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Exercise 10.3 (Bernstein basis to Monom basis)

$$\begin{pmatrix} B_0^n(t) \\ \vdots \\ B_n^n(t) \end{pmatrix} = C \begin{pmatrix} 1 \\ t \\ t \\ t^2 \\ \vdots \\ t^n \end{pmatrix} \text{ with } C_{i,j} = \tbinom{j}{i}\tbinom{n}{j}(-1)^{j-i}$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \tag{1}$$

$$\begin{aligned}
B_i^n(t) &= \sum_{j=0}^{n} C_{i,j} t^j \\
&= \sum_{j=0}^{n} \binom{j}{i}\binom{n}{j}(-1)^{j-i} t^j \overset{\binom{j}{i}=0 \text{ for } i>j}{=} \sum_{j=i}^{n} \frac{n!j!}{(n-j)!j!(j-i)!}(-1)^{j-i} t^{j-i} t^i \\
&= t^i \frac{n!}{i!(n-i)!} \sum_{j=i}^{n} \frac{(n-i)!}{(n-j)!(j-i)!}(-t)^{j-i}(1)^{n-j} \\
&\overset{j \to j+i}{=} t^i \binom{n}{i} \sum_{j=0}^{n-i} \frac{(n-i)!}{(n-i-j)!j!}(-t)^j (1)^{n-i-j} \\
&\overset{1}{=} \binom{n}{i} t^i (1-t)^{n-i}
\end{aligned}$$