12. Traffic Network Optimization

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Outline

- Network systems
  - Transportation
  - Electricity
  - Communications
- Network optimization
  - Network flow model
  - Shortest-path problem
  - Max-flow problem
  - Min-cost flow problem
- Linear programming formulation

Transportation: Planning

- Transportation planning is the process of defining future policies, goals, investments, and spatial planning designs to prepare for future needs to move people and goods to destinations.
  - Should we build a bridge here?
  - Should we build a shopping mall here?
  - Should we build subway here?
- To do the planning, we need to predict and optimize the consequent traffic flow.

Transportation: Traffic control

- During morning/evening peak hours, traffic managers may need to intervene people’s route choice.
  - Navigation tool
  - Traveler information
  - Congestion pricing
- Objective: distribute traffic flow to minimize congestion

Transportation: Logistics

- Logistic companies receive packages at stations.
- Packages flow between stations.
- Limited/costly resource for package transportation.
  - How to allocate delivery trucks?
  - How to allocate station capacities?
  - Road or rail or air?

Electricity distribution

- Power stations generate electricity
- Transformers adjust voltage
- Distribution lines transmit power
- Consumers receive power
- Flow of power/current

Communications network

- Flow of data packets between terminals
- Cabled or wireless connectivity
- Routers distribute data flow between terminals
- Latency, bandwidth

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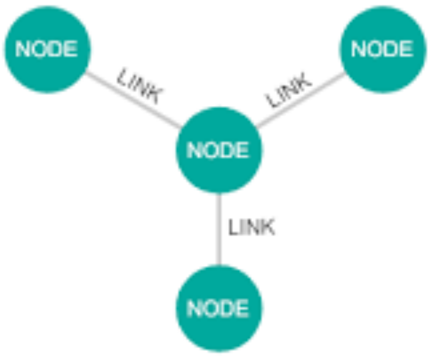
Min-cost flow problem

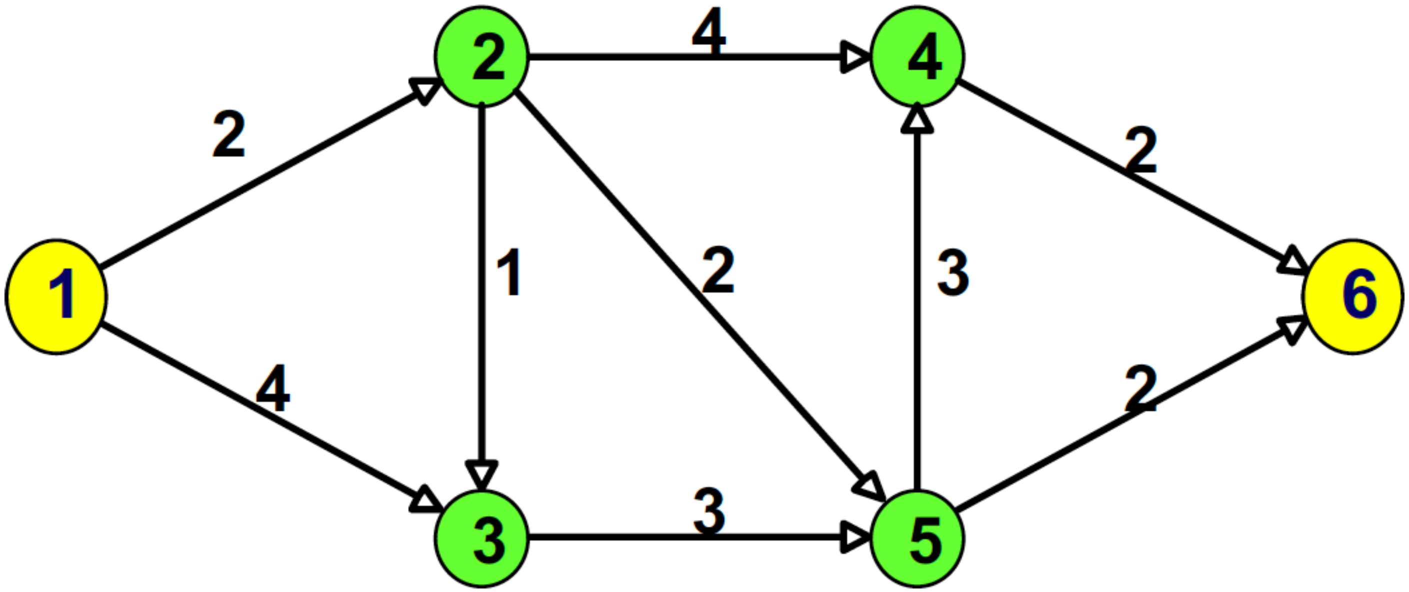
- Data:
  - Demand
  - Cost function
  - Capacity
- Decision variables:
  - Link flows
- Constraints:
  - Mass conservation
  - Link capacity
- Objective:
  - Minimize total flow cost

Network model

- Consider a network with nodes N and links E
  - Network also called graph
  - Nodes also called vertices (singular: vertex)
  - Links also called edges/arcs
- We use integers to label nodes
  - Node 1, 2,…
- We use pairs of integers to label links
  - Link (1,2), (2,3),…
- Directed link (i,j)
- Undirected link (i,j)



