Probability Review

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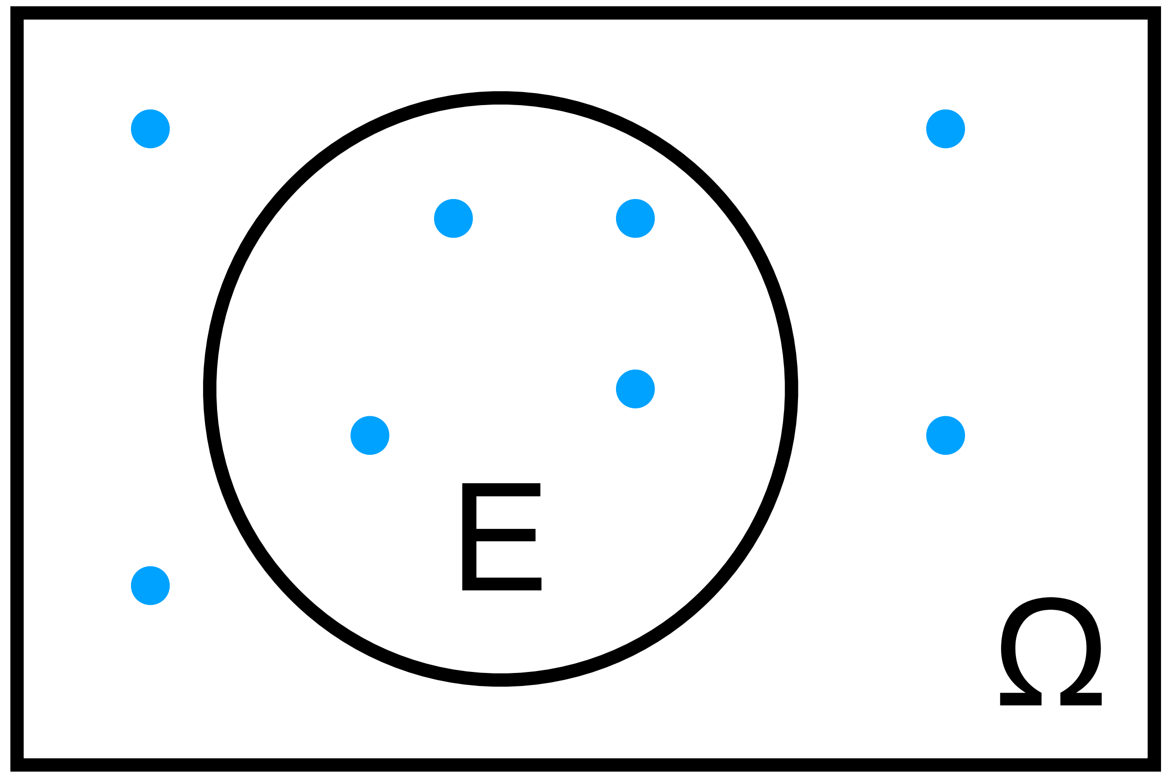
Objectives

- Create common language
- Remind of foundations
- Prepare for subsequent classes
- Bring everyone up to speed

Uniform Spaces

Equiprobable spaces

All outcomes equally likely



Simple formulas

Probability of

outcome $P(x) = \frac{1}{|\Omega|}$

event $P(E) = \frac{|E|}{|\Omega|}$

Die

$\Omega = \{1,2,3,4,5,6\}$

$|\Omega| = 6$

Event	Set	$\lvert Event \rvert$	$P(Event) = \frac{\lvert Event \rvert}{6}$
Even	{2,4,6}	3	$\frac{3}{6} = \frac{1}{2}$
Square	{1,4}	2	$\frac{2}{6} = \frac{1}{3}$

Probability that X=2, 4 ,or 6

Cards

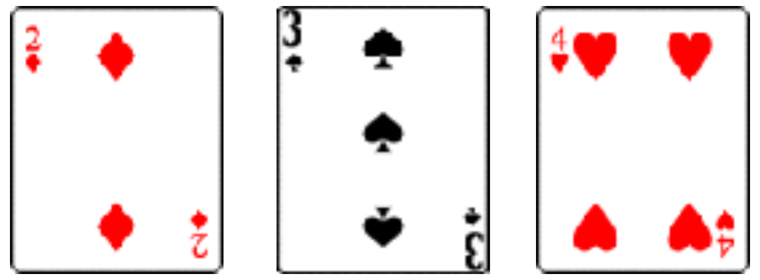
4 suits

Clubs ♣ Diamonds ♦ Hearts ♥ Spades ♠

13 ranks

Ace

Number cards



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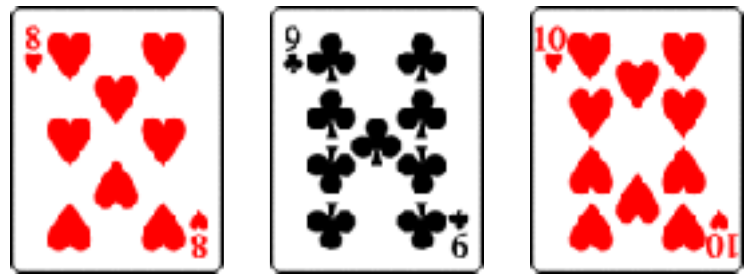


Face cards

No joker in poker

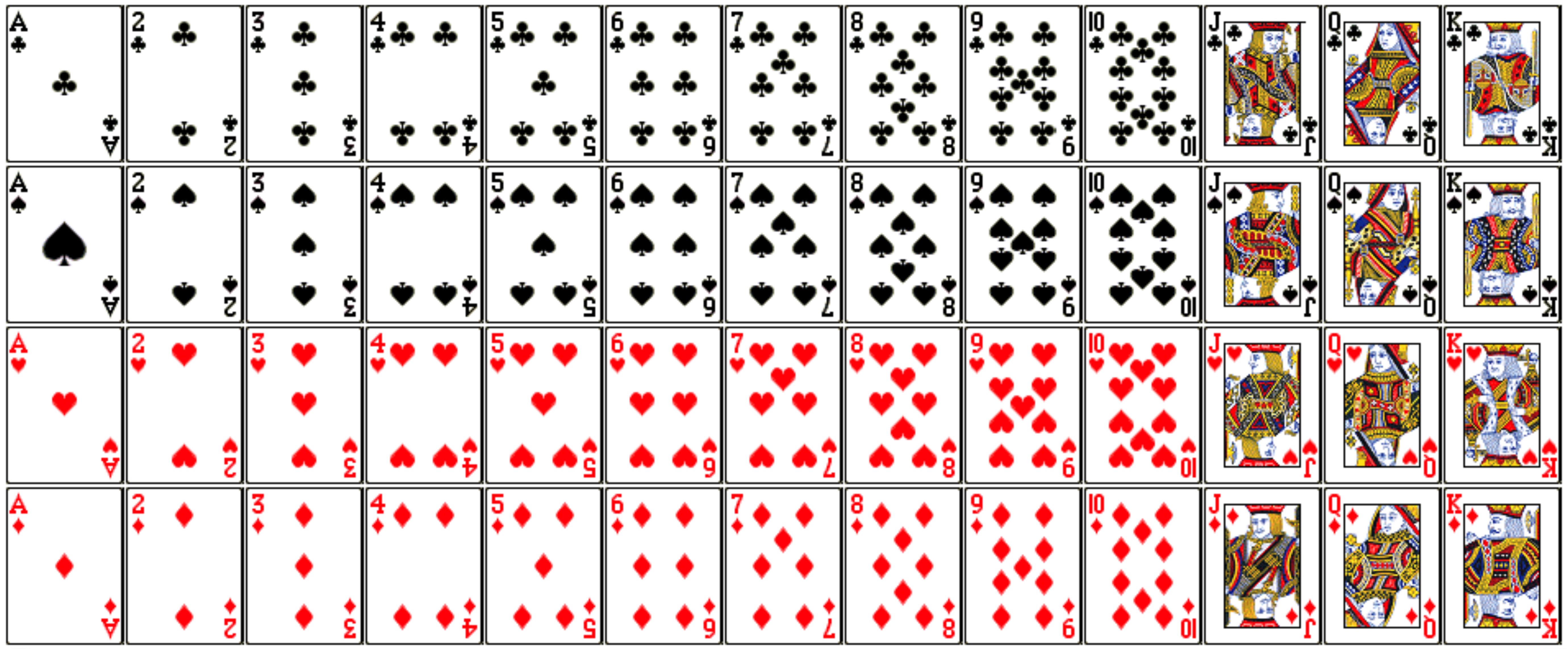
This course is no joker matter

Poker face

Deck

4 suits

♣ ♠ ♥ ♦



13 ranks

Cartesian product

cards ⊗ 13 X 4 52

Single Card

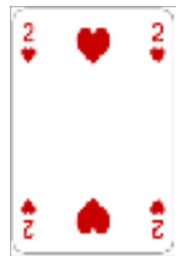
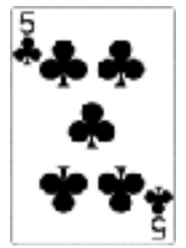
Select one card

