Homework 1

ECE4530J - Decision Making in Smart Cities Summer 2022

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Problem 1

a) What is a mapping? What is a function?

b) What is asymptotic convergence?

Answer:

a) Suppose there is a rule f for two non-empty sets X and Y, so that for each element x in X, according to the rule f, there is a unique element y corresponding to it in Y, then f is called from X to Y mapping.

Although in most cases the words function and map are used interchangeably, in some parts of mathematics the emphasis is different.

First, "map" is a generic word, the word "function" is used to map to $\mathbb{R}$ or $\mathbb{C}$. So mapping to R^n for example would not be called a function.

b) If there is a 1D LTI system

$$x[t+1] = ax[t] + bu[t].$$

and given initial condition $x[0]$, select $u[t]$ for $t = 0, 1, 2, \ldots$ such that

$$\lim_{t\to\infty} x[t] = 0$$

. It is called asymptotic convergence. That is to say, in the case of gradually increasing time, the data gradually tends to 0 and the change range is smaller and smaller.

Problem 2

Consider a one-dimensional linear time-invariant system

$$\dot{x} = ax + bu.$$

a. What is the system state? What is the system state space?

b. What is the control input?

c. Given the initial condition $x(0) = x_0$ and a linear controller $\mu(x) = -kx$, find $x(t)$.

d. When does the feedback-controlled system in c) converge?

Answer:

a. System state is variable x.

System state space is domain $\mathbb{R}$

b. Control input is variable u.

c. Since it is a CT system. We have

$$\frac{d}{dt}x(t) = ax(t) + bu(t)$$

and linear controller $\mu(x) = -kx$. So that we have

$$\frac{d}{dt}x(t) = ax(t) - bkx(t)$$

$$\begin{aligned} \mathrm{CT} : \dot{x}(t) &= ax(t) - bkx(t) = (a - bk)x(t) \\ x(t) &= x(0)e^{(a-bk)t} = x_0 e^{(a-bk)t}, \quad t > 0 \end{aligned}$$

d. the system is convergent if and only if

$$\mathrm{Re}(a + bk) < 0.$$

Problem 3

Consider an n-dimensional linear time-invariant system

$$\dot{x} = Ax + Bu.$$

a. What is the state space?

b. Suppose that $u \in \mathbb{R}^2$. What are the dimensions of matrices A and B ?

c. Discretize the system into discrete time with step size δ. You need to express $x((k+1)\delta)$ in terms of $A, B, \delta, x(k\delta), u(k\delta)$. To make the notation easier to read, you can use $x[k+1]$ instead of $x((k+1)\delta)$ to denote the discrete-time state and express $x[k+1]$ in terms of $x[k]$; note that you still need to consider the impact of δ.

d. Suppose that we use a linear controller $\mu(x) = -Kx$. Write the difference equation for the discretized system; i.e., write how to obtain $x[k+1]$ from $x[k]$. Find $x[k]$ in terms of A, B, K and the initial condition $x[0] = x_0 \in \mathbb{R}^n$.

e. Suppose that $u \in \mathbb{R}^n$. What are the domain and the range for μ ? Hint: the range will depend on the rank of K.

Answer:

a. State space is $\mathbb{R}^n$ since it is an n-dimensional LTI system.

b. dim $A = n \times n$ and dim $B = n \times 2$

c. We have

$$x[k+1] = Ax[k] + Bu[k]$$

we use the δ notation here, it can be expressed as

$$\begin{aligned} x((k+1)\delta) &= x(k\delta) + (Ax(k\delta) + Bu(k\delta))\delta \\ &= (1 + A\delta)x(k\delta) + B\delta u(k\delta). \end{aligned}$$

d. Since we have $\mu(x) = -Kx$, we can express the former equation as

$$\begin{aligned} x[k+1] &= (1 + A\delta)x[k] - KB\delta x[k] \\ &= (1 + A\delta - KB\delta)x[k] \end{aligned}$$

The solution can be given by induction:

$$x[k] = (1 + A\delta - KB\delta)^k x_0, \quad k = 0, 1, 2, \ldots$$

e. Since $u \in \mathbb{R}^n$ the dimension of K is $n \times n$, the domain and range is both $\mathbb{R}^n$