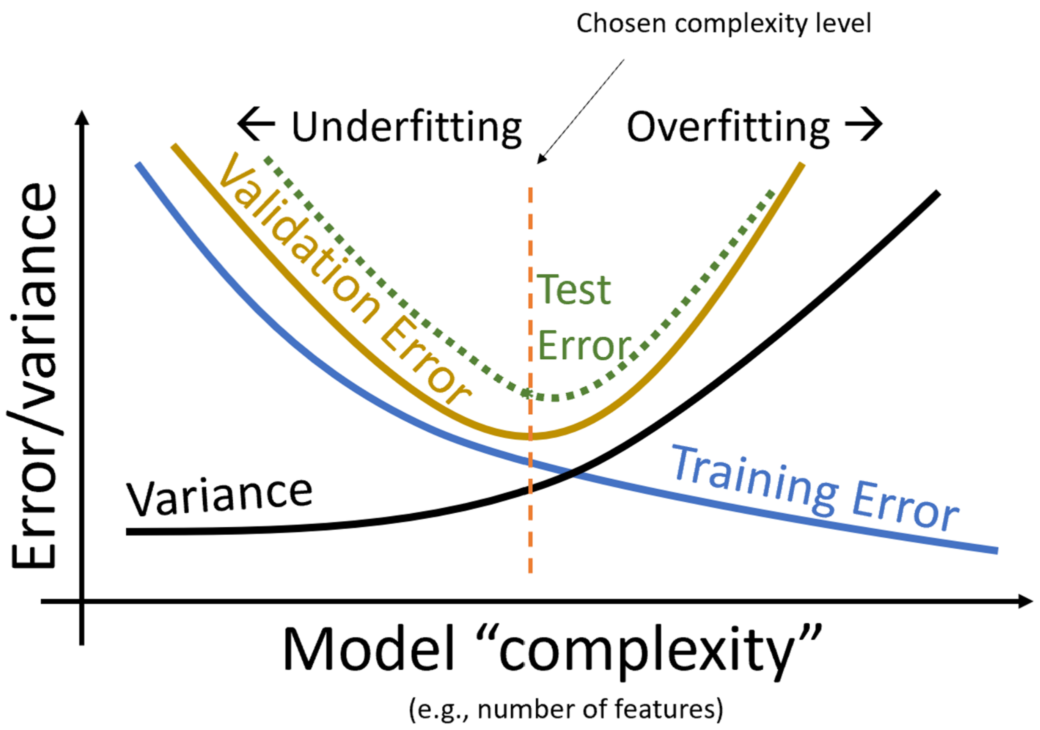
LECTURE 17

Random Variables

Numerical functions of random samples and their properties; sampling variability.

The Bias-Variance Tradeoff

What is the mathematical underpinning of this plot?



We'll come back to this…

Our Goal Today

Formalize the notions of **sample statistic, population parameter**.

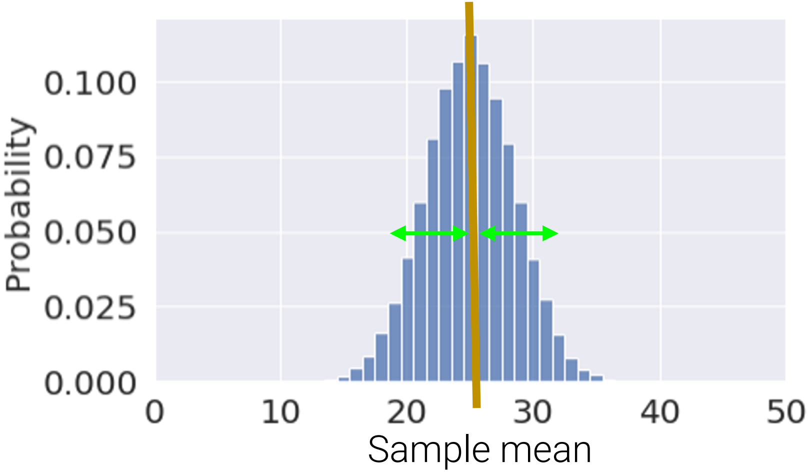
1. **sample mean** - the mean of your random sample.

`np.mean(data)`

2. The Central Limit Theorem: If you draw a large random sample with replacement, then, regardless of the population distribution, the probability distribution of the sample mean:

- Is roughly normal
- Is centered at the **population mean**
- Has an SD = $\dfrac{\textbf{population SD}}{\sqrt{\text{sample size}}}$

We will go over **just enough probability** to help you understand its implications for modeling.