Ω = { possible outcomes } = {  }

|Ω| = 52

Equiprobable

U

P () = P () = … = P ()

$$= \frac{1}{|\Omega|} = \frac{1}{52}$$

Events

Equiprobable spaces $P(E) = \frac{|E|}{|\Omega|}$

Event	Occurs	Elements	Size	Prob.	Intuition
Heart	Card is a heart				
King	Card is a king				
Face Card	Card is J, Q, or K				

Hands

Most card games

One card → Several cards hand

Order does not matter Only set { A, J, Q, K, 10} = { K, 10, Q, A, J}

cards in a hand varies Bridge 13 Canasta 11

Poker 5

Poker Hand Probability

Deck 52 cards

Hand Subset of 5

Ω = { possible hands } = { , , , …}

$$|\Omega| = \binom{52}{5} = \frac{52!}{5!\cdot 47!} = \frac{52\cdot 51\cdot \overset{10}{\cancel{50}}\cdot 49\cdot \overset{2}{\cancel{48}}}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = 2,598,960 \approx \text{2.6 million}$$

$\approx 50\cdot 50\cdot 10\cdot 100$

All hands equally likely

Equiprobable

U

P() ≈ 1/ 2.6 million

or any other hand

Once a day

2.6m days

7,000 years

one hand

↓

set of hands

“same suit”

Multiple Events

Often care about more than one event

Find probability of combination of such events

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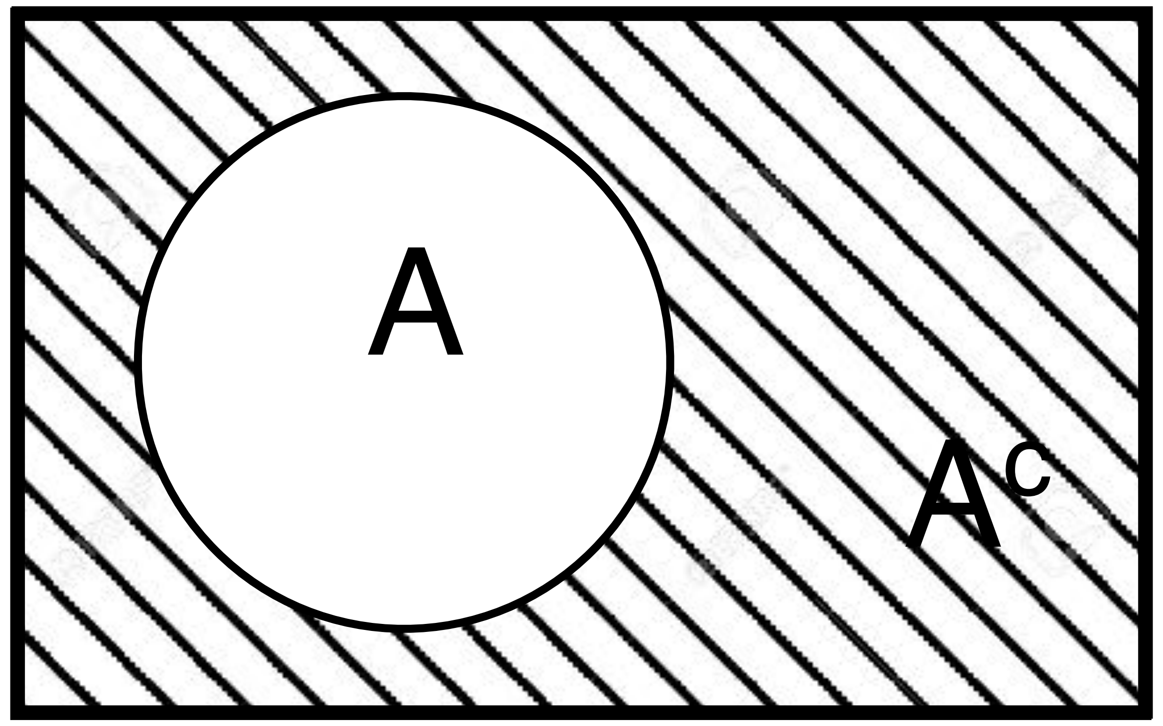
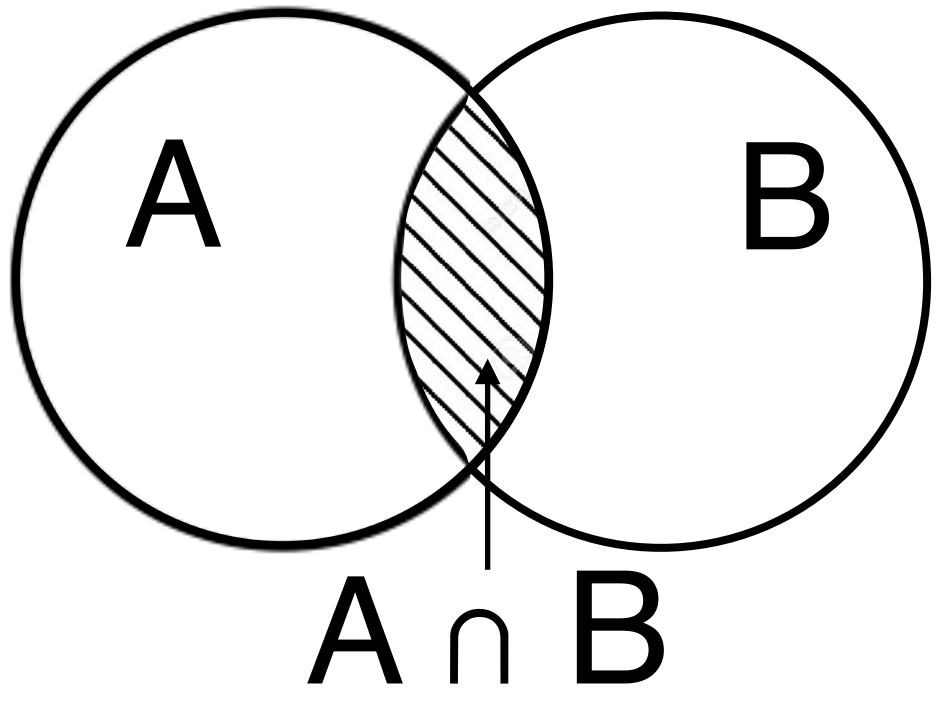
Set Operations

