# Assignment 5

**MATH 305 - Applied Complex Analysis**

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## Problem 1

The given closed polygonal arc can be expressed as a piecewise smooth arc:

$$\begin{cases} \alpha_1 = t(1+i), & t \in [0,1] \\ \alpha_2 = t + (2-t)i, & t \in [1,2] \\ \alpha_3 = -t, & t \in [-2,0] \end{cases}$$

The derivatives are:

$$\begin{cases} \alpha_1' = 1+i, & t \in [0,1] \\ \alpha_2' = 1-i, & t \in [1,2] \\ \alpha_3' = -1, & t \in [-2,0] \end{cases}$$

i) $f(z) = z$:

$$\begin{aligned}
\oint_\alpha z\,dz &= \int_{\alpha_1} z\,dz + \int_{\alpha_2} z\,dz + \int_{\alpha_3} z\,dz \\
&= \int_0^1 t(1+i)(1+i)\,dt + \int_1^2 (t+(2-t)i)(1-i)\,dt + \int_{-2}^0 (-t)(-1)\,dt \\
&= (1+i)^2 \int_0^1 t\,dt + (1-i)\int_1^2 (t + 2i - ti)\,dt + \int_{-2}^0 t\,dt \\
&= (2i)[\frac{1}{2}t^2]_0^1 + (1-i)[\frac{(1-i)}{2}t^2 + 2it]_1^2 + [\frac{1}{2}t^2]_{-2}^0 \\
&= i + (1-i)((2(1-i)+4i) - (\frac{(1-i)}{2} + 2i)) - 2 \\
&= i + (1-i)(2 - 2i + 4i - (\frac{1}{2} - \frac{1}{2}i + 2i)) - 2 \\
&= i + \frac{3}{2} + \frac{1}{2}i - \frac{3}{2}i + \frac{1}{2} - 2 \\
&= 0
\end{aligned}$$

ii)

$$\begin{aligned}
\ell(\alpha) &= \int_{\alpha_1} |\alpha_1'(t)|\,dt + \int_{\alpha_2} |\alpha_2'(t)|\,dt + \int_{\alpha_3} |\alpha_3'(t)|\,dt \\
&= \int_0^1 |1+i|\,dt + \int_1^2 |1-i|\,dt + \int_{-2}^0 |-1|\,dt \\
&= \sqrt{2}\int_0^1 dt + \sqrt{2}\int_1^2 dt + \int_{-2}^0 dt \\
&= \sqrt{2} + \sqrt{2}(2-1) + (0-(-2)) \\
&= \sqrt{2} + \sqrt{2} + 2 \\
&= 2(\sqrt{2}+1)
\end{aligned}$$

iii) Let $f(z) = \text{Re}(z) = x$:

$$\begin{aligned}
\int_\alpha f(z)\,dz &= \int_0^1 \text{Re}(t(1+i))(1+i)\,dt + \int_1^2 \text{Re}(t+(2-t)i)(1-i)\,dt + \int_{-2}^0 \text{Re}(-t)(-1)\,dt \\
&= (1+i)\int_0^1 t\,dt + (1-i)\int_1^2 t\,dt - \int_{-2}^0 -t\,dt \\
&= (1+i)[\frac{1}{2}t^2]_0^1 + (1-i)[\frac{1}{2}t^2]_1^2 + [\frac{1}{2}t^2]_{-2}^0 \\
&= \frac{1}{2}(1+i) + \frac{3}{2}(1-i) - 2 \\
&= \frac{1}{2} + \frac{1}{2}i + \frac{3}{2}1 - \frac{3}{2}i - 2 \\
&= 0 - i \\
&= -i
\end{aligned}$$

## Problem 2

i) Let $f(z) = \frac{1}{z^2}$, and let $\alpha(t) = -t + (1-t)i$, $t \in [0,1]$. We have the bound:

$$\Big|\int_\alpha f(z)\,dz\Big| \leq M(f)\ell(\alpha), \tag{1}$$

where:

$$\begin{aligned}
\ell(\alpha) &= \int_0^1 |-1-i|\,dt \\
&= \int_0^1 \sqrt{1^2+1^2}\,dt \\
&= \sqrt{2}
\end{aligned}$$

