Assignment 1

MATH 305 - Applied Complex Analysis

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1. A complex number, $z = x + yi$ has real part x and imaginary part y, where $x,\, y \in \mathbb{R}$. We define the addition of two complex numbers, $z = x + yi$, and $w = u + vi$, as:

$$z + w := (x + u) + i(y + v), \tag{1}$$

and the product of two complex numbers as:

$$zw := (xu - yv) + i(xv + yu) \tag{2}$$

i) Using Equation 1:

$$\begin{aligned}(2+3i) - (1-i) &= (2+3i) + (-1+i) \\ &= (2 + (-1)) + i(3+1) \\ &= 1 + 4i\end{aligned}$$

$\therefore$ Real part $= 1$, imaginary part $= 4$.

ii) The imaginary unit, i, is defined such that:

$$i^2 := -1 \tag{3}$$

$$\begin{aligned}\therefore i^3(1+i) &= i(i^2)(1+i) \\ &= i(-1)(1+i) \\ &= (0 - 1i)(1 + 1i) \\ &= [(0)(1) - (-1)(1)] + i[(0)(1) + (-1)(1)] \\ &= 1 + i(-1) \\ &= 1 - i\end{aligned}$$

$\therefore \mathrm{Re}(i^3(1+i)) = 1$, $\mathrm{Im}(i^3(1+i)) = -1$.

iii) The inverse of a complex number, z^{-1} is defined as:

$$z^{-1} := \frac{x - yi}{x^2 + y^2} \tag{4}$$

$$\begin{aligned}
\frac{2-2i}{4+3i} &= (2-2i)(4+3i)^{-1} \\
&= (2-2i)(\frac{4-3i}{4^2+3^2}) \\
&= (2-2i)(\frac{4-3i}{25}) \\
&= (2-2i)(\frac{4}{25}-\frac{3}{25}i) \\
&= [2(\frac{4}{25})-(-2)(-\frac{3}{25})]+i[2(-\frac{3}{25})+(-2)(\frac{4}{25})] \\
&= (\frac{8}{25}-\frac{6}{25})+i(-\frac{6}{25}-\frac{8}{25}) \\
&= \frac{2}{25}+i(-\frac{14}{25}) \\
&= \frac{2}{25}-\frac{14}{25}i
\end{aligned}$$

$\therefore \mathrm{Re}(\frac{2-2i}{4+3i}) = \frac{2}{25}$, $\mathrm{Im}(\frac{2-2i}{4+3i}) = -\frac{14}{25}$

iv) Using Equation 4, we can define $\frac{1}{i}$ as:

$$\frac{1}{i} := \frac{1}{0+1i} = \frac{0-i}{0^2+1^2} = -i \tag{5}$$

$$\begin{aligned}
\frac{2}{i}+\frac{i}{2} &= -2i+\frac{1}{2}i \\
&= -\frac{3}{2}i
\end{aligned}$$

$\therefore \mathrm{Re}(\frac{2}{i}+\frac{i}{2}) = 0$, $\mathrm{Im}(\frac{2}{i}+\frac{i}{2}) = -\frac{3}{2}$

v) Using Equation 5 and Equation 4:

$$\begin{aligned}
\frac{2+i}{1-i}+\frac{3+2i}{i} &= (2+i)(1-i)^{-1}+(-i)(3+2i)\\
&= (2+i)(\frac{1+i}{1^2+1^2})+(-2i^2-3i)\\
&= (2+i)(\frac{1}{2}+\frac{1}{2}i)+(-2(-1)-3i)\\
&= (2(\frac{1}{2})-1(\frac{1}{2}))+i(2(\frac{1}{2})+1(\frac{1}{2}))+(2-3i)\\
&= (1-\frac{1}{2})+i(1+\frac{1}{2})+(2-3i)\\
&= (\frac{1}{2}+\frac{3}{2}i)+(2-3i)\\
&= \frac{5}{2}-\frac{1}{2}i
\end{aligned}$$

$\therefore \mathrm{Re}(\frac{2+i}{1-i}+\frac{3+2i}{i})=\frac{5}{2}$, $\mathrm{Im}(\frac{2+i}{1-i}+\frac{3+2i}{i})=-\frac{1}{2}$