Complement

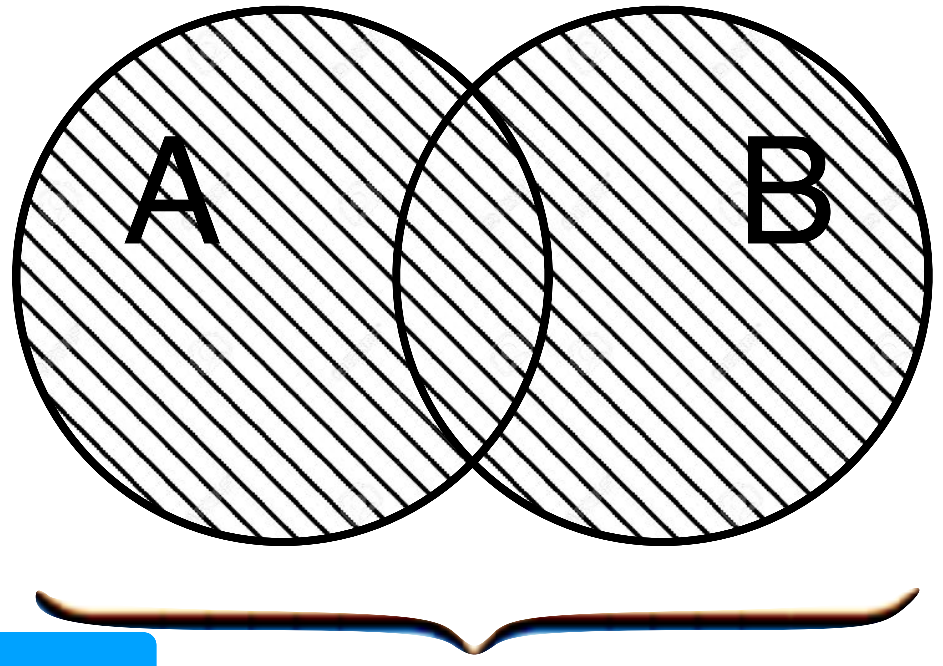


$P(A^c)$ $= 1 - P(A)$

Intersection



Union

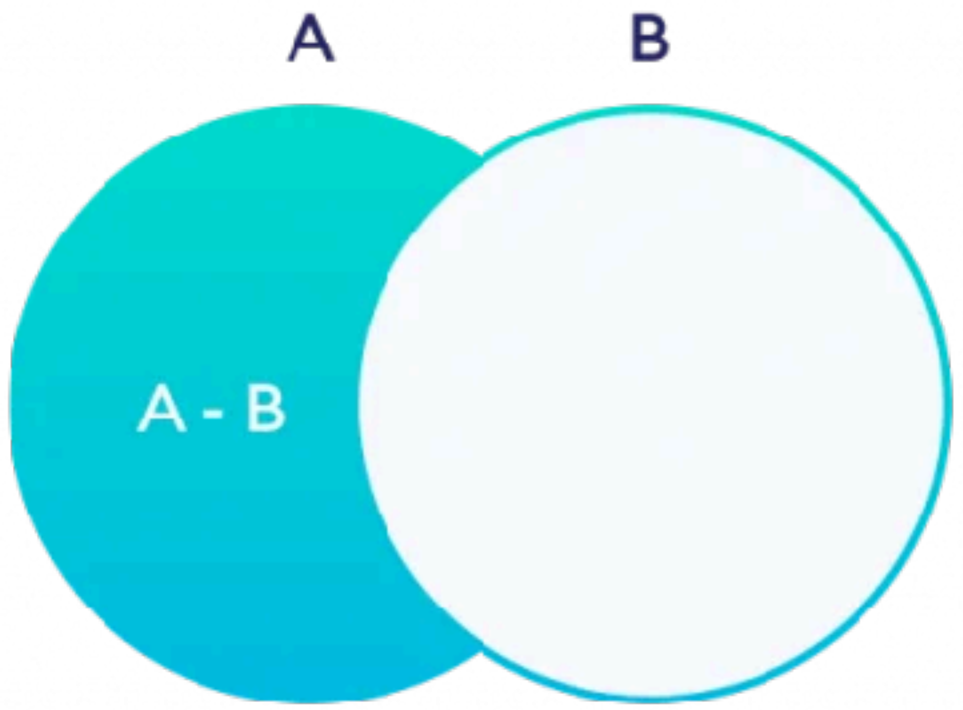


A ∪ B

Inclusion / Exclusion

$P(A \cup B)$ $= P(A) + P(B) - P(A \cap B)$

Set subtraction



$P(A - B)$ $= P(A) - P(A \cap B)$

Conditional Probability

E and F events

Conditional probability of F given E

$P(F \mid E)$

Fraction of times F occurs in experiments where E occurs

exp'ts → ∞

To estimate $P(F \mid E)$ take many samples, consider only experiments where E occurs, and calculate the fraction therein where F occurs too

