Kitaev Chain and Marajana Fermion

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1 Pauli Matrices

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

The effect of σ_z on the spin up and down states are as follow:

$$\sigma_z| \uparrow \rangle = -| \uparrow \rangle,$$
$$\sigma_z| \downarrow \rangle = -| \downarrow \rangle.$$

This shows the eigenstates of σ_z are $| \uparrow \rangle$ and $| \downarrow \rangle$.

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

The effect of σ_x on the spin up and down states are as follow:

$$\sigma_x| \uparrow \rangle = | \downarrow \rangle,$$
$$\sigma_x| \downarrow \rangle = | \uparrow \rangle.$$

If we define $| \rightarrow \rangle \equiv \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}$ and $| \leftarrow \rangle \equiv \frac{|\uparrow\rangle - |\downarrow\rangle}{\sqrt{2}}$. Then $| \rightarrow \rangle$ and $| \leftarrow \rangle$ are the eigenstates of σ_x. Moreover, σ_z flips the spin in the σ_x basis.

$$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

2 Transverse Field Ising Model

Paramagnetism: $\rightarrow \ \leftarrow \ \rightarrow \ \leftarrow \ \rightarrow$
Ferromagnetism: $\rightarrow \ \rightarrow \ \rightarrow \ \rightarrow \ \rightarrow$

Following the convention of [**1**], all the following discussion is done in the σ_x-basis. Here, J is the ferromagnetic exchange coupling in x-direction and h_z is a uniform transverse field in z-direction.

$$H_{TFIM} = -J\sum_{i=1}^{N-1}\sigma_i^x\sigma_{i+1}^x - h_z\sum_{i=1}^{N}\sigma_i^z. \tag{1}$$

This model has a $\mathbb{Z}_2$-symmetry (P_s) dfined as bellow:

$$P_s = \prod_{j=1}^{N}\sigma_j^z. \tag{2}$$

Since P_s flips the spin globally, $P_s^2 = 1$ as we are considering a 1D-chain. And $[H_{TFIM}, P_s] = 0$ by considering $P_s\sigma_i^x P_s = \sigma_i^x$ and $P_s\sigma_i^z P_s = -\sigma_i^x$.

2.1 Two-Fold Degeneracy of the Ground States

In this section we are going to consider the limit $h_z \to 0$, and $H \approx -J\sum_{i=1}^{N-1}\sigma_i^x\sigma_{i+1}^x$. Then, we will have two possible gound states (gound state degeneracy),

$$|\psi_{\rightarrow}\,\rangle \equiv |\rightarrow\quad\rightarrow\quad\rightarrow\quad\rightarrow\quad\rightarrow\rangle,$$

$$|\psi_{\leftarrow}\,\rangle \equiv |\leftarrow\quad\leftarrow\quad\leftarrow\quad\leftarrow\quad\leftarrow\rangle.$$

Assume we start with $|\psi_{\rightarrow}\,\rangle$. If we perturbate the state by introducing a very small h_z field, there will be tunneling between $|\psi_{\rightarrow}\,\rangle$ and $|\psi_{\leftarrow}\,\rangle$ via soliton(domain wall). This is schematically shown below:

$$|\psi_{\rightarrow}\rangle = |\rightarrow\quad\rightarrow\quad\rightarrow\quad\rightarrow\quad\rightarrow\rangle \Longrightarrow |\leftarrow\quad\rightarrow\quad\rightarrow\quad\rightarrow\quad\rightarrow\rangle \Longrightarrow |\leftarrow\quad\leftarrow\quad\rightarrow\quad\rightarrow\quad\rightarrow\rangle \Longrightarrow$$

$$|\leftarrow\quad\leftarrow\quad\leftarrow\quad\rightarrow\quad\rightarrow\rangle \Longrightarrow |\leftarrow\quad\leftarrow\quad\leftarrow\quad\leftarrow\quad\rightarrow\rangle \Longrightarrow |\leftarrow\quad\leftarrow\quad\leftarrow\quad\leftarrow\quad\leftarrow\rangle = |\psi_{\leftarrow}\rangle$$

