1 PROOF OF TYPE PRESERVATION

The structure of the theorem for type preservation and its proof are a little surprising. Intuitively, we would expect to show something like "if $\mathsf{e} : \mathsf{A}$ then $[\![\mathsf{e}]\!] : [\![\mathsf{A}]\!]$". We will ultimately prove this, but we need a stronger lemma first.

As an example of the of the ANF translation, consider the $\mathsf{snd}\ \mathsf{e}$ case: $[\![\mathsf{snd}\ \mathsf{e}]\!]\ \mathbf{K} \stackrel{\text{def}}{=} [\![\mathsf{e}]\!]\ \mathbf{let}\ \mathbf{x} = [\cdot]\ \mathbf{in}\ \mathbf{K}[\mathbf{snd}\ \mathbf{x}]$. Notice that the ANF translation is pushing the computation inside-out by translating the sub-expression e of the term $\mathsf{snd}\ \mathsf{e}$ and constructing a new continuation that expects the translated sub-expression $[\![\mathsf{e}]\!]$. Furthermore, the translation ensures that the original continuation $\mathbf{K}$ is applied to a target computation $\mathbf{snd}\ \mathbf{x}$, which, after some machine steps, is equivalent to the translation of the source expression $[\![\mathsf{snd}\ \mathsf{e}]\!]$.

These observations lead us to state the following lemma, which includes reasoning about the type of the continuation $\mathbf{K}$: If $\Gamma \vdash \mathsf{e} : \mathsf{A}$, and $[\![\Gamma]\!], \mathbf{\Gamma'} \vdash \mathbf{K} : (\mathbf{N} : [\![\mathsf{A}]\!]) \Rightarrow \mathbf{B}$ such that $[\![\Gamma]\!], \mathbf{\Gamma'} \vdash [\![\mathsf{e}]\!] \equiv \mathbf{N}$, then $[\![\Gamma]\!], \mathbf{\Gamma'} \vdash [\![\mathsf{e}]\!]\ \mathbf{K} : \mathbf{B}$. We require the continuation $\mathbf{K}$ and its type are not arbitrary, but have a particular relationship to the translation of the source term e. It expects a term that is *not* the translation of the source expression directly, but rather some target computation $\mathbf{N}$ that is equivalent to the translated source expression e under an extended context $\mathbf{\Gamma'}$.

Intuitively, this extended context $\mathbf{\Gamma'}$ contains information about new variables introduced through the ANF translation. In the case of dependent if, this extended context contains equivalences about the expression being eliminated. We then use [Reflect] to prove that the continuation is applied to an expression equivalent to the translation of the dependent if.

We first prove this stronger lemma (Lemma 1.1), towards our goal of proving type preservation. Wielding our propositions-as-types hat, we can view this lemma as in accumulator-passing style. The ANF translation takes a procedural accumulator (the continuation $\mathbf{K}$) and builds up the translation in an accumulator as a procedure from values to ANF terms. Our type preservation proof builds up a proof of correctness as an accumulator as well. The accumulator is a *proposition* that if the computation it receives is well typed, then composing the continuation with the computation is well typed.

Lemma 1.1.

(1) If $\vdash \Gamma$ *then* $\vdash [\![\Gamma]\!]$

(2) If $\Gamma \vdash \mathsf{e} : \mathsf{A}$*, and* $[\![\Gamma]\!], \mathbf{\Gamma'} \vdash \mathbf{K} : (\mathbf{N} : [\![\mathsf{A}]\!]) \Rightarrow \mathbf{B}$ *such that* $[\![\Gamma]\!], \mathbf{\Gamma'} \vdash [\![\mathsf{e}]\!] \equiv \mathbf{N}$*, then* $[\![\Gamma]\!], \mathbf{\Gamma'} \vdash [\![\mathsf{e}]\!]\ \mathbf{K} : \mathbf{B}$.

Proof. The proof is by induction on the mutually defined judgments $\vdash \Gamma$ and $\Gamma \vdash \mathsf{e} : \mathsf{A}$. Cases [Ax-Prop], [Lam], [App], [Snd], and [If] are given.

