Assignment 2

MATH 305 - Applied Complex Analysis

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1. For this problem, let $z = x + yi$, $w = u + vi$.

   i) LHS:

$$\begin{aligned}\overline{z+w} &= \overline{(x+yi)+(u+vi)} \\ &= \overline{(x+u)+(y+v)i} \\ &= (x+u)-(y+v)i\end{aligned}$$

   RHS:

$$\begin{aligned}\bar{z}+\bar{w} &= (x-yi)+(u-vi)i \\ &= (x+u)+(-y-v)i \\ &= (x+u)-(y+v)i \\ &= \text{LHS}\end{aligned}$$

$\square$

   ii) LHS:

$$\begin{aligned}\overline{zw} &= \overline{(x+yi)(u+vi)} \\ &= \overline{(xu-yv)+(xv+yu)i} \\ &= (xu-yv)-(xv+yu)i\end{aligned}$$

   RHS:

$$\begin{aligned}\bar{z}\bar{w} &= (x-yi)(u-vi) \\ &= (xu-yv)+(-xv-yu)i \\ &= (xu-yv)-(xy+yu)i \\ &= \text{LHS}\end{aligned}$$

$\square$

   iii) LHS:

$$\begin{aligned}|\bar{z}| &= |(x-yi)| \\ &= \sqrt{x^2+(-y)^2} \\ &= \sqrt{x^2+y^2}\end{aligned}$$

RHS:

$$\begin{aligned}
|z| &= |x + yi| \\
&= \sqrt{x^2 + y^2} \\
&= \text{LHS}
\end{aligned}$$

$\square$

iv) From Problem 1.3 above, we have seen that:

$$|z| = \sqrt{x^2 + y^2}$$

$$\begin{aligned}
|\operatorname{Re}(z)| &= |x| \\
\therefore |x| &\leq \sqrt{x^2 + y^2} \\
\Rightarrow |\operatorname{Re}(z)| &\leq |z| \\
|\operatorname{Im}(z)| &= |y| \\
\therefore |y| &\leq \sqrt{x^2 + y^2} \\
\Rightarrow |\operatorname{Im}(z)| &\leq |z|
\end{aligned}$$

$\square$

v) We have that:

$$\begin{aligned}
z\bar{w} &= (x + yi)(u - vi) \\
&= (xu + yv) + (-xv + yu)i
\end{aligned}$$

LHS:

$$\begin{aligned}
|z + w|^2 &= (z + w)\overline{z + w} \\
&= (z + w)(\bar{z} + \bar{w}), \text{ as proven in Problem 1.1} \\
&= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} \\
&= |z|^2 + z\bar{w} + w\bar{z} + |w|^2 \\
&= |z|^2 + |w|^2 + (x + yi)(u - vi) + (u + vi)(x - yi) \\
&= |z|^2 + |w|^2 + (xu + yv) + (-xv + yu)i + (ux + yv) + (-yu + xv)i \\
&= |z|^2 + |w|^2 + (xu + yv) + \cancel{(yu - xv)i} + (xu + yv) \cancel{- (yu - xv)i} \\
&= |z|^2 + |w|^2 + 2(xu + yv) \\
&= |z|^2 + |w|^2 + 2\operatorname{Re}(z\bar{w}), \text{ using the result above} \\
&= \text{RHS}
\end{aligned}$$

$\square$

vi) From Problem 1.5, we have:

$$\begin{aligned}
|z+w|^2 &= |z|^2 + |w|^2 + 2\,\mathrm{Re}(z\bar{w}) \\
|z+w|^2 &\leq |z|^2 + |w|^2 + 2|z\bar{w}|, \text{ from Problem 1.4} \\
|z+w|^2 &\leq |z|^2 + |w|^2 + 2|z||\bar{w}|, \text{ from Problem 1.2} \\
|z+w|^2 &\leq |z|^2 + |w|^2 + 2|z||w|, \text{ from Problem 1.3} \\
|z+w|^2 &\leq (|z| + |w|)^2, \text{ from } (a+b)^2 = (a^2 + 2ab + b^2) \\
|z+w| &\leq |z| + |w|, \text{ after taking the square root of both sides}
\end{aligned}$$

□

vii) We have that:

$$\begin{aligned}
|-z| &= |(-x - yi)| \\
&= \sqrt{(-x)^2 + (-y)^2} \\
&= \sqrt{x^2 + y^2} \\
&= |z|, \qquad (1)
\end{aligned}$$

and that the AM-GM inequality is:

$$\frac{a+b}{2} \geq \sqrt{ab}, \{a, b : a \in \mathbb{R}^+, b \in \mathbb{R}^+\} \qquad (2)$$