This shows the presence of the h_z field lifts the degeneracy between $|\psi_{\rightarrow}\,\rangle$ and $|\psi_{\leftarrow}\,\rangle$. The tunneling amplitude

$$t \sim e^{-\frac{N}{\xi}},$$

where ξ is the correlation length of the model as t should fall off expotentially in the distance the soliton has to propergate. And the effective hamiltonian can be written as

$$H_{eff} = \begin{bmatrix} 0 & t \\ t & 0 \end{bmatrix}.$$

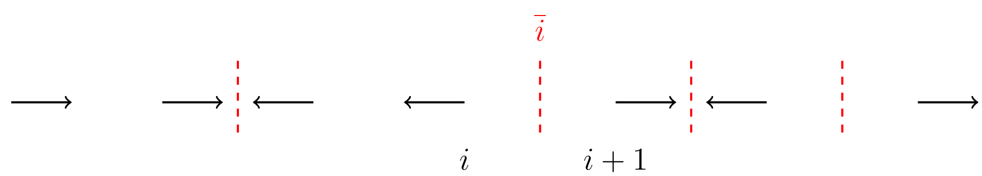
So for any finte size N, tunneling effect and therefore the splitting of ground states is negligible. And the two-fold degeneracy can only be recovered in the thermodynamic limit ($N \to \infty$). Or more explicitly in our case, we will not observe the $|\psi_{\leftarrow}\rangle$ state unless we wait for a very long time. This is not a good news, as the two-fold degeneracy of the system has the potential to encode quantum information and be a qubit.

Moreover, the loss of ground state degeneracy leads to spontaneous symmetry breaking. Because a system consists of entirely $|\psi_{\rightarrow}\rangle$ or $|\psi_{\leftarrow}\rangle$ breaks the $\{\sigma_i^x\} \leftrightarrow \{-\sigma_i^x\}$ symmetry of $H \approx -J\sum_{i=1}^{N-1}\sigma_i^x\sigma_{i+1}^x$, at least within the thermodynamic limit.

And if we add a small longitudinal field ($h_x \to 0$) , that is larger than the energy splitting between the states ($N \to \infty$), the symmetry is explicitly broken and h_x will fully split the degeneracy.

3 Domain Wall (Kink)

Paramagnetic Spins with Domain Walls and Indexed Labels:



Domain walls $\bar{i}$ at spin flips (red dashed lines)

This illustarion shows we can view paramagnet as several excitations (creation of domain walls) on a ferromagnet. And the change of indicies ($(i, i+1) \to \bar{i}$) shows a 2 to 1 map between the spin configuration and domain wall configuration.

The Domain wall variables are defined as the following:

$$\tau_{\bar{i}}^x \equiv \sigma_i^x\sigma_{i+1}^x = \begin{cases} +1 & \text{if no domain wall at } \bar{i} \\ -1 & \text{if no domain wall at } \bar{i}. \end{cases} \tag{3}$$

And the creation of the domain wall is described as below as $\tau_{\bar{i}}^z$ flips all the spin on the right of $\bar{i}$.

$$\tau_{\bar{i}}^z \equiv \prod_{j>\bar{i}} \sigma_j^z. \tag{4}$$

$\tau_{\bar{i}-1}^z\tau_{\bar{i}}^z = (\prod_{j>\bar{i}-1}\sigma_j^z)(\prod_{j>\bar{i}}\sigma_j^z) = \sigma_j^z$. This shows domain wall is localized.

$$H_{TFIM} = -J\sum_{i=1}^{N-1}\sigma_i^x\sigma_{i+1}^x - h_z\sum_{i=1}^{N}\sigma_i^z$$
$$\Big\Downarrow \text{ change of variables: } \{\sigma_i^x,\sigma_i^z\} \to \{\tau_{\bar{i}}^x,\tau_{\bar{i}}^z\}$$
$$H_{TFIM} = -J\sum_{\bar{i}=1}^{N-1}\tau_{\bar{i}}^x - h_z\sum_{\bar{i}=2}^{N-1}\tau_{\bar{i}-1}^z\tau_{\bar{i}}^z$$

Before changing the variables, in terms of pauli matrices (σ), the first term of H_{TFIM} favours ferrmomagentic state while the emergence of h_z favours the paramagnetic state. On the other hand, in terms of domain wall variables (τ), the first term favours paramagnetic state while the emergence of h_z favours the ferromagnetic state. Therefore, ferromagnet state of the domain walls correspond to the paramagnetic state of the spins and vice versa.