Case: [Ax-Prop]

We have that $\vdash \Gamma$, $\Gamma \vdash \mathsf{Prop} : \mathsf{Type}_0$, and $[\![\Gamma]\!], \mathbf{\Gamma'} \vdash \mathbf{K} : (\mathbf{N} : [\![\mathsf{Type}_0]\!]) \Rightarrow \mathbf{B}$ such that $[\![\Gamma]\!], \mathbf{\Gamma'} \vdash [\![\mathsf{Prop}]\!] \equiv \mathbf{N}$. We must show that $[\![\Gamma]\!], \mathbf{\Gamma'} \vdash [\![\mathsf{Prop}]\!]\ \mathbf{K} : \mathbf{B}$, that is, $[\![\Gamma]\!], \mathbf{\Gamma'} \vdash \mathbf{K}[\mathbf{Prop}] : \mathbf{B}$. This follows from Cut if we can show $[\![\mathsf{Prop}]\!]$ is equivalent to $\mathbf{Prop}$ and that $[\![\Gamma]\!], \mathbf{\Gamma'} \vdash \mathbf{Prop} : [\![\mathsf{Type}_0]\!]$. $[\![\mathsf{Prop}]\!]$ is equivalent to $\mathbf{Prop}$ by the ANF translation. Finally, since $[\![\mathsf{Type}_0]\!]$ is equivalent to $\mathbf{Type}_0$ by the ANF translation, we are trying to show $[\![\Gamma]\!], \mathbf{\Gamma'} \vdash \mathbf{Prop} : \mathbf{Type}_0$, which follows by [Ax-Prop].

Case: [Lam]

We have that $\vdash \Gamma$, $\Gamma \vdash [\![\lambda\, \mathsf{x} : \mathsf{A'}.\, \mathsf{e'}]\!] : \Pi\, \mathsf{x} : \mathsf{A'}.\, \mathsf{B'}$, and $[\![\Gamma]\!], \mathbf{\Gamma'} \vdash \mathbf{K} : (\mathbf{N} : [\![\Pi\, \mathsf{x} : \mathsf{A'}.\, \mathsf{B'}]\!]) \Rightarrow \mathbf{B}$ such that $[\![\Gamma]\!], \mathbf{\Gamma'} \vdash [\![\lambda\, \mathsf{x} : \mathsf{A'}.\, \mathsf{e'}]\!] \equiv \mathbf{N}$. We must show that $[\![\Gamma]\!] \vdash [\![\lambda\, \mathsf{x} : \mathsf{A'}.\, \mathsf{e'}]\!]\ \mathbf{K} : \mathbf{B}$, that is, $[\![\Gamma]\!] \vdash \mathbf{K}[\boldsymbol{\lambda}\, \mathbf{x} : [\![\mathsf{A'}]\!].\, [\![\mathsf{e'}]\!]] : \mathbf{B}$. This follows from Cut if we can show $[\![\lambda\, \mathsf{x} : \mathsf{A'}.\, \mathsf{e'}]\!]$ is equivalent to $\boldsymbol{\lambda}\, \mathbf{x} : [\![\mathsf{A'}]\!].\, [\![\mathsf{e'}]\!]$ and that $[\![\Gamma]\!], \Gamma \vdash \boldsymbol{\lambda}\, \mathbf{x} : [\![\mathsf{A'}]\!].\, [\![\mathsf{e'}]\!] : [\![\Pi\, \mathsf{x} : \mathsf{A'}.\, \mathsf{B'}]\!]$. $[\![\lambda\, \mathsf{x} : \mathsf{A'}.\, \mathsf{e'}]\!]$ is equivalent to $\boldsymbol{\lambda}\, \mathbf{x} : [\![\mathsf{A'}]\!].\, [\![\mathsf{e'}]\!]$ by the ANF translation.

