14. Facility Location

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Recap

- Path planning in Euclidean spaces
  - One-dimension
  - Two-dimension
- Path planning on networks
  - Minimum spanning tree problem
  - Traveling salesman problem
  - Chinese postman problem

Outline

- Introduction
- Facility location in Euclidean space
  - k-median problem
  - Coverage problem
- Facility location on networks
  - k-median problem
  - Center problem
  - Requirement problem

Ref: Larson R C, Odoni A R. Urban operations research[M]. 1981.

Background

- Extensive category of smart city service system problems.
- Concerned with determining good locations for the stationing of service vehicles or the construction of major facilities.
- These problems arise in the context of both routine and emergency services, but the objectives are usually different in the two cases.


Objectives

Different facilities lead to very different objectives.

Median problems

- A prespecified number of facilities must be located so as to minimize the average distance (or the average travel time or the average travel cost) to or from the facilities for the population of their users.
- Arise very often in the context of facility construction for delivery of nonemergency services.

Center problems

- A prespecified number of facilities must be located so as to minimize the maximum distance (or time or cost), to or from the facilities, that any user will have to travel.
- More applicable in the context of emergency urban services.

How smart city makes a difference

Information technology applies to...

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1-median

- Consider a finite set of points N in a 2D world.
- Every point is characterized by (i) its coordinate (x_k, y_k) and (ii) its demand d_k.
- Now we are building an EV charging station.
- Arbitrarily located in the 2D world.
- Infinite charging capability.

1-median

- Data:

$$(x_k, y_k), d_k, k = 1,2, \dots, N.$$

- Decision variables:

$$(p, q)$$

- Constraints:

$$(p, q) \in \mathbb{R}^2$$

- Objective:

$$\min \sum_{k=1}^{N} d_k \sqrt{(x_k - p)^2 + (y_k - q)^2}$$

k-median

- Consider a finite set of points N in a 2D world.
- Every point is characterized by (i) its coordinate (x_i, y_i) and (ii) its demand d_i.
- Now we are building k EV charging stations.
- Arbitrarily located in the 2D world.
- Infinite charging capability.

k-median

- Data:

$$(x_i, y_i), d_i, i = 1,2, \dots, N; k.$$

- Decision variables:

$$(p_j, q_j), j = 1,2, \dots, k.$$

- Constraints:

$$(p_j, q_j) \in \mathbb{R}^2$$

$$l(i,(p,q)) = \min_{j=1,\dots,k} \sqrt{\left(x_i - p_j\right)^2 + \left(y_i - q_j\right)^2}$$

- Objective:

$$\min_{p,q} \sum_{i=1}^{N} d_i l(i,(p,q)).$$

Requirement problem

- Consider a finite set of points N in a 2D world.
- Every point is characterized by (i) its coordinate (x_i, y_i) and (ii) its demand d_i.
- Now we are building some EV charging stations.
- Arbitrarily located in the 2D world.
- Infinite charging capability.

Requirement problem

- Data:

$$(x_i, y_i), d_i, i = 1,2, \dots, N; c.$$

- Decision variables:

$$(p_j, q_j), j = 1,2, \dots, k; k$$

- Constraints:

$$(p_j, q_j) \in \mathbb{R}^2$$

$$l(i,(p,q)) = \min_{j=1,\dots,k} \sqrt{\left(x_i - p_j\right)^2 + \left(y_i - q_j\right)^2}$$

$$k \in \mathbb{Z}_{>0}$$

- Objective:

$$\min_{p,q} \sum_{i=1}^{N} d_i l(i,(p,q)) + ck.$$

Max coverage

- Consider a finite set of points N in a 2D world.
- Every point is characterized by (i) its coordinate (x_k, y_k) and (ii) its demand d_k.
- Suppose that we are building a 5G station with a coverage radius of R.
- How to select the location of the station (p, q) such that it covers as much demand as possible?

