MATH 305:201, 2020W T2

# Homework set 9 — due March 26

**Problem 1.** Compute the residues at the singularities of the following functions:

$$\text{(i)} \quad f_1(z) = z^2 \mathrm{e}^{1/z}, \qquad \text{(ii)} \quad f_2(z) = \frac{\cos(2z)}{z(z-\pi/2)^2}.$$

(iii) Compute $\oint_{|z|=1} f_i(z)\mathrm{d}z$, for both $i = 1, 2$.

**Problem 2.** (i) Compute

$$\oint_{|z|=2\pi} \tan(z)\mathrm{d}z$$

(ii) Compute

$$I = \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{2x}}{\cosh(\pi x)}\mathrm{d}x$$

*Hint:* Use a rectangular closed curve with corners $R, R + \mathrm{i}, -R + \mathrm{i}, -R$

**Problem 3.** Let $P$ be a polynomial of degree $d$, and let $\alpha$ be a closed, positively oriented simple curve such that no zero of $P$ lies on $\alpha$. Show that

$$\frac{1}{2\pi\mathrm{i}} \oint_\alpha \frac{P'(z)}{P(z)}\mathrm{d}z = N,$$

where $0 \leq N \leq d$ is an integer counting the number of zeroes of $P$ inside $\alpha$ (with multiplicity: a zero of multiplicity $m$ is counted $m$ times)
*Hint:* One can write $P(z) = c(z - z_1)\cdots(z - z_d)$; why?

**Problem 4.** Show that $\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}$ by following the strategy presented in class but using $f(z) = \frac{1}{z^2}$ directly (no limit to be taken at the end of the argument). Adapt the computation to account for the fact that the integrand now has a pole of order three at $z = 0$.