Since $[\![\Pi\, \mathsf{x} : \mathsf{A'}.\, \mathsf{B'}]\!]$ is equivalent to $\boldsymbol{\Pi}\, \mathbf{x} : [\![\mathsf{A'}]\!].\, [\![\mathsf{B'}]\!]$ by the ANF translation, we are trying to show $[\![\Gamma]\!], \mathbf{\Gamma'} \vdash \boldsymbol{\lambda}\, \mathbf{x} : [\![\mathsf{A'}]\!].\, [\![\mathsf{e'}]\!] : \boldsymbol{\Pi}\, \mathbf{x} : [\![\mathsf{A'}]\!].\, [\![\mathsf{B'}]\!]$. This follows from [Lam] if we can show $[\![\Gamma]\!], \mathbf{\Gamma'}, \mathbf{x} : [\![\mathsf{A'}]\!] \vdash [\![\mathsf{e'}]\!] : [\![\mathsf{B'}]\!]$. By induction on the judgment $\Gamma, \mathsf{x} : \mathsf{A'} \vdash \mathsf{e'} : \mathsf{B'}$ with the empty continuation $[\![\Gamma]\!], \mathbf{\Gamma'_1} \vdash [\cdot] : ([\![\mathsf{e'}]\!] : [\![\mathsf{B'}]\!]) \Rightarrow [\![\mathsf{B'}]\!]$, where $\mathbf{\Gamma'_1} = \mathbf{\Gamma'}, [\![\mathsf{x} : \mathsf{A'}]\!] = \mathbf{\Gamma'}, \mathbf{x} : [\![\mathsf{A'}]\!]$, we obtain $[\![\Gamma]\!], \mathbf{\Gamma'}, \mathbf{x} : [\![\mathsf{A'}]\!] \vdash [\![\mathsf{e'}]\!] : [\![\mathsf{B'}]\!]$ as desired.

Case: [App]

We have that $\vdash \Gamma$, $\Gamma \vdash e_1\ e_2 : B'[x := e_2]$, and $\llbracket\Gamma\rrbracket, \Gamma' \vdash K : (N : \llbracket B'[x := e_2]\rrbracket) \Rightarrow B$ such that $\llbracket\Gamma\rrbracket, \Gamma' \vdash \llbracket e_1\ e_2\rrbracket \equiv N$. We must show that $\llbracket\Gamma\rrbracket, \Gamma' \vdash \llbracket e_1\ e_2\rrbracket\, K : B$, that is, $\llbracket\Gamma\rrbracket, \Gamma' \vdash \llbracket e_1\rrbracket\,(\mathbf{let}\, x_1 = [\cdot]\ \mathbf{in}\ \llbracket e_2\rrbracket\,(\mathbf{let}\, x_2 = [\cdot]\ \mathbf{in}\, K[x_1\ x_2])) : B$.

Let $K_1 = (\mathbf{let}\, x_1 = [\cdot]\ \mathbf{in}\ \llbracket e_2\rrbracket\,(\mathbf{let}\, x_2 = [\cdot]\ \mathbf{in}\, K[x_1\ x_2]))$. Our conclusion follows by induction on $\Gamma \vdash e_1 : \Pi\, x : A'.\, B'$, if we show $\llbracket\Gamma\rrbracket, \Gamma' \vdash K_1 : (\llbracket e_1\rrbracket : \llbracket \Pi\, x : A'.\, B'\rrbracket) \Rightarrow B$. By [K-Bind], we must show (1) $\llbracket\Gamma\rrbracket, \Gamma' \vdash \llbracket e_1\rrbracket : \llbracket \Pi\, x : A'.\, B'\rrbracket$ and (2) $\llbracket\Gamma\rrbracket, \Gamma', x_1 = \llbracket e_1\rrbracket \vdash \llbracket e_2\rrbracket\,(\mathbf{let}\, x_2 = [\cdot]\ \mathbf{in}\, K[x_1\ x_2]) : B$.