Even = {2, 4, 6}

$$P(2 \mid \text{Even}) = \frac{2}{6} = \frac{1}{3}$$

1					2						
$\boxed{2}$	1	3	6	4	$\boxed{2}$	5	4	3	6	5	1
1			2	3	4		5		6		

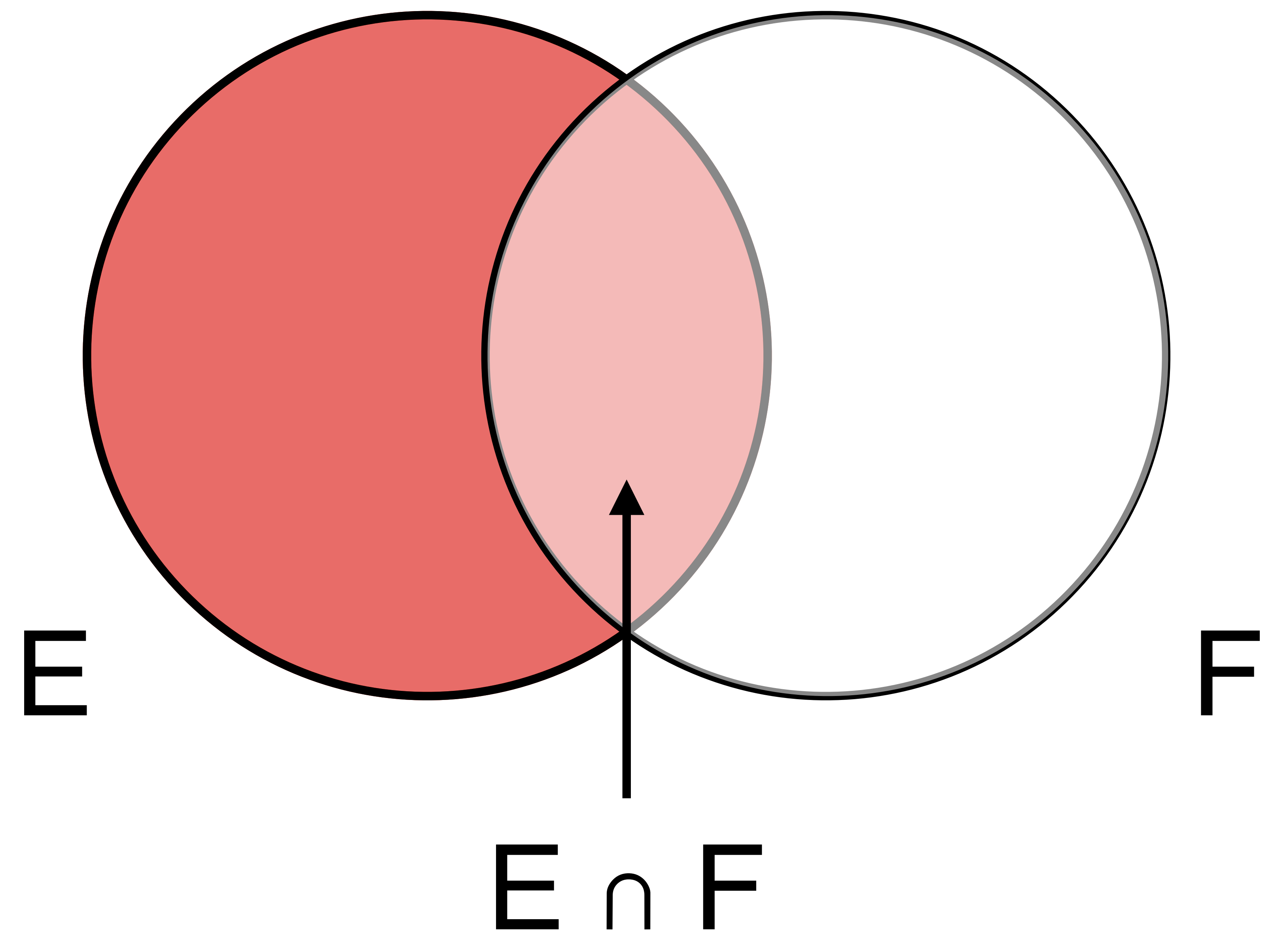
General Spaces

$$P(\,F \mid E\,) = P(\,X{\in}F \mid X{\in}E\,)$$

$$= P\,[\,X{\in}E \cap X{\in}F \;\mid\; X{\in}E\,]$$

$$= P\,[\,X \in E{\cap}F \;\mid\; X{\in}E\,]$$

$$= \frac{P(E \cap F)}{P(E)}$$



4-Sided Die

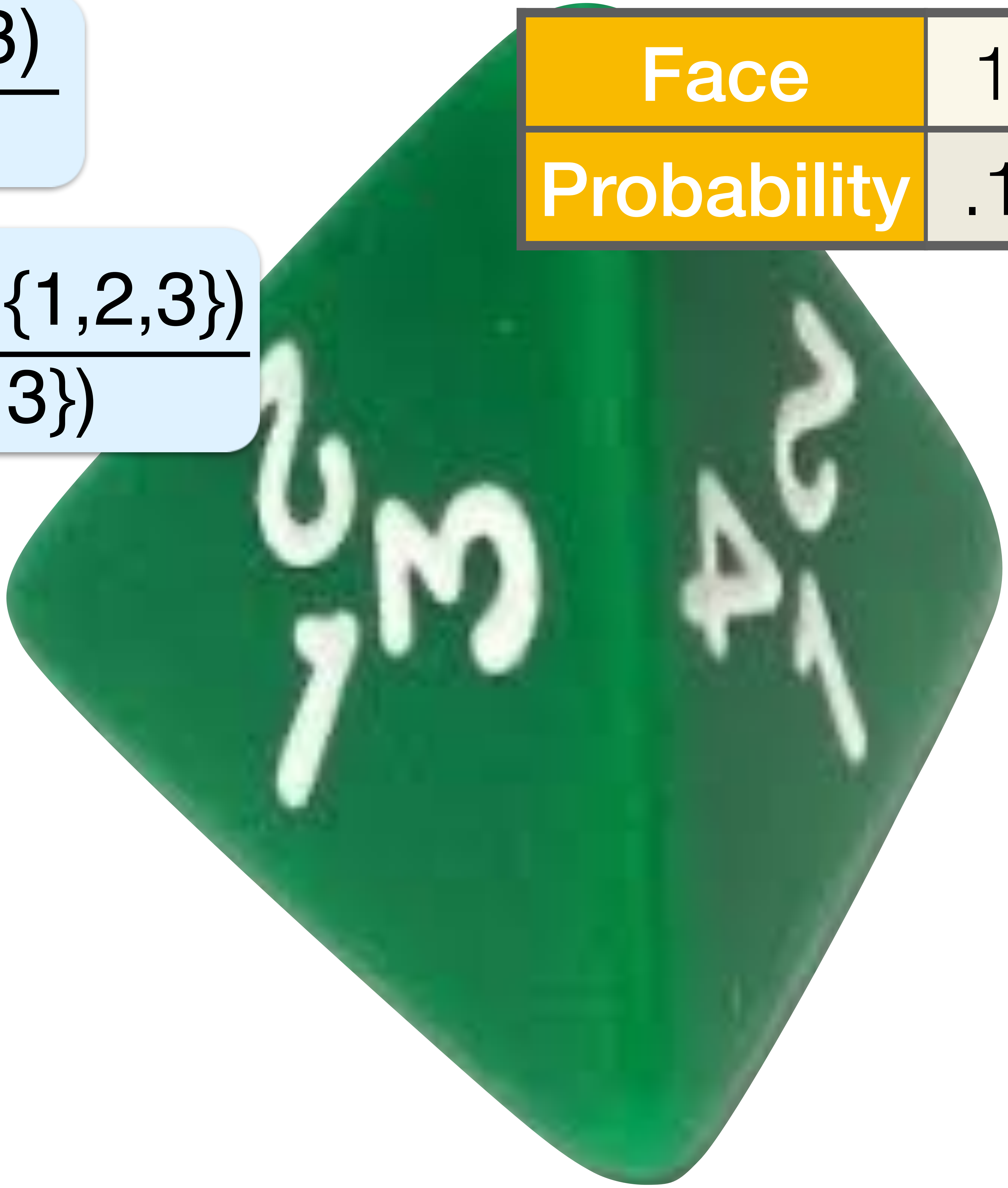
$$P(\geq 2 \mid \leq 3) = \frac{P(\geq 2 \cap \leq 3)}{P(\leq 3)}$$

$$= \frac{P(\{2,3,4\} \cap \{1,2,3\})}{P(\{1,2,3\})}$$

$$= \frac{P(\{2,3\})}{P(\{1,2,3\})}$$

$$= \frac{.5}{.6} = \frac{5}{6}$$

Face	1	2	3	4
Probability	.1	.2	.3	.4



Independence - Informal

Events E and F are statistically independent, or independent, denoted $E \perp\!\!\!\perp F$, if the occurrence of one does not affect the other's probability

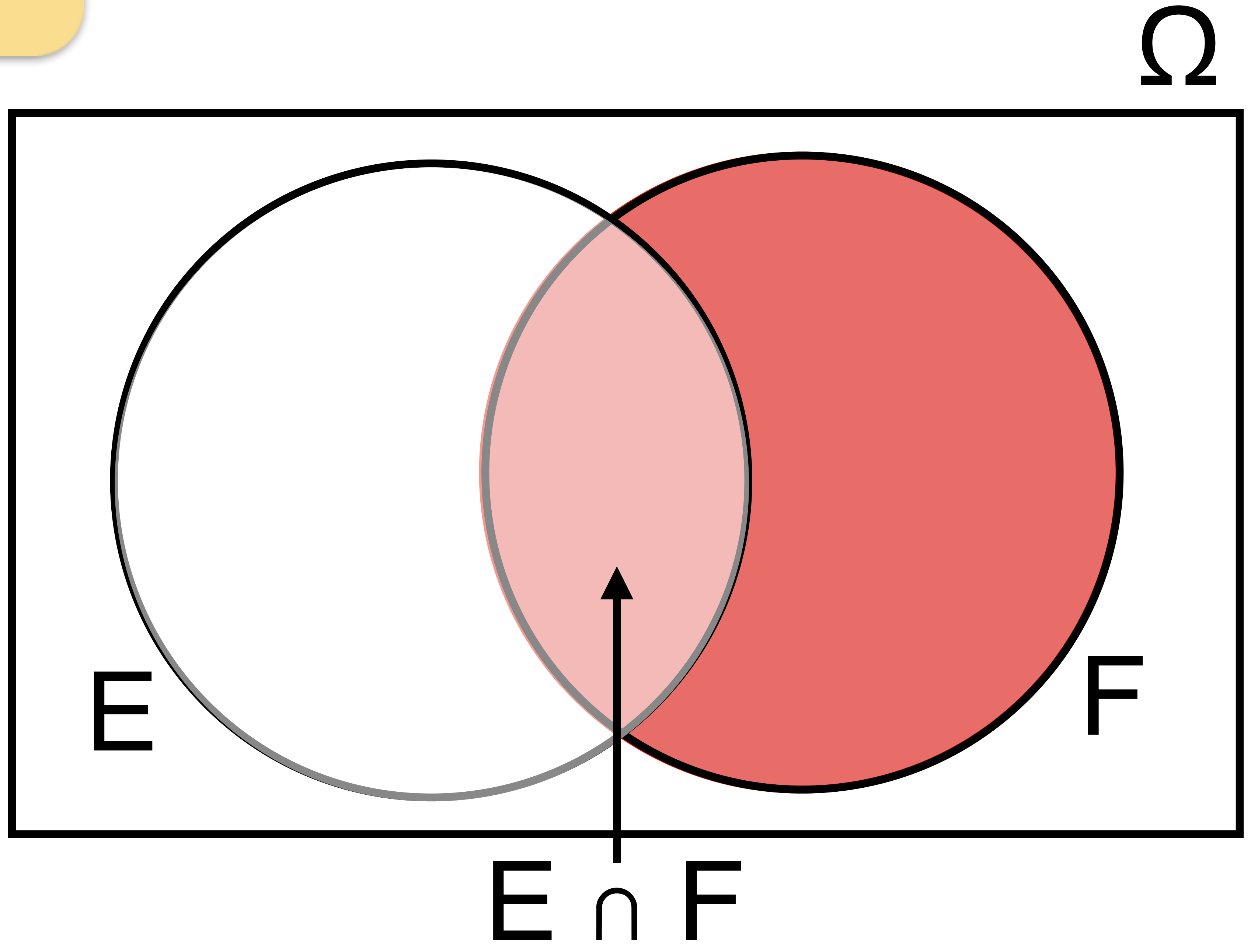
$P(F \mid E) = P(F)$

Visually

$P(F) = \dfrac{P(F)}{P(\Omega)}$ F as a fraction of Ω

=

$P(F \mid E) \triangleq \dfrac{P(E \cap F)}{P(E)}$ $E \cap F$ as a fraction of E



If E has occurred,
the probability that F occurs too is
same as F's original probability.

