MATH 305:201, 2020W T2

Homework set 2 — due January 29

Problem 1. Prove the following identities and bounds:

1. $\overline{z+w} = \overline{z} + \overline{w}$
2. $\overline{zw} = \overline{z}\,\overline{w}$
3. $|\overline{z}| = |z|$
4. $|\mathrm{Re}(z)| \leq |z|, \quad |\mathrm{Im}(z)| \leq |z|$
5. $|z+w|^2 = |z|^2 + |w|^2 + 2\mathrm{Re}(z\overline{w})$ (*Hint:* Recall that $|\zeta|^2 = \zeta\overline{\zeta}$)
6. Combine the above two results to derive the *triangle inequality*:
$$|z+w| \leq |z| + |w|,$$
valid for any $z, w \in \mathbb{C}$.
7. Deduce the inequality $|z-w| \geq ||z| - |w||$.

Problem 2. Derive the following trigonometric identities using the complex exponential:

$$\begin{aligned} &\text{(i)} && \sin(3\theta) = 3\cos^2(\theta)\sin(\theta) - \sin^3(\theta), \\ &\text{(ii)} && \sin(\theta - \psi) = \sin(\theta)\cos(\psi) - \cos(\theta)\sin(\psi). \end{aligned}$$

Problem 3. Sketch the domain $\Omega \subset \mathbb{C}$ and its image $f(\Omega)$ under the mapping f:

1. $\Omega = \{z \in \mathbb{C} : |z| < 1 \text{ and } \mathrm{Im}(z) > 0\}$ and $f(z) = 2\mathrm{e}^{\mathrm{i}\pi/2}z + (2 + 2\mathrm{i})$
2. $\Omega = \{z \in \mathbb{C} : -1 < \mathrm{Im}(z) < 1\}$ and $f(z) = \mathrm{e}^{\pi z/2}$
3. $\Omega = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ and $f(z) = (z-1)/(z+1)$

Problem 4. Let $\Omega \subset \mathbb{C}$ be a domain and let $f \in H(\Omega)$.

1. Assume that f is real-valued. Show that f is constant on Ω.
2. Assume now that both $f, \overline{f} \in H(\Omega)$. Show that f is constant on Ω.

Problem 5. Use the Cauchy-Riemann equations to show that the functions below are entire and compute their derivative:

1. $f : \mathbb{C} \to \mathbb{C}$ given by $f(x + \mathrm{i}y) = \mathrm{e}^{-2xy}\left(\cos(x^2 - y^2) + \mathrm{i}\sin(x^2 - y^2)\right)$
2. $g : \mathbb{C} \to \mathbb{C}$ given by $g(z) = (1/2)\left(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}\right)$