Today’s Roadmap

Random Variables and Distributions

Expectation and Variance

Sums of Random Variables

- Equality vs Identically Distributed vs. IID
- Properties of Expectation and Variance
- Covariance, Correlation

Bernoulli and Binomial Random Variables

Sample Statistics

- Sample Mean
- Central Limit Theorem

Derivations

[Terminology] Random Variable

Suppose we draw a random sample of size n from a population.

A **random variable** is a numerical function of a sample.

sample was drawn at random

value depends on how the sample came out

- Often denoted with uppercase “variable-like” letters (e.g. X, Y).
- Also known as a sample statistic, or **statistic.**
- Domain (input): all random samples of size n
- Range (output): number line

[Terminology] Random Variable

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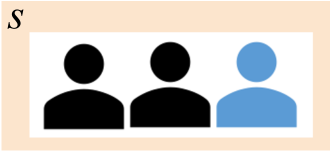
- Domain (input): all random samples of size n
- Range (output): number line

Suppose you draw a random sample s of size 3 from the following population:

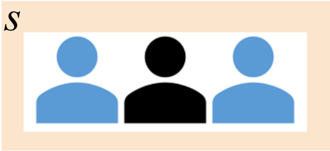
$X(s) = 2$

Define X = **# of blue people**.

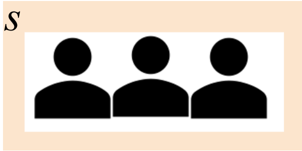
X is a random variable!



$X(s) = 1$



$X(s) = 2$



$X(s) = 0$

From Population to Distribution

	X(s)
0	3
1	4
2	4
3	6
4	8
...	...
79995	6
79996	6
79997	4
79998	6
...	...

X(s) from all possible samples

$$P(X = x) = \frac{\#\text{ times where } X = x}{\text{pop. size}}$$

x	P(X = x)
3	0.1
4	0.2
6	0.4
8	0.3

Probability Distribution Table

[Terminology] Distribution

The **distribution** of a random variable X is a description of how the total probability of 100% is split over all the possible values of X .

A distribution fully defines a random variable.

Assuming (for now) that X is discrete, i.e., has a finite number possible values:

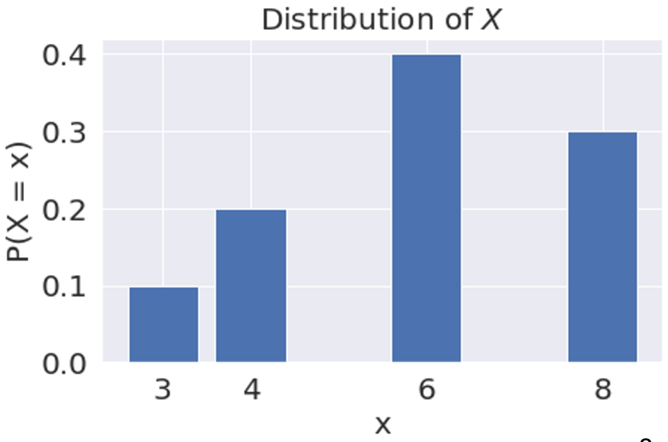
$$\boxed{P(X = x)}$$

The probability that random variable X takes on the value x .

$$\sum_{\text{all } x} P(X = x) = 1$$

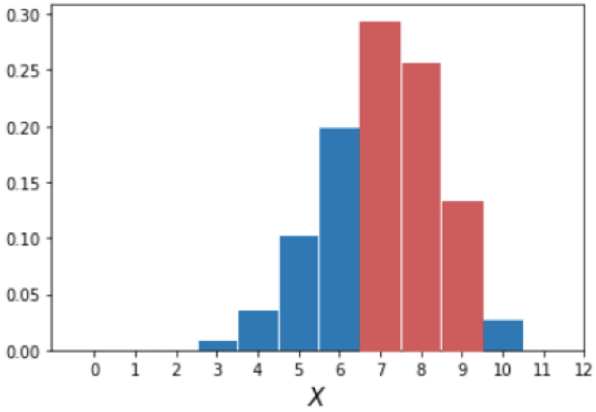
Probabilities must sum to 1.

x	P(X = x)
3	0.1
4	0.2
6	0.4
8	0.3



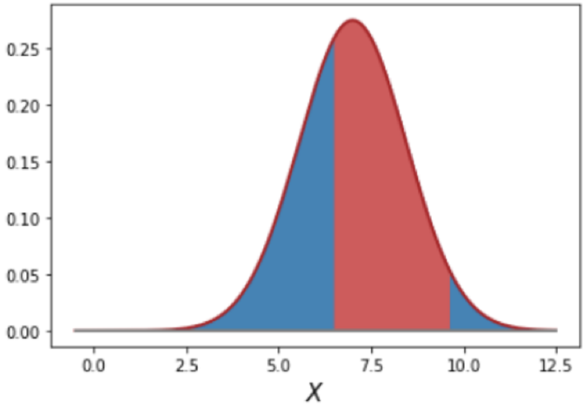
Probabilities are Areas of Histograms

Distribution of **discrete** random variable X



The area of the red bars is **P(7 <= X <= 9)**.

