$$
\begin{array}{llll}
\textit{Universes} & \mathrm{U} & ::= & \mathbf{Prop} \mid \mathbf{Type}_i \\
\textit{Values} & \mathrm{V} & ::= & \mathrm{x} \mid \mathrm{U} \mid \lambda\, \mathrm{x} : \mathrm{M}.\, \mathrm{M} \mid \Pi\, \mathrm{x} : \mathrm{M}.\, \mathrm{M} \mid \Sigma\, \mathrm{x} : \mathrm{M}.\, \mathrm{M} \mid \langle \mathrm{V}, \mathrm{V} \rangle \mid \mathbf{bool} \mid \mathbf{true} \mid \mathbf{false} \\
\textit{Computations} & \mathrm{N} & ::= & \mathrm{V} \mid \mathrm{V}\ \mathrm{V} \mid \mathbf{fst}\ \mathrm{V} \mid \mathbf{snd}\ \mathrm{V} \\
\textit{Configurations} & \mathrm{M} & ::= & \mathrm{N} \mid \mathbf{let}\ \mathrm{x} = \mathrm{N}\ \mathbf{in}\ \mathrm{M} \mid \mathbf{if}\ \mathrm{V}\ \mathbf{then}\ \mathrm{M}_1\ \mathbf{else}\ \mathrm{M}_2 \\
\textit{Continuations} & \mathrm{K} & ::= & [\cdot] \mid \mathbf{let}\ \mathrm{x} = [\cdot]\ \mathbf{in}\ \mathrm{M} \\
\textit{Environments} & \Gamma & ::= & \cdot \mid \Gamma, \mathrm{x} : \mathrm{A} \mid \Gamma, \mathrm{x} = \mathrm{V} \mid \Gamma, \mathrm{V} \equiv \mathrm{V}
\end{array}
$$

Fig. 1. ECC^A Syntax

$\boxed{\Gamma \vdash \mathrm{M} \rhd \mathrm{M}'}$

$$
\begin{array}{rcll}
\mathrm{x} & \rhd_\delta & \mathrm{V} & \text{where } \mathrm{x} = \mathrm{V} \in \Gamma \\
(\lambda\, \mathrm{x} : \mathrm{A}.\, \mathrm{M}_1)\ \mathrm{M}_2 & \rhd_\beta & \mathrm{M}_1[\mathrm{x} := \mathrm{M}_2] & \\
\mathbf{fst}\ \langle \mathrm{M}_1, \mathrm{M}_2 \rangle & \rhd_{\pi_1} & \mathrm{M}_1 & \\
\mathbf{snd}\ \langle \mathrm{M}_1, \mathrm{M}_2 \rangle & \rhd_{\pi_2} & \mathrm{M}_2 & \\
\mathbf{let}\ \mathrm{x} = \mathrm{N}\ \mathbf{in}\ \mathrm{M} & \rhd_\zeta & \mathrm{M}[\mathrm{x} := \mathrm{N}] & \\
\mathbf{if}\ \mathbf{true}\ \mathbf{then}\ \mathrm{M}_1\ \mathbf{else}\ \mathrm{M}_2 & \rhd_{\iota_1} & \mathrm{M}_1 & \\
\mathbf{if}\ \mathbf{false}\ \mathbf{then}\ \mathrm{M}_1\ \mathbf{else}\ \mathrm{M}_2 & \rhd_{\iota_2} & \mathrm{M}_2 &
\end{array}
$$

Fig. 2. ECC^A Single Step Reduction

Lemma 0.1 (Naturality). $\mathrm{K}\langle\langle \mathrm{M} \rangle\rangle \equiv \mathcal{ST}(\mathrm{K}[\mathrm{M}])$

Proof. By induction on M.

Case: $\mathrm{M} = \mathrm{N}$ Trivial.

Case: $\mathrm{M} = \mathbf{let}\ \mathrm{x} = \mathrm{N}\ \mathbf{in}\ \mathrm{M}'$

Must show that $\mathbf{let}\ \mathrm{x} = \mathrm{N}'\ \mathbf{in}\ \mathrm{K}\langle\langle \mathrm{M}' \rangle\rangle \equiv \mathcal{ST}(\mathrm{K}[\mathbf{let}\ \mathrm{x} = \mathrm{N}'\ \mathbf{in}\ \mathrm{M}'])$.