Max coverage

- Data:

$$(x_i, y_i), d_i, i = 1,2, \dots, N; R.$$

- Decision variables:

$$(p, q)$$

- Constraints:

$$I_i = \begin{cases} 1 & (x_i - p)^2 + (y_i - q)^2 \leq R^2 \\ 0 & o.w. \end{cases}$$

- Objective:

$$\max_{p,q} \sum_{i=1}^{N} d_i I_i \, .$$

Big-M constraint

- Translates a logical relation to an inequality.
- Consider a number $M \in \mathbb{R}_{>0}$.
- Define an auxiliary variable $w_i \in \{0,1\}$.

$$(x_i - p)^2 + (y_i - q)^2 - R^2 \leq M w_i.$$
$$I_i = 1 - w_i.$$

Full coverage

- Consider a finite set of points N in a 2D world.
- Every point is characterized by (i) its coordinate (x_k, y_k) and (ii) its demand d_k.
- Suppose that we are building 5G stations with a coverage radius of R.
- How many stations do we need to cover all points?
- How to select the location of the stations $(p_j\ , q_j)$ such that all demand is covered?

Max coverage

- Consider a continuous "field" of demand $\rho(x, y)$.
- ρ interpreted as density of demand.
- Suppose that we are building a 5G station with a coverage radius of R.
- How to select the location of the station (x, y) such that it covers as much demand as possible?

$$\max \int_{x,y:(x-p)^2+(y-q)^2 \leq R^2} \rho(x,y)\, dxdy$$
$$\text{s.t.} \quad (p,q) \in \mathbb{R}^2.$$

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Network model

- Network models of an urban or metropolitan area are particularly convenient for the discussion of facility location problems.
- Represent the various transportation arteries as links of a network and their intersections as nodes on it.
- Thus, travel is restricted to take place solely along the links and nodes of a network.
- A further assumption will be that demands for services will be generated only at a finite number of points, also designated as a set of nodes on the network.

Median Problems

- Suppose that we are planning 3 convenient stores in a community.
- Assume that the stores can only be located at intersections.
- We have complete demographic information.
- We want the stores to be as close to as many residents as possible.

Median Problems

- Let us consider an undirected network $G(N, E)$ with n nodes.
- Let $k \in \mathbb{Z}_{>0}$ and let us choose k distinct points on the graph G to be indicated as the set $X_k = \{x_1, x_2, \ldots, x_{k-1}, x_k\} \subset N$.
- We shall then indicate by $d(X_k, j)$ the minimum distance between any one of the points $x_i \in X_k$ and the node j on G; i.e.,
$$d(X_k, j) = \min_{x_i \in X_k} d(x_i, j).$$
- This notion of distance towards a set is common.

k-median

- Let $h_j \in \mathbb{R}_{\geq 0}$ be the weight of node j.
- Interpreted as population or demand.
- Given a set of facility locations X_k, the average travel distance is given by

$$J(X_k) = \sum_{j \in N} h_j d(X_k, j) \,.$$

- A set of points $X_k^* \subseteq N$ is a k-median of network $G(N, E)$ if

$$J(X_k^*) \leq J(X_k), \qquad \forall X_k \subseteq N.$$

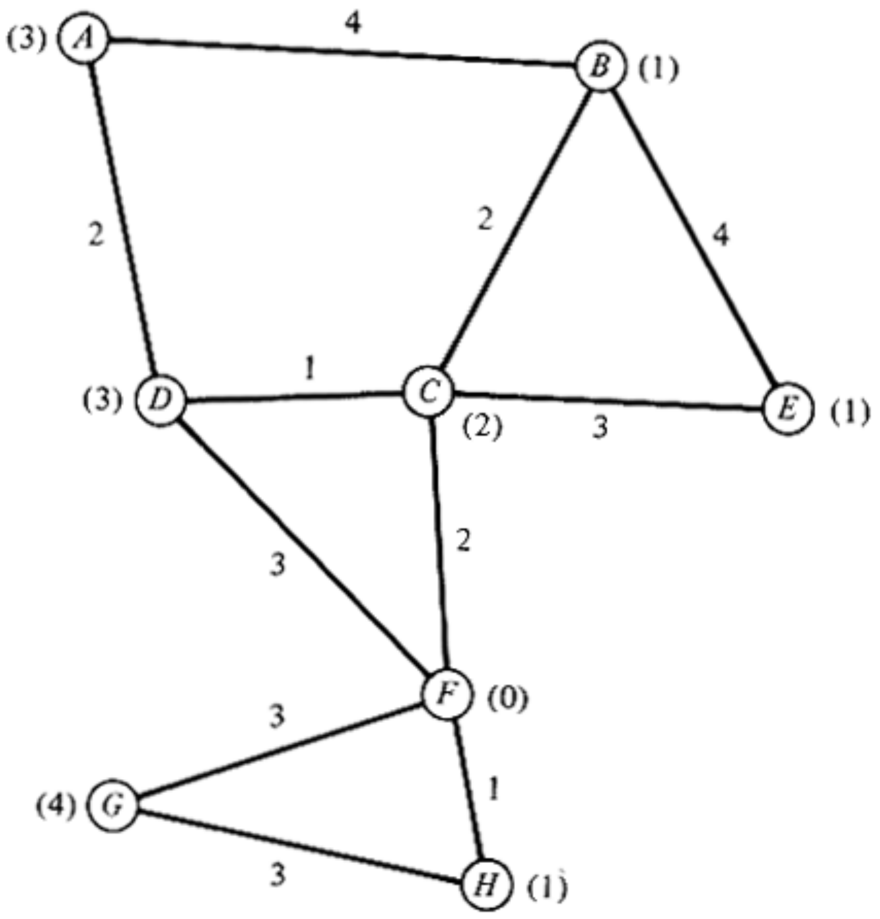
- "Center of mass".