Example



- Set of nodes

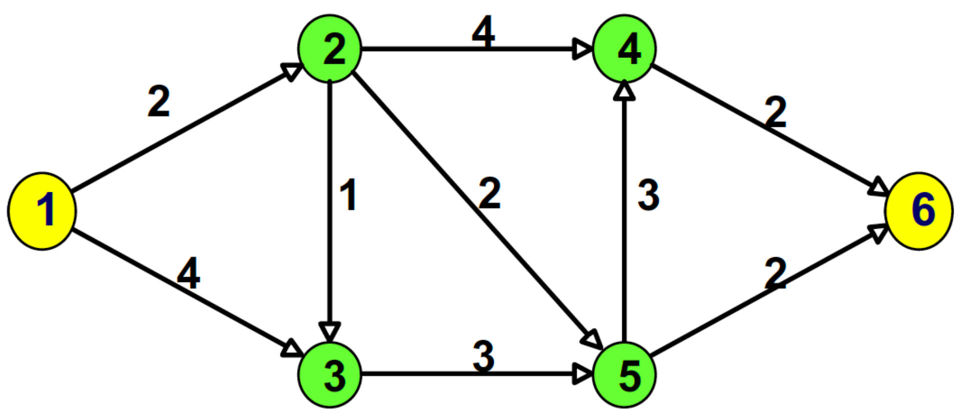
$$N = \{1,2,3,4,5,6\}.$$

- Set of links

$$E = \{(1,2), (1,3), (2,3), (2,4), (2,5), (3,5), (4,6), (5,4), (5,6)\}.$$

Adjacency matrix

	(1,2)	(1,3)	(2,3)	(2,4)	(2,5)	(3,5)	(4,6)	(5,4)	(5,6)
1	-1	-1							
2	1		-1	-1	-1				
3		1	1			-1			
4				1			-1	1	
5					1	1		-1	-1
6							1		1



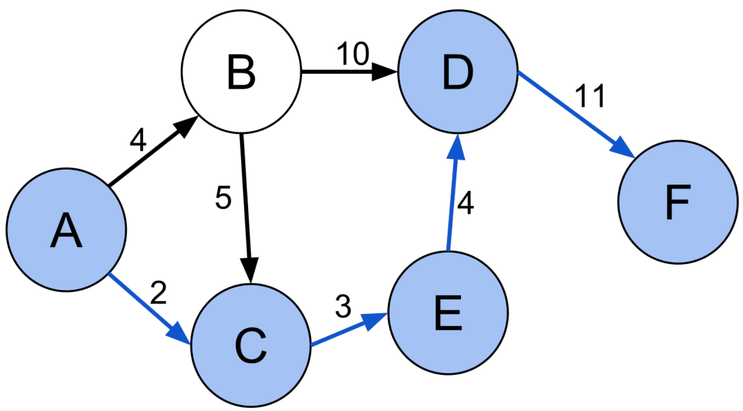
Demand & flow

- Each node i is associated with demand b_i
  - $b_i > 0$: traffic flows in (origin)
  - $b_i < 0$: traffic flows out (destination)
  - $b_i = 0$: traffic flows through (transmission)
- Each link (i, j) is associated with flow f_{ij}
  - $f_{ij} > 0$: flow from i to j
  - $f_{ij} < 0$: flow from j to i
- Mass conservation: inflow = outflow

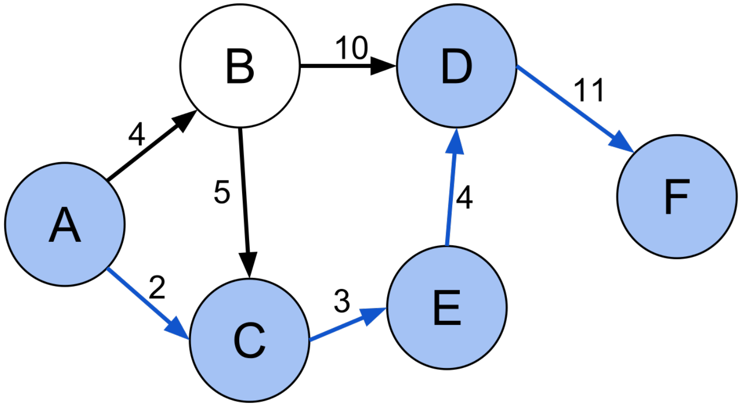
$$b_i + \sum_{j \in In(i)} f_{ji} = \sum_{j \in Out(i)} f_{ij}$$

Shortest path problem

- Shortest path problem is the problem of finding a path between two vertices (or nodes) in a graph such that the sum of the weights of its constituent edges is minimized.
- Can be formulated as a network flow optimization with unit demand and unit cost



Shortest path problem



- Decision variables:

$f_{AB}, f_{AC}, f_{BC}, f_{BD}, f_{CE}, f_{DF}, f_{ED}$

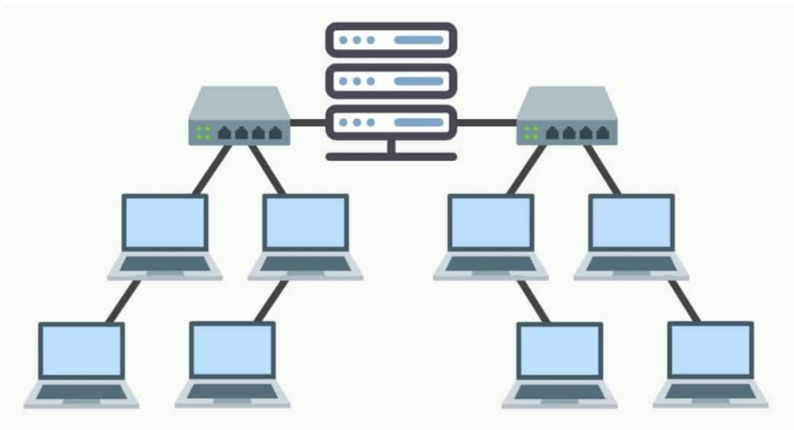
- Objective function:

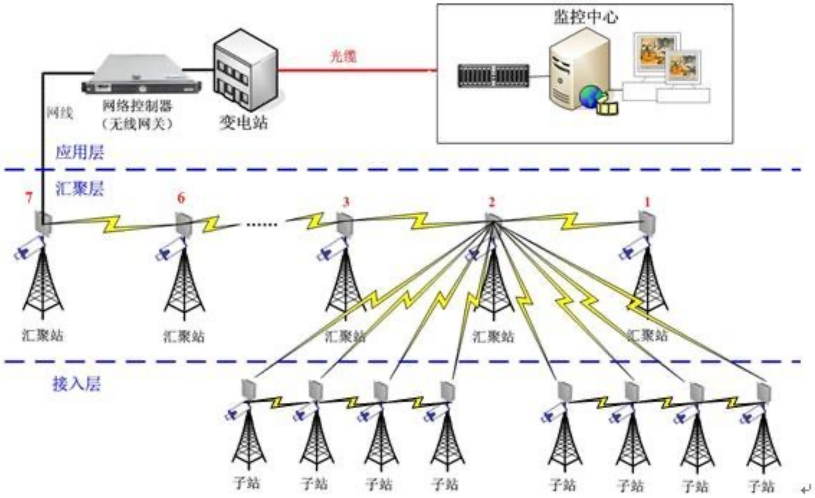
$$z = 4f_{AB} + 2f_{AC} + 5f_{BC} + 10f_{BD} + 3f_{CE} + 11f_{DF} + 4f_{ED}$$

- Constraints:

$1 = f_{AB} + f_{AC}, f_{AB} = f_{BD} + f_{BC}, f_{AC} + f_{BC} = f_{CE},$
$f_{BD} + f_{ED} = f_{DF}, f_{CE} = f_{ED}, -1 + f_{DF} = 0,$
all flows are non-negative.

Min-cost flow problem





Capacity

- Sometimes we impose an upper bound on flows
$$f_{ij} \leq \bar{f}_{ij}$$
- Capacity $\bar{f}_{ij}$ depends on
  - Road: # of lanes, type of surface, weather
  - Electricity: transmission link capacity
  - Communications: bandwidth
- $\bar{f}_{ij} < \infty$: capacitated problem (harder)
- $\bar{f}_{ij} = \infty$: uncapacitated problem (easier)

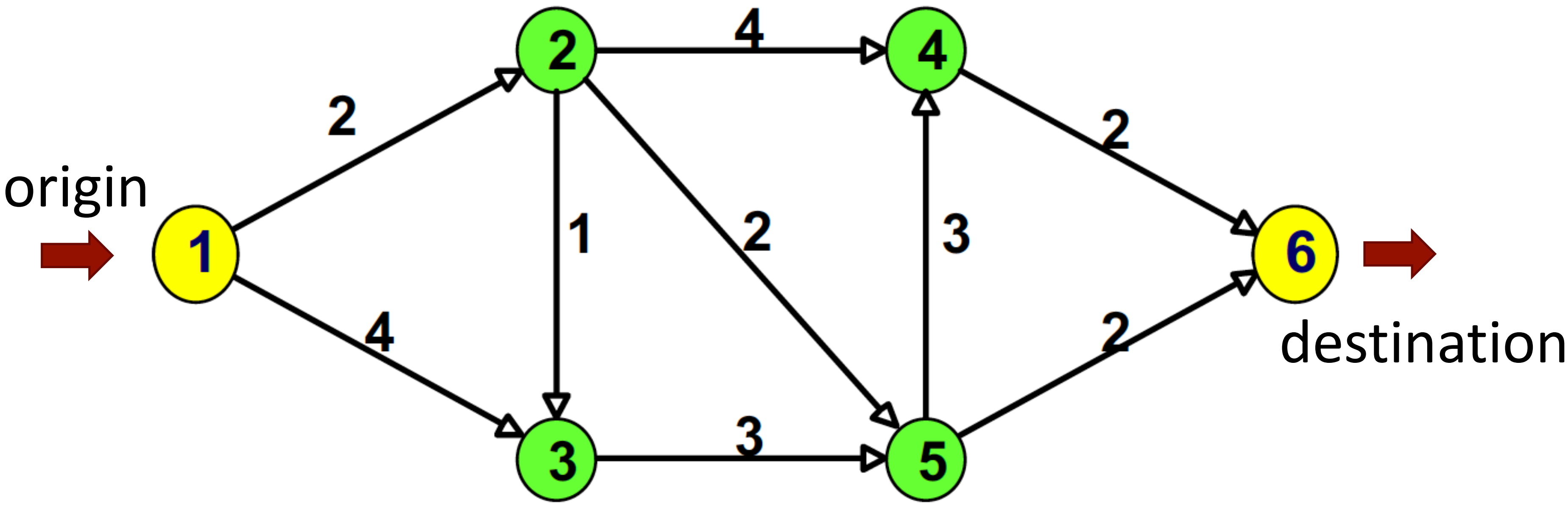
Max-flow problem

- Consider a network $G = (N, E)$
- Consider an origin (source) $s \in N$ and a destination (sink) $t \in N$.
- Link e has a capacity $u_e \in \mathbb{R}_{>0}$.

What is the maximal flow that can be transmitted from s to t?

