

Assignment 9

MATH 305 - Applied Complex Analysis

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Problem 1

i) f_1 has an essential singularity at $z_0 = 0$, with the Laurent series:

$$\begin{aligned} f_1(z) &= z^2 \left(1 + \frac{1}{z} + \frac{1}{2} \left(\frac{1}{z^2} \right) + \frac{1}{3!} \left(\frac{1}{z^3} \right) + \dots \right) \\ &= z^2 + z + \frac{1}{2} + \frac{1}{6} \left(\frac{1}{z} \right) + \dots \\ &= \dots + \frac{1}{6} z^{-1} + \dots \\ \Rightarrow a^{-1} &= \frac{1}{6} = \text{Res}(f_1; 0) \end{aligned}$$

ii) f_2 has a simple pole at $z_0 = 0$ and a pole of order 2 at $z_1 = \frac{\pi}{2}$. For the simple pole:

$$\begin{aligned} \text{Res}(f_2; 0) &= \lim_{z \rightarrow z_0} (z) \left(\frac{\cos(2z)}{z(z - \frac{\pi}{2})^2} \right) \\ &= \lim_{z \rightarrow 0} \left(\frac{\cos(2z)}{(z - \frac{\pi}{2})^2} \right) \\ &= \frac{1}{\frac{\pi^2}{4}} \\ &= \frac{4}{\pi^2} \end{aligned}$$

For the pole of order 2:

$$\begin{aligned} \text{Res}\left(f_2; \frac{\pi}{2}\right) &= \lim_{z \rightarrow \frac{\pi}{2}} \frac{d}{dz} \left((z - \frac{\pi}{2})^2 \left(\frac{\cos(2z)}{z(z - \frac{\pi}{2})^2} \right) \right) \\ &= \lim_{z \rightarrow \frac{\pi}{2}} \frac{d}{dz} \left(\frac{\cos(2z)}{z} \right) \\ &= \lim_{z \rightarrow \frac{\pi}{2}} - \left(\frac{2 \sin(2z)}{z} + \frac{\cos(2z)}{z^2} \right) \\ &= - \left(\frac{2 \sin(\pi)}{\frac{\pi}{2}} + \frac{\cos(\pi)}{\frac{\pi^2}{4}} \right) \\ &= - \left(-\frac{4}{\pi^2} \right) \\ &= \frac{4}{\pi^2} \end{aligned}$$

iii) a) The essential singularity $z_0 = 0$ lies in the unit circle. Laurent's Theorem then gives:

$$\begin{aligned}\oint_{|z|=1} f_1(z) &= 2\pi i a_{-1} = 2\pi i \operatorname{Res}(f_1; 0) \\ &= 2\pi i \left(\frac{1}{6}\right) \\ &= \frac{\pi}{3} i\end{aligned}$$

b) Both residues of f_2 are equal and lie in the unit circle. The Residue Theorem then gives:

$$\begin{aligned}\oint_{|z|=1} f_2(z) &= 2\pi i \left(\frac{4}{\pi^2} + \frac{4}{\pi^2}\right) \\ &= \frac{16}{\pi} i\end{aligned}$$

Problem 2

i) We have the identity $\tan(z) = \sin(z)/\cos(z)$, and we can therefore observe that $\tan(z)$ has 2 simple poles at $z_0 = \frac{\pi}{2}$ and $z_1 = \frac{3\pi}{2}$ that lie in the contour $|z| = 2\pi$. Then:

$$\begin{aligned}\operatorname{Res}(\tan(z); z_0) &= \frac{\sin(z_0)}{(\cos(z_0))'} = \frac{\sin\left(\frac{\pi}{2}\right)}{-\sin\left(\frac{3\pi}{2}\right)} \\ &= -1 \\ \operatorname{Res}(\tan(z); z_1) &= \frac{\sin(z_1)}{(\cos(z_1))'} = \frac{\sin\left(\frac{3\pi}{2}\right)}{-\sin\left(\frac{\pi}{2}\right)} \\ &= -1\end{aligned}$$

The Residue Theorem then gives:

$$\begin{aligned}\oint_{|z|=2\pi} \tan(z) &= 2\pi i(-1 + -1) \\ &= -4\pi i\end{aligned}$$

ii) We use the function $f(z) = \frac{e^{2z}}{\cosh(\pi z)}$. We have that:

$$\begin{aligned}f(z) &= \frac{e^{2z}}{\cosh(\pi z)} = \frac{e^{2z}}{\cos(i\pi z)} \\ &= \frac{e^{2z}}{\frac{1}{2}(e^{i(i\pi z)} + e^{-i(i\pi z)})}\end{aligned}$$

$$\begin{aligned} f(z) &= \frac{e^{2z}}{\frac{1}{2}(e^{-\pi z} + e^{\pi z})} \\ &= \frac{2e^{2z}}{e^{-\pi z}(1 + e^{2\pi z})} \\ &= \frac{2e^{(2+\pi)z}}{1 + e^{2\pi z}} \end{aligned}$$

We note that:

$$\begin{aligned} 1 + e^{2\pi z} &= 0 \text{ if and only if} \\ 2\pi z &= n\pi i, \forall n \in \mathbb{Z} \setminus \{0\} \\ \Rightarrow z &= \frac{n}{2}i, \end{aligned}$$

meaning $f(z)$ has singularities $\frac{n}{2}i$ for all positive non-zero integer values of n . We now consider the positively oriented contour as shown in Figure 1.

