Assignment 6

MATH 305 - Applied Complex Analysis

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Problem 1

The denominator can be expressed as $z^2 - z(1+i) = (z)(z-(1+i))$, i.e. there are two singularities at $z = 0$, and $z = 1+i$.

i) $\alpha(t) = B_2(1)$. We can partition $\alpha(t)$ into $\alpha_1(t)$, which contains the singularity $z = 0$, and $\alpha_2(t)$, which contains the singularity $z = 1+i$.

$$\begin{aligned}\oint_\alpha \frac{z-i+1}{z^2-z(1+i)}\,dz &= \oint_{\alpha_1} \frac{z-i+1}{z^2-z(1+i)}\,dz + \oint_{\alpha_2} \frac{z-i+1}{z^2-z(1+i)}\,dz \\ &= \oint_{\alpha_1} \frac{\frac{z-i+1}{z}}{z-(1+i)}\,dz + \oint_{\alpha_2} \frac{\frac{z-i+1}{z-(1+i)}}{z}\,dz\end{aligned}$$

For $\oint_{\alpha_1} \frac{(z-i+1)/z}{z-(1+i)}\,dz$, let $f_1(z) = \frac{z-i+1}{z}$, and $w_1 = 1+i$. Cauchy's Integral Theorem gives:

$$\begin{aligned}\oint_{\alpha_1} \frac{\frac{z-i+1}{z}}{z-(1+i)}\,dz &= 2\pi i\, f_1(1+i) \\ &= 2\pi i(\frac{(1+i)-i+1}{1+i}) \\ &= 2\pi i(\frac{2}{1+i}) \\ &= \frac{4\pi i}{1+i} \qquad (1)\end{aligned}$$

For $\oint_{\alpha_2} \frac{(z-i+1)/(z-(1+i))}{z}\,dz$, let $f_2(z) = \frac{z-i+1}{z-(1+i)}$, and $w_2 = 0$. Similarly, we have:

$$\begin{aligned}\oint_{\alpha_2} \frac{\frac{z-i+1}{z-(1+i)}}{z}\,dz &= 2\pi i\, f_2(0) \\ &= 2\pi i(\frac{1-i}{-(1+i)}) \\ &= \frac{2\pi i(i-1)}{1+i} \\ &= \frac{2\pi(-1-i)}{1+i} \\ &= \frac{-2\pi\cancel{(1+i)}}{\cancel{(1+i)}} \\ &= -2\pi\end{aligned}$$

$$\therefore \oint_\alpha \frac{z-i+1}{z^2-z(1+i)}\,dz = -2\pi + \frac{4\pi i}{1+i}$$

ii) Let $\beta = B_1(1+i)$. β only contains the singularity $z = 1+i$, so we can use result (1) above:

$$\oint_\beta \frac{z-i+1}{z^2-z(1+i)}\,dz = \oint_{\alpha_1} \frac{\frac{z-i+1}{z}}{z-(1+i)}\,dz = \frac{4\pi i}{1+i}$$

iii) Let $\gamma = B_{\frac{1}{2}}(2)$. γ does not contain either of the singularity points, so by Cauchy's Theorem:

$$\oint_\gamma \frac{z-i+1}{z^2 - z(1+i)}\, dz = 0$$

Problem 2

Let the countour be α_R:

$$\begin{cases} c_\epsilon(t) = \epsilon e^{i(\pi - t)}, & t \in [0, \pi], \\ c_1(t) = t, & t \in [\epsilon, R], \\ c_R(t) = Re^{it}, & t \in [0, \pi], \\ c_2(t) = t, & t \in [-R, -\epsilon], \end{cases} \Rightarrow \begin{cases} c'_\epsilon(t) = -i\epsilon e^{i(\pi - t)}, & t \in [0, \pi], \\ c'_1(t) = 1, & t \in [\epsilon, R], \\ c'_R(t) = iRe^{it}, & t \in [0, \pi], \\ c'_2(t) = 1, & t \in [-R, -\epsilon], \end{cases}$$

