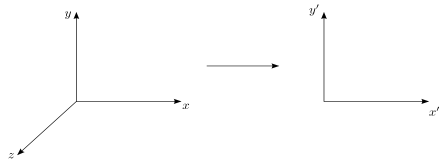
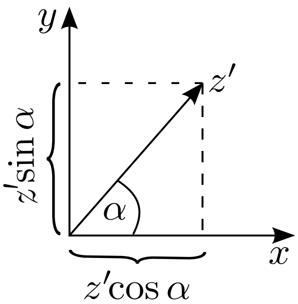
Exercise 4.1 (Parallel-perspective point-picking) (See code)

Exercise 4.2 (Cavalier perspective)



To get the point $p' = \begin{pmatrix} x' \\ y' \\ 0 \\ 1 \end{pmatrix}$ from the point $p = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$ we must divide z in half and project it on the x/y-axis and must add this to the x/y value. Projection:



The z' in the picture equals -0.5z $\Rightarrow p' = \begin{pmatrix} x - 0.5cos(\alpha)z \\ y - 0.5cos(\alpha)z \\ 0 \\ 1 \end{pmatrix}$

$$p' = Mp \Longleftrightarrow \begin{pmatrix} x - 0.5cos(\alpha)z \\ y - 0.5cos(\alpha)z \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & q \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} ax + by + cz + d \\ ex + fy + gz + h \\ ix + jy + kz + l \\ mx + ny + oz + q \end{pmatrix}$$

$$\Rightarrow M = \begin{pmatrix} 1 & 0 & -0.5\cos(\alpha) & 0 \\ 0 & 1 & -0.5\cos(\alpha) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

With $\alpha = 45°$ and $\cos(45°) = \sin(45°) = \frac{1}{\sqrt{2}}$ we get: $M = \begin{pmatrix} 1 & 0 & -\frac{1}{2\sqrt{2}} & 0 \\ 0 & 1 & -\frac{1}{2\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Exercise 4.3 (Perspective projection and lines) We want to understand the effects of perspective projection on lines in three dimensional space. Let $f > 0$ be some constant. We omit the discussion of all the tricks with the third component of the projection (which are relevant only for the z-buffer), and we also omit all the details related to hardware-specific screen sizes and sign-conventions. Instead, we assume without loss of generality, that the observer is located at 0 and is looking into the positive z-direction, projecting everything on a near-clipping plane at $z = f$. That means, that a point (x, y, z) is projected to $(fx/z, fy/z, f)$. Expressed as a matrix for homogeneous coordinates, this looks as follows:

$$\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & f & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} fx \\ fy \\ fz \\ z \end{bmatrix} \rightharpoonup \begin{bmatrix} fx/z \\ fy/z \\ f \end{bmatrix}$$

Now let $b \in \mathbb{R}^3$ be some base vector and $d \in \mathbb{R}^3$ be some dimension vector, we want to consider the straight line $\{b + td \mid t \in \mathbb{R}\}$. Applying the projection to the points of the straight line, we obtain:

$$\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & f & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} b_x + td_x \\ b_y + td_y \\ b_z + td_z \\ 1 \end{bmatrix} \rightharpoonup \begin{bmatrix} f\frac{b_x + td_x}{b_z + td_z} \\ f\frac{b_y + td_y}{b_z + td_z} \\ f \end{bmatrix}$$

Now we want to compute the derivative of the right-hand side with respect to t. Obviously, it is sufficient to consider only the first component (the second looks almost the same, the third is a constant). Let's also drop the factor f for a moment. It holds:

$$\begin{aligned} \frac{d}{dt}\frac{b_x + td_x}{b_z + td_z} &= \frac{d_x}{b_z + td_z} - \frac{(b_x + td_x)d_z}{(b_z + td_z)^2} \\ &= \frac{d_x(b_z + td_z) - (b_x + td_x)d_z}{(b_z + td_z)^2} \\ &= \frac{d_x b_z - b_x d_z}{(b_z + td_z)^2} \end{aligned}$$

For all three components together, it means:

$$\frac{d}{dt}\begin{bmatrix} f\frac{b_x + td_x}{b_z + td_z} \\ f\frac{b_y + td_y}{b_z + td_z} \\ f \end{bmatrix} = \frac{f}{(b_z + td_z)^2}\begin{bmatrix} d_x b_z - b_x d_z \\ d_y b_z - b_y d_z \\ 0 \end{bmatrix}.$$

Now we can make the following observations. First, notice that although the whole velocity vector is not constant, *it's direction is*: it is always the vector $(d_x b_z - b_x d_z, d_y b_z - b_y d_z, 0)$, scaled by some t-dependent factor. In particular, the projection of the line is contained in the line $\{(f b_x/b_z, f b_y/b_z, f) + q(d_x b_z - b_x d_z, d_y b_z - b_y d_z, 0) \mid q \in \mathbb{R}\}$. If one does not care too much about the fact that the original line is infinite, while the projected line is bounded by vanishing points, one could expressed it as " straight lines are projected to straight lines" (this is the statement (i)).

Now take a closer look at the direction vector $(d_x b_z - b_x d_z, d_y b_z - b_y d_z, 0)$. As long as d_z is not zero, this direction vector depends on the coordinates of the base point b_x and b_y, so that parallel lines with $d_z \neq 0$ will in general no longer be parallel after projection (this shows (ii)).

However, if d_z is zero, then the direction vector becomes just $b_z(d_x, d_y, 0)$ after projection, that is, it is a scaled version of the original vector (d_x, d_y, d_z). This means that parallel lines that are parallel to the projection plane remain parallel, and do not intersect in a vanishing point (iii).