With the above, and the triangle inequality from Problem 1.6, we have:

$$\begin{aligned}
|z-w| &\geq \Big||z|-|w|\Big| \\
|z-w|^2 &\geq (|z|-|w|)^2 \\
|z-w|^2 &\geq |z|^2 - 2|z||w| + |w|^2 \\
|z|^2 + |-w|^2 + 2\operatorname{Re}(z(-\overline{w})) &\geq |z|^2 - 2|z||w| + |w|^2, \text{ from Problem 1.5} \\
\cancel{|z|^2} + \cancel{|w|^2} + 2\operatorname{Re}((x+yi)(-(u-vi))) &\geq \cancel{|z|^2} - 2|z||w| + \cancel{|w|^2}, \text{ from Equation 1} \\
\cancel{2}\operatorname{Re}((x+yi)(-(u-vi))) &\geq -\cancel{2}|z||w| \\
-(xu+yv) &\geq -\sqrt{x^2+y^2}\sqrt{u^2+v^2} \\
(xu+yv) &\leq \sqrt{x^2+y^2}\sqrt{u^2+v^2} \\
(xu+yv)^2 &\leq (x^2+y^2)(u^2+v^2) \\
\cancel{(xu)^2} + 2xuyv + \cancel{(yv)^2} &\leq \cancel{(xu)^2} + (xv)^2 + (yu)^2 + \cancel{(yv)^2} \\
2xuyv &\leq (xv)^2 + (yu)^2 \\
\frac{(xv)^2+(yu)^2}{2} &\geq xvyu \\
\frac{(xv)^2+(yu)^2}{2} &\geq \sqrt{(xv)^2(yu)^2} \\
\equiv \frac{a+b}{2} &\geq \sqrt{ab},
\end{aligned}$$

where $a = (xv)^2$, $b = (yu)^2$. As the inequality reduces to the AM-GM inequality (which is known), the inequality $|z-w| \geq ||z|-|w||$ is valid. □

2. i) We have that $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$:

$$\begin{aligned}
\sin(3\theta) &= \operatorname{Im}(e^{i3\theta}) \\
&= \operatorname{Im}((e^{i\theta})^3) \\
&= \operatorname{Im}(\cos^3(\theta) + i^3(\sin^3(\theta)) + 3\cos(\theta)(i\sin(\theta))(\cos(\theta)+i\sin(\theta))) \\
&= \operatorname{Im}(cos^3(\theta) + ii^2(\sin^3(\theta)) + 3i\cos(\theta)\sin(\theta)(\cos(\theta)+i\sin(\theta))) \\
&= \operatorname{Im}(\cos^3(\theta) - i(\sin^3(\theta)) + 3i\cos^2(\theta)\sin(\theta) - 3\cos(\theta)\sin(\theta))) \\
&= \operatorname{Im}((\cos^3(\theta) - 3\cos(\theta)\sin(\theta)) + (3\cos^2(\theta)\sin(\theta) - \sin^3(\theta))i) \\
&= 3\cos^2(\theta)\sin(\theta) - \sin^3(\theta)
\end{aligned}$$

□

ii)

$$\begin{aligned}\sin(\theta-\psi) &= \operatorname{Im}(e^{i(\theta-\psi)})\\ &= \operatorname{Im}(e^{i\theta}e^{-i\psi})\\ &= \operatorname{Im}([\cos(\theta)+i\sin(\theta)][\cos(\psi)-i\sin(\psi)])\\ &= \operatorname{Im}(\cos(\theta)\cos(\psi)-i\cos(\theta)\sin(\psi)+i\sin(\theta)\cos(\psi)+\sin(\theta)\sin(\psi))\\ &= \sin(\theta)\cos(\psi)-\cos(\theta)\sin(\psi)\end{aligned}$$

$\square$

3. All sketches in this problem are vector (`.svg`) files, and can be zoomed in without loss in quality.

   i) The sketch for the domain $\Omega \subset \mathbb{C}$ is shown in Figure 1.