$$
\begin{array}{lr}
\mathbf{let}\ \mathrm{x} = \mathrm{N}'\ \mathbf{in}\ \mathrm{K}\langle\langle \mathrm{M}' \rangle\rangle & \\
\equiv \mathbf{let}\ \mathrm{x} = \mathrm{N}'\ \mathbf{in}\ \mathrm{K}[\mathrm{M}'] & \text{by induction} \quad (1) \\
\equiv \mathcal{ST}(\mathrm{K}[\mathrm{M}'][\mathrm{x} := \mathrm{N}']) & \text{by } [\equiv -\zeta] \quad (2) \\
= \mathcal{ST}(\mathrm{K}[\mathrm{M}'[\mathrm{x} := \mathrm{N}']]) & \text{by uniqueness of names} \quad (3) \\
\equiv \mathcal{ST}(\mathrm{K}[\mathbf{let}\ \mathrm{x} = \mathrm{N}'\ \mathbf{in}\ \mathrm{M}']) & \text{by } [\equiv -\zeta] \text{ and continuation congruence} \quad (4)
\end{array}
$$

Case: $\mathrm{M} = \mathbf{if}\ \mathrm{V}\ \mathbf{then}\ \mathrm{M}_1\ \mathbf{else}\ \mathrm{M}_2$

Must show that $\mathbf{if}\ \mathrm{V}\ \mathbf{then}\ \mathrm{K}\langle\langle \mathrm{M}_1 \rangle\rangle\ \mathbf{else}\ \mathrm{K}\langle\langle \mathrm{M}_2 \rangle\rangle \equiv \mathcal{ST}(\mathrm{K}[\mathbf{if}\ \mathrm{V}\ \mathbf{then}\ \mathrm{M}_1\ \mathbf{else}\ \mathrm{M}_2])$. ???

□

Lemma 0.2 (Compositionality). $\mathrm{K}'\langle\langle\, [\![e]\!]\ \mathrm{K} \rangle\rangle = [\![e]\!]\ \mathrm{K}'\langle\langle \mathrm{K} \rangle\rangle$

Proof. By induction on the structure of e. All value cases are trivial. The cases for non-values are all essentially similar, by definition of composition for continuations or configurations. We give some representative cases.

$\boxed{\Gamma \vdash M \equiv M}$

$$\frac{V_1 \equiv V_2 \in \Gamma}{\Gamma \vdash V_1 \equiv V_2}\ [\equiv\text{-Reflect}] \qquad \frac{\Gamma \vdash M \rhd M'}{\Gamma \vdash M \equiv M'}\ [\equiv\text{-Step}] \qquad \frac{\Gamma \vdash M_1 \equiv M_2}{\Gamma \vdash M[x := M_1] \equiv M[x := M_2]}\ [\equiv\text{-Subst}]$$

$$\frac{}{\Gamma \vdash M \equiv M}\ [\equiv\text{-Refl}] \qquad \frac{\Gamma \vdash M' \equiv M}{\Gamma \vdash M \equiv M'}\ [\equiv\text{-Symm}] \qquad \frac{\Gamma \vdash M_1 \equiv M' \qquad \Gamma \vdash M' \equiv M_2}{\Gamma \vdash M_1 \equiv M_2}\ [\equiv\text{-Trans}]$$

$$\frac{\Gamma \vdash A \equiv A' \qquad \Gamma, x : A \vdash M \equiv M'}{\Gamma \vdash \lambda\, x : A.\, M \equiv \lambda\, x : A'.\, M'}\ [\equiv\text{-Lam}]$$

$$\frac{\Gamma \vdash M_1 \equiv \lambda\, x : A.\, M \qquad \Gamma \vdash M_2 \equiv M' \qquad \Gamma, x : A \vdash M \equiv M'\ x}{\Gamma \vdash M_1 \equiv M_2}\ [\equiv\text{-}\eta_1]$$

$$\frac{\Gamma \vdash M_1 \equiv M' \qquad \Gamma \vdash M_2 \equiv \lambda\, x : A.\, M \qquad \Gamma, x : A \vdash M \equiv M'\ x}{\Gamma \vdash M_1 \equiv M_2}\ [\equiv\text{-}\eta_2] \qquad \frac{\Gamma \vdash V_1 \equiv V_1' \qquad \Gamma \vdash V_2 \equiv V_2'}{\Gamma \vdash V_1\ V_2 \equiv V_1'\ V_2'}\ [\equiv\text{-App}]$$

$$\frac{\Gamma \vdash A \equiv A' \qquad \Gamma, x : A \vdash B \equiv B'}{\Gamma \vdash \Pi\, x : A.\, B \equiv \Pi\, x : A'.\, B'}\ [\equiv\text{-Pi}] \qquad \frac{\Gamma \vdash V_1 \equiv V_1' \qquad \Gamma \vdash V_2 \equiv V_2' \qquad \Gamma \vdash A \equiv A'}{\Gamma \vdash \langle V_1, V_2 \rangle \text{ as } A \equiv \langle V_1', V_2' \rangle \text{ as } A'}\ [\equiv\text{-Pair}]$$

