Assignment 3

MATH 305 - Applied Complex Analysis

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Problem 1

(i) The determinate of the derivative of the vector field is given by:

$$\det\begin{pmatrix}\partial_x u(x,y) & \partial_y u(x,y)\\ \partial_x v(x,y) & \partial_y v(x,y)\end{pmatrix} = (\partial_x u(x,y))(\partial_y v(x,y)) - (\partial_y u(x,y))(\partial_x v(x,y))$$

As $f \in H(\Omega)$, the Cauchy-Riemann equations are satisfied:

$$\begin{cases}\partial_x u(x,y) = \partial_y v(x,y),\\ \partial_x v(x,y) = -\partial_y u(x,y),\end{cases}$$

and $f'(z)$ is given by:

$$f'(z) = \partial_x u(x,y) + i\partial_x v(x,y) = \partial_y v(x,y) - i\partial_y u(x,y)$$

This gives us:

$$\begin{aligned}|f'(z)|^2 &= (f'(z))\overline{(f'(z))}\\ &= (\partial_x u(x,y) + i\partial_x v(x,y))(\partial_x u(x,y) - i\partial_x v(x,y))\\ &= (\partial_x u(x,y))(\partial_x u(x,y)) + (\partial_x v(x,y))(\partial_x v(x,y))\\ &= (\partial_x u(x,y))(\partial_y v(x,y)) + (-\partial_y u(x,y))(\partial_x v(x,y)),\ \text{from (CR)}\\ &= (\partial_x u(x,y))(\partial_y v(x,y)) - (\partial_y u(x,y))(\partial_x v(x,y))\\ &= \det\begin{pmatrix}\partial_x u(x,y) & \partial_y u(x,y)\\ \partial_x v(x,y) & \partial_y v(x,y)\end{pmatrix}\end{aligned}$$

$\square$

(ii) We can express $f(z) = e^{2z}$ as:

$$\begin{aligned}f(z) &= (e^z)^2\\ &= (e^{x+yi})^2\\ &= (e^x(\cos(y) + i\sin(y)))^2\\ &= e^{2x}(\cos(y) + i\sin(y))^2\\ &= e^{2x}(\cos^2(y) + 2i\cos(y)\sin(y) - \sin^2(y))\\ &= e^{2x}(\cos(2y) + i\sin(2y)),\ \text{using the double angle trigonometric indentities}\\ &= e^{2x}\cos(2y) + ie^{2x}\sin(2y)\\ \Leftrightarrow \begin{pmatrix}u\\ v\end{pmatrix} &= \begin{pmatrix}e^{2x}\cos(2y)\\ e^{2x}\sin(2y)\end{pmatrix},\end{aligned}$$

which gives us $u(x,y) = e^{2x}\cos(2y)$, $v(x,y) = e^{2x}\sin(2y)$. The derivative of the vector field is then:

$$\begin{aligned}
\begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} &= \begin{pmatrix} 2e^{2x}\cos(2y) & -2e^{2x}\sin(2y) \\ 2e^{2x}\sin(2y) & 2e^{2x}\cos(2y) \end{pmatrix} \\
\therefore \det \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} &= (2e^{2x}\cos(2y))(2e^{2x}\cos(2y)) - (-2e^{2x}\sin(2y))(2e^{2x}\sin(2y)) \\
&= 4e^{4x}\cos^2(2y) + 4e^{4x}\sin^2(2y) \\
&= 4e^{4x}(\cos^2(2y) + \sin^2(2y)) \\
&= 4e^{4x}, \text{ from} \sin^2(\theta) + \cos^2(\theta) = 1
\end{aligned}$$