We also have:

$$\begin{aligned}
M(f) &= \max\{|f(z)| : z \in \alpha\} \\
&= \max\{|\frac{1}{z^2}| : z \in \alpha\} \\
&= \min\{|z^2| : z \in \alpha\}
\end{aligned}$$

We can observe that the smallest value of $|z^2|$ (along $\alpha$) occurs on the midpoint on the straight line from $i$ to $-1$, which is the point $z = -\frac{1}{2} + \frac{1}{2}i$. At this point:

$$
\begin{aligned}
|f(z)| &= \frac{1}{|(-\frac{1}{2}+\frac{1}{2}i)^2|} \\
&= \frac{1}{|-\frac{1}{2}i|} \\
&= \frac{1}{\frac{1}{2}} \\
\Rightarrow M(f) &= 2
\end{aligned}
$$

Using this result and Inequality (1) above, we obtain:

$$
\Big|\int_\alpha f(z)\,dz\Big| \leq 2\sqrt{2} = M(f)\ell(\alpha)
$$

□

ii) Let $\alpha_R(t) = Re^{it}$, $t \in [0, \pi]$, where $\alpha_R'(t) = iRe^{it}$:

$$
\begin{aligned}
\ell(\alpha) &= \int_0^\pi |iRe^{it}|\,dt \\
&= \int_0^\pi |iR(\cos(t) + i\sin(t))|\,dt \\
&= \int_0^\pi |iR\cos(t) - R\sin(t)|\,dt \\
&= \int_0^\pi \sqrt{R^2\cos^2(t) + R^2\sin^2(t)}\,dt \\
&= \int_0^\pi \sqrt{R^2(\cos^2(t) + \sin^2(t))}\,dt \\
&= \int_0^\pi R\,dt \\
&= R\pi
\end{aligned}
$$

Let $f_1(z) = e^{ikz} = e^{ik(x+iy)} = e^{-ky}e^{ikx}$:

$$
\begin{aligned}
M_1(f) &= \max\{|f_1(z)| \,:\, z \in \alpha\} \\
&= \max\{|e^{-ky}e^{ikx}| \,:\, z \in \alpha\} \\
&= \max\{e^{-ky} \,:\, z \in \alpha\}
\end{aligned}
$$

As $y \geq 0$, $k > 0$, if $ky$ is minimum (i.e. $ky = 0$), then $e^{-ky} = 1$; if $ky$ is maximum (i.e. $ky \to \infty$), then $e^{-ky} \to 0$. Therefore, we have that:

$$
M_1(f) = \max\{e^{-ky} \,:\, z \in \alpha\} = 1 \tag{2}
$$

Let $f_2(z) = \frac{1}{z^2+1}$:

$$\begin{aligned} M_2(f) &= \max\{|f_2(z)| \,:\, z \in \alpha\} \\ &= \max\{|\frac{1}{z^2+1}| \,:\, z \in \alpha\} \\ &= \min\{|z^2+1| \,:\, z \in \alpha\} \end{aligned}$$

Using the Inverse Triangle Inequality ($|\gamma - \eta| \geq ||\gamma| - |\eta||$, where $\gamma = z^2$, and $\eta = -1$:

$$\begin{aligned} &|z^2+1| \geq ||z^2| - |-1|| \\ &|z^2+1| \geq R^2 - 1 \\ &\frac{1}{|z^2+1|} \leq \frac{1}{R^2-1} \end{aligned} \tag{3}$$

