Homework set 10 — due Wednesday April 7

In this sheet, you can use without further justification that

$$\sin(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} z^{2j+1}.$$

Problem 1. Compute the real integral

$$I = \int_0^{2\pi} \frac{\sin^2(\varphi)}{5 + 4\cos(\varphi)} \, d\varphi.$$

Problem 2. Compute

$$\int_{|z|=\frac{3}{2}} \frac{z^2(z-1)}{\sin^2(\pi z)} \, dz$$

Problem 3. (i) Find the Laurent series of $f(z) = z^3 \sin(\frac{1}{2z})$ in $|z| > 0$. What kind of singularity is the point 0?
(ii) Determine the singular part of f in $C(2; 0, 1)$ and in $C(0; 0, \infty)$.
(iii) Compute

$$\oint_{|z|=1} z^3 \sin\left(\frac{1}{2z}\right) dz$$

Problem 4. Consider the function $f(z) = \frac{1}{1+z^2}$.
(i) Compute its Taylor series in the disk $|z - 1| < \sqrt{2}$.
(ii) Compute its Laurent series in the following two annuli

$$
\begin{aligned}
&\text{(a)} \quad C(i; 0, 2), \\
&\text{(b)} \quad C(i; 2, \infty).
\end{aligned}
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