- **Paramagnetic State** $(h_z >> 1): \quad <\tau_{\bar{i}}^z> \approx 1$
  This is called domain wall condensate and these states are excited by spin flip (σ_i^z).
- **Ferromagnetic State** $(h_z << 1): \quad <\sigma_i^x> \approx 1$
  This is called spin condensate and these states are excited by creation of domain wall ($\tau_{\bar{i}}^z$).

4 Fermions

Next, we try to link spins with fermions via Jordan-Wigner transformation which exploted the Fermi's exclusion principle. Schematically, this can be done as following:

$$|\uparrow\rangle \;\Leftrightarrow\; n_f = 0\;, \qquad |\downarrow\rangle \;\Leftrightarrow\; n_f = 1$$

We can map the 'up' spin state to an empty fermion state and the 'down' spin state to a state occupied by a single fermion. This is illusrated in figure 1.

4.1 Jordan Wigner Transformation

Write in terms of Majorana fermions (a_j, b_j), $\bar{i} = i + \frac{1}{2}$,

$$a_j = \sigma_j^x\tau_{\bar{i}}^z = \sigma_j^x(\prod_{j>\bar{i}}\sigma_j^z), \tag{5a}$$

$$b_j = \sigma_j^y\tau_{\bar{i}}^z = \sigma_j^y(\prod_{j>\bar{i}}\sigma_j^z). \tag{5b}$$

Since they are Majorana fermions, they are real ($a_j = a_j^\dagger, b_j = b_j^\dagger$) and they follow the fermionic anticommutation relatrion ($\{a_i,a_j\} = 2\delta_{ij}, \{b_i,b_j\} = 2\delta_{ij}, \{a_i,b_j\} = 0$).

Write in terms of Dirac fermions $(c_j^\dagger, c_j)$,

$$c_j^\dagger = \frac{1}{2}(a_j + ib_j), \tag{6a}$$

$$c_j = \frac{1}{2}(a_j - ib_j). \tag{6b}$$

Here, $c_j^\dagger$ acts as a creation operator, creating a spinless fermion on site j. And they follow the fermionic anticommutation relatrion ($\{c_i^\dagger, c_j\} = \delta_{ij}$).

Expanding $a_i b_i$ in terms of Pauli matrices:

$$a_i b_i = (\sigma_i^x \tau_{\bar{i}}^z)(\sigma_i^y \tau_{\bar{i}}^z) = (\sigma_i^x \sigma_i^y)(\tau_{\bar{i}}^z)^2 = i\sigma_i^z.$$

Here we have used the fact that the indices of $\tau_{\bar{i}}^z$ and σ_i^y does not overlap and therefore commute. Also, we use the relations $(\tau_{\bar{i}}^z)^2 = 1$ and $\sigma^x \sigma^y = i\sigma^z$.

Expanding $a_i b_i$ in terms of Dirac fermions:

$$a_i b_i = (c_i^\dagger + c_i)\frac{(c_i^\dagger - c_i)}{i} = i(c_i^\dagger c_i - c_i c_i^\dagger) = i(2n_i - 1),$$

where the number operator $n_i = c_i^\dagger c_i$ and the fermionic relation of the Dirac fermions $\{c_i^\dagger, c_i\}$ is used.

Comparing the expression of $a_i b_i$ written in Pauli matrices and Dirac fermions, we get

$$\sigma_i^z = 2n_i - 1.$$

This relation link the spin in σ^z axis and the occupation of Dirac fermion state (up to sign difference).