Unsurprising Independence

Fair coin

Toss twice

H_1 First toss heads

$P(H_1)=½$

H_2 Second toss heads

$P(H_2)=½$

$P(H_2 \mid H_1)$ Heads follows heads $=½$

$P(H_2 \mid H_1) = P(H_2)$ $H_1 \perp\!\!\!\perp H_2$

Unsurprising

Two separate coins

"Independent" experiments always $\perp\!\!\!\perp$

Can events in one experiment be independent?

$\perp\!\!\!\perp$ Informal $\rightarrow$ Formal

Informal

$$P(F) = P(F \mid E) \triangleq \frac{P(E \cap F)}{P(E)}$$

Asymmetric $P(E \mid F) \stackrel{?}{=} P(E)$

Undefined if P(E)=0

Formal

E and F are independent if $P(E \cap F) = P(E) \cdot P(F)$

Otherwise, dependent

The probability of the intersection is the product of the probabilities

$P(F \mid E) = P(F) \rightarrow P(E \mid F) = P(E)$ ✅

Symmetric ✅

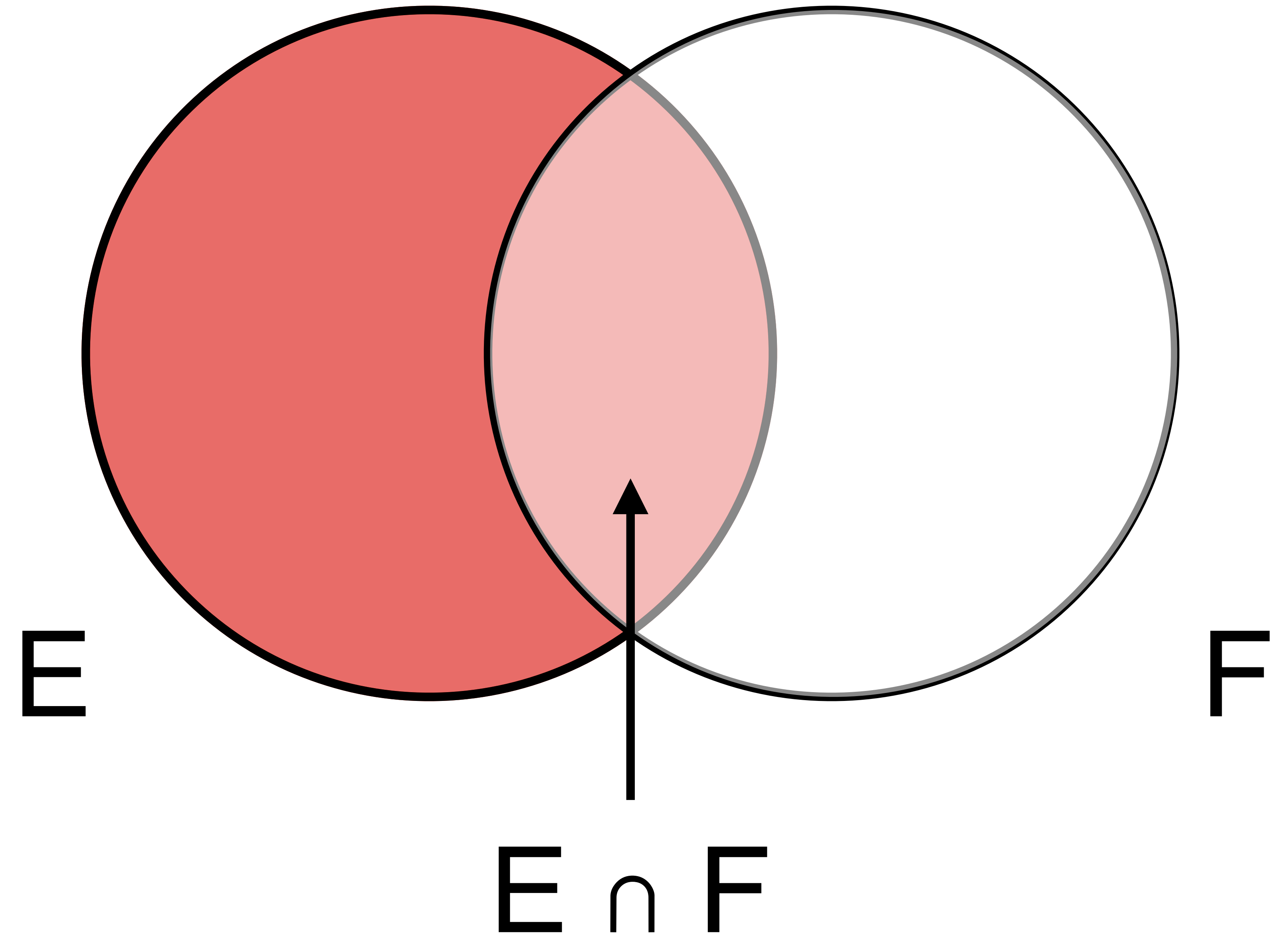
Applies when P(E) = 0 ✅

Product Rule

Conditional probability

$$P(F \mid E) = \frac{P(E \cap F)}{P(E)}$$

$$P(E \cap F) = P(E) \cdot P(F \mid E)$$



Product   Chain   Rule   X

Helps calculate regular (not conditional) probabilities

Sequential Selection

1 blue 2 red Balls 

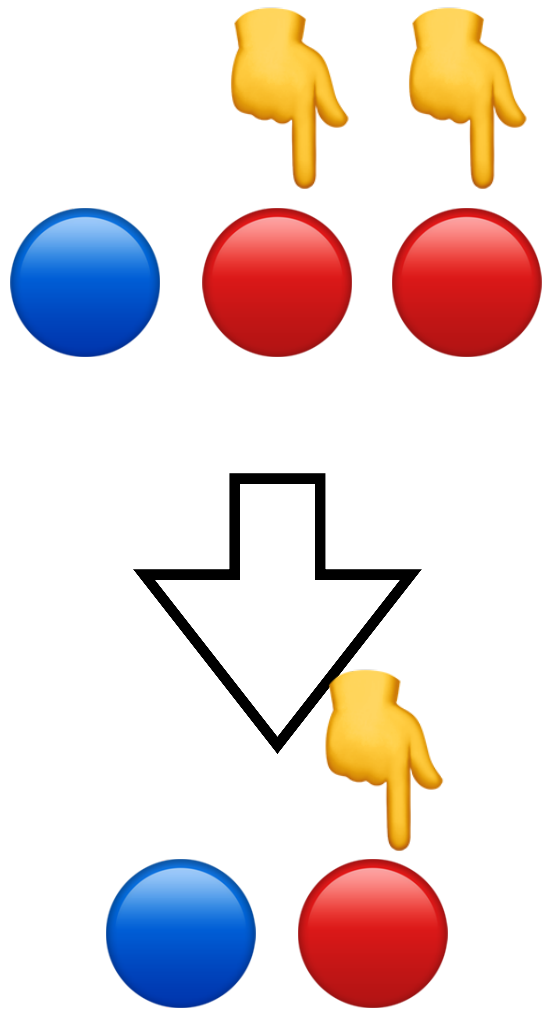
Draw 2 balls without replacement P(both red) ?

R_i i^{th} ball is red

P(both red) $= P(R_1, R_2)$

$= P(R_1) \cdot P(R_2 \mid R_1)$

$= \frac{2}{3} \cdot \frac{1}{2}$ $= \frac{1}{3}$



$P(R_1) = \frac{2}{3}$

$P(R_2 \mid R_1) = \frac{1}{2}$

General Product (Chain) Rule

For 3 events

$$P(E \cap F \cap G) \quad = P((E \cap F) \cap G)$$

X

$$= P(E \cap F) \cdot P(G \mid E \cap F)$$

X

= P(E) · P(F | E) · P(G | E∩F)

Similarly for more events

$$P(E_1 \cap E_2 \cap \ldots \cap E_n) = P(E_1) \cdot P(E_2 \mid E_1) \cdot \ldots \cdot P(E_n \mid E_1 \cap \ldots \cap E_{n-1}$$

The Birthday Paradox

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The Birthday Paradox

How many people does it take so two will likely share a birthday?