Now, we evaluate $f'(z)$:

$$\begin{aligned}
f'(z) &= 2e^{2x}\cos(2y) + i2e^{2x}\sin(2y) \\
\therefore |f'(z)|^2 &= (2e^{2x}\cos(2y) + i2e^{2x}\sin(2y))\overline{(2e^{2x}\cos(2y) + i2e^{2x}\sin(2y))} \\
&= (2e^{2x}\cos(2y) + i2e^{2x}\sin(2y))(2e^{2x}\cos(2y) - i2e^{2x}\sin(2y)) \\
&= 4e^{4x}\cos^2(2y) + 4e^{4x}\sin^2(2y) \\
&= 4e^{4x} \\
&= \det \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix},
\end{aligned}$$

which therefore shows that (i) is valid in the case $f(z) = e^{2z}$. □

(iii) The gradients of u and v, ∇u and ∇v, are:

$$\nabla u = \begin{pmatrix} \partial_x u(x,y) \\ \partial_y u(x,y) \end{pmatrix}, \nabla v = \begin{pmatrix} \partial_x v(x,y) \\ \partial_y v(x,y) \end{pmatrix}$$

We now take the inner product of these two vectors:

$$\begin{aligned}
\langle \nabla u, \nabla v \rangle &= \langle \begin{pmatrix} \partial_x u(x,y) \\ \partial_y u(x,y) \end{pmatrix}, \begin{pmatrix} \partial_x v(x,y) \\ \partial_y v(x,y) \end{pmatrix} \rangle \\
&= (\partial_x u(x,y))(\partial_x v(x,y)) + (\partial_y u(x,y))(\partial_y v(x,y))
\end{aligned}$$

Since $f \in H(\Omega)$, we can apply the Cauchy-Riemann equations:

$$\begin{aligned}
\therefore \langle \nabla u, \nabla v \rangle &= (\partial_x u(x,y))(\partial_x v(x,y)) + (\partial_y u(x,y))(\partial_y v(x,y)) \\
&= (\partial_x u(x,y))(-\partial_y u(x,y)) + (\partial_y u(x,y))(\partial_x u(x,y)) \\
&= 0
\end{aligned}$$

As the inner product of ∇u and ∇v is 0, we can conclude that ∇u and ∇v are everywhere orthogonal. $\square$

(iv)

$$\begin{aligned} f(z) &= z^2 \\ &= (x+yi)^2 \\ &= x^2 + 2xyi - y^2 \\ &= (x^2 - y^2) + (2xy)i \\ &\Rightarrow u(x,y) = x^2 - y^2,\ v(x,y) = 2xy \end{aligned}$$

∇u and ∇v are then:

$$\nabla u = \begin{pmatrix} 2x \\ -2y \end{pmatrix},\ \nabla v = \begin{pmatrix} 2y \\ 2x \end{pmatrix}$$

We then evaluate the inner product:

$$\begin{aligned} \langle \nabla u, \nabla v \rangle &= \langle \begin{pmatrix} 2x \\ -2y \end{pmatrix}, \begin{pmatrix} 2y \\ 2x \end{pmatrix} \rangle \\ &= (2x)(2y) + (-2y)(2x) \\ &= 4xy - 4xy \\ &= 0, \end{aligned}$$

which shows that ∇u and ∇v are orthogonal in the case of $f(z) = z^2$, which proves that (iii) is valid. $\square$

Problem 2

(i) As $f = u + iv \in H(\Omega)$, the Cauchy-Riemann equations must hold, and we can solve them to obtain $v(x,y)$. We have that $\partial_x u = 2(1-y)$ and $\partial_y u = -2x$, which gives:

$$\begin{aligned} \partial_x v &= -\partial_y u = 2x \\ \partial_y v &= \partial_x u = 2(1-y) \end{aligned}$$

Applying the antiderivative w.r.t x, we obtain:

$$\begin{aligned} v(x,y) &= x^2 + C(y) \\ \Rightarrow \partial_y v &= C'(y) = 2(1-y) \\ \Rightarrow C(y) &= 2y - y^2 + C,\ C \in \mathbb{R} \\ \Rightarrow &\boxed{v(x,y) = x^2 + 2y - y^2 + C} \end{aligned}$$

