LECTURE 10

Introduction to Modeling, SLR

Understanding the usefulness of models and the simple linear regression model

Today's Roadmap

Review: Regression Line, Correlation

What is a model?

The Modeling Process: Definitions

Loss Functions

Minimizing Average Loss (Empirical Risk)

Interpreting SLR: Slope, Anscombe's Quartet

Evaluating the Model: RMSE, Residual Plot

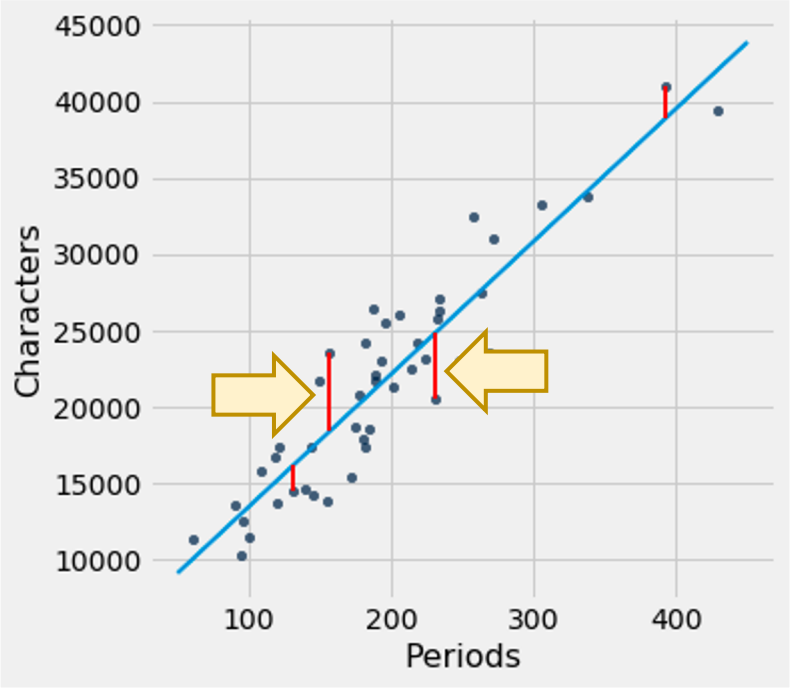
The Regression Line

> The **regression line** is the unique straight line that minimizes the **mean squared error** of estimation among all straight lines.

$$\text{slope} = r \cdot \frac{\text{SD of } y}{\text{SD of } x}$$

$$\text{intercept} = \text{average of } y - \text{slope} \cdot \text{average of } x$$

$$\textbf{residual} = \text{observed value} - \text{regression estimate}$$



For every chapter of the novel *Little Women*, Estimate the **# of characters** $\hat{y}$ based on the **# of periods** x in that chapter.

The Regression Line

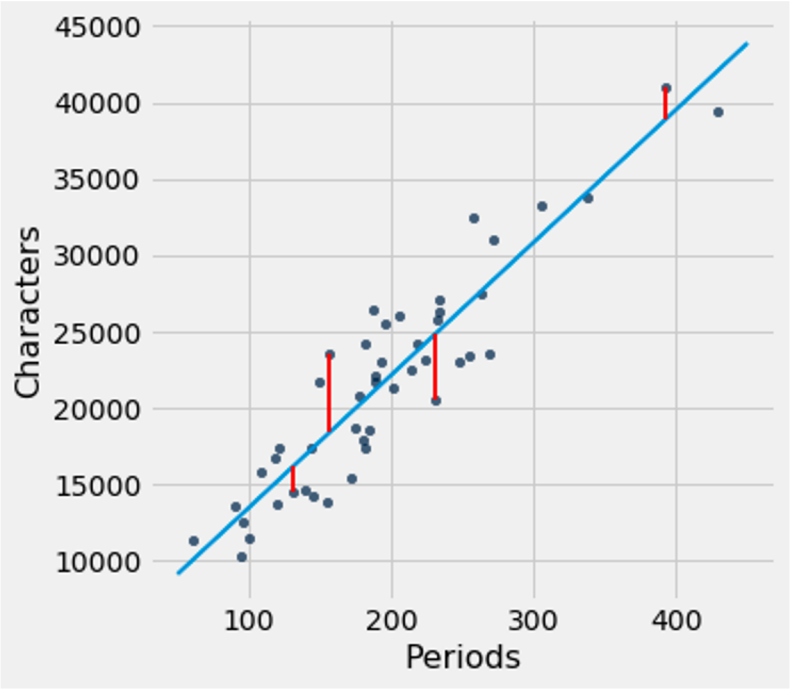
> The **regression line** is the unique straight line that minimizes the **mean squared error** of estimation among all straight lines.

correlation

$$\text{slope} = r \cdot \frac{\text{SD of } y}{\text{SD of } x}$$

$$\text{intercept} = \text{average of } y - \text{slope} \cdot \text{average of } x$$

$$\text{residual} = \text{observed value} - \text{regression estimate}$$



For every chapter of the novel *Little Women*, Estimate the **# of characters** $\hat{y}$ based on the **# of periods** x in that chapter.

