Exercise 6.2 (Converting into Euler's angles)

a)

Angles have negative sense of rotation.

$$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{pmatrix} \quad R_y(\theta) = \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

$$R_z(\psi) = \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
&R_x(\phi)R_y(\theta)R_z(\psi) \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} \cos(\theta)\cos(\psi) & \cos(\theta)\sin(\psi) & -\sin(\theta) \\ -\sin(\psi) & \cos(\psi) & 0 \\ \sin(\theta)\cos(\psi) & \sin(\theta)\sin(\psi) & \cos(\theta) \end{pmatrix} \\
&= \begin{pmatrix} \cos(\theta)\cos(\psi) & \cos(\theta)\sin(\psi) & -\sin(\theta) \\ \sin(\phi)\sin(\theta)\cos(\psi) - \cos(\phi)\sin(\psi) & \sin(\phi)\sin(\theta)\sin(\psi) + \cos(\phi)\cos(\psi) & \sin(\phi)\cos(\theta) \\ \cos(\phi)\sin(\theta)\cos(\psi) + \sin(\phi)\sin(\psi) & \cos(\phi)\sin(\theta)\sin(\psi) - \sin(\phi)\cos(\psi) & \cos(\phi)\cos(\theta) \end{pmatrix}
\end{aligned}$$

b) $R_x(\phi)R_y(\theta)R_z(\psi) = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$

$\Rightarrow$

$$\theta = -\arcsin(z_1) \tag{1}$$

- $\theta \neq \pm\frac{\pi}{2}$:

$$y_1 = \sin(\psi)\cos(\theta) \tag{2}$$

$$z_2 = \sin(\phi)\cos(\theta) \tag{3}$$

$$\cos(\arcsin(x)) = \sqrt{1 - \sin^2(\arcsin(x))} = \sqrt{1 - x^2} \tag{4}$$

$\Rightarrow$ The solutions for ψ and ϕ are the same except that where y_1 is in the solution for ψ there is z_2 in the solution for ϕ.

Equation 2,4 $\Rightarrow$

$$\psi = \arcsin\left(\frac{y_1}{\cos(\theta)}\right)$$
$$\psi = \arcsin\left(\frac{y_1}{\cos(\arcsin(z_1))}\right)$$
$$\psi = \arcsin\left(\frac{y_1}{\sqrt{1-z_1^2}}\right) \tag{5}$$

$\Rightarrow$

$$\phi = \arcsin\left(\frac{z_2}{\sqrt{1-z_1^2}}\right) \tag{6}$$

- $\theta = \pm\frac{\pi}{2}$
  Addition theorems:

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y) \tag{7}$$

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y) \tag{8}$$

$\theta = \frac{\pi}{2} \Leftrightarrow \sin(\theta) = 1$
With equation 8,7 and $\cos(\theta) = 0$ you get the following matrix

$$R = \begin{pmatrix} 0 & 0 & -1 \\ \sin(\phi-\psi) & \cos(\phi-\psi) & 0 \\ \cos(\phi-\psi)) & -\sin(\phi-\psi) & 0) \end{pmatrix}$$

$\Rightarrow$

$$\phi - \psi = \arcsin(x_2)$$
$$\phi = \arcsin(x_2) + \psi \tag{9}$$

$\theta = -\frac{\pi}{2} \Leftrightarrow \sin(\theta) = -1$
With equation 8,7 and $\cos(\theta) = 0$ you get the following matrix

$$R = \begin{pmatrix} 0 & 0 & 1 \\ -\sin(\phi+\psi) & \cos(\phi+\psi) & 0 \\ -\cos(\phi+\psi)) & -\sin(\phi+\psi) & 0) \end{pmatrix}$$

$\Rightarrow$

$$\phi + \psi = \arcsin(-x_2)$$
$$\phi = -(\arcsin(x_2) + \psi) \tag{10}$$