(ii) For a function $f = u + iv$ to be entire, then $\Delta u = \Delta v = 0$ in $\mathbb{C}$. When $v(x,y) = 3x^3y - 2x^2 + 5xy^2 - 1$:

$$\begin{aligned} \Delta v(x,y) &= \partial^2_{xx} v + \partial^2_{yy} v \\ &= \partial_x(9x^2y - 4x + 5y^2) + \partial_y(3x^3 + 10y) \\ &= 19xy - 4 + 10 \\ &= 19xy - 6, \end{aligned}$$

which is only 0 on the set of points where $\{xy = \frac{6}{19},\ x \in \mathbb{R},\ y \in \mathbb{R}\}$, which is only a subset of $\mathbb{C}$. Therefore, there is no **entire** function $f = u + vi$ with $v(x,y) = 3x^3y - 2x^2 + 5xy^2 - 1$. □

Problem 3

For this problem, let $z = x + yi$.

(i) The hyperbolic sine function can be expressed as:

$$\sinh(z) := \frac{1}{2}(e^z - e^{-z})$$

$$
\begin{aligned}
\therefore \sinh(2z) = i \Leftrightarrow \frac{1}{2}(e^{2z} - e^{-2z}) &= i \\
\frac{1}{2}e^{-2z}(e^{4z} - 1) &= i \\
e^{4z} - 1 &= 2ie^{2z} \\
e^{4z} - 2ie^{2z} - 1 &= 0 \\
(e^{2z} - i)^2 - i^2 - 1 &= 0 \\
(e^{2z} - i)^2 &= 0 \\
e^{2z} &= i \\
e^{2x+2yi} &= i \\
e^{2x}e^{2yi} &= i \\
e^{2x}e^{2yi} &= e^{\frac{\pi}{2}ni},\ n \in \mathbb{Z} \\
&\Rightarrow e^{2x} = 1,\ 2y = \frac{n\pi}{2} \\
&\Leftrightarrow x = 0,\ y = \frac{n\pi}{4}
\end{aligned}
$$

The set of solutions is therefore $\{\frac{n\pi}{4}i \,:\, n \in \mathbb{Z}\}$.

(ii) Cosine and sine can be expressed as:

$$
\cos(z) := \frac{1}{2}(e^{iz} + e^{-iz}),\ \sin(z) := \frac{1}{2i}(e^{iz} - e^{-iz})
$$

$$
\begin{aligned}
\therefore 2\cos(z) &= i\sin(z) \\
(e^{iz} + e^{-iz}) &= \frac{1}{2}(e^{iz} - e^{-iz}) \\
e^{iz} + e^{-iz} &= \frac{1}{2}e^{iz} - \frac{1}{2}e^{-iz} \\
\frac{1}{2}e^{iz} + \frac{3}{2}e^{-iz} &= 0 \\
e^{iz} + 3e^{-iz} &= 0 \\
e^{iz} &= -3e^{-iz} \\
e^{2iz} &= -3 \\
e^{-2y+2ix} &= 3e^{n\pi i} \\
e^{-2y}e^{2ix} &= 3e^{n\pi i} \\
&\Rightarrow e^{-2y} = 3,\ 2x = n\pi \\
&\Rightarrow -2y = \ln(3),\ x = \frac{n\pi}{2} \\
&\Rightarrow x = \frac{n\pi}{2},\ y = -\frac{1}{2}\ln(3)
\end{aligned}
$$

The set of solutions is therefore $\{\frac{n\pi}{2} - \frac{1}{2}i\ln(3) \,:\, n \in \mathbb{Z}\}$

