

Assignment 6

MATH 305 - Applied Complex Analysis

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Problem 1

The denominator can be expressed as $z^2 - z(1+i) = (z)(z - (1+i))$, i.e. there are two singularities at $z = 0$, and $z = 1+i$.

- i) $\alpha(t) = B_2(1)$. We can partition $\alpha(t)$ into $\alpha_1(t)$, which contains the singularity $z = 0$, and $\alpha_2(t)$, which contains the singularity $z = 1+i$.

$$\begin{aligned} \oint_{\alpha} \frac{z-i+1}{z^2 - z(1+i)} dz &= \oint_{\alpha_1} \frac{z-i+1}{z^2 - z(1+i)} dz + \oint_{\alpha_2} \frac{z-i+1}{z^2 - z(1+i)} dz \\ &= \oint_{\alpha_1} \frac{\frac{z-i+1}{z}}{z - (1+i)} dz + \oint_{\alpha_2} \frac{\frac{z-i+1}{z-(1+i)}}{z} dz \end{aligned}$$

For $\oint_{\alpha_1} \frac{(z-i+1)/z}{z-(1+i)} dz$, let $f_1(z) = \frac{z-i+1}{z}$, and $w_1 = 1+i$. Cauchy's Integral Theorem gives:

$$\begin{aligned} \oint_{\alpha_1} \frac{\frac{z-i+1}{z}}{z - (1+i)} dz &= 2\pi i f_1(1+i) \\ &= 2\pi i \left(\frac{(1+i) - i + 1}{1+i} \right) \\ &= 2\pi i \left(\frac{2}{1+i} \right) \\ &= \frac{4\pi i}{1+i} \end{aligned} \tag{1}$$

For $\oint_{\alpha_2} \frac{(z-i+1)/(z-(1+i))}{z} dz$, let $f_2(z) = \frac{z-i+1}{z-(1+i)}$, and $w_2 = 0$. Similarly, we have:

$$\begin{aligned} \oint_{\alpha_2} \frac{\frac{z-i+1}{z-(1+i)}}{z} dz &= 2\pi i f_2(0) \\ &= 2\pi i \left(\frac{1-i}{-(1+i)} \right) \\ &= \frac{2\pi i(i-1)}{1+i} \\ &= \frac{2\pi(-1-i)}{1+i} \\ &= \frac{-2\pi(1+i)}{(1+i)} \\ &= -2\pi \end{aligned}$$

$$\therefore \oint_{\alpha} \frac{z-i+1}{z^2 - z(1+i)} dz = -2\pi + \frac{4\pi i}{1+i}$$

- ii) Let $\beta = B_1(1+i)$. β only contains the singularity $z = 1+i$, so we can use result (1) above:

$$\oint_{\beta} \frac{z-i+1}{z^2 - z(1+i)} dz = \oint_{\alpha_1} \frac{\frac{z-i+1}{z}}{z - (1+i)} dz = \frac{4\pi i}{1+i}$$

iii) Let $\gamma = B_{\frac{1}{2}}(2)$. γ does not contain either of the singularity points, so by Cauchy's Theorem:

$$\oint_{\gamma} \frac{z-i+1}{z^2-z(1+i)} dz = 0$$

Problem 2

Let the contour be α_R :

$$\begin{cases} c_{\epsilon}(t) = \epsilon e^{i(\pi-t)}, & t \in [0, \pi], \\ c_1(t) = t, & t \in [\epsilon, R], \\ c_R(t) = R e^{it}, & t \in [0, \pi], \\ c_2(t) = t, & t \in [-R, -\epsilon], \end{cases} \Rightarrow \begin{cases} c'_{\epsilon}(t) = -i\epsilon e^{i(\pi-t)}, & t \in [0, \pi], \\ c'_1(t) = 1, & t \in [\epsilon, R], \\ c'_R(t) = iR e^{it}, & t \in [0, \pi], \\ c'_2(t) = 1, & t \in [-R, -\epsilon], \end{cases}$$

