Assignment 9

MATH 305 - Applied Complex Analysis

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Problem 1

i) f_1 has an essential singularity at $z_0 = 0$, with the Laurent series:

$$\begin{aligned}
f_1(z) &= z^2(1 + \frac{1}{z} + \frac{1}{2}(\frac{1}{z^2}) + \frac{1}{3!}(\frac{1}{z^3}) + \dots) \\
&= z^2 + z + \frac{1}{2} + \frac{1}{6}(\frac{1}{z}) + \dots \\
&= \dots + \frac{1}{6}z^{-1} + \dots \\
\Rightarrow a^{-1} &= \frac{1}{6} = \text{Res}(f_1\,;\,0)
\end{aligned}$$

ii) f_2 has a simple pole at $z_0 = 0$ and a pole of order 2 at $z_1 = \frac{\pi}{2}$. For the simple pole:

$$\begin{aligned}
\text{Res}(f_2\,;\,0) &= \lim_{z \to z_0}(z)(\frac{\cos(2z)}{z(z-\frac{\pi}{2})^2}) \\
&= \lim_{z \to 0}(\frac{\cos(2z)}{(z-\frac{\pi}{2})^2}) \\
&= \frac{1}{\frac{\pi^2}{4}} \\
&= \frac{4}{\pi^2}
\end{aligned}$$

For the pole of order 2:

$$\begin{aligned}
\text{Res}\Big(f_2\,;\,\frac{\pi}{2}\Big) &= \lim_{z \to \frac{\pi}{2}} \frac{d}{dz}((z-\frac{\pi}{2})^2(\frac{\cos(2z)}{z(z-\frac{\pi}{2})^2})) \\
&= \lim_{z \to \frac{\pi}{2}} \frac{d}{dz}(\frac{\cos(2z)}{z}) \\
&= \lim_{z \to \frac{\pi}{2}} -(\frac{2\sin(2z)}{z} + \frac{\cos(2z)}{z^2}) \\
&= -(\frac{2\sin(\pi)}{\frac{\pi}{2}} + \frac{\cos(\pi)}{\frac{\pi^2}{4}}) \\
&= -(-\frac{4}{\pi^2}) \\
&= \frac{4}{\pi^2}
\end{aligned}$$

iii) a) The essential singularity $z_0 = 0$ lies in the unit circle. Laurent's Theorem then gives:

$$\begin{aligned}\oint_{|z|=1} f_1(z) &= 2\pi i a_{-1} = 2\pi i \operatorname{Res}(f_1\,;\,0) \\ &= 2\pi i(\frac{1}{6}) \\ &= \frac{\pi}{3} i\end{aligned}$$

b) Both residues of f_2 are equal and lie in the unit circle. The Residue Theorem then gives:

$$\begin{aligned}\oint_{|z|=1} f_2(z) &= 2\pi i(\frac{4}{\pi^2} + \frac{4}{\pi^2}) \\ &= \frac{16}{\pi} i\end{aligned}$$

Problem 2

i) We have the identity $\tan(z) = \sin(z)/\cos(z)$, and we can therefore observe that $\tan(z)$ has 2 simple poles at $z_0 = \frac{\pi}{2}$ and $z_1 = \frac{3\pi}{2}$ that lie in the contour $|z| = 2\pi$. Then:

$$\begin{aligned}\operatorname{Res}(\tan(z)\,;\,z_0) &= \frac{\sin(z_0)}{(\cos(z_0))'} = \frac{\sin\left(\frac{\pi}{2}\right)}{-\sin\left(\frac{3\pi}{2}\right)} \\ &= -1 \\ \operatorname{Res}(\tan(z)\,;\,z_1) &= \frac{\sin(z_1)}{(\cos(z_1))'} = \frac{\sin\left(\frac{3\pi}{2}\right)}{-\sin\left(\frac{3\pi}{2}\right)} \\ &= -1\end{aligned}$$

The Residue Theorem then gives:

$$\begin{aligned}\oint_{|z|=2\pi} \tan(z) &= 2\pi i(-1 + -1) \\ &= -4\pi i\end{aligned}$$

ii) We use the function $f(z) = \frac{e^{2z}}{\cosh(\pi z)}$. We have that:

$$\begin{aligned}f(z) &= \frac{e^{2z}}{\cosh(\pi z)} = \frac{e^{2z}}{\cos(i\pi z)} \\ &= \frac{e^{2z}}{\frac{1}{2}(e^{i(i\pi z)} + e^{-i(i\pi z)})}\end{aligned}$$

$$
\begin{aligned}
f(z) &= \frac{e^{2z}}{\frac{1}{2}(e^{-\pi z}+e^{\pi z})} \\
&= \frac{2e^{2z}}{e^{-\pi z}(1+e^{2\pi z})} \\
&= \frac{2e^{(2+\pi)z}}{1+e^{2\pi z}}
\end{aligned}
$$