2. i) Using Equation 4:

$$\begin{aligned}
\frac{1-i}{2+i} &= (1-i)(2+1)^{-1}\\
&= (1-i)(\frac{2-i}{2^2+1^2})\\
&= \frac{1}{5}(1-i)(2-i)\\
&= \frac{1}{5}[(2-1)+i(-1-2)]\\
&= \frac{1}{5}-\frac{3}{5}i
\end{aligned}$$

We also have that:

$$|z| := \sqrt{x^2+y^2}, \tag{6}$$

for a complex number $z=x+yi$.

$$\begin{aligned}
\therefore |\frac{1-i}{2+1}| &= |\frac{1}{5}-\frac{3}{5}i|\\
&= \sqrt{(\frac{1}{5})^2+(-\frac{3}{5})^2}\\
&= \sqrt{\frac{2}{5}}
\end{aligned}$$

ii) We have that the conjugate of z is defined as:

$$\bar{z} := x-yi \tag{7}$$

$$\begin{aligned}\therefore (1-2i)\overline{(1-i)} &= (1-2i)(1+i)\\ &= (1+2)+i(1-2)\\ &= 3-i,\end{aligned}$$

which gives:

$$\begin{aligned}|(1-2i)\overline{(1-i)}| &= \sqrt{3^2+1^2}\\ &= \sqrt{10}\end{aligned}$$

iii) To convert a complex number from its rectangular to polar form:

$$z := |z|e^{i\theta}, \tag{8}$$

$$\theta := \arg(z) := \arctan\Big(\frac{y}{x}\Big) \tag{9}$$

$$\begin{aligned}\therefore (1-i)^{2021} &= [(\sqrt{1^2+1^2})e^{i(\arctan(\frac{-1}{1}))}]^{2021}\\ &= (\sqrt{2}e^{\frac{\pi}{4}i})^{2021}\\ &= (\sqrt{2})^{2021}e^{\frac{2021\pi}{4}i},\end{aligned}$$

$$\begin{aligned}i^{-2021} &= (0+i)^{-2021}\\ &= e^{-\frac{2021\pi}{2}i}\end{aligned}$$

We also have that, for two complex numbers z, w:

$$|zw| := |z||w| \tag{10}$$

$$\begin{aligned}\therefore |\frac{(1-i)^{2021}}{i^{2021}}| &= |(1-i)^{2021}(i)^{-2021}|\\ &= |(1-i)^{2021}||(i)^{-2021}|\\ &= (\sqrt{2})^{2021}(1)\\ &= (\sqrt{2})^{2021}\end{aligned}$$

iv) As $\frac{\pi}{2}$ has no imaginary part, it exists as only a point on the real axis:

$$\arg(\frac{\pi}{2}) = \mathrm{Arg}(\frac{\pi}{2}) = 0$$

v) Let $\tan(x) = -\frac{1}{\sqrt{3}}$. We have that $\arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$, and therefore $x = \frac{5\pi}{6}, \frac{11\pi}{6}, \forall x \in [0, 2\pi)$.

$$\Rightarrow \arg(\sqrt{3} - i) = \frac{5\pi}{6}, \frac{11\pi}{6}$$
$$\text{Arg}(\sqrt{3} - i) = \frac{5\pi}{6}, -\frac{\pi}{6}$$

3. For this problem, let $z = x + yi$, and $\zeta = a + bi$:

i) $|z - \zeta| = 2$ describes the set of points which are at a distance of 2 units from the point ζ, i.e. the points that lie on a circle centered about ζ of radius 2, as shown in Figure 1.

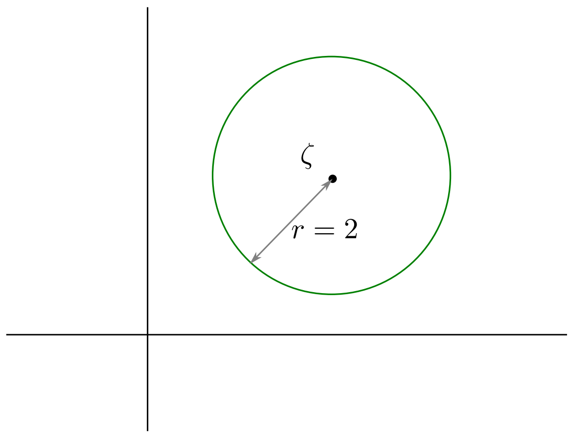

Figure 1

ii) We have that:

$$z^{-1} = \frac{x - yi}{x^2 + y^2}, \text{ and that:}$$
$$\bar{z} = x - yi$$

Combining the two, we have:

$$\frac{x - yi}{x^2 + y^2} = x - yi$$
$$x^2 + y^2 = 1$$
$$\Rightarrow |z|^2 = 1$$
$$\Rightarrow |z| = 1$$

which can be interpreted geometrically as the set of points that lie on the unit circle centered about the origin in the complex plane, as shown in Figure 2.

iii) $\text{Re}(z) = \frac{1}{2}$ geometrically describes the set of points that lie on the vertical line on the point $\frac{1}{2}$ in the complex plane, as shown in Figure 3.

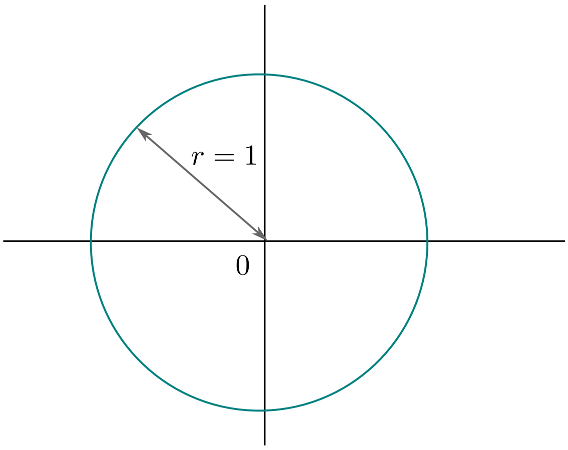

Figure 2

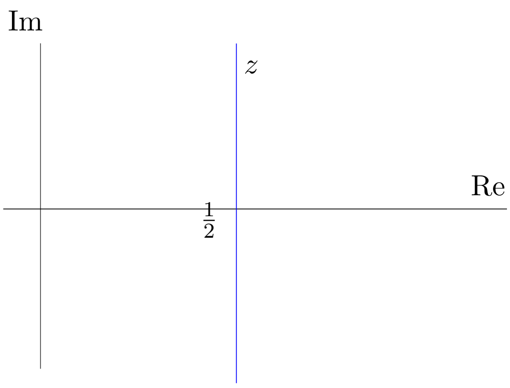

Figure 3

iv)

$$\begin{aligned} \operatorname{Im}(z) - 2\operatorname{Re}(z) &\leq 3 \\ y - 2x &\leq 3 \\ y &\leq 2x + 3 \end{aligned}$$

This set of points is geometrically described by the shaded region in Figure 4.

v) We have that:

$$\begin{aligned} z\bar{z} &= (x + yi)(x - yi) \\ &= (x^2 + y^2) - i(-xy + xy) \\ &= x^2 + y^2 \\ \therefore z\bar{z} \geq 1 &\equiv x^2 + y^2 \geq 1, \end{aligned}$$

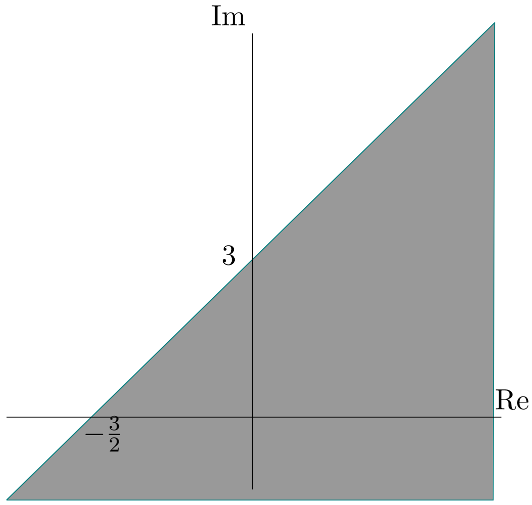

Figure 4

which geometrically describes the set of points that lie on or outside the unit circle centered about the origin, as shown in Figure 5.

