10 Control Review

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Outline

- Quiz 1 instructions
- Review of key points
  - Vehicle control problems
  - Traffic control problems
- Example questions
- Q&A

Quiz 1 instructions

- Wednesday, June 8, 2022
- 75 min in class (8:15-9:30AM BJT)
- Every one must turn on video
- Open-book: you can refer to relevant references, but you must not communicate with anyone during the quiz.
- Please always explain and justify your response; the final solution alone may not lead to full credit.
- Clearly define any notation that you use.
- Typeset is preferred. If you write, please ensure neat handwriting and clear scanning.

Quiz 1 format

- 5 problems
- 3 problems with 4 parts; 5 points for each part.
- 2 problems with 4 regular parts and 1 bonus part; 5 points for either type of parts.
- Total = 100+10
- Upper bounded by 100

Summer 2021 statistics

- 38 students
- Average: 88/100
- Highest: 100
- Lowest: 50*; 73
- Similar statistics expected for this semester.

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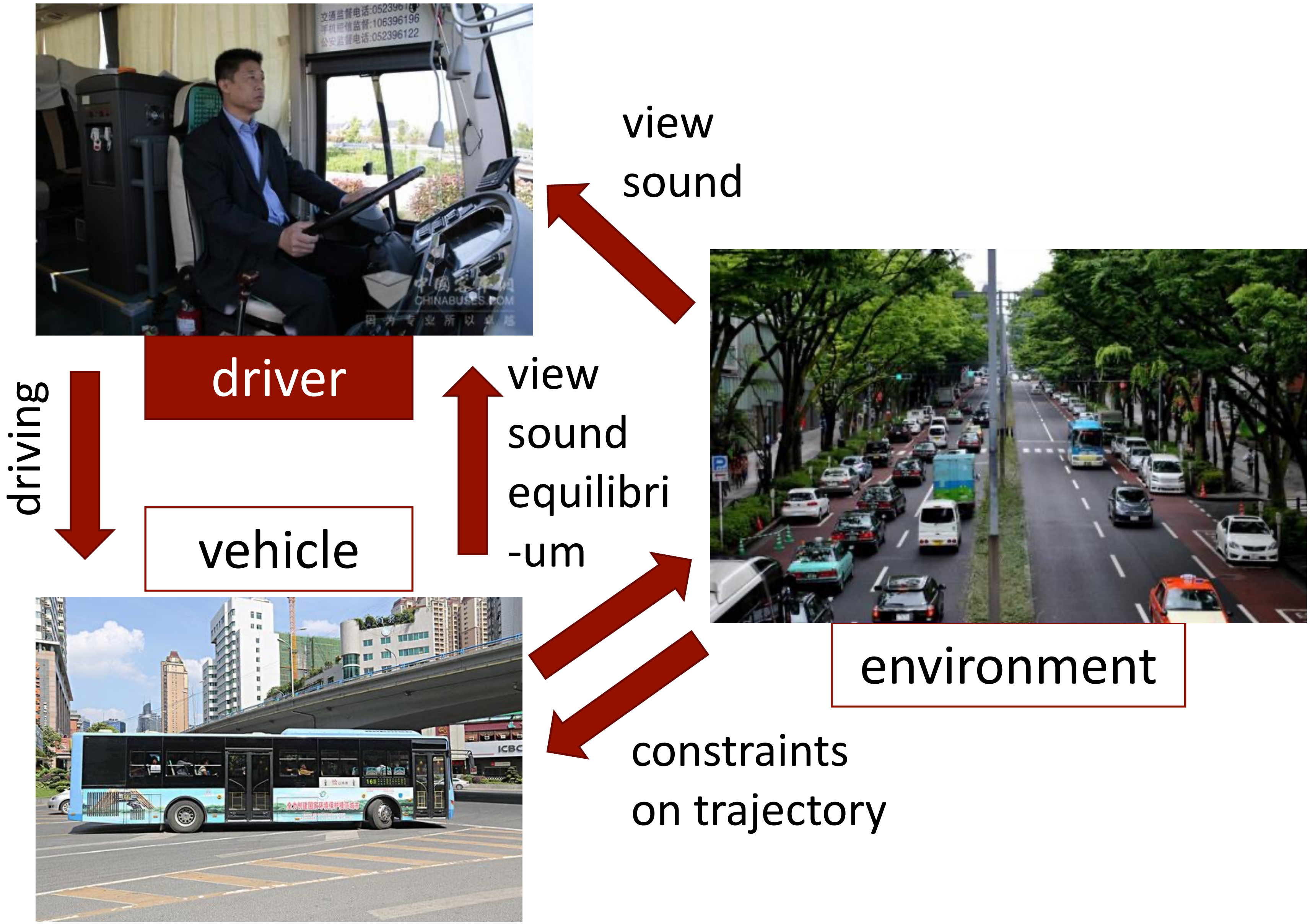
What does a driver do when he/she drives?