(iii) Let $w = z + i$. We then have:

$$\begin{aligned}(z-i)^4 &= (z+i)^4\\ w^4 &= \overline{w}^4\\ \Rightarrow w &= \overline{w}\\ \Rightarrow \mathrm{Im}(w) &= 0 \Rightarrow \mathrm{Im}(z+i) = 0\\ \Rightarrow \mathrm{Im}(x+yi+i) &= 0\\ \Rightarrow \mathrm{Im}((x+(y+1)i)) &= 0\\ \Rightarrow y+1 &= 0\\ \Rightarrow y &= -1\end{aligned}$$

The set of solutions is therefore $\{x - i \,:\, x \in \mathbb{R}\}$

Problem 4

(i) Let $z = x + yi$.

$$\begin{aligned}\sin(z) &= \frac{1}{2i}(e^{iz} - e^{-iz})\\ &= \frac{1}{2i}(e^{ix-y} - e^{-ix+y})\\ &= \frac{1}{2i}(e^{ix}e^{-y} - e^{-ix}e^{y})\\ &= \frac{1}{2i}(e^{-y}(\cos(x) + i\sin(x)) - e^{y}(\cos(x) - i\sin(x)))\\ &= \frac{1}{2i}(e^{-y}\cos(x) + ie^{-y}\sin(x) - e^{y}\cos(x) + ie^{y}\sin(x))\\ &= \frac{1}{2i}((e^{-y}\cos(x) - e^{y}\cos(x)) + i(e^{-y}sin(x) + e^{y}\sin(x)))\\ &= \frac{1}{2}\sin(x)(e^{-y} + e^{y}) - \frac{1}{2}i\cos(x)(e^{-y} - e^{y})\\ \Rightarrow \mathrm{Re}(\sin(z)) &= \frac{1}{2}\sin(x)(e^{y} + e^{-y})\\ &= \sin(x)(\frac{1}{2}(e^{y} + e^{-y}))\\ &= \sin(x)\cosh(y),\ \text{from}\ \cosh(z) = \frac{1}{2}(e^{z} + e^{-z}),\ \forall z \in \mathbb{C}\\ &= \sin(\mathrm{Re}(z))\cosh(\mathrm{Im}(z))\end{aligned}$$

$\square$

(ii) We have that:

$$\begin{aligned}\cos(z) &:= \frac{1}{2}(e^{iz} + e^{-iz}) \\ &= \frac{1}{2}(e^{ire^{i\theta}} + e^{-ire^{i\theta}}) \\ &= \frac{1}{2}(e^{ir(\cos(\theta)+i\sin(\theta))} + e^{-ir(\cos(\theta)+i\sin(\theta))}) \\ &= \frac{1}{2}(e^{ir\cos(\theta)}e^{-r\sin(\theta)} + e^{-ir\cos(\theta)}e^{r\sin(\theta)})\end{aligned}$$

Problem 5

The function $\mathrm{Log}(z)$ is defined as:

$$\mathrm{Log}(z) := \ln|z| + i\mathrm{Arg}(z)$$

(i)

$$\begin{aligned}|-1-i| &= \sqrt{(-1)^2 + (-1)^2} \\ &= \sqrt{2} \\ \mathrm{Arg}(-1-i) &= \arctan\left(\frac{-1}{-1}\right) \\ &= \frac{\pi}{4} \\ \boxed{\therefore \mathrm{Log}(-1-i) = \ln\left(\sqrt{2}\right) + \frac{\pi}{4}i}\end{aligned}$$

(ii)

$$\begin{aligned}|2e^{3\pi i}| &= 2 \\ \mathrm{Arg}(2e^{3\pi i}) &= 3\pi \mod 2\pi \\ &= 0, \text{ as } \mathrm{Arg}(z) \in (-\pi, \pi) \\ \boxed{\therefore \mathrm{Log}(2e^{3\pi i}) = \ln(2)}\end{aligned}$$

