MATH 305:201, 2020W T2

Homework set 6 — due March 05

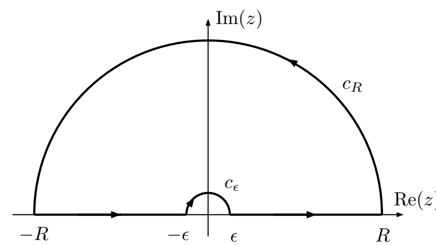
Problem 1. Compute

$$\oint_\alpha \frac{z - \mathrm{i} + 1}{z^2 - z(1+\mathrm{i})} \mathrm{d}z$$

along the following three circles, all positively oriented:
(i) The circle of radius 2 centred at $z_0 = 1$
(ii) The circle of radius 1 centred at $z_0 = 1 + \mathrm{i}$
(iii) The circle of radius 1/2 centred at $z_0 = 2$

Problem 2. Use the function $f(z) = \frac{\mathrm{e}^{\mathrm{i}z}}{z}$ and the contour sketched in the figure to show that

$$\int_0^\infty \frac{\sin(x)}{x} \mathrm{d}x = \frac{\pi}{2}.$$



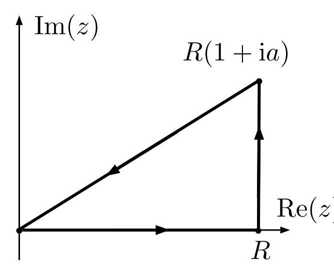
Hints: (a) Since the integrand is even and continuous at $x = 0$,

$$\int_0^\infty \frac{\sin(x)}{x} \mathrm{d}x = \frac{1}{2} \lim_{R\to\infty} \lim_{\epsilon\to 0} \left[\int_{-R}^{-\epsilon} \frac{\sin(x)}{x} \mathrm{d}x + \int_\epsilon^R \frac{\sin(x)}{x} \mathrm{d}x \right].$$

(b) On the two curves c_ϵ and c_R, you are allowed to exchange the order to integrating and taking limits.

Problem 3. Let $0 < a \le 1$. Use the contour sketched in the figure to show that

$$\int_0^\infty \mathrm{e}^{-(1+\mathrm{i}a)^2 t^2} \mathrm{d}t = \frac{\sqrt{\pi}}{2(1+\mathrm{i}a)}.$$



Conclude that

$$\int_0^\infty \cos(t^2) dt = \int_0^\infty \sin(t^2) dt = \sqrt{\frac{\pi}{8}}.$$

Hint: You can use that $\int_0^\infty \mathrm{e}^{-t^2} dt = \sqrt{\pi}/2$. While evaluating the integral along the vertical segment, you may find the bound $t^2 \le t$ useful, which is valid for $0 \le t \le 1$.

Problem 4. Let f, g be two holomorphic functions in a domain Ω. Let α be a simple closed curve in Ω such that its complete interior lies in Ω. Show that if $f(z) = g(z)$ for every $z \in \alpha$, then $f(z) = g(z)$ for all z in the interior of α.