How do human make decisions?

Human drivers have objectives:

- Arrive at destination as soon as possible
- Ensure no collision with other vehicles/bikes/pedestrians/obstacles…
- Drive smoothly to ensure comfort
- Avoid traffic rule violation
- Take as few actions as possible

Speed tracking

- We can formulate the linear motion of a vehicle as a dynamical system as follows.
- State variable: $v[t]$ = speed at time t.
- State space: $\mathbb{R}_{\geq 0}$ = domain for state variable.
- Control input (action): $u[t]$ = acceleration at time t.
- System dynamics:

$$v[t+1] = v[t] + u[t]\delta.$$

- δ = discrete time step.
- We consider discrete times (DT) $0,1,2, \ldots$ for now.
- *To define a dynamical system, you need to specify (1) state, (2) control input, (3) system dynamics.*

Speed tracking

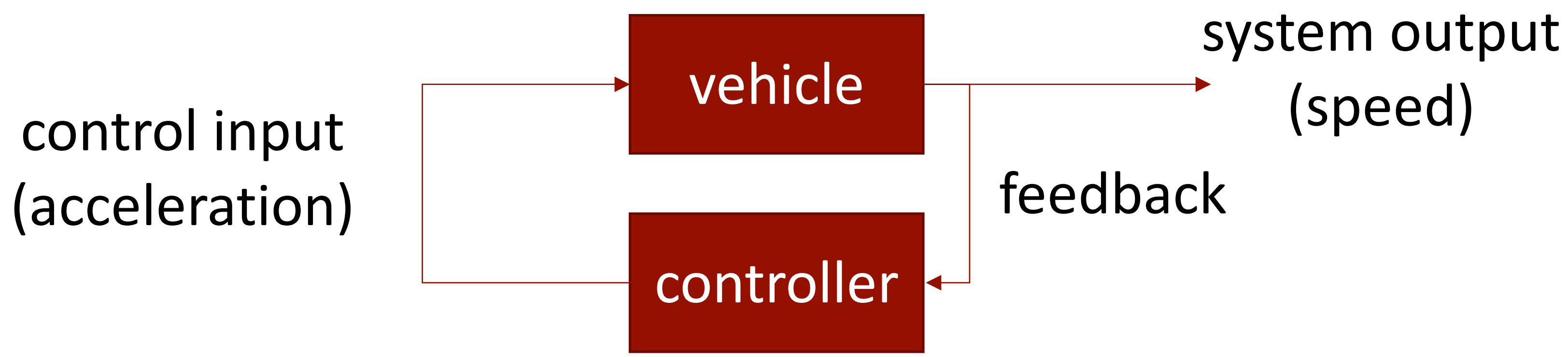
- Implementation of $u[t]$ is never perfect.
- To attain the desired acceleration, we need to go (at least) through the following:
- *Push the pedal -> inject gas -> generate engine torque -> power transmitted to axis -> tire force translated to propelling force -> acceleration.*
- None of the above legs can be perfectly measured or implemented.
- In other words, in practice we usually have

$$v[t+1] = v[t] + u[t]\delta + w[t],$$

- $w[t]$ is a noise term capturing unmodeled factors.