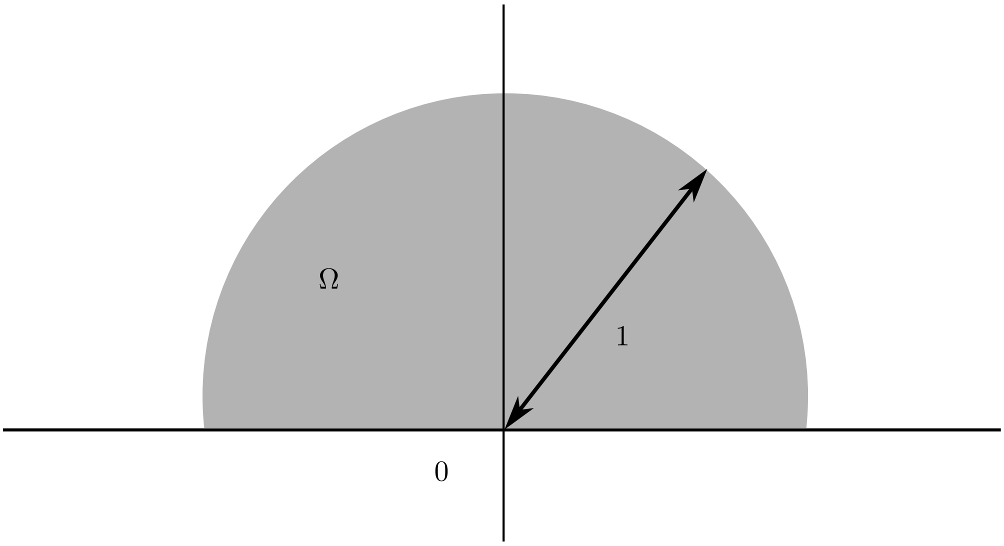

Figure 1

We can observe that the mapping f is a composite of three functions, i.e. $f(z) = (f_3 \circ f_2 \circ f_1)(z)$, where:

$$\begin{aligned} f_1(z) &= e^{i\frac{\pi}{2}}z, \text{ which represents a rotation of } \frac{\pi}{2},\\ f_2(w) &= 2w, \text{ which represents a dilation of the set by factor 2},\\ f_3(s) &= s+(2+2i), \text{ which represents a translation by } (2+2i),\end{aligned}$$

which can be verified by finding $f_3(f_2(f_1(z)))$:

$$\begin{aligned} f_3(f_2(f_1(z))) &= f_3(f_2(e^{i\frac{\pi}{2}})) \\ &= f_3(2e^{i\frac{\pi}{2}}) \\ &= 2e^{i\frac{\pi}{2}} + (2+2i) \\ &= f(z) \end{aligned}$$

$\square$

Therfore, the image of the domain Ω, $f(\Omega)$, is shown in Figure 2

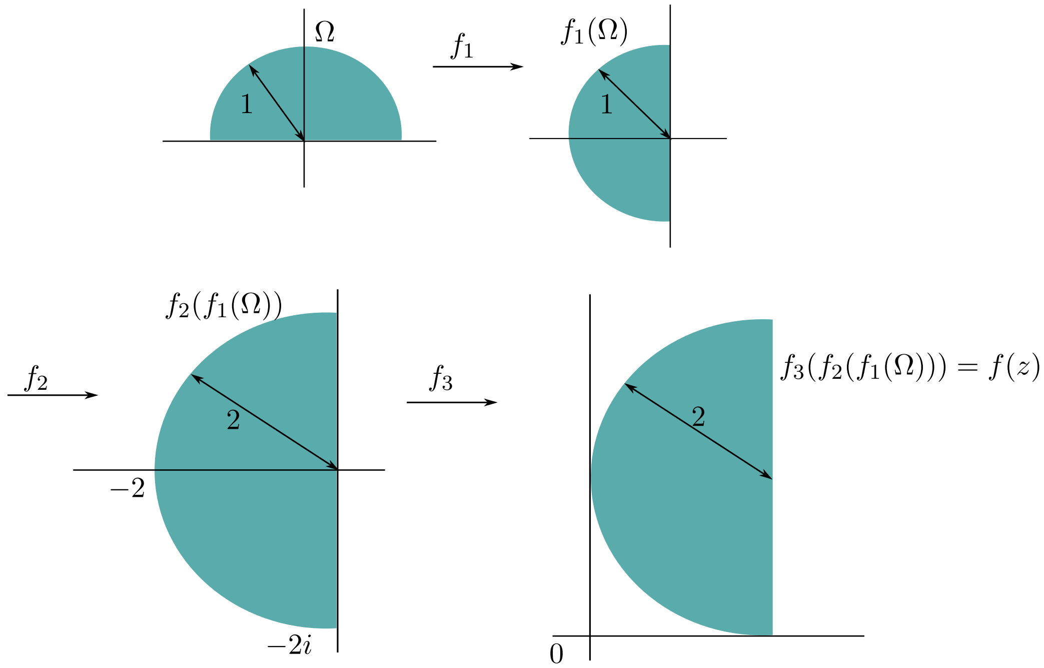

Figure 2

ii) The sketch for the domain $\Omega \subset \mathbb{C}$ is shown in Figure 3. We define the set $\{\zeta \in \mathbb{C} \,|\, \zeta = e^{\frac{\pi}{2}z},\ \text{for } -1 < \mathrm{Im}(z) < 1\}$:

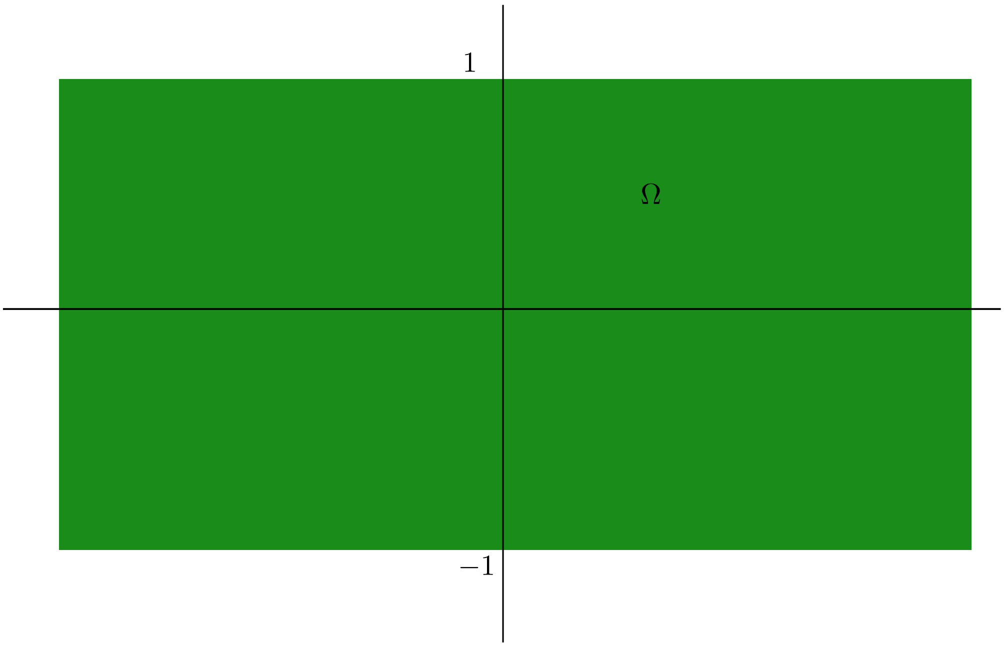

Figure 3

$$
\begin{aligned}
\zeta &= e^{\frac{\pi}{2}z} \\
&= (e^{z})^{\frac{\pi}{2}} \\
&= (e^{\operatorname{Re}(z)+\operatorname{Im}(z)i})^{\frac{\pi}{2}} \\
&= (e^{\operatorname{Re}(z)}\, e^{\operatorname{Im}(z)i})^{\frac{\pi}{2}} \\
&= e^{\frac{\pi}{2}\operatorname{Re}(z)}(e^{\frac{\pi}{2}i\operatorname{Im}(z)}) \\
&= e^{\frac{\pi}{2}\operatorname{Re}(z)}(\cos\Big(\frac{\pi}{2}\operatorname{Im}(z)\Big) + i\sin\Big(\frac{\pi}{2}\operatorname{Im}(z)\Big)) \\
&= e^{\frac{\pi}{2}\operatorname{Re}(z)}\cos\Big(\frac{\pi}{2}\operatorname{Im}(z)\Big) + ie^{\frac{\pi}{2}\operatorname{Re}(z)}\sin\Big(\frac{\pi}{2}\operatorname{Im}(z)\Big) \\
\Rightarrow \operatorname{Im}(\zeta) &= e^{\frac{\pi}{2}\operatorname{Re}(z)}\sin\Big(\frac{\pi}{2}\operatorname{Im}(z)\Big) \\
\sin\Big(\frac{\pi}{2}\operatorname{Im}(z)\Big) &= e^{-\frac{\pi}{2}\operatorname{Re}(z)}\operatorname{Im}(\zeta) \\
\therefore \operatorname{Im}(z) &= \frac{2}{\pi}\arcsin\Big(e^{-\frac{\pi}{2}\operatorname{Re}(z)}\operatorname{Im}(\zeta)\Big)
\end{aligned}
$$