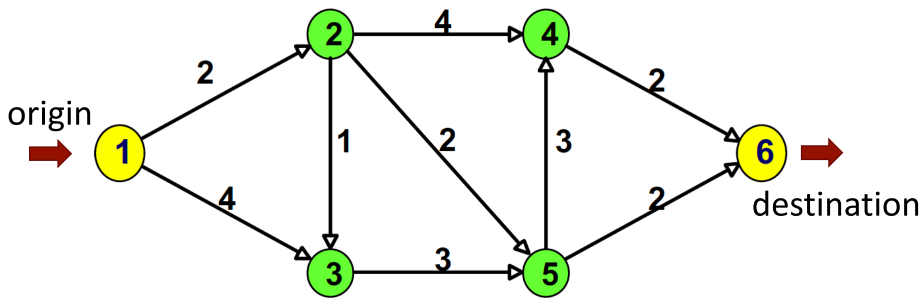
- Very practical question to ask in many engineering settings.
- Related keywords: capacity, throughput.

Example



Every link has a capacity of 10

Example



$$\max d$$

s.t. $d = f_{12} + f_{13}$

$f_{12} = f_{23} + f_{24} + f_{25}$

$f_{13} + f_{23} = f_{35}$

$f_{24} + f_{54} = f_{46}$

$f_{35} + f_{25} = f_{54} + f_{56}$

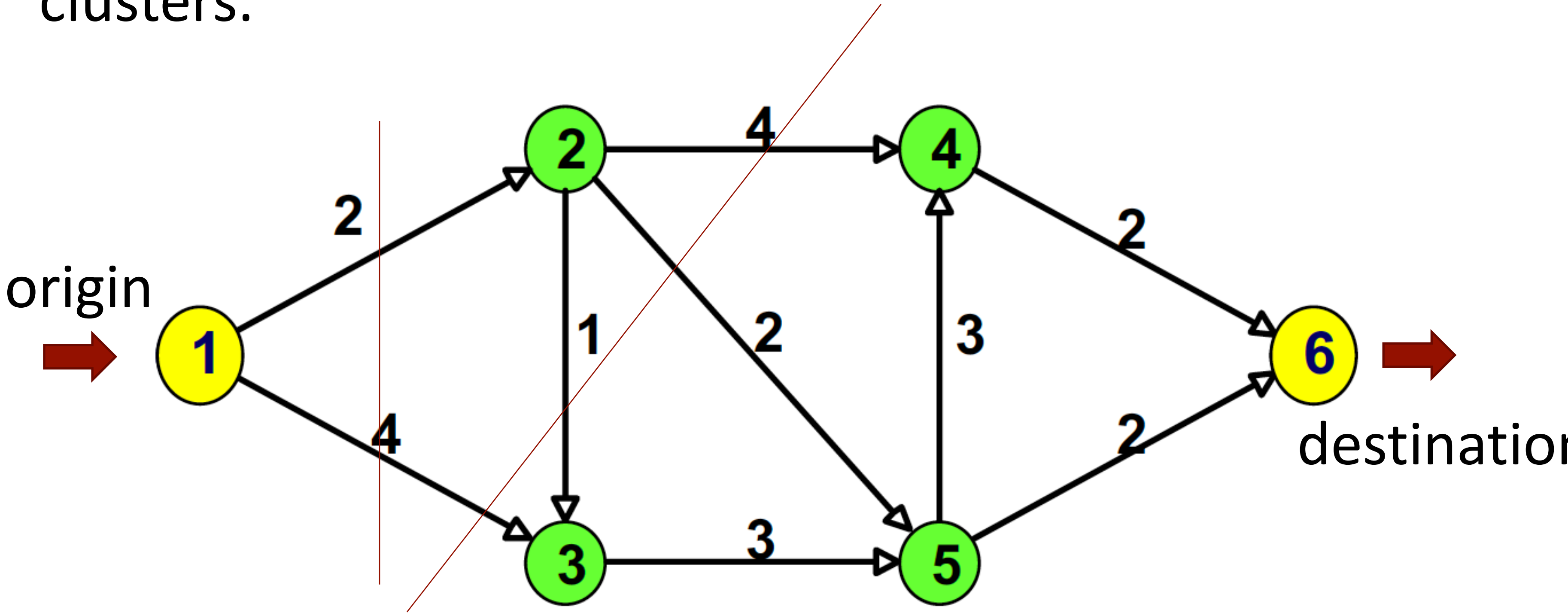
$f_{46} + f_{56} = d$

all flows are non-negative.

$f_{ij} \leq 10$ for all $(i, j) \in E$.