Correlation

> The **correlation** r is the average of the product of x and y, both measured in standard units.

$$r = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{\sigma_x}\right)\left(\frac{y_i - \bar{y}}{\sigma_y}\right)$$

Define the following:

$\mathcal{D} = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ data

$\bar{x}, \bar{y}$ means; σ_x, σ_y standard deviations

- x_i in standard units: $\frac{x_i - \bar{x}}{\sigma_x}$
- r is also known as Pearson's correlation coefficient.
- Side note: **covariance** is $\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = r\sigma_x\sigma_y$

Correlation

> The **correlation** r is the average of the product of x and y, both measured in standard units.

Define the following:

$\mathcal{D} = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ data

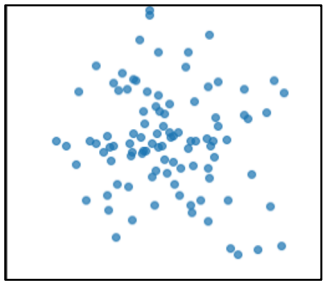
$\bar{x}, \bar{y}$ means; σ_x, σ_y standard deviations

$$r = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{x_i - \bar{x}}{\sigma_x} \right) \left(\frac{y_i - \bar{y}}{\sigma_y} \right)$$

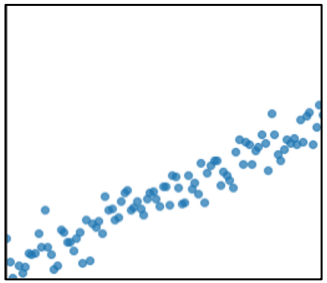
- x_i in standard units: $\frac{x_i - \bar{x}}{\sigma_x}$
- r is also known as Pearson's correlation coefficient.
- Side note: **covariance** is $\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) = r \sigma_x \sigma_y$

Correlation measures the strength of a **linear association** between two variables.

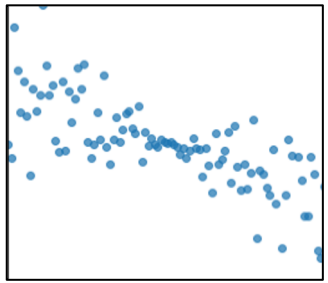
$|r| < 1$



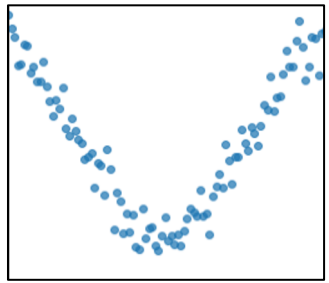
r = −0.121



r = 0.951



r = −0.723



⚠ **r = 0.056**

Expressing the Regression Line Mathematically

> The **regression line** is the unique straight line that minimizes the **mean squared error** of estimation among all straight lines.

Define the following:

$\mathcal{D} = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ data

$\bar{x}, \bar{y}$ means; σ_x, σ_y standard deviations;

r correlation coefficient

$$\hat{y} = \hat{a} + \hat{b}x$$ **regression line**

1. slope $= r \cdot \dfrac{\text{SD of } y}{\text{SD of } x}$

2. intercept $=$ average of y $-$ slope $\cdot$ average of x

3. residual $=$ observed value $-$ regression estimate

?

Rewrite each expression using math notation.

Expressing the Regression Line Mathematically

> The **regression line** is the unique straight line that minimizes the **mean squared error** of estimation among all straight lines.

Define the following:

$\mathcal{D} = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ data

$\bar{x}, \bar{y}$ means; σ_x, σ_y standard deviations;

r correlation coefficient

$$\hat{y} = \hat{a} + \hat{b}x$$ **regression line**

1. slope $= r \cdot \dfrac{\text{SD of } y}{\text{SD of } x}$

2. intercept $=$ average of y $-$ slope $\cdot$ average of x

3. residual $=$ observed value $-$ regression estimate

?

$$\hat{b} = r\frac{\sigma_y}{\sigma_x}$$

$$\hat{a} = \bar{y} - \hat{b}\bar{x}$$

$$e_i = y_i - \hat{y}_i$$

Error for the i-th datapoint

Today's Goal

> The **regression line** is the unique straight line that minimizes the **mean squared error** of estimation among all straight lines.

$$\hat{y} = \hat{a} + \hat{b}x$$

$$\hat{b} = r\frac{\sigma_y}{\sigma_x}$$

$$\hat{a} = \bar{y} - \hat{b}\bar{x}$$

$$e_i = y_i - \hat{y}_i$$

Goal: Derive and define everything on this slide!

What is a model?

Review: Simple Linear Regression and Correlation

What is a model?