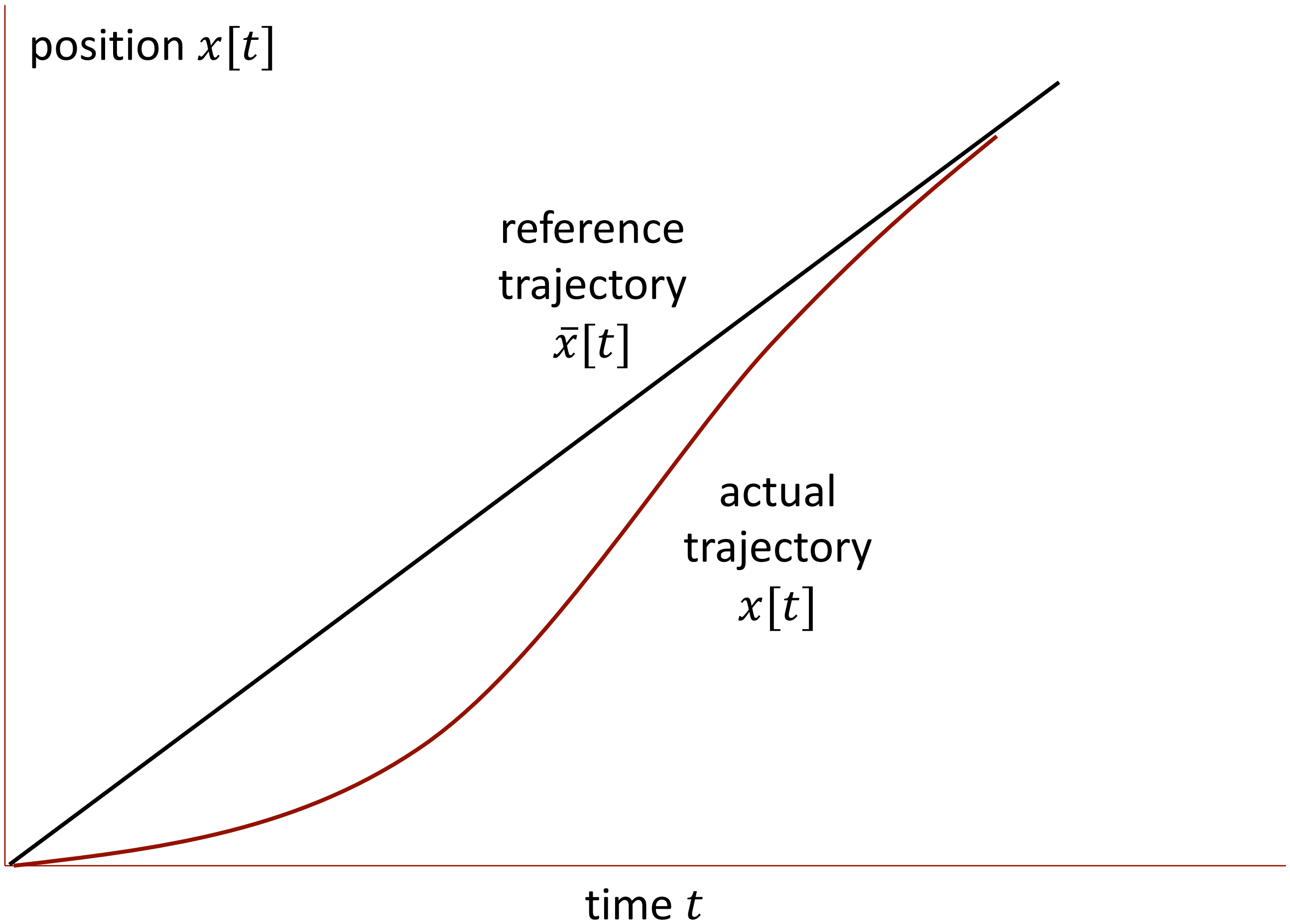
Speed tracking

- Closed-loop control is able to compensate for error and noise via feedback.



- Key idea: controller compares actual speed (output) with the desired speed (reference).
- If vehicle is faster than specified, then slow it down.
- If vehicle is slower than specified, then speed it up.
- How to quantify such intuition!

Trajectory tracking



Trajectory tracking

- Now we are ready to specify how the system evolves.
- State vector $[x[t], v[t]]^T$.
- Control input $u[t]$ (**same as speed tracking**)
- System dynamics:

$$\begin{bmatrix} x[t+1] \\ v[t+1] \end{bmatrix} = \begin{bmatrix} x[t] \\ v[t] \end{bmatrix} + \begin{bmatrix} v[t] \\ u[t] \end{bmatrix} \delta.$$

- δ = discrete time step size.
- This is the DT, state-space model for a vehicle.
- Since we are restricted to linear motion, this is called longitudinal control.

Trajectory tracking

- Since feedback control focuses on the deviation between actual and reference trajectories, we reformulate the model as follows.
- Tracking errors as states:

$$\tilde{x}[t] = x[t] - \bar{x}[t],$$
$$\tilde{v}[t] = v[t] - \bar{v}.$$

- Then, we have

$$\begin{aligned}
&\tilde{x}[t+1] = x[t+1] - \bar{x}[t+1] \\
&= (x[t] + v[t]\delta) - (\bar{x}[t] + \bar{v}\delta) \\
&= (x[t] - \bar{x}[t]) + (v[t] - \bar{v})\delta = \tilde{x}[t] + \tilde{v}[t]\delta, \\
&\tilde{v}[t+1] = v[t+1] - \bar{v} = (v[t] + u[t]\delta) - \bar{v} \\
&= (v[t] - \bar{v}) + u[t]\delta = \tilde{v}[t] + u[t]\delta.
\end{aligned}$$

Trajectory tracking

- Hence, the system is convergent if

$$\begin{bmatrix}\tilde{x}[t]\\ \tilde{v}[t]\end{bmatrix} \to \begin{bmatrix}0\\ 0\end{bmatrix}.$$