Hakimi's theorem

- Recall: we assumed that facilities must be located at nodes.
- Why can we do this?
- **Theorem (Hakimi)**: At least one set of k-medians exist solely on the nodes of G.
- The practical significance of this theorem is great.
- It states in effect that the search for the set of the k optimal locations for the k facilities can be limited to the node set of G (i.e., to a total of n points only) instead of the infinite number of points that lie on the links of G.

Example

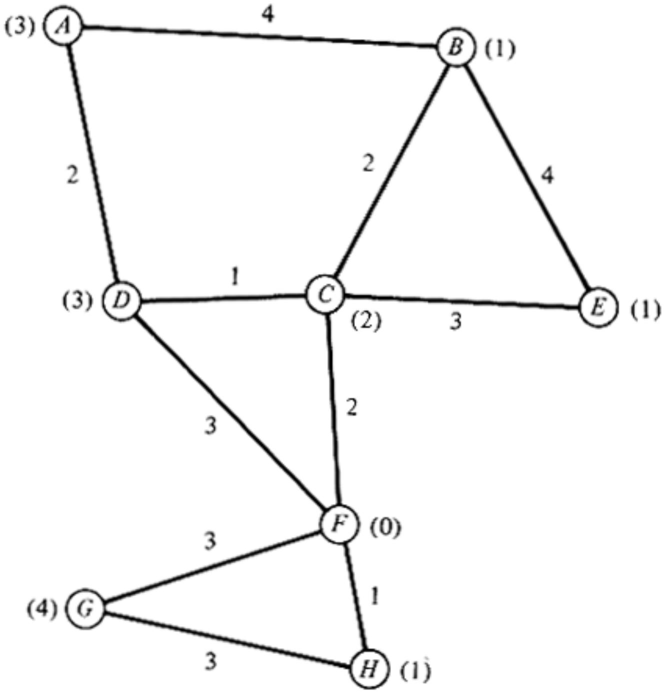
- Nodes A through H represent points where demands for service are generated and/or points where major roads in the area intersect.
- A single facility is to be located in the area and its prospective users will have to travel to the facility to partake of the service provided there.



Example

- We can compute the distance (shortest-path) matrix $[d\ (i,j)]$ for all pairs of nodes, i and j, of the graph.

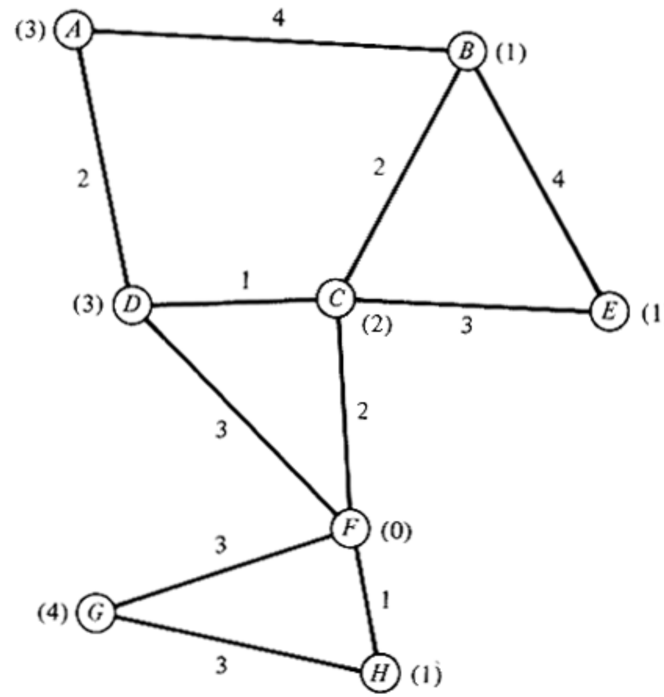
	A	B	C	D	E	F	G	H
A	0	4	3	2	6	5	8	6
B		0						
C			0					
D				0				
E					0			
F						0		
G							0	
H								0