We note that:

$$
\begin{aligned}
1+e^{2\pi z} = 0 \text{ if and only if} \\
2\pi z = n\pi i, \, \forall n \in \mathbb{Z}\backslash\{0\} \\
\Rightarrow z = \frac{n}{2} i,
\end{aligned}
$$

meaning $f(z)$ has singularities $\frac{n}{2}i$ for all positive non-zero integer values of n. We now consider the positively oriented countour as shown in Figure 1.

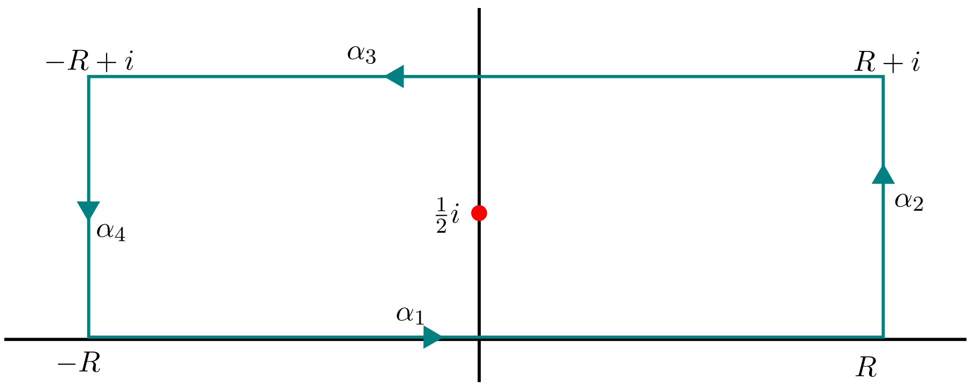

Figure 1

We assume (for now) that the subsequent poles of $f(z)$ (i.e. πi, $\frac{3\pi}{2}i$, ...) lie outside this contour. By the Residue Theorem, we have (for this rectangular contour):

$$
\begin{aligned}
\oint_{\alpha} f(z)\,dz &= \int_{\alpha_1} f(z)\,dz + \int_{\alpha_2} f(z)\,dz + \int_{\alpha_3} f(z)\,dz + \int_{\alpha_4} f(z)\,dz \\
&= 2\pi i \,\mathrm{Res}\left(f\,;\,\frac{1}{2}i\right) \\
&= 2\pi i \,\mathrm{Res}\left(\frac{2e^{(2+\pi)z}}{1+e^{2\pi z}}\,;\,\frac{1}{2}i\right) \\
&= 2\pi i \left(\frac{2e^{(2+\pi)z}}{2\pi e^{2\pi z}}\right)\Big|_{z=\frac{1}{2}i}
\end{aligned}
$$

$$\begin{aligned}\oint_{\alpha} f(z)\,dz &= \frac{2e^{(2+\pi)(\frac{1}{2}i)}}{e^{2\pi(\frac{1}{2}i)}}i \\ &= \frac{2e^{i+i\pi}}{e^{i\pi}}i \\ &= -2ie^{i}\end{aligned}$$

Let $\alpha_2(t) = R + ti$ for $t \in [0,1]$. This gives:

$$\begin{aligned}\ell(\alpha_2) &= \int_0^1 |i|\,dt \\ &= 1\end{aligned}$$

Therefore:

$$\left| \int_{\alpha_2} f(z)\,dz \right| \leq \ell(\alpha_2)\ \max\{|f(\alpha_2(t))|\ :\ t \in [0,1]\} \leq \frac{2e^{(2+\pi)R+(2+\pi)ti}}{1+e^{2\pi R}e^{2\pi ti}}$$

For large R, we can estimate that:

$$\frac{2e^{(2+\pi)R+(2+\pi)ti}}{1+e^{2\pi R}e^{2\pi ti}} \approx \frac{e^{2+\pi R}}{e^{2\pi R}}$$

Since $2\pi > 2+\pi$, we then have that:

$$\left| \int_{\alpha_2} f(z)\,dz \right| \leq \frac{e^{2+\pi R}}{e^{2\pi R}} \to 0,\ \text{ as } R \to \infty$$