Next, we try to write H_{TFIM} in terms of Majorana fermions and Dirac fermions. We do this by expanding the term $a_i b_{i+1}$, first in the Pauli matrices language,

$$a_i b_{i+1} = (\sigma_i^x \tau_{\bar{i}}^z)(\sigma_{i+1}^y \tau_{\overline{i+1}}^z) = \sigma_i^x (\sigma_{i+1}^z) \sigma_{i+1}^y = -i\sigma_i^x \sigma_{i+1}^x.$$

The seond equal sign used the fact that $\tau_{\overline{i+1}}^z$ has an extra σ_{i+1}^z comparing to $\tau_{\bar{i}}^z$ and the last equal sign used the relation $\sigma_{i+1}^z \sigma_{i+1}^y = -i\sigma_{i+1}^x$.

Expanding in terms of Dirac fermions,

$$a_i b_{i+1} = (c_i^\dagger + c_i)\frac{(c_{i+1}^\dagger - c_{i+1})}{i} = -i(c_i^\dagger c_{i+1}^\dagger - c_i^\dagger c_{i+1} + c_i c_{i+1}^\dagger - c_i c_{i+1})$$

Comparing the above two expressions, we get

$$\begin{aligned} \sigma_i^x \sigma_{i+1}^x &= c_i^\dagger c_{i+1}^\dagger - c_i^\dagger c_{i+1} + c_i c_{i+1}^\dagger - c_i c_{i+1} \\ &= c_{i+1}^\dagger c_i^\dagger + c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1} + c_i c_{i+1} \\ &= c_{i+1}^\dagger c_i^\dagger + c_{i+1}^\dagger c_i + h.c. \end{aligned}$$

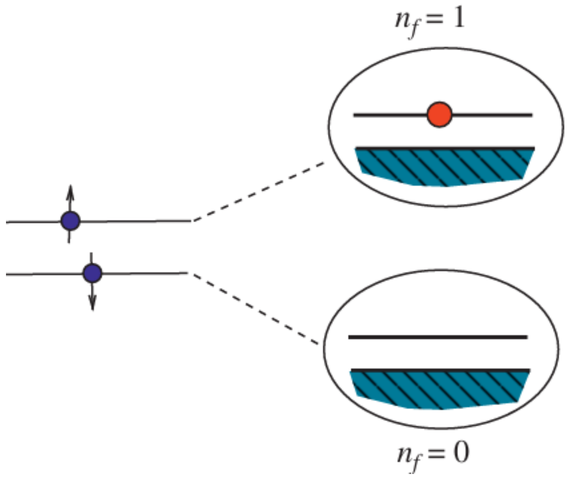


Figure 1: An illustraion of the mapping between spin state and the occupation of the fermion state (Reprinted from [**2**])

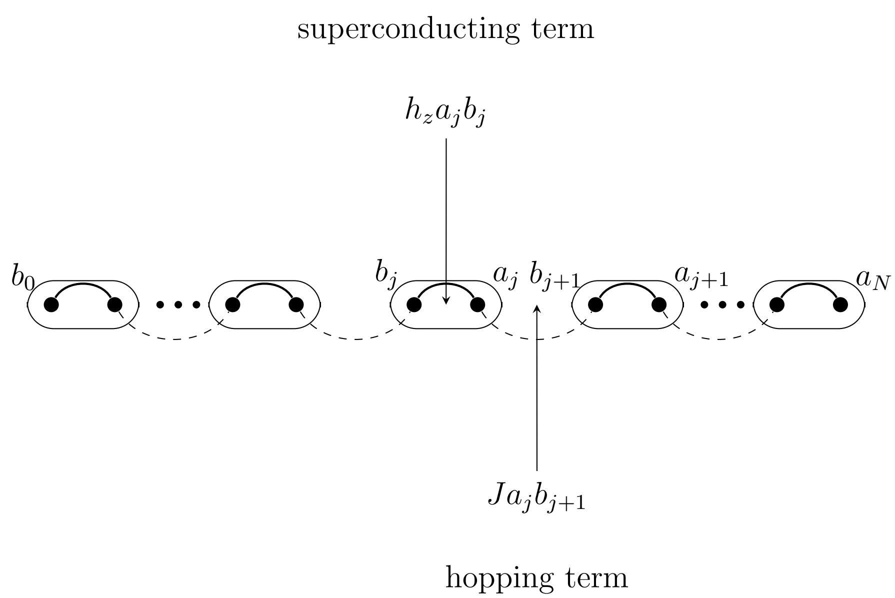
Now, we are ready to rewrite the hamiltonian interms of Majorana and Dirac fermions.

