \cos(\theta) \end{pmatrix} \\
&= \begin{pmatrix} \cos(\theta)\cos(\psi) & \cos(\theta)\sin(\psi) & -\sin(\theta) \\ \sin(\phi)\sin(\theta)\cos(\psi) - \cos(\phi)\sin(\psi) & \sin(\phi)\sin(\theta)\sin(\psi) + \cos(\phi)\cos(\psi) & \sin(\phi)\cos(\theta) \\ \cos(\phi)\sin(\theta)\cos(\psi) + \sin(\phi)\sin(\psi) & \cos(\phi)\sin(\theta)\sin(\psi) - \sin(\phi)\cos(\psi) & \cos(\phi)\cos(\theta) \end{pmatrix}
\end{aligned}$$

b) $R_x(\phi)R_y(\theta)R_z(\psi) = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$

$\Rightarrow$

$$\theta = -\arcsin(z_1) \tag{1}$$

- $\theta \neq \pm\frac{\pi}{2}$:

$$y_1 = \sin(\psi)\cos(\theta) \tag{2}$$

$$z_2 = \sin(\phi)\cos(\theta) \tag{3}$$

$$\cos(\arcsin(x)) = \sqrt{1 - \sin^2(\arcsin(x))} = \sqrt{1 - x^2} \tag{4}$$

$\Rightarrow$ The solutions for ψ and ϕ are the same except that where y_1 is in the solution for ψ there is z_2 in the solution for ϕ.
Equation 2,4 $\Rightarrow$

$$\psi = \arcsin\left(\frac{y_1}{\cos(\theta)}\right)$$

$$\psi = \arcsin\left(\frac{y_1}{\cos(\arcsin(z_1))}\right)$$

$$\psi = \arcsin\left(\frac{y_1}{\sqrt{1 - z_1^2}}\right) \tag{5}$$

$\Rightarrow$

$$\phi = \arcsin\left(\frac{z_2}{\sqrt{1-z_1^2}}\right) \tag{6}$$

- $\theta = \pm\frac{\pi}{2}$
  Addition theorems:

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y) \tag{7}$$

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y) \tag{8}$$

$\theta = \frac{\pi}{2} \Leftrightarrow \sin(\theta) = 1$
With equation 8,7 and $\cos(\theta) = 0$ you get the following matrix

$$R = \begin{pmatrix} 0 & 0 & -1 \\ \sin(\phi-\psi) & \cos(\phi-\psi) & 0 \\ \cos(\phi-\psi)) & -\sin(\phi-\psi) & 0) \end{pmatrix}$$

$\Rightarrow$

$$\begin{aligned} \phi - \psi &= \arcsin(x_2) \\ \phi &= \arcsin(x_2) + \psi \end{aligned} \tag{9}$$

$\theta = -\frac{\pi}{2} \Leftrightarrow \sin(\theta) = -1$
With equation 8,7 and $\cos(\theta) = 0$ you get the following matrix

$$R = \begin{pmatrix} 0 & 0 & 1 \\ -\sin(\phi+\psi) & \cos(\phi+\psi) & 0 \\ -\cos(\phi+\psi)) & -\sin(\phi+\psi) & 0) \end{pmatrix}$$

$\Rightarrow$

$$\begin{aligned} \phi + \psi &= \arcsin(-x_2) \\ \phi &= -(\arcsin(x_2) + \psi) \end{aligned} \tag{10}$$

Exercise 6.3 (Some basic properties of quaternions)
We denote [1] quaternions as $\bar{a} = (a_0, a)$ and assume that the vector-part a is a column vector. We sometimes write the whole quaternion vertically. Remember that the multiplication of quaternions is defined as follows:

$$\bar{a} \cdot \bar{b} = (a_0 b_0 - a^T b, a_0 b + b_0 a + a \times b).$$

a) We want to show that the multiplication of quaternions is associative (which is not obvious, because the quaternion product contains the cross product). The brute-force proof with all 3 quaternions simultaneously seems to be surprisingly nasty, so instead

[1] What's that "underscore"-thingie, how does one TeX it?

we embed the quaternions into a matrix algebra about which we know that it is associative (this allows us to deal with just two quaternions at a time).

