Assignment 6

MATH 305 - Applied Complex Analysis

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Problem 1

The denominator can be expressed as $z^2 - z(1+i) = (z)(z - (1+i))$, i.e. there are two singularities at $z = 0$, and $z = 1 + i$.

i) $\alpha(t) = B_2(1)$. We can partition $\alpha(t)$ into $\alpha_1(t)$, which contains the singularity $z = 0$, and $\alpha_2(t)$, which contains the singularity $z = 1 + i$.

$$\oint_\alpha \frac{z - i + 1}{z^2 - z(1+i)} \, dz = \oint_{\alpha_1} \frac{z - i + 1}{z^2 - z(1+i)} \, dz + \oint_{\alpha_2} \frac{z - i + 1}{z^2 - z(1+i)} \, dz$$

$$= \oint_{\alpha_1} \frac{\frac{z-i+1}{z}}{z - (1+i)} \, dz + \oint_{\alpha_2} \frac{\frac{z-i+1}{z-(1+i)}}{z} \, dz$$

For $\oint_{\alpha_1} \frac{(z-i+1)/z}{z-(1+i)} \, dz$, let $f_1(z) = \frac{z-i+1}{z}$, and $w_1 = 1 + i$. Cauchy's Integral Theorem gives:

$$\oint_{\alpha_1} \frac{\frac{z-i+1}{z}}{z - (1+i)} \, dz = 2\pi i \, f_1(1 + i)$$

$$= 2\pi i \left(\frac{(1+i) - i + 1}{1 + i} \right)$$

$$= 2\pi i \left(\frac{2}{1 + i} \right)$$

$$= \frac{4\pi i}{1 + i} \tag{1}$$

For $\oint_{\alpha_2} \frac{(z-i+1)/(z-(1+i))}{z} \, dz$, let $f_2(z) = \frac{z-i+1}{z-(1+i)}$, and $w_2 = 0$. Similarly, we have:

$$\oint_{\alpha_2} \frac{\frac{z-i+1}{z-(1+i)}}{z} \, dz = 2\pi i \, f_2(0)$$

$$= 2\pi i \left(\frac{1 - i}{-(1+i)} \right)$$

$$= \frac{2\pi i(i - 1)}{1 + i}$$

$$= \frac{2\pi(-1 - i)}{1 + i}$$

$$= \frac{-2\pi \cancel{(1+i)}}{\cancel{(1+i)}}$$

$$= -2\pi$$

$$\therefore \oint_\alpha \frac{z - i + 1}{z^2 - z(1+i)} \, dz = -2\pi + \frac{4\pi i}{1 + i}$$

ii) Let $\beta = B_1(1 + i)$. β only contains the singularity $z = 1 + i$, so we can use result (1) above:

$$\oint_\beta \frac{z - i + 1}{z^2 - z(1+i)} \, dz = \oint_{\alpha_1} \frac{\frac{z-i+1}{z}}{z - (1+i)} \, dz = \frac{4\pi i}{1 + i}$$

iii) Let $\gamma = B_{\frac{1}{2}}(2)$. γ does not contain either of the singularity points, so by Cauchy's Theorem:

$$\oint_\gamma \frac{z - i + 1}{z^2 - z(1 + i)}\, dz = 0$$

Problem 2

Let the countour be α_R:

$$\begin{cases} c_\epsilon(t) = \epsilon e^{i(\pi - t)}, & t \in [0, \pi], \\ c_1(t) = t, & t \in [\epsilon, R], \\ c_R(t) = Re^{it}, & t \in [0, \pi], \\ c_2(t) = t, & t \in [-R, -\epsilon], \end{cases} \Rightarrow \begin{cases} c'_\epsilon(t) = -i\epsilon e^{i(\pi - t)}, & t \in [0, \pi], \\ c'_1(t) = 1, & t \in [\epsilon, R], \\ c'_R(t) = iRe^{it}, & t \in [0, \pi], \\ c'_2(t) = 1, & t \in [-R, -\epsilon], \end{cases}$$

