Universität Hamburg

DER FORSCHUNG | DER LEHRE | DER BILDUNG

FAKULTÄT
FÜR MATHEMATIK, INFORMATIK
UND NATURWISSENSCHAFTEN

Instanton Calculus and $\mathcal{N} = 2$ Supersymmetry Yang-Mills Theory

Tsz Ying, Yeung

Masterarbeit im Studiengang Mathematical Physics
Universität Hamburg

First Supervisor: Prof. Dr. Elli Pomoni
Second Supervisor: Prof. Dr. Timo Weigand

2024

Summary

The goal of this work is to calculate the non-perturbative solution contribution due to instantons for Yang-Mills theory via path integral method. In particular, we are interested in the $\mathcal{N} = 2$ supersymmetry case, as we can compare our result with the prediction of Seiberg-Witten theory on instanton effect.

We starts by introducing instanton as solution of $SU(N)$ Yang-Mills theory after imposing the self-duality condition. Since Yang-Mills action is a second order partial differential equation, imposing self duality constraints allow us to solve the action via first order differential equations.

Next, we construct the instanton moduli space by first using zero-modes and collective coordinates, then we introduce ADHM construction for building the moduli space in the general scenarios. We also add a discussion about ADHM construction in string theory setting.

Then, we write the $\mathcal{N} = 1, 2$ supersymmetric zero-modes explicitly. After that we suggest a BRST algebra from studying the zero-modes. Combining with topological twisting of $\mathcal{N} = 2$ SUSY theory in flat space, we rewrite the multi-instanton action in BRST-close form.

After that we try to mitigate the small instanton singularity and non-compactness of the instanton moduli space. We introduce non-commutative space to regularize the moduli space. And we utilize localization to deal with the non-compactness of the moduli space. However, the critical points of the action are not isolated at this stage, and we have to further deform the moduli space by a T^2-rotation for localization to work properly.

Finally, we conclude the work by calculating the prepotential of one and two instantons explicitly using the method we derived. And we compare the result with the one predicted by Seiberg-Witten theory.

Contents

1 Introduction **4**

2 Zero-Modes and ADHM Consturction **6**
2.1 Zero-Modes 6
2.2 ADHM Construction 8
2.3 ADHM Construction in String Theory Setting 12

3 $\mathcal{N} = 1, 2$ Supersymmetric Zero-Modes **17**
3.1 $\mathcal{N} = 1$ supersymmetry 17
3.1.1 Gaugino zero-modes 17
3.2 $\mathcal{N} = 2$ supersymmetry 18
3.2.1 Scalar non-zero VEVs 18
3.2.2 Higgs boson zero-modes 19

4 Supersymmetry and BRST algebra of Instanton Moduli Space **21**
4.1 Algebraic Construction of BRST Transformation 21
4.2 Strategy of solving Instanton Measure 23

5 Regularization of the Moduli Space **24**
5.1 Small Instanton Singularity 24
5.2 ADHM Construction of Non-Commutative Instanton 25
5.3 SU(2) 1-instanton in Commutative $\mathbb{R}^4$ 27
5.4 $U(2)$ 1-instanton in Non-Commutative $\mathbb{R}^4$ 28

6 $\mathcal{N} = 2$ SQCD Multi-Instanton Action **29**
6.1 Topological Twist 29
6.2 Auxiliary fields for the ADHM constraints 31
6.3 Multi-instanton action 31
6.3.1 Rewriting the Multi-instanton action 32
6.4 Localization 33
6.4.1 Localization argument 33
6.4.2 Localization formula for supersymmetric path integrals 35
6.4.3 Fixed points of Instanton moduli space 36

7 Instanton contribution to $\mathcal{N} = 2$ Prepotential **37**
7.1 T^2-symmetry and localization 37
7.2 Critical-point equations and Young diagram 39
7.3 Tangent space of the Moduli space and Character of Superdeterminant 41

7.4 Adding masses to the $\mathcal{N} = 2$ and the $\mathcal{N} = 2^*$ $SU(N)$ supersymmetric theories 43
7.4.1 Contribution of Fundamental Matters 43
7.4.2 Contribution of Adjoint Matters 44
7.5 Explicit Examples of calculating Instanton contribution to Prepotential 45

8 Conclusion **48**

1 Introduction

Starting with the action of the SU(N) Yang-Mills theory in four-dimensional Euclidean space ($\mathbb{R}^4$),

$$S[A] = \frac{1}{2g^2}\int d^4x\ tr_N F_{mn}F_{mn} + i\theta k, \tag{1.1}$$

where $F_{mn} = \partial_m A_n - \partial_n A_m + [A_m, A_n]$ is the field strength, A_n are $N \times N$ anti-Hermitian gauge fields, and $m, n = 1, ..., 4$ are Lorentz indices.

The instanton number k is defined as

$$k = \frac{1}{16\pi^2}\int d^4x tr_N F_{mn}{}^*F_{mn}, \tag{1.2}$$

where ${}^*F_{mn} \equiv \frac{1}{2}\epsilon_{mnkl}F_{kl}$. The role of instanton number can be shown more explicitly if we expand the integrand,

$$\begin{aligned} F_{mn}{}^*F_{mn} &= 2\epsilon_{mnkl}(\partial_m A_n \partial_k A_l + 2\partial_m A_n A_k A_l + A_m A_n A_k A_l) \\ tr_N F_{mn}{}^*F_{mn} &= 2\partial_m tr_N \epsilon_{mnkl}(A_n\partial_k A_l + \frac{2}{3}A_n A_k A_l), \end{aligned} \tag{1.3}$$

where the last term vanishes due to cyclic property of trace.

Equation (1.3) shows that $F_{mn}{}^*F_{mn}$ is a total derivative. According to Stoke's theorem, the four-dimensional spacetime integral becomes an integral over S^3_∞-boundary at infinity, so k can count the map of $\partial\mathbf{R}^4 \cong \mathbf{S}^3_\infty \to$ SU(N).

In order to have finite action, the field strength F_{mn} has to approach zero at infinity faster than $|x|^{-2}$ so the gauge field asymptotically approach pure gauge $A_m = U^{-1}\partial_m U$ where $U \in$ SU(N). F_{mn} vanishes at large x also implies that $\partial_n A_m = -A_n A_m$. This leads to $k = \frac{1}{24\pi^2}\oint_{S^3_\infty} d\Omega_m \epsilon_{mnkl}[(U^{-1}\partial_n U)(U^{-1}\partial_k U)(U^{-1}\partial_l U)]$, where $d\Omega$ is the surface element on S^3_∞.

We observe that each point x on S^3_∞ corresponds to a gauge group element U. This again shows k counts the number of times S^3_∞ is winded around SU(N). Therefore, k is also called winding number which is a topological charge.

Also, observing that in the case of U(1) with $U = e^{i\alpha}$ and $\alpha \in \mathfrak{u}(1)$, k will be trivial in regular gauge as x tends to 0 at infinity.

Next, we try to solve the classical equation of motion ($D_n F^{mn} = 0$) for any given topological sector (k)

$$\begin{aligned} S &= \frac{1}{2g^2} \int d^4x \; trF^2 \\ &= \frac{1}{4g^2} \int d^4x \; tr(F \mp {}^*F)^2 \; \mp \; \frac{1}{2g^2} \int d^4x \; F \, {}^*F \\ &\geq \mp \frac{1}{2g^2} \int d^4x \; trF^*F = \frac{8\pi^2}{g^2}(\pm k). \end{aligned} \tag{1.4}$$

This shows that for each topological sector (k), the action is bounded below by $8\pi^2|k|/g^2$ which occurred when

$$F = \pm^* F. \tag{1.5}$$

These are the (anti) self-dual equations. Solutions to these first-order equations should minimize the action in a given topological sector and solve the classical equation. These solutions are know as (anti) instantons.

Moreover, since we would like the gauge potential A to fall faster at large distance for better convergence, most calculations are carried out in singular gauge in which A becomes pure gauge as $x \to 0$.

This can be shown more explicitly by using $k = 1$ SU(2) instanton as an example. Using result in [**1**],

$$A_n^{reg} = \frac{2\rho^2 (x - X)_m \sigma_{mn}}{(x - X)^2 + \rho^2} \tag{1.6}$$

$$A_n^{sing} = \frac{2\rho^2 (x - X)_m \bar{\sigma}_{mn}}{(x - X)^2((x - X)^2 + \rho^2)} \tag{1.7}$$

where $\sigma_{mn} = \frac{1}{4}(\sigma_m \bar{\sigma}_n - \sigma_n \bar{\sigma}_m)$ and $\bar{\sigma}_{mn} = \frac{1}{4}(\bar{\sigma_m} \sigma_n - \bar{\sigma_n} \sigma_m)$ are Lorentz generators, $\bar{\sigma}_n \equiv (-i\vec{\tau}, 1_{[2]x[2]})$ and τ^c , c=1,2,3 are the Pauli matrices. And X_n and ρ are the postion and the size of the instanton respectively. Note that $\sigma_{mn} = \frac{1}{2}\varepsilon_{mnkl}\sigma_{kl}$ and $\bar{\sigma}_{mn} = -\frac{1}{2}\varepsilon_{mnkl}\bar{\sigma}_{kl}$ are self-dual and anti-self-dual respectively.

Equation (1.8) shows that A_n^{sing} converges in a faster rate of $\frac{1}{x^3}$ than that of A_n^{reg} which falls at $\frac{1}{x}$. Moreover, placing the instanton in the origin ($X = 0$), A_n^{sing} has a singularity at $x = 0$ but this can be removed by gauge transformation. On the other hand, A_n^{reg} is regularized at $x = 0$ as long as $\rho \neq 0$. Notice that a singularity exsists at $\rho = 0$ that cannot be gauged away and the solution space has to be regularized. This singularity is called small instanton singularity which exists in arbitrary k and SU(N) gauge group. More will be discussed in later section (5.1).

2 Zero-Modes and ADHM Consturction

2.1 Zero-Modes

The zero-modes are physical fluctuations in field space that will not change the value of action. More explicitly, zero-modes are fluctuations (δA_n) of the solution that satisfy the following two criteria:

(1) self-duality condition : $D_m \delta A_n - D_n \delta A_m = *(D_m \delta A_n - D_n \delta A_m)$

(2) Orthogonal to gauge transformation $D_n \Omega$ (where Ω is an arbitrary function):
$\int d^4x \; tr\{D_n \Omega \; \delta A_n\} = 0 \quad \Rightarrow \quad D_n \delta A_n = 0 \quad \Rightarrow \bar{\not{D}}^{\dot{\alpha}\alpha} \delta A_n = 0$

Combining these two conditions, one obtains a single quaternion equation,

$$\bar{\not{D}}^{\dot{\alpha}\alpha} \delta A_{\alpha\dot{\beta}} = 0, \tag{2.1}$$

which is the covariant Weyl equation for $\delta A_{\alpha\dot{\beta}}$ in instanton background.

Use the simplest case, one instanton in SU(2), as an example. In equation (1.7), there are two types of collective coordinates which is the coordinates of moduli space or solution space of $F = *F$. These collective coordinates are four translations X_n and a scale size ρ.

Moreover, there exists a stability group that embeds the instanton solution as an SU(2) gauge group. This gives additional collective coordinates for the gauge orientation of instantons. All in all, there are three collective coordinates with respect to SU(2).

Then, denote $\{X^\alpha, \; \alpha = 1,..,8\}$ as a collection of collective coordinates. They can be related to zero-modes as follows

$$\delta_\alpha A_n = \frac{\partial A_n}{\partial X^\alpha} + D_n \Omega_\alpha. \tag{2.2}$$

We can associate a zero-mode with each collective coordinate by using equation(2.2) for $SU(N)$ and arbitrary k as well. However, it is not guaranteed that the other way round works (i.e. not every zero-mode can be integrated to give corresponding collective coordinate). This can happen when the gauge fields couple to other fields. Here, the number of zero-mode is equal to number of collective coordinates as the problem will be solved by addition of VEVs to scalar field (section 3.2). In general we get $4Nk$ zero-modes as predicted by index theorem ($dim(\tilde{\mathfrak{M}}) = 4Nk$).

Back to our example of single instanton in SU(2), we get eight zreo-modes. Following the discussion in [**5**], we study the metric of moduli space which is defined by the overlapping of zero-modes in the collective coordinate $X^\alpha,\ \alpha = 1, .., 4kN$,

$$g_{\alpha\beta} \equiv \frac{1}{2g^2} \int d^4x Tr(\delta_\alpha A_n)(\delta_\beta A_n). \tag{2.3}$$

Then, we divide the eight zero-modes into three classes using the three types of collective coordinates and calculate their metric individually:

(1) Translational zero-mode : S_{inst}

(2) Scale zero-mode : $2S_{inst}$

(3) Gauge orientation zero-mode : $2S_{inst}\rho^2$

where $S_{inst} = 8\pi^2|k|/g^2$. The gauge orientation mode is interesting as its metric value depends on the scale collective coordinate ρ^2. This shows the gauge group (SU(2) $\cong S^3$) from the gauge rotation combines with $\mathbf{R}^+$ from scale transformation ρ^2. Therefore, these zero-modes live in $S^3 \times \mathbf{R}^+ \cong \mathbf{R}^4\backslash\{0\}$.

Moreover, the fields in adjoint representation are invariant under the center $\mathbf{Z}_2 \subset$ SU(2). Therefore, the gauge rotation gives $\mathbf{R}^4/\mathbf{Z}_2$ space and the singularity of this orbifold is the small instanton singularity. This will be discussed later in section (5.1). Thus, the moduli space for single SU(2) instanton is

$$\mathfrak{M} = \mathbf{R}^4 \times\ (\mathbf{R}^4\backslash\{0\})/\mathbf{Z}_2, \tag{2.4}$$

where the first term comes from the translational modes and the latter from the scale and SU(2) gauge orientation.

From this simple example, we can see that the moduli space has an UV and IR divergence. Firstly, the $(\mathbf{R}^4\backslash\{0\})/\mathbf{Z}_2$ part shows the moduli space has an UV divergence, which can be regularized by introduction of non-commutative spacetime. This will be discussed more in chapter 5.
Secondly, the IR divergence can be seen in the $\mathbf{R}^4$ part of the moduli space already. This non-compactness can be dealt by introducing Ω-deformation which will be discussed in chapter 7.

2.2 ADHM Construction

For general N and instanton number k, it is hard to solve the self duality equation (1.5) directly and get the gauge potential (1.7) as we have done above for one SU(2) instanton. Instead we will utilize the algebra of the moduli space following the method introduced by Atiyah, Drinfeld, Hitchin and Manin (ADHM) [**1**].

We start by introducing an index assignment:

Instanton number indices [k] : $1 \leq i, j, l, ... \leq k$,

Colour indices [N]: $1 \leq u, v, ... \leq N$,

ADHM indices [N+2k]: $1 \leq \lambda, \mu, ... \leq N + 2k$,

Quaternionic (Weyl) indices [2]: $\alpha, \beta, \dot{\alpha}, \dot{\beta}, ... = 1, 2$,

Lorentz indices [4]: m,n,... = 1,2,3,4.

The covering group of the Lorentz group SO(4) is Spin(4) $\cong$ SU$(2)_L \times$ SU$(2)_R$. So a 4-vector x_n can be written as $x_{\alpha\dot{\alpha}}$, where $\alpha, \dot{\alpha}$ are spinor indices of SU$(2)_L$ and SU$(2)_R$ respectively. Writing explicitly

$$x_{\alpha\dot{\alpha}} = x_m \sigma_{m\alpha\dot{\alpha}} \quad , \qquad \bar{x}_{\alpha\dot{\alpha}} = x_m \bar{\sigma}^m_{\alpha\dot{\alpha}}, \tag{2.5}$$

where $\sigma_{m\alpha\dot{\alpha}}$ consists of four 2×2 matrices $\sigma_n = (i\vec{\tau}, 1_{[2]\times[2]})$ and $\bar{\sigma}^m_{\alpha\dot{\alpha}}$ consists of four $\bar{\sigma}_n \equiv \sigma_n^\dagger = (-i\vec{\tau}, 1_{[2]\times[2]})$.