The Modeling Process: Definitions

Loss Functions

Minimizing Average Loss on Data

Interpreting SLR: Slope

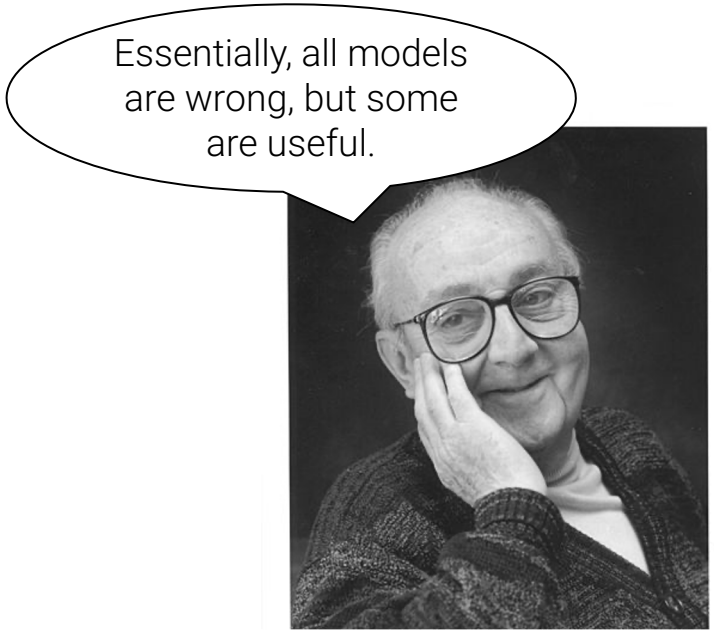
Evaluating the Model: RMSE, Residual Plot

What is a model?

A model is an **idealized representation** of a system.

Example:
We model the fall of an object on Earth as subject to a constant acceleration of 9.81 m/s^2 due to gravity.

- While this describes the behavior of our system, it is merely an approximation.
- It doesn't account for the effects of air resistance, local variations in gravity, etc.
- But in practice, it's accurate enough to be useful!



George Box, Statistician (1919-2013)

Known for	"All models are wrong" Response-surface methodology EVOP q-exponential distribution Box–Jenkins method Box–Cox transformation

Why do we build models?

Reason 1:

To understand **complex phenomena** occurring in the world we live in.

- What factors play a role in the growth of COVID-19?
- How do an object's velocity and acceleration impact how far it travels? (Physics: $d = d_0 + vt + \frac{1}{2}at^2$)

Often times, we care about creating models that are simple and interpretable, allowing us to understand what the relationships between our variables are.

Reason 2:

To make **accurate predictions** about unseen data.

- Can we predict if this email is spam or not?
- Can we generate a one-sentence summary of this 10-page long article?

Other times, we care more about making extremely accurate predictions, at the cost of having an uninterpretable model. These are sometimes called **black-box models**, and are common in fields like deep learning.

Most of the time, we want to strike a balance between interpretability and accuracy.

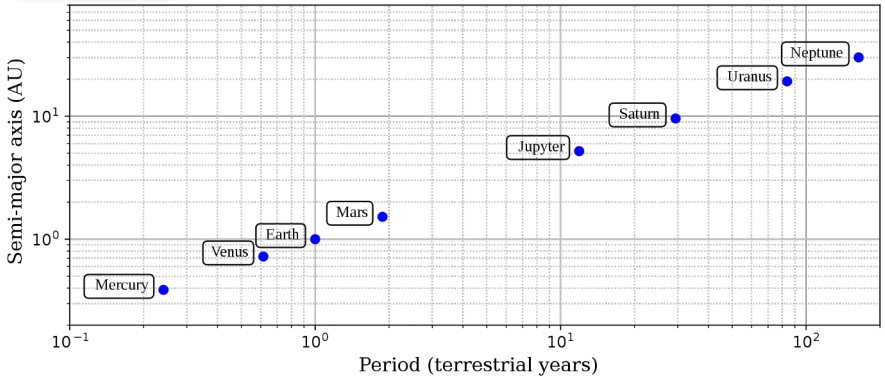
Two common types of models

Physical (mechanistic) models

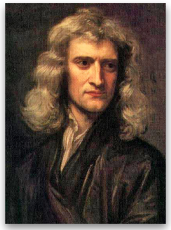
Laws that govern how the world works.

Kepler's Third Law of Planetary Motion (1619)
[Wikipedia]

$$T^2 \propto R^3$$

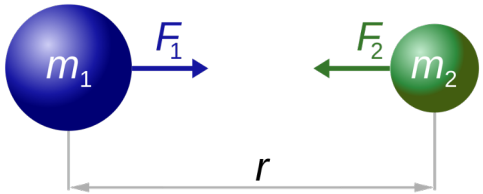


Newton's Laws: motion and gravitation (1687)
[Wikipedia]