Assume that every year has 365 days

Everyone is equally likely to be born on any given day

Probabilistically

Choose n random integers $\in \{1, \ldots, 365\}$ With replacement

S_n Event that two (or more) are the same

Find first n such that $P(S_n)$ is high

2 and 3 People

$P(S_2)$ = P(Second same as first) $= \frac{1}{365}$

All three share same birthday

Just first two share birthday

$P(S_3) = P(S,S,S) + P(S,S,D) + P(S,D,S) + P(D,S,S)$

$$= \frac{1}{365^2} + \frac{1}{365} \cdot \frac{364}{365} \cdot 3 \approx \frac{3}{365}$$

Whoa! What a coincidence! And how does he know my birthday??

Pointed in my direction, said both of us born on July 25!

Random Guy

Me

His mom

He pointed at her. All three of us shared same birthday

$$P = \frac{1}{365^2} \approx \frac{1}{130K}$$

What is first n s.t. $P(S_n)$ is high Say $\geq \frac{1}{2}$

Some think $n \approx 365$

In fact, much smaller

Total Probability

(Divide and Conquer)

Break into events

Use chain rule for each

Combine probabilities

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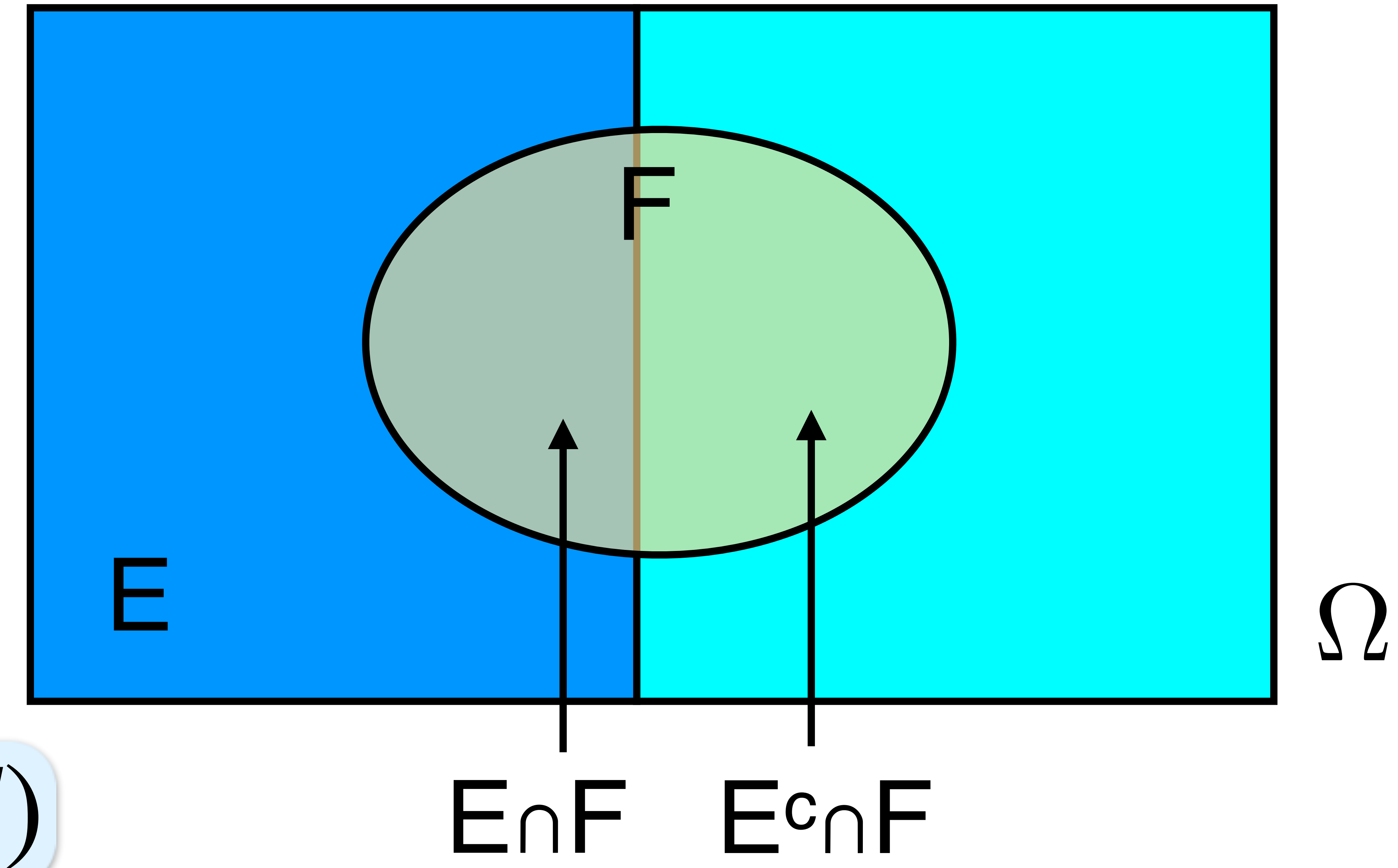
Law of Total Probability

E,F events   P(F)=?

$$F = (E \cap F) \;\cup\; (E^c \cap F)$$

$$P(F) = P(E \cap F) + P(E^c \cap F)$$

$$= P(E) \cdot P(F \mid E) + P(E^c) \cdot P(F \mid E^c)$$



Bayes' Rule

$P(F\,|\,E) \rightarrow P(E\,|\,F)$

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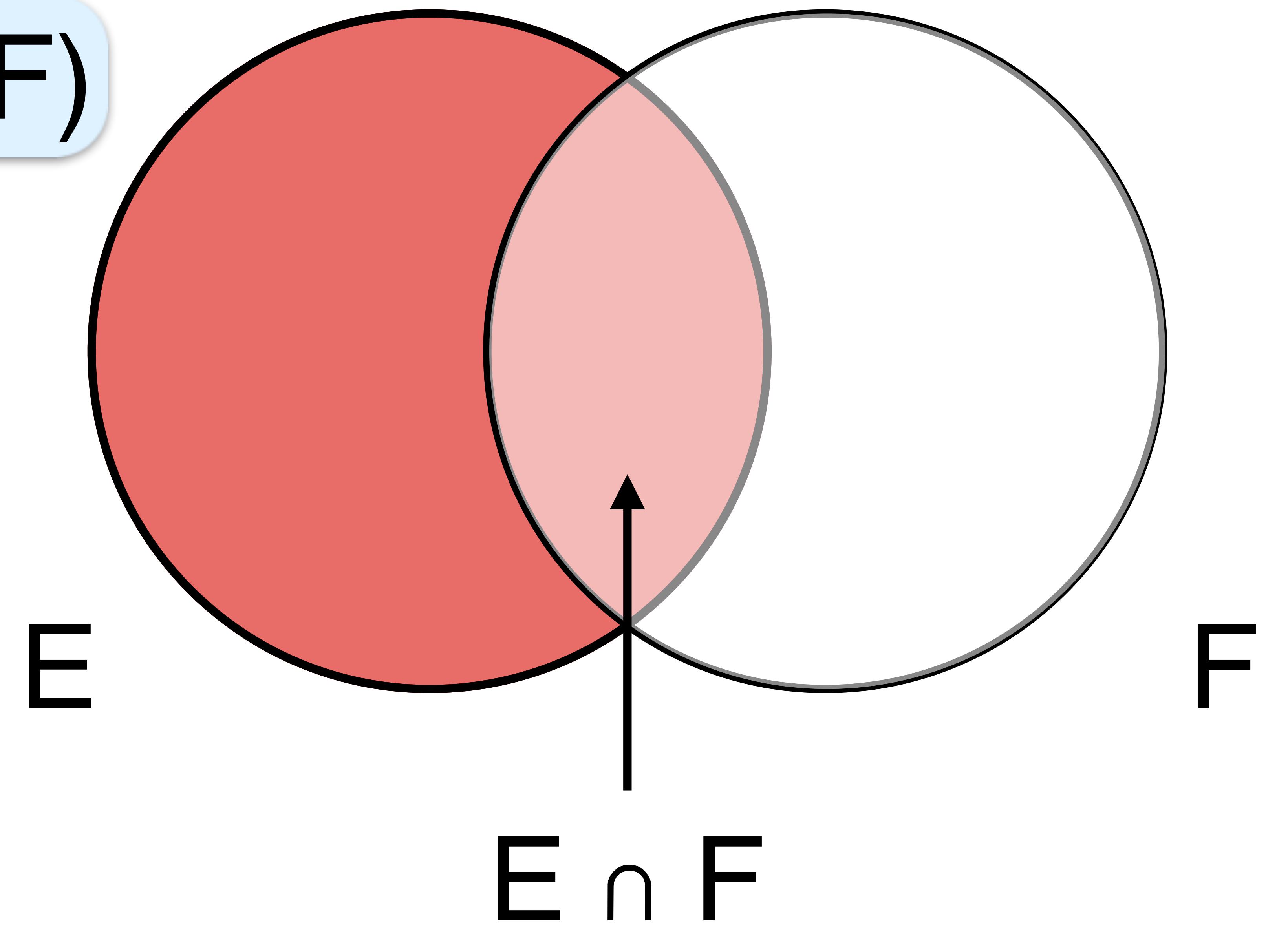
Bayes' Rule

