Homework 2

ECE4530J - Decision Making in Smart Cities Summer 2022

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Problem 1

Consider the trajectory tracking problem with acceleration being the control input:

$$\begin{bmatrix} x[t+1] \\ v[t+1] \end{bmatrix} = \begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x[t] \\ v[t] \end{bmatrix} + \begin{bmatrix} 0 \\ \delta \end{bmatrix} u[t]$$

Let $\bar{x}[t] = \bar{v}t$ be the reference trajectory to track.

a. Reformulate the state-space model with the position and speed errors being the state.

b. Using the reformulated model, find a linear controller that stabilizes the system.

c. Find a linear controller that destabilizes the system.

Answer:

a. Since we have tracking errors as states:

$$\tilde{x}[t] = x[t] - \bar{x}[t],$$
$$\tilde{v}[t] = v[t] - \bar{v}.$$

Then, we have

$$\begin{aligned} \tilde{x}[t+1] &= x[t+1] - \bar{x}[t+1] \\ &= (x[t] + v[t]\delta) - (\bar{x}[t] + \bar{v}\delta) \\ &= (x[t] - \bar{x}[t]) + (v[t] - \bar{v})\delta = \tilde{x}[t] + \tilde{v}[t]\delta, \\ \tilde{v}[t+1] &= v[t+1] - \bar{v} = (v[t] + u[t]\delta) - \bar{v} \\ &= (v[t] - \bar{v}) + u[t]\delta = \tilde{v}[t] + u[t]\delta. \end{aligned}$$

Hence, the system is convergent if

$$\begin{bmatrix} \tilde{x}[t] \\ \tilde{v}[t] \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence, we have

$$\begin{bmatrix} \tilde{x}[t] \\ \tilde{v}[t] \end{bmatrix} = \begin{bmatrix} 1 & \delta \\ -k_1\delta & 1 - k_2\delta \end{bmatrix}^t \begin{bmatrix} \tilde{x}[0] \\ \tilde{v}[0] \end{bmatrix}$$

b. The above formula clearly indicates that the system is convergent if and only if

$$\lim_{t\to\infty} \begin{bmatrix} 1 & \delta \\ -k_1\delta & 1 - k_2\delta \end{bmatrix}^t = 0.$$

.We have $\mu(x, v) = -k_1(x[t] - \bar{x}[t]) - k_2(v[t] - \bar{v})$ and since $\bar{x}[t] = \bar{v}t$. We have $\mu(x, v) = -k_1(x[t] - \bar{v}t) - k_2(v[t] - \bar{v})$

c. Since we need to let the system become non-linear, we can change the linear controller to $\mu(x, v) = -k_1(3x[t] - \bar{v}) - k_2(4v[t] - 5\bar{v})$

Problem 2

Consider a one-dimensional non-linear system

$$\dot{x} = a_1 x + a_2 x^2 + bu + c.$$

a. Use Taylor expansion to linearize the RHS of the dynamical equation in the neighborhood of $x = 0$.

b. For the linearized system, design a linear controller $\mu(x)$ that stabilizes the linearized system. Hint: a linear system $\dot{x} = \tilde{a}x$ is stable if and only if $\mathrm{Re}(\tilde{a}) < 0$.

c. For the continuous-time system, design a controller $\mu(x)$ such that, with $u = \mu(x)$, the RHS of the dynamical equation is linear. Hint: do not confuse this part with part a).

Answer:

a. Suppose that the relative speed x is not far away from the equilibrium speed 0 .

$$\begin{aligned}\dot{x} &\approx \frac{\partial}{\partial x}[a_1 x + a_2 x^2 + bu + c]_{x=0} x \\ &+ \frac{\partial}{\partial u}[a_1 x + a_2 x^2 + bu + c]_{x=0} u + c \\ &= a_1 x + bu + c\end{aligned}$$

b. Select a desired stabilized linear system, e.g., $\dot{x} = \tilde{a}x$

Design a controller to realize the above linear system:

$$\mu(x) = -\frac{a_1}{b}x + \tilde{a}x - \frac{c}{b}$$

c. Since we need to let the system become linear, the quadratic item should be zero. Exactly,

$$\mu(x) = -\frac{a_2}{b}x^2$$

Problem 3

Consider a two-vehicle platoon with states $\begin{bmatrix} x_1[t] \\ v_1[t] \end{bmatrix}, \begin{bmatrix} x_2[t] \\ v_2[t] \end{bmatrix}$. Vehicle 1 tracks a pre-specified trajectory $\bar{x}[t] = \bar{v}t, t = 0, 1, 2, \ldots$ Vehicle 2 follows vehicle 1 to keep a spacing of d to vehicle 1. The inputs are the engine torques $\tau_1[t]$ and $\tau_2[t]$. For i = 1,2, the propelling force is given by $\alpha\tau_i$, while the resistant force is given by βv_i^2. The vehicle masses are m_1, m_2, respectively.

a. Formulate Newton's second law for both vehicles.

b. Formulate the state-space model for both vehicles using absolute position and speed as the state.

c. Reformulate the model with the tracking/following errors being the state.

d. Construct a trajectory-tracking policy for vehicle 1, i.e., a function that maps tracking errors e_1, y_1 to τ_1. Explain why the policy will work. No mathematical proof needed.

e. Construct a vehicle-following algorithm for vehicle 2, i.e., a function that maps x_1, v_1, x_2, v_2 to τ_2. Explain why the policy will work. No mathematical proof needed.

Answer:

a. According to Newton's second law, we have

$$m\dot{v} = F_p - F_r$$

In this problem, we have $F_p = \alpha\tau_i$ and $F_r = \beta v_i^2$ As a result,

$$m_1\dot{v}_1 = \alpha\tau_1 - \beta v_1^2$$
$$m_2\dot{v}_2 = \alpha\tau_2 - \beta v_2^2$$

b. We need to use (s, v) to do the state-space model.

Absolute speeds: v_1, v_2

Absolute positions: s_1, s_2.

$$\begin{bmatrix} s_1[t+1] \\ v_1[t+1] \\ s_2[t+1] \\ v_2[t+1] \end{bmatrix} = \begin{bmatrix} v_1[t] \\ \frac{\alpha}{m_1}\tau_1[t] - \frac{\beta}{m_1}v_1^2[t] \\ v_2[t] \\ \frac{\alpha}{m_2}\tau_2[t] - \frac{\beta}{m_2}v_2^2[t] \end{bmatrix}$$

c. $x_1[t], y_1[t]$ (tracking errors) and $x_2[t], y_2[t]$ (following errors)

$$\begin{bmatrix} x_1[t+1] \\ y_1[t+1] \end{bmatrix} = \begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1[t] \\ y_1[t] \end{bmatrix} + \begin{bmatrix} 0 \\ \delta \end{bmatrix} u_1[t]$$

$$\begin{bmatrix} x_2[t+1] \\ y_2[t+1] \end{bmatrix} = \begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_2[t] \\ y_2[t] \end{bmatrix} + \begin{bmatrix} 0 \\ \delta \end{bmatrix} (u_2[t] - u_1[t])$$

d. Since we have

$$m_1\dot{v}_1 = \alpha\tau_1 - \beta v_1^2$$

Exactly,

$$\dot{v}_1 = \frac{\alpha\tau_1 - \beta v_1^2}{m_1}$$

and $e(t) = \bar{v} - v(t)$ Since $v = \bar{v} - e$, we have $\dot{v} = -\dot{e}$ and thus

$$-\dot{e} = \frac{\alpha\tau_1 - \beta(\bar{v_1} - e_1)^2}{m_1}$$

e.