(1) follows from induction on $\Gamma \vdash e_1 : \Pi\, x : A'.\, B'$ with the empty continuation $\llbracket\Gamma\rrbracket, \Gamma' \vdash [\cdot] : (\llbracket e_1\rrbracket : \llbracket \Pi\, x : A'.\, B'\rrbracket) \Rightarrow \llbracket \Pi\, x : A'.\, B'\rrbracket$.

(2) follows by induction on $\Gamma \vdash e_2 : A'$, if we can show $\llbracket\Gamma\rrbracket, \Gamma', x_1 = \llbracket e_1\rrbracket \vdash \mathbf{let}\, x_2 = [\cdot]\ \mathbf{in}\, K[x_1\ x_2] : (\llbracket e_2\rrbracket : \llbracket A'\rrbracket) \Rightarrow B$. By [K-Bind], we must show (2.1) $\llbracket\Gamma\rrbracket, \Gamma', x_1 = \llbracket e_1\rrbracket \vdash \llbracket e_2\rrbracket : \llbracket A'\rrbracket$ and (2.2) $\llbracket\Gamma\rrbracket, \Gamma', x_1 = \llbracket e_1\rrbracket, x_2 = \llbracket e_2\rrbracket \vdash K[x_1\ x_2] : B$.

(2.1) follows from induction on $\Gamma \vdash e_2 : A'$ with the empty continuation $\llbracket\Gamma\rrbracket, \Gamma', x_1 = \llbracket e_1\rrbracket \vdash [\cdot] : (\llbracket e_2\rrbracket : \llbracket A'\rrbracket) \Rightarrow \llbracket A'\rrbracket$.

(2.2) follows from Cut (modulo equivalence) if we can show (2.2.1) $\llbracket\Gamma\rrbracket, \Gamma', x_1 = \llbracket e_1\rrbracket, x_2 = \llbracket e_2\rrbracket \vdash x_1\ x_2 : \llbracket B'[x := e_2]\rrbracket$ and (2.2.2) $\llbracket\Gamma\rrbracket, \Gamma', x_1 = \llbracket e_1\rrbracket, x_2 = \llbracket e_2\rrbracket \vdash \llbracket e_1\ e_2\rrbracket \equiv x_1\ x_2$.

(2.2.1) By $\triangleright_\delta$, we are trying to show $\llbracket\Gamma\rrbracket, \Gamma', x_1 = \llbracket e_1\rrbracket, x_2 = \llbracket e_2\rrbracket \vdash \llbracket e_1\rrbracket\ \llbracket e_2\rrbracket : \llbracket B'[x := e_2]\rrbracket$. We have previously shown in (1) and (2.1) that $\llbracket\Gamma\rrbracket, \Gamma' \vdash \llbracket e_1\rrbracket : \Pi\, x : \llbracket A'\rrbracket.\, \llbracket B'\rrbracket$ and $\llbracket\Gamma\rrbracket, \Gamma', x_1 = \llbracket e_1\rrbracket \vdash \llbracket e_2\rrbracket : \llbracket A'\rrbracket$. Using these facts with [App] and weakening, we derive $\llbracket\Gamma\rrbracket, \Gamma', x_1 = \llbracket e_1\rrbracket, x_2 = \llbracket e_2\rrbracket \vdash x_1\ x_2 : \llbracket B'\rrbracket[x := \llbracket e_2\rrbracket]$. We have that $\llbracket B'\rrbracket[x := \llbracket e_2\rrbracket]$ is equivalent to $\llbracket B'[x := e_2]\rrbracket$ by Lemma 3.4 (Substitution), and we finally derive our conclusion by [Conv].