P(E), P(F)

Given P(F | E) (and a bit more) determine P(E | F)

$$P(E \mid F) = \frac{P(E) \cdot P(F \mid E)}{P(F)}$$



μ-proof

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E) \cdot P(F \mid E)}{P(F)}$$

Random Variables

Definition	Number valued random experiment			
Motivation	Numbers everywhere			
Advantages	Specify	Visualize	Functions	Operations
Types	Discrete	Countinuous		

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Expectation

In $n \to \infty$ samples　x will appear　$p(x) \cdot n$ times

$$\text{Average} = \frac{\sum_x [P(x) \cdot n] \cdot x}{n} \boxed{= \sum_x P(x) \cdot x \stackrel{\text{def}}{=} E(X)}$$

Expectation

Mean

Also denoted　E[X]　EX　μ_X　μ

Not random　Deterministic　Constant　Function of distribution

Yet determines the average of large random samples

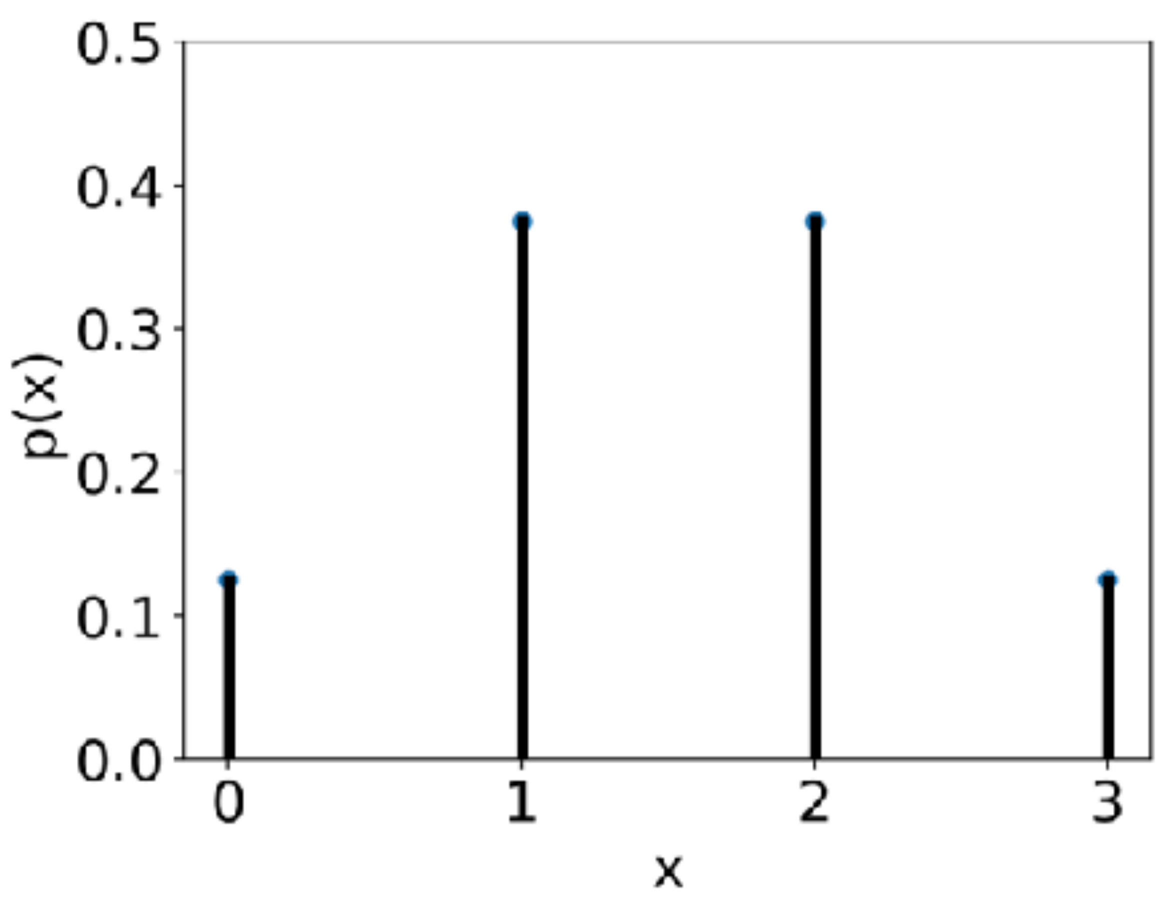
3 Coins

Toss a coin 3 times

X - # heads

E(X) = ?

x	outcomes	p(x)
0	ttt	⅛
1	tth,tht,htt	⅜
2	thh, hth, hht	⅜
3	hhh	⅛



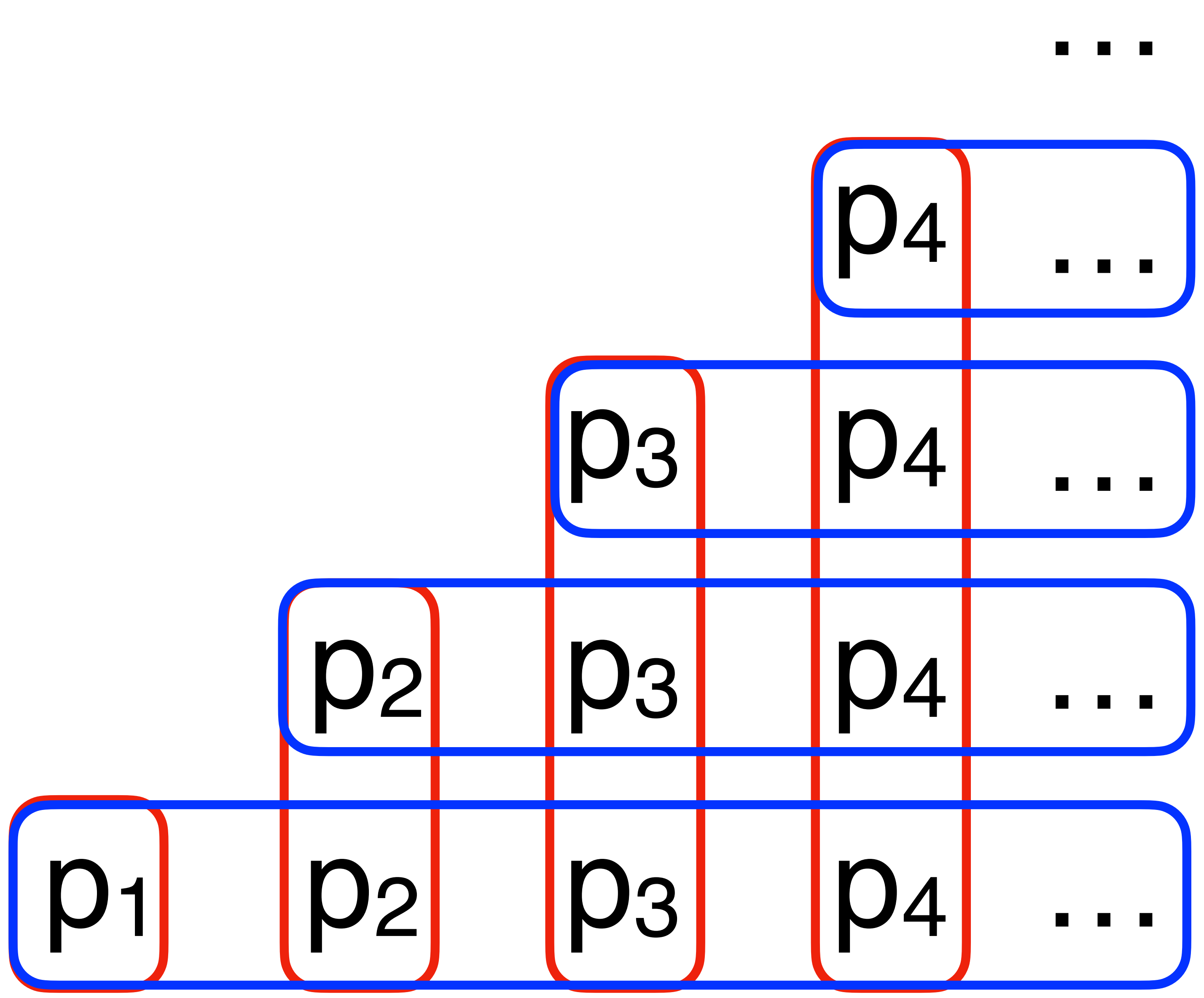
$\sum P(x) \cdot x$ $= \frac{1}{8} \cdot 0 + \frac{3}{8} \cdot 1 + \frac{3}{8} \cdot 2 + \frac{1}{8} \cdot 3$ $= 1.5$