$$\frac{\Gamma \vdash V \equiv V'}{\Gamma \vdash \text{fst } V \equiv \text{fst } V'}\ [\equiv\text{-Fst}] \qquad \frac{\Gamma \vdash V \equiv V'}{\Gamma \vdash \text{snd } V \equiv \text{snd } V'}\ [\equiv\text{-Snd}] \qquad \frac{\Gamma \vdash A \equiv A' \qquad \Gamma, x : A \vdash B \equiv B'}{\Gamma \vdash \Sigma\, x : A.\, B \equiv \Sigma\, x : A'.\, B'}\ [\equiv\text{-Sig}]$$

$$\frac{\Gamma \vdash N \equiv N' \qquad \Gamma, x = N \vdash M \equiv M'}{\Gamma \vdash \text{let } x = N \text{ in } M \equiv \text{let } x = N' \text{ in } M'}\ [\equiv\text{-Let}] \qquad \frac{\Gamma \vdash V \equiv V' \qquad \Gamma \vdash M_1 \equiv M_1' \qquad \Gamma \vdash M_2 \equiv M_2'}{\Gamma \vdash \text{if } V \text{ then } M_1 \text{ else } M_2 \equiv \text{if } V' \text{ then } M_1' \text{ else } M_2'}\ [\equiv\text{-If}]$$

$$\frac{V \equiv \text{true} \in \Gamma}{\Gamma \vdash \text{if } V \text{ then } M_1 \text{ else } M_2 \equiv M_1}\ [\equiv\text{-If-}\eta_1] \qquad \frac{V \equiv \text{false} \in \Gamma}{\Gamma \vdash \text{if } V \text{ then } M_1 \text{ else } M_2 \equiv M_2}\ [\equiv\text{-If-}\eta_2]$$

Fig. 3. ECC^A Equivalence

Case: $e = x$ Must show $K' \langle\!\langle K[x] \rangle\!\rangle = K' \langle\!\langle K[x] \rangle\!\rangle$, which is trivial.

Case: $e = \Pi\, x : A.\, B$ Must show that $K' \langle\!\langle K[\Pi\, x : [\![A]\!].\, [\![B]\!]] \rangle\!\rangle = K' \langle\!\langle K[\Pi\, x : [\![A]\!].\, [\![B]\!]] \rangle\!\rangle$, which is trivial. Note that we need not appeal to induction, since the recursive translation does not use the current continuation for values.

Case: $e = e_1\ e_2$ Must show that

$$\begin{aligned} & K' \langle\!\langle ([\![e_1]\!]\ (\text{let } x_1 = [\cdot] \text{ in } ([\![e_2]\!]\ \text{let } x_2 = [\cdot] \text{ in } K[x_1\ x_2]))) \rangle\!\rangle \\ & = ([\![e_1]\!]\ (\text{let } x_1 = [\cdot] \text{ in } ([\![e_2]\!]\ \text{let } x_2 = [\cdot] \text{ in } K' \langle\!\langle K \rangle\!\rangle [x_1\ x_2]))) \end{aligned}$$

The proof follows essentially from the definition of continuation composition.

$$\begin{aligned} & K' \langle\!\langle ([\![e_1]\!]\ (\text{let } x_1 = [\cdot] \text{ in } ([\![e_2]\!]\ \text{let } x_2 = [\cdot] \text{ in } K[x_1\ x_2]))) \rangle\!\rangle \\ = \quad & ([\![e_1]\!]\ K' \langle\!\langle (\text{let } x_1 = [\cdot] \text{ in } ([\![e_2]\!]\ \text{let } x_2 = [\cdot] \text{ in } K[x_1\ x_2])) \rangle\!\rangle) \end{aligned} \tag{5}$$

by the induction hypothesis applied to e_1

$$= \quad ([\![e_1]\!]\ (\mathbf{let}\ x_1 = [\cdot]\ \mathbf{in}\ K' \langle\!\langle ([\![e_2]\!]\ \mathbf{let}\ x_2 = [\cdot]\ \mathbf{in}\ K[x_1\ x_2]) \rangle\!\rangle)) \tag{6}$$

by definition of continuation composition

$$= \quad ([\![e_1]\!]\ (\mathbf{let}\ x_1 = [\cdot]\ \mathbf{in}\ ([\![e_2]\!]\ K' \langle\!\langle \mathbf{let}\ x_2 = [\cdot]\ \mathbf{in}\ K[x_1\ x_2] \rangle\!\rangle))) \tag{7}$$

by the induction hypothesis applied to e_2

$$= \quad ([\![e_1]\!]\ (\mathbf{let}\ x_1 = [\cdot]\ \mathbf{in}\ ([\![e_2]\!]\ \mathbf{let}\ x_2 = [\cdot]\ \mathbf{in}\ K' \langle\!\langle K \rangle\!\rangle [x_1\ x_2]))) \tag{8}$$

by definition of continuation composition

□

Lemma 0.3. *If* $\Gamma \vdash e \rhd e'$ *then* $[\![\Gamma]\!] \vdash [\![e]\!] \equiv [\![e']\!]$.