As the contour is closed, by Cauchy's Theorem:

$$\oint_{\alpha_R} f(z) dz = \oint_{\alpha_R} \frac{e^{iz}}{z} dz = 0, \quad (2)$$

where $f(z)$ can be expressed as:

$$\begin{aligned} f(z) &= \frac{e^{-y} e^{ix}}{z} \\ &= \frac{e^{-y} (\cos(x) + i \sin(x))}{z} \\ &= \frac{e^{-y}}{z} (\cos(x) + i \sin(x)) \end{aligned}$$

where $x = \operatorname{Re}(z)$, $y = \operatorname{Im}(z)$. Similarly with the segments c_{ϵ} and c_R :

$$\begin{aligned} c_{\epsilon}(t) &= \epsilon e^{i\pi} e^{-it} \\ &= -\epsilon (\cos(t) - i \sin(t)) \\ &= -\epsilon \cos(t) + i\epsilon \sin(t), \end{aligned}$$

$$\begin{aligned} c_R(t) &= R(\cos(t) + i \sin(t)) \\ &= R \cos(t) + iR \sin(t), \end{aligned}$$

For the segment c_ϵ :

$$\begin{aligned} \int_{c_\epsilon} f(z) dz &= \int_0^\pi \left(\frac{e^{-\epsilon \sin(t)}}{\epsilon e^{i(\pi-t)}} \right) (\cos(-\epsilon \cos(t)) + i \sin(-\epsilon \cos(t))) (-i\epsilon e^{i(\pi-t)}) dt \\ &= -i \int_0^\pi e^{-\epsilon \sin(t)} e^{-i\epsilon \cos(t)} dt \\ &= -i \int_0^\pi e^{-\epsilon(\sin(t)+i \cos(t))} dt \end{aligned} \quad (3)$$

For the segment c_1 :

$$\begin{aligned} \int_{c_1} f(z) dz &= \int_\epsilon^R \frac{e^{ix}}{x} dx \\ &= \int_\epsilon^R \frac{\cos(x)}{x} + i \frac{\sin(x)}{x} dx \\ &= \int_\epsilon^R \frac{\cos(x)}{x} dx + i \int_\epsilon^R \frac{\sin(x)}{x} dx \end{aligned} \quad (4)$$

For the segment c_R :

$$\begin{aligned} \int_{c_R} f(z) dz &= \int_0^\pi \left(\frac{e^{-R \sin(t)}}{R e^{it}} \right) (\cos(R \cos(t)) + i \sin(R \cos(t))) (iR e^{it}) dt \\ &= i \int_0^\pi e^{-R \sin(t)} e^{iR \cos(t)} dt \\ &= i \int_0^\pi e^{-R(\sin(t)-i \cos(t))} dt \end{aligned} \quad (5)$$

For the segment c_2 :

$$\begin{aligned} \int_{c_2} f(z) dz &= \int_{-R}^{-\epsilon} \frac{e^{ix}}{x} dx \\ &= \int_{-R}^{-\epsilon} \frac{\cos(x)}{x} + i \frac{\sin(x)}{x} dx \\ &= \int_{-R}^{-\epsilon} \frac{\cos(x)}{x} dx + i \int_{-R}^{-\epsilon} \frac{\sin(x)}{x} dx \end{aligned} \quad (6)$$

For (3), taking the limit as $\epsilon \rightarrow 0$ gives:

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} -i \int_0^\pi e^{-\epsilon(\sin(t)+i \cos(t))} dt &= -i \int_0^\pi e^{-0} dt \\ &= -i \int_0^\pi 1 dt \\ &= -\pi i \end{aligned} \quad (7)$$

Taking the limit of (5) as $R \rightarrow \infty$ gives:

$$\lim_{R \rightarrow \infty} i \int_0^\pi e^{-R(\sin(t)-i \cos(t))} dt = 0 \quad (8)$$

