MATH 305:201, 2020W T2

Homework set 5 — due Feb 26

Problem 1. Compute

$$\text{(i)} \int_\alpha z\mathrm{d}z, \qquad \text{(ii)}\ \ell(\alpha), \qquad \text{(iii)} \int_\alpha \mathrm{Re}(z)\mathrm{d}z,$$

along the polygonal arc $\alpha : 0 \to 1+\mathrm{i} \to 2 \to 0$.

Problem 2. (i) If α is the straight line from i to -1, show without computing the integral that

$$\left| \int_\alpha \frac{1}{z^2}\mathrm{d}z \right| \leq 2\sqrt{2}.$$

(ii) Let $k > 0$. Show that

$$\lim_{R\to+\infty} \left| \int_{\alpha_R} \frac{\mathrm{e}^{\mathrm{i}kz}}{1+z^2}\mathrm{d}z \right| = 0$$

where α_R is the positively oriented semi-circle of radius R in the upper half-plane centred at the origin.
(iii) Show that

$$\lim_{R\to+\infty} \int_{\gamma_R} \mathrm{e}^{-z^2}\mathrm{d}z = 0$$

where γ_R is vertical line segment from R to $R + \mathrm{i}h$, $h > 0$ fixed.

Problem 3. (i) Recall that Log denotes the principal value of the logarithm. Compute

$$\int_\alpha \mathrm{Log}(z)\mathrm{d}z$$

where α is the straight line from 1 to i.
(ii) Compute

$$\int_{\alpha_\pm} \bar{z}\mathrm{d}z$$

where α_+, resp. α_-, is the semi-circle running from 1 to -1 in the upper, resp. lower, half-plane.

Problem 4. Let $\alpha = a + \mathrm{i}b$ be any non-zero complex number, and let $f : \mathbb{R} \to \mathbb{C}$ be defined by $f(t) = \mathrm{e}^{\alpha t}$. Use the fact that $F(t) = \alpha^{-1}\mathrm{e}^{\alpha t}$ is an antiderivative of f to show that

$$\int \mathrm{e}^{at}\cos(bt)\mathrm{d}t = \frac{\mathrm{e}^{at}}{a^2+b^2}\left(a\cos(bt) + b\sin(bt)\right), \quad \int \mathrm{e}^{at}\sin(bt)\mathrm{d}t = \frac{\mathrm{e}^{at}}{a^2+b^2}\left(a\sin(bt) - b\cos(bt)\right).$$

Conclude that for $a > 0$,

$$\int_0^\infty \mathrm{e}^{-at}\cos(bt)\mathrm{d}t = \frac{a}{a^2+b^2}, \quad \int_0^\infty \mathrm{e}^{-at}\sin(bt)\mathrm{d}t = \frac{b}{a^2+b^2}.$$

Problem 5. Let g be continuous in a neighbourhood of z_0 and let $f : \mathbb{C} \setminus \{z_0\} \to \mathbb{C}$ be defined by

$$f(z) = \frac{a}{z - z_0} + g(z).$$

Let α_ϵ be any positively oriented semi-circle of radius $\epsilon > 0$ centred at z_0. Show that

$$\lim_{\epsilon \to 0} \int_{\alpha_\epsilon} f(z) \mathrm{d}z = \mathrm{i}\pi a.$$