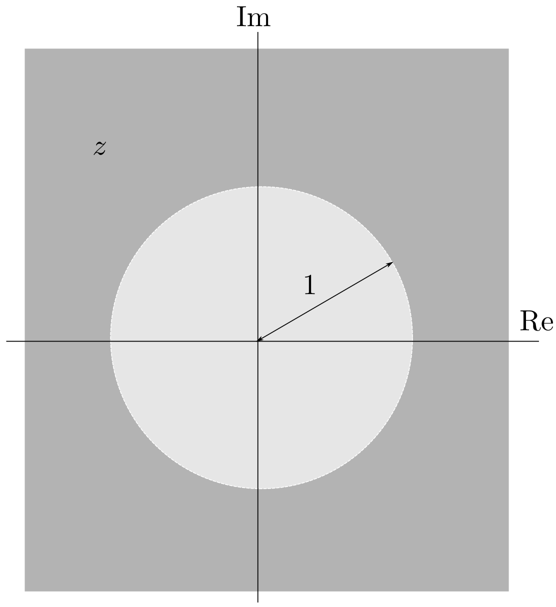

Figure 5

vi) The 5th roots of unity are:

$$\{1, e^{\frac{2\pi}{5}i}, e^{\frac{4\pi}{5}i}, e^{\frac{6\pi}{5}i}, e^{\frac{8\pi}{5}i}\}$$

Geometrically, this describes a set of 5 points on the unit circle, as shown in Figure 6.

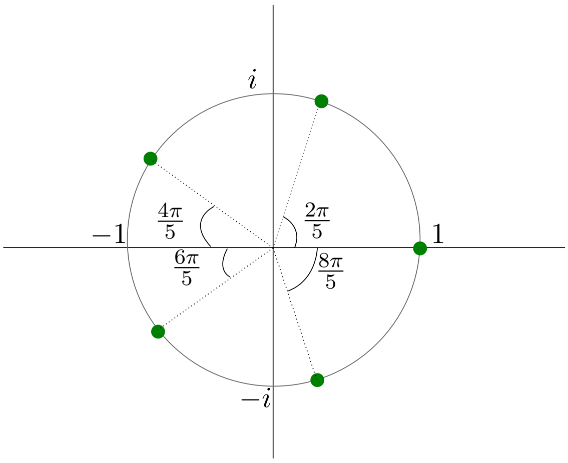

Figure 6

4. i) Let $z := x + yi$, then $\text{Re}(z) = x$, $\text{Im}(z) = y$:

$$\begin{aligned} iz &= (0+i)(x+yi) \\ &= (0-y) + i(0+x) \\ &= -y + xi \end{aligned}$$

We therefore have $\text{Re}(iz) = -y = -\text{Im}(z)$, and that $\text{Im}(iz) = x = \text{Re}(z)$. (Q.E.D)

ii) We can express i as $e^{\frac{\pi}{2}i}$ in the polar form. Therefore:

$$\begin{aligned} i^{4n} &= (e^{\frac{\pi}{2}})^{4n} \\ &= e^{2\pi ni} = 1, \end{aligned}$$

as all roots of 1 have the form $e^{2\pi ni}$, $\forall n \in \mathbb{N}$. We also have that:

$$\begin{aligned} i^2 &= -1 \\ i^3 &= i(i^2) = -i \end{aligned}$$

$$\begin{aligned} \therefore i^{4n} &= 1 \\ i^{4n+1} &= i^{4n}(i) = (1)(i) = i \\ i^{4n+2} &= i^{4n}(i^2) = (1)(-1) = -1 \\ i^{4n+3} &= i^{4n}(i^3) = 1(-i) = -i \text{ (Q.E.D.)} \end{aligned}$$