(2.2.2) By $\triangleright_\delta$, we are trying to show $\llbracket\Gamma\rrbracket, \Gamma', x_1 = \llbracket e_1\rrbracket, x_2 = \llbracket e_2\rrbracket \vdash \llbracket e_1\ e_2\rrbracket \equiv \llbracket e_1\rrbracket\ \llbracket e_2\rrbracket$. We find that the left-hand side of the equivalence can be converted as well:

$$
\begin{aligned}
&\llbracket e_1\ e_2\rrbracket \\
&\stackrel{\text{def}}{=} \llbracket e_1\rrbracket\ \mathbf{let}\, x_1 = [\cdot]\ \mathbf{in}\ \llbracket e_2\rrbracket\ \mathbf{let}\, x_2 = [\cdot]\ \mathbf{in}\, x_1\ x_2 && && (1)\\
&= \mathbf{let}\, x_1 = [\cdot]\ \mathbf{in}\ \llbracket e_2\rrbracket\ \mathbf{let}\, x_2 = [\cdot]\ \mathbf{in}\, x_1\ x_2 \langle\!\langle\, \llbracket e_1\rrbracket\,\rangle\!\rangle && \text{by Lemma 3.2 in TR} && (2)\\
&\equiv \mathrm{let}\, x_1 = \llbracket e_1\rrbracket\ \mathrm{in}\ \llbracket e_2\rrbracket\ \mathbf{let}\, x_2 = [\cdot]\ \mathbf{in}\, x_1\ x_2 && \text{by Lemma 2.5 in TR} && (3)\\
&\triangleright^*\ \llbracket e_2\rrbracket\ \mathbf{let}\, x_2 = [\cdot]\ \mathbf{in}\ \llbracket e_1\rrbracket\ x_2 && && (4)\\
&= \mathbf{let}\, x_2 = [\cdot]\ \mathbf{in}\ \llbracket e_1\rrbracket\ x_2 \langle\!\langle\, \llbracket e_2\rrbracket\,\rangle\!\rangle && \text{by Lemma 3.2 in TR} && (5)\\
&\equiv \mathrm{let}\, x_2 = \llbracket e_2\rrbracket\ \mathrm{in}\ \llbracket e_1\rrbracket\ x_2 && \text{by Lemma 2.5 in TR} && (6)\\
&\triangleright_\zeta\ \llbracket e_1\rrbracket\ \llbracket e_2\rrbracket && && (7)
\end{aligned}
$$

Case: [Snd]

We have that $\vdash \Gamma$, $\Gamma \vdash \mathrm{snd}\, e : B'[x := \mathrm{fst}\, e]$, and $\llbracket\Gamma\rrbracket, \Gamma' \vdash K : (N : \llbracket B'[x := \mathrm{fst}\, e]\rrbracket) \Rightarrow B$ such that $\llbracket\Gamma\rrbracket, \Gamma' \vdash \llbracket \mathrm{snd}\, e\rrbracket \equiv N$. We must show that $\llbracket\Gamma\rrbracket, \Gamma' \vdash \llbracket \mathrm{snd}\, e\rrbracket\, K : B$, that is, $\llbracket\Gamma\rrbracket, \Gamma' \vdash \llbracket e\rrbracket\,(\mathbf{let}\, x_1 = [\cdot]\ \mathbf{in}\, K[\mathbf{snd}\, x_1]) : B$. Our conclusion follows by induction on $\Gamma \vdash e : \Sigma\, x : A'.\, B'$, if we show $\llbracket\Gamma\rrbracket, \Gamma' \vdash \mathbf{let}\, x_1 = [\cdot]\ \mathbf{in}\, K[\mathbf{snd}\, x_1] : (\llbracket e\rrbracket : \llbracket \Sigma\, x : A'.\, B'\rrbracket) \Rightarrow B$. By [K-Bind], we must show (1) $\llbracket\Gamma\rrbracket, \Gamma' \vdash \llbracket e\rrbracket : \llbracket \Sigma\, x : A'.\, B'\rrbracket$ and (2) $\llbracket\Gamma\rrbracket, \Gamma', x_1 = \llbracket e\rrbracket \vdash K[\mathbf{snd}\, x_1] : B$.

(1) follows from induction on $\Gamma \vdash e : \Sigma\, x : A'.\, B'$ with the empty continuation $\llbracket\Gamma\rrbracket, \Gamma' \vdash [\cdot] : (\llbracket e\rrbracket : \llbracket \Sigma\, x : A'.\, B'\rrbracket) \Rightarrow \llbracket \Sigma\, x : A'.\, B'\rrbracket$.