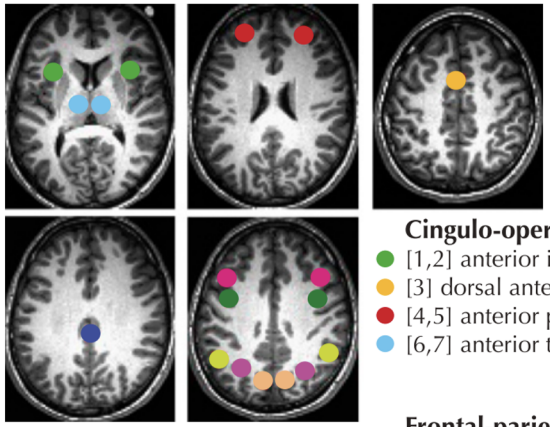
$$\mathbf{F} = m\mathbf{a}$$

$$F = G\frac{m_1 m_2}{r^2}$$



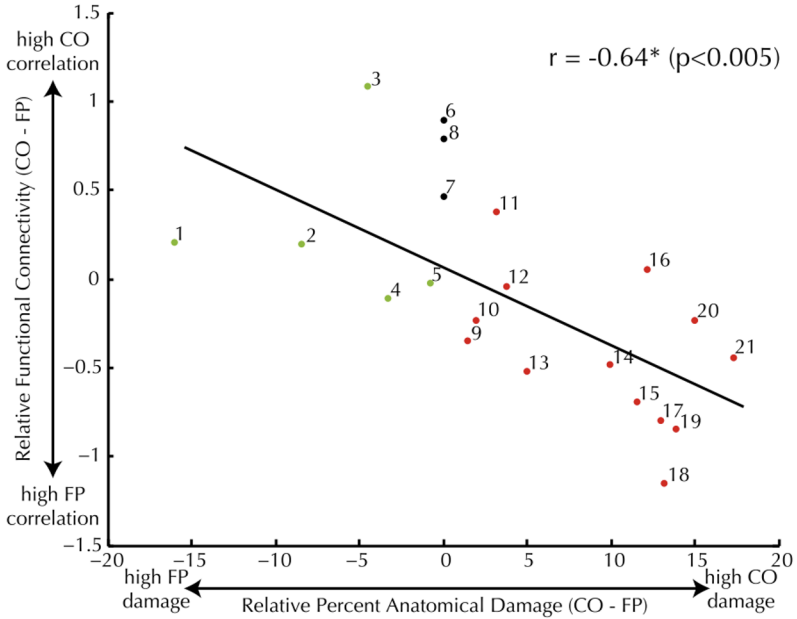
Statistical models

Relationships between variables found through data and statistical analysis.



Cingulo-opercular (CO)
- [1,2] anterior insula/frontal operculum (aI/
- [3] dorsal anterior cingulate (dACC)
- [4,5] anterior prefrontal cortex (aPFC)
- [6,7] anterior thalamus (ant thalamus)

Frontal-parietal (FP)
- [1,2] intraparietal sulcus (IPS)
- [3,4] frontal cortex
- [5,6] precuneus
- [7,8] intraparietal lobule (IPL)
- [9,10] dorsolateral prefrontal cortex (dlPFC
- [11] midcingulate



Nomura et al., PNAS 2010 [paper]

Physical Model or Statistical Model?

The Modeling Process: Definitions

Review: Simple Linear Regression and Correlation

What is a model?

The Modeling Process: Definitions

Loss Functions

Minimizing Average Loss on Data

Interpreting SLR: Slope

Evaluating the Model: RMSE, Residual Plot

Simple Linear Regression Model (SLR)

$$\hat{y} = a + bx$$

SLR is a **parametric model**, meaning we choose the "best" **parameters** for slope and intercept based on data.

What do we mean by "best"?
What's the difference between a, b and $\hat{a}, \hat{b}$?

Parametric Model Notation

y True outputs

$\hat{y}$ Predicted outputs

For data:

$$\mathcal{D} = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$$

The i-th datapoint is an **observation**:

- y_i is the i-th **output** (aka dependent variable)
- x_i is the i-th **feature** (aka independent variable)

$\hat{y}_i$ is the i-th **prediction** (aka estimation).

θ Model parameter(s)

$\hat{\theta}$ Optimal parameter(s), for some definition of optimal

$\hat{y} = a + bx$ Any linear model with parameters $\theta = (a, b)$

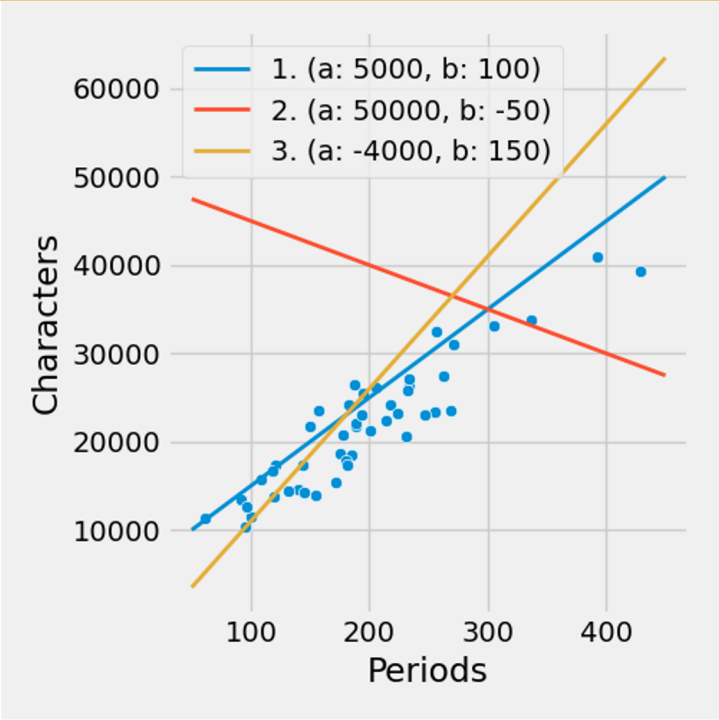
$\hat{y} = \hat{a} + \hat{b}x$ The "best" linear model with parameters $\hat{\theta} = (\hat{a}, \hat{b})$

Which θ is best?

Based on your interpretation of the data, which are the "optimal parameters" for this linear model?

$$\hat{y} = a + bx$$

$$\hat{a} = ? \quad \hat{b} = ?$$



We only had 3 values to choose from to find the optimal parameter. In practice, our parameter domain is all reals, i.e., $(a, b) \in \mathbb{R}^2$

For every chapter of the novel *Little Women*, Estimate the **# of characters** $\hat{y}$ based on the **# of periods** x in that chapter.