Taking the limit of (4)+(6) as $R \rightarrow \infty$ and $\epsilon \rightarrow 0$ gives:

$$\begin{aligned} & \lim_{R \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \left(\int_{\epsilon}^R \frac{\cos(x)}{x} dx + i \int_{\epsilon}^R \frac{\sin(x)}{x} dx + \int_{-R}^{-\epsilon} \frac{\cos(x)}{x} dx + i \int_{-R}^{-\epsilon} \frac{\sin(x)}{x} dx \right) \\ &= \lim_{R \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \left(\int_{\epsilon}^R \frac{\cos(x)}{x} dx + \int_{-R}^{-\epsilon} \frac{\cos(x)}{x} dx \right) + i \left(\lim_{R \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \left(\int_{\epsilon}^R \frac{\sin(x)}{x} dx + \int_{-R}^{-\epsilon} \frac{\sin(x)}{x} dx \right) \right) \\ &= \lim_{R \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \left(\int_{\epsilon}^R \frac{\cos(x)}{x} dx + \int_{-R}^{-\epsilon} \frac{\cos(x)}{x} dx \right) + 2i \int_0^{\infty} \frac{\sin(x)}{x} dx, \text{ by Hint (a)} \end{aligned} \quad (9)$$

Combining (7), (8), (9), and substituting into (2) then gives:

$$\begin{aligned} -\pi i + 0 + \lim_{R \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \left(\int_{\epsilon}^R \frac{\cos(x)}{x} dx + \int_{-R}^{-\epsilon} \frac{\cos(x)}{x} dx \right) + 2i \int_0^{\infty} \frac{\sin(x)}{x} dx &= 0 \\ \lim_{R \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \left(\int_{\epsilon}^R \frac{\cos(x)}{x} dx + \int_{-R}^{-\epsilon} \frac{\cos(x)}{x} dx \right) + 2i \int_0^{\infty} \frac{\sin(x)}{x} dx &= \pi i \end{aligned}$$

Taking $\text{Im}(\text{LHS}) = \text{Im}(\text{RHS})$ gives:

$$\begin{aligned} 2 \int_0^{\infty} \frac{\sin(x)}{x} dx &= \pi \\ \int_0^{\infty} \frac{\sin(x)}{x} dx &= \frac{\pi}{2} \end{aligned}$$

□

Problem 3

Let the contour $\gamma(t)$ be parametrized as follows:

$$\begin{cases} \gamma_1(t) = t, & t \in [0, R], \\ \gamma_2(t) = R(1 + it), & t \in [0, a], \\ \gamma_3(t) = t(1 + ia), & t \in [0, R], \end{cases} \Rightarrow \begin{cases} \gamma'_1(t) = 1, \\ \gamma'_2(t) = Ri, \\ \gamma'_3(t) = (1 + ia), \end{cases}$$

Let $f(z) = e^{-z^2}$. As $\gamma(t)$ is closed, by Cauchy's Theorem we have $\oint_{\gamma} f(z) dz = 0$:

$$\Rightarrow \int_0^R e^{-t^2} dt + \int_0^a e^{-R^2(1+it)^2} (Ri) dt - \underbrace{\int_0^R e^{-(t(1+ia)^2)} (1+ia) dt}_{} = 0 \quad (10)$$

We subtract this term as
the orientation is opposite
of the parametrization
above

Taking the limit of the left hand side of (10) as $R \rightarrow \infty$ gives:

$$\int_0^\infty e^{-t^2} dt + \int_0^a e^{-\infty}(Ri) dt - \int_0^\infty e^{-(t(1+ia)^2)}(1+ia) dt = \frac{\sqrt{\pi}}{2} + 0 - (1+ia) \int_0^\infty e^{-(1+ia)^2 t^2} dt$$

Substituting this result back into (10) gives:

$$\begin{aligned} \frac{\sqrt{\pi}}{2} - (1+ia) \int_0^\infty e^{-(1+ia)^2 t^2} dt &= 0 \\ (1+ia) \int_0^\infty e^{-(1+ia)^2 t^2} dt &= \frac{\sqrt{\pi}}{2} \\ \int_0^\infty e^{-(1+ia)^2 t^2} dt &= \frac{\sqrt{\pi}}{2(1+ia)} \end{aligned}$$