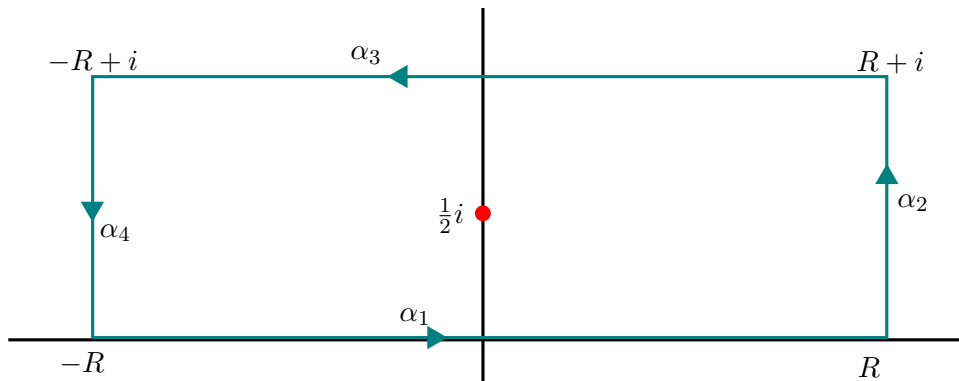


Figure 1

We assume (for now) that the subsequent poles of $f(z)$ (i.e. $\pi i, \frac{3\pi}{2}i, \dots$) lie outside this contour. By the Residue Theorem, we have (for this rectangular contour):

$$\begin{aligned} \oint_{\alpha} f(z) dz &= \int_{\alpha_1} f(z) dz + \int_{\alpha_2} f(z) dz + \int_{\alpha_3} f(z) dz + \int_{\alpha_4} f(z) dz \\ &= 2\pi i \operatorname{Res}\left(f; \frac{1}{2}i\right) \\ &= 2\pi i \operatorname{Res}\left(\frac{2e^{(2+\pi)z}}{1 + e^{2\pi z}}; \frac{1}{2}i\right) \\ &= 2\pi i \left(\frac{2e^{(2+\pi)z}}{2\pi e^{2\pi z}}\right) \Big|_{z=\frac{1}{2}i} \end{aligned}$$

$$\begin{aligned}\oint_{\alpha} f(z) dz &= \frac{2e^{(2+\pi)(\frac{1}{2}i)}}{e^{2\pi(\frac{1}{2}i)}} i \\ &= \frac{2e^{i+i\frac{\pi}{2}}}{e^{i\pi}} i \\ &= \frac{2i^2 e^i}{e^{i\pi}} \\ &= 2e^i\end{aligned}$$

Let $\alpha_2(t) = R + ti$ for $t \in [0, 1]$. This gives:

$$\begin{aligned}\ell(\alpha_2) &= \int_0^1 |i| dt \\ &= 1\end{aligned}$$

Therefore:

$$\left| \int_{\alpha_2} f(z) dz \right| \leq \ell(\alpha_2) \max\{|f(\alpha_2(t))| : t \in [0, 1]\} \leq \frac{2e^{(2+\pi)R+(2+\pi)ti}}{1 + e^{2\pi R} e^{2\pi ti}}$$

For large R , we can estimate that:

$$\frac{2e^{(2+\pi)R+(2+\pi)ti}}{1 + e^{2\pi R} e^{2\pi ti}} \approx \frac{e^{2+\pi R}}{e^{2\pi R}}$$

Since $2\pi > 2 + \pi$, we then have that:

$$\left| \int_{\alpha_2} f(z) dz \right| \leq \frac{e^{2+\pi R}}{e^{2\pi R}} \rightarrow 0, \text{ as } R \rightarrow \infty$$

We can conclude similarly that:

$$\left| \int_{\alpha_4} f(z) dz \right| \rightarrow 0 \text{ as } R \rightarrow \infty$$