$$
\begin{aligned}
H_{TFIM} &= -J\sum_{i=1}^{N-1}\sigma_i^x\sigma_{i+1}^x - h_z\sum_{i=1}^{N}\sigma_i^z \\
&\Downarrow \text{ Majorana Fermions } \{\sigma_i^x, \sigma_i^z\} \rightarrow \{a_i, b_i\} \\
H_F &= -i(J\sum_{i=1}^{N-1}a_i b_{i+1} - h_z\sum_{i=1}^{N}a_i b_i) \\
&\Downarrow \text{ Dirac Fermions } \{a_i, b_i\} \rightarrow \{c_i^\dagger, c_i\} \\
H_F &= -J\sum_{i=1}^{N-1}(c_{i+1}^\dagger c_i^\dagger + c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1} + c_i c_{i+1}) - h_z\sum_{i=1}^{N}(2n_i - 1)
\end{aligned}
$$

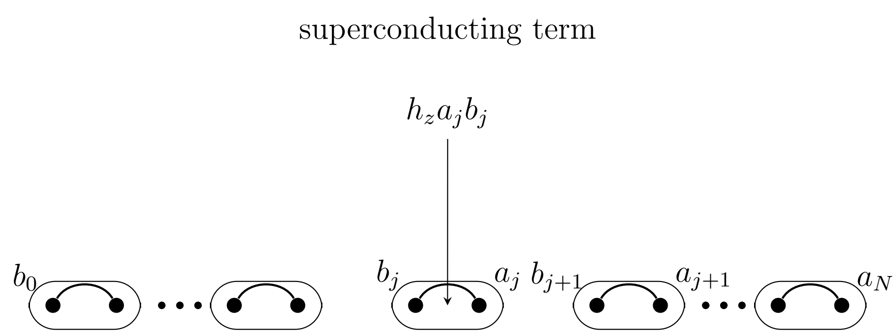
The Jordan Wigner Transformation transformed the spin-1/2 transverse Ising model into a spinless fermion model.

4.2 Majorana Chain

In the following illustrations, each dot corresponds to a Majorana fermion and a pair of the Majoranas (a_j, b_j) forms a box which corresponds to one Dirac fermion. Therefore, $2N$ real Majorana fermions correspond to N Dirac fermions.l



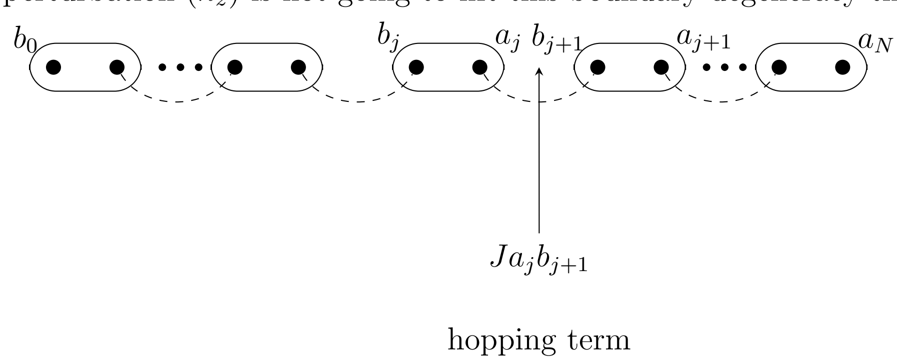
- $h_z >> 1$



- $h_z = 0$

  In this scenerio, the majoranas are disconnected and we are left with two zero modes on the edges ($[H, a_0] = [H, b_N] = 0$).

  And we are going to argue these two modes (gound state degeneracy) is more robust than the ground state degeneracy of the transverse Ising model. As small field perturbation (h_z) is not going to lift this boundary degeneracy this time.