Example

- We can compute the distance (shortest-path) matrix $[d\ (i,j)]$ for all pairs of nodes, i and j, of the graph.

$$
\begin{array}{c c}
\text{From} & \begin{array}{cccccccc} A & B & C & D & E & F & G & H \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array} &
\begin{bmatrix}
0 & 4 & 3 & 2 & 6 & 5 & 8 & 6 \\
4 & 0 & 2 & 3 & 4 & 4 & 7 & 5 \\
3 & 2 & 0 & 1 & 3 & 2 & 5 & 3 \\
2 & 3 & 1 & 0 & 4 & 3 & 6 & 4 \\
6 & 4 & 3 & 4 & 0 & 5 & 8 & 6 \\
5 & 4 & 2 & 3 & 5 & 0 & 3 & 1 \\
8 & 7 & 5 & 6 & 8 & 3 & 0 & 3 \\
6 & 5 & 3 & 4 & 6 & 1 & 3 & 0
\end{bmatrix} = [d(i,j)]
\end{array}
$$

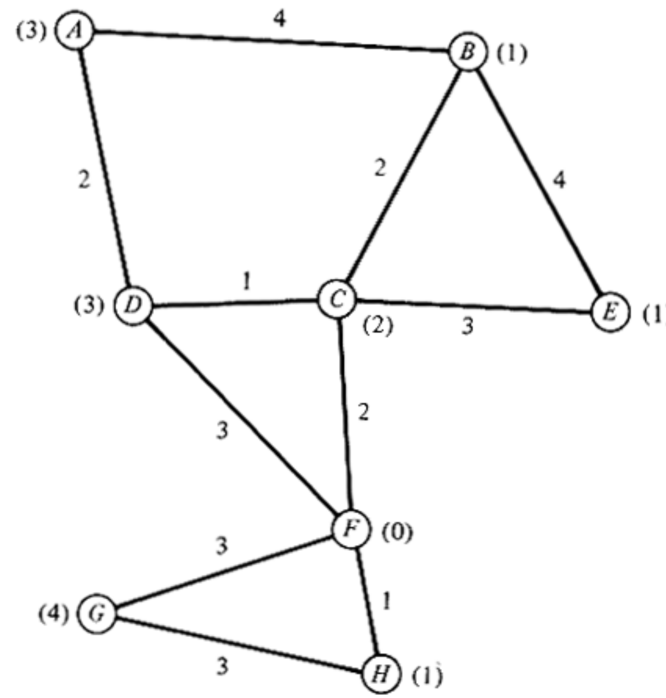


Example

- We next compute the terms $h_j d(i,j)$ by multiplying each column of the distance matrix by the weight of node j.

$$
\begin{array}{c} \text{From} \\ A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array}
\begin{array}{c} \begin{array}{cccccccc} A & B & C & D & E & F & G & H \end{array} \\
\begin{bmatrix}
0 & 4 & 3 & 2 & 6 & 5 & 8 & 6 \\
4 & 0 & 2 & 3 & 4 & 4 & 7 & 5 \\
3 & 2 & 0 & 1 & 3 & 2 & 5 & 3 \\
2 & 3 & 1 & 0 & 4 & 3 & 6 & 4 \\
6 & 4 & 3 & 4 & 0 & 5 & 8 & 6 \\
5 & 4 & 2 & 3 & 5 & 0 & 3 & 1 \\
8 & 7 & 5 & 6 & 8 & 3 & 0 & 3 \\
6 & 5 & 3 & 4 & 6 & 1 & 3 & 0
\end{bmatrix} \end{array} = [d(i,j)]
$$