This leaves us with:

$$\int_{\alpha_1} f(z) dz + \int_{\alpha_3} f(z) dz = -2ie^i$$

Let $\alpha_1(x) = x$ for $x \in [-R, R]$; $\alpha_3(x) = x + i$ for $x \in [R, -R]$:

$$\begin{aligned}&\int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)} (1) dx + \int_{\infty}^{-\infty} \frac{e^{2(x+i)}}{\frac{1}{2}(e^{\pi x + \pi i} + e^{-\pi x - \pi i})} (1) dx = 2e^i \\ &\int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)} dx - \int_{-\infty}^{\infty} \frac{e^{2x} e^{2i}}{\frac{1}{2}(e^{\pi x} e^{\pi i} + e^{-\pi x} e^{-\pi i})} dx = 2e^i \\ &\int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)} dx - e^{2i} \int_{-\infty}^{\infty} \frac{e^{2x}}{\frac{1}{2}e^{\pi i}(e^{\pi x} + e^{-\pi x})} dx = 2e^i\end{aligned}$$

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)} dx + e^{2i} \int_{-\infty}^{\infty} \frac{e^{2x}}{\frac{1}{2}(e^{\pi x} + e^{-\pi x})} dx &= 2e^i \\ \int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)} dx + e^{2i} \int_{-\infty}^{\infty} \frac{e^{-2x}}{\cosh(\pi x)} dx &= 2e^i \\ (1 + e^{2i}) \int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)} dx &= 2e^i \\ \int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)} dx &= \frac{2e^i}{1 + e^{2i}} \end{aligned}$$

Simplifying this expression further gives:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)} dx &= \frac{1}{\frac{1}{2}e^{-i}(1 + e^{2i})} \\ &= \frac{1}{\frac{1}{2}(e^{-i}e^i)} \\ &= \frac{1}{\frac{1}{2}(e^{-i(1)} + e^{i(1)})} \\ &= \frac{1}{\cos(1)} \\ &= \sec(1) \end{aligned}$$

Problem 3

Let $P(z) = c(z - z_1)(z - z_2) \dots (z - z_d)$. By the product rule ([according to Wikipedia](#)), we have:

$$P'(z) = c\left(\prod_{j=1}^d (z - z_j)\right)\left(\sum_{j=1}^d \frac{1}{z - z_j}\right) \quad (1)$$

Using similar notation, we can express $P(z)$ as:

$$P(z) = c \prod_{j=1}^d (z - z_j) \quad (2)$$

Taking (1)/(2) then gives:

$$\begin{aligned}\frac{P'(z)}{P(z)} &= \frac{c(\prod_{j=1}^d (z - z_j))(\sum_{j=1}^d \frac{1}{z - z_j})}{c \prod_{j=1}^d (z - z_j)} \\ &= \sum_{j=1}^d \frac{1}{z - z_j} \\ &= \frac{1}{z - z_1} + \frac{1}{z - z_2} + \frac{1}{z - z_3} + \dots + \frac{1}{z - z_d}\end{aligned}$$

The residue of $P'(z)/P(z)$ at z_j , for $j \in [1, d]$, is defined as the coefficient of the term(s) w.r.t z^{-1} , and we can observe that all terms in the above expression are z^{-1} terms. Therefore, we can conclude that the function $P'(z)/P(z)$ has d residues, and that:

$$\begin{cases} \operatorname{Res}\left(\frac{P'(z)}{P(z)}; z_j\right) = 1, & \text{if } z_j \in \operatorname{int}(\alpha), \\ \operatorname{Res}\left(\frac{P'(z)}{P(z)}; z_j\right) = 0, & \text{if } z_j \notin \operatorname{int}(\alpha), \end{cases} \quad (3)$$

By the residue theorem:

$$\oint_{\alpha} \frac{P'(z)}{P(z)} dz = 2\pi i \sum_{z_j \in \operatorname{int}(\alpha)} \operatorname{Res}\left(\frac{P'(z)}{P(z)}; z_j\right)$$

$$\begin{aligned}\frac{1}{2\pi i} \oint_{\alpha} \frac{P'(z)}{P(z)} dz &= \sum_{z_j \in \operatorname{int}(\alpha)} \operatorname{Res}\left(\frac{P'(z)}{P(z)}; z_j\right) \\ &= \sum_{z_j \in \operatorname{int}(\alpha)} 1, \text{ from (3)} \\ &= N,\end{aligned}$$

as it is given that N counts the number of zeros of $P \in \operatorname{int}(\alpha)$. □

Problem 4

Let $f(z) = \frac{1}{z^2}$. From the lecture notes, we then have:

$$\begin{aligned}\operatorname{Res}\left(\frac{\pi \cot(\pi z)}{z^2}; n\right) &= \frac{1}{n^2}, \forall n \in \mathbb{Z} \setminus \{0\} \\ \therefore \frac{1}{2\pi i} \oint_{\alpha} \frac{\pi \cot(\pi z)}{z^2} &= \sum_{j=-\infty}^{-1} \frac{1}{j^2} + \sum_{j=1}^{\infty} \frac{1}{j^2} + \operatorname{Res}\left(\frac{\pi \cot(\pi z)}{z^2}; 0\right)\end{aligned}$$