(iii) We have that $(-1-i\sqrt{3})^2 = (1+i\sqrt{3})^2 = 1+2i\sqrt{3}-3 = -2+2i\sqrt{3}$:

$$\begin{aligned}
\therefore |(-1-i\sqrt{3})^2| &= \sqrt{(-2)^2+(2\sqrt{3})^2} = 4 \\
\mathrm{Arg}(-1-i\sqrt{3})^2) &= \arctan\left(\frac{2\sqrt{3}}{-2}\right) \\
&= \arctan\left(\frac{\sqrt{3}}{-1}\right) \\
&= \frac{2\pi}{3} \\
\Rightarrow \mathrm{Log}((-1-i\sqrt{3})^2) &= \ln(4) + \frac{2\pi}{3}i = 2\ln(2) + 2(\frac{\pi}{3})i \\
&\boxed{\therefore \mathrm{Log}((-1-i\sqrt{3})^2) = 2(\ln(2) + \frac{\pi}{3}i)}
\end{aligned}$$

For $(-1-i\sqrt{3})$:

$$\begin{aligned}
|(-1-i\sqrt{3})| &= \sqrt{(-1)^2+(-\sqrt{3})^2} = 2 \\
\mathrm{Arg}(-1-i\sqrt{3}) &= \arctan\left(\frac{-\sqrt{3}}{-1}\right) = \frac{\pi}{3} \\
&\boxed{\therefore \mathrm{Log}(-1-i\sqrt{3}) = \ln(2) + \frac{\pi}{3}i}
\end{aligned}$$

We can therefore observe that:

$$\mathrm{Log}((-1-i\sqrt{3})^2) = 2\mathrm{Log}(-1-i\sqrt{3}),$$

which could imply that the rules for logarithms in the real domain could also apply in $\mathbb{C}$, namely the rule:

$$\log(x^n) = n\log(x),\ \{x \in \mathbb{R},\, n \in \mathbb{R} : x > 0\}$$

(iv) We have that:

$$\begin{aligned}
\frac{1}{z} &= z^{-1} \\
&= \frac{x - yi}{x^2 + y^2} \\
&= \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2} i \\
|\frac{1}{z}| &= \sqrt{\frac{x^2 + y^2}{(x^2 + y^2)^2}} \\
&= \sqrt{\frac{1}{x^2 + y^2}} \\
\mathrm{Arg}(\frac{1}{z}) &= \arctan\left(\frac{-\frac{y}{x^2+y^2}}{\frac{x}{x^2+y^2}}\right) \\
&= -\arctan\left(\frac{y}{x}\right) \\
\therefore \mathrm{Log}(\frac{1}{z}) &= \ln\left(\sqrt{\frac{1}{x^2 + y^2}}\right) - i\arctan\left(\frac{y}{x}\right) \\
&= \frac{1}{2}\ln\left(\frac{1}{x^2 + y^2}\right) - i\arctan\left(\frac{y}{x}\right) \\
&= -\frac{1}{2}\ln\left(x^2 + y^2\right) - i\arctan\left(\frac{y}{x}\right)
\end{aligned}$$

We also have that:

$$\begin{aligned}
|z| &= \sqrt{x^2 + y^2} \\
\mathrm{Arg}|z| &= \arctan\left(\frac{y}{x}\right) \\
\mathrm{Log}(z) &= \ln\left(\sqrt{x^2 + y^2}\right) + i\arctan\left(\frac{y}{x}\right) \\
&= \frac{1}{2}\ln\left(x^2 + y^2\right) + i\arctan\left(\frac{y}{x}\right) \\
&= -\mathrm{Log}(\frac{1}{z})
\end{aligned}$$

As $\mathrm{Log}(\frac{1}{z}) = -\mathrm{Log}(z)$, $\{z \in \mathbb{C} : |z| > 0\}$, the only $z \in \mathbb{C}$ where $\mathrm{Log}(\frac{1}{z}) \neq \mathrm{Log}(z)$ is where $\mathrm{Arg}(z)$ is discontinuous, that is along $(-\infty, 0]$.