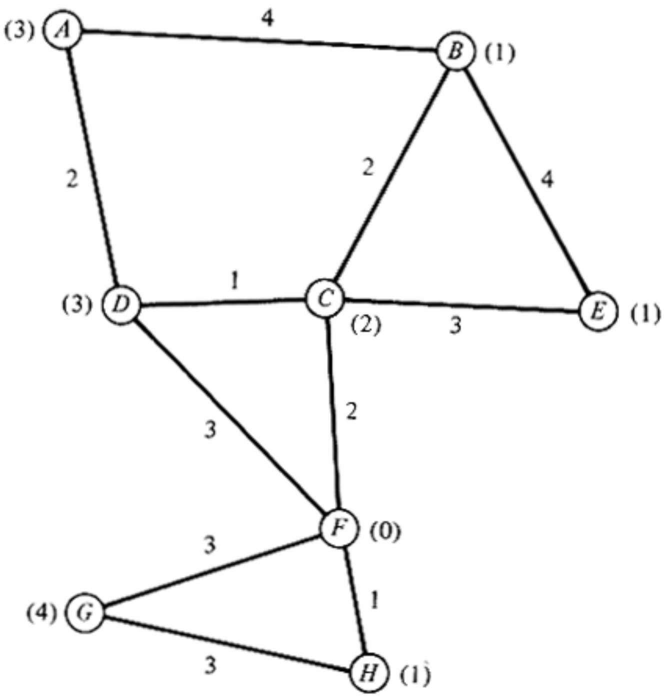
	A	B	C	D	E	F	G	H
A	0	4	6	6	6	0	32	6



Example

- We next compute the terms $h_j d(i,j)$ by multiplying each column of the distance matrix by the weight of node j.

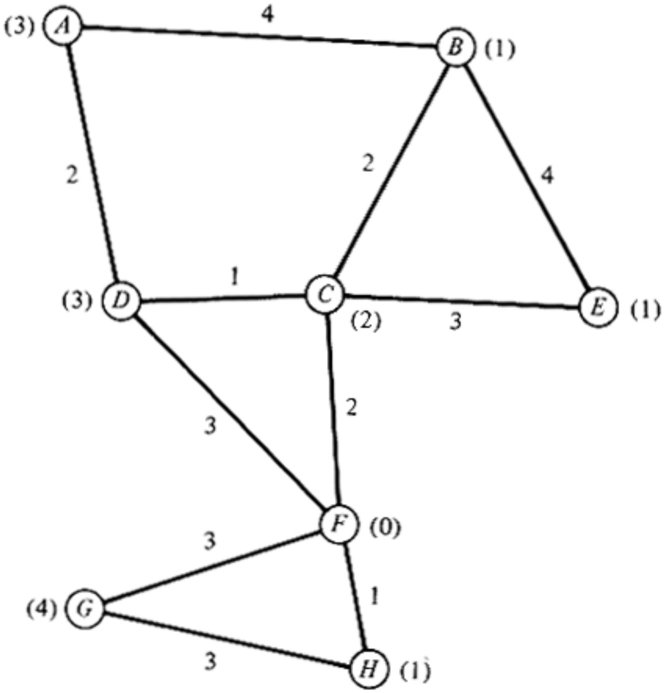
$$
[h_j d(i,j)] = \begin{array}{c}
 \\ A \\ B \\ C \\ D \\ E \\ F \\ G \\ H
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} A & B & C & D & E & F & G & H \end{array} \\
\begin{bmatrix}
0 & 4 & 6 & 6 & 6 & 0 & 32 & 6 \\
12 & 0 & 4 & 9 & 4 & 0 & 28 & 5 \\
9 & 2 & 0 & 3 & 3 & 0 & 20 & 3 \\
6 & 3 & 2 & 0 & 4 & 0 & 24 & 4 \\
18 & 4 & 6 & 12 & 0 & 0 & 32 & 6 \\
15 & 4 & 4 & 9 & 5 & 0 & 12 & 1 \\
24 & 7 & 10 & 18 & 8 & 0 & 0 & 3 \\
18 & 5 & 6 & 12 & 6 & 0 & 12 & 0
\end{bmatrix}
\end{array}
$$



