Assignment 8

MATH 305 - Applied Complex Analysis

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Problem 1

The Taylor series is given by:

$$f(w) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(w - z_0)^n,$$

where $w \in \mathbb{C}$, $z_0 \in \mathbb{C}$

i)

$$\begin{aligned}
\frac{1+z}{1-z} &= \frac{1+i+(z-i)}{1-i-(z-i)} \\
&= \frac{1+i+(z-i)}{(1-i)(1-\frac{z-i}{1-i})} \\
&= \frac{\frac{1+i+(z-i)}{(1-i)}}{1-\frac{z-i}{1-i}}
\end{aligned}$$

Using the geometric series then gives:

$$\begin{aligned}
\frac{1+z}{1-z} &= \frac{1+i+(z-i)}{(1-i)} \sum_{n=0}^{\infty} (\frac{z-i}{1-i})^n \\
&= \frac{1+i}{1-i}\frac{z-i}{1-i} \sum_{n=0}^{\infty} (\frac{z-i}{1-i})^n \\
&= \frac{1+i}{1-i} \sum_{n=0}^{\infty} (\frac{z-i}{1-i})^{n+1}
\end{aligned}$$

The function $(\frac{1+z}{1-z})^{-1} = \frac{1-z}{1+z}$ has a simple zero at $z = 1$. Therefore, $\frac{1+z}{1-z}$ has a simple pole at $z = 1$, which gives a radius of convergence:

$$\begin{aligned}
|i - 1| &= \sqrt{1^2 + 1^2} \\
&= \sqrt{2}
\end{aligned}$$

ii) We have (from lectures) that the Taylor series for trigonometric functions for the complex trigonometric functions are equivalent to those from the reals. Therefore:

$$z^4 \cos(3z) = z^4 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(3z)^{2n}$$

As the function $1/z^4 \cos(3z)$ has a zero only at infinity, the function is entire and the radius of convergence is infinite.

Problem 2

Let:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = \sum_{n=0}^{\infty} a_n(z-z_0)^n,\ a_n = \frac{f^{(n)}(z_0)}{n!}$$

Let also $z_0 = 0$. We then have:

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} a_n(z-z_0)^n \\
&= \sum_{n=0}^{\infty} a_n z^n \\
&= a_0 + a_1 z + \sum_{n=2}^{\infty} a_n z^n \\
zf'(z) &= z\sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1} \\
&= z\sum_{n=1}^{\infty} n a_n z^{n-1} \\
&= \sum_{n=1}^{\infty} n a_n z^n \\
&= a_1 + \sum_{n=2}^{\infty} n a_n z^n \\
f''(z) &= \sum_{n=2}^{\infty} n(n-1)a_n(z-z_0)^{n-2} \\
&= \sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} z^n \\
&= 2a_0 + 6a_1 + \sum_{n=2}^{\infty} (n+2)(n+1)a_{n+2} z^n
\end{aligned}$$

Where:

$$\begin{cases} a_0 = \frac{f(0)}{0!} = 1 \\ a_1 = \frac{f'(0)}{1!} = 0 \end{cases}$$

Plugging these back into the differential equation gives:

$$2 + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}z^n - \sum_{n=2}^{\infty} na_n z^n - 1 - \sum_{n=2}^{\infty} a_n z^n = 0$$
$$1 + \sum_{n=2}^{\infty}((n+2)(n+1)a_{n+2} - (n+1)a_n)z^n = 0$$

By comparing coefficients, we obtain the recursion relation:

$$\begin{aligned}(n+2)(n+1)a_{n+2} - (n+1)a_n &= 0 \\ (n+2)\cancel{(n+1)}a_{n+2} &= \cancel{(n+1)}a_n \\ a_{n+2} &= \frac{a_n}{n+2}\end{aligned}$$

Given that $a_0 = 1$, $a_1 = 0$, and the above result, we have:

$$\begin{aligned}a_2 &= \frac{a_0}{2} = \frac{1}{2} \\ a_3 &= \frac{a_1}{3} = 0 \\ a_4 &= \frac{a_2}{4} = \frac{1}{8}\end{aligned}$$

We can observe that, as all odd powered terms $a_{2k+1} = a_{2k-1}/(2k-1)$, $\forall k \in \mathbb{Z}$, $k \geq 1$ (i.e. each odd numbered term is a factor of the previous odd powered term), as the first odd powered term $a_1 = 0$, all subsequent odd powered terms will be 0. Meanwhile, the even powered terms have the recursion relation:

$$\begin{aligned}a_{2k+2} &= \frac{a_{2k}}{2k+2} \\ &= \frac{a_{2k-2}}{(2k+2)(2k-2)} \\ &= \frac{a_{2k-4}}{(2k+2)(2k-2)(2k-4)} \\ &= \dots\end{aligned}$$

Let $j = k + 1$. Then:

$$\begin{aligned} a_{2k+2} = a_{2j} &= \frac{a_{2j-2}}{2j-2} \\ &= \frac{a_{2j-4}}{(2j-2)(2j-4)} \\ &= \frac{a_{2j-6}}{(2j-2)(2j-4)(2j-6)} \\ &= \dots \\ &= \frac{a_{((2j)!!)}}{(2j)!!} \end{aligned}$$

As the first even powered term has a numerator of 1, all subsequent even powered terms will have a numerator of 1 as well. Therefore we can rewrite the Maclaurin series of $f(z)$ as:

$$\begin{aligned} f(z) &= 1 + 0 + \underbrace{\sum_{n=2}^{\infty} \frac{1}{(2n)!!} z^{2n}}_{\substack{\text{we can use} \\ 2n \text{ as only} \\ \text{the even} \\ \text{power terms} \\ \neq 0}} \\ &= 1 + \underbrace{\sum_{n=1}^{\infty} \frac{1}{(2n)!!} z^{2n}}_{\substack{\text{we can start} \\ \text{the series} \\ \text{from } n=1 \\ \text{as } a_n = 0}} \end{aligned}$$

$\square$

Problem 3

We can rewrite $f(z)$ as:

$$\begin{aligned} f(z) &= \frac{1}{1 - (-\text{Log}(1-z))} \\ &= \sum_{n=0}^{\infty} (-1)^n (\text{Log}(1-z))^n, \end{aligned}$$

using the geometric series.