The ADHM scheme is derived to solve the self-dual equation $F_{mn} = {}^*F_{mn}$. It starts with a $(N + 2k) \times 2k$ complex-valued matrix Δ which is linear in spacetime variable x_m,

$$\Delta_{[N+2k]\times[2k]}(x) \equiv \Delta_{[N+2k]\times[k]\times[2]}(x) = a_{[N+2k]\times[k]\times[2]} + b_{[N+2k]\times[k]\times\underline{[2]}}x_{[2]\times\underline{[2]}}. \tag{2.6}$$

By definition, the conjugate of Δ is defined by

$$\bar{\Delta}_{[2k]\times\underline{[N+2k]}} \equiv (\Delta_{[N+2k]\times[2k]})^{*}. \tag{2.7}$$

Writing in indicies, the above matrices can be written as

$$\Delta_{\lambda i\dot{\alpha}}(x) = a_{\lambda i\dot{\alpha}} + b^{\alpha}_{\lambda i}x_{\alpha\dot{\alpha}}\ , \qquad \bar{\Delta}^{\dot{\alpha}\lambda}_{i}(x) = \bar{a}^{\dot{\alpha}\lambda}_{i} + \bar{x}^{\dot{\alpha}\alpha}\bar{b}^{\lambda}_{\alpha i}. \tag{2.8}$$

The basis vector of the null-space of Δ can be assembled into a $(N+2k)\times N$ complex-valued matrix $U(x)$,

$$\bar{\Delta}_{[2k]\times\underline{[N+2k]}}U_{\underline{[N+2k]}\times[N]} = 0 = \bar{U}_{[N]\times\underline{[N+2k]}}, \Delta_{\underline{[N+2k]}\times[2k]} \tag{2.9}$$

where U is orthonormalized according to

$$\bar{U}_{[N]\times\underline{[N+2k]}}U_{\underline{[N+2k]}\times[N]} = 1_{[N]\times[N]}. \tag{2.10}$$

As mention in chapter 1, in $k=0$ sector, the antisymmetric pure gauge configuration solves the self-duality equation $F = \pm^{*}F$. The ADHM ansatz argues that pure gauge will continue to solve the self-duality equation in nonzero k sectors

$$A_{n[N]\times[N]} = \bar{U}_{[N]\times\underline{[N+2k]}}\partial_n U_{\underline{[N+2k]}\times[N]}. \tag{2.11}$$

with the following additional factorization condition:

$$\bar{\Delta}_{[2]\times[k]\times\underline{[N+2k]}}\Delta_{\underline{[N+2k]}\times[k]\times[2]} = 1_{[2]\times[2]}f^{-1}_{[k]\times[k]}, \tag{2.12}$$

where f is an arbitrary x-dependent $k\times k$ dimensional Hermitian matrix. Writing in indices,

$$\bar{\Delta}^{\dot{\beta}\lambda}_{i}\Delta_{\lambda i\dot{\alpha}} = \delta^{\dot{\beta}}_{\dot{\alpha}}(f^{-1})_{ij}. \tag{2.13}$$

Next, we would like to prove the ADHM ansatz. First, combination of the factorization condition and the null-space condition implies the following completeness relation

$$\begin{aligned}\mathcal{P} &\equiv U_{[N+2k]\times\underline{[N]}}\bar{U}_{\underline{[N]}\times[N+2k]} \\ &= 1_{[N+2k]\times[N+2k]} - \Delta_{[N+2k]\times\underline{[k]}\times\underline{[2]}}f_{\underline{[k]}\times\underline{[k]}}\bar{\Delta}_{\underline{[2]}\times\underline{[k]}\times[N+2k]},\end{aligned} \tag{2.14}$$

where $\mathcal{P}$ is actually a projection operator.

Then, using equations (2.8)-(2.13) and integration by part, we rewrite the field strength F_{mn} as follow:

$$\begin{aligned} F_{mn} &\equiv \partial_m A_n - \partial_n A_m + g[A_m, A_n] \\ &= \partial_{[m}(\bar{U}\partial_{n]}U) + g^{-1}(\bar{U}\partial_{[m}U)(\bar{U}\partial_{n]}U) \\ &= \partial_{[m}\bar{U}(1 - U\bar{U})\partial_{n]}U \\ &= \partial_{[m}\bar{U}\Delta f \bar{\Delta}\partial_{n]}U \\ &= \bar{U}\partial_{[m}\Delta f \partial_{n]}\bar{\Delta}U \\ &= \bar{U}b\sigma_{[m}\bar{\sigma}_{n]}f\bar{b}U \\ &= 4\bar{U}b\sigma_{mn}f\bar{b}U. \end{aligned} \tag{2.15}$$

Using the self-duality of σ_{mn}, equation (2.15) shows that $F = {}^*F$ and the ADHM theorem has been proved.

The above construction works for both $U(N)$ and $SU(N)$ gauge groups. While the pure gauge field constructed this way is not automatically traceless, one can restore the traceless property by gauge transformation $U \to Ug$, where $g \in U(1)$. Therefore, the distinction between $U(N)$ and $SU(N)$ only comes when matter fields are coupled in.

Combining equations (2.8) and (2.13), and noting f_{ij} is arbitrary, one can extract the three x-independent conditions which are called ADHM constraints

$$\bar{a}_i^{\dot{\alpha}\lambda} a_{\lambda j\dot{\beta}} = (\frac{1}{2}a\bar{a})_{ij}\delta_{\dot{\alpha}}^{\dot{\beta}} \quad \propto \quad \delta_{\dot{\alpha}}^{\dot{\beta}}, \tag{2.16a}$$

$$\bar{a}_i^{\dot{\alpha}\lambda} b_{\lambda j}^{\beta} = \bar{b}_i^{\beta\lambda} a_{\lambda j}^{\dot{\alpha}}, \tag{2.16b}$$

$$\bar{b}_{\beta j i}^{\lambda} b_{\lambda j}^{\beta} = (\frac{1}{2}b\bar{b})_{ij}\delta_{\alpha}^{\beta} \quad \propto \quad \delta_{\alpha}^{\beta}. \tag{2.16c}$$

Here, a and b comprise the collective coordinate which gives $4k(N+2k) = 4Nk + 8k^2$ of real degree of freedom. But according to index theorem, there are only $4Nk$ physical collective coordinates. This shows a and b consists a redundant $8k^2$ real degree of freedom.

This redundancy is eliminated first by noticing ADHM construction is unaffected by the following x-independent transformation

$$\Delta_{[N+2k]\times[k]\times[2]} \to \Lambda_{[N+2k]\times\underline{[N+2k]}}\Delta_{\underline{[N+2k]}\times\underline{[k]}\times[2]}B^{-1}_{\underline{[k]}\times[k]},$$

$$U_{[N+2k]\times[N]} \to \Lambda_{[N+2k]\times\underline{[N+2k]}}U_{\underline{[N+2k]}\times[N]},$$

$$f_{[k]\times[k]} \to B_{[k]\times\underline{[k]}}f_{\underline{[k]}\times\underline{[k]}}B^{\dagger}_{\underline{[k]}\times[k]},$$

provided $\Lambda \in U(N+2k)$ and $B \in Gl(k,\mathbf{C})$. Using these symmetries, we choose a representation in which b assumes canonical form. Then a and b become

$$b_{[N+2k]\times[2k]} = \begin{pmatrix} 0_{[N]\times[2k]} \\ 1_{[2k]\times[2k]} \end{pmatrix} \quad , \quad a_{[N+2k]\times[2k]} = \begin{pmatrix} \omega_{[N]\times[2k]} \\ a'_{[2k]\times[2k]} \end{pmatrix}. \tag{2.17}$$

We can write equation (2.17) more explicitly by splitting the ADHM index $\lambda \in [N+2k]$ into $\lambda = u + i\alpha \;\; , 1 \le u \le N, \;\; 1 \le i \le k, \;\; \alpha = 1,2.$

In this way, the top $N \times 2k$ submatices of a and b have row indexed by $u \in [N]$, while the bottom $2k \times 2k$ submatices have row indexed by $i\alpha \in [k] \times [2]$. So a and b become

$$a_{\lambda j\dot\alpha} = a_{(u+i\alpha)j\dot\alpha} = \begin{pmatrix} \omega_{uj\dot\alpha} \\ (a'_{\alpha\dot\alpha})_{ij} \end{pmatrix}, \tag{2.18a}$$

$$\bar a^{\dot\alpha\lambda}_j = \bar a^{\dot\alpha(u+i\alpha)}_j = \begin{pmatrix} \bar\omega^{\dot\alpha}_{ju} & (\bar{a}'_{\alpha\dot\alpha})_{ij} \end{pmatrix}, \tag{2.18b}$$

$$b^{\beta}_{\lambda j} = b^{\beta}_{(u+i\alpha)j} = \begin{pmatrix} 0 \\ \delta^{\beta}_{\alpha}\delta_{ij} \end{pmatrix}, \tag{2.18c}$$

$$\bar b^{\lambda}_{\beta j} = \bar b^{(u+i\alpha)}_{\beta j} = \begin{pmatrix} 0 & \delta^{\beta}_{\alpha}\delta_{ij} \end{pmatrix}. \tag{2.18d}$$

When b is in canonical form, the third ADHM constraint is satisfied automatically. So the ADHM constraints boil down to

$$tr_2(\tau^c \bar a a)_{ij} = 0, \tag{2.19a}$$

$$(a'^m)^{\dagger}_{ij} = a'^m_{ij}. \tag{2.19b}$$

We condense these constraints by using the 't Hooft symbols (η^c), $c = 1,2,3$ which are related to the Lorentz spinors (σ_{mn}) by $\sigma_{mn} \equiv i\bar\eta^c_{mn}\tau_c$. Finally, the ADHM constraints can be written as

$$\bar{\omega}\tau^c\omega - i\bar{\eta}^c_{mn}[a'_m, a'_n] = 0. \tag{2.20}$$

Moreover, equation (2.19) shows that $\bar{a}a$ is contracted with three Pauli matrices in the first constraint which eliminates $3k^2$ real degree of freedom. The second constraint eliminates further $4k^2$ real degree of freedom. So there is still k^2 redundant degree of freedom.

This is solved by noticing the canonical form of b is not only preserved by the symmetry groups $U(N+2k) \times Gl(k, \mathbf{C})$. It is also preserved by $U(k)$ subgroup of $U(N+2k) \times Gl(k, \mathbf{C})$ which is called the residual transformation,

$$\Delta_{[N+2k]\times[2k]} \to \begin{pmatrix} 1_{[N]\times[N]} & 0_{[2k]\times[N]} \\ 0_{[N]\times[2k]} & \mathcal{R}^\dagger_{[2k]\times[2k]} \end{pmatrix} \Delta_{[N+2k]\times[2k]} \mathcal{R}_{[2k]\times[2k]}, \tag{2.21}$$

where $\mathcal{R}_{[2k]\times[2k]} = \mathcal{R}_{ij}\delta^{\dot{\beta}}_{\dot{\alpha}}$ and $\mathcal{R}_{ij} \in U(k)$. In terms of ω and a', the residual transformation acts as

$$\omega^{\dot{\alpha}}_{ui} \to \omega^{\dot{\alpha}}_{uj} R_{ji} \quad , \quad (a'_{\beta\dot{\alpha}})_{ij} \to R^\dagger_{il}(a'_{\beta\dot{\alpha}})_{lp} R_{pj}. \tag{2.22}$$

Finally, a and b get $4Nk$ real degree of freedom which shows we can construct the physical moduli space $\mathfrak{M}_k$ in ADHM construction. However, the ADHM construction can also be viewed as the hyper-Kähler quotient of $\tilde{\mathfrak{M}} = \mathbf{R}^{4k(N+k)}$ by isometry group U(k).

The construction can be done in 2 steps. Starting from the mother space $\tilde{\mathfrak{M}} = \mathbf{R}^{4k(N+k)}$:

1. restrict to the level set $\mathfrak{N} \subset \tilde{\mathfrak{M}}$, space of all solutions to ADHM constraints
2. ordinary quotient of $\mathfrak{N}$ by residual U(k) action

$$\mathfrak{M}_k = T\mathfrak{N}/\mathrm{U}(k). \tag{2.23}$$

2.3 ADHM Construction in String Theory Setting

It is also possible to construct the instanton solution and moduli space in string theory setting. We start by noticing that in 3+1 dimension, D(p-4) branes in Dp-branes can be thought as instantons. Then, we consider a configuration of N Dp-branes and k D(p-4)-branes.

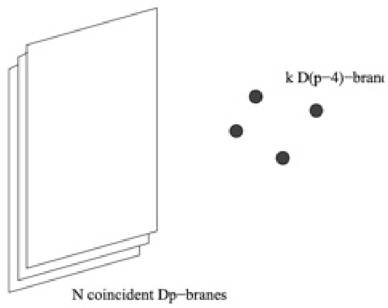

Figure 1: Instanton in string theory setting. Reprinted form [**5**].

We can construct the instanton solution by viewing the whole configuration in the perspective of Dp-branes. And we can construct the moduli space in the perspective of D(p-4)-branes. This is illustrated in figure 1.

Take $p = 3$ as an example. We first start in the point of view of the D3 branes. When considering the spectrum of massless open string between D-1 branes, a gauge theory is needed. We start with 32 supercharges in 10 dimension. The presence of D-1 branes imposes boundary conditions which break the amount of supersymmetry by half. A maximal SUSY gauge theory with 16 supercharges in 0 dimension is related to $\mathcal{N}$=4 super Yang-Mills (SYM) theory in 3+1 dimension by dimension reduction.

Moreover, since there are k D-1 branes, the Chan-Paton label gives a $U(k)$ symmetry among the branes. Combining this $U(k)$ symmetry with above, we should consider a $\mathcal{N}$=4 SYM theory in 3+1 dimension with a U(k) gauge symmetry.

	$SU(2)_\alpha$	$SU(2)_{\dot\alpha}$	$SU(4)_R$	$U(k)$	$U(N)$
$a'_{\alpha\dot\alpha}$	2	2	1	adj	1
χ^a	1	1	6	adj	1
$\mathcal{M}'^{\alpha\dot\alpha}$	2	1	4	adj	1
λ^a	1	2	4	adj	1

Table 1: Field contents in point of view of D3 branes.

The SU(2)$_\alpha$ and SU(2)$_{\dot\alpha}$ in table 1come from the Lorentz symmetry in 3+1 spacetime.

The D-1 brane is embedded in D3 brane and the D3 has a spacetime filling that can be thought as R-symmetry of the D-1 brane. While SU$(4)_R$ $\cong$ SO(6) comes from the rotation of the remaining 6 dimensions where the D3 brane is point-like. So the SU$(4)_R$ can be thought as R-symmetry of D3 brane rotating the 16 supercharges.

Moreover, $(a'_{\alpha\dot{\alpha}}, \mathcal{M}'^{\alpha\dot{\alpha}})$ and (χ^a , λ^a) are superpartners, where the fermions $(\mathcal{M}'^{\alpha\dot{\alpha}}, \lambda^a)$ live in fundamental representation of SU$(4)_R$. Focusing on the bosonic sector,

$$(\ a_{\alpha\dot{\alpha}} \ , \ \chi^a) \qquad \alpha,\dot{\alpha} = 1,2; \ \ \mathrm{a} = 1,...6$$

where $a_{\alpha\dot{\alpha}}$ forms an adjoint hypermultiplet and χ^a forms a vector multiplet in 3+1 dimension. For later use, it should be noted that χ^a lies in directions transverse to D3 brane. This will be considered later.

Now, we view in the point of view of D-1 branes and consider the spectrum of massless open strings between D-1 and D3 branes. These strings between D-1 and D3 branes impose additional set of boundary conditions and halves the amount of supersymmetry further to 8 supercharges. Dimension reduction again relates a maximal SUSY gauge theory with 8 supercharges in 0 dimension to $\mathcal{N}$=2 hypermultiplet in 3+1 dimension.

Moreover, the Chan-Paton label gives a U(N) symmetry among the branes, where this U(N) will be the flavor symmetry that rotates the hypermultiplets. Therefore, we should now consider a $\mathcal{N}$=2 gauge theory in 3+1 dimension with a $U(k)$ gauge symmetry and U(N) flavor symmetry.