Example

- Normalizing the quantities h_j by dividing by the total demand (= 15), we can find the average user travel distance associated with each of the eight candidate locations.

$$
[h_j d(i,j)] = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array}
\begin{array}{c} \begin{array}{cccccccc} A & B & C & D & E & F & G & H \end{array} \\
\begin{bmatrix}
0 & 4 & 6 & 6 & 6 & 0 & 32 & 6 \\
12 & 0 & 4 & 9 & 4 & 0 & 28 & 5 \\
9 & 2 & 0 & 3 & 3 & 0 & 20 & 3 \\
6 & 3 & 2 & 0 & 4 & 0 & 24 & 4 \\
18 & 4 & 6 & 12 & 0 & 0 & 32 & 6 \\
15 & 4 & 4 & 9 & 5 & 0 & 12 & 1 \\
24 & 7 & 10 & 18 & 8 & 0 & 0 & 3 \\
18 & 5 & 6 & 12 & 6 & 0 & 12 & 0
\end{bmatrix} \end{array}
$$



- $\overline{D}_A = (0 + 4 + 6 + 6 + 6 + 0 + 32 + 6)/15$

Example

- Normalizing the quantities h_j by dividing by the total demand (= 15), we can find the average user travel distance associated with each of the eight candidate locations.
- The optimum location for the facility is at node C, and the associated average travel distance is 2.67 km.

Facility located at	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*
Total travel distance	60	62	40	43	78	50	70	59
Average travel distance $\left(=\frac{\text{total travel distance}}{15}\right)$	4.0	4.13	2.67	2.87	5.2	3.33	4.67	3.93

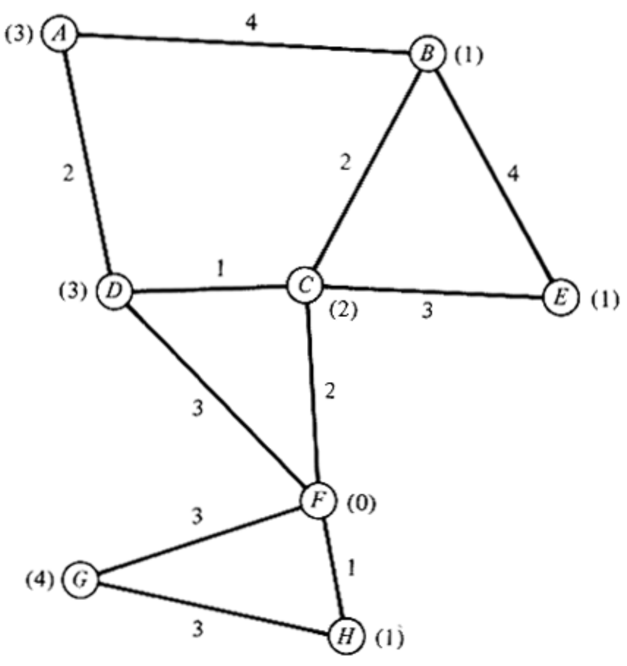
Example

- What would now happen if two facilities were desired?
- To solve the 2-median problem we can still take advantage of Hakimi's theorem and consider only sets of points composed of two nodes.
- With a total of eight nodes, there are $\binom{8}{2} = 28$ possibilities.
- Demands from each node will be "assigned" to the facility closest to it (i.e., the one that requires the least amount of travel).

$$A, B: \quad 0 + 0 + 4 + 6 + 4 + 0 + 28 + 5 = 47$$
$$C, D: \quad 6 + 2 + 0 + 0 + 3 + 0 + 20 + 3 = 34$$
$$D, G: \quad 6 + 3 + 2 + 0 + 4 + 0 + 0 + 3 = 18$$