Max-flow min-cut theorem

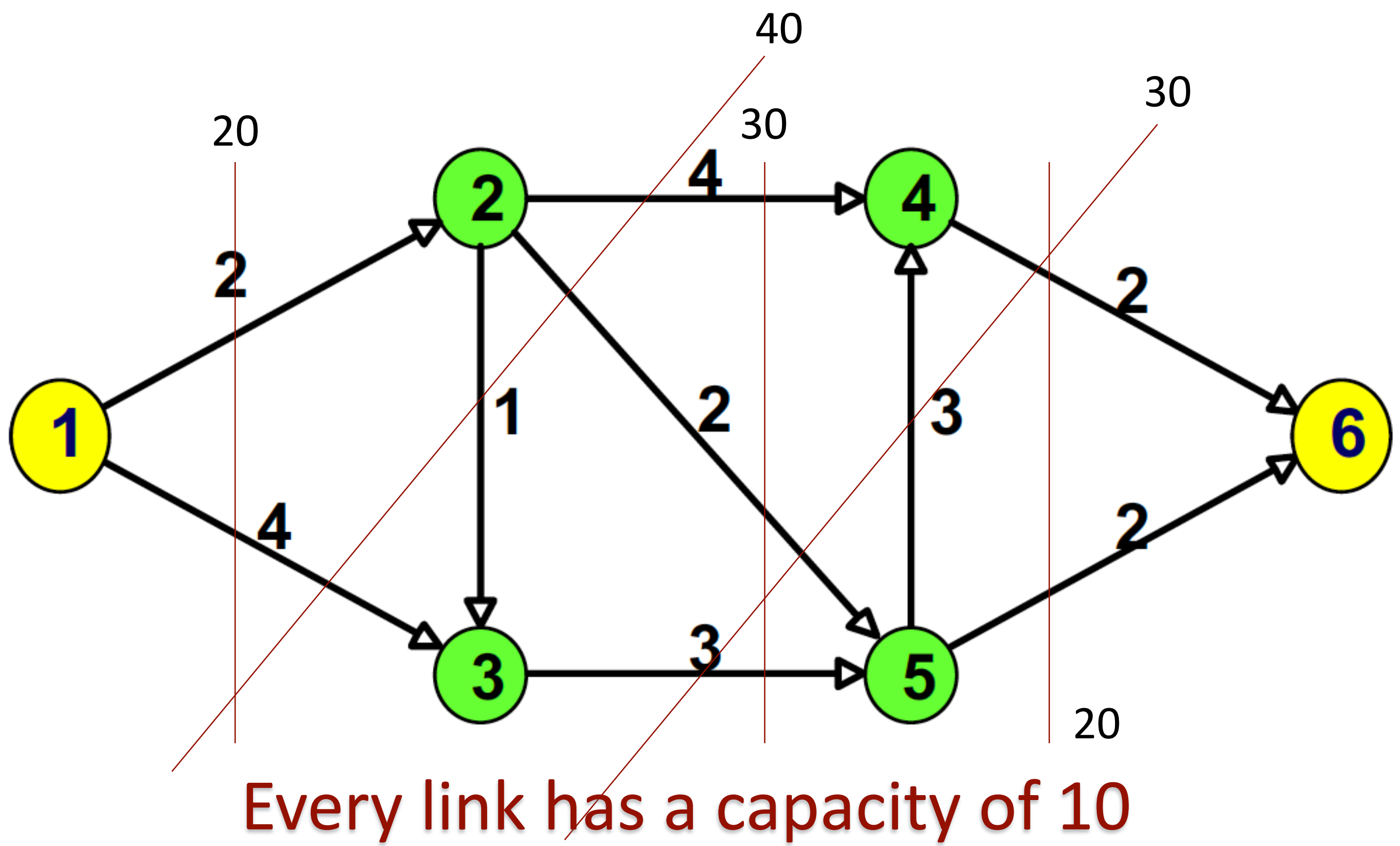
- **Theorem**: Max flow = min cut.
- Cut = a set of links that partition the nodes into two clusters.



Every link has a capacity of 10

Max-flow min-cut theorem

- Cut can also mean the total capacity of the set of links.
- Note: only count from one side to another side.



Every link has a capacity of 10

Sports elimination problem

- Given the standings in a sports league at some point during the season, which teams have been mathematically eliminated from winning the league?
- A simplification: a game cannot end as a draw.
- Basketball, baseball, etc.
- Max-flow problem...

排名	球队	场数	胜/负	胜率	近况
1	广东	5	5/0	100%	5连胜
2	广厦	4	4/0	100%	4连胜
3	山西	5	4/1	80%	3连胜
4	辽宁	5	4/1	80%	1连胜
5	浙江	5	4/1	80%	2连胜
6	深圳	5	4/1	80%	4连胜
7	北京	4	3/1	75%	2连胜
8	上海	5	3/2	60%	1连胜

Sports elimination problem

- This is a quite non-trivial problem:
- For a particular team to win the league, the team in question themselves have to win as many games as possible.
- At the same time, all the other teams can “collaborate” to produce results favorable to the team in question.
- For larger leagues, we cannot eyeball whether such collaboration will work.

国际

国足成功晋级12强赛 英媒：半个亚洲来“帮忙”

2016年03月30日 12:49:20 来源： 参考消
网

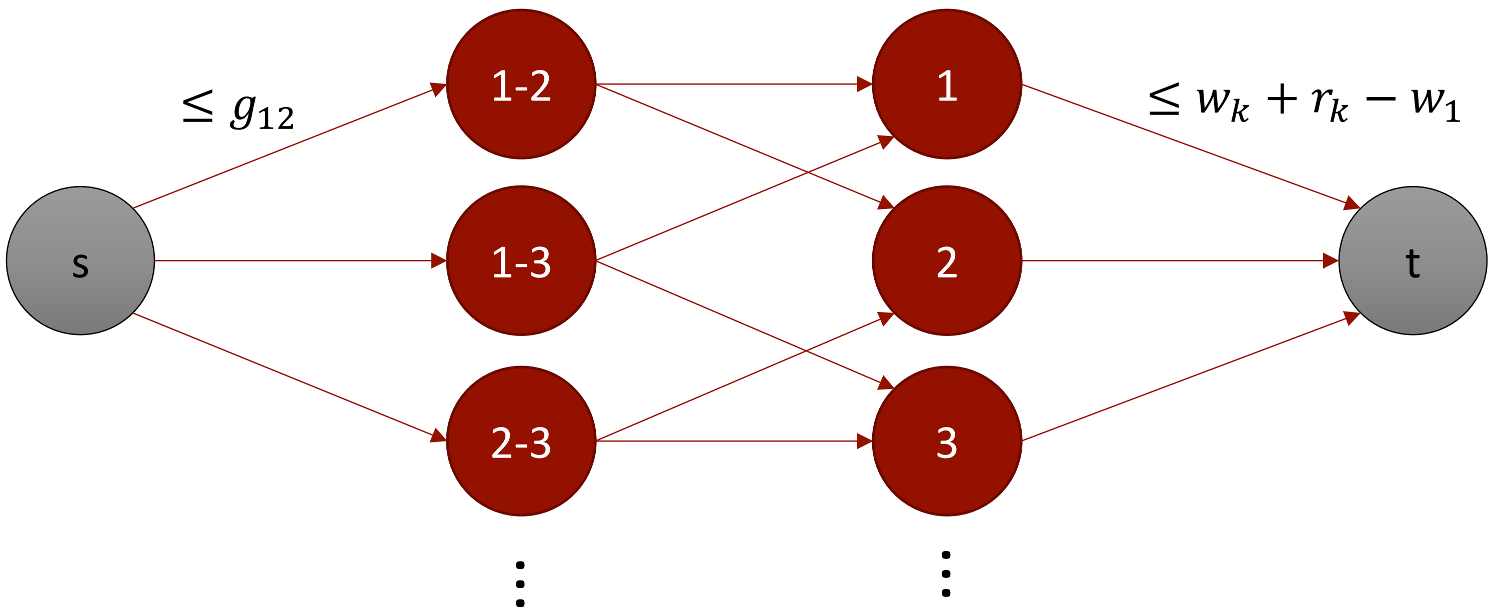
高洪波在赛后说，这只是中国队“正常发挥”的一场比赛。（资料图片：英国广播公司网站）