	$SU(2)_\alpha$	$SU(2)_{\dot{\alpha}}$	$SU(2)_R$	$U(k)$	$U(N)$
$\omega_{\dot{\alpha}}$	1	2	1	k	N
$\tilde{\omega}_\alpha$	1	2	1	k	$\bar{N}$
μ^A	1	1	2	$\bar{k}$	N
$\tilde{\mu}^A$	1	1	2	k	$\bar{N}$

Table 2: Field contents in point of view of D-1 branes.

The above table shows the fields involved, where $A = 1, 2$ is fundamental indices of $SU(2)_R$. Here, $(\omega_\alpha,\ \mu^A)$ and $(\tilde{\omega}_{\dot{\alpha}},\ \tilde{\mu}^A)$ are superpartners. Again, the fermions live in fundamental representation of SU$(2)_R$.

Comparing tables 1 and 2, addition of N D3 branes breaks half of SU$(4)_R$. More explicitly, SU$(4)_R$ $\cong$ SU$(2)_L$ $\times$ SU$(2)_R$ where SU$(2)_L$ is broken going from $\mathcal{N}$=4 to $\mathcal{N}$=2. And the bosonic sector (ω_α, $\tilde{\omega}_{\dot{\alpha}}$) forms N fundamental hypermultiplets in 3+1 dimension.

Also, string from D3 to D-1 branes are in $(N, \bar{k})$ of $U(N) \times U(k)$ and strings from D-1 to D3 branes are in $(\bar{N}, k)$ of $\mathrm{U}(N) \times \mathrm{U}(k)$.

Now, we construct the moduli space. First, from the point of view of D-1 branes, χ^a=0 as the instantons lie in the D3 brane. Moreover, to construct the moduli space, we require the scaler potential $V = 0$. With these two constraints , a solution space can be constructed:

$$\mathfrak{M}_{Higgs} \cong (\chi^a = 0, V = 0)/U(k) \tag{2.24}$$

$U(k)$ is modded out since it is a gauge symmetry, and this space is called "Higgs Branch" of D-1 theory.

Here we write $\mathfrak{M}_{Higgs}$ more explicitly. First, the constraint $\chi^a = 0$ kills all but the D-terms and F-terms in scalar potential (V),

$$\begin{aligned} &\mathrm{D-term}: \; g^2 Tr(\sum_{n=1}^{N} \omega\omega^\dagger - \tilde{\omega}^\dagger\tilde{\omega} + [Z, Z^+] + [W, W^+])^2, \\ &\mathrm{F-term}: \; g^2 Tr|\sum_{n=1}^{N} \omega\tilde{\omega} + [Z, W]|^2, \end{aligned} \tag{2.25}$$

where $Z = a'_1 + ia'_2$ and $W = a'_3 - ia'_4$.

Now, we count the dimension of $\mathfrak{M}_{Higgs}$. Consider only the bosonic sector, according to table 2, ω and $\tilde{\omega}$ give $4kN$ real degrees of freedom while Z and W give $4k^2$ real degrees of freedom by table 1. But equation (2.25) shows D-terms and F-terms impose k^2 and $2k^2$ real constraints respectively.

Finally, modding out the $U(k)$ loses k^2 real degrees of freedom. Therefore, $\dim(\mathfrak{M}_{Higgs}) = 4kN + 4k^2 - k^2 - 2k^2 - k^2 = 4kN$ which is equal to $\dim(\mathfrak{M}_k)$ where $\mathfrak{M}_k$ is the moduli space of instanton as shown in section 2.2.

Furthermore, the D and F-flatness conditions ($V = 0$) give

$$\sum_{n=1}^{N} \omega_n^\dagger \sigma^i \omega_n - i[a'_\mu, a'_\mu]\bar{\eta}^i_{\mu\nu} = 0, \tag{2.26}$$

which are precisely the ADHM constraints.

All in all, $\mathfrak{M}_{Higgs}$ has the correct dimension and gives the ADHM constraints. So the claim of ADHM construction is

$$\mathfrak{M}_{Higgs} \cong \mathfrak{M}_k.$$

To summarize, the instanton moduli space is identified with Higgs branch of a gauge theory with 8 supercharges in 0 dimension consisting of 2 adjoint hypermultiplets Z and W, and N fundemental hypermultiplets ω and $\tilde{\omega}$.

The space $\mathfrak{M}_{Higgs}$ is also a hyper-Kähler quotient. First notice an auxiliary $U(k)$ theory defines a metric on Higgs branch. The Euclidean space $\mathbb{R}^{4k(N+k)} \equiv \tilde{\mathfrak{M}}$ is parametrized by the fields $\omega, \tilde{\omega}$, Z, and W. The metric is defined as

$$ds^2 = 8\pi^2 tr(|d\omega|^2 + |dZ|^2 + |dW|^2) = 8\pi^2 tr(|d\omega|^2 + |da'|^2). \qquad (2.27)$$

Then, the Higgs branch poses 2 restrictions on $\tilde{\mathfrak{M}}$, namely

(1) Tangent to $V = 0$ hyperspace

(2) Orthogonal to $U(k)$ action

For (1), if we name this hyperspace as $\mathfrak{N} \subset \tilde{\mathfrak{M}}$, then the vectors satisfying (1) live in T$\mathfrak{N}$. Furthermore, $V = 0$ is equivalent to satisfying the D and F-flatness conditions and they give $3k^2$ constraints. Then, taking quotient of T$\mathfrak{N}$ by $U(k)$ action satisfies (2) which gives further k^2 constraints. Therefore, the dimension of the quotient space $\mathfrak{M}_k$ is

$$dim(\mathfrak{M}_k) = dim(\tilde{\mathfrak{M}}) - 4k^2 = 4k(N + k) - 4k^2 = 4Nk$$

which gives the correct dimension of instanton moduli space.

Recall that we are considering a configuration with 8 conserved supercharges and [7] shows that a Kähler quotient with 8 conserved supercharges is a hyper-Kähler quotient. Therefore, the instanton moduli space is a hyper-Kähler quotient.

3 $\mathcal{N}=1,2$ Supersymmetric Zero-Modes

3.1 $\mathcal{N}=1$ supersymmetry

3.1.1 Gaugino zero-modes

For more general calculations, it is useful to write the zero-modes in ADHM variables instead of collective coordinates. First, we would like to show that $\lambda_\alpha(\mathcal{M})$ solves the covariant Weyl equation ($\bar{D}^{\dot{\alpha}\alpha}\lambda_\alpha(\mathcal{M})=0$), and $\lambda_\alpha(\mathcal{M})$ is thus a zero-mode

$$\lambda_\alpha(\mathcal{M}) \equiv \bar{U}\mathcal{M}f\bar{b}_\alpha U - \bar{U}b_\alpha f\bar{\mathcal{M}}U, \tag{3.1}$$

where $\mathcal{M}$ is a $(N+2k)\times k$ matrix in Grassmann collective coordinate.

First, we can use the following results from [**1**], for any $J(x)$

$$\partial_n f = -f\partial_n(\frac{1}{2}\bar{\Delta}^{\dot{\alpha}}\Delta_{\dot{\alpha}})f, \tag{3.2}$$

$$\begin{aligned} D_n(\bar{U}JU) &= \partial_n(\bar{U}JU) + [A_n, \bar{U}JU] \\ &= \bar{U}\partial_n JU - \bar{U}b^\alpha\sigma_{n\alpha\dot{\alpha}}f\bar{\Delta}^{\dot{\alpha}}JU - \bar{U}J\Delta_{\dot{\alpha}}f\bar{\sigma}_n^{\dot{\alpha}\alpha}\bar{b}_\alpha U, \end{aligned} \tag{3.3}$$

substituting equations (3.2) and (3.3) into equation (3.1) give

$$\bar{D}^{\dot{\alpha}\alpha}\lambda_\alpha(\mathcal{M}) = 2\bar{U}b^\alpha f(\bar{\Delta}^{\dot{\alpha}}\mathcal{M} + \mathcal{M}\Delta_{\dot{\alpha}})f\bar{b}_\alpha U. \tag{3.4}$$

This shows $\bar{D}^{\dot{\alpha}\alpha}\lambda_\alpha(\mathcal{M})$ can be zero if

$$\bar{\Delta}^{\dot{\alpha}}\mathcal{M} + \mathcal{M}\Delta_{\dot{\alpha}} = 0. \tag{3.5}$$

Then, substituting $\Delta_{\lambda i\dot{\alpha}}(x) = a_{\lambda i\dot{\alpha}} + b^\alpha_{\lambda i}x_{\alpha\dot{\alpha}}$ into equation (3.5) gives the fermionic constraints for $\mathcal{M}$. These constraints are the "spin 1/2" superpartners of the original "spin 1" ADHM constraints:

$$\begin{aligned} \bar{\mathcal{M}}_i^\lambda a_{\lambda i\dot{\alpha}} &= -\bar{a}^\lambda_{i\dot{\alpha}}\mathcal{M}_{\lambda j}, \\ \bar{\mathcal{M}}_i^\lambda b^\alpha_{\lambda i} &= \bar{b}_i^{\alpha\lambda}\mathcal{M}_{\lambda j}. \end{aligned} \tag{3.6}$$

Similar to section (2.1), we can write b in canonical form and decompose the ADHM index $\lambda = u + i\beta$ to further solve equation (3.6)

$$\mathcal{M}_{\lambda i} \equiv \mathcal{M}_{(u+l\beta)i} = \begin{pmatrix} \mu_{ui} \\ (\mathcal{M}'_\beta)_{li} \end{pmatrix}, \tag{3.7}$$

$$\bar{\mathcal{M}}_i^\lambda \equiv \bar{\mathcal{M}}_i^{(u+l\beta)} = (\bar{\mu}_{iu}\ ,\ (\mathcal{M}'^\beta)_{il}). \tag{3.8}$$

So the second "spin 1/2" constraint collapses to

$$\bar{\mathcal{M}}'^{\alpha} = \mathcal{M}'^{\alpha}. \tag{3.9}$$

Then the constraints can be condensed into

$$\bar{\mathcal{M}} a_{\dot{\alpha}} + \bar{a}_{\dot{\alpha}} \mathcal{M} \equiv \bar{\mu}\omega_{\dot{\alpha}} + \bar{\omega}_{\dot{\alpha}}\mu + [\mathcal{M}'^{\alpha}, a'_{\alpha\dot{\alpha}}] = 0. \tag{3.10}$$

3.2 $\mathcal{N} = 2$ supersymmetry

3.2.1 Scalar non-zero VEVs

Instanton is a semi-classical saddle point of pure Yang-Mills action and it is not the dominated contribution when the gauge coupling is large. This is expected since Yang-Mills theory is asymptotically free to start with. When the gauge coupling constant runs, it is small at high energy and becomes large at low energy. Therefore, instanton only works in a weakly-coupled phase (where VEVs $\gg$ dynamical scale Λ).

Writing the VEVs (v) more explicitly

$$v = \mathrm{diag}(v_1, v_2, ..., v_N), \tag{3.11}$$

where $\sum_{a=1}^{N} v_a = 0$.

Unfortunately, pure $\mathcal{N} = 2$ theories are strongly coupled in the IR when $v_a = 0$. This problem can be avoided by the Higgs mechanism which breaks the gauge symmetry to an abelian subgroup, yielding a Coulomb phase.

More explicitly, the $\mathcal{N} = 2$ theories have two real adjoint-valued scalar fields. The classical potentials have flat directions where these two scalar fields develop VEVs and break the SU(N) gauge group to its maximal abelian subgoups $U(1)^{N-1}$ by Higgs mechanism. Then the diagonal components of all fields remain massless while the off-diagonal components gain mass $|v_a - v_b|$.

Finally, if we choose VEVs $|v_a - v_b| \gg \Lambda$ for all $a, b \in N$ where Λ is the dynamically generated scale, the theory will be weakly-coupled in all length and instanton calculus should be reliable.

Take SU(2) as an example, consider a complex scalar field $\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$, and a Higgs potential $V(\phi)$ of the form $V(\phi) = -\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2$. If μ^2 is larger than zero the scalar field develop a VEV $<\phi> = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}$ where $v^2 = \frac{\mu^2}{\lambda}$. And we see that Higgs mechanism spontaneously break the SU(2) symmetry to U(1).

3.2.2 Higgs boson zero-modes

Again, we would like to show the following equalities (equations (3.12)-(3.14)) are the solution of covariant Laplace equation with bi-fermion source $D^2\Phi = \lambda(\mathcal{M}^A)\lambda(\mathcal{M}^B)$, with the boundary condition $\lim_{x\to\infty}\Phi(x) = v$.

$$\Phi = -\frac{1}{4}\bar{U}\mathcal{M}^A f\bar{\mathcal{M}}^B U + \bar{U}\mathcal{A}U, \tag{3.12}$$

where

$$\mathcal{A} = \begin{pmatrix} v & 0 & . & . & . & 0 \\ . & & & & & \\ . & & & \mathcal{A}' & & \\ . & & & & & \\ 0 & & & & & \end{pmatrix}, \tag{3.13}$$

$$\mathcal{A}' = \mathbf{L}^{-1}(\bar{\omega}^{\dot{\alpha}}v\ \omega_{\dot{\alpha}}) + \frac{1}{4}\bar{\mathcal{M}}^A\mathcal{M}^B, \tag{3.14}$$

is a $k \times k$ matrix.

Expanding the right hand-side of the equation gives,

$$\begin{aligned}\lambda(\mathcal{M}^A)\lambda(\mathcal{M}^B) = &-\bar{U}\mathcal{M}^A f\bar{b}_\alpha\mathcal{P}\mathcal{M}^B f\bar{b}_\alpha U + \bar{U}\mathcal{M}^A f b^\alpha\mathcal{P}\bar{b}_\alpha f\bar{\mathcal{M}}^B U \\ &+ \bar{U}b_\alpha f\bar{\mathcal{M}}^A\mathcal{P}\mathcal{M}^B f\bar{b}^\alpha U - \bar{U}b_\alpha f\bar{\mathcal{M}}^A\mathcal{P}\mathcal{M}^B f b^\alpha f\bar{\mathcal{M}}^B U.\end{aligned} \tag{3.15}$$

We can use the following results again from [**1**], for any $J(x)$:

$$\begin{aligned}D^2(\bar{U}JU) = &-4\bar{U}\{b^\alpha f\bar{b}_\alpha, J\} + 4\bar{U}b^\alpha f\bar{\Delta}^{\dot{\alpha}}J\Delta_{\dot{\alpha}}f\bar{b}_\alpha U \\ &+ \bar{U}\partial^2 JU - 2\bar{U}b^\alpha f\sigma_{n\alpha\dot{\alpha}}\bar{\Delta}^{\dot{\alpha}}\partial_n JU + 2\bar{U}\partial_n J\Delta_{\dot{\alpha}}\bar{\sigma}_n^{\dot{\alpha}\alpha}f\bar{b}_\alpha U.\end{aligned} \tag{3.16}$$

Substituing equation (3.19) with $J = \frac{1}{4}\mathcal{M}^A f C'$, we expand the first term of

$D^2\Phi$:

$$D^2(\frac{1}{4}\bar{U}\mathcal{M}^A f\mathcal{M}^B U) = -\bar{U}\{b^\alpha f\bar{b}_\alpha, \mathcal{M}^A\mathcal{M}^B\} + \bar{U}b^\alpha f\bar{\Delta}^{\dot{\alpha}}\mathcal{M}^A f\mathcal{M}^B\Delta_{\dot{\alpha}} f\bar{b}_\alpha U$$
$$+ \frac{1}{4}\bar{U}\partial^2\mathcal{M}^A f\mathcal{M}^B U - \frac{1}{2}\bar{U}b^\alpha f\sigma_{n\alpha\dot{\alpha}}\bar{\Delta}^{\dot{\alpha}}\partial_n\mathcal{M}^A f\mathcal{M}^B U$$
$$+ \frac{1}{2}\bar{U}\partial_n\mathcal{M}^A f\mathcal{M}^B\Delta_{\dot{\alpha}}\bar{\sigma}_n^{\dot{\alpha}\alpha} f\bar{b}_\alpha U. \tag{3.17}$$