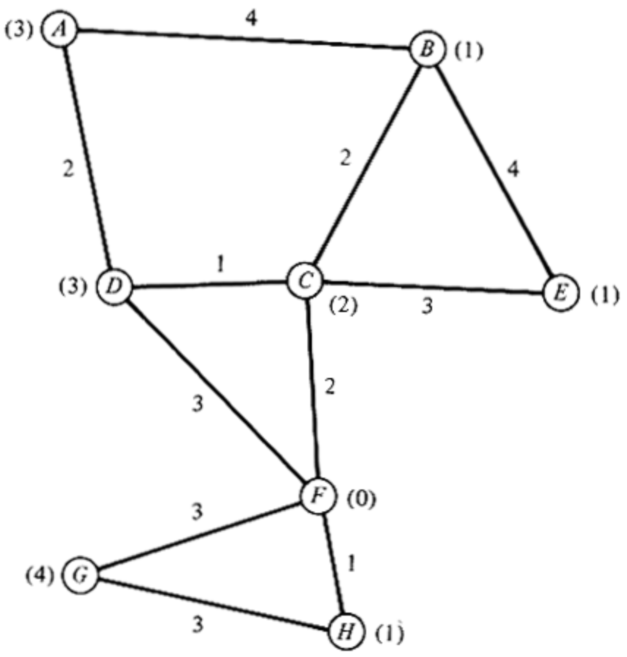
Example

- By exhaustive consideration of all possibilities, we can reach the conclusion that the solution of the 2-median problem consists of locations at nodes D and G for a total distance of 18 units (or an average distance of 1.2 km).
- Under this solution, demand from nodes A, B, C, D, and E (total of 10 units of demand) is assigned to the facility at D while demand from G and H (total of 5 units) is assigned to G.



Example

- Thus, the facility at D assumes double the load of the facility at G.
- Note also that despite the considerable overall reduction in the average travel distance, service users from nodes B, C, and E now have to travel farther than they had to with a single facility.



Supplementary facilities

- It often happens, particularly with regard to urban transportation services, that important service facilities in a given area are severely congested due to high demand.
- It may then be deemed desirable to establish a number of secondary facilities, whose sole purpose is to "preprocess" the prospective users of the primary facilities.
- Those users who choose (or, for that matter, are compelled) to pass through the secondary facilities first, presumably receive some type of "reward" either in the form of faster access to the primary facilities or in the form of faster service once they get there or both.

Supplementary facilities

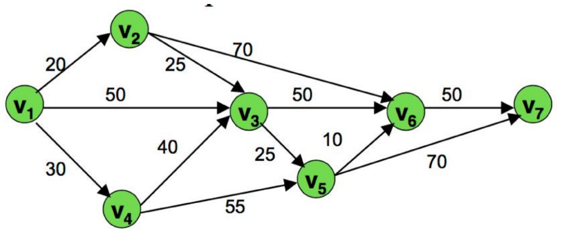
- A good example of this type of setup is the often-discussed concept of constructing secondary remote terminals for airports.
- Prospective air passengers will congregate, will go through the check-in procedures, and will then be transferred to the airport.

Supplementary facilities

- Problems such as these, in which primary facilities already exist (and quite often are less than optimally located) and secondary facilities must be established to provide supportive services, are known as "supporting facility" problems.
- When the location of the supporting facilities must be determined so as to minimize the average travel distance (or time or cost) to all users, then, by analogy to all the above, we have the "supporting k-medians" problems.

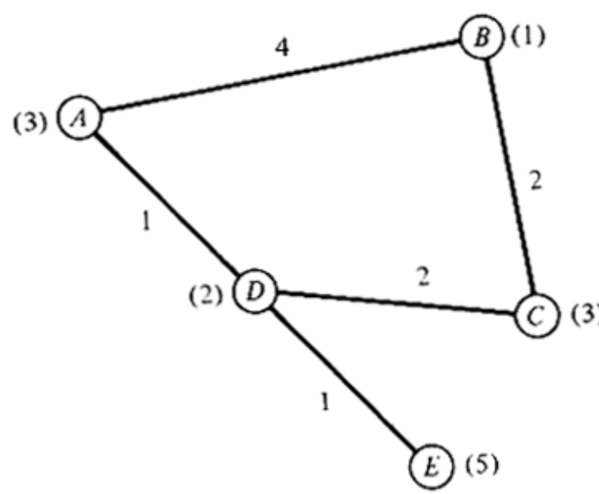
Directed networks

- With a few modifications (and some changes in the definitions), essentially the same results could have been obtained for directed graphs (i.e., for cases where some or all of an area's transportation links are used for one-way travel).
- It is important to realize that, with directed graphs, it makes a difference whether the average distance to be minimized is the distance *to* the facilities, *from* the facilities, or the *round-trip* distance.