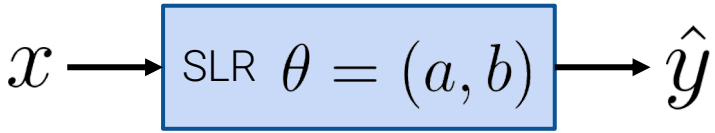
Simple Linear Regression: Our First Model

Simple Linear Regression Model (SLR)

$$\hat{y} = a + bx$$

SLR is a **parametric model**, meaning
we choose the "best" **parameters**
for slope and intercept based on data.

- We often express θ as a single parameter vector.
- x is **not** a parameter! It is input to our model.



- Note that the true relationship between x and y is usually non-linear.
  This is why $\hat{y}$ (and not y) appears in our **estimated linear model** expression.

- Other parametric models we'll see soon: $\hat{y} = \theta$ $\hat{y} = x^T\theta$ $\hat{y} = \frac{1}{1 + \exp(-x^T\vec{\theta})}$
- Note: Not all statistical models have parameters! KDEs are non-parametric models.

The Modeling Process

1. Choose a model ✅

How should we represent the world?

$$\hat{y} = a + bx$$

SLR model

2. Choose a loss function

How do we quantify prediction error?

3. Fit the model

How do we choose the best parameters of our model given our data?

$$\hat{y} = \hat{a} + \hat{b}x$$

4. Evaluate model performance

How do we evaluate whether this process gave rise to a good model?

Reflect

Loss Functions

The Modeling Process

1. Choose a model ✅

How should we represent the world?

$$\hat{y} = a + bx$$

SLR model

2. Choose a loss function

How do we quantify prediction error?

3. Fit the model

How do we choose the best parameters of our model given our data?

4. Evaluate model performance

How do we evaluate whether this process gave rise to a good model?

Loss Functions

We need some metric of how “good” or “bad” our predictions are.

A **loss function** characterizes the cost, error, or **fit**
resulting from a particular choice of model or model parameters.

- Loss quantifies how bad a prediction is for a **single** observation.
- If our prediction $\hat{y}$ is **close** to the actual value y, we want **low loss**.
- If our prediction $\hat{y}$ is far from the actual value y, we want high loss.

$$L(y, \hat{y})$$

There are many definitions of loss functions!

The choice of loss function:

- Affects the accuracy and computational cost of estimation.
- Depends on the estimation task:
  - Are outputs quantitative or qualitative?
  - Do we care about outliers?
  - Are all errors equally costly? (e.g., false negative on cancer test)

L2 and L1 Loss

Squared Loss

$$L(y, \hat{y}) = (y - \hat{y})^2$$

- Widely used
- Also called “L2 loss”
- Reasonable:
  - $\hat{y} = y$ → good prediction→ good fit → no loss
  - $\hat{y}$ far from y → bad prediction → bad fit → *lots of loss*

Absolute Loss

$$L(y, \hat{y}) = |y - \hat{y}|$$

- Sounds worse than it is
- Also called “L1 loss”
- Reasonable:
  - $\hat{y} = y$ → good prediction→ good fit → no loss
  - $\hat{y}$ far from y → bad prediction → bad fit → *some loss*

L2 and L1 Loss for SLR

Squared Loss (L2 Loss)

$$L(y, \hat{y}) = (y - \hat{y})^2$$

For an SLR model $\hat{y} = a + bx$:

$$L(y, \hat{y}) = (y - (a + bx))^2$$

Absolute Loss (L1 Loss)

$$L(y, \hat{y}) = |y - \hat{y}|$$

For an SLR model $\hat{y} = a + bx$:

1. What is the SLR L1 Loss?

2. Why don't we directly use residual error as the loss function? $e = (y - \hat{y})$

3. Which loss function is better: L1 or L2?

L2 and L1 Loss for SLR

Squared Loss (L2 Loss)

$$L(y, \hat{y}) = (y - \hat{y})^2$$

For an SLR model $\hat{y} = a + bx$:

$$L(y, \hat{y}) = (y - (a + bx))^2$$

Absolute Loss (L1 Loss)

$$L(y, \hat{y}) = |y - \hat{y}|$$

For an SLR model $\hat{y} = a + bx$:

$$L(y, \hat{y}) = |y - (a + bx)|$$

Why don't we directly use residual error as the loss function? $e = (y - \hat{y})$

- This unfortunately treats "negative" predictions and "positive" predictions differently.
- Predicting 16 when the true value is 15 should be penalized the same as predicting 14.

Which loss function is better: L1 or L2?

We'll compare tradeoffs next lecture.
Today we'll focus on Squared (L2) Loss.

Empirical Risk is Average Loss over Data

We care about how bad our model's predictions are for our entire data set, not just for one point. A natural measure, then, is of the **average loss** (aka **empirical risk**) across all points.

Given data $\mathcal{D} = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$:

$$R(\theta) = \frac{1}{n} \sum_{i=1}^{n} L(y_i, \hat{y}_i)$$

Average loss is a function of the parameter θ because **our data do not change**. What defines how well our model works is our choice of θ , which determines $\hat{y}$.