英媒称，2018年世界杯亚洲区预选赛第二阶段最后一轮小组赛，中国男足国家队主场以2-0战

Sports elimination problem

- Consider a set of N teams.
- At the current time, team i has won w_i games.
- Between teams i and j, g_{ij} games will be played.
- We want to know whether a particular team, say k, can still win the title.
- Suppose that team k has won w_k games and still has r_k games to play.
- For team k to win the league, no other teams can win more than $w_k + r_k$ games.
- That is, every team i cannot win $w_k + r_k - w_i$ games.

Sports elimination problem



Sports elimination problem

- If the max flow is equal to $\sum_{i<j} g_{ij}$, then team k can still win the title.
- Otherwise, team k can no longer win the title.
- Extension: how about soccer (association football)?
- Main difference: draw is allowed.

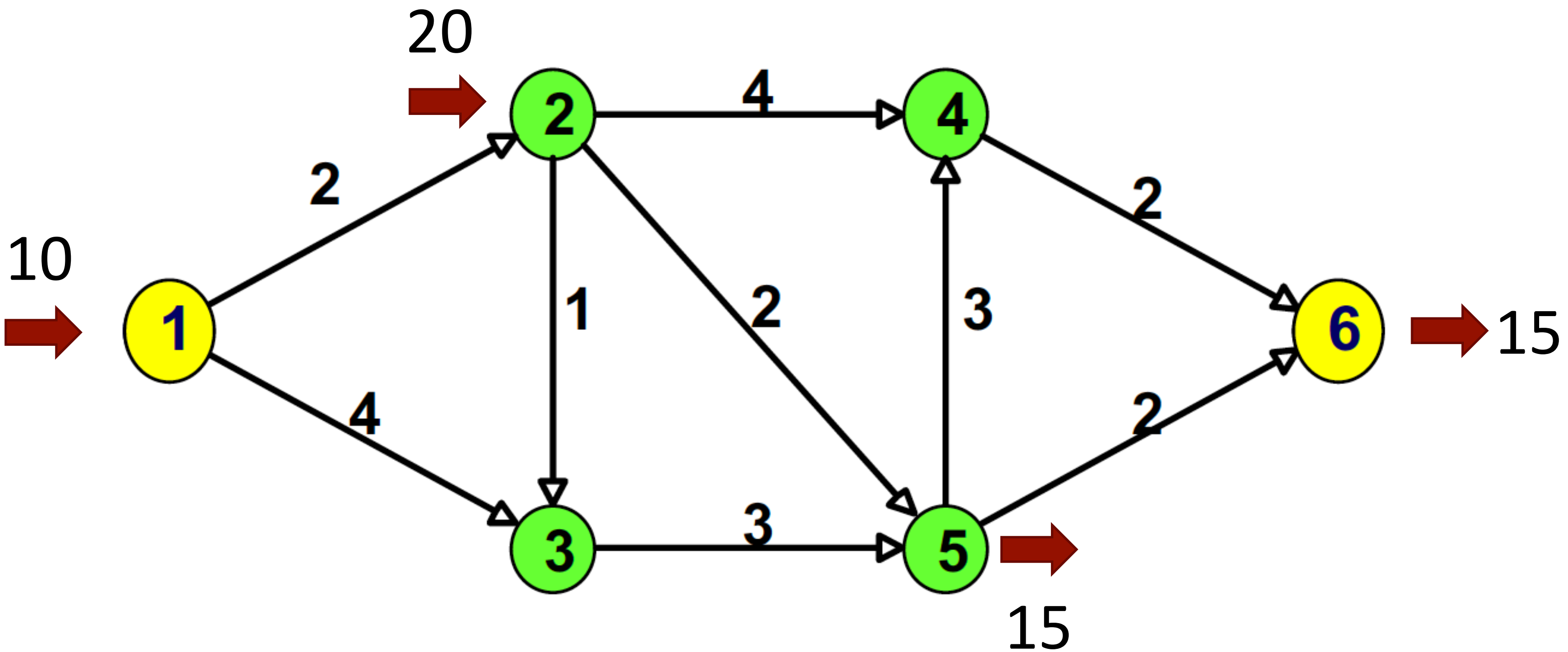
1. Before the 1990s, 1 win gives 2 points, and 1 draw gives 1 point. 1 Loss gives no point.
2. Since the 1990s, 1 win gives 3 points, 1 draw gives 1 point, and 1 loss gives no point.