Proof. By cases on $\Gamma \vdash e \rhd e'$. We give the key cases.

Case: $\Gamma \vdash x \rhd_\delta e'$

We must show that $[\![\Gamma]\!] \vdash [\![x]\!] \equiv [\![e']\!]$

We know that $x = e' \in \Gamma$, and by definition $x = [\![e']\!] \in [\![\Gamma]\!]$, so the goal follows by [$\equiv$-δ].

Case: $\Gamma \vdash \lambda\, x : A.\, e_1\ e_2 \rhd_\beta e_1[x := e_2]$

We must show $[\![\Gamma]\!] \vdash [\![(\lambda\, x : A.\, e_1)\ e_2]\!] \equiv [\![e_1[x := e_2]]\!]$

$$\begin{aligned}
&[\![\lambda\, x : A.\, e_1\ e_2]\!] && && \\
&= [\![\lambda\, x : A.\, e_1]\!]\ \mathbf{let}\ x_1 = [\cdot]\ \mathbf{in}\ [\![e_2]\!]\ \mathbf{let}\ x_2 = [\cdot]\ \mathbf{in}\ x_1\ x_2 && && (9)\\
&= \mathbf{let}\ x_1 = (\boldsymbol{\lambda}\, x : [\![A]\!].\, [\![e_1]\!])\ \mathbf{in}\ [\![e_2]\!]\ \mathbf{let}\ x_2 = [\cdot]\ \mathbf{in}\ x_1\ x_2 && && (10)\\
&\equiv [\![e_2]\!]\ \mathbf{let}\ x_2 = [\cdot]\ \mathbf{in}\ \boldsymbol{\lambda}\, x : [\![A]\!].\, [\![e_1]\!]\ \ x_2 && \text{by } \equiv\text{-}\zeta && (11)\\
&= \mathbf{let}\ x_2 = [\cdot]\ \mathbf{in}\ (\boldsymbol{\lambda}\, x : [\![A]\!].\, [\![e_1]\!])\ x_2 \langle\!\langle\, [\![e_2]\!]\, \rangle\!\rangle && \text{by Lemma 0.2} && (12)\\
&\equiv \mathrm{let}\ x_2 = [\![e_2]\!]\ \ \mathrm{in}\ (\boldsymbol{\lambda}\, x : [\![A]\!].\, [\![e_1]\!])\ x_2 && \text{by Lemma 0.1} && (13)\\
&\equiv (\boldsymbol{\lambda}\, x : [\![A]\!].\, [\![e_1]\!])\ \ [\![e_2]\!] && \text{by } \equiv\text{-}\zeta && (14)\\
&\equiv \mathcal{ST}([\![e_1]\!][x := [\![e_2]\!]]) && \text{by } \equiv\text{-}\beta && (15)\\
&\equiv [\![e_1[x := e_2]]\!] && \text{by Substitution} && (16)
\end{aligned}$$

□

Lemma 0.4. *If* $\Gamma \vdash e \rhd^* e'$ *then* $[\![\Gamma]\!] \vdash [\![e]\!] \equiv [\![e']\!]$

Proof. By induction on the structure of $\Gamma \vdash e \rhd^* e'$.

Case: [Red-Refl], trivial.

Case: [Red-Trans], by Lemma 0.3, the induction hypothesis, and [$\equiv$-Trans].

Case: [Red-Cong-Let]

We have $\Gamma \vdash \mathrm{let}\ x = e\ \mathrm{in}\ e_1 \rhd^* \mathrm{let}\ x = e'\ \mathrm{in}\ e_2$, $\Gamma \vdash e \rhd^* e'$, and $\Gamma, x = e \vdash e_1 \rhd^* e_2$.

We must show that $[\![\Gamma]\!] \vdash [\![\mathrm{let}\ x = e_1\ \mathrm{in}\ e]\!] \equiv [\![\mathrm{let}\ x = e_1\ \mathrm{in}\ e']\!]$.

This follows by induction and [$\equiv$-Let].

□