CIS 1600 22F

Thomas Zeuthen

Recitation Number: 211

Homework 14H

Collaborators: None

Q1. For R to be a valid equivalence relation it must be reflexive, symmetric, and transitive.

Since R is both symmetric and anti-symmetric, aRb implies $a = b$. If R is symmetric then aRb implies bRa exists, and since R is antisymmetric if both aRb and bRa then $a = b$.

Since R is reflexive this means that for all $a \in A$ aRa exists.

Therefore there exists an element in R for each element in A and no others since the elements must equal each other.

Therefore, $|R| = |A|$.

Q2. Let our sample space be the all strings of capital letters of any length.

Let I_i be a random variable that represent the number of letters until and including the ith appearance of I.

I_i is a geometric random variable since each letter is an independent bernoulli trial with probability of success $\frac{1}{26}$

After each success there is a probability of $\frac{1}{26}$ that the next letter is a j.

Let X be a random variable that represents the number of letters until I and L.

Therefore the expected value of a geometric random varable is one over p so 26^2

Q3. (a) Bethany divides all of the pawns into two equal size groups. If the number of pawns is odd one group will have exactly one more pawn than the others. She does this each turn.

(b) Let Rajiv chose between X and Y randomly with probability one half of chosing either set.

Let X be a random variable representing the number of pawns that make it to row n.

Let X_i be an indicator random variable that equals one if the ith pawn makes it to row n and 0 otherwise.

Consider $P(X_i = 1)$. A pawn can only advance if it is chosen. If a pawn is chosen it has a $\frac{1}{2}$ chance of advancing and a $\frac{1}{2}$ chance of being removed since it is equally likely to be in each set. A pawn is selected at most n times since after n times it either makes it to row n or has to leave the board. Rajiv randomly selects each set indendent from the last so the $P(X_i = 1) \leq \frac{1}{2^n}$

Since X_i is an indicator random variable, $E(X_i) = P(X_i = 1)$

$$X = \sum_{i=1}^{k} X_i =$$

By LOE, $E(X) \leq \sum_{i=1}^{k} \frac{1}{2^n}$

$$E(X) \leq \frac{k}{2^n}$$

Since $k < 2^n$, $E(X) < 1$.

Since the expected value of the number of pieces that make it to the other side is less than one, there must exist an scenario where 0 pieces make it to row n since there cannot be a negative number of pieces that make it. Therefore if $k < 2^n$, then there exists a winning strategy for Rajiv.

Q4. (a) The sample space can be represented by a string of length n with an A at position i if the ith TA is on team Goose or B otherwise.

A is a binomial random variable since it represents a series of n bernouli trials for each of the n TAs with a probability of success(joining the team) of p.

Therefore the PMF of A is $P(A = k) = \binom{n}{k} p^k (1-p)^n - k$

(b) Use the same sample space as part a. Since A is a bionially distributed, $E(A) = np$

(c) Use the same sample space as part a. Since A is binomially distributed, Var(A) $= np(1-p)$

(d) Use the same sample space as part a.

Given the team choice of the first TA, we want the other n-1 tas to chose the opposite team. Since each TA choses independetly of one another, this is the probability the TA choses the other team to the power n-1.

Using LOTP, the probability is equal to $p(1-p)^{n-1} + (1-p)p^{n-1}$

(e) Use the same sample space as part a.

This is the probability that there is a team with exactly one player is $P(A = 1 \cup A = n-1)$, since these two events are clearly disjoint we can add the probabilities using the PMF from part a.

$= \binom{n}{1} p(1-p)^{n-1} + \binom{n}{n-1} p^{n-1}(1-p)$

(f) We know $A + B = n$

$B = A - D$

$A + A - D = n$

$D = 2A - n$

By LOE, $E(D) = 2E(A) - E(n) = 2np - n$

Using LOE, Var(D)= Var $(2A - n) = E((2A-n)^2) - E(2A-n)^2$

$= E(4A^2 - 4An + n^2) - (4n^2p^2 - 4n^2p + n^2)$

$= 4E(A^2)) - 4n^2p + n^2 - 4E(A)^2 + 4n^2p - n^2$

$= 4(E(A^2) - E(A)^2)$

$= 4$Var(A)

$= 4np(1-p)$

Q5.

- Let X_i be an indicator random variable that equals 1 if question i was asked and 0 otherwise.

  $(\frac{7}{8})^{i-1}$, since a TA that is not Kristina must answer the question so there are 7 out of the 8 possible TAs, and each TA is independently chosen fro each question. For the question to be asked each of the previous i-1 questions must have been answered by the TA.

  Since X_i is an indicator random variable, $E(X_i) = P(X_i = 1)$

  $$X = \sum_{i=1}^{n} X_i$$

  By LOE, $E(X) = \sum_{i=1}^{n} (\frac{7}{8})^{i-1}$

  This equal to the infinite geometric series with first term one and common ration seven eights, minus the infinite geometric series with first term seven eights to the n-1 and common ration seven eights.

  Using the formula for an infinite geometric series we get

  $$= \frac{1}{\frac{1}{8}} - \frac{(\frac{7}{8})^{n+1}}{\frac{1}{8}}$$

- Var(X) $= E(X^2) - E(X)^2$

  $$E(X^2) = \sum_{k=1}^{n} k^2 P(X = k)$$

  $$= \sum_{k=1}^{n} k^2 (\frac{7}{8})^{k-1}) \frac{1}{8}$$

  $$\text{Var(X)} = = \sum_{k=1}^{n} k^2 (\frac{7}{8})^{k-1}) \frac{1}{8} - (\frac{1}{\frac{1}{8}} - \frac{(\frac{7}{8})^{n+1}}{\frac{1}{8}})^2$$

Q6. Let X $= \{1, 2, 3, 4\}$

Let Y $= \{1, 2, 3, 4, 5\}$

$f = \{(1, 4), (2, 1), (3, 2), (4, 3)\}$

$g = \{(1, 1), (2, 2).(3, 3), (4, 4), (5, 4)\}$

$g \circ f$ is a bijection since it maps all 4 elements in X to each of the four elements in X such that no two elements map to the same value.

gf is not an identity function since one goes to 4 so it is not an identity funciton.

f is not a surrjection. No element in f maps to 5 which is in the codomain of f.

g is not injective. Both 4 and 5 in g map to 4.

Y has 5 elements.