- [Not required] Can you formulate it as a max flow problem?

Min-cost flow

- When we assign flows, some patterns are better than the others.
- Let's start with the simplest case: linear cost
- Suppose that link (i,j) has a flow of f_{ij}
- Then, the cost on link (i,j) is $c_{ij}f_{ij}$, where $c_{ij} > 0$ is the cost per unit flow
- This is a rather simplistic model that ignores any congestion effect.

Example



Example

- Decision variables:

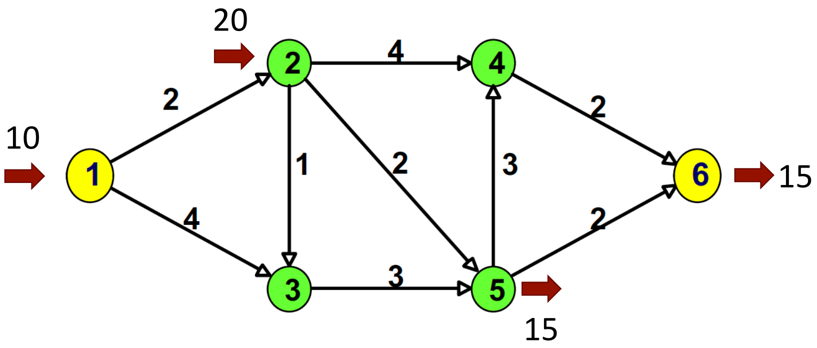
$f_{12}, f_{13}, f_{23}, f_{24}, f_{25}, f_{35}, f_{54}, f_{56}, f_{46}$

- Objective function:

$$z = 2f_{12} + 4f_{13} + f_{23} + 4f_{24} + 2f_{25} + 3f_{35} + 3f_{54} + 2f_{56} + 2f_{46}$$

- Constraints:

1. flow conservation
2. link capacity
3. non-negativity



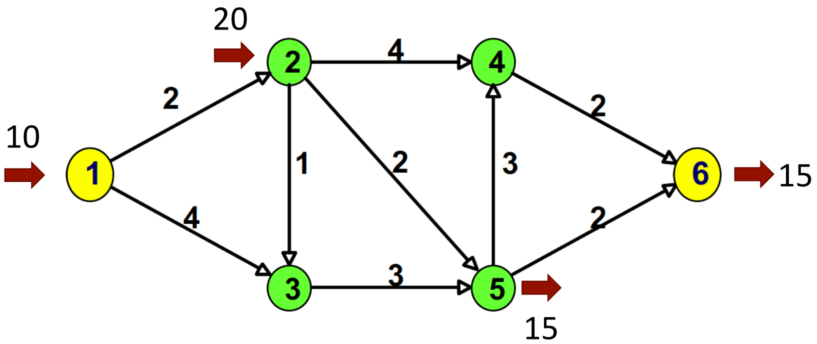
Example

$$\min\ z = 2f_{12} + 4f_{13} + f_{23} + 4f_{24} + 2f_{25} + 3f_{35} + 3f_{54} + 2f_{56} + 2f_{46}$$

s.t. flow conservation

link capacity

non-negativity



Min-cost flow

- min $\sum_{(i,j)\in E} c_{ij} f_{ij}$
- s.t. $b_i + \sum_{j\in In(i)} f_{ji} = \sum_{j\in Out(i)} f_{ij}$ for all $i \in N$

  $0 \le f_{ij} \le \bar{f}_{ij}$ for all $(i,j) \in E$
- Linear programming!
- Can be solved very efficiently
- If $\bar{f}_{ij} = \infty$ for each link (i,j), i.e. if the problem is uncapacitated, we simply allocate demands to the "nearest" destinations.

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  - Min-cost flow problem
- Linear programming formulation

Overview

- If both the constraints and the objective function are linear, the optimization problem is a linear programming (LP).
- Variables must be real-valued.
- In addition, the feasible set has to be closed.
- LP is the simplest yet a very powerful optimization model.
- Extensive theory has been developed on this topic, which provides insights into the structure of optimization problems and algorithms.

Overview

- One has to note that linear programming is an abstract idealization of real problems.
- We have to make assumptions and approximations to keep the problem tractable.
- A good model does not necessarily have to give remarkably accurate prediction value, but provides relative effects of alternative decisions.
- The optimal solution is optimal with regards to the underlying model.
- It is the best decision to make if certain information is given and we relate it in a certain manner.

Generic form

- However, there is no guarantee that this decision will give the best result when implemented, since there are too many uncertainties in a real problem.
- An LP gives insights rather than numerical predictions.
- A standard formulation of LP is as follows.
- Let $x = (x_1, x_2, \dots, x_n)$ be the vector of decision variables
- c_j be the coefficient of x_j in the objective function,
- a_{ij} be the coefficient of x_j in the ith constraint,
- b_i be the right-hand side of the ith constraint.