(2) follows by Cut (modulo equivalence) if we can show (2.1) $\llbracket\Gamma\rrbracket, \Gamma', x_1 = \llbracket e\rrbracket \vdash \mathbf{snd}\, x_1 : \llbracket B'[x := \mathrm{fst}\, e]\rrbracket$ and (2.2) $\llbracket\Gamma\rrbracket, \Gamma', x_1 = \llbracket e\rrbracket \vdash \llbracket \mathrm{snd}\, e\rrbracket \equiv \mathbf{snd}\, x_1$.

(2.1) By $\rhd_\delta$, we are trying to show $[\![\Gamma]\!], \Gamma', x_1 = [\![e]\!] \vdash \mathbf{snd}\ [\![e]\!] : [\![B'[x := \text{fst } e]]\!]$. We have previously shown in (1) that $[\![\Gamma]\!], \Gamma' \vdash [\![e]\!] : \Sigma\, x : [\![A']\!].\, [\![B']\!]$. Using this fact with [Snd] and weakening, we derive that $[\![\Gamma]\!], \Gamma', x_1 = [\![e]\!] \vdash \mathbf{snd}\ [\![e]\!] : [\![B']\!][x := \mathbf{fst}\ [\![e]\!]]$. We can derive our conclusion by [Conv] if we can show that $[\![B'[x := \text{fst } e]]\!]$ is equivalent to $[\![B']\!][x := \mathbf{fst}\ [\![e]\!]]$.
By Lemma 3.4 (Substitution), we have that $[\![B'[x := \text{fst } e]]\!]$ is equivalent to $[\![B']\!][x := [\![\text{fst } e]\!]]$. Focusing on $[\![\text{fst } e]\!]$, we have:

$$
\begin{aligned}
& [\![\text{fst } e]\!] & & \\
& \stackrel{\text{def}}{=} [\![e]\!]\ (\mathbf{let}\ x_1 = [\cdot]\ \mathbf{in}\ \mathbf{fst}\ x_1) & & (8)\\
& = \mathbf{let}\ x_1 = [\cdot]\ \mathbf{in}\ \mathbf{fst}\ x_1 \langle\!\langle\, [\![e]\!]\, \rangle\!\rangle & \text{by Lemma 3.2 in TR} \quad & (9)\\
& \equiv \text{let } x_1 = [\![e]\!]\ \text{ in } \mathbf{fst}\ x_1 & \text{by Lemma 2.5 in TR} \quad & (10)\\
& \rhd_\zeta\ \mathbf{fst}\ [\![e]\!] & & (11)
\end{aligned}
$$

Thus we derive that $[\![B'[x := \text{fst } e]]\!]$ is equivalent to $[\![B']\!][x := \mathbf{fst}\ [\![e]\!]]$.
(2.2) By $\rhd_\delta$, we are trying to show $[\![\Gamma]\!], \Gamma', x_1 = [\![e]\!] \vdash [\![\text{snd } e]\!] \equiv \mathbf{snd}\ [\![e]\!]$. Focusing on $[\![\text{snd } e]\!]$, we have:

$$
\begin{aligned}
& [\![\text{snd } e]\!] & & \\
& \stackrel{\text{def}}{=} [\![e]\!]\ (\mathbf{let}\ x_1 = [\cdot]\ \mathbf{in}\ \mathbf{snd}\ x_1) & & (12)\\
& = \mathbf{let}\ x_1 = [\cdot]\ \mathbf{in}\ \mathbf{snd}\ x_1 \langle\!\langle\, [\![e]\!]\, \rangle\!\rangle & \text{by Lemma 3.2 in TR} \quad & (13)\\
& \equiv \text{let } x_1 = [\![e]\!]\ \text{ in } \mathbf{snd}\ x_1 & \text{by Lemma 2.5 in TR} \quad & (14)\\
& \rhd_\zeta\ \mathbf{snd}\ [\![e]\!] & & (15)
\end{aligned}
$$

