Assignment 10

MATH 305 - Applied Complex Analysis

Zhi Chuen Tan (65408361)

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THE UNIVERSITY OF BRITISH COLUMBIA
Mathematics
Faculty of Science

Problem 1

Let:

$$\sin(\varphi) = \frac{1}{2i}(z - \frac{1}{z}),\ z = e^{i\varphi},\ dz = ie^{i\varphi}d\varphi = iz\,d\varphi$$
$$\cos(\varphi) = \frac{1}{2}(z + \frac{1}{z}),\ z = e^{i\varphi},\ dz = ie^{i\varphi}d\varphi = iz\,d\varphi$$

The real integral can then be rewritten as:

$$\begin{aligned}
I &= \oint_{|z|=1} \frac{(\frac{1}{2i}(z-\frac{1}{z}))^2}{5+4(\frac{1}{2}(z+\frac{1}{z}))}\,\frac{dz}{iz} \\
&= \oint_{|z|=1} \frac{\frac{1}{4(i)^2}(z-\frac{1}{z})^2}{5+2(z+\frac{1}{z})}\,\frac{dz}{iz} \\
&= -\frac{1}{4i}\oint_{|z|=1} \frac{(z-\frac{1}{z})^2}{z(5+2(z+\frac{1}{z}))}\,dz \\
&= -\frac{1}{4i}\oint_{|z|=1} \frac{(\frac{1}{z})^2(z^2-1)^2}{z(5+2z+\frac{2}{z})}\,dz \\
&= -\frac{1}{4i}\oint_{|z|=1} \frac{(z^2-1)^2}{2z^2(z+\frac{1}{2})(z+2)}\,dz
\end{aligned}$$

Let $f(z) = \frac{(z^2-1)^2}{2z^2(z+\frac{1}{2})(z+2)}$. We can observe that f has simple poles $-\frac{1}{2}$ and -2, and a pole of order 2 at $z_0 = 0$ As the pole $z_0 = -2$ lies outside the unit circle, we only have to consider the Residue of f at the simple pole at $-\frac{1}{2}$ and double pole at $z_0 = 0$:

$$\begin{aligned}
\operatorname{Res}\left(f\,;\,-\frac{1}{2}\right) &= \lim_{z\to-\frac{1}{2}}(z+\frac{1}{2})f(z) \\
&= \lim_{z\to-\frac{1}{2}}\frac{(z^2-1)^2}{2z^2(z+2)} \\
&= \frac{3}{4}
\end{aligned}$$

$$\begin{aligned}
\operatorname{Res}(f\,;\,0) &= \lim_{z\to0}\frac{d}{dz}z^2f(z) \\
&= \lim_{z\to0}\frac{d}{dz}\frac{(z^2-1)^2}{2(z+\frac{1}{2})(z+2)} \\
&= \lim_{z\to0}\frac{4z^5+15z^4+8z^3-10z^2-12z-5}{(z+2)^2(2z+1)^2}
\end{aligned}$$

$$\text{Res}(f\,;\,0) = -\frac{5}{4}$$

The Residue theorem then gives us:

$$\begin{aligned} I &= -\frac{1}{4i}(2\pi i)(\frac{3}{4} - \frac{5}{4}) \\ &= -\frac{\pi}{2}(-\frac{1}{2}) \\ &= \frac{\pi}{4} \end{aligned}$$

Problem 2

The function $f(z) = \frac{z^2(z-1)}{\sin^2(\pi z)}$ has singularities $\forall z = 0, \pm 1, \pm 2, \pm 3, \ldots$. To compute the integral using the Residue theorem, we only need to consider the singularities at $z_0 = 0, \pm 1$, as the remaining singularities lie outside the contour $|z| = \frac{3}{2}$. For $z_0 = 0$, we can use the small angle approximation that $\sin(\pi z) \approx \pi z$, $\forall\, |z| \ll 1$. Therefore, as $z \to 0$ (i.e. z becomes infinitesimally small):

$$\begin{aligned} \lim_{z\to 0} f(z) &\approx \frac{z^2(z-1)}{(\pi z)^2} \\ &= \frac{z^2(z-1)}{\pi^2 z^2} \\ &= \frac{(z-1)}{\pi^2} \\ &= -\frac{1}{\pi^2} \end{aligned}$$

As the limit converges, $z_0 = 0$ is a removable singularity, and we therefore have that:

$$\text{Res}(f\,;\,0) = 0$$

Using the Taylor series of $\sin(z)$ about $z = \pi z_0$, we have that:

$$\sin^2(\pi z_0) = \sin^2(\pi z_0) + \pi \sin(2\pi z_0)(z - z_0) + \pi^2 \cos(2\pi z_0)(z - z_0)^2 - 2\pi^3 \sin(2\pi z_0)(z - z_0)^3 - \ldots$$

At $z_0 = 1$, which is a simple pole (simple zero divided by zero of order two), we then have:

$$\begin{aligned} \text{Res}(f\,;\,1) &= \lim_{z\to 1}(z-1)\frac{z^2(z-1)}{\pi^2(z-1)^2 - \ldots} \\ &= \lim_{z\to 1}\cancel{(z-1)}\frac{z^2\cancel{(z-1)}}{\cancel{(z-1)^2}(\pi^2 - \frac{8}{3!}\pi^4(z-1) + \ldots)} \end{aligned}$$

$$\begin{aligned}
\operatorname{Res}(f\,;\,1) &= \lim_{z\to 1} \frac{z^2}{\pi^2 + \frac{8}{3!}\pi^4(z-1) + \mathcal{O}(z-1)} \\
&= \frac{1}{\pi^2}
\end{aligned}$$