After massaging the terms, we see that the right and left hand-side do not match exactly

$$D^2(\frac{1}{4}\bar{U}\mathcal{M}^A f\mathcal{M}^B U) = \lambda(\mathcal{M}^A)\lambda(\mathcal{M}^B) - \bar{U}b_\alpha f\bar{\mathcal{M}}^A\mathcal{M}^B f\bar{b}_\alpha U. \tag{3.18}$$

The second term of ϕ will deal with the extra term. This time by substituing equation (3.19) with $J = \mathcal{A}$ and $\partial_n\phi = 0$, we get

$$D^2[\bar{U}\begin{pmatrix} v & 0 \\ 0 & \mathcal{A}' \end{pmatrix} U] = 4\bar{U}b^\alpha[-\{f,\varphi\} + f\bar{\Delta}^{\dot{\alpha}}\begin{pmatrix} v & 0 \\ 0 & \mathcal{A}' \end{pmatrix}\Delta_{\dot{\alpha}} f]\bar{b}_\alpha U. \tag{3.19}$$

In order to cancel the extra term, the following equality has to be satisfied:

$$\begin{aligned} -4f\bar{\mathcal{M}}^A\mathcal{M}^B f &\equiv \{f,\mathcal{A}'\} + f\bar{\Delta}^{\dot{\alpha}}\begin{pmatrix} v & 0 \\ 0 & \varphi 1_{[2]\times[2]} \end{pmatrix}\Delta_{\dot{\alpha}} f \\ &= \{f,\mathcal{A}'\} + f(\bar{\omega}^{\dot{\alpha}} v\ \omega_{\dot{\alpha}} - \mathbf{L}\mathcal{A}' + \{\mathcal{A}', f^{-1}\})f \\ &= f(\bar{\omega}^{\dot{\alpha}} v\ \omega_{\dot{\alpha}} - \mathbf{L}\mathcal{A}')f. \end{aligned} \tag{3.20}$$

Therefore, we get the "spin 0"constraint below:

$$\mathbf{L}\cdot\mathcal{A}' = (\bar{\omega}^{\dot{\alpha}}\ v\ \ \omega_{\dot{\alpha}}) + \frac{1}{4}\bar{\mathcal{M}}^A\mathcal{M}^B. \tag{3.21}$$

$\mathbf{L}$ is an operator on $k \times k$ Hermitian matrix of the form,

$$\begin{aligned} \mathbf{L}\ \cdot\ \Omega\ &\equiv \frac{1}{2}\{\bar{\omega}^{\dot{\alpha}}\omega_{\dot{\alpha}}, \Omega\} + \frac{1}{2}\bar{a}'^{\dot{\alpha}\alpha}a'_{\dot{\alpha}\alpha} - \bar{a}'^{\dot{\alpha}\alpha}\Omega a'_{\dot{\alpha}\alpha} + \frac{1}{2}\Omega\bar{a}'^{\dot{\alpha}\alpha}a'_{\dot{\alpha}\alpha} \\ &= \frac{1}{2}\{\bar{\omega}^{\dot{\alpha}}\omega_{\dot{\alpha}}, \Omega\} + [a'_n, [a'_n, \Omega]]. \end{aligned} \tag{3.22}$$

4 Supersymmetry and BRST algebra of Instanton Moduli Space

4.1 Algebraic Construction of BRST Transformation

To study the symmetries of the zero-modes, we start by performing an infinitesimal scaler variation(s) of the bosonic ADHM constraints ($\bar{\Delta}\Delta = f^{-1} = (\bar{\Delta}\Delta)^T$),

$$(s\bar{\Delta})\Delta + \bar{\Delta}(s\Delta) = [(s\bar{\Delta})\Delta]^T + [\bar{\Delta}(s\Delta)]^T. \tag{4.1}$$

Then, we guess the solution of $s\Delta$, that will satisfy this constraint, has the following form

$$s\Delta = \mathcal{M} - \mathcal{C}\Delta, \tag{4.2}$$

where $\mathcal{M}$ is defined before in gaugino zero-mode. Here $s\Delta$ is a $(N+2k)\times 2k$ complex-valued matrix while $\mathcal{M}$ is a matrix with $(N+2k)\times 2k$ real Grassmann variables. So this equation has the correct dimension.

Next, we derive the expression and constraints of $\mathcal{C}$ from equation (4.2). From analyzing the dimension of the solution, $\mathcal{C}$ has to be a $(N+2k)\times(N+2k)$ matrix. The structure of $\mathcal{C}$ is constrained by Δ and $\mathcal{M}$, and thus by the "spin 1" and "spin 1/2" constraints. So the most general expression satisfying all conditions is

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_{00} & 0 & . & . & . & 0 \\ . & & & & & \\ . & & & \mathcal{C}' & & \\ . & & & & & \\ 0 & & & & & \end{pmatrix}, \tag{4.3}$$

where $\mathcal{C}'$ is a real antisymmetric $k\times k$ matrix, $\bar{\mathcal{C}}' = -\mathcal{C}'$. After that, the constraints of $\mathcal{C}$ can be found by substituting (4.2) into (4.1)

$$\bar{\Delta}(\mathcal{C}+\bar{\mathcal{C}})\Delta = [\bar{\Delta}(\mathcal{C}+\bar{\mathcal{C}})\Delta]^T, \tag{4.4}$$

and the constraint is

$$\mathbf{L}\cdot(\mathcal{C}'+\bar{\mathcal{C}}') = \bar{\omega}^{\dot{\alpha}}(\mathcal{C}_{00}+\bar{\mathcal{C}}_{00})\omega_{\dot{\alpha}}, \tag{4.5}$$

where $\mathbf{L}$ is an invertible operator defined before in equation (3.25). As a result, $-\bar{\mathcal{C}_{00}} = \mathcal{C}_{00}$ follows the antisymmetry of $\mathcal{C}'$.

Then, we perform $s-$variation of the fermionic "spin 1/2" constraints ($\bar{\Delta}\mathcal{M} + \mathcal{M}\Delta = 0$)

$$(s\bar{\Delta})\mathcal{M} + \bar{\Delta}(s\mathcal{M}) = [(s\bar{\Delta})\mathcal{M} + \bar{\Delta}(s\mathcal{M})]^T. \tag{4.6}$$

Substituting (4.2) into it, gives

$$(\bar{\mathcal{M}} + \bar{\Delta}\mathcal{C})\mathcal{M} - [(\bar{\mathcal{M}} + \bar{\Delta}\mathcal{C})\mathcal{M}]^T = [\bar{\Delta}(s\mathcal{M})]^T - \bar{\Delta}(s\mathcal{M}). \tag{4.7}$$

Same as before, we guess the solution of $s\mathcal{M}$ satisfying the above equation has the following form

$$s\mathcal{M} = \mathcal{A}\Delta - \mathcal{C}\mathcal{M}, \tag{4.8}$$

where $\mathcal{A}$ is defined before in equation (3.24). The constraints of $\mathcal{A}$ can be found by substituting (4.8) into (4.7)

$$\bar{\Delta}\mathcal{A}\Delta - (\bar{\Delta}\mathcal{A}\Delta)^T = (\bar{\mathcal{M}}\mathcal{M})^T - \bar{\mathcal{M}}\mathcal{M}, \tag{4.9}$$

and the constraint is

$$\mathbf{L} \cdot \mathcal{A}' = (\bar{\omega}^{\dot{\alpha}} v \ \omega_{\dot{\alpha}}) + [\bar{\mathcal{M}}\mathcal{M} - (\bar{\mathcal{M}}\mathcal{M})^T], \tag{4.10}$$

which is the same as the constraint of $\mathcal{A}$ found before by solving covariant Laplace equation.

Next, we would like the operator s to be nilpotent, such that it is a BRST operator

$$s^2\Delta = (\mathcal{A} - s\mathcal{C} - \mathcal{C}\mathcal{C})\Delta \equiv 0, \tag{4.11}$$

$$s^2\mathcal{M} = (\mathcal{A} - s\mathcal{C} - \mathcal{C}\mathcal{C})\mathcal{M} + (s\mathcal{A} + [\mathcal{C}, \mathcal{A}])\Delta \equiv 0. \tag{4.12}$$

Finally, since $\mathcal{A}'$ is a supersymmetry invariant, the BRST variation of the super-constraints of $\mathcal{A}$ (4.9) should be 0,

$$\bar{\Delta}(s\mathcal{A} + [\mathcal{C}, \mathcal{A}])\Delta - [\bar{\Delta}(s\mathcal{A} + [\mathcal{C}, \mathcal{A}])\Delta]^T = 0, \tag{4.13}$$

which is satisfied because of equation (4.12).

To sum up, the BRST algebra of the ADHM variables on instanton moduli space are

$$\begin{aligned} s\Delta &= \mathcal{M} - \mathcal{C}\Delta, \\ s\mathcal{M} &= \mathcal{A}\Delta - \mathcal{C}\mathcal{M}, \\ s\mathcal{A} &= -[\mathcal{C}, \mathcal{A}], \\ s\mathcal{C} &= \mathcal{A} - \mathcal{C}\mathcal{C}. \end{aligned} \tag{4.14}$$

Since we are working in the instanton moduli space which has a redundant symmetry $U(k)$, every expression in this space has to be covariant with respect to this $U(k)$. And the above BRST algebra suggests that $Q = s + \mathcal{C}$ can act as covariant derivative for the moduli space where $\mathcal{C}$ is the $U(k)$-connection.

$$\begin{aligned} Q\Delta &= \mathcal{M}, \\ Q\mathcal{M} &= \mathcal{A}\Delta, \\ Q\mathcal{A} &= 0, \\ Q\mathcal{C} &= \mathcal{A}. \end{aligned} \tag{4.15}$$

Observing the above algebra, Q acts as a supersymmetry operator for the ADHM variables in the moduli space. And $\mathcal{A}$ is the field strength or curvature of the connection $\mathcal{C}$.

4.2 Strategy of solving Instanton Measure

In order to perform integration over zero-modes, one has to integrate over collective coordinates instead of bosonic zero-modes, which will give rise to a bosonic Jacobian.

All in all, the integration is done by projecting the fields A, ψ, ϕ onto the zero-modes subspace depending on $\Delta, \mathcal{M}, \mathcal{A}, \mathcal{C}$. We choose $(\{\Delta_i\}\{\mathcal{M}_i\})$, $i = 1, ..., 4Nk$, as a basis of ADHM coordinates on $\mathfrak{M}_k$ since all the fields can be written in terms of them. Equation (4.14) can be written as

$$\mathcal{M}_i = s\Delta_i + (\mathcal{C}\Delta)_i, \tag{4.16}$$

where the coefficients of the expansion are totally antisymmetric. One can show $\mathcal{M}_i$ and $s\Delta_i$ are related by linear transformation K_{ij} by substituting $\mathcal{C} = \begin{pmatrix} \mathcal{C}_{00} & 0 \\ 0 & \mathcal{C}' \end{pmatrix}$ into $s\Delta = \mathcal{M} - \mathcal{C}\Delta$,

$$\mathcal{M}_i = K_{ij}(\Delta) s\Delta_j. \tag{4.17}$$

Expanding this to p fields can be written as

$$\begin{aligned} \mathcal{M}_{i_1}\mathcal{M}_{i_2}.....\mathcal{M}_{i_p} &= K_{i_1j_1}K_{i_2j_2}....K_{i_pj_p} s\Delta_{j_1} s\Delta_{j_2}....s\Delta_{j_p} \\ &= \varepsilon_{j_1....j_p} K_{i_1j_1}K_{i_2j_2}....K_{i_pj_p} s^p\Delta \\ &= \varepsilon_{j_1....j_p} (detK) s^p\Delta, \end{aligned} \tag{4.18}$$

where $s^p \equiv s\Delta_1....s\Delta_p$.

A generic function (g) on the zero-mode subspace has the expansion

$$
\begin{aligned}
g(\Delta, \mathcal{M}) = g_0(\Delta) + g_{i_1}(\Delta)\mathcal{M}_{i_1} + \frac{1}{2!}g_{i_1 i_2}(\Delta)\mathcal{M}_{i_1}\mathcal{M}_{i_2} + ...\\
+ \frac{1}{p!}g_{i_1 i_2 .....i_p}(\Delta)\mathcal{M}_{i_1}\mathcal{M}_{i_2}.....\mathcal{M}_{i_p},
\end{aligned}
\tag{4.19}
$$

Substituting equation (4.18) to (4.19) and performing integration on the moduli space gives

$$
\begin{aligned}
\int_{\mathfrak{M}_k} g(\Delta, \mathcal{M}) &= \frac{1}{p!}\int_{\mathfrak{M}_k} g_{i_1 i_2 .....i_p}(\Delta)\mathcal{M}_{i_1}\mathcal{M}_{i_2}.....\mathcal{M}_{i_p}\\
&= \int_{\mathfrak{M}_k} s^p \Delta |detK| g_{i_1 i_2 .....i_p}(\Delta).
\end{aligned}
\tag{4.20}
$$

Here, $|\mathrm{det}K|$ is the instanton integration measure for $\mathcal{N} = 2$ SYM theory. Instead of obtaining it as a ratio of bosonic and fermionic zero-modes Jacobian, it can also be calculated through studying the connection $\mathcal{C}$.

However, adopting this method involves solving the ADHM constraints, and in practice it can only be done for $k = 1, 2$. So instead, we will introduce the constraints by hand as a δ-function in the action accompanied by a supermultiplet of Lagrange multiplier. This will be done explicitly in section (6.2).

5 Regularization of the Moduli Space

5.1 Small Instanton Singularity

As shown in the introduction, there exists singularities in the moduli space preventing $\mathfrak{M}_k$ to be a smooth manifold. The singularities are points on the level set $\mathfrak{N}$ where some subgroups of the auxiliary group $U(k)$ do not act freely.

Using $k = 1$ as an example. The auxiliary group $U(1)$ acts as rotation $\omega_{\dot\alpha} = e^{i\phi}\omega_{\dot\alpha}$ where $\phi \in U(1)$. This $U(1)$ does not act freely at $\omega_{\dot\alpha} = 0$. In this case, by identifying $a'_n = -X_n \in \mathbf{R}^4$, the ADHM constraints reduce to $\bar\omega^{\dot\alpha}\omega_{\dot\beta} = \rho^2\delta^{\dot\alpha}_{\dot\beta}$ where ρ is identified as the size of instanton. Therefore, there is a localized singularity at $\rho = 0$ as advertised before.

For $k > 1$, the small singularities are points where $\omega = 0$ and all the non-trace parts of a' vanish. In this limit, the subgroup of $U(k)$ does not act freely. This corresponds to the extreme case where the whole $U(k)$ is fixed. First, we start with the case with one fixed $U(1)$ subgroup. Here, the k-instanton configuration looks like a smooth $(k-1)$-instanton with a single instanton shrink to point in $\mathbb{R}^4$. In other words, $\mathcal{M}_k \to \mathcal{M}_{k-1} \times \mathbb{R}^4$. In general, there can be r fixed subgroups $U(1)^r \subset U(k)$. And the partial compactification of the moduli space becomes

$$\mathcal{M}_k \to \mathcal{M}_{k-r} \times Sym^r\mathbb{R}^4, \tag{5.1}$$

where $Sym^r\mathbb{R}^4$ is the symmetric product of r points in $\mathbb{R}^4$. The maximally degeneracy occurs when the whole $U(k)$ is fixed. This is where all k instantons shrink to a point in $\mathbb{R}^4$. This point is the apex of the hyper-Kähler cone.

