Instanton Calculus and N = 2 Supersymmetry Yang-Mills Theory

TSZ YING, YEUNG

MASTERARBEIT IM STUDIENGANG MATHEMATICAL PHYSICS UNIVERSITAT HAMBURG

26 Nov 2024

1) **Introduction**

- Starting with the action of the SU(N) Yang-Mills theory in four-dimensional Euclidean space

$$S[A] = \frac{1}{2g^2}\int d^4x \; tr_N F_{mn}F_{mn} + i\theta k,$$

where $F_{mn} = \partial_m A_n - \partial_n A_m + [A_m, A_n]$ is the field strength, A_n are $N \times N$ anti-Hermitian gauge fields, and $m, n = 1, ..., 4$ are Lorentz indices.

Instanton Number

$$k = \frac{1}{16\pi^2}\int d^4x tr_N \overbrace{F_{mn} {}^*F_{mn}}^{\textit{Total derivative}}$$

- The dual is defined as ${}^*F_{mn} \equiv \frac{1}{2}\epsilon_{mnkl}F_{kl}$
- k counts the number of times S^3_∞ is winded around SU(N)
- k is also called the winding number

(Anti) Self-Dual Equation

Substitute the classical equation of motion ($D_n F^{mn} = 0$) for any given topological sector (k)

$$
\begin{aligned}
S &= \frac{1}{2g^2} \int d^4x \; tr F^2 \\
&= \frac{1}{4g^2} \int d^4x \; tr(F \mp {}^*F)^2 \; \mp \; \frac{1}{2g^2} \int d^4x \; F \, {}^*F \\
&\geq \mp \frac{1}{2g^2} \int d^4x \; tr F^* F = \frac{8\pi^2}{g^2}(\pm k)
\end{aligned}
$$

- Action is bounded below by $8\pi^2|k|/g^2$ which occurred when

$$F = \pm^* F$$

- (Anti) instantons are solutions to these first-order equations

2.1) **Zero-Modes**

Zero-modes are fluctuations (δA_n) of the solution that satisfy the following two criteria:

1. self-duality condition : $D_m \delta A_n - D_n \delta A_m = *(D_m \delta A_n - D_n \delta A_m)$
2. Orthogonal to gauge transformation $D_n \Omega$ (where Ω is an arbitrary function):
   $\int d^4x \; tr\{D_n \Omega \; \delta A_n\} = 0 \quad \Rightarrow \quad D_n \delta A_n = 0 \quad \Rightarrow \not{D}^{\dot{\alpha}\alpha} \delta A_n = 0$

- Combining these two conditions,

$$\bar{\not{D}}^{\dot{\alpha}\alpha} \delta A_{\alpha\dot{\beta}} = 0$$

- The covariant Weyl equation for $\delta A_{\alpha\dot{\beta}}$ in instanton background

Example: One instanton in SU(2)

$$A_n^{reg} = \frac{2\rho^2 (x-X)_m \sigma_{mn}}{(x-X)^2+\rho^2}$$

$$A_n^{sing} = \frac{2\rho^2 (x-X)_m \bar{\sigma}_{mn}}{(x-X)^2((x-X)^2+\rho^2)}$$

- four translations X_n
- scale size ρ
- stability group that embeds the instanton solution as an SU(2) gauge group

$$g_{\alpha\beta} \equiv \frac{1}{2g^2}\int d^4x Tr(\delta_\alpha A_n)(\delta_\beta A_n)$$

$$\delta_\alpha A_n = \frac{\partial A_n}{\partial X^\alpha} + D_n \Omega_\alpha$$

$\{X^\alpha,\ \alpha = 1,..,8\}$ as a collection of collective coordinates.

(1) Translational zero-mode : S_{inst}

(2) Scale zero-mode : $2S_{inst}$

(3) Gauge orientation zero-mode : $2S_{inst}\rho^2$

$$S_{inst} = 8\pi^2|k|/g^2$$

- gauge group ($\mathrm{SU}(2) \cong S^3$) from the gauge rotation

  $\mathbf{R}^+$ from scale transformation ρ^2

$+$

$$S^3 \times \mathbf{R}^+ \cong \mathbf{R}^4 \backslash \{0\}$$

- fields are invariant under the center $\mathbf{Z}_2 \subset \mathrm{SU}(2)$

⇒ gauge rotation gives $\mathbf{R}^4/\mathbf{Z}_2$ space

$$\mathfrak{M} = \mathbf{R}^4 \times \ (\mathbf{R}^4 \backslash \{0\})/\mathbf{Z}_2$$

1. **UV divergence**

   the $\mathbf{R}^4 \backslash \{0\}\ /\mathbf{Z}_2$ part has a singularity

2. **IR divergence**

   the $\mathbf{R}^4$ part is non-compact

Instanton number indices [k] : $1 \le i, j, l, ... \le k$,

Colour indices [N]: $1 \le u, v, ... \le N$,

ADHM indices [N+2k]: $1 \le \lambda, \mu, ... \le N + 2k$,

Quaternionic (Weyl) indices [2]: $\alpha, \beta, \dot{\alpha}, \dot{\beta}, ... = 1, 2$,

Lorentz indices [4]: m,n,... = 1,2,3,4.

2.2) ADHM Construction

- The ADHM scheme is derived to solve the self-dual equation $F_{mn} = {}^*F_{mn}$
- A $(N+2k) \times 2k$ complex-valued matrix Δ which is linear in spacetime variable x_m

$$\Delta_{[N+2k]\times[2k]}(x) \equiv \Delta_{[N+2k]\times[k]\times[2]}(x) = a_{[N+2k]\times[k]\times[2]} + b_{[N+2k]\times[k]\times\underline{[2]}} x_{[2]\times\underline{[2]}}$$

- By definition, the conjugate of Δ is defined by

$$\bar{\Delta}_{[2k]\times\underline{[N+2k]}} \equiv \left(\Delta_{[N+2k]\times[2k]}\right)^*$$

- Writing in indicies, the above matrices can be written as

$$\Delta_{\lambda i\dot{\alpha}}(x) = a_{\lambda i\dot{\alpha}} + b^{\alpha}_{\lambda i} x_{\alpha\dot{\alpha}} \qquad \bar{\Delta}^{\dot{\alpha}\lambda}_{i}(x) = \bar{a}^{\dot{\alpha}\lambda}_{i} + \bar{x}^{\dot{\alpha}\alpha} \bar{b}^{\lambda}_{\alpha i}$$

- Null-space condition

$$\bar{\Delta}_{[2k]\times\underline{[N+2k]}} U_{\underline{[N+2k]}\times[N]} = 0 = \bar{U}_{[N]\times\underline{[N+2k]}}, \Delta_{\underline{[N+2k]}\times[2k]}$$

- U is orthonormalized according to

$$\bar{U}_{[N]\times\underline{[N+2k]}} U_{\underline{[N+2k]}\times[N]} = 1_{[N]\times[N]}$$

ADHM Ansatz

Instanton number indices [k] : $1 \le i, j, l, ... \le k$,

Colour indices [N]: $1 \le u, v, ... \le N$,

ADHM indices [N+2k]: $1 \le \lambda, \mu, ... \le N + 2k$,

Quaternionic (Weyl) indices [2]: $\alpha, \beta, \dot{\alpha}, \dot{\beta}, ... = 1, 2$,

Lorentz indices [4]: m,n,... = 1,2,3,4.

ADHM ansatz argues that pure gauge will continue to solve the self-duality equation in nonzero k sectors

$$A_{n[N]\times[N]} = \bar{U}_{[N]\times\underline{[N+2k]}}\partial_n U_{\underline{[N+2k]}\times[N]}$$

1. Factorization condition

$$\bar{\Delta}_{[2]\times[k]\times\underline{[N+2k]}}\Delta_{\underline{[N+2k]}\times[k]\times[2]} = 1_{[2]\times[2]}f^{-1}_{[k]\times[k]}$$

where f is an arbitrary x-dependent $k \times k$ dimensional Hermitian matrix. Writing in indices,

$$\bar{\Delta}_i^{\dot{\beta}\lambda}\Delta_{\lambda i\dot{\alpha}} = \delta^{\dot{\beta}}_{\dot{\alpha}}(f^{-1})_{ij}$$

2. Completeness relation
combination of factorization condition and null-space condition

$$\begin{aligned}\mathcal{P} &\equiv U_{[N+2k]\times\underline{[N]}}\bar{U}_{\underline{[N]}\times[N+2k]} \\ &= 1_{[N+2k]\times[N+2k]} - \Delta_{[N+2k]\times\underline{[k]}\times\underline{[2]}}f_{\underline{[k]}\times\underline{[k]}}\bar{\Delta}_{\underline{[2]}\times\underline{[k]}\times[N+2k]}\end{aligned}$$

where $\mathcal{P}$ is actually a projection operator

Instanton number indices [k] : $1 \leq i, j, l, \ldots \leq k$,

Colour indices [N]: $1 \leq u, v, \ldots \leq N$,

ADHM indices [N+2k]: $1 \leq \lambda, \mu, \ldots \leq N + 2k$,

Quaternionic (Weyl) indices [2]: $\alpha, \beta, \dot\alpha, \dot\beta, \ldots = 1, 2$,

Lorentz indices [4]: m,n,... $=$ 1,2,3,4.