Case: [If]
We have that $\vdash \Gamma$, $\Gamma \vdash \text{if } e \text{ then } e_1 \text{ else } e_2 : B'[x' := e]$, and $[\![\Gamma]\!], \Gamma' \vdash \mathbf{K} : (\mathbf{N} : [\![B'[x' := e]]\!]) \Rightarrow \mathbf{B}$ such that $[\![\Gamma]\!], \Gamma' \vdash [\![\text{if } e \text{ then } e_1 \text{ else } e_2]\!] \equiv \mathbf{N}$. We must show that $[\![\Gamma]\!], \Gamma' \vdash [\![\text{if } e \text{ then } e_1 \text{ else } e_2]\!]\ \mathbf{K} : \mathbf{B}$, that is, $[\![\Gamma]\!], \Gamma' \vdash [\![e]\!]\ (\mathbf{let}\ x = [\cdot]\ \mathbf{in}\ \mathbf{if}\ x\ \mathbf{then}\ [\![e_1]\!]\ \mathbf{K}\ \mathbf{else}\ [\![e_2]\!]\ \mathbf{K}) : \mathbf{B}$. Let $\mathbf{K}_1 = \mathbf{let}\ x = [\cdot]\ \mathbf{in}\ \mathbf{if}\ x\ \mathbf{then}\ [\![e_1]\!]\ \mathbf{K}\ \mathbf{else}\ [\![e_2]\!]\ \mathbf{K}$. Our conclusion follows by induction on $\Gamma \vdash e : \text{bool}$, if we show $[\![\Gamma]\!], \Gamma' \vdash \mathbf{K}_1 : ([\![e]\!] : [\![\text{bool}]\!]) \Rightarrow \mathbf{B}$. By [K-Bind], we must show (1) $[\![\Gamma]\!], \Gamma' \vdash [\![e]\!] : [\![\text{bool}]\!]$ and (2) $[\![\Gamma]\!], \Gamma', x = [\![e]\!] \vdash \mathbf{if}\ x\ \mathbf{then}\ [\![e_1]\!]\ \mathbf{K}\ \mathbf{else}\ [\![e_2]\!]\ \mathbf{K} : \mathbf{B}$.
(1) follows from induction on $\Gamma \vdash e : \text{bool}$ with the empty continuation $[\![\Gamma]\!], \Gamma' \vdash [\cdot] : ([\![e]\!] : [\![\text{bool}]\!]) \Rightarrow [\![\text{bool}]\!]$.
(2) By [If], we now must show (2.1) $[\![\Gamma]\!], \Gamma', x = [\![e]\!] \vdash \mathbf{B} : \mathbf{U}$, (2.2) $[\![\Gamma]\!], \Gamma', x = [\![e]\!] \vdash x : \mathbf{bool}$ (2.3) $[\![\Gamma]\!], \Gamma', x = [\![e]\!], x \equiv \mathbf{true} \vdash [\![e_1]\!]\ \mathbf{K} : \mathbf{B}$ (2.4) $[\![\Gamma]\!], \Gamma', x = [\![e]\!], x \equiv \mathbf{false} \vdash [\![e_2]\!]\ \mathbf{K} : \mathbf{B}$
(2.1) should follow from induction on $\Gamma \vdash B : U$ and the empty continuation.
(2.2) By $\rhd_\delta$ and the fact that $[\![\text{true}]\!] = \mathbf{true}$, we can show that $[\![\Gamma]\!], \Gamma', x = [\![e]\!] \vdash [\![e]\!] : \mathbf{true}$ following the same method done above in (1).
(2.3) This follows by induction on $\Gamma \vdash e_1 : B'[x' := \text{true}]$ if we can show (2.3.1) $[\![\Gamma]\!], \Gamma', x = [\![e]\!], x \equiv \mathbf{true} \vdash \mathbf{K} : ([\![e_1]\!] : [\![B'[x' := \text{true}]]\!]) \Rightarrow \mathbf{B}$ and (2.3.2) $[\![\Gamma]\!], \Gamma', x = [\![e]\!], x \equiv \mathbf{true} \vdash [\![\text{if } e \text{ then } e_1 \text{ else } e_2]\!] \equiv [\![e_1]\!]$.
(2.3.1) follows from our earlier assumption and [Conv] if we can show that $[\![\Gamma]\!], \Gamma', x = [\![e]\!], x \equiv \mathbf{true} \vdash [\![B'[x' := e]]\!] \equiv [\![B'[x' := \text{true}]]\!]$. By Substitution, we are trying to show $[\![\Gamma]\!], \Gamma', x = [\![e]\!], x \equiv \mathbf{true} \vdash [\![B']\!][x' := [\![e]\!]] \equiv [\![B']\!][x' := \mathbf{true}]$. By [Reflect] and **??**, symmetry and transitivity, we can derive that under this extended context $[\![\Gamma]\!], \Gamma', x = [\![e]\!], x \equiv \mathbf{true} \vdash [\![e]\!] \equiv \mathbf{true}$, and thus we can conclude $[\![\Gamma]\!], \Gamma', x = [\![e]\!], x \equiv \mathbf{true} \vdash [\![B']\!][x' := [\![e]\!]] \equiv [\![B']\!][x' := \mathbf{true}]$ (by what exactly though???).
(2.3.2) follows by reducing the left-hand side of the expression, and the fact that we derived $[\![e]\!] \equiv \mathbf{true}$ in (2.3.1.):