5.2 ADHM Construction of Non-Commutative Instanton

Let x^n be the spatial coordinate, non-commutative (NC) space is defined by

$$[x^m, x^n] = i\theta^{mn}, \tag{5.2}$$

where θ^{mn} is a real antisymmetric tensor, in 4-dimensional Eucledian space, it takes the canonical form

$$\theta^{mn} = \left(\begin{array}{cc|cc} 0 & -\theta_1 & 0 & 0 \\ \theta_1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -\theta_2 \\ 0 & 0 & \theta_2 & 0 \end{array}\right), \tag{5.3}$$

where the real numbers θ_1, θ_2 are called non-commutative parameters. In commutative space, θ_1, θ_2 equal to 0 thus $\theta^{mn} = 0$. In complex coordinates $(z_1\ ,\ z_2)$, we can rewrite x^n as $z_1 = x^2 + ix^1$ and $z_2 = x^4 + ix^3$, then the non-commutativity can be expressed as

$$[z_1, \bar{z}_1] = 2\theta_1 \quad , \quad [z_2, \bar{z}_2] = 2\theta_2. \tag{5.4}$$

Now we can start the ADHM construction in NC space. First, we rewrite the ADHM datas in terms of quadruple of complex matrices (B_1, B_2, I, J) by using relation $a = \begin{pmatrix} \bar{I} & J \\ \bar{B}_2 & -B_1 \\ \bar{B}_1 & B_2 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 0 \\ \mathbf{I}_k & 0 \\ 0 & \mathbf{I}_k \end{pmatrix}$ where $B_{1,2}$ are $k \times k$

matrices, and I and J are $k \times N$ and $N \times k$ matrices respectively.

Then by equation (2.6), $\Delta(x)$ becomes

$$\Delta(x) = a + b(x) = \begin{pmatrix} \bar{I} & J \\ \bar{z}_2 - \bar{B}_2 & -(z_1 - B_1) \\ \bar{z}_1 - \bar{B}_1 & z_2 - B_2 \end{pmatrix}, \tag{5.5}$$

and the ADHM constraints become

$$\mu_{\mathbb{R}} \equiv [B_1, \bar{B}_1] + [B_2, \bar{B}_2] + I\bar{I} - \bar{J}J = \zeta, \tag{5.6a}$$

$$\mu_{\mathbb{C}} \equiv [B_1, B_2] + IJ = 0. \tag{5.6b}$$

They can also be written in terms of $\{\omega, a'\}$, then the modified ADHM constraints are

$$\bar{\omega}\tau^c\omega - i\bar{\eta}^c_{mn}[a'_m, a'_n] = \zeta^c, \tag{5.7}$$

where $\zeta \equiv -[z_1, \bar{z}_1] - [z_2, \bar{z}_2] = -2(\theta_1 + \theta_2)$ is a non-negative constant.

If $\zeta = 0$ (i.e. $-\theta_1 = \theta_2$), then the constraint collapses to commutative one. For example, this can happen when antisymmetric tensor θ^{mn} is anti-self-dual (e.g. $\theta^{12} = -\theta^{34}$). So only the self-dual part of ζ contributes in anti-instanton background and vice versa.

On the other hand, if $\zeta \neq 0$, we can always chose the orientation such that $\theta_1 < 0$ and $\theta_2 < 0$ to ensure $\zeta > 0$. Moreover, in the perspective of low energy effective theory of D-1 branes, ζ is the Fayet-Iliopoulous (FI) parameter.

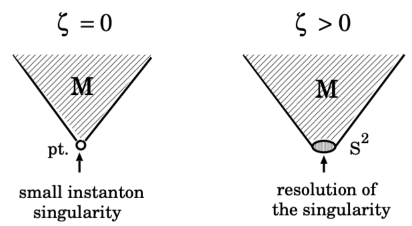

Figure 2: One $U(2)$ instanton moduli space. Reprinted from [**4**].

Figure 2 shows the role of ζ in the geometry of instanton moduli space. $\zeta = 0$ is the case of instanton on commutative $\mathbb{R}^4$ and the moduli space contains the small instanton singularity. When $\zeta > 0$, the small instanton singularity is resolved and replaced by a new class of smooth instantons, $U(1)$ instantons.

This can be shown explicitly by solving the simplest case in both spaces, namely $SU(2)$ 1-instanton in commutative $\mathbb{R}^4$ and $U(2)$ 1-instanton at origin of NC space.

5.3 SU(2) 1-instanton in Commutative $\mathbb{R}^4$

$B_{1,2}$ are chosen as arbitrary complex number $\alpha_{1,2}$. Then I , J can be found by solving the constraints with $\zeta = 0$ using real number ρ

$$B_1 = \alpha_1 \quad , \quad B_2 = \alpha_2 \quad , \quad I = (\rho \ , 0) \quad , \quad J = \begin{pmatrix} 0 \\ \rho \end{pmatrix} . \tag{5.8}$$

Then using $\bar{\Delta} U = 0$, $\alpha_1 = b_2 + ib_1$, and $\alpha_2 = b_4 + ib_3$, we can find U

$$U = \frac{1}{\sqrt{\phi}} \begin{pmatrix} \bar{e_n}(x_n - b_n) & \\ -\rho & 0 \\ 0 & -\rho \end{pmatrix} \quad , \quad \phi = |x - b|^2 + \rho^2, \tag{5.9}$$

where the normalization factor ϕ can be found by $\bar{U} U = 1$. Finally, the instanton solution can be constructed

$$A_n = \bar{U} \partial_n U = \frac{2\rho^2 (x - b)_m \sigma_{mn}}{(x - b)^2 + \rho^2}, \tag{5.10}$$

which is the same regular gauge solution shown in equation (1.7).

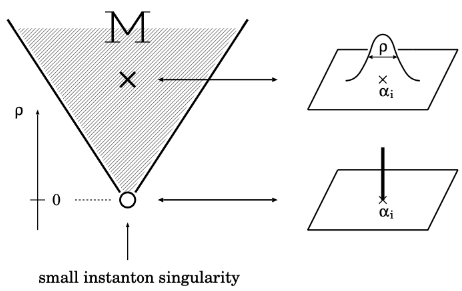

Figure 3: Instanton moduli space and configuration in commutative space. Reprinted from [**4**].

Recall that the moduli space of $SU(2)$ 1-instanton (2.4) is $\mathfrak{M} = \mathbf{R}^4 \times (\mathbf{R}^4 \backslash \{0\})/\mathbf{Z}_2$. And figure 3 shows as $\rho \to 0$ in commutative $\mathbb{R}^4$, the solution approaches the small instanton singularity.

5.4 $U(2)$ 1-instanton in Non-Commutative $\mathbb{R}^4$

Same as before, $B_{1,2}$ are chosen as arbitrary complex number $\alpha_{1,2}$. Then we solve the constraints for nonzero ζ. This gives

$$B_1 = \alpha_1 \quad , \quad B_2 = \alpha_2 \quad , \quad I = (\sqrt{\rho^2 + \zeta}\ , 0) \quad , \quad J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}. \tag{5.11}$$

Here, only I is deformed by non-commutativity and this will keep the configuration smooth in $\rho \to 0$ limit where the ADHM datas become

$$B_1 = \alpha_1 \quad , \quad B_2 = \alpha_2 \quad I = (\sqrt{\zeta}\ , 0) \quad , \quad J = 0. \tag{5.12}$$

And the ADHM constraints (5.6a and 5.6b) become

$$I\bar{I} - \bar{J}J = \zeta > 0, \tag{5.13a}$$

$$IJ = 0. \tag{5.13b}$$

Equation (5.13a) shows I, J are in $\mathbf{S}^{2N-1}$. Modding out the $U(1)$ residual symmetry gives a complex projective space $\mathbb{CP}^{N-1}$ with size ζ. Finally, (5.13b) indicates that I and J are orthogonal to each other so we should consider the cotangent space of $\mathbb{CP}^{N-1}$, namely $T^*\mathbb{CP}^{N-1}$. Therefore, the moduli in NC space is

$$\mathfrak{M}^{NC} = \mathbf{R}^4 \times T^*\mathbb{CP}^{N-1}. \tag{5.14}$$

Since the size of zero section of $\mathbb{CP}^{N-1}$ is ζ, the singularity is resolved. The effect can be demonstrated in figure 4 where the moduli space contains no singularity and is therefore regularized.

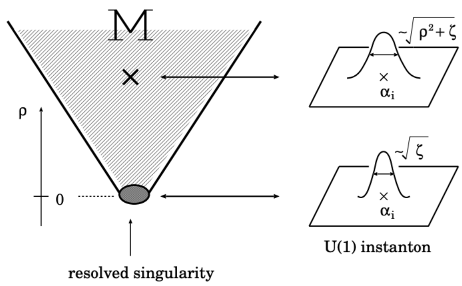

Figure 4: Instanton modli space and configuration in non-commutative space. Reprinted from [**4**].

In particular, the moduli space of $U(2)$ 1-instanton is $\mathbf{R}^4 \times T^*\mathbf{S}^2$. This space inherits a hyper-Kähler metric known as Eguchi-Hanson metric [**5**],

$$ds^2_{EH} = (1 - 4\zeta^2/\rho^2)^{-1} d\rho^2 + \frac{\rho^2}{4}(\sigma_1^2 + \sigma_2^2 + (1 - 4\zeta^2/\rho^4)\sigma_3^2), \tag{5.15}$$

where σ_i are SU(2)-invariant one form, which in polar angles $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and $0 \leq \psi \leq 2\pi$ are

$$\begin{aligned}
\sigma_1 &= -\sin\psi d\theta + \cos\psi \sin\theta d\phi, \\
\sigma_2 &= \cos\psi d\theta + \sin\psi \sin\theta d\phi, \\
\sigma_3 &= d\psi + \cos\theta d\phi.
\end{aligned} \tag{5.16}$$

Studying the metric, the first term $(1 - 4\zeta^2/\rho^2)^{-1}$ shows that as we approach the origin, the scale size is cut at $\rho^2 = 2\zeta$, and the singularity is resolved to zero section of $\mathbf{S}^2$.

All in all, this section shows that the instanton moduli space can be regularized in NC space setting. This allows some mathematical operations such as localization to take place.

6 $\mathcal{N} = 2$ SQCD Multi-Instanton Action

6.1 Topological Twist

The classical global symmetry group of the $\mathcal{N} = 2$ SUSY theory in flat space is

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{SU}(2)_A, \tag{6.1}$$

where $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ is the Lorentz group and $\mathrm{SU}(2)_A$ is the automorphism group of supercharges. Moreover, the $\mathrm{SU}(2)_A$ comes form breaking the SU(4), $\mathcal{N} = 4$ R-symmetry group to $\mathcal{N} = 2$ R-symmetry group as described in section 2.3.

The topological twist involves replacing the $\mathrm{SU}(2)_R$ of the rotation group with $\mathrm{SU}(2)'_R$ which is a diagonal subgroup of $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_A$. The symmetry group of the twisted theory becomes

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)'_R. \tag{6.2}$$

The twisted group $\mathrm{SU}(2)'_R$ mixes up the spacetime symmetry and R-symmetry, and the $\mathcal{N}=2$ SUSY charges ($\mathcal{Q}$) are decomposed into a scalar (Q), vector (Q_n), and self-dual antisymmetric tensor (Q_{mn}) with respect to $\mathrm{SU}(2)'_R$

$$\begin{aligned} \bar{\mathcal{Q}}_{\dot{\alpha}I} &\to Q \oplus Q_{mn}, \\ \mathcal{Q}_{\alpha I} &\to Q_n, \end{aligned} \tag{6.3}$$

where $I=1,2$ are indices of $SU(2)_A$. This scalar charge is the covariant derivative of the moduli space described in equation (4.15). Q transforms the field content of $\mathcal{N}=2$ ($A_n, \lambda_\alpha, \phi$) as follows

$$\begin{aligned} QA_n &= \lambda_\alpha, \\ Q\lambda_\alpha &= -D_n\phi, \\ Q\phi &= 0. \end{aligned} \tag{6.4}$$

This shows Q is not nilpotent but nilpotent modulo gauge transformation with parameter ϕ

$$\begin{aligned} Q^2A_n &= -D_n\phi, \\ Q^2\lambda_\alpha &= -[\phi,\lambda_\alpha], \\ Q^2\phi &= 0. \end{aligned} \tag{6.5}$$

A nilpotent BRST charge s can be obtained by introducing a gauge symmetry s_g and a ghost field c

$$\begin{aligned} s_gA &= -Dc, \\ s_g\lambda &= -c\lambda, \\ s_g\phi &= -c\phi, \\ s_gc &= -\frac{1}{2}cc. \end{aligned} \tag{6.6}$$

We define the BRST charge as $s \equiv s_g + Q$. Requiring $s^2=0$, we get $Qc=\phi$, and the BRST charge in equation (4.14) is recovered

$$\begin{aligned} sA &= \lambda - Dc, \\ s\lambda &= -c\lambda - D\phi, \\ s\phi &= -c\phi, \\ sc &= -\frac{1}{2}cc + \phi. \end{aligned} \tag{6.7}$$

The last line of equation (6.7) shows that the scalar field ϕ is the curvature of the connection c, similar to equation (4.15).

Moreover, it should be noted that the fermionic fields are affected by the twist

$$\begin{aligned} \mu^{\dot{A}} &\to (\frac{i\pi}{\sqrt{2}})^{\frac{1}{2}}\mu^{\dot{A}}, \\ \mathcal{M}'^{\alpha\dot{A}} &\to (\frac{i\pi}{\sqrt{2}})^{\frac{1}{2}}\mathcal{M}'_n\bar{\sigma}_n^{\alpha\dot{A}}, \end{aligned} \tag{6.8}$$

where $\alpha = \dot{A} = 1, 2$ since we are working in $\mathcal{N} = 2$. Therefore, the fermionic constraints in equation (3.14) are modified accordingly

$$\begin{aligned} \bar{\omega}^{\dot{\alpha}}\mu_{\dot{\alpha}} - \bar{\mu}^{\dot{\alpha}}\omega_{\dot{\alpha}} - 2[a'_n, \mathcal{M}'_n] = 0, \\ \bar{\omega}^{\dot{\alpha}}\tau^c\mu_{\dot{\alpha}} + \bar{\mu}^{\dot{\alpha}}\tau^c\omega_{\dot{\alpha}} - 2i\bar{\eta}^c_{mn}[a'_n, \mathcal{M}'_n] = 0. \end{aligned} \tag{6.9}$$

The first line is calculated by considering twisting of μ and the second line by considering the twisting of $\mathcal{M}'$ together with $\sigma_{mn} \equiv i\bar{\eta}^c_{mn}\tau_c$ and $\tau^2 = 1$.

6.2 Auxiliary fields for the ADHM constraints

First, let us recall all the ADHM constraints, namely the bosonic constraints deformed by non-commutative spacetime equation (5.7) and the topological twisted fermionic constraints equation (6.9).

$$\begin{aligned} \bar{\omega}\tau^c\omega - i\bar{\eta}^c_{mn}[a'_m, a'_n] = \zeta^c, \\ \bar{\omega}^{\dot{\alpha}}\mu_{\dot{\alpha}} - \bar{\mu}^{\dot{\alpha}}\omega_{\dot{\alpha}} - 2[a'_n, \mathcal{M}'_n] = 0, \\ \bar{\omega}^{\dot{\alpha}}\tau^c\mu_{\dot{\alpha}} + \bar{\mu}^{\dot{\alpha}}\tau^c\omega_{\dot{\alpha}} - 2i\bar{\eta}^c_{mn}[a'_n, \mathcal{M}'_n] = 0. \end{aligned} \tag{6.10}$$

Then, we associate auxiliary field D_c to the bosonic constraints and D_c has a superpartner ψ_c. ϕ generator of $U(k)$ residual group, under which the constraints are invariant. ϕ has a superpartner η. This η is associated to superconformal invariance broken by the presence of VEVs. The doublets (D_c, ψ_c) and $(\bar{\phi}, \eta)$ are transformed under the scalar supercharge Q as

$$\begin{aligned} &Q\psi_c = D_c, \quad QD_c = [\phi, \psi_c], \\ &Q\bar{\phi} = \eta, \qquad Q\eta = [\phi, \bar{\phi}], \\ &Q\phi = 0, \qquad Qv = Q\bar{v} = 0, \\ &Q\zeta^c = 0. \end{aligned} \tag{6.11}$$

6.3 Multi-instanton action

Summarizing all the ingredients, we are ready to write the cohomological action implementing the ADHM constraints

(1) The ADHM constraints, equation (6.10)

(2) Auxiliary fields associating the the constraints, equation (6.11)

(3) Turning on the VEVs v, section (3.2.1)

$$\begin{aligned} S = tr_k Q[(\bar{\phi}+\bar{v})(\bar{\omega}^{\dot{\alpha}}\mu_{\dot{\alpha}} - \bar{\mu}^{\dot{\alpha}}\omega_{\dot{\alpha}} - 2[a'_n, \mathcal{M}'_n]) + \frac{1}{g_0^2}\eta[\phi,\bar{\phi}] \\ +\psi_c(\bar{\omega}\tau^c\omega - i\bar{\eta}^c_{mn}[a'_m, a'_n] - \zeta^c) - \frac{1}{g_0^2}\psi_c D_c]. \end{aligned} \tag{6.12}$$

Finally, we get a $\mathcal{Q}$-close pure $\mathcal{N}=2$ SYM action.