ADHM Constraints

$$\bar{a}_i^{\dot\alpha\lambda} a_{\lambda j \dot\beta} = (\frac{1}{2} a\bar{a})_{ij} \delta^{\dot\beta}_{\dot\alpha} \quad \propto \quad \delta^{\dot\beta}_{\dot\alpha},$$

$$\bar{a}_i^{\dot\alpha\lambda} b^{\beta}_{\lambda j} = \bar{b}_i^{\beta\lambda} a^{\dot\alpha}_{\lambda j},$$

$$\bar{b}^{\lambda}_{\beta j i} b^{\beta}_{\lambda j} = (\frac{1}{2} b\bar{b})_{ij} \delta^{\beta}_{\alpha} \quad \propto \quad \delta^{\beta}_{\alpha}.$$

a and b comprise the collective coordinate which gives $4k(N+2k) = \boxed{4Nk} + \boxed{8k^2}$ of real degree of freedom

Index Theorem

Redundant degree of freedom

Instanton number indices [k] : $1 \leq i, j, l, \ldots \leq k$,

Colour indices [N]: $1 \leq u, v, \ldots \leq N$,

ADHM indices [N+2k]: $1 \leq \lambda, \mu, \ldots \leq N + 2k$,

Quaternionic (Weyl) indices [2]: $\alpha, \beta, \dot{\alpha}, \dot{\beta}, \ldots = 1, 2$,

Lorentz indices [4]: m,n,... = 1,2,3,4.

$$\Delta_{[N+2k]\times[k]\times[2]} \to \Lambda_{[N+2k]\times\underline{[N+2k]}}\Delta_{\underline{[N+2k]}\times\underline{[k]}\times[2]}B^{-1}_{\underline{[k]}\times[k]}$$

$$U_{[N+2k]\times[N]} \to \Lambda_{[N+2k]\times\underline{[N+2k]}}U_{\underline{[N+2k]}\times[N]}$$

$$f_{[k]\times[k]} \to B_{[k]\times\underline{[k]}}f_{\underline{[k]}\times\underline{[k]}}B^{\dagger}_{\underline{[k]}\times[k]}$$

$$\Lambda \in U(N+2k) \text{ and } B \in Gl(k, \mathbf{C})$$

b assumes canonical form

$$b_{[N+2k]\times[2k]} = \begin{pmatrix} 0_{[N]\times[2k]} \\ 1_{[2k]\times[2k]} \end{pmatrix}$$

$$a_{[N+2k]\times[2k]} = \begin{pmatrix} \omega_{[N]\times[2k]} \\ a'_{[2k]\times[2k]} \end{pmatrix}$$

$$\lambda \in [N+2k] \to \lambda = u + i\alpha$$

$$1 \leq u \leq N$$
$$1 \leq i \leq k, \quad \alpha = 1, 2$$

$$a_{\lambda j\dot{\alpha}} = a_{(u+i\alpha)j\dot{\alpha}} = \begin{pmatrix} \omega_{uj\dot{\alpha}} \\ (a'_{\alpha\dot{\alpha}})_{ij} \end{pmatrix},$$

$$\bar{a}^{\dot{\alpha}\lambda}_{j} = \bar{a}^{\dot{\alpha}(u+i\alpha)}_{j} = \begin{pmatrix} \bar{\omega}^{\dot{\alpha}}_{ju} & (\bar{a}'_{\alpha\dot{\alpha}})_{ij} \end{pmatrix},$$

$$b^{\beta}_{\lambda j} = b^{\beta}_{(u+i\alpha)j} = \begin{pmatrix} 0 \\ \delta^{\beta}_{\alpha}\delta_{ij} \end{pmatrix},$$

$$\bar{b}^{\lambda}_{\beta j} = \bar{b}^{(u+i\alpha)}_{\beta j} = \begin{pmatrix} 0 & \delta^{\beta}_{\alpha}\delta_{ij} \end{pmatrix}.$$

Instanton number indices [k] : $1 \le i, j, l, \ldots \le k$,

Colour indices [N]: $1 \le u, v, \ldots \le N$,

ADHM indices [N+2k]: $1 \le \lambda, \mu, \ldots \le N + 2k$,

Quaternionic (Weyl) indices [2]: $\alpha, \beta, \dot{\alpha}, \dot{\beta}, \ldots = 1, 2$,

Lorentz indices [4]: m,n,... = 1,2,3,4.

ADHM Constraints

$3k^2$

$$tr_2(\tau^c \bar{a} a)_{ij} = 0,$$

$$(a'^m)^\dagger_{ij} = a'^m_{ij}.$$

$4k^2$

't Hooft symbols η^c, $c = 1, 2, 3$

Lorentz spinors $\sigma_{mn} \equiv i\bar{\eta}^c_{mn}\tau_c$

$$\bar{\omega}\tau^c\omega - i\bar{\eta}^c_{mn}[a'_m, a'_n] = 0.$$

$\cancel{k^2}$

U(k) residual symmetry

Hyper – Kähler Qotient

The construction can be done in 2 steps. Starting from the mother space $\tilde{\mathfrak{M}} = \mathbf{R}^{4k(N+k)}$:

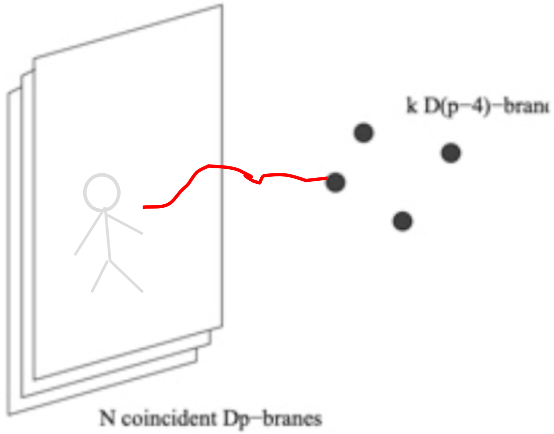
1. restrict to the level set $\mathfrak{N} \subset \tilde{\mathfrak{M}}$, space of all solutions to ADHM constraints

2. ordinary quotient of $\mathfrak{N}$ by residual U(k) action

$$\mathfrak{M}_k = T\mathfrak{N}/\mathrm{U}(k).$$

2.3) ADHM Construction in String Theory Setting

p=3



32 supercharges in 10-dimension

D-1 brane imposes boundary condition

16 supercharges in 0-dimension

dimension reduction

$\mathcal{N}$=4 SYM theory in 3+1 dimension

D3 brane point of view

+ *U(k) symmetry among the D-1 branes*

$\mathcal{N}$=4 SYM theory
in 3+1 dimension
with a U(k) gauge symmetry

Lorentz symmetry in 3+1 spacetime

r-symmetry of the D3 brane rotating 16 supercharges

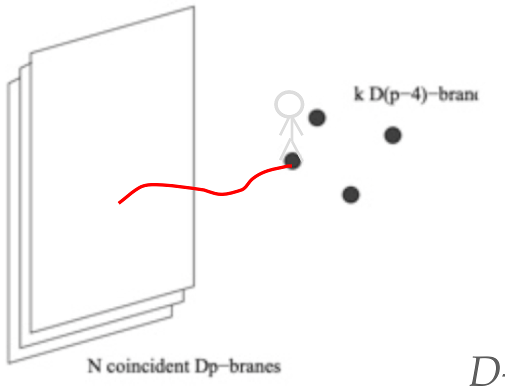
bosons

fermions

	$SU(2)_\alpha$	$SU(2)_{\dot\alpha}$	$SU(4)_R$	$U(k)$	$U(N)$
$a'_{\alpha\dot\alpha}$	2	2	1	*adj*	1
χ^a	1	1	6	*adj*	1
$\mathcal{M}'^{\alpha\dot\alpha}$	2	1	4	*adj*	1
λ^a	1	2	4	*adj*	1