Generic form

$$
\begin{aligned}
\min \quad & c_1x_1 + c_2x_c + \cdots + c_nx_n \\
s.t. \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
& \quad \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m.
\end{aligned}
$$

- Or, in matrix form,

$$
\begin{aligned}
\min \quad & \mathbf{c}^T\mathbf{x} \\
s.t. \quad & \mathbf{A}\mathbf{x} = \mathbf{b}
\end{aligned}
$$

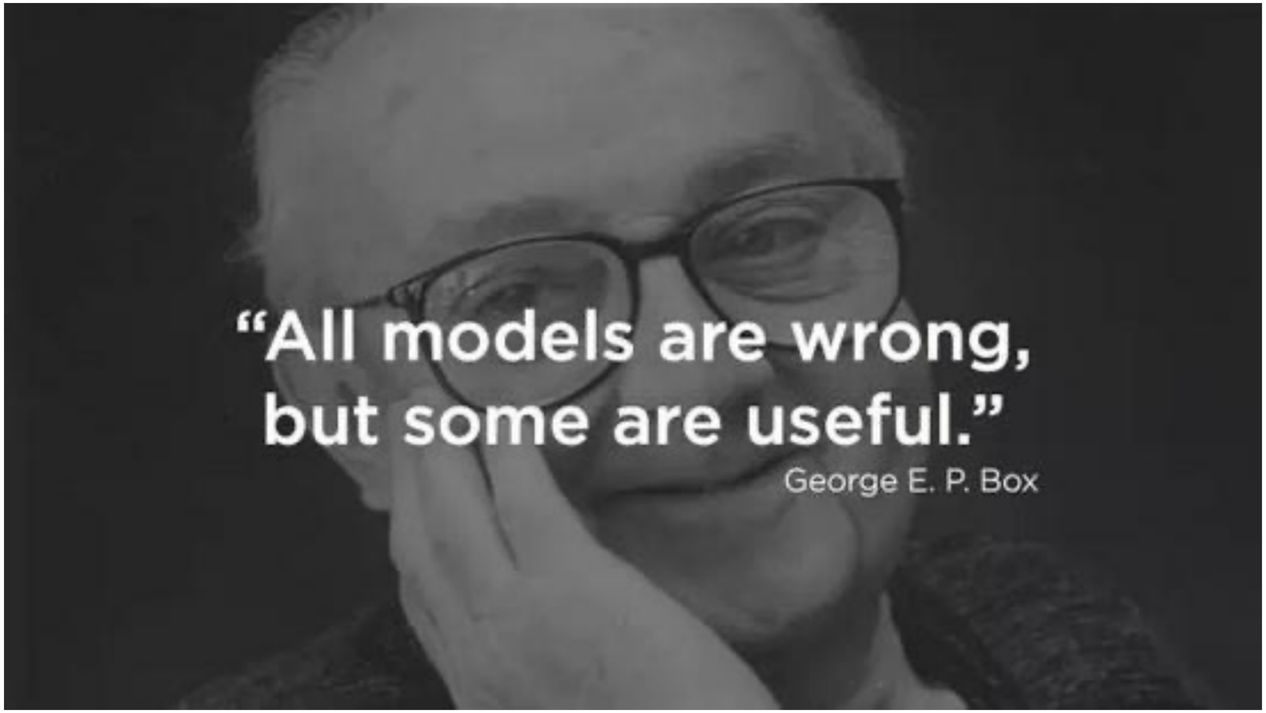
Implicit assumptions

There are four implicit assumptions of LP.

1. The effect (on objective function and constraints) of an activity is proportional to its level.
2. The impact of activities can be simply summed and no interacting effect is considered.
3. Decision variables are real-valued and thus infinitely divisible.
4. Parameters and model are deterministic and all uncertainties are ignored.

Formulating LPs

- Formulating an optimization problem is more of an art than of a science; no systematic method is available.
- A good formulation should involve small number of variables and constraints, but with a sparse $\boldsymbol{A}$ matrix.



Standard form

- For notational simplicity and software programming convenience, we should write an LP in the standard form:

1. The objective function is minimized;
2. All constraints are equalities;
3. All right-hand sides of constraints are non-negative;
4. All decision variables are non-negative.

$$
\begin{aligned}
\min \quad & \mathbf{c}^T \mathbf{x} \\
s.t. \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\
& \mathbf{x} \geq 0
\end{aligned}
$$

Standard form

Every LP can be converted to the standard form in five steps.

1. Negate the objective function if it is originally maximization. One has to keep in mind that the ultimate optimal value is the negative of that of the standard-form problem.
2. Add slack/surplus variables to the left-hand side of inequality constraints. Slack/surplus variables are non-negative. ($x \geq 0 \Leftrightarrow x - y = 0, y \geq 0$)
3. Flip the sign of those constraints with negative right-hand sides.
4. Substitute free variables, i.e. variables that are not non-negative, with $x_j = x_j^+ - x_j^-$.

Note that some of the steps will impact the representation from previous steps.

LP in practice

- LP is widely used in engineering systems, including transportation, telecommunication, manufacturing, medicine, and software.
- Real engineering problems require extensive expertise from researchers to be formulated as a mathematical problem.

Feasible solution

$$\begin{aligned} \min \quad & \mathbf{c}^T \mathbf{x} \\ s.t. \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

- A feasible solution is a solution x such that
$$Ax = b, x \geq 0.$$
- Intuitively, a feasible solution is a solution that satisfies all constraints.
- If an LP has a feasible solution, the LP is feasible.
- Otherwise, the LP is infeasible.
- Hence, the first step for solving an LP is to determine its feasibility.

Optimal solution

$$\begin{aligned} \min \quad & \mathbf{c}^T\mathbf{x} \\ s.t. \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

- An optimal solution is a feasible solution x^* such that
$$c^T x^* \geq c^T x$$
for all feasible solution x.
- Intuitively, an optimal solution is a solution that leads to the optimal objective value; may not be unique!
- **Note**: not every feasible LP has an optimal solution.

$$\min x$$
$$\text{s.t. } x \in \mathbb{R}$$

no optimal solution

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