Then, we can rewrite the covariant derivative relations (4.15) by using $(\omega, a'), (\mu, \mathcal{M}'), (v, \phi)$ instead of $\Delta, \mathcal{M}, \mathcal{A}$,

$$\begin{aligned} &Qa'_n = \mathcal{M}'_n, && Q\mathcal{M}'_n = [\phi, a'_n], \\ &Q\omega_{\dot{\alpha}} = \mu_{\dot{\alpha}}, && Q\mu_{\dot{\alpha}} = -\omega_{\dot{\alpha}}(\phi+v), \\ &Q\bar{\omega}^{\dot{\alpha}} = \bar{\mu}^{\dot{\alpha}}, && Q\bar{\mu}^{\dot{\alpha}} = (\phi+v)\bar{\omega}^{\dot{\alpha}}. \end{aligned} \tag{6.13}$$

Here, g_0^2 acts as a "gauge-fixing" parameter. If $g_0^2 \to \infty$, the ADHM constraints will be implemented as Dirac delta function instead of a Gaussian weight being spread out.

Equation (6.13) can be shown by expanding the equations in (4.15). For example,

$$\begin{aligned} Q\mathcal{M} &= \mathcal{A}\Delta \\ &= \begin{pmatrix} v & 0 \\ 0 & \phi \end{pmatrix} \begin{pmatrix} \omega \\ a' \end{pmatrix} = \begin{pmatrix} v\omega_1 - \omega_m \mathcal{A}'_{m1} & .... & v\omega_k - \omega_m\mathcal{A}'_{mk} \\ & [\phi, a'] & \end{pmatrix}. \end{aligned} \tag{6.14}$$

6.3.1 Rewriting the Multi-instanton action

For the convenience of later section, we rewrite the action and its $\mathcal{Q}$-relations in terms of (B_1, B_2, I, J) and their fermionic partners $(\mathcal{M}_1, \mathcal{M}_2, \mu_I, \mu_J)$ as in section (5.2). The BRST relations in equation (6.13) become

$$\begin{aligned} &QI = \mu_I, && Q\mu_I = \phi I - Iv, \\ &QJ = \mu_J, && Q\mu_J = -\phi J + vJ, \\ &QB_1 = \mathcal{M}_1, && Q\mathcal{M}_1 = [\phi, B_1], \\ &QB_2 = \mathcal{M}_2, && Q\mathcal{M}_2 = [\phi, B_2], \end{aligned} \tag{6.15}$$

and the ADHM constraints are

$$\begin{aligned}\boldsymbol{\varepsilon}_{\mathbb{R}} &\equiv [B_1, \bar{B}_1] + [B_2, \bar{B}_2] + I\bar{I} - \bar{J}J - \zeta = 0,\\ \boldsymbol{\varepsilon}_{\mathbb{C}} &\equiv [B_1, B_2] + IJ = 0.\end{aligned} \tag{6.16}$$

Furthermore, we rename ψ_c and D_c to $\vec{\chi} = (\chi_{\mathbb{R}}, \chi_{\mathbb{C}})$ and $\vec{H} = (H_{\mathbb{R}}, H_{\mathbb{C}})$ respectively, and condense $\mathcal{M}_1, \mathcal{M}_2$ and B_1, B_2 to $\mathcal{M}_{\hat{l}}, B_{\hat{l}}$ with $\hat{l} = 1, 2$ respectively. The action in equation (6.12) becomes

$$\begin{aligned}S = tr_k Q\{&[\mu_I(I^\dagger\bar{\phi} - \bar{v}I^\dagger) + \mu_J(\bar{\phi}J^\dagger - J^\dagger\bar{v}) + \mathcal{M}_{\hat{l}}[\bar{\phi}, B_{\hat{l}}^\dagger]] + h.c.\\ &+ \chi_{\mathbb{R}}\boldsymbol{\varepsilon}_{\mathbb{R}} + \chi_{\mathbb{C}}\boldsymbol{\varepsilon}_{\mathbb{C}} + \frac{1}{g_0^2}(\eta[\phi, \bar{\phi}] + \vec{H}\cdot\vec{\chi})\},\end{aligned} \tag{6.17}$$

and the $\mathcal{Q}$-relations become

$$\begin{aligned}&Q\vec{\chi} = \vec{H}, &&Q\vec{H} = [\phi, \vec{\chi}],\\ &Q\bar{\phi} = \eta, &&Q\eta = [\phi, \bar{\phi}],\\ &Q\phi = 0, &&Qv = Q\bar{v} = 0.\end{aligned} \tag{6.18}$$

Finally, we group the fields as $B_{\hat{s}} \equiv (I, J^\dagger, B_{\hat{l}})$ and $\Psi_{\hat{s}} \equiv (\mu_I, \mu_J^\dagger, \mathcal{M}_{\hat{l}})$ with $\hat{s} = 1, .., 4$. The action becomes

$$S = tr_k Q\{\frac{1}{g_0^2}(\eta[\phi, \bar{\phi}] + \vec{H}\cdot\vec{\chi}) - i\vec{\varepsilon}\cdot\vec{\chi} - \frac{1}{2}\sum_{\hat{s}=1}^{4}[\Psi_{\hat{s}}^\dagger(\bar{\phi})\cdot B_{\hat{s}} + \Psi_{\hat{s}}(\bar{\phi})\cdot B_{\hat{s}}^\dagger]\}. \tag{6.19}$$

6.4 Localization

In the following section, we will introduce the concept of localization and try to show that the path integral with respect to the action (6.19) can be be solved by localization.

6.4.1 Localization argument

Let Q be a fermionic symmetry generator such that $Q^2 = 0$ or $Q^2 = B$ where B is the bosonic symmetry generator. Suppose the action S is preserved by Q

$$QS = 0, \tag{6.20}$$

and we would like to compute the expectation value of Q-closed gauge invariant operators $\mathcal{O}$ ($Q\mathcal{O} = 0$), $< \mathcal{O} >= \int_{\mathcal{M}} \mathcal{D}X\mathcal{O}e^{-S[X]}$ across the field space $\mathcal{M}$.

First, we would like to show that path integrals with insertions of Q-exact observables depend only on the Q-cohomology class of the inserted operators ($<\mathcal{O}> = <\mathcal{O}+Q\mathcal{O}>$). Since $QS=0$,

$$
\begin{aligned}
<Q\mathcal{O}> &= \int_{\mathcal{M}} \mathcal{D}X(Q\mathcal{O})e^{-S[X]} \\
&= \int_{\mathcal{M}} \mathcal{D}XQ(\mathcal{O}e^{-S[X]}) \\
&= 0.
\end{aligned}
\tag{6.21}
$$

The integral is a total derivative in the field space which vanishes if there are no boundary terms. This happens provided that the integrand decays fast enough.

Then, we would like to deform the path integral by adding a Q-variation of a B-invariant fermionic functional ($tQV[X]$) to the action,

$$
<\mathcal{O}>= \int_{\mathcal{M}} \mathcal{D}X\mathcal{O}e^{-S[X]-tQV[X]} \quad \text{with} \quad BV \equiv Q^2V = 0, \tag{6.22}
$$

where we also assume $QV \geq 0$. The B-invariance of $V[X]$ is needed such that the deformed observables $\mathcal{O}e^{-S[X]-tQV[X]}$ is in the same Q-cohomology class as the original observables $\mathcal{O}e^{-S[X]}$. Thus, by equation (6.21), the deformed observables should have same expectation value as the original ones.

Finally, we would like to show that the expectation value of the deformed observables are t-independent,

$$
\begin{aligned}
\frac{d}{dt} <\mathcal{O}> &= \frac{d}{dt}\int_{\mathcal{M}} \mathcal{D}X\mathcal{O}e^{-S-tQV} \\
&= -\int_{\mathcal{M}} \mathcal{D}X(QV)\mathcal{O}e^{-S-tQV} \\
&= -\int_{\mathcal{M}} \mathcal{D}XQ(V\mathcal{O}e^{-S-tQV}) + \int_{\mathcal{M}} \mathcal{D}V\mathcal{O}(Qe^{-S-tQV}).
\end{aligned}
\tag{6.23}
$$

The first term vanishes as this is a total derivative and there are no boundary terms. The second terms is equal to zero due to the B-invariance of V ($BV \equiv Q^2V = 0$) and the action is preserved by supercharge Q ($QS[X] = 0$).

Now, we would like to show how the path integral is localized. The path integral is

$$
Z \equiv \int_{\mathcal{M}} \mathcal{D}Xe^{-S[X]-tQV[X]}. \tag{6.24}
$$

Suppose at $t = 0$, the path integral is $Z(0)$ with contour X_0. If there exists some V such that $QV[X] \geq 0$ along the contour, then in the limit $t \to \infty$, all contributions from $QV[X] > 0$ are infinitely suppressed. Therefore, $Z(\infty) = Z(0)$ and the path integral is localized to the path X_0 where $QV[X_0] = 0$.

6.4.2 Localization formula for supersymmetric path integrals

To evaluate the path integral, we expand around X_0,

$$X = X_0 + \frac{1}{\sqrt{t}}\delta X. \tag{6.25}$$

As $t \to \infty$, we Taylor expand the action around X_0,

$$\begin{aligned} S[X] + tQV[X] = S[X_0] &+ tQV[X_0] \\ &+ \left(\frac{1}{\sqrt{t}}\frac{\delta S}{\delta X} + \sqrt{t}\int \frac{\delta(QV[X])}{\delta X}\Big|_{X=X_0}\right)(\delta X) \\ &+ \left(\frac{1}{2t}\frac{\delta^2 S}{\delta X^2} + \frac{1}{2}\iint \frac{\delta^2(QV[X])}{\delta X^2}\Big|_{X=X_0}\right)(\delta X)^2. \end{aligned} \tag{6.26}$$

The expansion is "1-loop exact" since higher order terms in the Taylor expansion are vanished when $t \to \infty$ as they all contain negative power of t.

In (6.26), $tQV[X_0] = 0$ and $\int \frac{\delta(QV[X])}{\delta X}\Big|_{X=X_0} = 0$ as $QV[X]$ is localized at X_0. Also, $\frac{1}{\sqrt{t}}\frac{\delta S}{\delta X}$ and $\frac{1}{2t}\frac{\delta^2 S}{\delta X^2}$ are both equal to zero because of the negative power of t. Therefore, (6.26) becomes

$$S[X] + tQV[X] = S[X_0] + \frac{1}{2}\iint \frac{\delta^2(QV[X])}{\delta X^2}\Big|_{X=X_0}(\delta X)^2. \tag{6.27}$$

Substituting equation back to the path integral and integrate out δX that are normal to the fixed points X_0 by Gaussian integration, the path integral becomes

$$\begin{aligned} Z &= \int_{\mathcal{M}} \mathcal{D}X e^{-(S[X_0] + \frac{1}{2}\iint \frac{\delta^2(QV[X])}{\delta X^2}\big|_{X=X_0}(\delta X)^2)} \\ &= \sum_{X_0} \mathcal{D}X_0\ e^{-S[X_0]} \frac{1}{\mathrm{SDet}[\frac{\delta^2(QV[X_0])}{\delta X_0^2}]}. \end{aligned} \tag{6.28}$$

The SDet is the superdeterminent which is the ratio of bosonic and fermionic determinants or measure on space of fixed points $\{X_0\}$.

Localization is useful as it reduces the dimension of the path integral that one needs to calculate. Moreover, one can choose different V and work in different localization scheme. These schemes differ by integral of Q-exact observables but they should be vanished due to (6.21). Therefore, the results of different schemes should agree.

6.4.3 Fixed points of Instanton moduli space

Going back to equation (6.19), when we go on-shell, the last part of the action remains

$$Q(\sum_{\hat{s}=1}^{4}[\Psi_{\hat{s}}^{\dagger}\cdot B_{\hat{s}}+\Psi_{\hat{s}}\cdot B_{\hat{s}}^{\dagger}]). \tag{6.29}$$

Using relations (6.15), it can be written as

$$Q\sum_{\hat{s}=1}^{4}[\Psi_{\hat{s}}^{\dagger}\cdot(Q\Psi_{\hat{s}}^{\dagger})^{\dagger}+\Psi_{\hat{s}}\cdot(Q\Psi_{\hat{s}})^{\dagger}]. \tag{6.30}$$

Then we would like to find the fixed points X_0,

$$\begin{aligned} QV\equiv Q\sum_{\hat{s}=1}^{4}[\Psi_{\hat{s}}^{\dagger}\cdot(Q\Psi_{\hat{s}}^{\dagger})^{\dagger}+\Psi_{\hat{s}}\cdot(Q\Psi_{\hat{s}})^{\dagger}] \\ =\sum_{\hat{s}=1}^{4}[|Q\Psi_{\hat{s}}^{\dagger}|^2+|Q\Psi_{\hat{s}}|^2+\Psi_{\hat{s}}^{\dagger}\cdot(Q^2\Psi_{\hat{s}}^{\dagger})^{\dagger}+\Psi_{\hat{s}}(Q^2\Psi_{\hat{s}})^{\dagger}], \end{aligned} \tag{6.31}$$

the last two terms vanish since $Q^2=B$ is a bsonic charge. And, $QV=0$ only when

$$\begin{aligned} Q\Psi_{\hat{s}}=0=Q\Psi_{\hat{s}}^{\dagger}, \\ \Psi_{\hat{s}}=0=\Psi_{\hat{s}}^{\dagger}, \end{aligned} \tag{6.32}$$

these fixed points coincide with the BPS equation.

Writing explicitly with relations (6.15), the fixed points of the mulit-instanton action are

$$\begin{aligned} Q\mu_I=\phi I-Iv=0, \\ Q\mu_J=-\phi J+vJ=0, \\ Q\mathcal{M}_1=[\phi,B_1]=0, \\ Q\mathcal{M}_2=[\phi,B_2]=0. \end{aligned} \tag{6.33}$$

7 Instanton contribution to $\mathcal{N}=2$ Prepotential

7.1 T^2-symmetry and localization

The "fixed points" found in (6.33) are actually critical surfaces rather than points which are still difficult to solve. Fortunately, we can introduce further symmetry to simplify the critical surfaces into isolated points. This symmetry describes two independent rotations on the $x_1 - x_2$ and the $x_3 - x_4$ planes. Or in complex coordinate the z_1 and the z_2 planes with $z_1 = x^2 + ix^1$ and $z_2 = x^4 + ix^3$. Writing explicitly, the group elements describing the rotations are

$$T^2 = (t_1, t_2) \qquad \text{with} \qquad t_1 = e^{i\epsilon_1} \text{ and } t_2 = e^{i\epsilon_2}, \tag{7.1}$$

then the rotations become

$$(z_1\ ,\ z_2) \to (t_1 z_1\ ,\ t_2 z_2). \tag{7.2}$$

In order to modify the charge Q with the newly introduced T^2 transformation, we introduce a new charge Q_ϵ [**11**]

$$Q_\epsilon \equiv Q + \Omega_{mn} x^m Q^n, \tag{7.3}$$

where Q^n is the vector in equation (6.2) which comes from decomposing the $\mathcal{N}=2$ supercharge $\mathcal{Q}$ when preforming topological twist. And Ω_{mn} is the matrix of infinitesimal rotation of $\mathbb{R}^4$. In the coordinate system of $\mathbb{R}^4$, Ω_{mn} has a canonical form

$$\Omega_{mn} = \begin{pmatrix} 0 & \epsilon_1 & 0 & 0 \\ -\epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_2 \\ 0 & 0 & -\epsilon_2 & 0 \end{pmatrix}. \tag{7.4}$$

In summary, the charge Q_ϵ is parameterized by the following elements

(1) ϕ_I from the residual group $U(k)$

(2) v_l from breaking the gauge group by vevs through Higgs mechanism in Columb branch $SU(N) \supset U(1)^{N-1}$

(3) ϵ_1 , ϵ_2 from $SO(4) \supset U(1)_{\epsilon_{1,2}}$ representing gravitational deformations

Here, the vevs v and the generator of $U(k)$ residual symmetry ϕ are diagonalized to be v_l with $l = 1, ..., N$ and ϕ_I with $I = 1, ..., k$ respectively.