$\alpha,\dot\alpha = 1,2;\quad \mathrm{a} = 1,\ldots 6$

Table 1: Field contents in point of view of D3 branes.



p=3

16 supercharges
in 0-dimension

D3 brane imposes boundary condition

8 supercharges
in 0-dimension

dimension reduction

$\mathcal{N}$=2 hypermultiplet
in 3+1 dimension

D-1 brane point of view

+ *U(N) symmetry among the D3 branes*

+ *U(k) symmetry among the D-1 branes*

	$SU(2)_\alpha$	$SU(2)_{\dot\alpha}$	$SU(2)_R$	$U(k)$	$U(N)$
$\omega_{\dot\alpha}$	1	2	1	$\bar{k}$	N
$\tilde\omega_\alpha$	1	2	1	k	$\bar{N}$
μ^A	1	1	2	$\bar{k}$	N
$\tilde\mu^A$	1	1	2	k	$\bar{N}$

A=1,2

D3 to D-1

D-1 to D3

Table 2: Field contents in point of view of D-1 branes.

$$\mathrm{SU}(4)_R \cong \cancel{\mathrm{SU}(2)_L} \times \mathrm{SU}(2)_R$$

addition of N D3 branes

$\boxed{\begin{array}{l}\mathcal{N}\text{=2 hypermultiplet}\\ \text{in 3+1 dimension}\\ \text{with a U}(k)\text{ gauge symmetry}\\ \text{and a U}(N)\text{ flavor symmetry}\end{array}}$

- from the point of view of D-1 branes, χ^a=0 as the instantons lie in the D3 brane

- we require the scaler potential $V = 0$

"Higgs Branch" of D-1 theory

$$\mathfrak{M}_{Higgs} \cong (\chi^a = 0, V = 0)/U(k)$$

k^2

$$\mathrm{D-term:}\ g^2 Tr(\sum_{n=1}^{N} \omega\omega^\dagger - \tilde{\omega}^\dagger\tilde{\omega} + [Z, Z^+] + [W, W^+])^2,$$

k^2

$$\mathrm{F-term:}\ g^2 Tr|\sum_{n=1}^{N} \omega\tilde{\omega} + [Z, W]|^2,$$

$2k^2$

where $Z = a'_1 + ia'_2$ and $W = a'_3 - ia'_4$.

Bosonic sector:

ω and $\tilde{\omega}$ give $4kN$ real degrees of freedom

Z and W give $4k^2$ real degrees of freedom

D and F-flatness conditions

$$\mathfrak{M}_{Higgs} \cong \mathfrak{M}_k$$

$$\boxed{\sum_{n=1}^{N} \omega_n^\dagger \sigma^i \omega_n - i[a'_\mu, a'_\mu]\bar{\eta}^i_{\mu\nu} = 0}$$

ADHM constraints

$$\boxed{\dim(\mathfrak{M}_{Higgs}) = 4kN + 4k^2 - k^2 - 2k^2 - k^2 = 4kN}$$

3) Regularization of the Moduli Space

Non-Commutative Spacetime

$$[x^m, x^n] = i\theta^{mn}$$

$$\theta^{mn} = \left(\begin{array}{cc|cc} 0 & -\theta_1 & 0 & 0 \\ \theta_1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -\theta_2 \\ 0 & 0 & \theta_2 & 0 \end{array}\right)$$

$$\xrightarrow[z_2 = x^4 + ix^3]{z_1 = x^2 + ix^1}$$

$$[z_1, \bar{z}_1] = 2\theta_1 \quad , \quad [z_2, \bar{z}_2] = 2\theta_2.$$

Rewrite the ADHM Data

$$\Delta(x) = a + b(x) = \begin{pmatrix} \bar{I} & J \\ \bar{z}_2 - \bar{B}_2 & -(z_1 - B_1) \\ \bar{z}_1 - \bar{B}_1 & z_2 - B_2 \end{pmatrix}$$

$$a = \begin{pmatrix} \bar{I} & J \\ \bar{B}_2 & -B_1 \\ \bar{B}_1 & B_2 \end{pmatrix}$$

$$b = \begin{pmatrix} 0 & 0 \\ \mathbf{I}_k & 0 \\ 0 & \mathbf{I}_k \end{pmatrix}$$

$B_{1,2}$ are k x k matrices

I is N x k matrices

J is k x N matrices

ADHM Constraints

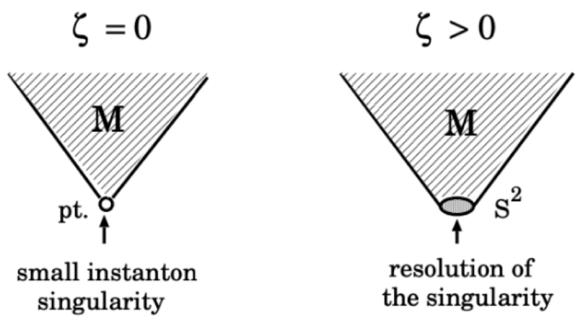
$$\mu_{\mathbb{R}} \equiv [B_1, \bar{B}_1] + [B_2, \bar{B}_2] + I\bar{I} - \bar{J}J = \zeta,$$

$$\mu_{\mathbb{C}} \equiv [B_1, B_2] + IJ = 0.$$

$$\bar{\omega}\tau^c\omega - i\bar{\eta}^c_{mn}[a'_m, a'_n] = \zeta^c$$

$$\boxed{\zeta \equiv -[z_1, \bar{z}_1] - [z_2, \bar{z}_2] = -2(\theta_1 + \theta_2)}$$

Fayet-Iliopoulous (FI) parameter



Example: $U(2)$ 1-instanton in Non-Commutative $\mathbb{R}^4$

$$B_1 = \alpha_1 \ , \quad B_2 = \alpha_2 \ , \quad I = (\sqrt{\rho^2+\zeta}\ , 0) \ , \quad J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

$\downarrow \ \rho \to 0$

$$B_1 = \alpha_1 \ , \quad B_2 = \alpha_2 \quad I = (\sqrt{\zeta}\ , 0) \ , \quad J = 0$$

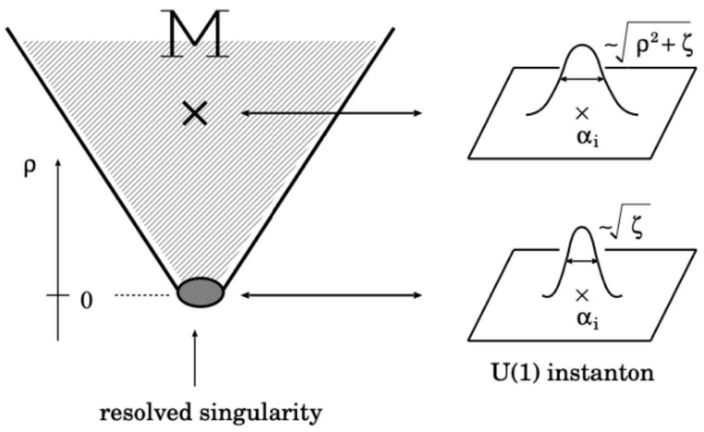
$\downarrow$

$$\mu_{\mathbb{R}} \equiv [B_1, \bar{B}_1] + [B_2, \bar{B}_2] + I\bar{I} - \bar{J}J = \zeta,$$

$$\mu_{\mathbb{C}} \equiv [B_1, B_2] + IJ = 0.$$

$$I\bar{I} - \bar{J}J = \zeta > 0,$$

$$IJ = 0.$$



$\Rightarrow$ scale size is cut at $\rho^2 = 2\zeta$

4.1) Gaugino Zero-modes

$\lambda_\alpha(\mathcal{M})$ solves the covariant Weyl equation $(\bar{D}^{\dot{\alpha}\alpha}\lambda_\alpha(\mathcal{M}) = 0)$

$$\boxed{\lambda_\alpha(\mathcal{M}) \equiv \bar{U}\mathcal{M}f\bar{b}_\alpha U - \bar{U}b_\alpha f\bar{\mathcal{M}}U}$$

where $\mathcal{M}$ is a $(N+2k)\times k$ matrix in Grassmann collective coordinate.