As the contour is closed, by Cauchy's Theorem:

$$\oint_{\alpha_R} f(z)\, dz = \oint_{\alpha_R} \frac{e^{iz}}{z}\, dz = 0, \tag{2}$$

where $f(z)$ can be expressed as:

$$\begin{aligned} f(z) &= \frac{e^{-y}e^{ix}}{z} \\ &= \frac{e^{-y}(\cos(x) + i\sin(x))}{z} \\ &= \frac{e^{-y}}{z}(\cos(x) + i\sin(x)) \end{aligned}$$

where $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$. Similarly with the segments c_ϵ and c_R:

$$\begin{aligned} c_\epsilon(t) &= \epsilon e^{i\pi} e^{-it} \\ &= -\epsilon(\cos(t) - i\sin(t)) \\ &= -\epsilon\cos(t) + i\epsilon\sin(t), \end{aligned}$$

$$\begin{aligned} c_R(t) &= R(\cos(t) + i\sin(t)) \\ &= R\cos(t) + iR\sin(t), \end{aligned}$$

For the segment c_ϵ:

$$\begin{aligned}\int_{c_\epsilon} f(z)\,dz &= \int_0^\pi \left(\frac{e^{-\epsilon\sin(t)}}{\cancel{\epsilon e^{i(\pi-t)}}}\right)(\cos(-\epsilon\cos(t)) + i\sin(-\epsilon\cos(t)))(-i\cancel{\epsilon e^{i(\pi-t)}})\,dt \\ &= -i\int_0^\pi e^{-\epsilon\sin(t)}e^{-i\epsilon\cos(t)}\,dt \\ &= -i\int_0^\pi e^{-\epsilon(\sin(t)-i\cos(t))}\,dt \end{aligned} \tag{3}$$

For the segment c_1:

$$\begin{aligned}\int_{c_1} f(z)\,dz &= \int_\epsilon^R \frac{e^{ix}}{x}\,dx \\ &= \int_\epsilon^R \frac{\cos(x)}{x} + i\frac{\sin(x)}{x}\,dx \\ &= \int_\epsilon^R \frac{\cos(x)}{x}\,dx + i\int_\epsilon^R \frac{\sin(x)}{x}\,dx \end{aligned} \tag{4}$$

For the segment c_R:

$$\begin{aligned}\int_{c_R} f(z)\,dz &= \int_0^\pi \left(\frac{e^{R\sin(t)}}{\cancel{Re^{it}}}\right)(\cos(R\cos(t)) + i\sin(R\cos(t)))(i\cancel{Re^{it}})\,dt \\ &= i\int_0^\pi e^{R\sin(t)}e^{iR\cos(t)}\,dt \\ &= i\int_0^\pi e^{R(\sin(t)+i\cos(t))}\,dt \end{aligned} \tag{5}$$

For the segment c_2:

$$\begin{aligned}\int_{c_2} f(z)\,dz &= \int_{-R}^{-\epsilon} \frac{e^{ix}}{x}\,dx \\ &= \int_{-R}^{-\epsilon} \frac{\cos(x)}{x} + i\frac{\sin(x)}{x}\,dx \\ &= \int_{-R}^{-\epsilon} \frac{\cos(x)}{x}\,dx + i\int_{-R}^{-\epsilon} \frac{\sin(x)}{x}\,dx \end{aligned} \tag{6}$$

For (3), taking the limit as $\epsilon \to 0$ gives:

$$\begin{aligned}\lim_{\epsilon\to 0} -i\int_0^\pi e^{-\epsilon(\sin(t)-i\cos(t))}\,dt &= -i\int_0^\pi e^{-0}\,dt \\ &= -i\int_0^\pi 1\,dt \\ &= -\pi i \end{aligned} \tag{7}$$

Taking the limit of (5) as $R \to \infty$ gives:

$$\lim_{R\to\infty} i\int_0^\pi e^{R(\sin(t)+i\cos(t))}\,dt = 0 \tag{8}$$

Taking the limit of (4)+(6) as $R \to \infty$ and $\epsilon \to 0$ gives:

$$\begin{aligned}
&\lim_{R\to\infty}\lim_{\epsilon\to 0}\left(\int_\epsilon^R \frac{\cos(x)}{x}\,dx + i\int_\epsilon^R \frac{\sin(x)}{x}\,dx + \int_{-R}^{-\epsilon} \frac{\cos(x)}{x}\,dx + i\int_{-R}^{-\epsilon} \frac{\sin(x)}{x}\,dx\right) \\
&= \lim_{R\to\infty}\lim_{\epsilon\to 0}\left(\int_\epsilon^R \frac{\cos(x)}{x}\,dx + \int_{-R}^{-\epsilon} \frac{\cos(x)}{x}\,dx\right) + i\left(\lim_{R\to\infty}\lim_{\epsilon\to 0}\left(\int_{-R}^{-\epsilon} \frac{\sin(x)}{x}\,dx\right)\right) \\
&= \lim_{R\to\infty}\lim_{\epsilon\to 0}\left(\int_\epsilon^R \frac{\cos(x)}{x}\,dx + \int_{-R}^{-\epsilon} \frac{\cos(x)}{x}\,dx\right) + 2i\int_0^\infty \frac{\sin(x)}{x}\,dx, \text{ by Hint (a)} \qquad (9)
\end{aligned}$$

Combining (7), (8), (9), and substituting into (2) then gives:

$$\begin{aligned}
-\pi i + 0 + \lim_{R\to\infty}\lim_{\epsilon\to 0}\left(\int_\epsilon^R \frac{\cos(x)}{x}\,dx + \int_{-R}^{-\epsilon} \frac{\cos(x)}{x}\,dx\right) + 2i\int_0^\infty \frac{\sin(x)}{x}\,dx &= 0 \\
\lim_{R\to\infty}\lim_{\epsilon\to 0}\left(\int_\epsilon^R \frac{\cos(x)}{x}\,dx + \int_{-R}^{-\epsilon} \frac{\cos(x)}{x}\,dx\right) + 2i\int_0^\infty \frac{\sin(x)}{x}\,dx &= \pi i
\end{aligned}$$

Taking Im(LHS) = Im(RHS) gives:

$$\begin{aligned}
2\int_0^\infty \frac{\sin(x)}{x}\,dx &= \pi \\
\int_0^\infty \frac{\sin(x)}{x}\,dx &= \frac{\pi}{2}
\end{aligned}$$

$\square$

Problem 3

Let the countour $\gamma(t)$ be parametrized as follows:

$$\begin{cases} \gamma_1(t) = t, & t \in [0, R], \\ \gamma_2(t) = R(1+it), & t \in [0, a], \\ \gamma_3(t) = t(1+ia), & t \in [0, R], \end{cases} \Rightarrow \begin{cases} \gamma_1'(t) = 1, \\ \gamma_2'(t) = Ri, \\ \gamma_3'(t) = (1+ia), \end{cases}$$

Let $f(z) = e^{-z^2}$. As $\gamma(t)$ is closed, by Cauchy's Theorem we have $\oint_\gamma f(z)\,dz = 0$:

$$\Rightarrow \int_0^R e^{-t^2}\,dt + \int_0^a e^{-R^2(1+it)^2}(Ri)\,dt \underbrace{- \int_0^R e^{-(t(1+ia)^2)}(1+ia)\,dt}_{\substack{\text{We subtract this term as} \\ \text{the orientation is opposite} \\ \text{of the parametrization} \\ \text{above}}} = 0 \qquad (10)$$

Taking the limit of the left hand side of (10) as $R \to \infty$ gives:

$$\int_0^\infty e^{-t^2}\,dt + \int_0^a e^{-\infty}(Ri)\,dt - \int_0^\infty e^{-(t(1+ia)^2)}(1+ia)\,dt = \frac{\sqrt{\pi}}{2} + 0 - (1+ia)\int_0^\infty e^{-(1+ia)^2t^2}\,dt$$

Substituting this result back into (10) gives:

$$\begin{aligned}\frac{\sqrt{\pi}}{2} - (1+ia)\int_0^\infty e^{-(1+ia)^2t^2}\,dt &= 0\\ (1+ia)\int_0^\infty e^{-(1+ia)^2t^2}\,dt &= \frac{\sqrt{\pi}}{2}\\ \int_0^\infty e^{-(1+ia)^2t^2}\,dt &= \frac{\sqrt{\pi}}{2(1+ia)}\end{aligned}$$

