CIS 320 - HW7

Thomas Zeuthen
Collaborators: Megan Yang

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1. (a) Consider a directed graph with three vertices s, v, and t.

    There are 2 edges one from s to v and one from v to t.

    If the lower capacity of s to v is say 10 and the capacity of v to t is less than 10 say 5 then there are no feasible flows since the flow into v is at least 10 while the flow out of v is at most 5 which violates one of the flow conditions.

   (b) Given the feasible flow we can run a typical Max flow algorithm starting from that feasible flow like normal but with a modified residual graph as follows.

    In the residual graph back edges would now have capacity $f(e) - l(e)$ instead of $l(e)$ so that you do not push back so much flow that the edge has less than $l(e)$ flow going through it. Forward edges still have capacity $c(e) - f(e)$.

    Running the Edmonds-Karp algorithm as normal with this residual graph will find the max flow in time $O(nm^2)$.

   (c) First we will show that this expression is a lower bound on the value of the max flow in G.

    Assume for contradiction that there is a flow greater than this value, that is for every cut $S, V - S$

    $$\sum_{(u,v)u \in S, v \notin S} c(u, v) - \sum_{(u,v)u \notin S, v \in S} l(u, v) \text{ is less than } f.$$

    We know from class that f is at least the flow into S minus the flow out of S (for any cut S) (if this was not in class this was proved in the textbook). The flow out of S is at most the sum of the capacities of the edges out of S since that is the most flow all of the edges out of S can hold. The flow into S is at least the sum of the lower bounds of the edges going into S since that is the least amount of flow these edges can hold.

    Therefore the expression above provides an upper bound on f. Since it is true for all cuts S we can say f is upper bounded by the cut that minimizes this expression.

    Now it remains to show that f can equal this value.

    Consider the cut made in the residual graph G_f after our algorithm described in part b terminates.

    Let S be the cut that is formed where S is the vertices reachable from s in G_f when the algorithm terminates and T is the set of all other vertices. (we know this cut forms because the algorithm terminates when the graph is disconnected).

Consider the value of the flow in and out of S when the algorithm terminates. Since there are no edges out of S in G_f that means the flow value on each of these edges is equal to $c(e)$ since that is the value of the flow out of these edges (since the edges in the residual graph have capacity $c(e) - f(e)$.

Since there are no edges into S in G_f that means the value of the flow on these edges must be $l(e)$ since the capcity of these edges are $f(e) - l(e)$ to make these 0 $f(e)$ must be 0.

Therefore the sum of the flow out of S is the capacities out of S and the sum of the flows into S is the sum of the capcities into S so f will reach this value.

Since f will be equal to the expression above and it is an upper value on f this is the value of the max flow. Since the proof relies on the residual graph described in b it proves our algorithm to be correct.

2. (a) Run Edomonds Karp on $\tilde{G}$ to find the max flow. Then to get a feasable flow in G for every edge e, $f(e) = f(\tilde{e} + l(e))$ where $f(\tilde{e})$ is the flow in the corresponding edge in $\tilde{G}$.

    Then we iterate through all of the edges and vertices in G to check if the flow in $\tilde{G}$ has value of L as defined in the problem. if it does not return there is no such flow. Otherwise output the result above.

   (b) First we will show that the flow meets the capacity constraints. Clearly it is greater than the lower bound on the edge since $f(\tilde{e})$ has to be positive and we add it to $l(e)$.

    It also cannot be greater than $c(e)$, since the capacity on every edge in $\tilde{G}$ is at most $c(e) - l(e)$ when we add $l(e)$ units of flow to it it will not be greater than $c(e)$.

    Now we will show the flow conservation properties are met.

    Consdier an arbitrary vertex v in $\tilde{G}$. Since the value of the flow is L, based on how the edge are defined the flow into V is the sum of the lower bounds of all edges going into c (given from all edges going to v from s') plus the sum of the value of the flow of all edges going into v that are not s'.

    This is equal to the sum of the flow of all edges going out of v since $\tilde{G}$ is a valid flow. This is equal to the sum of the lower bounds of all edges going out of v (given by all edges (v, t') since the value of the flow is T) plus the sum of the flow on all edges going out of v in G.

    This shows that the flow going in and out of any vertex in G will be the same as well since for every edge we add the lower bound of that edge plus the value of the flow given in $\tilde{G}$ and outof the edge we have the value of the lower boudn of each edge plus the value of the flow given in $\tilde{G}$.

   (c) Suppose we have a valid flow in G.

    This means that for all verteces v the sum of the values of the flows in $\tilde{G}$ going into v plus the lower bound of all edges going into is equal to the sum of the values of the flows is equal to the sum of the lower bound of all the edges going out of v plus the sum of the flows in $\tilde{G}$ going out of v for flow conservation to hold.

    Since the sum of the flows going into and out of $\tilde{G}$ are clearly the same in $\tilde{G}$, we can consider the edges in $\tilde{G}$ that go from s' to v. Clearly the sum of these values has to equal the sum of the lower bounds of all the edges into v and the sum of the edges from v to t' has to equal the sum of the lower bounds of these edges since these ar the only values for which flow conservation holds (since it holds for G as stated above).

    Therefore the value of the flow for any edge s' to v is the lower bound of the corresponding edge. This means that the flow of $\tilde{G}$ is equal to L since the sum of the value of all the flows out of the source is clearly equal to L.