4.3 Diagonalization

4.3.1 Fourier Transformation

$$c_k = \frac{1}{\sqrt{N}} \sum_j c_j e^{-ikj} \qquad , \qquad c_j = \frac{1}{\sqrt{N}} \sum_k c_k e^{ikj} \tag{7}$$

where the $c_0 = c_{2N}$.

$$H = -J \sum_{i=1}^{N-1} (c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i^\dagger + c_i c_{i+1}) - h_z \sum_{i=1}^{N} (2n_i - 1)$$

- $c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}$

$$\begin{aligned}
&= \frac{1}{N} \sum_j \sum_{k,k'} [c_k^\dagger c_{k'} e^{-ik(j+1)} e^{ik'j} \ + \ c_k^\dagger c_{k'} e^{ik(j+1)} e^{-ik'j}] \\
&= \frac{1}{N} \sum_j \sum_{k,k'} [e^{ij(-k+k')} (c_k^\dagger c_{k'} e^{-ik} \ + \ c_k^\dagger c_{k'} e^{ik})] \\
&= \sum_k c_k^\dagger c_k (e^{-ik} + e^{ik}) \\
&= \sum_k n_k (2\cos k)
\end{aligned}$$

In the third equal sign, $\frac{1}{N} \sum_{k,k'} e^{ij(-k+k')} = \delta_{k,k'}$ is used.

- $c_{i+1}^\dagger c_i^\dagger + c_i c_{i+1}$

$$\begin{aligned}
&= \frac{1}{N} \sum_j \sum_{k,k'} [c_k^\dagger c_{k'}^\dagger e^{-ik(j+1)} e^{-ik'j} \ + \ c_k c_{k'} e^{ik(j+1)} e^{ik'j}] \\
&= \frac{1}{N} \sum_j \sum_{k,k'} [e^{ij(k+k')} (c_k^\dagger c_{k'}^\dagger e^{-ik} \ + \ c_k c_{k'} e^{ik})] \\
&= \sum_k (c_k^\dagger c_{-k}^\dagger e^{-ik} \ + \ c_k c_{-k} e^{ik}) \\
&= \sum_k [c_k^\dagger c_{-k}^\dagger (\frac{e^{-ik} - e^{ik}}{2}) \ + \ c_k c_{-k} (\frac{e^{ik} - e^{-ik}}{2})] \\
&= \sum_k i \sin k (c_{-k}^\dagger c_k^\dagger + c_{-k} c_k)
\end{aligned}$$

In the third equal sign, $\frac{1}{N} \sum_{k,k'} e^{ij(k+k')} = \delta_{-k,k'}$ is used. For the fourth equal sign, from observing $c_k^\dagger c_{-k}^\dagger = -c_{-k}^\dagger c_k^\dagger$, we deduce e^{-ik} can be written as $\frac{e^{-ik} - e^{ik}}{2}$.

Combing the terms together, we get

$$H = -2J \sum_k (h_z + \cos k) n_k - i \sin k (c_{-k}^\dagger c_k^\dagger + c_{-k} c_k) - h_z N. \tag{8}$$

The second term of the hamiltonian mixes the k and $-k$ modes therefore breaking the $\mathbb{Z}_2$-symmetry of the fermionic parity operator.

4.3.2 Bogoliubov Transformation

Let $\Psi_k = \begin{pmatrix} c_{-k} \\ c_k^\dagger \end{pmatrix}$, and $\Psi_k^\dagger = \begin{pmatrix} c_{-k}^\dagger & c_k \end{pmatrix}^T$, then we can write the hamiltonian as follow

$$H_F = -J \sum_k \Psi_k^\dagger \begin{pmatrix} h_z + \cos k & i \sin k \\ i \sin k & -(h_z + \cos k) \end{pmatrix} \Psi_k - h_z N. \tag{9}$$

$\Psi_k = \begin{pmatrix} c_{-k} \\ c_k^\dagger \end{pmatrix} = [\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Psi_{-k}]^T = (\sigma^x \Psi_{-k})^T$, this shows Ψ_k and Ψ_{-k} are still not independent.