The average loss of a model tells us how well it fits the given data.
We want to **find the parameter(s) that minimize average loss** to best predict the data.

Empirical Risk is Average Loss over Data

We care about how bad our model's predictions are for our entire data set, not just for one point.

A natural measure, then, is of the **average loss** (aka **empirical risk**) across all points.

Given data $\mathcal{D} = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$:

$$R(\theta) = \frac{1}{n} \sum_{i=1}^{n} L(y_i, \hat{y}_i)$$

The colloquial term for average loss depends on which loss function we choose.

L2 loss → **Mean squared error (MSE)** $\frac{1}{n} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$

L1 loss → **Mean absolute error (MAE)** $\frac{1}{n} \sum_{i=1}^{n} |y_i - \hat{y}_i|$

The Modeling Process

1. Choose a model ✅

How should we represent the world?

$$\hat{y} = a + bx$$ SLR model

2. Choose a loss function ✅

How do we quantify prediction error?

$$L(y, \hat{y}) = (y - \hat{y})^2$$ Squared loss

$$R(a, b) = \frac{1}{n} \sum_{i=1}^{n} (y_i - (a + bx_i))^2$$ MSE for SLR

3. Fit the model

How do we choose the best parameters of our model given our data?

4. Evaluate model performance

How do we evaluate whether this process gave rise to a good model?

The combination of model + loss that we focus on today is known as **least squares regression**.

Minimizing Average Loss (Empirical Risk) on Data

The Modeling Process

1. Choose a model ✅

How should we represent the world?

$$\hat{y} = a + bx$$ SLR model

2. Choose a loss function ✅

How do we quantify prediction error?

$$L(y, \hat{y}) = (y - \hat{y})^2$$ Squared loss

3. Fit the model

How do we choose the best parameters of our model given our data?

$$R(a, b) = \frac{1}{n} \sum_{i=1}^{n} (y_i - (a + bx_i))^2$$

We want to find $\hat{a}, \hat{b}$ that minimize this **objective function**.

4. Evaluate model performance

How do we evaluate whether this process gave rise to a good model?

Minimizing MSE for the SLR model

Objective function: In optimization theory, the function to minimize.

Find the values of a, b that minimize the average squared loss (MSE) for the SLR model:

$$R(a,b) = \frac{1}{n}\sum_{i=1}^{n}(y_i - (a + bx_i))^2$$

Note: The optimal parameters $a = \hat{a}, b = \hat{b}$ will also minimize this simplified function:

$$\sum_{i=1}^{n}(y_i - (a + bx_i))^2$$

- This is a quadratic function of two unknowns.
- Remember: The data (x_i, y_i) are known and—for our purposes—fixed.

Optimize this simplified objective with calculus!

Step 1 of 2: Fix b and minimize with respect to a

1. Rewrite the function:

$$\sum_{i=1}^{n}(y_i-(a+bx_i))^2=\sum_{i=1}^{n}(y_i-bx_i-a))^2$$

2. Differentiate with respect to a:

$$\frac{\partial}{\partial a}\sum_{i=1}^{n}(y_i-bx_i-a)^2=\sum_{i=1}^{n}\frac{\partial}{\partial a}(y_i-bx_i-a)^2$$

Derivative of sum is sum of derivatives

$$=\sum_{i=1}^{n}2(y_i-bx_i-a)(-1)$$

Chain rule

$$=-2\sum_{i=1}^{n}(y_i-bx_i-a)$$

Simplify constants

3. Set equal to 0:

$$0=-2\sum_{i=1}^{n}(y_i-bx_i-a)$$

4. Finally, rearrange and solve for $\hat{a}$:

$$0=-2\sum_{i=1}^{n}(y_i-bx_i-a)$$

$$=\sum_{i=1}^{n}(y_i-bx_i-a)$$

$$=\sum_{i=1}^{n}y_i-b\sum_{i=1}^{n}x_i-na$$

Pull out scalars

$$na=\sum_{i=1}^{n}y_i-b\sum_{i=1}^{n}x_i$$

$$\hat{a}=\frac{1}{n}\sum_{i=1}^{n}y_i-b\cdot\frac{1}{n}\sum_{i=1}^{n}x_i$$

Divide by n

$$\hat{a}=\bar{y}-b\bar{x}$$

Step 2 of 2: Plug in $\hat{a}$ and minimize with respect to b

Our expression for $\hat{a}$: $\boxed{\hat{a} = \bar{y} - \hat{b}\bar{x}}$

1. Plug in $\hat{a}$ to our objective function:

$$\sum_{i=1}^{n}(y_i - (a + bx_i))^2 = \sum_{i=1}^{n}(y_i - (\bar{y} - b\bar{x} + bx_i))^2$$

$$= \sum_{i=1}^{n}(y_i - \bar{y} - b(x_i - \bar{x}))^2$$

2. Differentiate with respect to b:

$$\sum_{i=1}^{n}\frac{\partial}{\partial b}(y_i - \bar{y} - b(x_i - \bar{x}))^2$$ Derivative of sum is sum of derivatives

$$= \sum_{i=1}^{n} 2 \cdot (y_i - \bar{y} - b(x_i - \bar{x})) \cdot (-1) \cdot (x_i - \bar{x})$$ Chain rule

$$= -2\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y} - b(x_i - \bar{x}))$$ Simplify constants