Let $n = 505$. Then, $i^{2021} = i^{4(505)+1} \equiv i^{4n+1}$, which gives:

$$i^{2021} = i, \text{ and}$$

$$i^{-2021} = \frac{1}{i} = \frac{-i}{1} = -i$$

iii) By substituting $z_1 = i$ into the equation:

$$\begin{aligned}(i-1)z_1^2 - 4z_1 - 1 + 5i &= (i-1)(i)^2 - 4(i) - 1 + 5i \\ &= -(i-1) - 4i - 1 + 5i \\ &= -i + 1 - 1 + i \\ &= 0,\end{aligned}$$

and by substituting $z_2 = -2 - 3i$ into the equation:

$$\begin{aligned}(i-1)z_2^2 - 4z_2 - 1 + 5i &= (i-1)(-2-3i)^2 - 4(-2-3i) - 1 + 5i \\ &= (-1+i)(-2-3i)(-2-3i) + 8 + 12i - 1 + 5i \\ &= [(2+3) + (3-2)i](-2-3i) + 7 + 17i \\ &= (5+i)(-2-3i) + 7 + 17i \\ &= (-10+3) + (-15-2)i + 7 + 17i \\ &= -7 - 17i + 7 + 17i \\ &= 0,\end{aligned}$$

it is verified that $z_1 = i$, $z_2 = -2 - 3i$ are solutions to the equation

$$(i-1)z^2 - 4z - 1 + 5i = 0$$

5. For $D \gg d$, $\theta' \approx \theta$, and the lines $r_+(x)$, $r_-(x)$, r are approximately parallel. If we zoom into this portion of the figure (as seen in Figure 8), we can observe that:

$$\begin{aligned}r_+ &= r - \frac{1}{2}\sin(\theta) \\ r_- &= r + \frac{1}{2}\sin(\theta)\end{aligned}$$

Therefore:

$$\begin{aligned}u(x,t) &= u_+(x,t) + u_-(x,t) \\ &= \frac{A}{r}(e^{i(kr_+(x)-\omega t)} + e^{i(kr_-(x)-\omega t)}) \\ &= \frac{A}{r}(e^{-i\omega t})(e^{i(kr_+(x))} + e^{i(kr_-(x))}), \text{ where} \\ e^{i(kr_+(x))} + e^{i(kr_-(x))} &= e^{ik(r-\frac{1}{2}\sin(\theta))} + e^{ik(r+\frac{1}{2}\sin(\theta))} \\ &= e^{ikr-\frac{1}{2}ik\sin(\theta)} + e^{ikr+\frac{1}{2}ik\sin(\theta)} \\ &= e^{ikr}(e^{-\frac{1}{2}ik\sin(\theta)} + e^{\frac{1}{2}ik\sin(\theta)}) \\ &= [2cos(\frac{1}{2}k\sin(\theta))]e^{ikr}\end{aligned}$$

This gives the overall expression:

$$u(x,t) = \frac{2A}{r}[cos(\frac{1}{2}k\sin(\theta))](e^{i(kr-\omega t)}),$$

from which we can extract $|u(x,t)| = \frac{A}{r}[2cos(\frac{1}{2}k\sin(\theta))]$, as per Equation 8. Then, we have:

$$\begin{aligned} I(x,t) &= |u(x,t)|^2 \\ &= (\frac{2A}{r}[cos(\frac{1}{2}k\sin(\theta))])^2 \\ &= \frac{4A^2}{r^2}\cos^2(\frac{1}{2}k\sin(\theta)), \end{aligned}$$

where $r = \frac{D}{\cos(\theta)}$, as shown in Figure 7. This gives:

$$\begin{aligned} I(x,t) &= 4A^2(\frac{\cos(\theta)}{D})^2\cos^2(\frac{1}{2}k\sin(\theta)) \\ &= \frac{4A^2}{D^2}\cos^2(\theta)\cos^2(\frac{1}{2}k\sin(\theta)) \\ &= \frac{4A^2}{D^2}\cos^2(\theta)\cos^2(\frac{1}{2}(\frac{2\pi}{\lambda})\sin(\theta)), \text{ as it is given that } \lambda = \frac{2\pi}{k} \\ &= \frac{4A^2}{D^2}\cos^2(\theta)\cos^2\left(\frac{\pi}{\lambda}\sin(\theta)\right) \end{aligned}$$