In other words, the ADHM constraints (6.16) are not only preserved by the residual group $U(k)$ but also by the maximal torus $\mathcal{T} = U(1)^{N-1} \times U(1)_{\epsilon_{1,2}}$. And the transformations of the ADHM data are modified accordingly,

$$\begin{aligned}
&Q_\epsilon I = \mu_I, && Q_\epsilon \mu_I = \phi_I I - I v_l, \\
&Q_\epsilon J = \mu_J, && Q_\epsilon \mu_J = -\phi_I J + v_l J + (\epsilon_1 + \epsilon_2) J, \\
&Q_\epsilon B_1 = \mathcal{M}_1, && Q_\epsilon \mathcal{M}_1 = [\phi_I, B_1] + \epsilon_1 B_1, \\
&Q_\epsilon B_2 = \mathcal{M}_2, && Q_\epsilon \mathcal{M}_2 = [\phi_I, B_2] + \epsilon_2 B_2.
\end{aligned} \tag{7.5}$$

While the transformation of the auxiliary fields become

$$\begin{aligned}
&Q_\epsilon \chi_{\mathbb{R}} = H_{\mathbb{R}}, && Q_\epsilon H_{\mathbb{R}} = [\phi_I, \chi_{\mathbb{R}}], \\
&Q_\epsilon \chi_{\mathbb{C}} = H_{\mathbb{C}}, && Q_\epsilon H_{\mathbb{C}} = [\phi_I, \chi_{\mathbb{C}}] + (\epsilon_1 + \epsilon_2) \chi_{\mathbb{C}}, \\
&Q_\epsilon \bar{\phi}_I = \eta_I, && Q_\epsilon \eta_I = [\phi_I, \bar{\phi}_I], \\
&Q_\epsilon \phi_I = 0, && Q_\epsilon v = Q_\epsilon \bar{v} = 0.
\end{aligned} \tag{7.6}$$

Using relationships (7.5) and (7.6), we can write the vector field Q^* that generates the transformation Q_ϵ

$$\begin{aligned}
Q^* = &\mu_I \frac{\partial}{\partial I} + \mu_J \frac{\partial}{\partial J} + \mathcal{M}_1 \frac{\partial}{\partial B_1} + \mathcal{M}_2 \frac{\partial}{\partial B_2} + H_{\mathbb{R}} \frac{\partial}{\partial \chi_{\mathbb{R}}} + H_{\mathbb{C}} \frac{\partial}{\partial \chi_{\mathbb{C}}} \\
&+ \eta \frac{\partial}{\partial \bar{\phi}} + [\phi, \chi_{\mathbb{R}}] \frac{\partial}{\partial H_{\mathbb{R}}} + [\phi, \chi_{\mathbb{C}}] \frac{\partial}{\partial H_{\mathbb{C}}} + [\phi, \bar{\phi}] \frac{\partial}{\partial \eta} \\
&+ (\phi I - I v) \frac{\partial}{\partial \mu_I} + (-\phi J + v J + \epsilon_1 + \epsilon_2) J \frac{\partial}{\partial \mu_J} \\
&+ ([\phi, B_1] + \epsilon_1 B_1) \frac{\partial}{\partial \mathcal{M}_1} + ([\phi, B_2] + \epsilon_2 B_2) \frac{\partial}{\partial \mathcal{M}_2} \\
= &(Q^*)^i_{\mathcal{B}} \frac{\partial}{\partial \mathcal{B}^i} + (Q^*)^i_{\mathcal{F}} \frac{\partial}{\partial \mathcal{F}^i}.
\end{aligned} \tag{7.7}$$

In the last line, the bosons $\mathcal{B} = (I, J, B_1, B_2, H_{\mathbb{R}}, H_{\mathbb{C}}, \bar{\phi})$ and the fermions $\mathcal{F} = (\mu_I, \mu_J, \mathcal{M}_1, \mathcal{M}_2, \chi_{\mathbb{R}}, \chi_{\mathbb{C}}, \eta)$ are grouped together respectively.

Observing (7.5), we see that ϕ , the generator of the $U(k)$, is Q_ϵ-invariance. We can diagonize $\phi = (\varphi_1, ..., \varphi_k)$, and define $\phi_{IJ} \equiv \varphi_I - \varphi_J$ with $I, J =$

$1,...,k$. This diagonization results in a Jocobian called Vandermonde determinant $\prod_{I<J}(\varphi_I-\varphi_J)^2$. Plugging this into the localization formula (6.28), we get

$$\begin{aligned} Z_k &= \int_{\mathcal{M}^\varsigma} \frac{D\phi}{U(k)} D\mathcal{B}D\mathcal{F}e^{-S} \\ &= \int_{\mathcal{M}^\varsigma} \prod_{I=1}^{k} d\phi_I \frac{\prod_{I<J}(\varphi_I-\varphi_J)^2}{Sdet\mathcal{L}} \\ &\equiv \sum_{X_0} \frac{1}{Sdet\hat{\mathcal{L}}_{X_0}} \end{aligned} \tag{7.8}$$

where $Sdet(\mathcal{L}) \equiv Sdet \begin{pmatrix} \frac{\partial(Q^*)^i_{\mathcal{B}}}{\partial\mathcal{F}^j} & \frac{\partial(Q^*)^i_{\mathcal{B}}}{\partial\mathcal{B}^j} \\ \frac{\partial(Q^*)^i_{\mathcal{F}}}{\partial\mathcal{F}^j} & \frac{\partial(Q^*)^i_{\mathcal{F}}}{\partial\mathcal{B}^j} \end{pmatrix}$. In the second line we use the fact that the $\mathcal{N}=2$ instanton action (6.19) vanishes on the critical points.

7.2 Critical-point equations and Young diagram

After the introduction of Q_ϵ (7.5) and change of variable of in (7.8), the critical-point equations (6.33) become

$$\begin{aligned} Q_\epsilon\mu_I &= (\varphi_I - v_l)I_{I,l} = 0, \\ Q_\epsilon\mu_J &= (-\varphi_I + \epsilon_1 + \epsilon_2 + v_l)J_{l,I} = 0, \\ Q_\epsilon\mathcal{M}_1 &= (\varphi_I - \varphi_J + \epsilon_1)B_{1,IJ} = 0, \\ Q_\epsilon\mathcal{M}_2 &= (\varphi_I - \varphi_J + \epsilon_2)B_{2,IJ} = 0. \end{aligned} \tag{7.9}$$

Then, using the result in [**10**], the condition ($\xi > 0$) in equation (5.7) allows us to apply stability condition resulting in $J_{l,I}=0$. The fixed points are therefore reduced to be the combination of

$$\begin{aligned} (\varphi_I - v_l)I_{I,l} &= 0, \\ (\varphi_I - \varphi_J + \epsilon_1)B_{1,IJ} &= 0, \\ (\varphi_I - \varphi_J + \epsilon_2)B_{2,IJ} &= 0. \end{aligned} \tag{7.10}$$

In other words, φ_I has to be a linear combination of $\{v_l\ ,\ \varphi_J-\epsilon_1\ ,\ \varphi_J-\epsilon_2\}$.

Next, we would like to associate each critical point φ_I to a set of N Young Tableaux $(Y_1, Y_2, ..., Y_N)$. To do this, we introduce a partition to each critical point indicie $I \in 1, 2, ..., k$ as follow

$$k \to k_1 + k_2 + ... + k_l + ... + k_N, \tag{7.11}$$

up to the $U(k)$ auxiliary symmetry. Each $I \in \{1, 2, ..., k\}$ can be associated to $l \in \{1, 2, ..., N\}$ in the way shown in figure 5 with with $k_l \geq 0$. In this way, we can distribute $k = \sum_l k_l$ boxes between Y_l's.

$$\Big\{\underbrace{1,2,\ldots,k_1}_{l=1},\underbrace{k_1+1,\ldots,k_1+k_2}_{l=2},\ldots,$$
$$\ldots,\underbrace{k_1+\cdots+k_{l-1}+1,\ldots,k_1+\cdots+k_l}_{l},\ldots,\ldots,\underbrace{k_1+\cdots+k_{N-1}+1,\ldots,k}_{l=N}\Big\}$$

Figure 5: Describing how $I \in \{1, 2, ..., k\}$ is related to a given l by a map l_I. Reprinted from [**1**].

The boxes in diagram Y_l can be labeled either by instanton index I_l or by a pair of integer (i_l, j_l) denoting the vertical and horizontal position in a Young diagram respectively. We further denote ν_{i_l}, ν'_{j_l} be the length of i_l-th row and j_l-th column respectively.

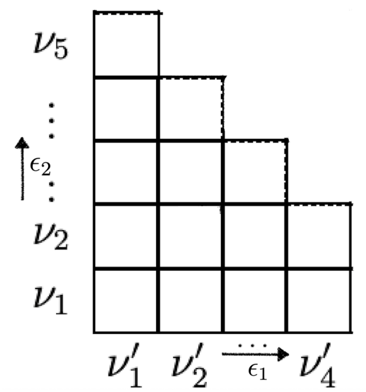

Figure 6: An example of Young diagram with $\nu_1 = 4$ and $\nu'_1 = 5$. Reprinted from [**8**].

In tis way, the soution to (7.10) can be written as

$$\varphi_{I_l} \equiv \varphi_{i_l j_l} = v_l - (j_l - 1)\epsilon_1 - (i_l - 1)\epsilon_2. \tag{7.12}$$

In the context of Young digram, ϵ_1 and ϵ_2 are denoting the horizontal and vertical direction respectively. Therefore, when $\varphi_{i_l j_l} = v_l$ is a nontrivial solution, the boxes on the pair (i_l, j_l)'s both right and up directions are nontrivial solutions as well.

We can now plug the fixed points (7.12) into the localization equation (7.8) to solve for the path integral. However, this calculation is difficult to preform. Instead we follow the approach of [**8**], [**9**] and [**10**], where we try to find the character of the superdeterminant. The character is defined as $\chi \equiv \sum_i (-)^F e^{i\lambda_i}$ where λ_i is the eigenvalue of $\hat{\mathcal{L}}_{X_0}$ and $(-)^F = \pm 1$ depends on the grading by (7.7).

7.3 Tangent space of the Moduli space and Character of Superdeterminant

First, we have to study the ADHM datas (B_1, B_2, I, J) more closely. Suppose $V = \mathbb{C}^k$ and $W = \mathbb{C}^N$, then figure 7 shows the relationship of the ADHM datas. Moreover, we incorporate the action of T^2-symmetry by introducing

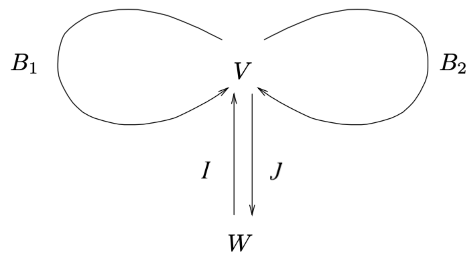

Figure 7: Relationship between ADHM datas. Reprinted from [**10**].

a doublet Q where $\bigwedge^2 Q = t_1 t_2 = e^{i(\epsilon_1+\epsilon_2)}$. Then we can summarise the ADHM datas and the ADHM constraints $(\chi_{\mathbb{R}}, \chi_{\mathbb{C}})$ as maps of vector fields in the following way

$$\begin{aligned} B_1, B_2 &: \quad Hom(V, Q \otimes V), \\ I &: \quad Hom(W, V), \\ J &: \quad Hom(V, \bigwedge^2 Q \otimes W), \\ \chi_{\mathbb{R}} &: \quad Hom(V, V), \\ \chi_{\mathbb{C}} &: \quad Hom(V, V) \otimes \bigwedge^2 Q. \end{aligned} \tag{7.13}$$

Now, we would like to consider the tangent space of the moduli space $T\mathfrak{M}_k$ which is spanned by $(\delta B_1, \delta B_2, \delta I, \delta J)$. And they should up to $U(k)$ transformation satisfy

(1) d_1: infinitesimal gauge transformation thus they should not be in Im d_1

(2) d_2: linearized ADHM equations thus in Ker d_2

Therefore, $T\mathfrak{M}_k \equiv$ Ker d_2 / Im d_1 that can be viewed as the first cohomology group of the complex shown in figure 8.

$$
Hom(V,V) \xrightarrow{d_1} \begin{array}{c} Hom(V,Q\otimes V) \\ \oplus \\ Hom(W,V) \\ \oplus \\ Hom(V,\bigwedge^2 Q\otimes W) \end{array} \xrightarrow{d_2} Hom(V,V)\otimes \bigwedge{}^2 Q
$$

Figure 8: Complex associating to the quotient Ker d_2 / Im d_1. Reprinted form [**8**].

Studying the complex, we find the tangent space can be written as

$$
\begin{aligned}
T\mathfrak{M}_k &= Hom(V,Q\otimes V) + Hom(W,V) + Hom(V,\bigwedge{}^2 Q\otimes W) \\
&\qquad - Hom(V,V) - Hom(V,V)\otimes \bigwedge{}^2 Q \\
&= V^*\otimes V\otimes (Q - \bigwedge{}^2 Q - 1) + W^*\otimes V + V^*\otimes W\otimes \bigwedge{}^2 Q.
\end{aligned} \tag{7.14}
$$

If we introduce generators for the elements in $U(1)^{N-1}$ to be $T_{v_l} = e^{iv_l}$ and that in $U(1)^2$ to be $T_1 = e^{i\epsilon_1}$ and $T_2 = e^{i\epsilon_2}$ and, we write V as $V = e^{i\varphi_I}$. Then, at the critical point, from (7.14) the supertrace of $\mathcal{L}$ can be written as

$$
\begin{aligned}
\chi &= V^*\times V\times[(T_1+T_2) - T_1T_2 - 1] + W^*\times V + V^*\times W\times T_1T_2 \\
&= V^*\times V\times[(T_1-1)\times(1-T_2)] + W^*\times V + V^*\times W\times T_1T_2
\end{aligned} \tag{7.15}
$$

with

$$
\begin{aligned}
V &= \sum_{l=1}^{N}\sum_{j_l=1}^{\nu_{i_l}}\sum_{i_l=1}^{\nu'_{j_l}} {T_1}^{-j_l+1}{T_2}^{-i_l+1}T_{v_l}, \\
W &= \sum_{l=1}^{N} T_{v_l}.
\end{aligned} \tag{7.16}
$$

Then, by using the result in [**10**], one gets

$$\chi = \sum_{l,m}^{N} \sum_{s \in Y_j} (T_{v_{lm}} \ T_1^{-h(s)} T_2^{v(s)+1} + T_{v_{lm}} \ T_1^{h(s)+1} T_2^{-v(s)}), \tag{7.17}$$

where

$$\begin{aligned} h(s) &= \nu_{i_l} - j_l, \\ v(s) &= (\nu'_{j_l} + \nu_{i_l} - j_l + 1) - i_l \ \equiv \ \tilde{\nu}'_{j_l} - i_l. \end{aligned} \tag{7.18}$$

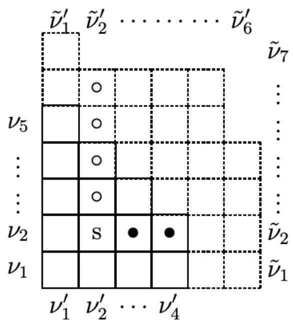

Figure 9: The nontrivial solution of position s. Reprinted from [**10**].