Now, we have:

$$
\begin{aligned}
-1 < \operatorname{Im}(z) < 1 \Leftrightarrow -1 < \frac{2}{\pi}\arcsin\Big(e^{-\frac{\pi}{2}\operatorname{Re}(z)}\operatorname{Im}(\zeta)\Big) < 1 \\
-\frac{\pi}{2} < \arcsin\Big(e^{-\frac{\pi}{2}\operatorname{Re}(z)}\operatorname{Im}(\zeta)\Big) < \frac{\pi}{2} \\
-\sin\Big(\frac{\pi}{2}\Big) < e^{-\frac{\pi}{2}\operatorname{Re}(z)}\operatorname{Im}(\zeta) < \sin\Big(\frac{\pi}{2}\Big) \\
-e^{\frac{\pi}{2}\operatorname{Re}(z)} < \operatorname{Im}(\zeta) < e^{\frac{\pi}{2}\operatorname{Re}(z)},
\end{aligned}
$$

which gives the image of f as:

$$f(\Omega) = \{\zeta \in \mathbb{C} \,|\, -e^{\frac{\pi}{2}\operatorname{Re}(z)} < \operatorname{Im}(\zeta) < e^{\frac{\pi}{2}\operatorname{Re}(z)}\},$$

which has the sketch as shown in Figure 4.

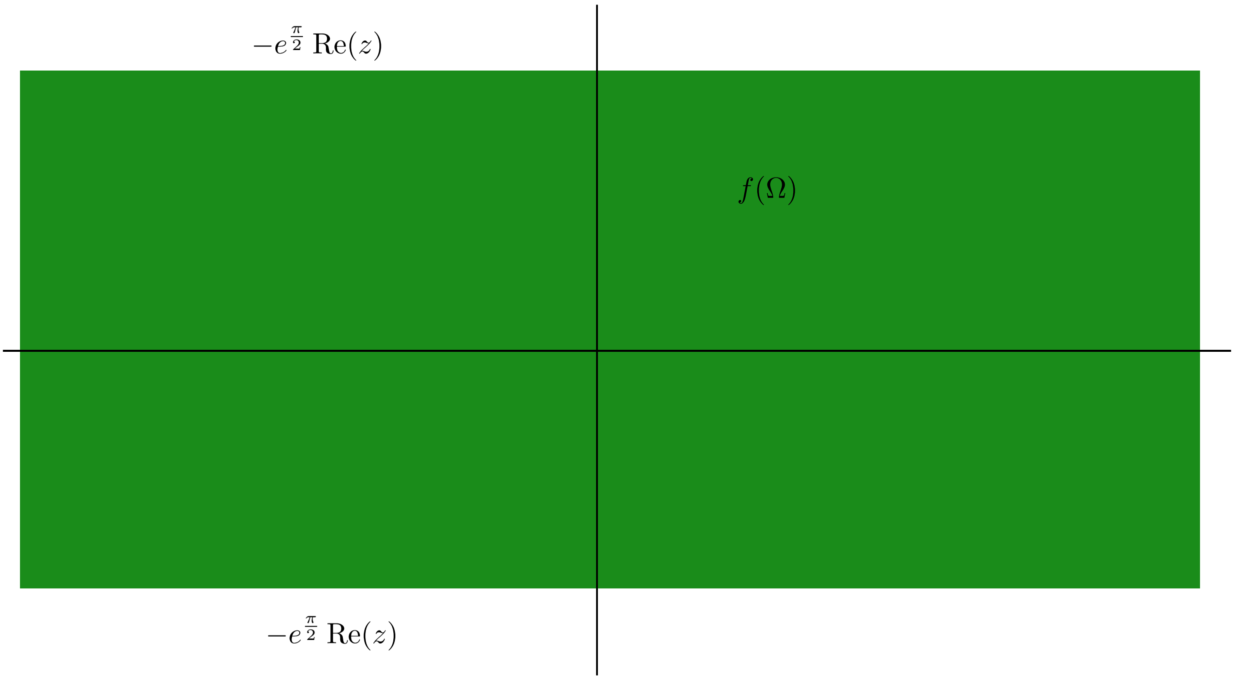

Figure 4

iii) The sketch for the domain $\Omega \subset \mathbb{C}$ is shown in Figure 5. We define a set $\{\zeta \in \mathbb{C} \,|\, \zeta = \frac{z-1}{z+1}, \operatorname{Re}(z) > 0\}$. Let $\operatorname{Re}(z) = x$, $\operatorname{Im}(z) = y$, $\operatorname{Re}(\zeta) = u$, $\operatorname{Im}(\zeta) = v$:

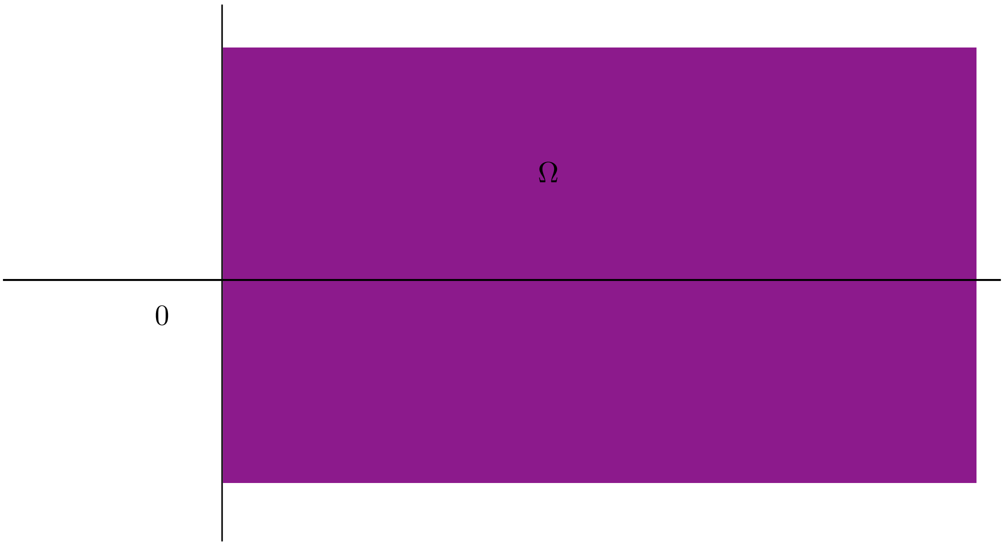

Figure 5

$$
\begin{aligned}
\zeta &= \frac{z-1}{z+1} \\
z &= \frac{1+\zeta}{\zeta-1} \\
&= \frac{(1+u)+vi}{(u-1)+vi} \\
&= ((1+u)+vi)((u-1)+vi)^{-1} \\
&= ((1+u)+vi)\left(\frac{u-1}{(u-1)^2+v^2} - i\frac{v}{(u-1)^2+v^2}\right) \\
&= \left(\frac{(u+1)(u-1)}{(u-1)^2+v^2} + \frac{v^2}{(u-1)^2+v^2}\right) + \dots \\
\therefore \operatorname{Re}(z) &= \left(\frac{(u+1)(u-1)}{(u-1)^2+v^2} + \frac{v^2}{(u-1)^2+v^2}\right),
\end{aligned}
$$

which now gives us:

$$
\begin{aligned}
\operatorname{Re}(z) > 0 \Leftrightarrow \left(\frac{(u+1)(u-1)}{(u-1)^2+v^2} + \frac{v^2}{(u-1)^2+v^2}\right) &> 0 \\
\frac{(u+1)(u-1)}{\cancel{(u-1)^2+v^2}} > -\frac{v^2}{\cancel{(u-1)^2+v^2}} \\
(u+1)(u-1) > -v^2 \\
u^2-1 > -v^2 \\
u^2+v^2 > 1 \\
\sqrt{u^2+v^2} > 1 \\
\Rightarrow |\zeta| > 1,
\end{aligned}
$$

as $|\zeta| = \sqrt{u^2+v^2}$, which gives the image:

$$f(\Omega) = \{\zeta \in \mathbb{C} \mid |\zeta| > 1\},$$

which has a sketch as shown in Figure 6.