- With the linear controller $u[t] = -k_1(x[t] - \bar{x}[t]) - k_2(v[t] - \bar{v})$, we have

$$\begin{bmatrix}\tilde{x}[t+1]\\ \tilde{v}[t+1]\end{bmatrix} = \begin{bmatrix}1 & \delta\\ 0 & 1\end{bmatrix}\begin{bmatrix}\tilde{x}[t]\\ \tilde{v}[t]\end{bmatrix} + \begin{bmatrix}0\\ \delta\end{bmatrix}\begin{bmatrix}-k_1 & -k_2\end{bmatrix}\begin{bmatrix}\tilde{x}[t]\\ \tilde{v}[t]\end{bmatrix}$$

$$= \begin{bmatrix}1 & \delta\\ -k_1\delta & 1-k_2\,\delta\end{bmatrix}\begin{bmatrix}\tilde{x}[t]\\ \tilde{v}[t]\end{bmatrix}.$$

- Hence, we have

$$\begin{bmatrix}\tilde{x}[t]\\ \tilde{v}[t]\end{bmatrix} = \begin{bmatrix}1 & \delta\\ -k_1\delta & 1-k_2\,\delta\end{bmatrix}^t \begin{bmatrix}\tilde{x}[0]\\ \tilde{v}[0]\end{bmatrix}$$

initial condition

Longitudinal control

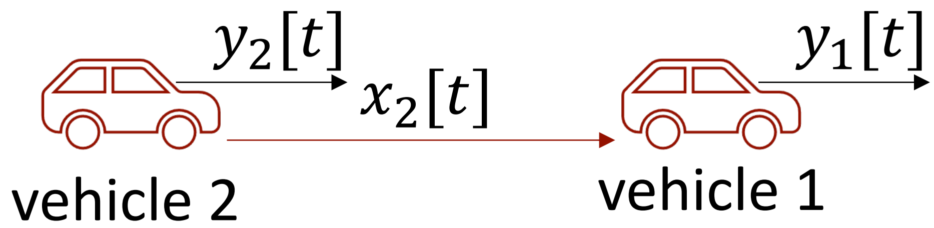
$$\dot{v} = -av^2 + (b_1 T_e - b_2 T_b) - c.$$

Main messages

- We can tune the speed by playing with the torques
- We can consider the total torque
$$T = b_1 T_e - b_2 T_b, \qquad T_e \geq 0, \qquad T_b \geq 0.$$
- One T is selected, we translate it to T_e or T_b.
- Note that either $T_e = 0$ or $T_b = 0$.
$$\dot{v} = -av^2 + T - c.$$
- We can reformulate the longitudinal dynamics in the standard state-space representation.

Platooning

State variables:



- Vehicle 1:
  - $x_1 \in \mathbb{R}$: deviation from reference position
  - $v_1 \in \mathbb{R}$: speed; or $y_1 \in \mathbb{R}$: relative speed w.r.t. reference speed.
- Vehicle 2:
  - $x_2 \in \mathbb{R}$: deviation from reference position (see below)
  - $v_2 \in \mathbb{R}$: speed; or $y_2 \in \mathbb{R}$: relative speed w.r.t. reference

Control objective:

- Speed: we want both vehicles to travel at speed $\bar{v}$.
- Spacing: we want a spacing of d between two vehicles.

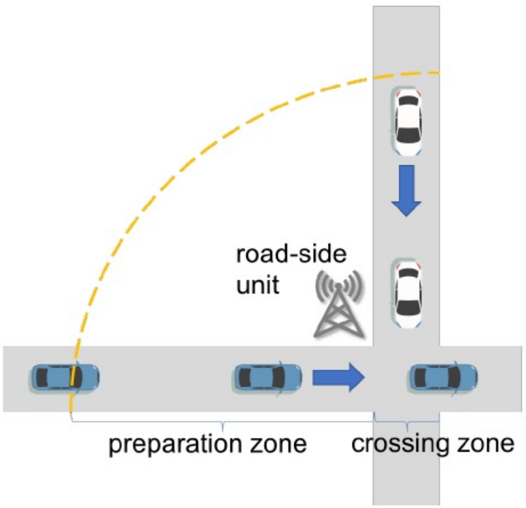
Platooning

- State variable: $x_2[t], y_2[t]$
- $x_2[t] = 0$ if exactly d away from vehicle 1.
- Control input: $u_2[t]$ = acceleration; input space = $\mathbb{R}$
- Note that $x_2[t+1] = x_2[t] + y_2[t]\delta$.
- Dynamic equation

$$\begin{bmatrix} x_2[t+1] \\ y_2[t+1] \end{bmatrix} = \begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_2[t] \\ y_2[t] \end{bmatrix} + \begin{bmatrix} 0 \\ \delta \end{bmatrix} (u_2[t] - u_1[t]).$$

- Equilibrium state: $x_2 = 0, y_2 = 0$.
- That is, spacing = d and speed = $\bar{v}$.
- We proceed as if this is trajectory tracking.

