Assignment 7

MATH 305 - Applied Complex Analysis

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Problem 1

For $g(z)$:

$$|g(z)| = \frac{|f(z)|}{3|z|^2}$$

We consider the boundary conditions of $f(z)$. $\forall\, z$ such that $|z| = 2$, we have:

$$|g(z)| \leq \frac{12}{3(2)^2} = \frac{12}{12} = 1,$$

and $\forall\, z$ such that $|z| = 1$, we have:

$$|g(z)| \leq \frac{3}{3(1)^2} = \frac{3}{3} = 1,$$

which gives the upper bound for $g(z)$ on the boundary of the open annulus as:

$$|g(z)| \leq 1$$

By the Maximum Modulus Principle, as the domain of $\{z \in \mathbb{C} : 1 < |z| < 2\}$ is bounded and $g(z)$ extends to the boundary of the annuus, then:

$$|g(z)| \leq 1$$

in the entire open annulus. This gives:

$$\begin{aligned} |g(z)| = \frac{|f(z)|}{3|z|^2} &\leq 1 \\ |f(z)| &\leq 3|z|^2 \end{aligned}$$

$\square$

Problem 2

The claim that all eigenvalues of A have modulus larger than 2 is equivalent to claiming that $c_A(t) = -t^3 + t^2 + 4t - 24$ has no zeros in the disc $B_2(0)$. By Rouché's Theorem, we then have that:

$$|f(z) - 2| < 2,\ \forall z \in B_2(0)$$

To find the eigenvalues of A, we let $c_A(t) = 0$:

$$\begin{aligned} -t^3 + t^2 + 4t - 24 &= 0 \\ \frac{1}{12}t^3 - \frac{1}{12}t^2 - \frac{1}{3}t + 2 &= 0 \end{aligned}$$

We then let $f(z) = \frac{1}{12}t^3 - \frac{1}{12}t^2 - \frac{1}{3}t + 2$. By the Triangle Inequality:

$$|f(z) - 2| = |\frac{1}{12}t^3 - \frac{1}{12}t^2 - \frac{1}{3}t| \leq \frac{1}{12}|t|^3 - \frac{1}{12}|t|^2 - \frac{1}{3}|t|$$

On the disc $B_2(0)$, $|z| = 2$. This gives:

$$|f(z) - 2| \leq \frac{1}{12}(2^3) - \frac{1}{12}(2^2) - \frac{1}{3}(2) = -\frac{1}{3} < 2,$$

which satisfies Rouché's Theorem, i.e. c_A has no zeros in the disc $B_2(0)$, i.e. all eigenvalues of A have modulus larger than 2. □

Problem 3

i) Let $w = a + bi$, a, $b \in \mathbb{R}$:

$$\begin{aligned}
\left|\frac{z-w}{1-\overline{w}z}\right| &= \frac{|1-w|}{|1-\overline{w}(1)|} \\
&= \frac{|1-w|}{|1-\overline{w}|} \\
&= \frac{|1-(a+bi)|}{|1-(a-bi)|} \\
&= \frac{|(1-a)-bi|}{|(1-a)+bi|} \\
&= \frac{\cancel{\sqrt{(1-a)^2+b^2}}}{\cancel{\sqrt{(1-a)^2+b^2}}} \\
&= 1
\end{aligned}$$

□

ii) The domain of the function $\left|\frac{z-w}{1-\overline{w}z}\right|$ is bounded by $|w| < 1$ and $|z| < 1$. At the boundary $|z| = 1$, we have the result in (i). Then, by the Maximum Modulus Principle, the modulus of the function achieves its maximum on the boundary $|w| < 1$, $|z| < 1$, and therefore:

$$\left|\frac{z-w}{1-\overline{w}z}\right| < 1, \{z \in \mathbb{C} : |z| < 1\}$$

□

Problem 4

By the Maximum Modulus Principle, since $\phi(x, y)$ is bounded by $x^2 + y^2 \leq 1$ and extends to the boundary of this constraint, then $\phi(x, y)$ reaches its maximum on the boundary $x^2 + y^2 = 1$. On the boundary of the constraint:

$$\begin{aligned}
x^2 &= 1 - y^2 \\
\therefore \varphi(y) &= (1 - y^2 - y^2 - 1)^2 + 4(1 - y^2)y^2 \\
&= 4y^4 + 4y^2 - 4y^4 \\
&= 4y^2 \\
\Rightarrow \varphi'(y) &= 8y,
\end{aligned}$$

which gives the critical point of φ as $y = 0$. The endpoints of φ are ± 1, as that is the range of y values for the constraint. This gives:

$$\begin{cases} \varphi(-1) = 4 \\ \varphi(0) = 0 \\ \varphi(1) = 4 \end{cases}$$

Corresponding x values are:

$$\begin{cases} y = -1: & x = 0 \\ y = 0: & x = \pm 1 \\ y = 1: & x = 0 \end{cases}$$

This gives:

$$\begin{cases} f(0, -1) = 4 \\ f(1, 0) = 0 \\ f(-1, 0) = 0 \\ f(0, 1) = 4 \end{cases}$$

From this, we can observe that the maximum of the function $\phi(x, y)$ is attained at $x = 0$, $y = \pm 1$. $\square$

Problem 5

Consider $e^{f(z)}$:

$$\begin{aligned}
e^{f(z)} &= e^{\operatorname{Re}(f(z)) + i \operatorname{Im}(f(z))} \\
&= e^{\operatorname{Re}(f(z))} e^{i \operatorname{Im}(f(z))} \\
&= e^{\operatorname{Re}(f(z))} (\cos(f(z)) + i \sin(f(z)))
\end{aligned}$$

As f is entire and $\operatorname{Re}(f(z))$ is bounded $\forall z \in \mathbb{C}$, by Liouville's Theorem $e^{\operatorname{Re}(f(z))}$ and therefore $e^{f(z)}$ is constant, which implies that $f(z)$ is constant as well. $\square$