Exercise 9.3 (De Casteljau algorithm) Be

$$P_i^0 = \sum_{j=0}^{0} P_{i+j}^0 \binom{0}{j} t^j (1-t)^{0-j} = P_i^0 \tag{1}$$

An we could use the recursive definition

$$P_i^r = (1-t)P_i^{r-1} + tP_{i+1}^{r-1} \tag{2}$$

To start an induction assuming that

$$P_i^k = \sum_{j=0}^{k} P_{i+j}^0 \binom{k}{j} t^j (1-t)^{k-j} \tag{3}$$

is valid we need to show that this form is valid for $k+1$.

$$\begin{align}
P_i^{k+1} &= (1-t)P_i^k + tP_{i+1}^k \tag{4}\\
&= \sum_{j=0}^{k} P_{i+j}^0 \binom{k}{j} t^j (1-t)^{k+1-j} + \sum_{j=0}^{k} P_{i+1+j}^0 \binom{k}{j} t^{j+1} (1-t)^{k-j} \tag{5}\\
&= P_i^0 \binom{k}{0} (1-t)^{k+1} + P_{i+k+1}^0 \binom{k}{k} t^{k+1} + \sum_{j=1}^{k} P_{i+j}^0 \left(\binom{k}{j} + \binom{k}{j-1} \right) t^j (1-t)^{k+1-j} \tag{6}\\
&= P_i^0 \binom{k}{0} (1-t)^{k+1} + P_{i+k+1}^0 \binom{k}{k} t^{k+1} + \sum_{j=1}^{k} P_{i+j}^0 \binom{k+1}{j} t^j (1-t)^{k+1-j} \tag{7}\\
&= \sum_{j=0}^{k+1} P_{i+j}^0 \binom{k+1}{j} t^j (1-t)^{k+1-j} \tag{8}
\end{align}$$

Using that

$$\begin{align}
\binom{k}{j} + \binom{k}{j-1} &= \frac{k!}{(k-j)!j!} \frac{k!}{(k+1-j)!(j-1)!} \tag{9}\\
&= \frac{k!(j+k+1-j)}{(k+1-j)!j!} = \binom{k+1}{j} \tag{10}\\
& \tag{11}
\end{align}$$

Exercise 9.4 (Tangent)

$$
\begin{aligned}
x(t) &= \sum_{i=0}^{n} B_i^n(t) P_i \\
\partial_t &:= \frac{\partial}{\partial_t} \\
\binom{n}{i} &= \frac{n!}{i!(n-i)!} \\
\partial_t x(t) &= \sum_{i01}^{n} \partial_t B_i^n(t) P_i \\
&= \sum_{i=0}^{n} \partial_t \left(\binom{n}{i} t^i (1-t)^{n-i} \right) P_i \\
&= \sum_{i=1}^{n} \left(\binom{n}{i} i t^{i-1} (1-t)^{n-i} \right) P_i - \sum_{i=0}^{n} \left(\binom{n}{i} t^i (n-i)(1-t)^{n-i-1} \right) P_i \\
&= n \left(\sum_{i=1}^{n} \frac{(n-1)!}{(i-1)!(n-i)!} t^{i-1} (1-t)^{n-i} P_i - \sum_{i=0}^{n-1} \frac{(n-1)!}{i!(n-i-1)!} t^i (1-t)^{n-i-1} P_i \right) \\
&= n \left(\sum_{i=0}^{n-1} \frac{(n-1)!}{i!(n-(i+1))!} t^i (1-t)^{n-(i+1)} P_{i+1} - \sum_{i=0}^{n-1} \binom{n-1}{i} t^i (1-t)^{(n-1)-i} P_i \right) \\
&= n \left(\sum_{i=0}^{n-1} \binom{n-1}{i} t^i (1-t)^{(n-1)-i} P_{i+1} - \sum_{i=0}^{n-1} \binom{n-1}{i} t^i (1-t)^{(n-1)-i} P_i \right) \\
&= n \sum_{i=0}^{n-1} B_i^{n-1}(t) (P_{i+1} - P_i) \\
n' &= n - 1 \\
\bar{P}_i &= (P_{i+1} - P_i) n \\
\dot{x}\,(t) &= \sum_{i=0}^{n'} B_i^{n'} \,\bar{P}_i \\
\dot{x}\,(0) &= \bar{P}_0 = n(P_1 - P_0) \\
\dot{x}\,(1) &= \bar{P}_{n'} = n(P_n - P_{n-1})
\end{aligned}
$$

Tangent in $x(0)$: $g(s) \ = P_0 + \dot{x}\,(0) l$
Tangent in $x(1)$: $g'(s) = P_n + \dot{x}\,(1) l$
$P_1 = g(\frac{1}{n}) \Rightarrow$ Tangent in $x(0)$ points in direction of $P_1 - P_0$
$P_{n-1} = g'(-\frac{1}{n}) \Rightarrow$ Tangent in $x(1)$ points in direction of $P_{n-1} - P_n$