Trajectory for multiple vehicles



- Consider a collection of vehicles

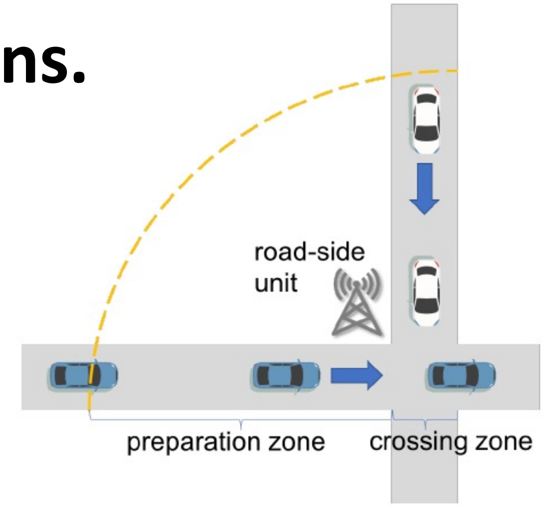
$$1,2,\ldots,n.$$

- Each vehicle arrive at time

$$s_1, s_2, \ldots, s_n.$$

- Crossing times $t_1, t_2, \ldots, t_n$.
- Initial speeds $\phi_1, \phi_2, \ldots, \phi_n$.
- For each vehicle i, we need to select the time series of acceleration $u_i[t]$ for $s_i \leq t < t_i$.
- Similar to the single-vehicle, problem, but with significant differences in terms of formulation and presentation!

Trajectory for multiple vehicles



First complication 1: multiple directions.

- For ease of presentation, consider two orthogonal orbits only.
- Only keeping straight; no turning.
- Label these two directions as 1 & 2.
- So, we need to adjust the notations:
- For direction $k \in \{1,2\}$, there are n_k vehicles with

  Arrival times: $s_1^k, s_2^k, \dots, s_{n_k}^k$;

  Crossing times: $t_1^k, t_2^k, \dots, t_{n_k}^k$;

  Initial speeds: $\phi_1^k, \phi_2^k, \dots, \phi_{n_k}^k$.

Trajectory for multiple vehicles

First complication 2: safe distance.

- For vehicles from the same direction

$$x_{i-1}^k[t] - x_i^k[t] \geq d + hv_i^k[t],$$

for all $i = 1,2, \ldots, n_k$ and for $k = 1,2,$

for all t: $s_i^k \leq t \leq t_{i-1}^k$.

- For vehicles on different orbits,

$$\left|x_i^1[t] - x_j^2[t]\right| \geq d'$$

for all $1 \leq i \leq n_1$ and $1 \leq j \leq n_2$,

for all t: $\max\left\{t_i^1 - T, t_j^2 - T\right\} \leq t \leq \min\left\{t_i^1, t_j^2\right\}$.

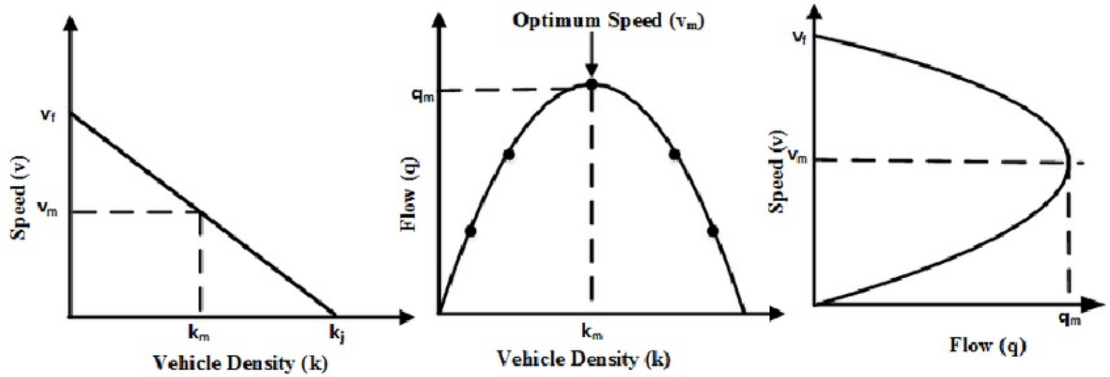
- Need to know how to write the index conditions!