We try to find the eigenstate of H_F by using Bogoliubov transformation. Assume $\Psi_k = U\Omega_k$ where $\Omega_k = \begin{pmatrix} d_{-k} \\ d_k^\dagger \end{pmatrix}$. And U is an unitary transformation that satisfies

$$-U^\dagger J \begin{pmatrix} h_z + \cos k & i \sin k \\ i \sin k & -(h_z + \cos k) \end{pmatrix} U = \begin{pmatrix} \frac{\varepsilon_k}{2} & 0 \\ 0 & \frac{-\varepsilon_k}{2} \end{pmatrix}$$

where $\varepsilon_k = 2J\sqrt{(h_z + \cos k)^2 + \sin^2 k}$.

Then

$$\begin{aligned}
& - J \sum_k (\Omega_k^\dagger U^\dagger) \begin{pmatrix} h_z + \cos k & i \sin k \\ i \sin k & -(h_z + \cos k) \end{pmatrix} (U\Omega_k) \\
&= \sum_k \begin{pmatrix} d_{-k}^\dagger & d_k \end{pmatrix} \begin{pmatrix} \frac{\varepsilon_k}{2} & 0 \\ 0 & \frac{-\varepsilon_k}{2} \end{pmatrix} \begin{pmatrix} d_{-k} \\ d_k^\dagger \end{pmatrix} \\
&= \sum_k \begin{pmatrix} \frac{\varepsilon_k}{2} d_{-k}^\dagger & -\frac{\varepsilon_k}{2} d_k \end{pmatrix} \begin{pmatrix} d_{-k} \\ d_k^\dagger \end{pmatrix} \\
&= \sum_k (\frac{\varepsilon_k}{2} d_{-k}^\dagger d_{-k} - \frac{\varepsilon_k}{2} d_k d_k^\dagger) \\
&= \sum_k (\frac{\varepsilon_k}{2} d_{-k}^\dagger d_{-k} - \frac{\varepsilon_k}{2} (1 - d_k^\dagger d_k)) \\
&= \sum_k \varepsilon_k (d_k^\dagger d_k - \frac{1}{2})
\end{aligned}$$

Finally, we can write H_F in terms of the normal modes,

$$H_F = \sum_k \varepsilon_k (d_k^\dagger d_k - \frac{1}{2}) + h_z N. \tag{10}$$

Only the edge majorana modes contribute to H_F.

5 Kitaev Chain

In order to realize the model with physical electrons, we message the coefficient of the hamiltonian of spinless fermion model, and write down a spin-polarized 1-D superconductor hamiltonian.

- **Hopping amplitude** (w) :
  This is the prefactor of $\sum_{i=1}^{N-1}(c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1})$, where $J \to w$.
- **Superconducting gap** (Δ, Δ^*) :
  This is the prefactor of $\sum_{i=1}^{N-1}(c_{i+1}^\dagger c_i^\dagger + c_i c_{i+1})$, where $J \to \Delta, \Delta^*$.
- **Chemical potential** (μ) :
  This is the prefactor of $\sum_{i=1}^{N}(2n_i - 1)$, where $h_z \to -\mu/2$.

$$\begin{aligned}
H_F &= -J \sum_{i=1}^{N-1} \left(c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i^\dagger + c_i c_{i+1} \right) - h_z \sum_{i=1}^{N} (2n_i - 1) \\
&\Downarrow \quad (J \to w, \Delta, \Delta^* \quad , \qquad h_z \to -\mu/2) \\
H_F &= -w \sum_{i=1}^{N-1} \left(c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1} \right) + \sum_{i=1}^{N-1} \left(\Delta^* c_{i+1}^\dagger c_i^\dagger + \Delta c_i c_{i+1} \right) + \mu \sum_{i=1}^{N} (c_i^\dagger c_i - \frac{1}{2})
\end{aligned}$$

References

[1] Kitaev, A and Laumann, C, "Topological phases and quantum computation", arXiv: 0904.2771v1, 2009.

[2] Uchicago," Simple examples of second quantization", https://home.uchicago.edu/dtson/phys411/Jordan-Wigner.pdf, 2021.