Assignment 9

MATH 305 - Applied Complex Analysis

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Problem 1

i) f_1 has an essential singularity at $z_0 = 0$, with the Laurent series:

$$\begin{aligned} f_1(z) &= z^2(1 + \frac{1}{z} + \frac{1}{2}(\frac{1}{z^2}) + \frac{1}{3!}(\frac{1}{z^3}) + \dots) \\ &= z^2 + z + \frac{1}{2} + \frac{1}{6}(\frac{1}{z}) + \dots \\ &= \dots + \frac{1}{6}z^{-1} + \dots \\ \Rightarrow a^{-1} &= \frac{1}{6} = \text{Res}(f_1\,;\,0) \end{aligned}$$

ii) f_2 has a simple pole at $z_0 = 0$ and a pole of order 2 at $z_1 = \frac{\pi}{2}$. For the simple pole:

$$\begin{aligned} \text{Res}(f_2\,;\,0) &= \lim_{z\to z_0}(z)(\frac{\cos(2z)}{z(z-\frac{\pi}{2})^2}) \\ &= \lim_{z\to 0}(\frac{\cos(2z)}{(z-\frac{\pi}{2})^2}) \\ &= \frac{1}{\frac{\pi^2}{4}} \\ &= \frac{4}{\pi^2} \end{aligned}$$

For the pole of order 2:

$$\begin{aligned} \text{Res}\Big(f_2\,;\,\frac{\pi}{2}\Big) &= \lim_{z\to\frac{\pi}{2}} \frac{d}{dz}((z-\frac{\pi}{2})^2(\frac{\cos(2z)}{z(z-\frac{\pi}{2})^2})) \\ &= \lim_{z\to\frac{\pi}{2}} \frac{d}{dz}(\frac{\cos(2z)}{z}) \\ &= \lim_{z\to\frac{\pi}{2}} -(\frac{2\sin(2z)}{z} + \frac{\cos(2z)}{z^2}) \\ &= -(\frac{2\sin(\pi)}{\frac{\pi}{2}} + \frac{\cos(\pi)}{\frac{\pi^2}{4}}) \\ &= -(-\frac{4}{\pi^2}) \\ &= \frac{4}{\pi^2} \end{aligned}$$

iii) a) The essential singularity $z_0 = 0$ lies in the unit circle. Laurent's Theorem then gives:

$$\oint_{|z|=1} f_1(z) = 2\pi i a_{-1} = 2\pi i \operatorname{Res}(f_1\,;\,0)$$
$$= 2\pi i(\frac{1}{6})$$
$$= \frac{\pi}{3} i$$

b) Both residues of f_2 are equal and lie in the unit circle. The Residue Theorem then gives:

$$\oint_{|z|=1} f_2(z) = 2\pi i(\frac{4}{\pi^2} + \frac{4}{\pi^2})$$
$$= \frac{16}{\pi} i$$

Problem 2

i) We have the identity $\tan(z) = \sin(z)/\cos(z)$, and we can therefore observe that $\tan(z)$ has 2 simple poles at $z_0 = \frac{\pi}{2}$ and $z_1 = \frac{3\pi}{2}$ that lie in the contour $|z| = 2\pi$. Then:

$$\operatorname{Res}(\tan(z)\,;\,z_0) = \frac{\sin(z_0)}{(\cos(z_0))'} = \frac{\sin\left(\frac{\pi}{2}\right)}{-\sin\left(\frac{3\pi}{2}\right)}$$
$$= -1$$
$$\operatorname{Res}(\tan(z)\,;\,z_1) = \frac{\sin(z_1)}{(\cos(z_1))'} = \frac{\sin\left(\frac{3\pi}{2}\right)}{-\sin\left(\frac{3\pi}{2}\right)}$$
$$= -1$$

The Residue Theorem then gives:

$$\oint_{|z|=2\pi} \tan(z) = 2\pi i(-1 + -1)$$
$$= -4\pi i$$

ii) We use the function $f(z) = \frac{e^{2z}}{\cosh(\pi z)}$. We have that:

$$f(z) = \frac{e^{2z}}{\cosh(\pi z)} = \frac{e^{2z}}{\cos(i\pi z)}$$
$$= \frac{e^{2z}}{\frac{1}{2}(e^{i(i\pi z)} + e^{-i(i\pi z)})}$$

$$\begin{aligned} f(z) &= \frac{e^{2z}}{\frac{1}{2}(e^{-\pi z} + e^{\pi z})} \\ &= \frac{2e^{2z}}{e^{-\pi z}(1 + e^{2\pi z})} \\ &= \frac{2e^{(2+\pi)z}}{1 + e^{2\pi z}} \end{aligned}$$

We note that:

$$\begin{aligned} &1 + e^{2\pi z} = 0 \text{ if and only if} \\ &\quad 2\pi z = n\pi i, \forall n \in \mathbb{Z}\backslash\{0\} \\ &\quad \Rightarrow z = \frac{n}{2}i, \end{aligned}$$

meaning $f(z)$ has singularities $\frac{n}{2}i$ for all positive non-zero integer values of n. We now consider the positively oriented countour as shown in Figure 1.