"spin 1" ADHM constraints:

$$\bar{\mathcal{M}}_i^\lambda a_{\lambda i\dot{\alpha}} = -\bar{a}_{i\dot{\alpha}}^\lambda \mathcal{M}_{\lambda j},$$
$$\bar{\mathcal{M}}_i^\lambda b_{\lambda i}^\alpha = \bar{b}_i^{\alpha\lambda}\mathcal{M}_{\lambda j}.$$

$$\mathcal{M}_{\lambda i} \equiv \mathcal{M}_{(u+l\beta)i} = \begin{pmatrix} \mu_{ui} \\ (\mathcal{M}'_\beta)_{li} \end{pmatrix},$$

$$\bar{\mathcal{M}}_i^\lambda \equiv \bar{\mathcal{M}}_i^{(u+l\beta)} = (\bar{\mu}_{iu}\ ,\ (\mathcal{M}'^\beta)_{il}).$$

$$\bar{\mathcal{M}}'^\alpha = \mathcal{M}'^\alpha.$$

$$\bar{\mathcal{M}}a_{\dot{\alpha}} + \bar{a}_{\dot{\alpha}}\mathcal{M} \equiv \bar{\mu}\omega_{\dot{\alpha}} + \bar{\omega}_{\dot{\alpha}}\mu + [\mathcal{M}'^\alpha, a'_{\alpha\dot{\alpha}}] = 0.$$

4.2) Higgs Boson Zero-modes

Scalar non-zero vacuum expectation value (VEVs)

Yang-Mills theory is asymptotically free $\rightarrow$ instanton(semi-classical saddle point of pure Yang-Mills action) only works in a weakly-coupled phase (where VEVs $\gg$ dynamical scale Λ)

Writing the VEVs (v) more explicitly

$$v = \mathrm{diag}(v_1, v_2, ..., v_N),$$

where $\sum_{a=1}^{N} v_a = 0$.

- pure $\mathcal{N} = 2$ theories are strongly coupled in the IR
- Higgs mechanism which breaks the gauge symmetry to an abelian subgroup, yielding a Coulomb phase

$\Downarrow$

choose VEVs $|v_a - v_b| \gg \Lambda$ for all $a, b \in N$

solution of covariant Laplace equation with bi-fermion source:

$$D^2\Phi = \lambda(\mathcal{M}^A)\lambda(\mathcal{M}^B)$$

with the boundary condition $\lim_{x\to\infty}\Phi(x) = v$

$$\Phi = -\frac{1}{4}\bar{U}\mathcal{M}^A f \bar{\mathcal{M}}^B U + \bar{U}\mathcal{A}U,$$

where

$$\mathcal{A} = \begin{pmatrix} v & 0 & . & . & . & 0 \\ . & & & & & \\ . & & & \mathcal{A}' & & \\ . & & & & & \\ 0 & & & & & \end{pmatrix},$$

$$\mathcal{A}' = \mathbf{L}^{-1}(\bar{\omega}^{\dot{\alpha}} v\ \omega_{\dot{\alpha}}) + \frac{1}{4}\bar{\mathcal{M}}^A\mathcal{M}^B,$$

is a $k\times k$ matrix.

"spin 0"constraint:

$$\mathbf{L}\cdot\mathcal{A}' = (\bar{\omega}^{\dot{\alpha}}\ v\ \ \omega_{\dot{\alpha}}) + \frac{1}{4}\bar{\mathcal{M}}^A\mathcal{M}^B.$$

$\mathbf{L}$ is an operator on $k\times k$ Hermitian matrix of the form,

$$\begin{aligned}\mathbf{L}\cdot\Omega &\equiv \frac{1}{2}\{\bar{\omega}^{\dot{\alpha}}\omega_{\dot{\alpha}},\Omega\} + \frac{1}{2}\bar{a}'^{\dot{\alpha}\alpha}a'_{\dot{\alpha}\alpha} - \bar{a}'^{\dot{\alpha}\alpha}\Omega a'_{\dot{\alpha}\alpha} + \frac{1}{2}\Omega\bar{a}'^{\dot{\alpha}\alpha}a'_{\dot{\alpha}\alpha} \\ &= \frac{1}{2}\{\bar{\omega}^{\dot{\alpha}}\omega_{\dot{\alpha}},\Omega\} + [a'_n,[a'_n,\Omega]].\end{aligned}$$

5) BRST Algebra of Instanton Moduli Space

$$(s\bar{\Delta})\Delta + \bar{\Delta}(s\Delta) = [(s\bar{\Delta})\Delta]^T + [\bar{\Delta}(s\Delta)]^T$$

infinitesimal scaler variation(s) of the bosonic ADHM constraints
$(\bar{\Delta}\Delta = f^{-1} = (\bar{\Delta}\Delta)^T)$,

- "spin 1" constraints
- "spin 1/2" constraints
- "spin 0" constraints
- requires s to be nilpotent

$$\begin{aligned} s\Delta &= \mathcal{M} - \mathcal{C}\Delta, \\ s\mathcal{M} &= \mathcal{A}\Delta - \mathcal{C}\mathcal{M}, \\ s\mathcal{A} &= -[\mathcal{C}, \mathcal{A}], \\ s\mathcal{C} &= \mathcal{A} - \mathcal{C}\mathcal{C}. \end{aligned}$$

BRST Algebra

covariant derivative of the moduli space

$Q = s + \mathcal{C}$

$$\begin{aligned} Q\Delta &= \mathcal{M}, \\ Q\mathcal{M} &= \mathcal{A}\Delta, \\ Q\mathcal{A} &= 0, \\ Q\mathcal{C} &= \mathcal{A}. \end{aligned}$$

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_{00} & 0 & . & . & . & 0 \\ . & & & & & \\ . & & & \mathcal{C}' & & \\ . & & & & & \\ 0 & & & & & \end{pmatrix}$$

Q : supersymmetry operator for the ADHM variables in the moduli space
$\mathcal{C}$: $U(k)$-connection of the covariant derivative
$\mathcal{A}$: field strength or curvature of the connection $\mathcal{C}$

where $\mathcal{C}'$ is a real antisymmetric $k \times k$ matrix, $\bar{\mathcal{C}}' = -\mathcal{C}'$

6) $\mathcal{N}=2$ Multi-Instanton Action

Topological Twist

The classical global symmetry group of the $\mathcal{N}=2$ SUSY theory in flat space is

$$\underbrace{\mathrm{SU}(2)_L}_{\text{Lorentz group}} \times \mathrm{SU}(2)_R \times \mathrm{SU}(2)_A$$

automorphism of supercharge

diagonal $\mathrm{SU}(2)'_R \subset \mathrm{SU}(2)_R \times \mathrm{SU}(2)_A$

$$\boxed{\mathrm{SU}(2)_L \times \mathrm{SU}(2)'_R}$$

$\mathcal{N}=2$ SUSY charges

scalar

antisymmetric tensor

$$\bar{Q}_{\dot{\alpha}I} \to Q \oplus Q_{mn}$$

$$Q_{\alpha I} \to Q_n$$

vector

$I = 1,2$ are indices of $SU(2)_A$

$s \equiv s_g + Q.$

field content of $\mathcal{N}=2$ $(A_n, \lambda_\alpha, \phi)$

$$s_g A = -Dc,$$
$$s_g \lambda = -c\lambda,$$
$$s_g \phi = -c\phi,$$
$$s_g c = -\frac{1}{2}cc.$$

$$sA = \lambda - Dc,$$
$$s\lambda = -c\lambda - D\phi,$$
$$s\phi = -c\phi,$$
$$sc = -\frac{1}{2}cc + \phi.$$

Auxiliary Fields of the ADHM Constraints

$$\bar{\omega}\tau^c\omega - i\bar{\eta}^c_{mn}[a'_m, a'_n] = \zeta^c$$
$$\bar{\omega}^{\dot{\alpha}}\mu_{\dot{\alpha}} - \bar{\mu}^{\dot{\alpha}}\omega_{\dot{\alpha}} - 2[a'_n, \mathcal{M}'_n] = 0$$
$$\bar{\omega}^{\dot{\alpha}}\tau^c\mu_{\dot{\alpha}} + \bar{\mu}^{\dot{\alpha}}\tau^c\omega_{\dot{\alpha}} - 2i\bar{\eta}^c_{mn}[a'_n, \mathcal{M}'_n] = 0$$

