CIS 1600 22F

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Homework 5T

Recitation Number: 211

Q1. (a) This proof is false. The mistake is that the induction step is backwards. They assume the claim is true for $k+1$ and reason back to the induction hypothesis. However, they need to use the induction hypothesis to show the claim is true for $k+1$.

(b) This proof is false. This mistake is made in the induction step. The induction hypothesis applies for 5^j where j is an integer between 1 and k. the p5^k and 5^{k-1}. The proof erroneously applies the induction hypothesis by assuming 5^k and 5^{k-1} are valid values of j. However, this is not the case for $k = 1$ since $k - 1 = 0$ falling outside of the induction hypothesis.

(c) This proof is valid.

(d) This proof is false. The mistake is made in the induction step. During the application of the induction hypothesis the proof assumes that $5 * 1 = 0$. However, we cannot assume $k = 1$ is inside the range for the induction hypothesis since $0 \leq j \leq k$. For example if $k = 0$, 5^1 would represent the value for $k+1$ which is outside the bounds of the induction hypothesis.

(e) This proof is false. The mistake is made in the induction step. The proof assumes that $max(p', q') = k+1$ and hence $max(p'-1, p'-1) = k$. However, this cannot be assumed since this is what the induction step is trying to prove.

Q2. Using the binomial theorem we get

$$(5x+5)^k = \sum_{r=0}^{k} \binom{k}{r} (5x)^{k-r} 5^r$$

$$= \sum_{r=0}^{k} \binom{k}{r} 5^k x^{k-r}$$

Therefore the coefficient on each term is $\binom{k}{r}5^k$ for some integer r between 0 and k.

We now need to find the number of distinct coefficients. Since every coefficient has 5^k we need to determine which values of r make $\binom{k}{r}$

We can start at $r = 0$. Since $\binom{n}{k} = \binom{n}{n-k}$ we can eliminate $\binom{k}{k}$ since it is the same as $\binom{k}{0}$. We can do the same for $r = 1$ and $r = k-1$, and so on increasing r by one. We need to find the upper bound for r or the last value we will consider. For the upper bound we will consider two cases for k.

Case 1: k is even.
This means that $k/2$ is an integer and $k - k/2 = k/2$ so the final value of r that would be considered is $k/2$. Since 0 is the first case there are $k/2+1$ distinct coefficients.

Case 2: k is odd.
This means that $k/2$ is not an integer. Consider $\lfloor k/2 \rfloor$.
$k - \lfloor k/2 \rfloor = \lceil k/2 \rceil$
This means that $\lfloor k/2 \rfloor$ is the upper bound of r, since $\lfloor k/2 \rfloor + 1 = \lceil k/2 \rceil$, so you have eliminated all options greater than $\lfloor k/2 \rfloor$, from the process above.
Since 0 is the lowest case there are $\lfloor k/2 \rfloor + 1$ distinct coefficients.

Since for the even case $k/2$ is an integer, $\lfloor k/2 \rfloor = k/2$ so for both cases there are $\lfloor k/2 \rfloor + 1$ distinct coefficients.

When dividing by $\lfloor k/2 \rfloor$, there are clearly $\lfloor k/2 \rfloor$ possible remainders from 1 to $\lfloor k/2 \rfloor - 1$. Therefore by the pigeonhole principle if there are $\lfloor k/2 \rfloor + 1$ distinct coefficients at least 2 of them will have the same remainder when divided by $\lfloor k/2 \rfloor$.

Q3. We will prove the claim using strong induction on n.

Induction Hypothesis: Suppose the claim is true $\forall x$ such that $x \in \mathbf{Z}$ and $0 \leq x \leq k$ for some non-negative integer k, that is $T_x = 4^x + (-1)^x$.

Base Case:

$n = 0$
$T_0 = 4^0 + (-1)^0 = 1 + 1 = 2$
Holds because $T_0 = 2$

$n = 1$
$t_1 = 4^1 + (-1)^1 = 4 - 1 = 3$
Holds because $T_1 = 3$

Induction Step: We want to show the claim is true for $k+1$, that is $T_{k+1} = 4^{k+1} + (-1)^{k+1}$

By definition $T_{k+1} = 3T_{k+1-1} + 4T_{k+1-2}$

$= 3T_k + 4T_{k-1}$

$= 3(4^k + (-1)^k) + 4(4^{k-1} + (-1)^{k-1})$ (using the induction hypothesis)

$= 3(4^k) + 3(-1)^k + 4(4^{k-1}) + 4(-1)^{k-1}$

$= 3(4^k) + 4^k + 3(-1)^k + 4(-1)^{k-1}$

$= (3+1)(4^k) + 3(-1)^k + (-4)(-1)^k$

$= 4(4^k) + (3-4)(-1)^k$

$= 4^{k+1} + (-1)(-1)^k$

$= 4^{k+1} + (-1)^{k+1}$