At $z_0 = -1$, which is a pole of order two. Let the residue at this point be μ. By the Residue theorem, we then have that:

$$\begin{aligned}
\oint_{|z|=\frac{3}{2}} f(z)\,dz &= 2\pi i(0 + \frac{1}{\pi^2} + \mu) \\
&= \frac{2i}{\pi} + 2\pi i\mu
\end{aligned}$$

Problem 3

i) The point $z_0 = 0$ is an essential singularity. The Laurent series of $f(z)$ is:

$$\begin{aligned}
f(z) &= z^3(\frac{1}{2z} - \frac{1}{3!(2z)^3} + \frac{1}{5!(2z)^5} - \ldots) \\
&= \frac{z^2}{2} - \frac{z^3}{3!(2z)^3} + \frac{z^3}{5!(2z)^5} - \ldots \\
&= \frac{z^2}{2} - \frac{z^3}{2^3 3! z^3} + \frac{z^3}{2^5 5! z^5} - \ldots \\
&= \frac{z^2}{2} - \frac{1}{2^3 3!} + \frac{1}{2^5 5! z^2} - \ldots \\
&= \frac{1}{2}z^2 - \frac{1}{2^3 3!} + \frac{1}{2^5 5!}z^{-2} - \ldots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}(2n+1)!} z^{-2(n-1)}
\end{aligned}$$

ii) a) For the annulus $C(2;0,1)$, we have $0 < |z-2| < 1$, which is away from the essential singularity, therefore f does not have a singular part in $C(2;0,1)$.

b) For the annulus $C(0;0,\infty)$, the singular part will be the series:

$$\begin{aligned}
f(z) &= \sum_{n=2}^{\infty} \frac{(-1)^n}{2^{2n+1}(2n+1)!} z^{-2(n-1)} \\
&= \frac{1}{2^5 5!}z^{-2} - \frac{1}{2^7 7!}z^{-4} + \frac{1}{2^9 9!}z^{-6} - \ldots
\end{aligned}$$

iii) From the Laurent series above, we observe that all odd-powered terms vanish in the expression, i.e. $a_{-1} = 0$. By the Residue theorem, we obtain the result:

$$\oint_{|z|=1} z^3 \sin\left(\frac{1}{2z}\right) dz = 0$$

Problem 4

i) We can express $f(z)$ as:

$$\begin{aligned} f(z) &= \frac{1}{z^2+1} \\ &= \frac{1}{(z-i)(z+i)} \\ &= \frac{1}{z-i} \cdot \frac{1}{z+i} \end{aligned}$$

ii) a) We have $0 < |z-i| < 2$. Since $1/(z+i)$ is analytic at $z_0 = i$:

$$\begin{aligned} \frac{1}{z+i} &= \frac{1}{(z-i)+2i} \\ &= \frac{1}{2i} \cdot \frac{1}{(1+\frac{z-i}{2i})} \end{aligned}$$

We can verify that $|z-i|/|2i| < 1$:

$$\frac{|z-i|}{|2i|} < \frac{2}{2} = 1$$

We can therefore use the geometric series:

$$\begin{aligned} \frac{1}{z+i} &= \frac{1}{2i} \cdot \frac{1}{(1-(-\frac{z-i}{2i}))} \\ &= \frac{1}{2i} \sum_{n=0}^{\infty} (-\frac{z-i}{2i})^n \\ \therefore f(z) &= \frac{1}{z-i} \cdot \frac{1}{z+i} = \frac{1}{z-i} \cdot \frac{1}{2i} \sum_{n=0}^{\infty} (-\frac{z-i}{2i})^n \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^{n-1}}{(2i)^{n+1}} \end{aligned}$$

b) We have $|z-i|>2$. Since $1/(z-i)$ is analytic at $z_0=-i$:

$$\frac{1}{z-i}=\frac{1+\frac{2i}{z-i}}{z-i}\cdot\frac{1}{1+\frac{2i}{z-i}}$$

We can verify that $|2i|/|z-i|<1$:

$$\begin{aligned}\frac{|z-i|}{|2i|}&>\frac{2}{2}=1\\ \Rightarrow\frac{|2i|}{|z-i|}&<1\end{aligned}$$

We can therefore use the geometric series:

$$\begin{aligned}\frac{1}{z-i}&=\frac{1+\frac{2i}{z-i}}{z-i}\cdot\frac{1}{1-(-\frac{2i}{z-i})}\\ &=\frac{1+\frac{2i}{z-i}}{z-i}\sum_{n=0}^{\infty}(-\frac{2i}{z-i})^n\\ &=(\frac{1}{z-i}+\frac{2i}{(z-i)^2})\sum_{n=0}^{\infty}(-\frac{2i}{z-i})^n\\ &=\sum_{n=0}^{\infty}(\frac{(-1)^n(2i)^n}{(z-i)^{n+1}}+\frac{(-1)^n(2i)^{n+1}}{(z-i)^{n+2}})\\ \therefore f(z)&=\frac{1}{z+i}\sum_{n=0}^{\infty}(\frac{(-1)^n(2i)^n}{(z-i)^{n+1}}+\frac{(-1)^n(2i)^{n+1}}{(z-i)^{n+2}})\end{aligned}$$