4. i) Since $f \in H(\Omega)$, the Cauchy-Riemann equations:

$$
\begin{cases}
\partial_x u(x,y) = \partial_y v(x,y), \\
\partial_x v(x,y) = -\partial_y u(x,y),
\end{cases}
$$

must be satisfied. As f is real valued, it will be of the form:

$$f = u(x,y) + iv(x,y),$$

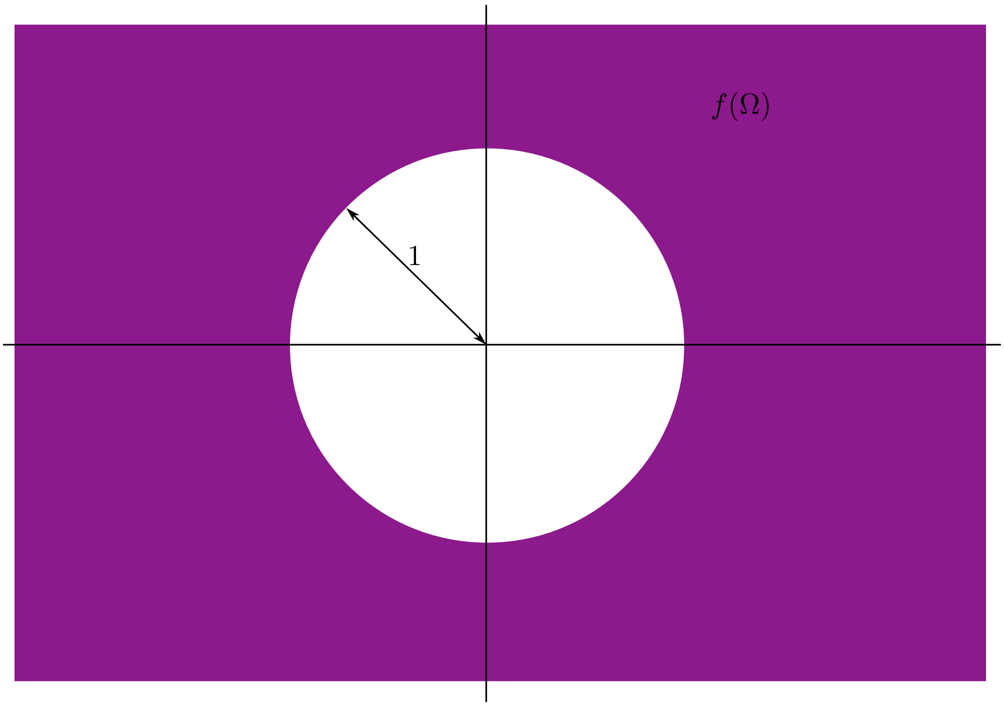

Figure 6

where $v(x, y) = 0$. Therefore, we have that:

$$\partial_x v(x, y) = 0,$$
$$\partial_y v(x, y) = 0,$$

and since the Cauchy-Riemann equations must be satisfied, we must also assert that:

$$\partial_x u(x, y) = \partial_y v(x, y) = 0$$
$$\partial_y u(x, y) = -\partial_x v(x, y) = 0$$

As both $\partial_x u(x, y)$ and $\partial_y u(x, y)$ are zero, we must have that $u(x, y)$ is constant with respect to both x and y, and since f is real-valued as well, we can conclude that:

$$f = u(x, y), \text{ where } u(x, y) = a, \, \{a \in \mathbb{R}\}, \text{ for } f \in \Omega$$

$\square$

ii) Let $f = u + vi$, $\bar{f} = u - vi$. As both $f, \bar{f} \in H(\Omega)$, they must both satisfy the Cauchy-

Riemann equations. For $\bar{f}$, we have:

$$\partial_x u = -\partial_y v$$
$$-\partial_x v = -\partial_y u$$

Similarly, for f we have:

$$\partial_x u = \partial_y v$$
$$\partial_x v = -\partial_y u$$

As both these sets of equations must be true, we can observe that:

$$\partial_x u = \partial_y v = -\partial_y v = 0,$$

as only 0 can satisfy the equation $a = -a$. Similarly, we can also obtain that

$$\partial_x v = \partial_y u = -\partial_y u = 0$$

As $\partial_x u = \partial_x v = \partial_y u = \partial_y v = 0$, we can conclude that $u(x, y)$ and $v(x, y)$ are independent of x, y, and that f is therefore constant on Ω. $\square$

5. The Cauchy-Riemann equations are:

$$\begin{cases} \partial_x u(x,y) = \partial_y v(x,y), \\ \partial_x v(x,y) = -\partial_y u(x,y), \end{cases}$$

where $z = x + yi$, $f(z) = u + vi$. If the Cauchy-Riemann equations are satisfied, we have that:

$$f'(z) = \partial_x u(x,y) + i\partial_x v(x,y) = \partial_y v(x,y) - i\partial_y u(x,y)$$

i) We have that $u(x,y) = e^{-2xy}\cos(x^2 - y^2)$, $v(x,y) = e^{-2xy}\sin(x^2 - y^2)$:

$$\begin{aligned} \therefore \partial_x u(x,y) &= -2ye^{-2xy}\cos(x^2 - y^2) - 2xe^{-2xy}(\sin(x^2 - y^2) \\ \partial_y v(x,y) &= -2xe^{-2xy}\sin(x^2 - y^2) + (-2y)e^{-2xy}\cos(x^2 - y^2) \\ &= -2ye^{-2xy}\cos(x^2 - y^2) - 2xe^{-2xy}\sin(x^2 - y^2) \\ \Rightarrow \partial_y v(x,y) &= \partial_x u(x,y), \text{ which satisfies the first Cauchy-Riemann equation} \end{aligned}$$

$$\begin{aligned}
\partial_x v(x,y) &= -2ye^{-2xy}\sin\left(x^2-y^2\right) + 2xe^{-2xy}\cos\left(x^2-y^2\right) \\
\partial_y u(x,y) &= -2xe^{-2xy}\cos\left(x^2-y^2\right) + 2ye^{-2xy}\sin\left(x^2-y^2\right) \\
&= -(2xe^{-2xy}\cos\left(x^2-y^2\right) - 2ye^{-2xy}\sin\left(x^2-y^2\right)) \\
&= -\partial_x v(x,y), \text{ which satisfies the second Cauchy-Riemann equation}
\end{aligned}$$

As both Cauchy-Riemann equations are satisfied $\forall x$, y, the function:

$$f(x+iy) = e^{-2xy}(\cos\left(x^2-y^2\right) + i\sin\left(x^2-y^2\right))$$

is entire. $\square$

Its derivative is therefore:

$$\begin{aligned}
f'(z) &= \partial_x u(x,y) + i\partial_x v(x,y) \\
&= [-2ye^{-2xy}\cos\left(x^2-y^2\right) - 2xe^{-2xy}(\sin\left(x^2-y^2\right)] \\
&+ i[-2ye^{-2xy}\sin\left(x^2-y^2\right) + 2xe^{-2xy}\cos\left(x^2-y^2\right)] \\
&= -2e^{-2xy}(y\cos\left(x^2-y^2\right) + x\sin\left(x^2-y^2\right)) + i2e^{-2xy}(x\cos\left(x^2-y^2\right) - y\sin\left(x^2-y^2\right))
\end{aligned}$$

ii) We have that:

$$\begin{aligned}
g(z) &= \frac{1}{2}(e^{iz} + e^{-iz}) \\
&= \frac{1}{2}(e^{ix+i^2y} + e^{-(ix+i^2y)}) \\
&= \frac{1}{2}(e^{ix-y} + e^{-ix+y}) \\
&= \frac{1}{2}(e^{ix}e^{-y} + e^{-ix}e^{y}) \\
&= \frac{1}{2}(e^{-y}(\cos(x) + i\sin(x)) + e^{y}(\cos(x) - i\sin(x))) \\
&= \frac{1}{2}(e^{-y} + e^{y})\cos(x) + \frac{1}{2}i(e^{-y} - e^{y})\sin(x),
\end{aligned}$$

which gives:

$$u = \frac{1}{2}(e^{-y} + e^{y})\cos(x),\; v = \frac{1}{2}(e^{-y} - e^{y})\sin(x)$$

Now, we calculate:

$$\begin{aligned}
\partial_x u &= -\frac{1}{2}(e^{-y} + e^{y}) \sin(x) \\
\partial_y v &= \frac{1}{2}(-e^{-y} - e^{y}) \sin(x) \\
&= -\frac{1}{2}(e^{-y} + e^{y}) \sin(x) \\
&= \partial_x u \\
&\Rightarrow \text{ the first Cauchy-Riemann equation is satisfied} \\
\partial_x v &= \frac{1}{2}(e^{-y} - e^{y}) \cos(x) \\
\partial_y u &= \frac{1}{2}(-e^{-y} + e^{y}) \cos(x) \\
&= -\frac{1}{2}(e^{-y} - e^{y}) \cos(x) \\
&= -\partial_x v \\
&\Rightarrow \text{ the second Cauchy-Riemann equation is satisfied} \\
&\Rightarrow \text{ the function } g(z) \text{ is entire}
\end{aligned}$$

□

We can therefore compute its derivative:

$$\begin{aligned}
g'(z) &= \partial_x u + i\partial_x v \\
&= -\frac{1}{2}(e^{-y} + e^{y}) \sin(x) + \frac{1}{2}i(e^{-y} - e^{y}) \cos(x)
\end{aligned}$$