$$
\begin{aligned}
& [\![\text{if e then } e_1 \text{ else } e_2]\!] \\
& \stackrel{\text{def}}{=} [\![e]\!] \, (\mathbf{let\ x} = [\cdot] \ \mathbf{in\ if\ x\ then}\ [\![e_1]\!]\ \mathbf{else}\ [\![e_2]\!]) && && (16) \\
& = \mathbf{let\ x} = [\cdot] \ \mathbf{in\ if\ x\ then}\ [\![e_1]\!]\ \mathbf{else}\ [\![e_2]\!] \langle\!\langle [\![e]\!] \rangle\!\rangle && \text{by Lemma 3.2 in TR} && (17) \\
& \equiv \text{let x} = [\![e]\!] \ \text{in } \mathbf{if\ x\ then}\ [\![e_1]\!]\ \mathbf{else}\ [\![e_2]\!] && \text{by Lemma 2.5 in TR} && (18) \\
& \rhd_{\zeta}\ \mathbf{if}\ [\![e]\!]\ \mathbf{then}\ [\![e_1]\!]\ \mathbf{else}\ [\![e_2]\!] && && (19) \\
& \rhd^{*}\ [\![e_1]\!] && \text{by } [\![e]\!] \equiv \mathbf{true} && (20)
\end{aligned}
$$

Thus we obtain our goal of $[\![\Gamma]\!], \mathbf{\Gamma'}, \mathbf{x} = [\![e]\!], \mathbf{x} \equiv \mathbf{true} \vdash [\![\text{if e then } e_1 \text{ else } e_2]\!] \equiv [\![e_1]\!]$.
(2.4) follows analogously (but we derive that $[\![e]\!] \equiv \mathbf{false}$).

□

Lemma 1.2. *If* $\mathbf{\Gamma} \vdash \mathbf{x} : \mathbf{A}$, $\mathbf{\Gamma} \vdash \mathbf{e} : \mathbf{A}$, *and* $\mathbf{x} = \mathbf{e} \in \mathbf{\Gamma}$, *then* $\mathbf{\Gamma} \vdash \mathbf{x} \equiv \mathbf{e}$.

Proof. We have that $\mathbf{\Gamma} \vdash \mathbf{x} \rhd^{*} \mathbf{e}$ by $\rhd_{\delta}$, and $\mathbf{\Gamma} \vdash \mathbf{e} \rhd^{*} \mathbf{e}$ by [Refl]. Then, by [≡], $\mathbf{\Gamma} \vdash \mathbf{x} \equiv \mathbf{e}$. □