Single center

- Consider a rural area with five towns, shown as nodes A through E, connected by a rather sparse transportation network with link lengths in miles also indicated.
- Town populations in thousands are listed in parentheses next to each node.



Single center

- The towns have entered into a cooperative agreement to obtain joint fire protection for certain types of fires.
- They are planning to build a firehouse where a single special-purpose fire engine, yet to be purchased, will be stationed.
- A considerable amount of discussion has led to the conclusion that the location of the firehouse must be such as to minimize the farthest distance that the fire engine will ever have to travel in responding to a fire alarm.
- This, indeed, is a quite reasonable objective for an emergency-type service such as the fire department.

Single center

- Suppose that the location of the firehouse were restricted to be at one of the five cooperating towns.
- By obtaining the minimum distance matrix $[d(i,j)]$ for the given network, we can find the node that minimizes the maximum distance to all other nodes.
- The optimum location for the facility is town C with a maximum distance of 3 miles to both town A and town E.

$$
[d(i,j)] = \begin{array}{c} \text{From} \backslash \text{To} \\ A \\ B \\ C \\ D \\ E \end{array}
\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 4 & 3 & 1 & 2 \\ 4 & 0 & 2 & 4 & 5 \\ 3 & 2 & 0 & 2 & 3 \\ 1 & 4 & 2 & 0 & 1 \\ 2 & 5 & 3 & 1 & 0 \end{bmatrix} \end{array}
\quad
\begin{array}{c} \text{Maximum in Row} \\ 4 \\ 5 \\ 3 \\ 4 \\ 5 \end{array}
$$

Single center

- Let $G(N, A)$ be an undirected network
- Let $x \in G$ be any point on the network; possibly on a link!
- Then, we denote the distance between x and the node of G which is farthest away from it as
$$m(x) = \max_{j \in N} d(x, j) .$$
- We call x^* a vertex center if
$$m(x^*) \leq m(x), \qquad \forall x \in N;$$
- We call x^* an absolute center if
$$m(x^*) \leq m(x), \qquad \forall x \in G;$$
- **Note**: vertex center may or may not coincide with the absolute center!

Multiple centers

- $Let\ G(N, A)$ be an undirected network and let $X_k = \{x_1, x_2, ..., X_k\}$ be a set of k points on G.
- We shall use, as before,

$$d\left(X_j, j\right) = \min_{x_i \in X_k} d(x_i, j)$$

- That is, $d\left(X_j, j\right)$ is the minimum distance between any one of the points $x_i \in X_k$ and the node j on G.
- A set of k points X_k^* on G is a set of *unconstrained* (or *absolute*) k-centers of G, if for every set $X_k \subset G$,

$$m(X_k^*) \leq m(X_k), \qquad \forall X_k \subset G,$$

where

$$m(X_k) = \max_{j \in N} d(X_k, j).$$

Requirements Problems

- So far we have addressed urban facility location problems of the type:

  "Where should I locate k facilities to maximize (or minimize) some (given) objective function ?"

- Very often, however, the question will be asked in quite different terms:

  "We would like to achieve certain standards of performance. What is then the smallest (or least costly) number of facilities that we need, and where should these facilities be located to achieve these standards ?"

Requirements Problems

- Suppose that we require 95% of emergent medical calls to be reached within 10 minutes.
- We thus now require a set of locations such that no potential users are more than 10 minutes away from at least one of them.
- This we recognize as a problem very similar to the k-centers problem, but k is not given!

Requirements Problems

- A certain *primary objective* is stated, usually in the form of a requirement for compliance with the performance standards set by an administrative, legislative, or other body.
- Some restrictions on the potentially acceptable facility locations are also specified. These restrictions are often due to local considerations/conditions or to special requirements of the facilities to be constructed.
- One or more secondary objectives are also often specified, usually in terms of cost.

Requirements Problems

$$
\begin{aligned}
\text{min} \quad & ck \\
\text{s.t.} \quad & m(X_k) \le L, \\
& X_k \subset G, \\
& k \in \mathbb{Z}_{>0}.
\end{aligned}
$$

- Such a problem is way easier to formulate than to solve...
- Nonlinear, nonconvex, non-structural.
- You can hardly expect an analytical solution.
- Brutal force may not work either.
- Need heuristic algorithms...

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