MATH 305:201, 2020W T2

Homework set 3 — due Feb 05

Problem 1. Let $\Omega \subset \mathbb{C}$ be a domain and $f : \Omega \to \mathbb{C}$ be holomorphic in Ω. Denote $f = u + \mathrm{i}v$ and identify (u, v) with the real vector field $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} u(x,y) \\ v(x,y) \end{smallmatrix}\right)$.
(i) Recall that the derivative of a vector field is given by the matrix $\left(\begin{smallmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{smallmatrix}\right)$. Prove that

$$\det \begin{pmatrix} \partial_x u(x,y) & \partial_y u(x,y) \\ \partial_x v(x,y) & \partial_y v(x,y) \end{pmatrix} = \left|f'(z)\right|^2, \qquad \text{for all } z = x + \mathrm{i}y \in \Omega.$$

(ii) Check the validity of (i) in the case of the function $f(z) = \mathrm{e}^{2z}$.
(iii) Prove that the gradients of u and v are everywhere orthogonal.
(iv) Check the validity of (iii) in the case of the function $f(z) = z^2$.

Problem 2. (i) Let $\Omega = \mathbb{C}$, and let $u(x, y) = 2x(1 - y) + 1$; Find a function $v(x, y)$ such that $f = u + \mathrm{i}v$ is holomorphic in Ω, and express f in terms of z
(ii) Show that there is no entire function $f = u + \mathrm{i}v$ with $v(x, y) = 3x^3y - 2x^2 + 5xy^2 - 1$.

Problem 3. Find all solutions of the following equations:
(i) $\sinh(2z) = \mathrm{i}$
(ii) $2\cos(z) = \mathrm{i}\sin(z)$
(iii) $(z - \mathrm{i})^4 = (z + \mathrm{i})^4$

Problem 4. (i) Show that $\mathrm{Re}(\sin(z)) = \sin(\mathrm{Re}(z))\cosh(\mathrm{Im}(z))$
(ii) Show that $|\cos(z)|$ tends to $+\infty$ as $|z| \to +\infty$ along a straight line through 0 with non-zero slope. *Hint:* Write $z = r\mathrm{e}^{\mathrm{i}\theta}$ with $\theta \neq n\pi$ fixed and let $r \to \infty$

Problem 5. (i) Compute $\mathrm{Log}(-1 - \mathrm{i})$
(ii) Compute $\mathrm{Log}(2\mathrm{e}^{3\pi\mathrm{i}})$
(iii) Compute $\mathrm{Log}((-1 - \mathrm{i}\sqrt{3})^2)$ and compare it with $2\mathrm{Log}(-1 - \mathrm{i}\sqrt{3})$
(iv) Find a $z \in \mathbb{C}$ such that $\mathrm{Log}(1/z) \neq -\mathrm{Log}(z)$