Trajectory for multiple vehicles

$$
\begin{aligned}
\text{min} \quad & \textstyle\sum_{t=0}^{T} \sum_k \sum_i \left(u_i^k[t]\right)^2 \\
\text{s.t.} \quad & x_i^k[t+1] = x_i^k[t] + v_i^k[t]\delta, \forall i, \forall t, \forall k, \\
& v_i^k[t+1] = v_i^k[t] + u_i^k[t]\delta, \forall i, \forall t, \forall k, \\
& x_{i-1}^k[t] - x_i^k[t] \geq d + h v_i^k[t], \forall i, \forall t, \forall k, \\
& \left| x_i^1[t] - x_j^2[t] \right| \geq d', \forall i, \forall j, \forall t, \\
& 0 \leq v_i^k[t] \leq \bar{v},\ -\bar{a} \leq u_i^k[t] \leq \bar{a}, \forall i, \forall t, \forall k, \\
& x_i^k\left[s_i^k\right] = 0,\ v_i^k\left[s_i^k\right] = \phi_i^k, \forall i, \forall k, \\
& x_i^k\left[t_i^k\right] = L, \forall i, \forall k.
\end{aligned}
$$

Greenshields model

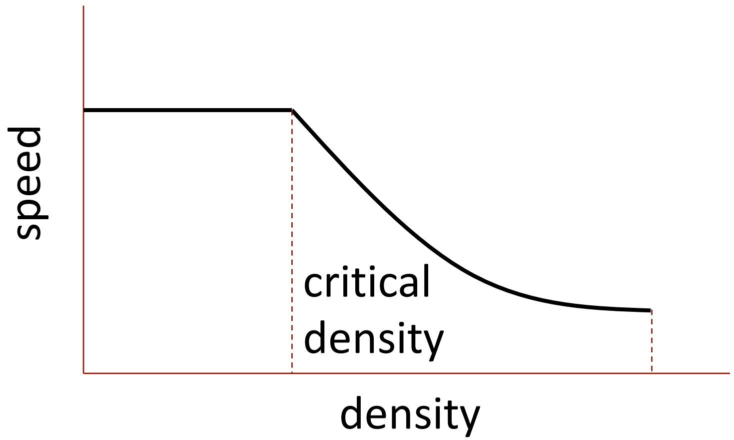
- Fundamental assumption: $v = v_0(1 - \rho/\bar{\rho})$
- v_0 = free-flow speed, $\bar{\rho}$ = jam (maximal) density

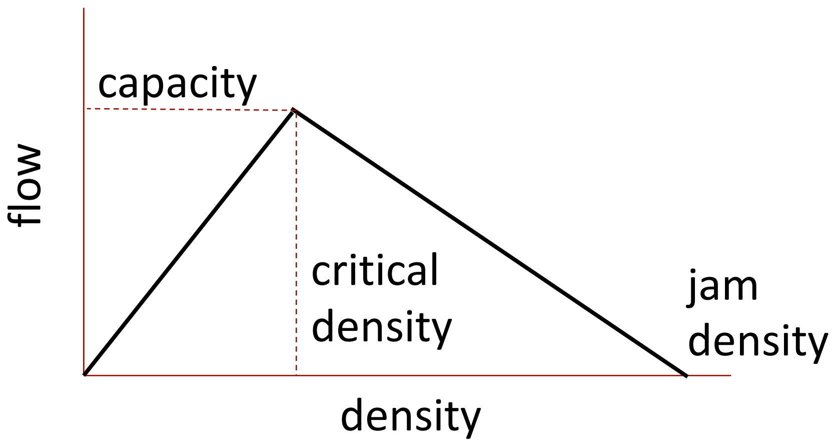


- Traffic & world in 1930s…

Triangular model

- Greenshields model is problematic in the low-density regime.
- At a low density, speed is not affected by density.
- Speed is affected until the density passes a threshold, called critical density.
- Hence, we have a modification as follows:



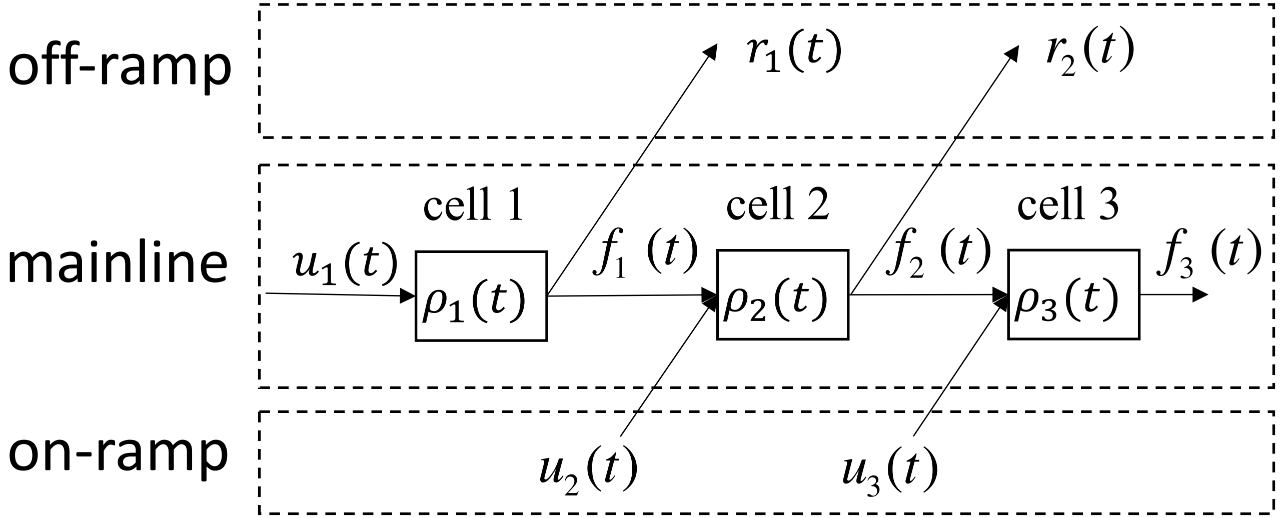


Cell transmission model

- Dynamic equation: mass conservation

$$\rho_k(t+1) = \rho_k(t) + \frac{1}{l_k}\big(f_{k-1}(t) + u_k(t) - f_k(t) - r_k(t)\big)$$

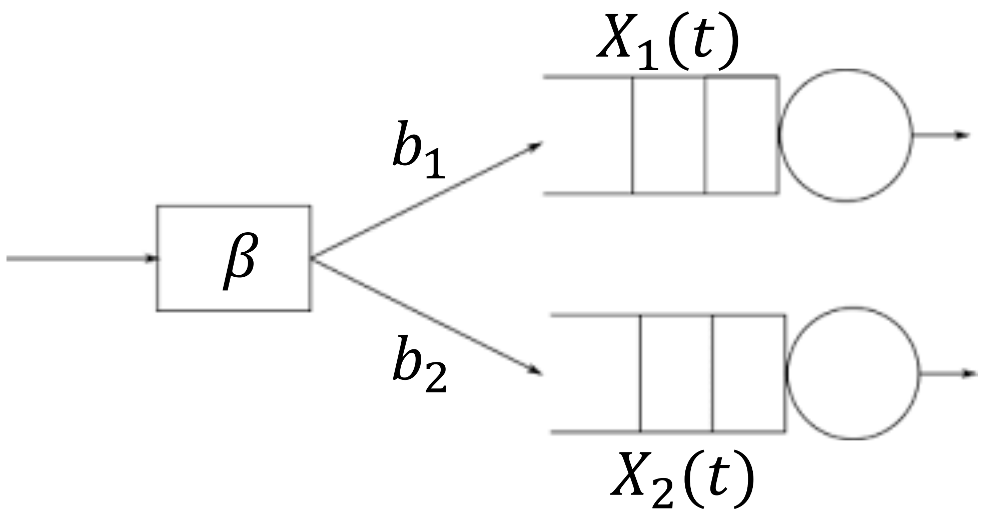
- l_k = length of cell k
- Nonlinear dynamical system with state $\rho[t] \in [0, \bar{\rho}]^n$.