We can observe that:

$$\begin{aligned} \frac{1}{(-n)^2} &= \frac{1}{n^2}, \forall n \in \mathbb{Z} \setminus \{0\} \\ \therefore \frac{1}{2\pi i} \oint_{\alpha} \frac{\pi \cot(\pi z)}{z^2} &= 2 \sum_{j=1}^{\infty} \frac{1}{j^2} + \text{Res} \left(\frac{\pi \cot(\pi z)}{z^2}; 0 \right) \end{aligned} \quad (4)$$

For the residue at the pole of order 3 at $z_0 = 0$:

$$\begin{aligned} \text{Res} \left(\frac{\pi \cot(\pi z)}{z^2}; 0 \right) &= \frac{1}{(3-1)!} \lim_{z \rightarrow 0} \frac{d^{3-1}}{dz^{3-1}} z^3 \frac{\pi \cot(\pi z)}{z^2} \\ &= \frac{1}{2} \lim_{z \rightarrow 0} \frac{d^2}{dz^2} \pi z \cot(\pi z) \\ &= \frac{1}{2} \lim_{z \rightarrow 0} \frac{d^2}{dz^2} \frac{\pi z \cos(\pi z)}{\sin(\pi z)} \\ &= \frac{1}{2} \lim_{z \rightarrow 0} \frac{d^2}{dz^2} \frac{\pi z \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\pi z)^{2n}}{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (\pi z)^{2n+1}} \\ &= \frac{1}{2} \lim_{z \rightarrow 0} \frac{d^2}{dz^2} \frac{\cancel{\pi z} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\pi z)^{2n}}{\cancel{\pi z} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (\pi z)^{2n}} \\ &= \frac{1}{2} \lim_{z \rightarrow 0} \frac{d^2}{dz^2} \frac{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\pi z)^{2n}}{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (\pi z)^{2n}} \\ &= \frac{1}{2} \lim_{z \rightarrow 0} \frac{d^2}{dz^2} \frac{1 - \frac{1}{2}\pi^2 z^2 + \frac{1}{4!}\pi^4 z^4 - \frac{1}{6!}\pi^6 z^6 + \dots}{1 - \frac{1}{3!}\pi^2 z^2 + \frac{1}{5!}\pi^4 z^4 - \frac{1}{7!}\pi^6 z^6 + \dots} \\ &= \frac{1}{2} \lim_{z \rightarrow 0} \frac{d^2}{dz^2} \left(1 - \frac{1}{3}\pi^2 z^2 + \dots \right), \text{ by long division} \\ &= \frac{1}{2} \lim_{z \rightarrow 0} \frac{d}{dz} \left(-\frac{2}{3}(\pi^2 z) + \dots \right) \\ &= \frac{1}{2} \lim_{z \rightarrow 0} \left(-\frac{2}{3}\pi^2 + \dots \right) \\ &= \frac{1}{2} \left(-\frac{2}{3}\pi^2 \right) \\ &= -\frac{\pi^2}{3} \end{aligned}$$

Combining this result with (4) gives:

$$\begin{aligned}\frac{1}{2\pi i} \oint_{\alpha} \frac{\pi \cot(\pi z)}{z^2} &= 2 \sum_{j=1}^{\infty} \frac{1}{j^2} - \frac{\pi^2}{3} \\ \oint_{\alpha} \frac{\pi \cot(\pi z)}{z^2} &= 4\pi i \sum_{j=1}^{\infty} \frac{1}{j^2} - \frac{2\pi^3}{3} i\end{aligned}$$

We use a square contour α with side length $2N + 1$. This gives us $\ell(\alpha) = 4(2N + 1) = 8N + 4$. As $|\cot(\pi z)|$ is bounded away from the real integers:

$$\therefore \left| \oint_{\alpha} \frac{\pi \cot(\pi z)}{z^2} \right| \leq \frac{c_N(8N + 4)}{N^2} \underbrace{\approx \frac{1}{N}}_{\substack{\text{for} \\ \text{large} \\ N}} \rightarrow 0 \text{ as } N \rightarrow \infty$$

where c_N is some constant obtained from $|\cot(\pi z)|$. This gives us:

$$\begin{aligned}4\pi i \sum_{j=1}^{\infty} \frac{1}{j^2} - \frac{2\pi^3}{3} i &= 0 \\ 4\pi i \sum_{j=1}^{\infty} \frac{1}{j^2} &= \frac{2\pi^3}{3} i \\ 2 \sum_{j=1}^{\infty} \frac{1}{j^2} &= \frac{\pi^2}{3} \\ \sum_{j=1}^{\infty} \frac{1}{j^2} &= \frac{\pi^2}{6}\end{aligned}$$

□