Lecture 3 Proof of Black-Scholes Formula

1. Replication of derivative dynamics

We have $dS_t = \mu S_t\, dt + \sigma S_t\, dW_t^{\mathbb{P}}$

$dB_t = r B_t\, dt$

$$dF(t,S_t) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S_t}dS_t + \frac{1}{2}\frac{\partial^2 F}{\partial S_t^2}(dS_t)^2$$

$$= \left(\frac{\partial F}{\partial t} + \mu S_t\frac{\partial F}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 F}{\partial S_t^2}\right)dt + \sigma S_t\frac{\partial F}{\partial S_t}dW_t^{\mathbb{P}} \quad ①$$

we want replicate this dynamic by a combination of Stock and Bond. Assume we have a portfolio $V(t)$ with Z_t shares of Stock and $(V_t - Z_t\cdot S_t)$ units of Bond.

Then, we have

$$dV_t = Z_t\cdot dS_t + r(V_t - Z_t\cdot S_t)\cdot dt$$

$$\Rightarrow dV_t = Z_t\cdot(\mu S_t\, dt + \sigma S_t\, dW_t^{\mathbb{P}}) + rV_t\, dt - rZ_tS_t\, dt$$

$$= \underline{Z_t\cdot\mu S_t\, dt} + Z_t\cdot\sigma S_t\, dW_t \underline{+ rV_t\, dt} \underline{- rZ_tS_t\, dt}$$

$$= (Z_t\cdot\mu S_t + rV_t - rZ_tS_t)dt + Z_t\cdot\sigma S_t\cdot dW_t^{\mathbb{P}} \quad ②$$

To make sure $dV_t = dF(t,S_t)$ we need

$$\begin{cases} Z_t\cdot\cancel{\sigma S_t} = \cancel{\sigma S_t}\frac{\partial F}{\partial S_t} \\ Z_t\mu S_t + rV_t - rZ_tS_t = \frac{\partial F}{\partial t} + \mu S_t\frac{\partial F}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 F}{\partial S_t^2} \end{cases}$$

$$\Rightarrow Z_t = \frac{\partial F}{\partial S_t} \qquad \cancel{\frac{\partial F}{\partial S_t}\mu S_t} - rV_t - r\frac{\partial F}{\partial S_t}\cdot S_t = \frac{\partial F}{\partial t} + \cancel{\mu S_t\frac{\partial F}{\partial S_t}} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 F}{\partial S_t^2}$$

$\boxed{\mu \text{ is gone}}$. we also have $V_t = F_t$

$$\Rightarrow \begin{cases} \frac{\partial F}{\partial t} + rS_t\frac{\partial F}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 F}{\partial S_t^2} - rF = 0 \\ F = \Phi(S_T) \end{cases}$$

This replication process gives us $\Delta_t = Z_t = \frac{\partial F}{\partial S_t}$ and a PDE with boundary condition.

2. Feynman-Kac Method

To solve this PDE, let's propose a new fictional Stock process. $dS_t = rS_t\, dt + \sigma S_t\, dW_t^{\mathbb{Q}}$ under $\mathbb{Q}$-measure.

Using this Stock dynamics, we define new process $\boxed{e^{-rt}\cdot F(t,S_t)} = A_t$

$$\Rightarrow dA_t = -re^{-rt}\cdot F\cdot dt + e^{-rt}\cdot dF$$

$$= -r\cdot e^{-rt}\cdot F\cdot dt + e^{-rt}\left[\left(\frac{\partial F}{\partial t} + rS_t\frac{\partial F}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 F}{\partial S_t^2}\right)dt + \sigma S_t\frac{\partial F}{\partial S_t}\cdot dW_t\right]$$

$$= e^{-rt}\left[\frac{\partial F}{\partial t} + rS_t\frac{\partial F}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 F}{\partial S_t^2} - rF\right]dt + e^{-rt}\cdot\sigma S_t\frac{\partial F}{\partial S_t}\cdot dW_t$$

$$= e^{-rt}\cdot\sigma S_t\frac{\partial F}{\partial S_t}dW_t$$

integrate from t to T

$$e^{-rT}\cdot F(T,S_T) - e^{-rt}\cdot F(t,S_t) = \int_t^T e^{-ru}\cdot\sigma S_u\frac{\partial F}{\partial S_u}dW_u$$

Let's take expectation on both side at time t.

$$\mathbb{E}_t\left[e^{-rT}\cdot F(T,S_T)\right]-e^{-rt}\cdot F(t,S_t)=\underbrace{\mathbb{E}_t\left[\int_t^T e^{-ru}\sigma S_u\frac{\partial F}{\partial S_u}dW_u\right]}_{=0}$$

$$\Rightarrow F(t,S_t)\cdot e^{-rt}=\mathbb{E}_t\left[e^{-rT}\cdot F(T,S_T)\right]$$

we also have $F(T,S_T)=\Phi(S_T)$

Thus $F(t,S_t)\cdot e^{-rt}=\mathbb{E}_t\left[e^{-rT}\cdot\Phi(S_T)\right]$

$$\boxed{F(t,S_t)=\mathbb{E}_t\left[e^{-r(T-t)}\cdot\Phi(S_T)\right]}$$

we see that original PDE problem reduced to an equivalent expectation of random variable problem.