□

From this:

$$\begin{aligned} \int_0^\infty e^{-(1+2ia-a^2)t^2} dt &= \frac{\sqrt{\pi}}{2(1+ia)} \\ \int_0^\infty e^{-((1-a^2)t^2+2iat^2)} dt &= \frac{\sqrt{\pi}}{2(1+ia)} \\ \int_0^\infty e^{(a^2-1)t^2-2iat^2} dt &= \frac{\sqrt{\pi}}{2} \left(\frac{1-ia}{1+a^2} \right) \\ \int_0^\infty e^{(a^2-1)t^2} e^{-i2at^2} dt &= \frac{\sqrt{\pi}}{2(1+a^2)} - i \frac{a\sqrt{\pi}}{2(1+a^2)} \\ \int_0^\infty e^{(a^2-1)t^2} (\cos(2at^2) - i \sin(2at^2)) dt &= \frac{\sqrt{\pi}}{2(1+a^2)} - i \frac{a\sqrt{\pi}}{2(1+a^2)} \end{aligned}$$

Taking $\text{Re}(\text{LHS})=\text{Re}(\text{RHS})$:

$$\int_0^\infty e^{(a^2-1)t^2} \cos(2at^2) dt = \frac{\sqrt{\pi}}{2(1+a^2)}$$

Let $a = 1$:

$$\begin{aligned} \int_0^\infty e^{(1-1)t^2} \cos(2t^2) dt &= \frac{\sqrt{\pi}}{2(1+1)} \\ \int_0^\infty \cos(2t^2) dt &= \frac{\sqrt{\pi}}{4} \end{aligned}$$

Let $u = \sqrt{2}t$, $du = \sqrt{2} dt$:

$$\begin{aligned} \therefore \int_0^\infty \cos\left(2\left(\frac{u}{\sqrt{2}}\right)^2\right) \frac{du}{\sqrt{2}} &= \frac{\sqrt{\pi}}{4} \\ \int_0^\infty \cos\left(2\left(\frac{u^2}{2}\right)\right) \frac{du}{\sqrt{2}} &= \frac{\sqrt{\pi}}{4} \\ \frac{1}{\sqrt{2}} \int_0^\infty \cos(u^2) du &= \frac{\sqrt{\pi}}{4} \\ \int_0^\infty \cos(u^2) du &= \frac{\sqrt{2\pi}}{4} \\ \int_0^\infty \cos(u^2) du &= \sqrt{\frac{2\pi}{16}} \quad \text{8} \\ \int_0^\infty \cos(u^2) du &= \sqrt{\frac{\pi}{8}} \\ \Rightarrow \int_0^\infty \cos(t^2) dt &= \sqrt{\frac{\pi}{8}} \end{aligned}$$

□

Taking $\text{Im}(\text{LHS}) = \text{Im}(\text{RHS})$:

$$\int_0^\infty e^{(a^2-1)t^2} \sin(2at^2) dt = \frac{a\sqrt{\pi}}{2(1+a^2)}$$

Let $a = 1$:

$$\begin{aligned} \int_0^\infty e^{(1-1)t^2} \sin(2t^2) dt &= \frac{\sqrt{\pi}}{2(1+1)} \\ \int_0^\infty \sin(2t^2) dt &= \frac{\sqrt{\pi}}{4} \end{aligned}$$

Similarly, let $u = \sqrt{2}t$, $du = \sqrt{2} dt$:

$$\begin{aligned} \frac{1}{\sqrt{2}} \int_0^\infty \sin(u^2) du &= \frac{\sqrt{\pi}}{4} \\ \int_0^\infty \sin(u^2) du &= \frac{\sqrt{2\pi}}{4} \\ \int_0^\infty \sin(u^2) du &= \sqrt{\frac{2\pi}{16}} \quad \text{8} \\ \int_0^\infty \sin(u^2) du &= \sqrt{\frac{\pi}{8}} \\ \Rightarrow \int_0^\infty \sin(t^2) dt &= \sqrt{\frac{\pi}{8}} = \int_0^\infty \cos(t^2) dt \end{aligned}$$

□

In both cases, we treat u as a 'dummy variable' and just substitute $u = t$ back into the expression.

Problem 4

Since $f(z) = g(z) \forall z \in \alpha$, we have that $|f(z) - g(z)| = 0$ on the curve α . By the Maximum Modulus Principle, $|f(z) - g(z)|$ reaches its maximum on the boundary of the bounded domain (in this case α). Therefore, there is no point in the interior of α such that $|f(z) - g(z)| > 0$. As the modulus of any complex number cannot be negative, this means that

$$|f(z) - g(z)| = 0 \Rightarrow f(z) = g(z), \forall z \in \text{int}(\alpha)$$

□