ADHM Constraints after deformed by non-commutative spacetime and topological twisting

D_c : auxiliary field of the bosonic constraints
ψ_c : superpartner of D_c
ϕ : generator of $U(k)$ residual group
η : superpartner of ϕ (superconformal invariance broken by the presence of VEV)

$$\begin{aligned} &Q\psi_c = D_c, \quad && QD_c = [\phi, \psi_c], \\ &Q\bar{\phi} = \eta, && Q\eta = [\phi, \bar{\phi}], \\ &Q\phi = 0, && Qv = Q\bar{v} = 0, \\ &Q\zeta^c = 0. && \end{aligned}$$

Cohomological action of the ADHM Constraints

(1) The ADHM constraints

(2) Auxiliary fields associating the the constraints

(3) Turning on the VEVs v

$$S = tr_k Q[(\bar{\phi}+\bar{v})(\bar{\omega}^{\dot{\alpha}}\mu_{\dot{\alpha}} - \bar{\mu}^{\dot{\alpha}}\omega_{\dot{\alpha}} - 2[a'_n, \mathcal{M}'_n]) + \frac{1}{g_0^2}\eta[\phi,\bar{\phi}]$$
$$+\psi_c(\bar{\omega}\tau^c\omega - i\bar{\eta}^c_{mn}[a'_m, a'_n] - \zeta^c) - \frac{1}{g_0^2}\psi_c D_c]$$

$$\begin{aligned} Qa'_n &= \mathcal{M}'_n, & Q\mathcal{M}'_n &= [\phi, a'_n],\\ Q\omega_{\dot{\alpha}} &= \mu_{\dot{\alpha}}, & Q\mu_{\dot{\alpha}} &= -\omega_{\dot{\alpha}}(\phi+v),\\ Q\bar{\omega}^{\dot{\alpha}} &= \bar{\mu}^{\dot{\alpha}}, & Q\bar{\mu}^{\dot{\alpha}} &= (\phi+v)\bar{\omega}^{\dot{\alpha}}. \end{aligned}$$

$$\xrightarrow[(\mathcal{M}_1, \mathcal{M}_2, \mu_I, \mu_J)]{(B_1, B_2, I, J)}$$

$$\begin{aligned} QI &= \mu_I, & Q\mu_I &= \phi I - Iv,\\ QJ &= \mu_J, & Q\mu_J &= -\phi J + vJ,\\ QB_1 &= \mathcal{M}_1, & Q\mathcal{M}_1 &= [\phi, B_1],\\ QB_2 &= \mathcal{M}_2, & Q\mathcal{M}_2 &= [\phi, B_2], \end{aligned}$$

$$S = tr_k Q[(\bar{\phi} + \bar{v})(\bar{\omega}^{\dot{\alpha}}\mu_{\dot{\alpha}} - \bar{\mu}^{\dot{\alpha}}\omega_{\dot{\alpha}} - 2[a'_n, \mathcal{M}'_n]) + \frac{1}{g_0^2}\eta[\phi, \bar{\phi}]$$
$$+\psi_c(\bar{\omega}\tau^c\omega - i\bar{\eta}^c_{mn}[a'_m, a'_n] - \zeta^c) - \frac{1}{g_0^2}\psi_c D_c]$$

$$\begin{aligned} Q\vec{\chi} &= \vec{H}, & Q\vec{H} &= [\phi, \vec{\chi}], \\ Q\bar{\phi} &= \eta, & Q\eta &= [\phi, \bar{\phi}], \\ Q\phi &= 0, & Qv &= Q\bar{v} = 0. \end{aligned}$$

$\psi_c \to \vec{\chi} = (\chi_{\mathbb{R}}, \chi_{\mathbb{C}})$
$D_c \to \vec{H} = (H_{\mathbb{R}}, H_{\mathbb{C}})$
$\mathcal{M}_1, \mathcal{M}_2$ and $B_1, B_2 \to \mathcal{M}_{\hat{l}}, B_{\hat{l}}$ with $\hat{l} = 1, 2$

$$\begin{aligned} \boldsymbol{\varepsilon}_{\mathbb{R}} &\equiv [B_1, \bar{B}_1] + [B_2, \bar{B}_2] + I\bar{I} - \bar{J}J - \zeta = 0, \\ \boldsymbol{\varepsilon}_{\mathbb{C}} &\equiv [B_1, B_2] + IJ = 0. \end{aligned}$$

$$S = tr_k Q\{[\mu_I(I^\dagger\bar{\phi} - \bar{v}I^\dagger) + \mu_J(\bar{\phi}J^\dagger - J^\dagger\bar{v}) + \mathcal{M}_{\hat{l}}[\bar{\phi}, B^\dagger_{\hat{l}}]] + h.c.$$
$$+ \chi_{\mathbb{R}}\boldsymbol{\varepsilon}_{\mathbb{R}} + \chi_{\mathbb{C}}\boldsymbol{\varepsilon}_{\mathbb{C}} + \frac{1}{g_0^2}(\eta[\phi, \bar{\phi}] + \vec{H}\cdot\vec{\chi})\},$$

$\Psi_{\hat{s}} \equiv (\mu_I, \mu_J^\dagger, \mathcal{M}_{\hat{l}})$

$B_{\hat{s}} \equiv (I, J^\dagger, B_{\hat{l}})$

$\hat{s} = 1, .., 4.$

$$\boxed{S = tr_k Q\{\frac{1}{g_0^2}(\eta[\phi, \bar{\phi}] + \vec{H}\cdot\vec{\chi}) - i\vec{\boldsymbol{\varepsilon}}\cdot\vec{\chi} - \frac{1}{2}\sum_{\hat{s}=1}^{4}[\Psi^\dagger_{\hat{s}}(\bar{\phi})\cdot B_{\hat{s}} + \Psi_{\hat{s}}(\bar{\phi})\cdot B^\dagger_{\hat{s}}]\}}$$

Localization

Localization Argument

Q be a fermionic symmetry generator

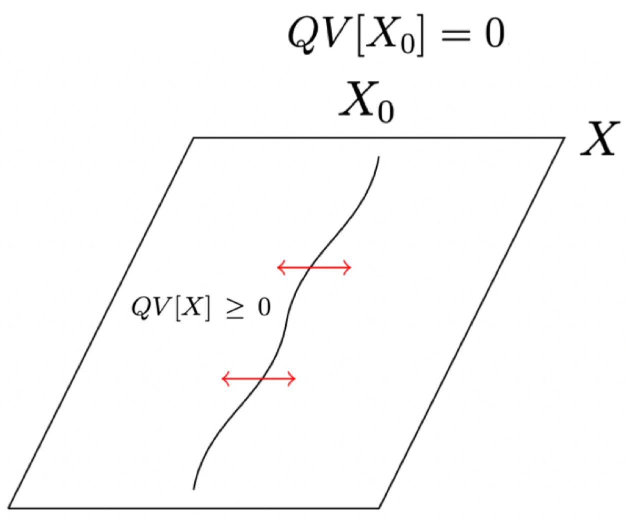
$Q^2 = 0$

$Q^2 = B$ where B is the bosonic symmetry generator

$QS = 0$

Q-variation of a B-invariant fermionic functional $(tQV[X])$

$BV \equiv Q^2 V = 0$



$$\boxed{Z \equiv \int_{\mathcal{M}} \mathcal{D}X e^{-S[X]-tQV[X]}}$$

$$\frac{\partial Z}{\partial t} = 0$$

This path integral is localized

Taylor expand around X_0

$$X = X_0 + \frac{1}{\sqrt{t}}\delta X$$

$$S[X] + tQV[X] = S[X_0] + \frac{1}{2}\iint \frac{\delta^2(QV[X])}{\delta X^2}\Big|_{X=X_0}(\delta X)^2$$

- substitute to the path integral
- integrate out δX that are normal to the fixed points X_0

$$Z = \int_{\mathcal{M}} \mathcal{D}X e^{-(S[X_0]+\frac{1}{2}\iint \frac{\delta^2(QV[X])}{\delta X^2}\big|_{X=X_0}(\delta X)^2)}$$

$$\boxed{= \sum_{X_0} \mathcal{D}X_0\, e^{-S[X_0]} \frac{1}{\mathrm{SDet}[\frac{\delta^2(QV[X_0])}{\delta X_0^2}]}}$$

