Assignment 4

MATH 305 - Applied Complex Analysis

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Problem 2

We assume the ice block has infinite length and lies on the complex plane as shown in Figure 1.

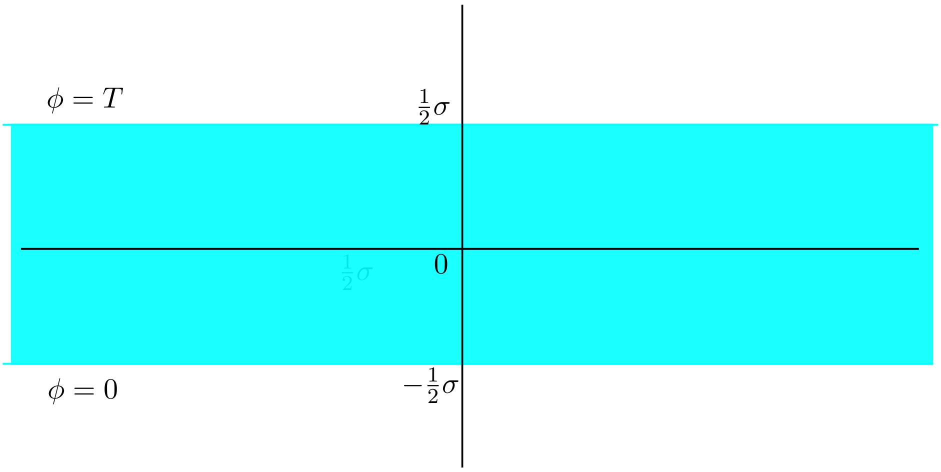

Figure 1

Let $\phi(x,y) = A\ln|z| + iB\text{Arg}(z)$. Applying the boundary conditions at $\phi = T$ and $\phi = 0$, gives:

$$\begin{cases} A\ln\left(\frac{1}{2}\sigma\right) + iB\frac{\pi}{2} = T \\ A\ln\left(\frac{1}{2}\sigma\right) - iB\frac{\pi}{2} = 0 \end{cases}$$
$$\Rightarrow \begin{cases} A = \frac{T}{2\ln\left(\frac{1}{2}\sigma\right)} \\ B = -\frac{T}{\pi}i \end{cases}$$

This gives the temperature distribution:

$$\phi(z) = \frac{T}{2\ln\left(\frac{1}{2}\sigma\right)}\ln|z| + \frac{T}{\pi}\text{Arg}(z)$$

Problem 3

(i) $\text{Log}(\frac{z-1}{z-2})$ has a branch cut on:

$$\begin{cases} \text{Im}(\frac{z-1}{z-2}) = 0 \\ \text{Re}(\frac{z-1}{z-2}) \leq 0 \end{cases}$$

$$\begin{aligned}
\Rightarrow \frac{z-1}{z-2} &= ((x-1)+yi)((x-2)+yi)^{-1} \\
&= \frac{((x-1)+yi)((x-2)-yi)}{(x-2)^2+y^2} \\
&= \frac{(x-1)(x-2)-iy(x-1)+iy(x-2)+y^2}{(x-2)^2+y^2} \\
&= \frac{x^2-3x+2+y^2}{(x-2)^2+y^2} + iy\frac{(x-1)-(x-2)}{(x-2)^2+y^2} \\
&= \frac{x^2-3x+2+y^2}{(x-2)^2+y^2} + i\frac{y}{(x-2)^2+y^2}
\end{aligned}$$

This gives the system of equations:

$$\begin{aligned}
&\begin{cases} y = 0 \\ x^2-3x+2+y^2 \leq 0 \end{cases} \\
&\Rightarrow x^2-3x+2 \leq 0 \\
&(x-1)(x-2) \leq 0 \\
&1 \leq x \leq 2
\end{aligned}$$

$\therefore \Omega = \mathbb{C} \setminus \{\mathrm{Re}(z) \in [1,2]\}$

(ii) We can write:

$$\begin{aligned}
(z^2-1)^{\frac{1}{2}} &= (-1)^{\frac{1}{2}}(1-z^2)^{\frac{1}{2}} \\
&= i(1-z^2)^{\frac{1}{2}} \\
&= ie^{\frac{1}{2}\mathrm{Log}(1-z^2)},
\end{aligned}$$

which has a branch cut on the set:

$$\begin{cases} \mathrm{Im}(1-z^2) = 0 \\ \mathrm{Re}(1-z^2) \leq 0 \end{cases}$$

We have that:

$$\begin{aligned}
1-z^2 &= 1-(x+yi)^2 \\
&= 1-(x^2+2xyi-y^2) \\
&= 1-x^2+y^2-2xyi \\
&\Rightarrow \begin{cases} -2xy = 0 \\ 1-x^2+y^2 \leq 0 \end{cases},
\end{aligned}$$

which indicates $y = 0$ or $x = 0$. If $x = 0$:

$$\begin{aligned} 1 + y^2 &\leq 0 \\ y^2 &\leq -1, \end{aligned}$$

which cannot be true, as $y \in \mathbb{R}$. Therefore, y must be 0:

$$\begin{aligned} 1 - x^2 &\leq 0 \\ x^2 &\geq 1 \\ &\Rightarrow x \leq -1,\ x \geq 1, \end{aligned}$$

which lies on the beyond the unit circle centered about 0 in $\mathbb{C}$. Therefore, any branch of $(z^2-1)^{\frac{1}{2}}$ within the disc $B_1(0)$ is holomorphic in $B_1(0)$, e.g. the branch $\{\mathrm{Im}(z) \in (-1,1)\}$.

Problem 4

(i) The contour can be parametrized as $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$, where the initial point of α_j is the end-point of α_{j-1}:

$$\begin{cases} \alpha_1 = re^{i(\pi - t)}, & t = [0, \pi], \\ \alpha_2 = t, & t = [r, R], \\ \alpha_3 = Re^{it}, & t = [0, \pi], \\ \alpha_4 = t, & t = [-R, -r], \end{cases}$$

(ii) In polar coordinates, we have that $x = r\cos(\theta)$ and $y = r\sin(\theta)$:

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Leftrightarrow \frac{r^2\cos^2(\theta)}{a^2} + \frac{r^2\sin^2(\theta)}{b^2} &= 1 \\ \frac{a^2r^2\sin^2(\theta) + b^2r^2\cos^2(\theta)}{a^2b^2} &= 1 \\ a^2r^2\sin^2(\theta) + b^2r^2\cos^2(\theta) &= a^2b^2 \\ r^2(a^2\sin^2(\theta) + b^2\cos^2(\theta)) &= a^2b^2 \\ r^2 &= \frac{a^2b^2}{a^2\sin^2(\theta) + b^2\cos^2(\theta)} \\ r &= \sqrt{\frac{a^2b^2}{a^2\sin^2(\theta) + b^2\cos^2(\theta)}} \end{aligned}$$

The contour can therefore be parametrized as:

$$\alpha(t) = \frac{ab}{\sqrt{a^2\sin^2(t) + b^2\cos^2(t)}},\ t \in [0, 2\pi]$$