MATH 305:201, 2020W T2

Homework set 4 — due Feb 12

Problem 1. *Optional, will not be marked.*
Let $\mathrm{Log}(\zeta)$ and $\zeta^{1/2}$ be the principal branches of the logarithm and the square root.
(i) Show that

$$w = -\mathrm{i}\mathrm{Log}\left(\mathrm{i}z + (1-z^2)^{\frac{1}{2}}\right) \tag{1}$$

is such that $\sin(w) = z$.
(ii) Let $w(z)$ be defined as above. Show that $w'(z) = \frac{1}{(1-z^2)^{1/2}}$.
(iii) Show that if z is real and in $[-1,1]$, then w is real and between $[-\pi/2, \pi/2]$.
Hint. Where does $\mathrm{i}z + (1-z^2)^{\frac{1}{2}}$ lie in the complex plane?
Remark. In other words, the expression in (1) is defined for all $z \in \mathbb{C} \setminus \{0\}$ and agrees with the standard arcsin on the real interval $[-1,1]$; we say that (1) defines a holomorphic extension of the usual arcsine function.

Problem 2. Use complex analysis to find the temperature distribution in a layer of ice of thickness σ on a frozen lake, given that the water temperature is 0°C and the air is at T°C.

Problem 3. (i) Determine the set Ω on which $\mathrm{Log}(\frac{z-1}{z-2})$ is holomorphic.
(ii) Find a branch of $(z^2-1)^{\frac{1}{2}}$ which is holomorphic in the disc $B_1(0)$.
Hint. Write $(z^2-1) = (-1)(1-z^2)$.

Problem 4. Find a (piecewise) smooth parametrization of the following contours:
(i) The curve in the figure to the right
(ii) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (where $z = x + \mathrm{i}y$), running once from $z_\mathrm{i} = a$ to itself with the positive orientation.
Hint. Use polar coordinates and express r as a function of the angle θ along the ellipse.