Routing for parallel queues

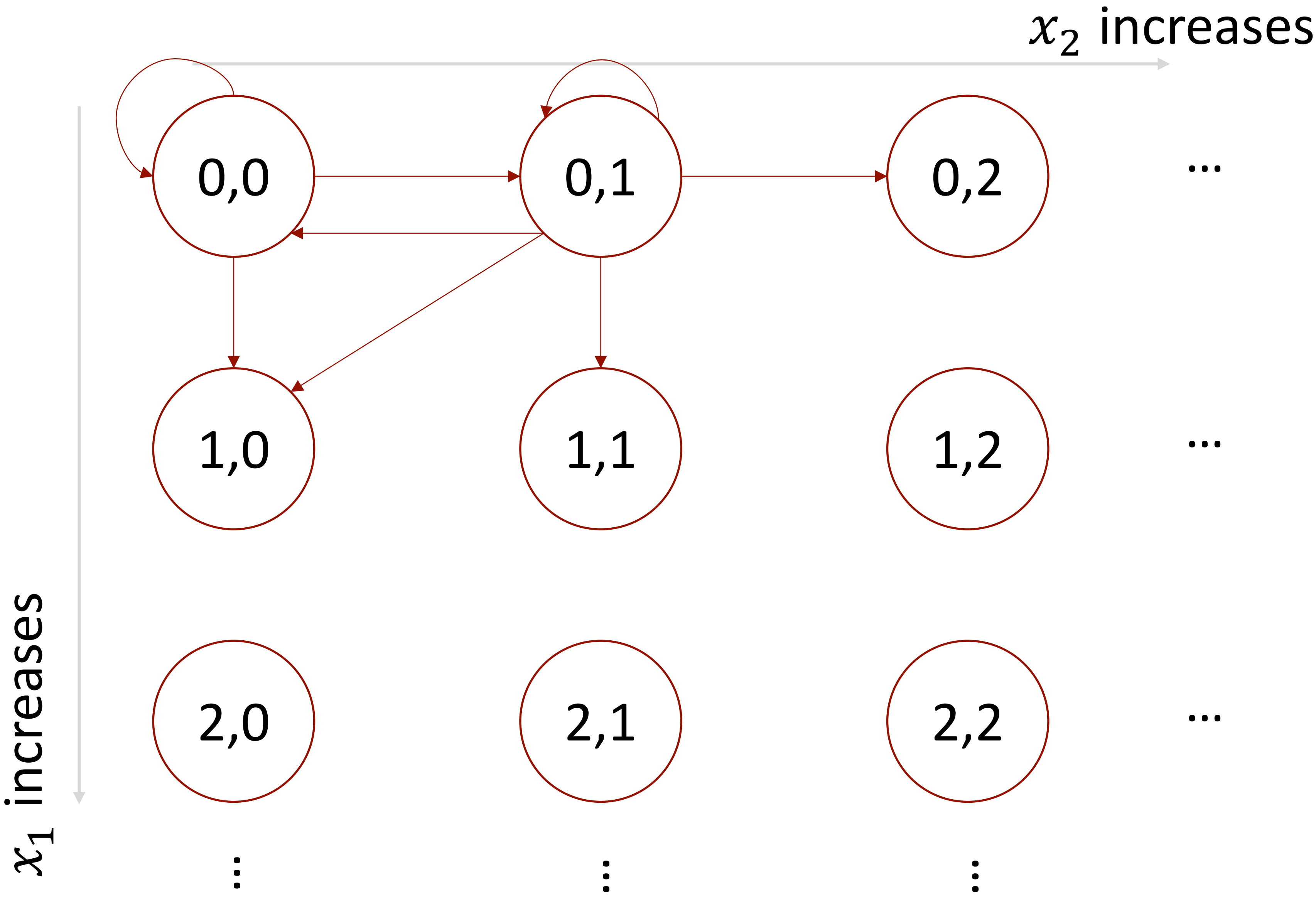
- Customers arrive at a router with rate λ
- Router assigns customer to one of two parallel queues
- State: $X[t] = [X_1[t] X_2[t]]^T \in \mathbb{Z}_{\geq 0}^2$
- Routing policy:

$$\beta: \mathbb{Z}_{\geq 0}^2 \to [0,1]^2 \text{ or } \beta: \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \text{ s.t. } b_1 + b_2 = 1.$$



Bernoulli (static) routing:
$\beta(x)$ independent of x.
Dynamic routing:
$\beta(x)$ dependent of x

State-transition diagram (Bernoulli routing)



State-transition diagram (JSQ routing)

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Problem 1

- What are some pros and cons of closed-loop control compared with open-loop control? Please name 2 pros and 2 cons.
- Why introducing distributed energy sources (DERs) can reduce the total transmission cost on a smart grid?

Problem 2

Consider the 1-dimensional autonomous vehicle (AV) control problem. Suppose that the dynamic equation is

$$\dot{v} = bT_e - a_1 v - a_2 v^2,$$

where v is the speed of the AV, T_e is the engine torque, a_1, a_2, b are positive constants. For ease of presentation, we assume that $v \geq 0$. We can directly control T_e. Our objective is to attain speed $\bar{v}$.

- Is this a linear or nonlinear system?
- Define asymptotic convergence in terms of v and $\bar{v}$.
- Suppose that we use a linear controller $T_e = k_1(v - \bar{v}) + k_2$. Should k be positive or negative?
- Suppose that we have an adaptive controller $T_e = \kappa v$, where κ can adjust itself with respect to the environment. Do you expect $|\kappa|$ to increase or decrease as the number of passengers increases?

Problem 3

Consider 3 parallel queues at 3 parallel servers. Suppose that the total demand is $\lambda = 0.3$ and the servers have service rates $\mu_1 = \mu_2 = 0.2, \mu_3 = 0.1$.

- (5 points) If we use Bernoulli routing with probabilities $b = [b_1, b_2, b_3]^T$, what are the constraints for b?
- (5 points) For the optimal (i.e., queue-minimizing) Bernoulli routing probabilities $b^* = [b_1^*, b_2^*, b_3^*]^T$, do you expect b_1^* to be less than, equal to, or greater than b_2^*? How about b_1^* vs. b_3^*?
- (5 points) Suppose that we use "join-the-shortest-queue" (JSQ) policy. What are some pros and cons of this policy compared with Bernoulli routing?

Q&A