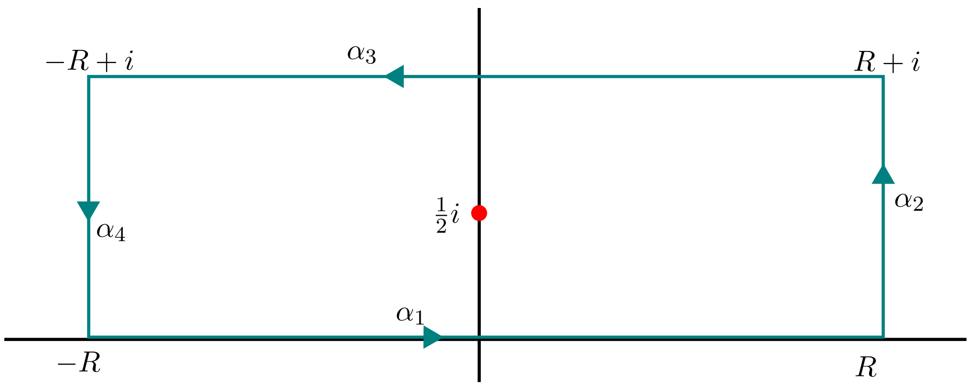

Figure 1

We assume (for now) that the subsequent poles of $f(z)$ (i.e. πi, $\frac{3\pi}{2}i$, ...) lie outside this contour. By the Residue Theorem, we have (for this rectangular contour):

$$\begin{aligned} \oint_{\alpha} f(z)\,dz &= \int_{\alpha_1} f(z)\,dz + \int_{\alpha_2} f(z)\,dz + \int_{\alpha_3} f(z)\,dz + \int_{\alpha_4} f(z)\,dz \\ &= 2\pi i \operatorname{Res}\left(f\,;\, \frac{1}{2}i\right) \\ &= 2\pi i \operatorname{Res}\left(\frac{2e^{(2+\pi)z}}{1 + e^{2\pi z}}\,;\, \frac{1}{2}i\right) \\ &= 2\pi i \left(\frac{2e^{(2+\pi)z}}{2\pi e^{2\pi z}}\right)\Big|_{z=\frac{1}{2}i} \end{aligned}$$

$$\begin{aligned}\oint_\alpha f(z)\,dz &= \frac{2e^{(2+\pi)(\frac{1}{2}i)}}{e^{2\pi(\frac{1}{2}i)}}i \\ &= \frac{2e^{i+i\frac{\pi}{2}}}{e^{i\pi}}i \\ &= \frac{2i^2e^i}{e^{i\pi}} \\ &= 2e^i\end{aligned}$$

Let $\alpha_2(t) = R + ti$ for $t \in [0, 1]$. This gives:

$$\begin{aligned}\ell(\alpha_2) &= \int_0^1 |i|\,dt \\ &= 1\end{aligned}$$

Therefore:

$$\Big|\int_{\alpha_2} f(z)\,dz\Big| \leq \ell(\alpha_2)\ \max\{|f(\alpha_2(t))|\ :\ t \in [0,1]\} \leq \frac{2e^{(2+\pi)R+(2+\pi)ti}}{1+e^{2\pi R}e^{2\pi ti}}$$

For large R, we can estimate that:

$$\frac{2e^{(2+\pi)R+(2+\pi)ti}}{1+e^{2\pi R}e^{2\pi ti}} \approx \frac{e^{2+\pi R}}{e^{2\pi R}}$$

Since $2\pi > 2+\pi$, we then have that:

$$\Big|\int_{\alpha_2} f(z)\,dz\Big| \leq \frac{e^{2+\pi R}}{e^{2\pi R}} \to 0,\ \text{ as } R \to \infty$$