Using equitation (7.8), we have

$$\begin{aligned} Z_k &= \sum_{X_0} \frac{1}{Sdet \hat{\mathcal{L}}_{X_0}} \\ &= \sum_{\{Y_l\}} \prod_{l,m}^{N} \prod_{s \in Y_l} \frac{1}{E(s)(E(s) - \epsilon_1 - \epsilon_2)}, \end{aligned} \tag{7.19}$$

where

$$E(s) = v_{lm} - \epsilon_1 h(s) + \epsilon_2 (v(s) + 1). \tag{7.20}$$

7.4 Adding masses to the $\mathcal{N} = 2$ and the $\mathcal{N} = 2^*$ $SU(N)$ supersymmetric theories

7.4.1 Contribution of Fundamental Matters

Due to the presence of N_F fundamental hypermultiplets, the action gets an extra term [**8**]

$$S_{hyp} = -Qtr_k[\tilde{h}\mathcal{K} - \tilde{\mathcal{K}}h], \tag{7.21}$$

where $(\mathcal{K}, \tilde{\mathcal{K}})$ are $k \times N_F$ and $N_F \times k$ matrices respectively. They denote the fermionic collective coordinates for the fundamental matter fields and have no ADHM constraints acting on them. Their bosonic auxiliary fields are $(h, \tilde{h})$. And they transform as

$$\begin{aligned} Q\mathcal{K} &= h, \qquad Qh = (\phi + m_f)\mathcal{K}, \\ Q\tilde{\mathcal{K}} &= \tilde{h}, \qquad Q\tilde{h} = -\tilde{\mathcal{K}}(\phi + m_f), \end{aligned} \tag{7.22}$$

with m_f being the mass of f-th favour.

These transformations modify the vector field Q^* in (7.7), by adding the terms

$$h\frac{\partial}{\partial \mathcal{K}} + (\phi + m_f)\mathcal{K}\frac{\partial}{\partial h}. \tag{7.23}$$

The critical points (7.12) remain the same as the extra terms in (7.23) can set to zero by setting $h = 0$ for generic masses. Then, the extra contribution $(\delta\chi)$ added to the supertrace (7.17) is

$$\delta\chi = -T_{m_f} \times V = -\sum_{l}^{N} \sum_{s \in Y_l} T_{a_l} T_1^{-j_l+1} T_2^{-i_l+1} T_{m_f}, \tag{7.24}$$

with $T_{m_f} = e^{im_f}$ which is an element of $U(1)^{N_f}$, the maximal tours of $U(N_f)$. Finally, considering the contribution of N_f hypermultiplets, the partition function (7.19) becomes

$$Z_k = \sum_{\{Y_l\}} \prod_{l,m}^{N} \prod_{s \in Y_l} \frac{F(s)}{E(s)(E(s) - \epsilon_1 - \epsilon_2)}, \tag{7.25}$$

with

$$F(s) = \prod_{f=1}^{N_f} (\varphi_{i_l j_l} + m_f). \tag{7.26}$$

7.4.2 Contribution of Adjoint Matters

To study adjoint matters, we have to start with considering the global symmetry of $\mathcal{N} = 4$ theory

$$SU(2)_L \times SU(2)_R \times SO(6) \quad \cong \quad SU(2)_L \times SU(2)_R \times SU(4)_A, \tag{7.27}$$

where $SU(4)_A$ is the automorphism group of $\mathcal{N} = 4$ supersymmetry algebra.

Then, in the $\mathcal{N}=2$ language, we deform the theory by adding mass to the $\mathcal{N}=2$ hypermultiplet in adjoint representation while keeping the $\mathcal{N}=2$ vector multiplet massless. This gives a pure $\mathcal{N}=2$ SYM theory with adjoint masses or a $\mathcal{N}=2^*$ theory. Therefore, we consider the following subgroup of $SU(4)_A$

$$SU(2)_A \times U(1)_m \quad \subset \quad SU(4)_A. \tag{7.28}$$

The global symmetry of $\mathcal{N}=2^*$ theory is thus

$$SU(2)_L \times SU(2)_R \times SU(2)_A \times U(1)_m, \tag{7.29}$$

then we preform topological twist as in section (6.1), the global symmetry becomes

$$SU(2)_L \times SU(2)'_R \times U(1)_m, \tag{7.30}$$

where $SU(2)'_R$ is the diagonal subgroup of $SU(2)_R \times SU(2)_A$.

Comparing to $\mathcal{N}=2$ theory, the global symmetry of $\mathcal{N}=2^*$ has an extra $U(1)_m$. This $U(1)_m$ contributes to the maximal torus which becomes $\mathcal{T}=U(1)^{N-1}\times U(1)_{\epsilon_{1,2}}\times U(1)_m$. Moreover, the eigenvalue is shifted by $-m$ comparing to $\mathcal{N}=2$ theory.

One can identify m, the parameter of $U(1)_m$, with the mass of the $\mathcal{N}=2^*$ hypermultiplet. The modified partition function reads [**8**],

$$Z_k = \sum_{\{Y_l\}}\prod_{l,m}^{N}\prod_{s\in Y_l}\frac{(E(s)-m)(E(s)+m-\epsilon_1-\epsilon_2)}{E(s)(E(s)-\epsilon_1-\epsilon_2)}. \tag{7.31}$$

7.5 Explicit Examples of calculating Instanton contribution to Prepotential

In order to calculate some explicit examples of using the partition function (7.31) with more convinience, we introduce the following definitions:

$$\begin{aligned} f(x) &= \frac{(x-m)(x+m-\epsilon_1-\epsilon_2)}{x(x-\epsilon_1-\epsilon_2)} \quad , \quad g(x) = \frac{1}{x(x-\epsilon_1-\epsilon_2)} \\ T_\alpha(x) &= \prod_{\alpha\neq\beta} f(v_{\alpha\beta}+x) \quad , \quad S_\alpha(x) = \prod_{\alpha\neq\beta} g(v_{\alpha\beta}+x), \end{aligned} \tag{7.32}$$

where $\alpha, \beta = \{1, 2, ..., N\}$ and $v_{\alpha\beta} = v_\alpha - v_\beta$.

And the partition function becomes,

$$Z_k = \sum_{\{Y_l\}} \prod_{l,m}^{N} \prod_{s \in Y_l} f(E(s)). \tag{7.33}$$

- $\boldsymbol{k = 1}$

  The Young diagram for one instanton is:
  $Y_\alpha = \square$, $Y_{\beta \neq \alpha} = \{\emptyset\}$.

  The above diagram will give us $v(s) = h(s) = 0$ for $l = \alpha$ and $v(s) = -1, h(s) = 0$ for $l \neq \alpha$. Then summing up both cases will give

$$Z_1 = \sum_\alpha f(\epsilon_2) T_\alpha(0). \tag{7.34}$$

- $\boldsymbol{k = 2}$

  (I) $Y_\alpha = \square$, $Y_\beta = \square$, $Y_{\gamma \neq \alpha, \beta} = \{\emptyset\}$

  The above diagram will give $v(s) = -1, h(s) = 0$ for $l = \alpha, m \neq \alpha, \beta$ substituting into (7.31) gives $\frac{T_\alpha(0)}{f(v_{\alpha\beta})}$. Then, $l = m = \alpha$ leads to $v(s) = h(s) = 0$ and contributes $f(\epsilon_2)$ to partition function. And the case $l = \alpha, m = \beta$ also leads to $v(s) = h(s) = 0$ but gives $f(v_{\alpha\beta} + \epsilon_2)$ this time. Finally, we have to consider the above contributions with $\alpha \leftrightarrow \beta$ as well. Summing up all six cases give

$$Z_2^I = \frac{1}{2} \sum_{\alpha \neq \beta} f(\epsilon_2)^2 (v_{\alpha\beta} + \epsilon_2)(v_{\beta\alpha} + \epsilon_2) \frac{T_\alpha(0) T_\beta(0)}{f(v_{\alpha\beta}) f(v_{\beta\alpha})}. \tag{7.35}$$

  (II) $\mathrm{Y}_\alpha = \square\square$, $Y_{\beta \neq \alpha} = \{\emptyset\}$

  This time when $l = m = \alpha$, $v(s) = 0$ while $h(s) = 0, 1$, then (7.31) gives $f(\epsilon_2) f(\epsilon_2 - \epsilon_1)$. And when $m \neq l = \alpha$, $v(s) = -1$

and $h(s) = 0, 1$, this contributes $T_\alpha(0)T_\alpha(-\epsilon_1)$ to the partition function. Summing up both cases will give

$$Z_2^{II} = \sum_\alpha f(\epsilon_2)f(\epsilon_2 - \epsilon_1)T_\alpha(0)T_\alpha(-\epsilon_1) \tag{7.36}$$

(III) $\mathrm{Y}_\alpha = \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}\ , Y_{\beta\neq\alpha} = \{\emptyset\}$

This is the transposition of the previous diagram. When $l = m = \alpha$, $v(s) = -1, -2$ while $h(s) = -1$, (7.31) gives $f(\epsilon_1)f(\epsilon_1 - \epsilon_2)$. And when $m \neq l = \alpha$, $v(s) = -1, -2$ and $h(s) = 0$, this contributes $T_\alpha(0)T_\alpha(-\epsilon_2)$ to the partition function. Summing up both cases will give

$$Z_2^{III} = \sum_\alpha f(\epsilon_1)f(\epsilon_1 - \epsilon_2)T_\alpha(0)T_\alpha(-\epsilon_2) \tag{7.37}$$

The partition function of two instantons (Z_2) is the sum of contributions from all three diagrams ($Z_2^I + Z_2^{II} + Z_2^{III}$). Finally we are ready to calculate the prepotential by setting $\epsilon_1 = \hbar = -\epsilon_2$ and using the identification in [**9**]:

$$Z(v, \epsilon_1, \epsilon_2) = \exp\left(\frac{\mathcal{F}^{inst} + O(\epsilon_1, \epsilon_2)}{\epsilon_1\epsilon_2}\right), \tag{7.38}$$

one can extract the instanton contribution to the prepotential ($\mathcal{F}^{inst}$).

- $\boldsymbol{k = 1}$

  Substituting (7.34) into (7.38), one gets

$$\begin{aligned}\mathcal{F}_1 &= -\lim_{\hbar\to 0}\hbar^2 Z_1 \\ &= m^2\sum_\alpha T_\alpha(0).\end{aligned} \tag{7.39}$$

- $\boldsymbol{k = 2}$

  Substituting (7.35), (7.36) and (7.37) into (7.38), one gets

$$\begin{aligned}\mathcal{F}_2 &= -\lim_{\hbar\to 0}\hbar^2\left(Z_2 - \frac{1}{2}Z_1^2\right) \\ &= \sum_\alpha\left(\frac{1}{4}m^4 T_\alpha T_\alpha'' - \frac{3}{2}m^2 T_\alpha^2\right) \\ &\quad + m^4\sum_{\alpha\neq\beta}T_\alpha T_\beta\left(\frac{1}{v_{\alpha\beta}} - \frac{1}{2(v_{\alpha\beta} - m)^2} - \frac{1}{2(v_{\alpha\beta} + m)^2}\right),\end{aligned} \tag{7.40}$$

with $T_\alpha = T_\alpha(0)$. Moreover, the result of pure $\mathcal{N} = 2$ theories can be calculated by replacing $f(x), T_\alpha(x)$ to $g(x), S_\alpha(x)$. Finally, these results agree with the ones predicted by Seiberg-Witten theory [**9**].

8 Conclusion

We have achieved the goal of calculating the instanton contribution of Yang-Mills theory in $\mathcal{N} = 2$ supersymmetry for $k = 1, 2$ and compare the results with predictions of Seiberg-Witten theory. We choose to approach the problem by studying the moduli space of instanton, during which many techniques in mathematical and physical calculations are employed.

First, ADHM algebra transforms the problem into a set of ADHM constraints. This helps the process of studying the $\mathcal{N} = 1, 2$ zero modes and later deriving BRST relations between them tremendously. After that we can incorporate the BRST charge into the ADHM constraints by topological twist of the global symmetry group of $\mathcal{N} = 2$ theory in flat space.

Then we use a ζ deformation of the moduli space to regularize the small instanton singularity. This ζ arises from considering a non-commutative space.

Finally, non-compactness of the moduli space requires the most effort to mitigate. Localization is the main technique we utilize to deal with the problem. In addition to the BRST charge we have prepared, we need to further deform the moduli space by exploiting the T^2-symmetry. This is done to isolate the critical points we got from applying localization on the multi-instanton action. Then we associate each critical point to a set of young tableaux. Instead of solving the localization formula directly, we approach the problem by finding the character of the superdeterminant in the localization formula. This leads us to study the complex associating to the tangent space of moduli space.

Acknowledgement

I would like to express my deepest gratitude to my supervisor Elli Pomoni for her invaluable patience and guidance. I am also extremely grateful to Xinyu Zhang, who generously provided his knowledge and expertise. Thanks should also go to Tianyi Yan for his feedbacks and moral support.

Finally, I would be remiss in not mentioning my parents and my partner. Their belief in me has kept my spirits and motivation during this endeavor.

References

[1] V. V. K. Nick Dorey, Timothy J. Hollowood and M. P. Mattis, *"The calculus of many instantons"*. arXiv:hep-th/0206063v1

[2] Valentin V. Khoze, Michael P.Mattis and Matthew J.Slater, *"The Instanton Hunter's Guide to Supersymmetric SU(N) Gauge Theories"* arXiv:hep-th/9804009v2

[3] Diego Bellisai, Francesco Fucito, Alessandro Tanzini and Gabriele Travaglini, *"Instanton Calculus, Topological Field Theories and N = 2 Super Yang-Mills Theories"* arXiv:hep-th/0003272v1

[4] Furuuchi Kazuyuki, *"Topological charge of $U(1)$ instanton on Noncommutative $\mathbb{R}^4$"*. arXiv:hep-th/0010006v1

[5] David Tong, *"TASI Lectures on Solitons"*. arXiv:hep-th/0509216v5

[6] Francesco Fucito, Jose F. Morales and Alessandro Tanzini, *"D-instanton probes of $\mathcal{N} = 2$ non-conformal geometries"*. arXiv:hep-th/0106061v2

[7] N. J. Hitchin , A. Karlhede , U. Lindstro and M. Roček, *"Hyperkahler Metrics and Supersymmetry"*. Commun. Math. Phys. 108, 535-589 (1987)

[8] Ugo Bruzzo, Francesco Fucito , Jose F. Morales and Alessandro Tanzini, *"Multi-Instanton Calculus and Equivariant Cohomology"*. arXiv:hep-th/0211108v4

[9] Nikita A. Nekrasov, *"Seiberg-Witten Prepotential form Instanton Counting "*. arXiv:hep-th/0206161v1

[10] H.Nikajima, *"Lectures on Hilbert Schemes of Points on Surfaces "*. American Mathematical Society, University Lectures Series v.18 (1999)

[11] Nikita A. Nekrasov, Andrei Okounkov *"Seiberg-Witten Theory form and Random Partitions"*. arXiv:hep-th/0306238v2

Hiermit versichere ich an Eides statt, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Hilfsmittel – insbesondere keine im Quellenverzeichnis nicht benannten Internet-Quellen – benutzt habe. Alle Stellen, die wörtlich oder sinngemäß aus Veröffentlichungen entnommen wurden, sind als solche kenntlich gemacht. Ich versichere weiterhin, dass ich die Arbeit vorher nicht in einem anderen Prüfungsverfahren eingereicht habe.

I hereby declare and affirm that this thesis for the master's degree program Intelligent Adaptive Systems is my own work and that I have not used any other sources other than those indicated—especially Internet sources that are not specifically listed in the bibliography. All passages from publications which have been cited literally or summarized are marked accordingly. I further guarantee that I have not previously submitted this thesis in another examination procedure and the written version submitted corresponds to that on the electronic storage medium.

Tsz Ying, Yeung
Hamburg, 21/10/2024