The SDet is the superdeterminent which is the ratio of bosonic and fermionic determinants or measure on space of fixed points $\{X_0\}$

Action

$$Q(\sum_{\hat{s}=1}^{4}[\Psi_{\hat{s}}^{\dagger}\cdot B_{\hat{s}} + \Psi_{\hat{s}}\cdot B_{\hat{s}}^{\dagger}])$$

fixed points

$$Q\Psi_{\hat{s}} = 0 = Q\Psi_{\hat{s}}^{\dagger},$$
$$\Psi_{\hat{s}} = 0 = \Psi_{\hat{s}}^{\dagger},$$

BPS equations

$$\boxed{\begin{aligned} Q\mu_I &= \phi I - Iv = 0 \\ Q\mu_J &= -\phi J + vJ = 0 \\ Q\mathcal{M}_1 &= [\phi, B_1] = 0 \\ Q\mathcal{M}_2 &= [\phi, B_2] = 0 \end{aligned}}$$

Fixed points of multi-instanton action

7) Instanton Contribution to $\mathcal{N} = 2$ Prepotential

T^2 – symmetry and localization

The "fixed points" are actually critical surfaces rather than points which are still difficult to solve

→ exploit further symmetry to simplify the critical surfaces into isolated points

Two independent rotations on the $x_1 - x_2$ and the $x_3 - x_4$ planes

$$T^2 = (t_1, t_2) \quad \text{with} \quad t_1 = e^{i\epsilon_1} \text{ and } t_2 = e^{i\epsilon_2}$$

$$\boxed{Q_\epsilon \equiv Q + \Omega_{mn} x^m Q^n}$$

$$\Omega_{mn} = \begin{pmatrix} 0 & \epsilon_1 & 0 & 0 \\ -\epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_2 \\ 0 & 0 & -\epsilon_2 & 0 \end{pmatrix}$$

1. ϕ_I from the residual group $U(k)$

2. v_l from breaking the gauge group by vevs through Higgs mechanism in Columb branch $SU(N) \supset U(1)^{N-1}$

3. ϵ_1 , ϵ_2 from $SO(4) \supset U(1)_{\epsilon_{1,2}}$ representing gravitational deformations

maximal torus
$\mathcal{T} = U(1)^{N-1} \times U(1)_{\epsilon_{1,2}}$

Transformation of ADHM data become

$$
\begin{aligned}
&Q_\epsilon I = \mu_I, && Q_\epsilon \mu_I = \phi_I I - I v_l,\\
&Q_\epsilon J = \mu_J, && Q_\epsilon \mu_J = -\phi_I J + v_l J + (\epsilon_1 + \epsilon_2) J,\\
&Q_\epsilon B_1 = \mathcal{M}_1, && Q_\epsilon \mathcal{M}_1 = [\phi_I, B_1] + \epsilon_1 B_1,\\
&Q_\epsilon B_2 = \mathcal{M}_2, && Q_\epsilon \mathcal{M}_2 = [\phi_I, B_2] + \epsilon_2 B_2.
\end{aligned}
$$

Transformation of the auxiliary fields become

$$
\begin{aligned}
&Q_\epsilon \chi_{\mathbb{R}} = H_{\mathbb{R}}, && Q_\epsilon H_{\mathbb{R}} = [\phi_I, \chi_{\mathbb{R}}],\\
&Q_\epsilon \chi_{\mathbb{C}} = H_{\mathbb{C}}, && Q_\epsilon H_{\mathbb{C}} = [\phi_I, \chi_{\mathbb{C}}] + (\epsilon_1 + \epsilon_2)\chi_{\mathbb{C}},\\
&Q_\epsilon \bar{\phi}_I = \eta_I, && Q_\epsilon \eta_I = [\phi_I, \bar{\phi}_I],\\
&Q_\epsilon \phi_I = 0, && Q_\epsilon v = Q_\epsilon \bar{v} = 0.
\end{aligned}
$$

Vector field Q^* that generates the transformation Q_ϵ

$$
Q^* = (Q^*)^i_{\mathcal{B}} \frac{\partial}{\partial \mathcal{B}^i} + (Q^*)^i_{\mathcal{F}} \frac{\partial}{\partial \mathcal{F}^i}
$$

$$
\mathcal{B} = (I, J, B_1, B_2, H_{\mathbb{R}}, H_{\mathbb{C}}, \bar{\phi})
$$

$$
\mathcal{F} = (\mu_I, \mu_J, \mathcal{M}_1, \mathcal{M}_2, \chi_{\mathbb{R}}, \chi_{\mathbb{C}}, \eta)
$$

$Q_\epsilon \phi_I = 0 \longrightarrow \phi$, the generator of the $U(k)$, is Q_ϵ-invariance

Utilize the freedom

diagonize $\phi = (\varphi_1, ..., \varphi_k)$

$\phi_{IJ} \equiv \varphi_I - \varphi_J$ with $I, J = 1, ..., k$

The change of coordinate gives a Jacobian

Vandermonde determinant $\prod_{I<J}(\varphi_I - \varphi_J)^2$

$$Z_k = \int_{\mathcal{M}^\zeta} \frac{D\phi}{U(k)} D\mathcal{B} D\mathcal{F} e^{-S}$$
$$= \int_{\mathcal{M}^\zeta} \prod_{I=1}^{k} d\phi_I \frac{\prod_{I<J}(\varphi_I - \varphi_J)^2}{Sdet\mathcal{L}}$$
$$\boxed{\equiv \sum_{X_0} \frac{1}{Sdet\hat{\mathcal{L}}_{X_0}}}$$

$$Sdet(\mathcal{L}) \equiv Sdet \begin{pmatrix} \frac{\partial (Q^*)^i_{\mathcal{B}}}{\partial \mathcal{F}^j} & \frac{\partial (Q^*)^i_{\mathcal{B}}}{\partial \mathcal{B}^j} \\ \frac{\partial (Q^*)^i_{\mathcal{F}}}{\partial \mathcal{F}^j} & \frac{\partial (Q^*)^i_{\mathcal{F}}}{\partial \mathcal{B}^j} \end{pmatrix}$$

Critical-point equations and Young Diagram

$$Q_\epsilon \mu_I = (\varphi_I - v_l) I_{I,l} = 0$$

$$\cancel{Q_\epsilon \mu_J = (-\varphi_I + \epsilon_1 + \epsilon_2 + v_l) J_{l,I} = 0}$$

$$Q_\epsilon \mathcal{M}_1 = (\varphi_I - \varphi_J + \epsilon_1) B_{1,IJ} = 0$$

$$Q_\epsilon \mathcal{M}_2 = (\varphi_I - \varphi_J + \epsilon_2) B_{2,IJ} = 0$$

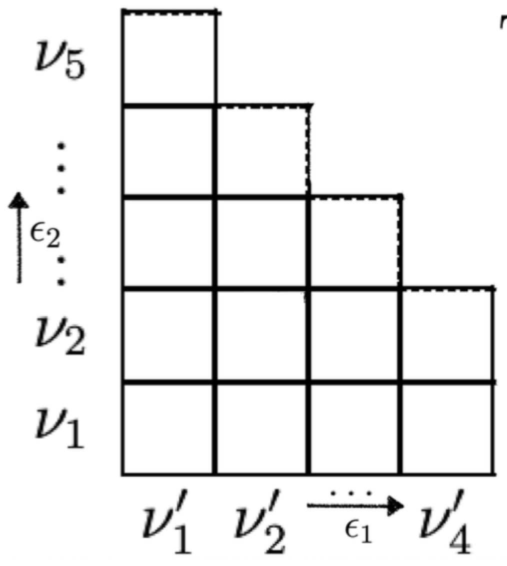
$$\longrightarrow$$

$$\boxed{\begin{aligned} (\varphi_I - v_l) I_{I,l} &= 0 \\ (\varphi_I - \varphi_J + \epsilon_1) B_{1,IJ} &= 0 \\ (\varphi_I - \varphi_J + \epsilon_2) B_{2,IJ} &= 0 \end{aligned}}$$

Associate each critical point φ_I to a set of N Young Tableaux $(Y_1, Y_2, \ldots.., Y_N)$