Combining Equations Equation 2, Equation 3, we have:

$$\begin{aligned} \Big| \int_{\alpha_R} \frac{e^{ikz}}{1+z^2}\, dz \Big| &\leq R\pi(\frac{1}{R^2-1}) \\ \Big| \int_{\alpha_R} \frac{e^{ikz}}{1+z^2}\, dz \Big| &\leq (\frac{R\pi}{R^2-1}) \\ \Rightarrow \lim_{R\to+\infty} \Big| \int_{\alpha_R} \frac{e^{ikz}}{1+z^2}\, dz \Big| &= 0 \end{aligned}$$

$\square$

iii) Let $\gamma_R(t) = R + it$, $t \in [0, h]$, $\gamma_R'(t) = i$:

$$\begin{aligned} \therefore \int_{\gamma_R} e^{-z^2}\, dz &= \int_0^h e^{-(R+it)^2}(i)\, dt \\ &= i\int_0^h e^{-(R^2+(2iR-1)t)}\, dt \\ &= i\int_0^h e^{-R^2}e^{(1-2iR)t}\, dt \\ &= ie^{-R^2}\int_0^h e^{(1-2iR)t}\, dt \\ &= ie^{-R^2}[\frac{1}{(1-2iR)}e^{(1-2iR)t}]_0^h \\ &= ie^{-R^2}(\frac{1}{(1-2hi)}e^{(1-2hi)t} - 1) \end{aligned}$$

$$\therefore \lim_{R\to+\infty} \int_{\gamma_R} e^{-z^2}\, dz = \lim_{R\to+\infty} ie^{-R^2}(\frac{1}{(1-2hi)}e^{(1-2hi)t} - 1) = 0$$

$\square$

## Problem 3

i) $f(z) = \text{Log}(z)$ has an antiderivative:

$$F(z) = z\text{Log}(z) - z$$

Let $\alpha(t) = 1 + (i-1)t$, $t \in [0,1]$ be a curve, $\alpha \in \Omega$. As $f$ has an antiderivative $F$ everywhere in $\mathbb{C} \setminus \{(\text{Re}(z) \in (-\infty, 0]) \cap (\text{Im}(z) = 0)\}$ (i.e. away from the branch cut), we can conclude that:

$$\begin{aligned}
\int_\alpha f(z)\,dz &= F(z_f) - F(z_i) \\
&= F(\alpha(1)) - F(\alpha(0)) \\
&= F(i) - F(1) \\
&= (i\text{Log}(i) - i) - (\text{Log}(1) - 1) \\
&= (i(i\frac{\pi}{2}) - i) - (-1) \\
&= 1 - \frac{\pi}{2} - i,
\end{aligned}$$

as the curve $\alpha(t)$ is away from the branch cut.

ii) Let $f(z) = \overline{z}$, Let $\alpha_+ = e^{it}$, $t \in [0,\pi]$, $\alpha_+' = ie^{it}$, and let $\alpha_- = e^{-it}$, $t \in [0,\pi]$, $\alpha_-' = -ie^{it}$.
For $\alpha_+$:

$$\begin{aligned}
\int_{\alpha_+} f(z)\,dz &= \int_0^\pi (ie^{it})\overline{e^{it}}\,dt \\
&= \int_0^\pi (ie^{it})e^{-it}\,dt \\
&= i\int_0^\pi\,dt \\
&= i\pi
\end{aligned}$$

For $\alpha_-$:

$$\begin{aligned}
\int_{\alpha_-} f(z)\,dz &= \int_0^\pi (-ie^{-it})\overline{e^{-it}}\,dt \\
&= \int_0^\pi (-ie^{-it})e^{it}\,dt \\
&= -i\int_0^\pi\,dt \\
&= -i\pi
\end{aligned}$$