Distribution of **continuous** random variable Y

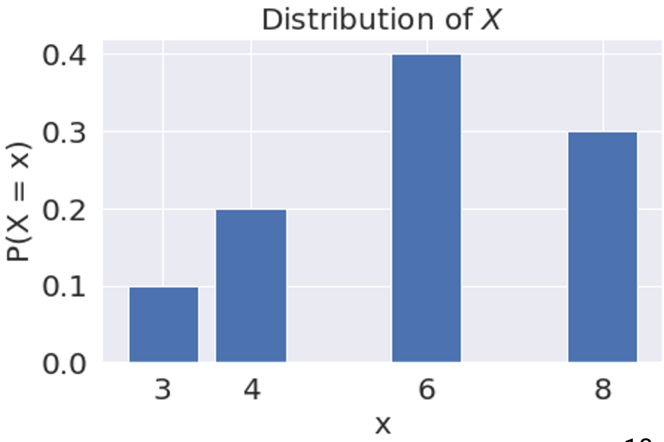


The red area under the curve is **P(6.8 <= Y <= 9.5)**.

Compute the following probabilities for the random variable X.

1. $P(X = 4) =$ 0.2
2. $P(X < 6) =$ $0.1 + 0.2 = 0.3$
3. $P(X \leq 6) =$ $0.1 + 0.2 + 0.4 = 0.7$
4. $P(X = 7) =$ 0
5. $P(X \leq 8) =$ 1

x	P(X = x)
3	0.1
4	0.2
6	0.4
8	0.3



Common Random Variables

Bernoulli(p)

- Takes on value 1 with probability p, and 0 with probability 1 - p
- AKA the “indicator” random variable.

Binomial(n, p)

- Number of 1s in n independent Bernoulli(p) trials
- Probabilities given by the binomial formula (Lecture 2)

We’ll go over these in detail.
The rest are provided for your reference.

Uniform on a finite set of values

- Probability of each value is 1 / (size of set)
- For example, a standard die

Uniform on the unit interval(0, 1)

- Density is flat on (0, 1) and 0 elsewhere

Normal(μ, σ^2)

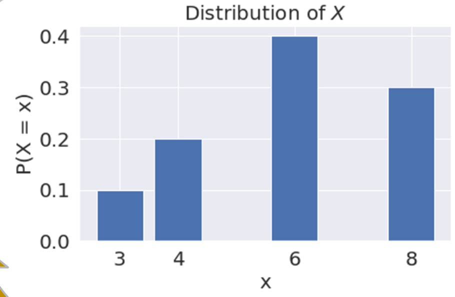
The numbers in parentheses are the **parameters** of a random variable, which are constants. Parameters define a random variable’s shape (i.e., distribution) and its values.

From Distribution to (Simulated) Population

?

X(s) from all possible samples?

$$P(X = x) = \frac{\#\text{ times where } X = x}{\text{pop. size}}$$



Given a random variable's distribution, how could we **generate/simulate** a population?

x	P(X = x)
3	0.1
4	0.2
6	0.4
8	0.3

Probability Distribution Table

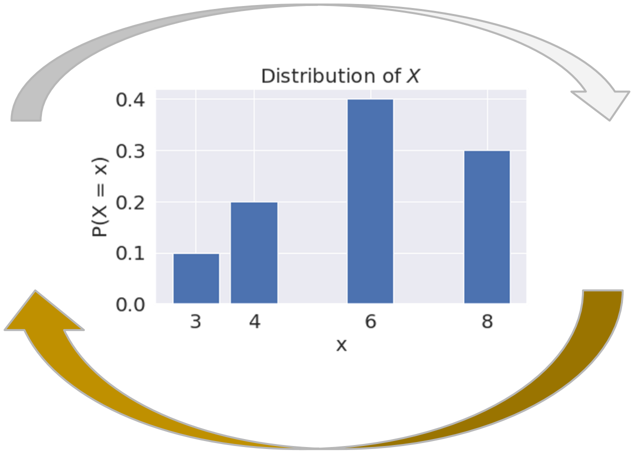
From Distribution to (Simulated) Population

	X(s)
0	3
1	4
2	4
3	6
4	8
...	...
79995	6
79996	6
79997	4
79998	6

... ...