$\square$

From this:

$$\begin{aligned}\int_0^\infty e^{-(1+2ia-a^2)t^2}\,dt &= \frac{\sqrt{\pi}}{2(1+ia)}\\ \int_0^\infty e^{-((1-a^2)t^2+2iat^2)}\,dt &= \frac{\sqrt{\pi}}{2(1+ia)}\\ \int_0^\infty e^{(a^2-1)t^2-2iat^2}\,dt &= \frac{\sqrt{\pi}}{2}(\frac{1-ia}{1+a^2})\\ \int_0^\infty e^{(a^2-1)t^2}e^{-i2at^2}\,dt &= \frac{\sqrt{\pi}}{2(1+a^2)} - i\frac{a\sqrt{\pi}}{2(1+a^2)}\\ \int_0^\infty e^{(a^2-1)t^2}(\cos(2at^2) - i\sin(2at^2))\,dt &= \frac{\sqrt{\pi}}{2(1+a^2)} - i\frac{a\sqrt{\pi}}{2(1+a^2)}\end{aligned}$$

Taking Re(LHS)=Re(RHS):

$$\int_0^\infty e^{(a^2-1)t^2}\cos(2at^2)\,dt = \frac{\sqrt{\pi}}{2(1+a^2)}$$

Let $a = 1$:

$$\begin{aligned}\int_0^\infty e^{(1-1)t^2}\cos(2t^2)\,dt &= \frac{\sqrt{\pi}}{2(1+1)}\\ \int_0^\infty \cos(2t^2)\,dt &= \frac{\sqrt{\pi}}{4}\end{aligned}$$

Let $u = \sqrt{2}t$, $du = \sqrt{2}\,dt$:

$$\therefore \int_0^\infty \cos\left(2(\frac{u}{\sqrt{2}})^2\right)\frac{du}{\sqrt{2}} = \frac{\sqrt{\pi}}{4}$$
$$\int_0^\infty \cos\left(2(\frac{u^2}{2})\right)\frac{du}{\sqrt{2}} = \frac{\sqrt{\pi}}{4}$$
$$\frac{1}{\sqrt{2}}\int_0^\infty \cos\big(u^2\big)\,du = \frac{\sqrt{\pi}}{4}$$
$$\int_0^\infty \cos\big(u^2\big)\,du = \frac{\sqrt{2\pi}}{4}$$
$$\int_0^\infty \cos\big(u^2\big)\,du = \sqrt{\frac{\cancel{2}\pi}{\cancel{16}\,8}}$$
$$\int_0^\infty \cos\big(u^2\big)\,du = \sqrt{\frac{\pi}{8}}$$
$$\Rightarrow \int_0^\infty \cos\big(t^2\big)\,dt = \sqrt{\frac{\pi}{8}}$$

$\square$

Taking Im(LHS)=Im(RHS):

$$\int_0^\infty e^{(a^2-1)t^2}\sin\big(2at^2\big)\,dt = \frac{a\sqrt{\pi}}{2(1+a^2)}$$

Let $a = 1$:

$$\int_0^\infty e^{(1-1)t^2}\sin\big(2t^2\big)\,dt = \frac{\sqrt{\pi}}{2(1+1)}$$
$$\int_0^\infty \sin\big(2t^2\big)\,dt = \frac{\sqrt{\pi}}{4}$$

Similarly, let $u = \sqrt{2}t$, $du = \sqrt{2}\,dt$:

$$\frac{1}{\sqrt{2}}\int_0^\infty \sin\big(u^2\big)\,du = \frac{\sqrt{\pi}}{4}$$
$$\int_0^\infty \sin\big(u^2\big)\,du = \frac{\sqrt{2\pi}}{4}$$
$$\int_0^\infty \sin\big(u^2\big)\,du = \sqrt{\frac{\cancel{2}\pi}{\cancel{16}\,8}}$$
$$\int_0^\infty \sin\big(u^2\big)\,du = \sqrt{\frac{\pi}{8}}$$
$$\Rightarrow \int_0^\infty \sin\big(t^2\big)\,dt = \sqrt{\frac{\pi}{8}} = \int_0^\infty \cos\big(t^2\big)\,dt$$

$\square$

In both cases, we treat u as a 'dummy variable' and just substitute $u = t$ back into the expression.

Problem 4

Since $f(z) = g(z)\,\forall z \in \alpha$, we have that $|f(z) - g(z)| = 0$ on the curve α. By the Maximum Modulus Principle, $|f(z) - g(z)|$ reaches its maximum on the boundary of the bounded domain (in this case α). Therefore, there is no point in the interior of α such that $|f(z) - g(z)| > 0$. As the modulus of any complex number cannot be negative, this means that

$$|f(z) - g(z)| = 0 \Rightarrow f(z) = g(z),\ \forall z \in \text{int}(\alpha)$$

□