We can conclude similarly that:

$$\left| \int_{\alpha_4} f(z)\,dz \right| \to 0 \text{ as } R \to \infty$$

This leaves us with:

$$\int_{\alpha_1} f(z)\,dz + \int_{\alpha_3} f(z)\,dz = -2ie^{i}$$

Let $\alpha_1(x) = x$ for $x \in [-R,R]$; $\alpha_3(x) = x+i$ for $x \in [R,-R]$:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)}(1)\,dx + \int_{\infty}^{-\infty} \frac{e^{2(x+i)}}{\frac{1}{2}(e^{\pi x+\pi i}+e^{-\pi x-\pi i})}(1) &= -2ie^{i} \\ \int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)}\,dx - \int_{-\infty}^{\infty} \frac{e^{2x}e^{2i}}{\frac{1}{2}(e^{\pi x}e^{\pi i}+e^{-\pi x}e^{-\pi i})}\,dx &= -2ie^{i} \\ \int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)}\,dx - e^{2i}\int_{-\infty}^{\infty} \frac{e^{2x}}{\frac{1}{2}e^{\pi i}(e^{\pi x}+e^{-\pi x})}\,dx &= -2ie^{i}\end{aligned}$$

$$\int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)}\, dx + e^{2i} \int_{-\infty}^{\infty} \frac{e^{2x}}{\frac{1}{2}(e^{\pi x} + e^{-\pi x})}\, dx = -2ie^{i}$$
$$\int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)}\, dx + e^{2i} \int_{-\infty}^{\infty} \frac{e^{-2x}}{\cosh(\pi x)}\, dx = -2ie^{i}$$
$$(1 + e^{2i}) \int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)}\, dx = -2ie^{i}$$
$$\int_{-\infty}^{\infty} \frac{e^{2x}}{\cosh(\pi x)}\, dx = -\frac{2ie^{i}}{1 + e^{2i}}$$

Problem 3

Let $P(z) = c(z - z_1)(z - z_2)\ldots(z - z_d)$. By the product rule (according to Wikipedia), we have:

$$P'(z) = c(\prod_{j=1}^{d}(z - z_j))(\sum_{j=1}^{d}\frac{1}{z - z_j}) \tag{1}$$

Using similar notation, we can express $P(z)$ as:

$$P(z) = c\prod_{j=1}^{d}(z - z_j) \tag{2}$$

Taking (1)/(2) then gives:

$$\begin{aligned}
\frac{P'(z)}{P(z)} &= \frac{c(\cancel{\prod_{j=1}^{d}(z - z_j)})(\sum_{j=1}^{d}\frac{1}{z-z_j})}{c\cancel{\prod_{j=1}^{d}(z - z_j)}} \\
&= \sum_{j=1}^{d}\frac{1}{z - z_j} \\
&= \frac{1}{z - z_1} + \frac{1}{z - z_2} + \frac{1}{z - z_3} + \ldots \frac{1}{z - z_d}
\end{aligned}$$

The residue of $P'(z)/P(z)$ at z_j, for $j \in [1, d]$, is defined as the coefficient of the term(s) w.r.t z^{-1}, and we can observe that all terms in the above expression are z^{-1} terms. Therefore, we can conclude that the function $P'(z)/P(z)$ has d residues, and that:

$$\begin{cases} \mathrm{Res}\left(\frac{P'(z)}{P(z)}\,;\, z_j\right) = 1, & \text{if } z_j \in \mathrm{int}(\alpha), \\ \mathrm{Res}\left(\frac{P'(z)}{P(z)}\,;\, z_j\right) = 0, & \text{if } z_j \notin \mathrm{int}(\alpha), \end{cases} \tag{3}$$

By the residue theorem:

$$\oint_{\alpha} \frac{P'(z)}{P(z)}\, dz = 2\pi i \sum_{z_j \in \mathrm{int}(\alpha)} \mathrm{Res}\left(\frac{P'(z)}{P(z)}\,;\, z_j\right)$$

$$\begin{aligned}\frac{1}{2\pi i}\oint_{\alpha}\frac{P'(z)}{P(z)}\,dz &= \sum_{z_j\in\text{int}(\alpha)}\text{Res}\left(\frac{P'(z)}{P(z)}\,;\,z_j\right)\\ &= \sum_{z_j\in\text{int}(\alpha)} 1,\ \text{from (3)}\\ &= N,\end{aligned}$$

as it is given that N counts the number of zeros of $P \in \text{int}(\alpha)$. $\square$