As the contour is closed, by Cauchy's Theorem:

$$\oint_{\alpha_R} f(z)\, dz = \oint_{\alpha_R} \frac{e^{iz}}{z}\, dz = 0, \tag{2}$$

where $f(z)$ can be expressed as:

$$\begin{aligned} f(z) &= \frac{e^{-y}e^{ix}}{z} \\ &= \frac{e^{-y}(\cos(x) + i\sin(x))}{z} \\ &= \frac{e^{-y}}{z}(\cos(x) + i\sin(x)) \end{aligned}$$

where $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$. Similarly with the segments c_ϵ and c_R:

$$\begin{aligned} c_\epsilon(t) &= \epsilon e^{i\pi} e^{-it} \\ &= -\epsilon(\cos(t) - i\sin(t)) \\ &= -\epsilon\cos(t) + i\epsilon\sin(t), \end{aligned}$$

$$\begin{aligned} c_R(t) &= R(\cos(t) + i\sin(t)) \\ &= R\cos(t) + iR\sin(t), \end{aligned}$$

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For the segment c_ϵ:

$$\int_{c_\epsilon} f(z)\,dz = \int_0^\pi \left(\frac{e^{-\epsilon\sin(t)}}{\epsilon e^{i(\pi-t)}}\right)(\cos(-\epsilon\cos(t)) + i\sin(-\epsilon\cos(t)))(-i\epsilon e^{i(\pi-t)})\,dt$$

$$= -i\int_0^\pi e^{-\epsilon\sin(t)}e^{-i\epsilon\cos(t)}\,dt$$

$$= -i\int_0^\pi e^{-\epsilon(\sin(t)-i\cos(t))}\,dt \tag{3}$$

For the segment c_1:

$$\int_{c_1} f(z)\,dz = \int_\epsilon^R \frac{e^{ix}}{x}\,dx$$

$$= \int_\epsilon^R \frac{\cos(x)}{x} + i\frac{\sin(x)}{x}\,dx$$

$$= \int_\epsilon^R \frac{\cos(x)}{x}\,dx + i\int_\epsilon^R \frac{\sin(x)}{x}\,dx \tag{4}$$

For the segment c_R:

$$\int_{c_R} f(z)\,dz = \int_0^\pi \left(\frac{e^{R\sin(t)}}{Re^{it}}\right)(\cos(R\cos(t)) + i\sin(R\cos(t)))(iRe^{it})\,dt$$

$$= i\int_0^\pi e^{R\sin(t)}e^{iR\cos(t)}\,dt$$

$$= i\int_0^\pi e^{R(\sin(t)+i\cos(t))}\,dt \tag{5}$$

For the segment c_2:

$$\int_{c_2} f(z)\,dz = \int_{-R}^{-\epsilon} \frac{e^{ix}}{x}\,dx$$

$$= \int_{-R}^{-\epsilon} \frac{\cos(x)}{x} + i\frac{\sin(x)}{x}\,dx$$

$$= \int_{-R}^{-\epsilon} \frac{\cos(x)}{x}\,dx + i\int_{-R}^{-\epsilon} \frac{\sin(x)}{x}\,dx \tag{6}$$

For (3), taking the limit as $\epsilon \to 0$ gives:

$$\lim_{\epsilon\to 0} -i\int_0^\pi e^{-\epsilon(\sin(t)-i\cos(t))}\,dt = -i\int_0^\pi e^{-0}\,dt$$

$$= -i\int_0^\pi 1\,dt$$

$$= -\pi i \tag{7}$$

Taking the limit of (5) as $R \to \infty$ gives:

$$\lim_{R\to\infty} i\int_0^\pi e^{R(\sin(t)+i\cos(t))}\,dt = 0 \tag{8}$$

Taking the limit of (4)+(6) as $R \to \infty$ and $\epsilon \to 0$ gives:

$$\lim_{R\to\infty} \lim_{\epsilon\to 0} \left(\int_\epsilon^R \frac{\cos(x)}{x}\, dx + i \int_\epsilon^R \frac{\sin(x)}{x}\, dx + \int_{-R}^{-\epsilon} \frac{\cos(x)}{x}\, dx + i \int_{-R}^{-\epsilon} \frac{\sin(x)}{x}\, dx \right)$$

$$= \lim_{R\to\infty} \lim_{\epsilon\to 0} \left(\int_\epsilon^R \frac{\cos(x)}{x}\, dx + \int_{-R}^{-\epsilon} \frac{\cos(x)}{x}\, dx \right) + i \left(\lim_{R\to\infty} \lim_{\epsilon\to 0} \left(\int_{-R}^{-\epsilon} \frac{\sin(x)}{x}\, dx \right) \right)$$

$$= \lim_{R\to\infty} \lim_{\epsilon\to 0} \left(\int_\epsilon^R \frac{\cos(x)}{x}\, dx + \int_{-R}^{-\epsilon} \frac{\cos(x)}{x}\, dx \right) + 2i \int_0^\infty \frac{\sin(x)}{x}\, dx, \text{ by Hint (a)} \qquad (9)$$