## Problem 4

Since $F(t) = \alpha^{-1}e^{\alpha t}$ is an antiderivative of $f(t) = e^{\alpha t} = e^{at}e^{ibt}$:

$$\begin{aligned}
\int e^{at}e^{ibt}\,dt &= \alpha^{-1}e^{\alpha t}\\
\int e^{at}(\cos(bt) + i\sin(bt))\,dt &= \frac{a-bi}{a^2+b^2}(e^{at}(\cos(bt)+i\sin(bt)))\\
\int e^{at}\cos(bt)\,dt + i\int e^{at}\sin(bt)\,dt &= \frac{a-bi}{a^2+b^2}(e^{at}(\cos(bt)+i\sin(bt)))\\
\int e^{at}\cos(bt)\,dt + i\int e^{at}\sin(bt)\,dt &= \frac{e^{at}}{a^2+b^2}(a\cos(bt)+ai\sin(bt)-b\cos(bt)+b\sin(bt))\\
\int e^{at}\cos(bt)\,dt + i\int e^{at}\sin(bt)\,dt &= \frac{e^{at}}{a^2+b^2}((a\cos(bt)+b\sin(bt))+i(a\sin(bt)-b\cos(bt)))\\
\int e^{at}\cos(bt)\,dt + i\int e^{at}\sin(bt)\,dt &= \frac{e^{at}}{a^2+b^2}(a\cos(bt)+b\sin(bt))+i\frac{e^{at}}{a^2+b^2}(a\sin(bt)-b\cos(bt))
\end{aligned}$$

Taking Re(LHS) = Re(RHS), and Im(LHS) = Im(RHS), gives:

$$\begin{aligned}
\int e^{at}\cos(bt)\,dt &= \frac{e^{at}}{a^2+b^2}(a\cos(bt)+b\sin(bt))\\
\int e^{at}\sin(bt)\,dt &= \frac{e^{at}}{a^2+b^2}(a\sin(bt)-b\cos(bt))
\end{aligned}$$

$\square$

For $a > 0$:

$$\begin{aligned}
\int_0^\infty e^{-at}\cos(bt)\,dt &= \frac{e^{-at}}{(-a)^2+b^2}(-a\cos(bt)+b\sin(bt))\Big|_0^\infty\\
&= 0 - (\frac{1}{a^2+b^2}(-a))\\
&= \frac{a}{a^2+b^2}
\end{aligned}$$

$\square$

$$\begin{aligned}
\int_0^\infty e^{-at}\sin(bt)\,dt &= \frac{e^{-at}}{(-a)^2+b^2}(-a\sin(bt)-b\cos(bt))\Big|_0^\infty\\
&= 0 - (\frac{1}{a^2+b^2})(-b)\\
&= \frac{b}{a^2+b^2}
\end{aligned}$$

$\square$

## Problem 5

Let $\alpha_\epsilon = z_0 + \epsilon e^{it}$, $t \in [0, \pi]$. This gives:

$$\begin{aligned}
f(\alpha_\epsilon(t)) &= \frac{a}{(z_0 + \epsilon e^{it}) - z_0} + g(\epsilon e^{it} - z_0), \\
&= \frac{a}{\epsilon e^{it}} + g(\epsilon e^{it} - z_0), \\
\alpha'_\epsilon(t) &= i\epsilon e^{it} \\
\Rightarrow f(\alpha_\epsilon(t))\alpha'_\epsilon(t) &= (\frac{a}{\epsilon e^{it}} + g(\epsilon e^{it} - z_0))(i\epsilon e^{it}) \\
&= ai + i\epsilon e^{it} g(\epsilon e^{it} - z_0)
\end{aligned}$$

This gives:

$$\begin{aligned}
\int_{\alpha_\epsilon} f(z)\, dz &= \int_0^\pi ai + i\epsilon e^{it} g(\epsilon e^{it} - z_0)\, dt \\
&= ai \int_0^\pi dt + i\epsilon \int_0^\pi e^{it} g(\epsilon e^{it} - z_0)\, dt \\
&= ai\pi + i\epsilon \int_0^\pi e^{it} g(\epsilon e^{it} - z_0)\, dt
\end{aligned}$$

Taking the limit then yields:

$$\begin{aligned}
\lim_{\epsilon \to 0} \int_{\alpha_\epsilon} f(z)\, dz &= ai\pi + i(0) \int_0^\pi e^{it} g(-z_0)\, dt \\
&= i\pi a
\end{aligned}$$

$\square$