We can conclude similarly that:

$$\Big|\int_{\alpha_4} f(z)\,dz\Big| \to 0 \text{ as } R \to \infty$$

This leaves us with:

$$\int_{\alpha_1} f(z)\,dz + \int_{\alpha_3} f(z)\,dz = -2ie^i$$

Let $\alpha_1(x) = x$ for $x \in [-R, R]$; $\alpha_3(x) = x + i$ for $x \in [R, -R]$:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)}(1)\,dx + \int_{\infty}^{-\infty} \frac{e^{2(x+i)}}{\frac{1}{2}(e^{\pi x+\pi i}+e^{-\pi x-\pi i})}(1) &= 2e^i \\ \int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)}\,dx - \int_{-\infty}^{\infty} \frac{e^{2x}e^{2i}}{\frac{1}{2}(e^{\pi x}e^{\pi i}+e^{-\pi x}e^{-\pi i})}\,dx &= 2e^i \\ \int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)}\,dx - e^{2i}\int_{-\infty}^{\infty} \frac{e^{2x}}{\frac{1}{2}e^{\pi i}(e^{\pi x}+e^{-\pi x})}\,dx &= 2e^i\end{aligned}$$

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)}\,dx + e^{2i}\int_{-\infty}^{\infty} \frac{e^{2x}}{\frac{1}{2}(e^{\pi x}+e^{-\pi x})}\,dx &= 2e^{i} \\
\int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)}\,dx + e^{2i}\int_{-\infty}^{\infty} \frac{e^{-2x}}{\cosh(\pi x)}\,dx &= 2e^{i} \\
(1+e^{2i})\int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)}\,dx &= 2e^{i} \\
\int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)}\,dx &= \frac{2e^{i}}{1+e^{2i}}
\end{aligned}
$$

Simplifying this expression further gives:

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)}\,dx &= \frac{1}{\frac{1}{2}e^{-i}(1+e^{2i})} \\
&= \frac{1}{\frac{1}{2}(e^{-i}+e^{i})} \\
&= \frac{1}{\frac{1}{2}(e^{-i(1)}+e^{i(1)})} \\
&= \frac{1}{\cos(1)} \\
&= \sec(1)
\end{aligned}
$$

Problem 3

Let $P(z) = c(z-z_1)(z-z_2)\dots(z-z_d)$. By the product rule (according to Wikipedia), we have:

$$
P'(z) = c\Big(\prod_{j=1}^{d}(z-z_j)\Big)\Big(\sum_{j=1}^{d}\frac{1}{z-z_j}\Big) \tag{1}
$$

Using similar notation, we can express $P(z)$ as:

$$
P(z) = c\prod_{j=1}^{d}(z-z_j) \tag{2}
$$

Taking (1)/(2) then gives:

$$
\begin{aligned}
\frac{P'(z)}{P(z)} &= \frac{c\cancel{(\prod_{j=1}^{d}(z-z_j))}(\sum_{j=1}^{d}\frac{1}{z-z_j})}{c\cancel{\prod_{j=1}^{d}(z-z_j)}} \\
&= \sum_{j=1}^{d}\frac{1}{z-z_j} \\
&= \frac{1}{z-z_1}+\frac{1}{z-z_2}+\frac{1}{z-z_3}+\dots\frac{1}{z-z_d}
\end{aligned}
$$

The residue of $P'(z)/P(z)$ at z_j, for $j \in [1, d]$, is defined as the coefficient of the term(s) w.r.t z^{-1}, and we can observe that all terms in the above expression are z^{-1} terms. Therefore, we can conclude that the function $P'(z)/P(z)$ has d residues, and that:

$$
\begin{cases}
\operatorname{Res}\left(\frac{P'(z)}{P(z)}\,;\,z_j\right) = 1, & \text{if } z_j \in \operatorname{int}(\alpha), \\
\operatorname{Res}\left(\frac{P'(z)}{P(z)}\,;\,z_j\right) = 0, & \text{if } z_j \notin \operatorname{int}(\alpha),
\end{cases} \tag{3}
$$

By the residue theorem:

$$
\oint_\alpha \frac{P'(z)}{P(z)}\,dz = 2\pi i \sum_{z_j\in\operatorname{int}(\alpha)} \operatorname{Res}\left(\frac{P'(z)}{P(z)}\,;\,z_j\right)
$$

$$
\begin{aligned}
\frac{1}{2\pi i}\oint_\alpha \frac{P'(z)}{P(z)}\,dz &= \sum_{z_j\in\operatorname{int}(\alpha)} \operatorname{Res}\left(\frac{P'(z)}{P(z)}\,;\,z_j\right) \\
&= \sum_{z_j\in\operatorname{int}(\alpha)} 1, \text{ from (3)} \\
&= N,
\end{aligned}
$$

as it is given that N counts the number of zeros of $P \in \operatorname{int}(\alpha)$. □

Problem 4

Let $f(z) = \frac{1}{z^2}$. From the lecture notes, we then have:

$$
\begin{aligned}
\operatorname{Res}\left(\frac{\pi\cot(\pi z)}{z^2}\,;\,n\right) &= \frac{1}{n^2},\ \forall n \in \mathbb{Z}\backslash\{0\} \\
\therefore \frac{1}{2\pi i}\oint_\alpha \frac{\pi\cot(\pi z)}{z^2} &= \sum_{j=-\infty}^{-1}\frac{1}{j^2} + \sum_{j=1}^{\infty}\frac{1}{j^2} + \operatorname{Res}\left(\frac{\pi\cot(\pi z)}{z^2}\,;\,0\right)
\end{aligned}
$$