(*) Notice, we have a very general result here, since we have not yet said anything about the nature of the derivative at all.

This means that the result here applies to any derivative with underlying S_t, with any functional form.

Let's restate the result

$$\boxed{F(t,S_t)=\mathbb{E}_t\left[e^{-r(T-t)}\cdot\Phi(X_T)\right]}$$

where $\Phi(X_T)$ is the terminal payoff of any particular derivative.

3. Now we can say something about European call. Only thing we need is $\Phi(X_T)$

and $\Phi(X_T)=(S_T-K)^+$ for call option

Thus $F(t,S_t)=\mathbb{E}_t\left[e^{-r(T-t)}\cdot(S_T-K)^+\right]$

$$=\int_{-\infty}^{+\infty}e^{-r(T-t)}(S_T-K)^+\,dF(S_t)$$

we can use LOTUS to convert this expectation to:

$$F(t,S_t)=\int_{S_T>K}e^{-r(T-t)}\left(S_t\cdot e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}\,z}-K\right)f_z(z)dz$$

Since if $S_T<K$, the value would be 0 $\quad z\sim N(0,1)$

$$F(t,S_t)=\int_C^{+\infty}e^{-r(T-t)}\cdot S_t\cdot e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}\,z}\cdot f(z)dz-\int_C^{+\infty}e^{-r(T-t)}K\cdot f(z)dz$$

$\Uparrow$

$$\boxed{\begin{aligned}&S_t\cdot e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}\,z}>K\Rightarrow(r-\tfrac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}\,z>\ln\left(\tfrac{K}{S_t}\right)\\&\Rightarrow z>\frac{\ln\left(\frac{K}{S_t}\right)-(r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}=C\end{aligned}}$$

① Second term is simple

$$e^{-r(T-t)}\cdot K\cdot\int_C^{+\infty}f(z)dz=e^{-r(T-t)}\cdot K\cdot(1-N(C))$$

where $N(C)$ is cumulative distribution from $-\infty\to C$

Notice $1-N(C)=N(-C)$ due to symmetry

Second term $= e^{-r(T-t)} \cdot K \cdot N(d_-)$

where $\boxed{d_- = -c = \dfrac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}}$

② First term

$$\int_c^{+\infty} e^{-r(T-t)} \cdot S_t \cdot e^{\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma\sqrt{T-t}\,z} f(z)\,dz$$

$$= S_t \cdot e^{-\frac{\sigma^2}{2}(T-t)} \int_c^{+\infty} e^{\sigma\sqrt{T-t}\,z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz$$

$$\text{Integral} = \int_c^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(z^2 - 2\sigma\sqrt{T-t}\,z\right)}\,dz$$

$$= \int_c^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left[\left(z-\sigma\sqrt{T-t}\right)^2 - \sigma^2(T-t)\right]}\,dz$$

$$= e^{\frac{\sigma^2}{2}(T-t)} \int_c^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left[\left(z-\sigma\sqrt{T-t}\right)^2\right]}\,dz$$

$$\text{First term} = S_t \cdot \int_c^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(z-\sigma\sqrt{T-t}\right)^2}\,dz$$

Let $z - \sigma\sqrt{T-t} = Y$

$$\text{First term} = S_t \int_{c'}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Y^2}\,dY$$

lower band

$$z = c = \frac{\ln\left(\frac{K}{S_t}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$Y = c - \sigma\sqrt{T-t} = \frac{\ln\left(\frac{K}{S_t}\right) - \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = c'$$

Thus, first term $= S_t\left(1 - N(c')\right) = S_t \cdot N(-c')$

$$d_+ = -c' = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\ldots}$$

Let $d_+ = -c' = \dfrac{\quad}{\sigma\sqrt{T-t}}$

We have proved Black-Scholes formula

$$C(t, S_t) = S_t \cdot N(d_+) - e^{-r(T-t)} \cdot K \cdot N(d_-)$$

where
$$\begin{cases} d_+ = \dfrac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_- = \dfrac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = d_+ - \sigma\sqrt{T-t} \end{cases}$$

$\square$.