Problem 4

Let $f(z) = \frac{1}{z^2}$. From the lecture notes, we then have:

$$\begin{aligned}\text{Res}\left(\frac{\pi\cot(\pi z)}{z^2}\,;\,n\right) &= \frac{1}{n^2},\ \forall n\in\mathbb{Z}\backslash\{0\}\\ \therefore \frac{1}{2\pi i}\oint_{\alpha}\frac{\pi\cot(\pi z)}{z^2} &= \sum_{j=-\infty}^{-1}\frac{1}{j^2}+\sum_{j=1}^{\infty}\frac{1}{j^2}+\text{Res}\left(\frac{\pi\cot(\pi z)}{z^2}\,;\,0\right)\end{aligned}$$

We can observe that:

$$\begin{aligned}\frac{1}{(-n)^2} &= \frac{1}{n^2},\ \forall n\in\mathbb{Z}\backslash\{0\}\\ \therefore \frac{1}{2\pi i}\oint_{\alpha}\frac{\pi\cot(\pi z)}{z^2} &= 2\sum_{j=1}^{\infty}\frac{1}{j^2}+\text{Res}\left(\frac{\pi\cot(\pi z)}{z^2}\,;\,0\right)\end{aligned} \tag{4}$$

For the residue at the pole of order 3 at $z_0 = 0$:

$$\begin{aligned}\text{Res}\left(\frac{\pi\cot(\pi z)}{z^2}\,;\,0\right) &= \frac{1}{(3-1)!}\lim_{z\to 0}\frac{d^{3-1}}{dz^{3-1}}z^3\frac{\pi\cot(\pi z)}{z^2}\\ &= \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\pi z\cot(\pi z)\\ &= \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\frac{\pi z\cos(\pi z)}{\sin(\pi z)}\\ &= \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\frac{\pi z\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(\pi z)^{2n}}{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(\pi z)^{2n+1}}\\ &= \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\frac{\cancel{\pi z}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(\pi z)^{2n}}{\cancel{\pi z}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(\pi z)^{2n}}\\ &= \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\frac{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(\pi z)^{2n}}{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(\pi z)^{2n}}\end{aligned}$$

$$\begin{aligned}
\text{Res}\left(\frac{\pi\cot(\pi z)}{z^2}\,;\,0\right) &= \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\frac{1-\frac{1}{2}\pi^2z^2+\frac{1}{4!}\pi^4z^4-\frac{1}{6!}\pi^6z^6+\ldots}{1-\frac{1}{3!}\pi^2z^2+\frac{1}{5!}\pi^4z^4-\frac{1}{7!}\pi^6z^6+\ldots} \\
&= \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}(1-\frac{1}{3}\pi^2z^2+\ldots),\ \text{by long division} \\
&= \frac{1}{2}\lim_{z\to 0}\frac{d}{dz}(-\frac{2}{3}(\pi^2 z)+\ldots) \\
&= \frac{1}{2}\lim_{z\to 0}(-\frac{2}{3}\pi^2+\ldots) \\
&= \frac{1}{\cancel{2}}(-\frac{\cancel{2}}{3}\pi^2) \\
&= -\frac{\pi^2}{3}
\end{aligned}$$

Combining this result with (4) gives:

$$\begin{aligned}
\frac{1}{2\pi i}\oint_\alpha \frac{\pi\cot(\pi z)}{z^2} &= 2\sum_{j=1}^{\infty}\frac{1}{j^2}-\frac{\pi^2}{3} \\
\oint_\alpha \frac{\pi\cot(\pi z)}{z^2} &= 4\pi i\sum_{j=1}^{\infty}\frac{1}{j^2}-\frac{2\pi^3}{3}i
\end{aligned}$$

We use a square contour α with side length $2N+1$. This gives us $\ell(\alpha)=4(2N+1)=8N+4$. As $|\cot(\pi z)|$ is bounded away from the real integers:

$$\therefore \left|\oint_\alpha \frac{\pi\cot(\pi z)}{z^2}\right| \le \frac{c_N(8N+4)}{N^2}\underbrace{\approx\frac{1}{N}}_{\substack{\text{for}\\ \text{large}\\ \text{N}}}\to 0 \text{ as } N\to\infty$$

where c_N is some constant obtained from $|\cot(\pi z)|$. This gives us:

$$\begin{aligned}
4\pi i\sum_{j=1}^{\infty}\frac{1}{j^2}-\frac{2\pi^3}{3}i &= 0 \\
4\pi i\sum_{j=1}^{\infty}\frac{1}{j^2} &= \frac{2\pi^3}{3}i \\
2\sum_{j=1}^{\infty}\frac{1}{j^2} &= \frac{\pi^2}{3}
\end{aligned}$$

$$\sum_{j=1}^{\infty}\frac{1}{j^2}=\frac{\pi^2}{6} \quad \square$$