We can observe that:

$$\frac{1}{(-n)^2} = \frac{1}{n^2}, \forall n \in \mathbb{Z} \backslash \{0\}$$
$$\therefore \frac{1}{2\pi i} \oint_{\alpha} \frac{\pi \cot(\pi z)}{z^2} = 2 \sum_{j=1}^{\infty} \frac{1}{j^2} + \operatorname{Res}\left(\frac{\pi \cot(\pi z)}{z^2} ; 0\right) \tag{4}$$

For the residue at the pole of order 3 at $z_0 = 0$:

$$\begin{aligned}
\operatorname{Res}\left(\frac{\pi \cot(\pi z)}{z^2} ; 0\right) &= \frac{1}{(3-1)!} \lim_{z \to 0} \frac{d^{3-1}}{dz^{3-1}} z^3 \frac{\pi \cot(\pi z)}{z^2} \\
&= \frac{1}{2} \lim_{z \to 0} \frac{d^2}{dz^2} \pi z \cot(\pi z) \\
&= \frac{1}{2} \lim_{z \to 0} \frac{d^2}{dz^2} \frac{\pi z \cos(\pi z)}{\sin(\pi z)} \\
&= \frac{1}{2} \lim_{z \to 0} \frac{d^2}{dz^2} \frac{\pi z \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\pi z)^{2n}}{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (\pi z)^{2n+1}} \\
&= \frac{1}{2} \lim_{z \to 0} \frac{d^2}{dz^2} \frac{\cancel{\pi z} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\pi z)^{2n}}{\cancel{\pi z} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (\pi z)^{2n}} \\
&= \frac{1}{2} \lim_{z \to 0} \frac{d^2}{dz^2} \frac{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\pi z)^{2n}}{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (\pi z)^{2n}} \\
&= \frac{1}{2} \lim_{z \to 0} \frac{d^2}{dz^2} \frac{1 - \frac{1}{2}\pi^2 z^2 + \frac{1}{4!}\pi^4 z^4 - \frac{1}{6!}\pi^6 z^6 + \dots}{1 - \frac{1}{3!}\pi^2 z^2 + \frac{1}{5!}\pi^4 z^4 - \frac{1}{7!}\pi^6 z^6 + \dots} \\
&= \frac{1}{2} \lim_{z \to 0} \frac{d^2}{dz^2} (1 - \frac{1}{3}\pi^2 z^2 + \dots), \text{ by long division} \\
&= \frac{1}{2} \lim_{z \to 0} \frac{d}{dz} (-\frac{2}{3}(\pi^2 z) + \dots) \\
&= \frac{1}{2} \lim_{z \to 0} (-\frac{2}{3}\pi^2 + \dots) \\
&= \frac{1}{\cancel{2}} (-\frac{\cancel{2}}{3}\pi^2) \\
&= -\frac{\pi^2}{3}
\end{aligned}$$

Combining this result with (4) gives:

$$\frac{1}{2\pi i}\oint_{\alpha}\frac{\pi\cot(\pi z)}{z^2} = 2\sum_{j=1}^{\infty}\frac{1}{j^2} - \frac{\pi^2}{3}$$
$$\oint_{\alpha}\frac{\pi\cot(\pi z)}{z^2} = 4\pi i\sum_{j=1}^{\infty}\frac{1}{j^2} - \frac{2\pi^3}{3}i$$

We use a square contour α with side length $2N+1$. This gives us $\ell(\alpha) = 4(2N+1) = 8N+4$. As $|\cot(\pi z)|$ is bounded away from the real integers:

$$\therefore \left|\oint_{\alpha}\frac{\pi\cot(\pi z)}{z^2}\right| \leq \frac{c_N(8N+4)}{N^2} \approx \underbrace{\frac{1}{N}}_{\text{for large N}} \to 0 \text{ as } N\to\infty$$

where c_N is some constant obtained from $|\cot(\pi z)|$. This gives us:

$$4\pi i\sum_{j=1}^{\infty}\frac{1}{j^2} - \frac{2\pi^3}{3}i = 0$$
$$4\pi i\sum_{j=1}^{\infty}\frac{1}{j^2} = \frac{2\pi^3}{3}i$$
$$2\sum_{j=1}^{\infty}\frac{1}{j^2} = \frac{\pi^2}{3}$$
$$\sum_{j=1}^{\infty}\frac{1}{j^2} = \frac{\pi^2}{6}$$

$\square$