First, we express quaternion multiplication as a matrix product:

$$\bar{a} \cdot \bar{b} = \begin{bmatrix} a_0 & -a^T \\ a & (a_0 I + [a]_\times) \end{bmatrix} \begin{bmatrix} b_0 \\ b \end{bmatrix},$$

and write $M_{\bar{a}}$ for the 4x4-matrix. It is not immediately obvious how the product of two such matrices relates to the product of quaternions. For the following computation we have to remember:

- $[a \times b]_\times = ba^T - ab^T$ (This is the BAC-CAB rule in matrix form)
- $[a]_\times [b]_\times = ba^T - a^T bI$ (This is again the BAC-CAB rule, up to some sign changes and permutations)

Using these BAC-CAB-rules in the lower right corner of the matrix, we compute:

$$\begin{aligned} M_{\bar{a}} M_{\bar{b}} &= \begin{bmatrix} a_0 & -a^T \\ a & a_0 I + [a]_\times \end{bmatrix} \begin{bmatrix} b_0 & -b^T \\ b & b_0 I + [b]_\times \end{bmatrix} \\ &= \begin{bmatrix} (a_0 b_0 - a^T b) & (-a_0 p^T - b_0 a^T - a^T [b]_\times) \\ (a_0 b + b_0 a + [a]_\times b) & -ab^T + a_0 b_0 I + a_0 [b]_\times + b_0 [a]_\times + [a]_\times [b]_\times \end{bmatrix} \\ &= \begin{bmatrix} (a_0 b_0 - a^T b) & -(a_0 p^T + b_0 a^T + a \times b) \\ (a_0 b + b_0 a + [a]_\times b) & (a_0 p_0 - a^T b) I + [a_0 b + b_0 a + a \times b]_\times \end{bmatrix} \\ &= M_{\bar{a}\bar{b}}. \end{aligned}$$

Now using this, for arbitrary quaternions $\bar{a}, \bar{b}, \bar{c}$ we can easily verify:

$$(\bar{a}\bar{b})\bar{c} = (M_{\bar{a}\bar{b}})\,\bar{c} = (M_{\bar{a}} M_{\bar{b}})\,\bar{c} = M_{\bar{a}}\,(M_{\bar{b}}\bar{c}) = \bar{a}(\bar{b}\bar{c}),$$

because we already know that matrix product is associative. ■

b) Despite being completely trivial, the following computation is actually kind of important, because it shows us that $\bar{a}\bar{a}^*$ is actually a real number:

$$\bar{a}\bar{a}^* = (a_0^2 - \|a\|^2, 0) = \bar{a}^*\bar{a}$$

This in turn allows us to define inverses of quaternions analogously to the inverses of complex numbers. ■

c) Notice: $-p \times q = (-q) \times (-p)$. Therefore:

$$\begin{aligned} (\bar{p}\bar{q})^* &= (p_0 q_0 - p^T q, -q_0 p - p_0 q - p \times q) \\ &= (q_0, -q)(p_0, -p) \\ &= \bar{q}^* \bar{p}^* \end{aligned}$$

The same in matrix-notation introduced above (bars omitted):

$$M_{(ab)^*} = M_{ab}^T = (M_a M_b)^T = M_b^T M_a^T = M_{b^*} M_{a^*}.$$

■

Exercise 6.4 (Two-sheeted covering of $SO(3)$) We want to show that for every unit quaternion $\bar{q}$ the rotation induced by $\bar{q}$ is the same as the rotation induced by $-\bar{a}$.

There are many possible ways. The simplest one is as follows: quaternions commute with all real numbers, in particular with ± 1, therefore for each $\bar{p} = (0, v)$ it holds:

$$(-\bar{q})\bar{p}(-\bar{q})^* = (-1)^2 \bar{q}\bar{p}\bar{q}^*. = \bar{q}\bar{p}\bar{q}^*.$$

A somewhat less simple possibility is to take a look at the resulting rotation matrix (for components $\bar{q} = (a, b, c, d)$):

$$\begin{bmatrix} a^2 + b^2 - c^2 - d^2 & -2ad + 2bc & 2ac + 2bd \\ 2ad + 2bc & a^2 + c^2 - b^2 - d^2 & -2ab + 2cd \\ -2ac + 2bd & 2ab + 2cd & a^2 + d^2 - b^2 - c^2 \end{bmatrix}$$

All entries of this matrix are actually polynomials in a, b, c, d consisting only of terms with degree exactly 2, therefore the factor $(-1)^2$ just cancels out everywhere.

Another possibility is to consider the Rodrigues-rotation matrix for the angle/axis (ϕ, z) associated with the quaternion $\bar{q}$ and to remember that $\cos$ is symmetric and $\sin$ antisymmetric:

$$\begin{aligned} R_{(-\phi,-z)} &= \cos(-\phi)I + \sin(-\phi)[-z]_\times + (1 - \cos(-\phi))(-z)(-z)^T \\ &= \cos(\phi)I + (-1)^2 \sin(\phi)[z]_\times + (1 - \cos(\phi))(-1)^2 zz^T \\ &= R_{(\phi,z)} \end{aligned}$$

■