Combining (7), (8), (9), and substituting into (2) then gives:

$$-\pi i + 0 + \lim_{R\to\infty} \lim_{\epsilon\to 0} \left(\int_\epsilon^R \frac{\cos(x)}{x}\, dx + \int_{-R}^{-\epsilon} \frac{\cos(x)}{x}\, dx \right) + 2i \int_0^\infty \frac{\sin(x)}{x}\, dx = 0$$

$$\lim_{R\to\infty} \lim_{\epsilon\to 0} \left(\int_\epsilon^R \frac{\cos(x)}{x}\, dx + \int_{-R}^{-\epsilon} \frac{\cos(x)}{x}\, dx \right) + 2i \int_0^\infty \frac{\sin(x)}{x}\, dx = \pi i$$

Taking $\text{Im}(\text{LHS}) = \text{Re}(\text{RHS})$ gives:

$$2 \int_0^\infty \frac{\sin(x)}{x}\, dx = \pi$$

$$\int_0^\infty \frac{\sin(x)}{x}\, dx = \frac{\pi}{2}$$

$\square$

Problem 3

Let the countour $\gamma(t)$ be parametrized as follows:

$$\begin{cases} \gamma_1(t) = t, & t \in [0, R], \\ \gamma_2(t) = R(1 + it), & t \in [0, a], \\ \gamma_3(t) = t(1 + ia), & t \in [0, R], \end{cases} \Rightarrow \begin{cases} \gamma_1'(t) = 1, \\ \gamma_2'(t) = Ri, \\ \gamma_3'(t) = (1 + ia), \end{cases}$$

Let $f(z) = e^{-z^2}$. As $\gamma(t)$ is closed, by Cauchy's Theorem we have $\oint_\gamma f(z)\, dz = 0$:

$$\Rightarrow \int_0^R e^{-t^2}\, dt + \int_0^a e^{-R^2(1+it)^2}(Ri)\, dt - \underbrace{\int_0^R e^{-(t(1+ia)^2)}(1 + ia)\, dt}_{\substack{\text{We subtract this term as} \\ \text{the orientation is opposite} \\ \text{of the parametrization} \\ \text{above}}} = 0 \qquad (10)$$

Taking the limit of the left hand side of (10) as $R \to \infty$ gives:

$$\int_0^\infty e^{-t^2}\,dt + \int_0^a e^{-\infty}(Ri)\,dt - \int_0^\infty e^{-(t(1+ia)^2)}(1+ia)\,dt = \frac{\sqrt{\pi}}{2} + 0 - (1+ia)\int_0^\infty e^{-(1+ia)^2t^2}\,dt$$

Substituting this result back into (10) gives:

$$\frac{\sqrt{\pi}}{2} - (1+ia)\int_0^\infty e^{-(1+ia)^2t^2}\,dt = 0$$

$$(1+ia)\int_0^\infty e^{-(1+ia)^2t^2}\,dt = \frac{\sqrt{\pi}}{2}$$

$$\int_0^\infty e^{-(1+ia)^2t^2}\,dt = \frac{\sqrt{\pi}}{2(1+ia)}$$

$\square$

Problem 4

Since $f(z) = g(z)\,\forall z \in \alpha$, we have that $|f(z) - g(z)| = 0$ on the curve α. By the Maximum Modulus Principle, $|f(z) - g(z)|$ reaches its maximum on the boundary of the bounded domain (in this case α). Therefore, there is no point in the interior of α such that $|f(z) - g(z)| > 0$. As the modulus of any complex number cannot be negative, this means that

$$|f(z) - g(z)| = 0 \Rightarrow f(z) = g(z),\ \forall z \in \text{int}(\alpha)$$

$\square$