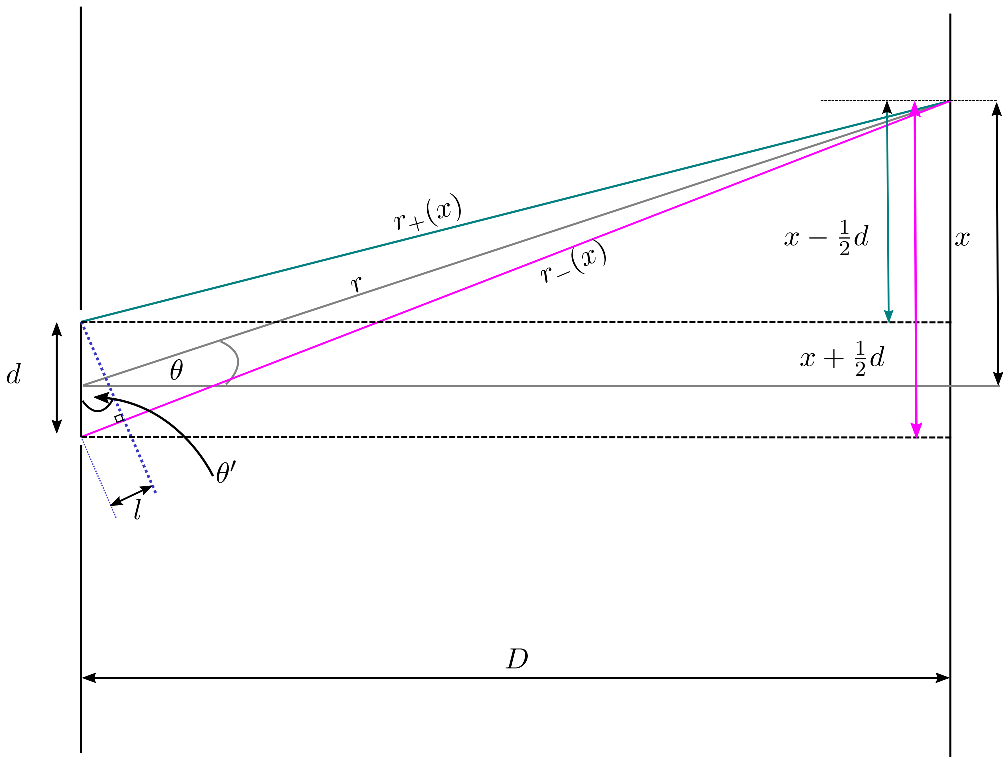

Figure 7: This is a vector diagram, so it can be magnified!

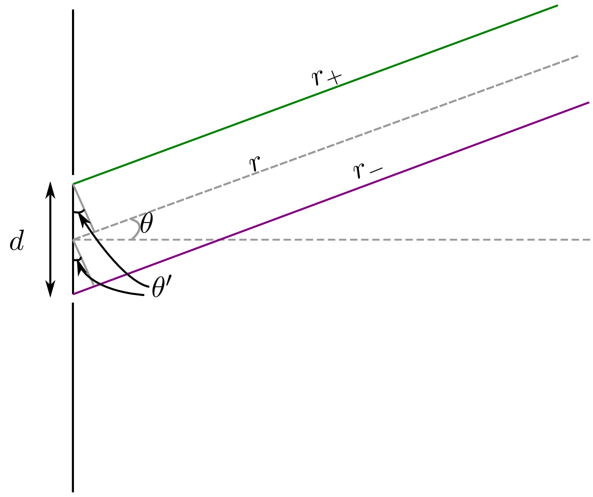

Figure 8: Zoomed into the region near the slit. This is a vector diagram, so it can be magnified too!

For small values of θ, we can make the approximation:

$$\tan(\theta) \approx \sin(\theta) \approx \theta$$

We have that $\tan\theta = \frac{x}{D}$, which can be approximated to be θ from above, as $\frac{x}{D}$ is small. Together with the fact that lines of maximal intensity occur when the path difference between the two waves (i.e. l in Figure 7) is an integer multiple of the light's wavelength, we can derive:

$$\begin{aligned}
d\sin(\theta) &\approx d\theta \approx n\lambda \\
\lambda &\approx \frac{d\theta}{n} \\
&\approx \frac{dx}{Dn} \\
x &\approx \frac{Dn}{d}\lambda \\
\therefore \bar{\lambda} &\approx x_{n+1} - x_n \\
&\approx \frac{D(n+1)}{d}\lambda - \frac{Dn}{d}\lambda \\
&\approx \frac{D\lambda}{d}(n+1-n) \\
&\approx \frac{D}{d}\lambda \text{ (Q.E.D.)}
\end{aligned}$$