Introduce a partition to each critical point indicie $I \in 1, 2, \ldots., k$ as follow

$$k \rightarrow k_1 + k_2 + \ldots + k_l + \ldots. + k_N$$



$l \in \{1, 2, ..., N\}$

The boxes in diagram Y_l can be labeled by (i_l, j_l)

vertical position of Young diagram

horizontal position of Young diagram

ν_{i_l} : length of i_l-th row
ν'_{j_l} : length of j_l-th column

$$\boxed{\varphi_{I_l} \equiv \varphi_{i_l j_l} = v_l - (j_l - 1)\epsilon_1 - (i_l - 1)\epsilon_2}$$

$\varphi_{i_l j_l} = v_l$ is a nontrivial solution

ϵ_1 : horizontal direction (right)
ϵ_2 : vertical direction (up)

the boxes on the pair (i_l, j_l)'s both right and up directions are also nontrivial solutions

Character of Superdeterminant

$$\chi \equiv \sum_i (-)^F e^{i\lambda_i}$$

λ_i is the eigenvalue of $\hat{\mathcal{L}}_{X_0}$

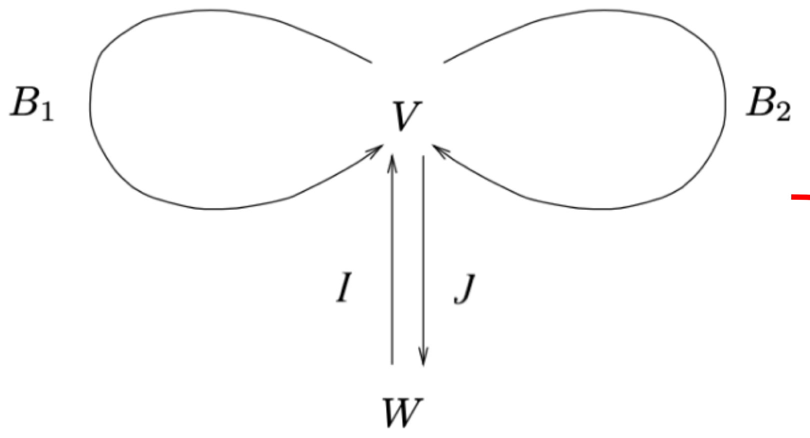
$(-)^F = \pm 1$

Tangent Space of Moduli Space and Character of Superdeterminant

<u>ADHM Data Space</u>

$V = \mathbb{C}^k$

$W = \mathbb{C}^N$



ADHM data and
ADHM constraints
as map of vector fields

$$
\begin{array}{rcl}
B_1, B_2 & : & Hom(V, Q \otimes V), \\
I & : & Hom(W, V), \\
J & : & Hom(V, \bigwedge^2 Q \otimes W), \\
\chi_{\mathbb{R}} & : & Hom(V, V), \\
\chi_{\mathbb{C}} & : & Hom(V, V) \otimes \bigwedge^2 Q
\end{array}
$$

incorporate the action of T^2-symmetry

$$\longrightarrow \quad \bigwedge^2 Q = t_1 t_2 = e^{i(\epsilon_1 + \epsilon_2)}$$

Tangent Space of Moduli Space

$T\mathfrak{M}_k$ spanned by $(\delta B_1, \delta B_2, \delta I, \delta J)$

(1) d_1: infinitesimal gauge transformation

(2) d_2: linearized ADHM equations thus in Ker d_2

$$T\mathfrak{M}_k \equiv \text{Ker } d_2 \,/\, \text{Im } d_1$$

$$Hom(V,V) \xrightarrow{d_1} \begin{array}{c} Hom(V, Q\otimes V) \\ \oplus \\ Hom(W,V) \\ \oplus \\ Hom(V, \bigwedge^2 Q \otimes W) \end{array} \xrightarrow{d_2} Hom(V,V)\otimes \bigwedge{}^2 Q$$

First cohomology group

$$
\begin{aligned}
T\mathfrak{M}_k &= Hom(V, Q \otimes V) + Hom(W, V) + Hom(V, \bigwedge^2 Q \otimes W) \\
&\quad - Hom(V, V) - Hom(V, V) \otimes \bigwedge^2 Q \\
&= V^* \otimes V \otimes (Q - \bigwedge^2 Q - 1) + W^* \otimes V + V^* \otimes W \otimes \bigwedge^2 Q
\end{aligned}
$$

$T_{v_l} = e^{iv_l}$: generators for the elements in $U(1)^{N-1}$

$T_1 = e^{i\epsilon_1}$, $T_2 = e^{i\epsilon_2}$: generators for the elements in $U(1)^2$

$V = e^{i\varphi_I}$

supertrace of $\mathcal{L}$

$$
\begin{aligned}
\chi &= V^* \times V \times [(T_1 + T_2) - T_1 T_2 - 1] + W^* \times V + V^* \times W \times T_1 T_2 \\
&= V^* \times V \times [(T_1 - 1) \times (1 - T_2)] + W^* \times V + V^* \times W \times T_1 T_2
\end{aligned}
$$

with

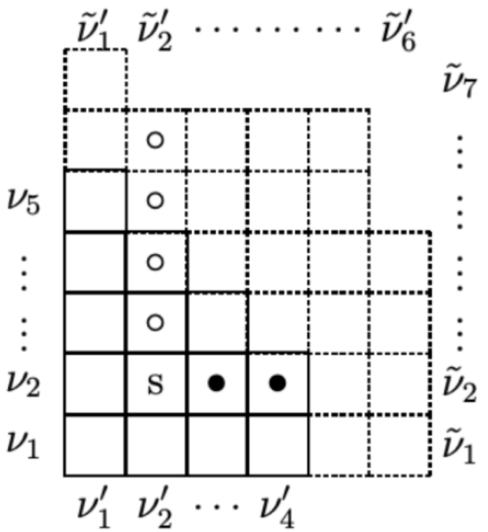
$$
V = \sum_{l=1}^{N} \sum_{j_l=1}^{\nu_{i_l}} \sum_{i_l=1}^{\nu'_{j_l}} T_1{}^{-j_l+1} T_2{}^{-i_l+1} T_{v_l},
$$

$$
W = \sum_{l=1}^{N} T_{v_l}
$$

$$\chi = \sum_{l,m}^{N} \sum_{s \in Y_j} (T_{v_{lm}}\ T_1^{-h(s)} T_2^{v(s)+1} + T_{v_{lm}}\ T_1^{h(s)+1} T_2^{-v(s)}),$$

where

$$h(s) = \nu_{i_l} - j_l,$$
$$v(s) = (\nu'_{j_l} + \nu_{i_l} - j_l + 1) - i_l \ \equiv \ \tilde{\nu}'_{j_l} - i_l$$



$$Z_k = \sum_{X_0} \frac{1}{Sdet \hat{\mathcal{L}}_{X_0}}$$
$$= \sum_{\{Y_l\}} \prod_{l,m}^{N} \prod_{s \in Y_l} \frac{1}{E(s)(E(s) - \epsilon_1 - \epsilon_2)},$$

where

$$E(s) = v_{lm} - \epsilon_1 h(s) + \epsilon_2 (v(s) + 1)$$

Contribution of Fundamental Matters

$$S_{hyp} = -Qtr_k[\tilde{h}\mathcal{K} - \tilde{\mathcal{K}}h]$$

$\mathcal{K}\ :\ k \times N_F$ matrices
$\tilde{\mathcal{K}}\ :\ N_F \times k$ matrices

fermionic collective coordinate of fundamental matter fields

presence of N_F fundamental hypermultiplets

$$Q\mathcal{K} = h, \qquad Qh = (\phi + m_f)\mathcal{K},$$
$$Q\tilde{\mathcal{K}} = \tilde{h}, \qquad Q\tilde{h} = -\tilde{\mathcal{K}}(\phi + m_f)$$

