MATH 305:201, 2020W T2

Homework set 7 — due March 12

Problem 1. Let f be holomorphic in the neighbourhood of the open annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$. Assume that $|f(z)| \leq 12$ for all z such that $|z| = 2$ and that $|f(z)| \leq 3$ for all z such that $|z| = 1$. By considering the function $g(z) = f(z)/3z^2$, prove that

$$|f(z)| \leq 3|z|^2.$$

Problem 2. A 3×3 matrix A has a characteristic polynomial

$$c_A(t) = -t^3 + t^2 + 4t - 24.$$

Prove that all eigenvalues of A have modulus larger than 2.

Problem 3. Let $w \in \mathbb{C}$ be such that $|w| < 1$.
(i) Show that if $z \in \mathbb{C}$ is such that $|z| = 1$ then

$$\left| \frac{z - w}{1 - \bar{w}z} \right| = 1.$$

(ii) Use (i) to show that

$$\left| \frac{z - w}{1 - \bar{w}z} \right| < 1$$

for all $z \in \mathbb{C}$ is such that $|z| < 1$. *Hint:* Why is the condition $|w| < 1$ needed?

Problem 4. Show that the maximum of the function $\phi(x, y) = (x^2 - y^2 - 1)^2 + 4x^2y^2$, subject to the constraint $x^2 + y^2 \leq 1$, is attained at $x = 0, y = \pm 1$.

Problem 5. Let f be entire. Assume that $\mathrm{Re}(f(z))$ is bounded above for all $z \in \mathbb{C}$. Conclude that $f(z)$ is a constant function.
Hint. Consider $\mathrm{e}^{f(z)}$.