heads ranges from 0 to 3, on average 1.5 ✅

Expectation via "Right" CDF

$X \in \mathbb{N}$ $\quad p_i = P(X = i)$

$$EX = \sum_{i=0}^{\infty} i p_i$$



$$= p_1 + 2p_2 + 3p_3 + \ldots$$

$$= (p_1 + p_2 + p_3 + \ldots) + (p_2 + p_3 + \ldots) + (p_3 + \ldots) + \ldots$$

$$= P(X \geq 1) + P(X \geq 2) + P(X \geq 3) + \ldots$$

$X \in \mathbb{N}$ $\quad EX = \sum_{i=1}^{\infty} P(X \geq i)$

Same # additions
No multiplications!

Expectations of Modified Variables

(Law of The Unconscious Statistician)

Expected value of g(X)

Two steps Y=g(X) P(Y=y) $E[Y] = \sum_y y \cdot p(y)$

One step $E[g(X)] = \sum_x g(x) \cdot p(x)$ Law of the Unconscious Statistician

Linearity of Expectations $E[aX + b] = a \cdot E[X] + b$

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Expectation of a Square

X

x	-2	-1	0	1	2
$p(x)$	⅕	⅕	⅕	⅕	⅕

$Y = X^2$

y	0	1	4
$p(y)$	⅕	⅖	⅖

Two steps

① $p(y)$

$$P(Y = 0) = P(X^2 = 0) = P(X = 0) = \tfrac{1}{5}$$

$$P(Y = 1) = P(X^2 = 1) = P(X \in \{-1, 1\}) = \tfrac{2}{5}$$

$$P(Y = 4) = P(X^2 = 4) = P(X \in \{-2, 2\}) = \tfrac{2}{5}$$

② $E(Y)$

$$E(Y) = \tfrac{1}{5} \cdot 0 + \tfrac{2}{5} \cdot 1 + \tfrac{2}{5} \cdot 4 = 10/5 = 2$$

One step

$E(Y)$

$$= \frac{1}{5} \cdot (-2)^2 + \frac{1}{5} \cdot (-1)^2 + \frac{1}{5} \cdot 0^2 + \frac{1}{5} \cdot 1^2 + \frac{1}{5} \cdot 2^2 = 2$$

Joint Distribution

Simple extension

	Sample space	Variable name	Probability	Abbreviation	properties	Distribution name
One	$\mathcal{X}$	X	p(X=x)	p(x)	$p(x) \geq 0$ $\sum_x p(x) = 1$	Probability mass function (pmf)
Two	$\mathcal{X} \times \mathcal{Y}$	X, Y	p((X,Y)=(x,y))	p(x,y)	$p(x,y) \geq 0$ $\sum_{x,y} p(x,y) = 1$	Joint pmf

Linearity of Expectations

X, Y Random variables

a, b, c Constants

LoE

$$E(aX + bY + c) = a \cdot E(X) + b \cdot E(Y) + c$$

Similarly for more variables

The Indicator Method

Expectation calculation can be hard

Sometimes can use linearity of expectations

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E\,X_i$$

The indicator method

Greatly simplifies calculation

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Birthdays

n students | D = # distinct birthdays | n=3 | 1 | 365 | 1 | D = 2

365 days | Equally likely | E(D)=?

n	E(D)
1	1
2	$P(1)\cdot 1 + P(2)\cdot 2 = \frac{1}{365}\cdot 1 + \frac{364}{365}\cdot 2 = 2 - \frac{1}{365}$
3	$P(1)\cdot 1 + P(2)\cdot 2 + P(3)\cdot 3 = \frac{1}{365^2}\cdot 1 + 3\cdot\frac{364}{365^2}\cdot 2 + \frac{364\cdot 363}{365^2}\cdot 3 = \ldots$
4	Anyone?

Indicator Variables

d=1,…,365

$$I_d = \begin{cases} 1 & \text{some student born on day } d \\ 0 & \text{no student born on day } d \end{cases}$$

n students

$$P(I_d = 0) = \left(1 - \frac{1}{365}\right)^n = \left(\frac{364}{365}\right)^n$$

$$P(I_d = 1) = 1 - \left(\frac{364}{365}\right)^n$$

$$I_d \sim B_{1-\left(\frac{364}{365}\right)^n}$$

$$E(I_d) = 1 - \left(\frac{364}{365}\right)^n$$

D # distinct birthdays $= \sum_{d=1}^{365} I_d$

$$E(D) = \sum_{d=1}^{365} E(I_d) = 365 \cdot \left(1 - \left(\frac{364}{365}\right)^n\right)$$

n=1 $E(D) = 1$ ✅

n=2 $E(D) = 365 \cdot \left(1 - \frac{364}{365}\right) \cdot \left(1 + \frac{364}{365}\right) = 2 - \frac{1}{365}$ ✅

Variance

How much a random variable differs from its mean

Standard deviation

Examples

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Variance

Expected squared difference between X and its mean

$$V(X) = E\left[(X - \mu)^2 \right] \quad = E\,(X - \mu)^2$$

$V[X]$ V

Standard deviation std

$$\sigma_X = +\sqrt{V(X)}$$

Constants

Functionals of distribution

Affine Transformation

$V(aX + b) = V(aX) = a^2V(X)$

$\sigma_{ax+b} = |a|\sigma_x$

Fancy	Real	Modification	Expectation	Variance	Std
Translation	Addition	$X+b$	$E(X) + b$	$V(X)$	σ_x
Scaling	Multiplication	$a \cdot X$	$a \cdot E(X)$	$a^2 \cdot V(x)$	$\lvert a \rvert \cdot \sigma_x$
Affine Transformation	+ & x	$a \cdot X+b$	$a \cdot E(X) + b$	$a^2 \cdot V(x)$	$\lvert a \rvert \cdot \sigma_x$

Alternative Variance Expression

$$V(X) = E\,(X-\mu)^2$$

$$\mu = E\,X$$

$$= E\,(X^2 - 2\mu X + \mu^2)$$

LoE

$$= E\,(X^2) - E\,2\mu X + E\,\mu^2$$

2, μ - constants

LoE

$$= E\,(X^2) - 2\mu E\,X + \mu^2$$

$$= E(X^2) - 2\mu^2 + \mu^2$$

$$= E(X^2) - \mu^2$$

$$= E(X^2) - (E\,X)^2$$

Expectation of square
Minus
Square of expectation