Alternative Answers for Workshop 6 Problem 4

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We are asked to show that

if x^* is a **local maximum** of $\ln f(x)$ then it is also a local maximum of $f(x)$.

Defining $g(x) = \ln f(x)$, this is **equivalent** to the statement

if x^* is a **local maximum** of $g(x)$ then it is also a local maximum of $e^{g(x)}$.

(It is equivelant because $f(x) = e^{\ln f(x)}$.)

And since an **exponential** function is an **increasing** function, it follows that $e^{g(x)}$ has the same **ups and downs** as $g(x)$.

So $g(x)$ and $e^{g(x)}$ will share the same maxima (and minima!).

We can also show this with calculus.

We are given that $g'(x^*) = 0$, $g''(x^*) < 0$.

Since $f(x) = e^{g(x)}$, by the **chain rule**, we have

$$f'(x) = e^{g(x)} \cdot g'(x) \tag{1}$$

In particular

$$f'(x^*) = e^{g(x^*)} \cdot g'(x^*) = 0 \quad .$$

So we know right away that x^* is a **critical point** of $f(x)$.

To see whether it is a maximum or minimum, we need to express $f''(x)$ in terms of $g(x)$ and its derivatives.

Applying the **product rule** to Eq. (1), we have

$$f''(x) = (e^{g(x)} \cdot g'(x))' = (e^{g(x)})' \cdot g'(x) + e^{g(x)} \cdot g''(x) \tag{2}$$

Using Eq. (1) again we have

$$f''(x) = e^{g(x)} \cdot g'(x) \cdot g'(x) + e^{g(x)} \cdot g''(x) = e^{g(x)} \cdot g'(x)^2 + e^{g(x)} \cdot g''(x) \tag{3}$$

Factoring out $e^{g(x)}$

$$f''(x) = e^{g(x)}(g'(x)^2 + g''(x)) \quad . \tag{3}$$

Plugging-in $x = x^*$ we get

$$f''(x^*) \; = \; e^{g(x^*)}(g'(x^*)^2 \; + g''(x^*)) \quad . \tag{4}$$

But, we already know that $g'(x^*) = 0$, so

$$f''(x^*) \; = \; e^{g(x^*)}(0^2 \; + g''(x^*)) \; = \; e^{g(x^*)}(0 \; + g''(x^*)) \; = \; e^{g(x^*)} \cdot g''(x^*) \quad . \tag{5}$$

Since $e^{anything}$ is always **positive**, and by assumption $g''(x^*) < 0$, and since *the product of a positive and a negative is a negative* , we proved that $f''(x^*) < 0$. Combined with the above fact that $f'(x^*) = 0$, this shows that x^* is also a local maximum of $f(x) = e^{g(x)}$.