3. Set equal to 0:

$$0 = -2\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y} - b(x_i - \bar{x}))$$

4. Finally, rearrange and solve for $\hat{b}$:

$$0 = \left[\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) - b(x_i - \bar{x})^2\right]$$ Distributive prop.

$$= \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) - b\sum_{i=1}^{n}(x_i - \bar{x})^2$$ Separate sums

Add in constants

$$= n\sigma_x\sigma_y\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{\sigma_x}\right)\left(\frac{y_i - \bar{y}}{\sigma_y}\right)\right) - bn\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$= nr\sigma_x\sigma_y - bn\sigma_x^2$$ Definitions of r, σ_x

$$b\sigma_x^2 = r\sigma_x\sigma_y$$

$$\hat{b} = \frac{r\sigma_x\sigma_y}{\sigma_x^2} = r\frac{\sigma_y}{\sigma_x}$$

$$\boxed{\hat{b} = r\frac{\sigma_y}{\sigma_x}}$$

The Modeling Process

1. Choose a model ✅

How should we represent the world?

$$\hat{y} = a + bx$$ SLR model

2. Choose a loss function ✅

How do we quantify prediction error?

$$L(y, \hat{y}) = (y - \hat{y})^2$$ Squared loss

$$R(a, b) = \frac{1}{n} \sum_{i=1}^{n} (y_i - (a + bx_i))^2$$

3. Fit the model ✅

How do we choose the best parameters of our model given our data?

$$\hat{y} = \hat{a} + \hat{b}x$$

$$\hat{a} = \bar{y} - \hat{b}\bar{x}$$

$$\hat{b} = r\frac{\sigma_y}{\sigma_x}$$

4. Evaluate model performance

How do we evaluate whether this process gave rise to a good model?

Galton demo

$$\hat{y} = \hat{a} + \hat{b}x$$ regression line

$$r = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{\sigma_x}\right)\left(\frac{y_i - \bar{y}}{\sigma_y}\right)$$

$$\hat{b} = r\frac{\sigma_y}{\sigma_x}$$

$$\hat{a} = \bar{y} - \hat{b}\bar{x}$$

Interpreting SLR: Slope

The Modeling Process

1. Choose a model ✅

How should we represent the world?

$$\hat{y} = a + bx$$

SLR model

2. Choose a loss function ✅

How do we quantify prediction error?

$$L(y, \hat{y}) = (y - \hat{y})^2$$

Squared loss (for today)

$$R(a, b) = \frac{1}{n} \sum_{i=1}^{n} (y_i - (a + bx_i))^2$$

3. Fit the model ✅

How do we choose the best parameters of our model given our data?

$$\hat{y} = \hat{a} + \hat{b}x$$

$$\hat{a} = \bar{y} - \hat{b}\bar{x}$$

$$\hat{b} = r\frac{\sigma_y}{\sigma_x}$$

4. Evaluate model performance

How do we evaluate whether this process gave rise to a good model?

Interpreting the least squares linear regression model

You may sometimes hear the prediction task defined as: "**regressing** y on x."

Suppose we fit a model that predicts a Chihuahua's weight (in pounds) given its length (in inches).

$$\hat{y} = \hat{a} + \hat{b}x \qquad \hat{a} = \bar{y} - \hat{b}\bar{x} \quad \hat{b} = r\frac{\sigma_y}{\sigma_x}$$

$$\text{predicted weight} = 3 + 2 \cdot \text{length}$$

Interpreting the least squares linear regression model

You may sometimes hear the prediction task defined as: "**regressing** y on x."

Suppose we fit a model that predicts a Chihuahua's weight (in pounds) given its length (in inches).

$$\hat{y} = \hat{a} + \hat{b}x \qquad \hat{a} = \bar{y} - \hat{b}\bar{x} \quad \hat{b} = r\frac{\sigma_y}{\sigma_x}$$

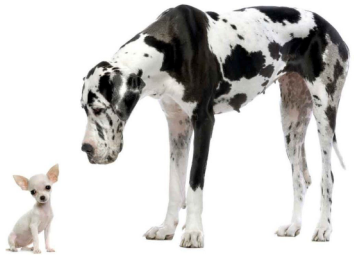
$$\text{predicted weight} = 3 + 2 \cdot \text{length}$$

Interpreting the slope?

By definition, the slope measures the increase in y (pounds) for a 1 unit increase in x (1 inch).

1. Does this mean that if a chihuahua in the dataset grows 1 inch, we estimate that they will get 2 pounds heavier? What does it actually mean?

Predicting on wildly different data?



Chihuahuas (left) range from 3-6 pounds, and 9.5-15 inches in length.
Great Danes (right) range from 110-175 pounds, and 35.5-43 inches in length.

2. Should we use this model to predict the weight of Great Danes (a much bigger dog)?

Interpreting the least squares linear regression model

You may sometimes hear the prediction task defined as: "**regressing** y on x."

Suppose we fit a model that predicts a Chihuahua's weight (in pounds) given its length (in inches).

$$\hat{y} = \hat{a} + \hat{b}x \qquad \hat{a} = \bar{y} - \hat{b}\bar{x} \quad \hat{b} = r\frac{\sigma_y}{\sigma_x}$$

$$\text{predicted weight} = 3 + 2 \cdot \text{length}$$

Interpreting the slope?