$(h, \tilde{h})$: superpartner of $(\mathcal{K}, \tilde{\mathcal{K}})$

$T_{m_f} = e^{im_f}$: generator of $U(1)^{N_f}$

$$\delta\chi = -T_{m_f} \times V = -\sum_{l}^{N}\sum_{s\in Y_l} T_{a_l}T_1^{-j_l+1}T_2^{-i_l+1}T_{m_f}$$

modify the vector field Q^*

extra contribution to the supertrace

$$h\frac{\partial}{\partial\mathcal{K}} + (\phi + m_f)\mathcal{K}\frac{\partial}{\partial h}$$

m_f being the mass of f-th favour

$$Z_k = \sum_{\{Y_l\}}\prod_{l,m}^{N}\prod_{s\in Y_l}\frac{F(s)}{E(s)(E(s)-\epsilon_1-\epsilon_2)}$$

with

$$F(s) = \prod_{f=1}^{N_f}(\varphi_{i_l j_l} + m_f)$$

Contribution of Adjoints Matters

global symmetry of $\mathcal{N}=4$ theory

$SU(4)_A$ is the automorphism group of $\mathcal{N}=4$ supersymmetry algebra

$$SU(2)_L \times SU(2)_R \times SO(6) \quad \cong \quad SU(2)_L \times SU(2)_R \times SU(4)_A$$

add mass to the $\mathcal{N}=2$ adjoint hypermultiplet
keep the $\mathcal{N}=2$ vector multiplet massles

$$SU(2)_A \times U(1)_m \quad \subset \quad SU(4)_A$$

global symmetry of $\mathcal{N}=2^*$ theory

$$SU(2)_L \times SU(2)_R \times SU(2)_A \times U(1)_m$$

topological twist

$$SU(2)_L \times SU(2)'_R \times U(1)_m$$

Maximal torus

$\mathcal{T} = U(1)^{N-1} \times U(1)_{\epsilon_{1,2}} \times U(1)_m$

$SU(2)'_R$ is the diagonal subgroup of $SU(2)_R \times SU(2)_A$

$$\boxed{Z_k = \sum_{\{Y_l\}} \prod_{l,m}^{N} \prod_{s \in Y_l} \frac{(E(s)-m)(E(s)+m-\epsilon_1-\epsilon_2)}{E(s)(E(s)-\epsilon_1-\epsilon_2)}}$$

Examples of Calculating Instanton Contribution to Prepotential

$$f(x) = \frac{(x-m)(x+m-\epsilon_1-\epsilon_2)}{x(x-\epsilon_1-\epsilon_2)} \quad , \quad g(x) = \frac{1}{x(x-\epsilon_1-\epsilon_2)}$$

$$T_\alpha(x) = \prod_{\alpha\neq\beta} f(v_{\alpha\beta}+x) \quad , \quad S_\alpha(x) = \prod_{\alpha\neq\beta} g(v_{\alpha\beta}+x)$$

pure $\mathcal{N} = 2$ theories

$$\alpha, \beta = \{1, 2, ..., N\}$$

$$v_{\alpha\beta} = v_\alpha - v_\beta$$

$$\boxed{Z_k = \sum_{\{Y_l\}} \prod_{l,m}^{N} \prod_{s\in Y_l} f(E(s))}$$

$$E(s) = v_{lm} - \epsilon_1 h(s) + \epsilon_2(v(s)+1)$$

$$f(x) = \frac{(x-m)(x+m-\epsilon_1-\epsilon_2)}{x(x-\epsilon_1-\epsilon_2)}$$

$$T_\alpha(x) = \prod_{\alpha \neq \beta} f(v_{\alpha\beta} + x)$$

- $\boldsymbol{k = 1}$

The Young diagram for one instanton is:

$Y_\alpha = \square$, $Y_{\beta\neq\alpha} = \{\emptyset\}$

$$l = \alpha : \ v(s) = h(s) = 0 \longrightarrow f(\epsilon_2)$$

$$l \neq \alpha : \ v(s) = -1, h(s) = 0 \longrightarrow T_\alpha(0)$$

$$\boxed{Z_1 = \sum_\alpha f(\epsilon_2) T_\alpha(0)}$$

$$E(s) = v_{lm} - \epsilon_1 h(s) + \epsilon_2 (v(s) + 1)$$

$$f(x) = \frac{(x-m)(x+m-\epsilon_1-\epsilon_2)}{x(x-\epsilon_1-\epsilon_2)}$$

$$T_\alpha(x) = \prod_{\alpha \neq \beta} f(v_{\alpha\beta} + x)$$

- $\boldsymbol{k = 2}$

$$\text{(I)} \quad Y_\alpha = \square \,,\; Y_\beta = \square \,,\; Y_{\gamma \neq \alpha,\beta} = \{\emptyset\}$$

$$l = \alpha, m \neq \alpha, \beta : \; v(s) = -1, h(s) = 0 \longrightarrow \frac{T_\alpha(0)}{f(v_{\alpha\beta})}$$

$$l = m = \alpha \qquad : v(s) = h(s) = 0 \longrightarrow f(\epsilon_2)$$

$$l = \alpha, m = \beta \qquad : v(s) = h(s) = 0 \longrightarrow f(v_{\alpha\beta} + \epsilon_2)$$

$$\alpha \leftrightarrow \beta$$

$$Z_2^I = \frac{1}{2} \sum_{\alpha \neq \beta} f(\epsilon_2)^2 (v_{\alpha\beta} + \epsilon_2)(v_{\beta\alpha} + \epsilon_2) \frac{T_\alpha(0) T_\beta(0)}{f(v_{\alpha\beta}) f(v_{\beta\alpha})}$$

$$(\mathrm{II})\ \ \mathrm{Y}_\alpha = \begin{array}{|c|c|}\hline & \\ \hline \end{array}\ , \ Y_{\beta\neq\alpha} = \{\emptyset\} \quad \xleftrightarrow{\ \textit{transposition}\ } \quad (\mathrm{III})\ \ \mathrm{Y}_\alpha = \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}\ , \ Y_{\beta\neq\alpha} = \{\emptyset\}$$

$$E(s) = v_{lm} - \epsilon_1 h(s) + \epsilon_2 (v(s)+1)$$

$$f(x) = \frac{(x-m)(x+m-\epsilon_1-\epsilon_2)}{x(x-\epsilon_1-\epsilon_2)}$$

$$T_\alpha(x) = \prod_{\alpha\neq\beta} f(v_{\alpha\beta}+x)$$

$$f(\epsilon_2)f(\epsilon_2-\epsilon_1)$$

$$l = m = \alpha \, : \, v(s) = 0 \ , \ h(s) = 0, 1$$
$$m \neq l = \alpha \, : \, v(s) = -1 \ , \ h(s) = 0, 1$$

$$T_\alpha(0)T_\alpha(-\epsilon_1)$$

$$l = m = \alpha \, : \, v(s) = -1, -2 \ , \ h(s) = -1$$
$$m \neq l = \alpha \, : \, v(s) = -1, -2 \ , \ h(s) = 0$$

$$f(\epsilon_1)f(\epsilon_1-\epsilon_2)$$

$$T_\alpha(0)T_\alpha(-\epsilon_2)$$

$$Z_2^{II} = \sum_\alpha f(\epsilon_2)f(\epsilon_2-\epsilon_1)T_\alpha(0)T_\alpha(-\epsilon_1)$$

$$Z_2^{III} = \sum_\alpha f(\epsilon_1)f(\epsilon_1-\epsilon_2)T_\alpha(0)T_\alpha(-\epsilon_2)$$

$$Z_2 = Z_2^I + Z_2^{II} + Z_2^{III}$$

extract the instanton contribution to the prepotential ($\mathcal{F}^{inst}$)

> $\epsilon_1 = \hbar = -\epsilon_2$

$$Z(v, \epsilon_1, \epsilon_2) = \exp(\frac{\mathcal{F}^{inst} + O(\epsilon_1, \epsilon_2)}{\epsilon_1 \epsilon_2})$$

- $\boldsymbol{k = 1}$

$$\mathcal{F}_1 = -\lim_{\hbar \to 0} \hbar^2 Z_1 = m^2 \sum_\alpha T_\alpha(0)$$

- $\boldsymbol{k = 2}$

$$T_\alpha = T_\alpha(0)$$

$$\begin{aligned}\mathcal{F}_2 &= -\lim_{\hbar \to 0} \hbar^2 (Z_2 - \frac{1}{2} Z_1^2) \\ &= \sum_\alpha (\frac{1}{4} m^4 T_\alpha T''_\alpha - \frac{3}{2} m^2 T_\alpha^2) \\ &\quad + m^4 \sum_{\alpha \neq \beta} T_\alpha T_\beta (\frac{1}{v_{\alpha\beta}} - \frac{1}{2(v_{\alpha\beta} - m)^2} - \frac{1}{2(v_{\alpha\beta} + m)^2})\end{aligned}$$

Thank You very much !