By definition, the slope measures the increase in y (pounds) for a 1 unit increase in x (1 inch).

1. Does this mean that if a chihuahua in the dataset grows 1 inch, we estimate that they will get 2 pounds heavier? **No!**

- The model we created shows **association**, not causation.
- The data we collected is a snapshot of several chihuahuas at one instance of time (**cross-sectional**), not snapshots of chihuahuas over time (**longitudinal**).

Slope interpretation: If two chihuahuas have a 1 inch height difference, their estimated weight difference is 2 lbs.

Interpreting the least squares linear regression model

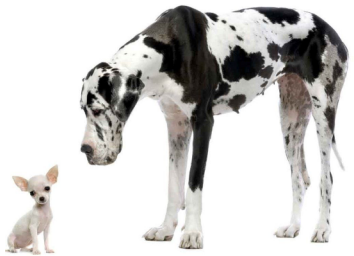
You may sometimes hear the prediction task defined as: "**regressing** y on x."

Suppose we fit a model that predicts a Chihuahua's weight (in pounds) given its length (in inches).

$$\hat{y} = \hat{a} + \hat{b}x \qquad \hat{a} = \bar{y} - \hat{b}\bar{x} \quad \hat{b} = r\frac{\sigma_y}{\sigma_x}$$

$$\text{predicted weight} = 3 + 2 \cdot \text{length}$$

Predicting on wildly different data?



Chihuahuas (left) range from 3-6 pounds, and 9.5-15 inches in length.
Great Danes (right) range from 110-175 pounds, and 35.5-43 inches in length.

- We have no indication that the weight vs. length relationship for Great Danes are the same as Chihuahuas.
- Great Danes' weights and lengths are well outside of the range of weights and lengths we fit our model on.

If the new data we test our model on looks nothing like the data we fit our model on, there's no guarantee that our model will be any good**.**

We will formalize this notion in a few lectures.

2. Should we use this model to predict the weight of Great Danes (a much bigger dog)? **No!**

Evaluating the Model: RMSE, Residual Plot

Evaluating models

What are some ways to determine if our model was a good fit to our data?

1. **Visualize data, compute statistics**

   Plot original data.

   Compute column means, standard deviation.

   If we want to fit a linear model, compute correlation r.

2. **Performance metrics**

   Root mean square error (RMSE)

$$\text{RMSE} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}$$

   - It is the square root of MSE, which is the average los
     that we've been minimizing to determine optimal model parameters.
   - RMSE is in the same units as y.
   - A lower RMSE indicates more "accurate" predictions (lower "average loss" across data)

3. **Visualization**

   Look at a residual plot of $e_i = y_i - \hat{y}_i$ to visualize the difference between actual and predicted y values.

Demo Slides

Four mysterious datasets + Least Squares

Ideal model evaluation steps, in order:

1. **Visualize original data, compute statistics**
2. **Performance Metrics**
   For our simple linear least square model, use RMSE (we'll see more metrics later)
3. **Residual Visualization**

It is tempting to only look at step 2. But you need to always visualize!!!!

Visualize, then quantify!

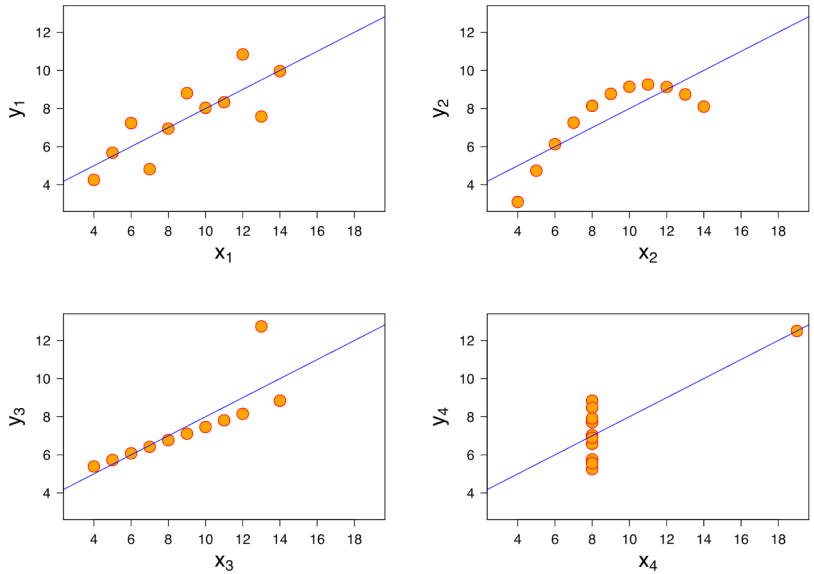
Anscombe's quartet refers to the following four sets of points on the right.

- They each have the same mean of x, mean of y, SD of x, SD of y, and r value.
- Since our optimal Least Squares SLR model only depends on those quantities, they all have the **same regression line**.

$$\bar{x} = 9, \bar{y} = 7.501$$

$$\sigma_x = 3.162, \sigma_y = 1.937$$

$$r = 0.816$$



However, only one of these four sets of data makes sense to model using SLR.

Before modeling, you should always visualize your data first!