X(s) from many, many **(simulated)** samples

$$P(X = x) = \frac{\#\text{ times where } X = x}{\text{pop. size}}$$



Simulate: Randomly pick values of X according to its distribution

np.random.choice or **df.sample**

x	P(X = x)
3	0.1
4	0.2
6	0.4
8	0.3

Probability Distribution Table

Expectation and Variance

Descriptive Properties of Random Variables

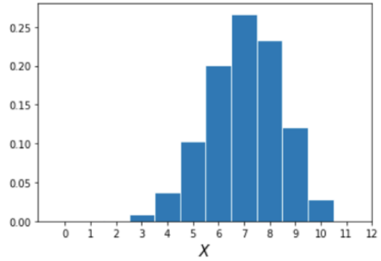
There are several ways to describe a random variable:

Table of all samples s, $X(s)$

Distribution Table

$P(X = x)$

Histogram



Definition (fully describes)

Random Variable X

Properties (summarizes)

Expectation

the “average value” of X

$$\mathbb{E}[X]$$

Variance

the “spread” of X

$$\mathrm{Var}(X)$$

The expectation and variance of a random variable are **numerical summaries** of X.

They are numbers and are not random!

Definition of Expectation

The **expectation** of a random variable X is the **weighted average** of the values of X, where the weights are the probabilities of the values.

Two equivalent ways to apply the weights:

1. One sample at a time: $$\mathbb{E}[X] = \sum_{\text{all samples } s} X(s)P(s)$$

1. One possible value at a time: $$\mathbb{E}[X] = \sum_{\text{all possible } x} xP(X = x)$$

More common (we are usually given the distribution, not all possible samples)

Expectation is a number, not a random variable!

- It is analogous to the **average** (same units as the random variable).
- It is the center of gravity of the probability histogram.
- It is the long run average of the random variable, if you simulate the variable many times.

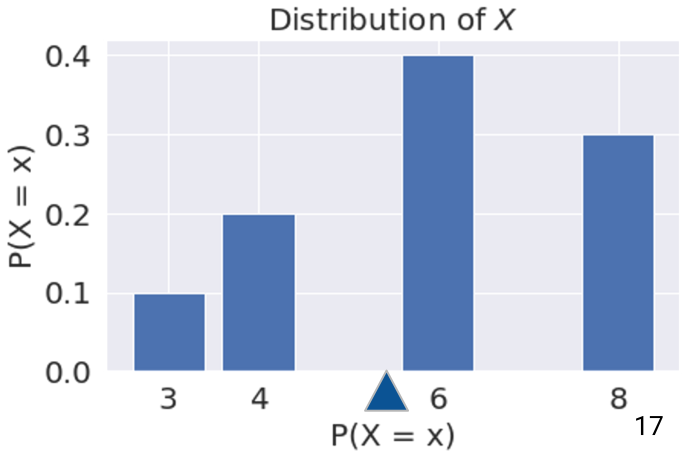
Example

$$\mathbb{E}[X] = \sum_x xP(X = x)$$

Consider the random variable X we defined earlier.

x	P(X = x)
3	0.1
4	0.2
6	0.4
8	0.3

$$\begin{aligned}\mathbb{E}[X] &= \sum_x x \cdot P(X = x) \\ &= 3 \cdot 0.1 + 4 \cdot 0.2 + 6 \cdot 0.4 + 8 \cdot 0.3 \\ &= 0.3 + 0.8 + 2.4 + 2.4 \\ &= 5.9\end{aligned}$$



Note, $E[X] = 5.9$ is not a possible value of X!
It is an average.

The expectation of X does not need to be a value of X.

Definition of Variance

Variance is the **expected squared deviation from the expectation** of X.

$$\text{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$$

- The units of the variance are the square of the units of X.
- To get back to the right scale, use the **standard deviation** of X: $\text{SD}(X) = \sqrt{\text{Var}(X)}$

Variance is a number, not a random variable!

- The main use of variance is to **quantify chance error**. How far away from the expectation could X be, just by chance?

By Chebyshev's inequality :

- No matter what the shape of the distribution of X is, the vast majority of the probability lies in the interval "expectation plus or minus a few SDs."

Variance: Alternate Calculation

$\mathrm{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$

There's a more convenient form of variance:

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

- Proof (involves expanding the square and properties of expectation/summations): link
- Useful in Mean Squared Error calculations
  - If X is centered (i.e. E[X] = 0), then $E[X^2]$ = Var(X)
- When computing variance by hand, often used instead of definition.

Rolling a Die

Let X be the outcome of a single die roll.
X is a random variable.

$$P(X = x) = \begin{cases} 1/6 & \text{if } x \in \{1,2,3,4,5,6\} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X] = \sum_x x P(X = x)$$
$$\begin{aligned}\text{Var}(X) &= \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] \\ &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2\end{aligned}$$

(definitions/properties)

1. What is the expectation, $E[X]$?

2. What is the variance, $\text{Var}(X)$?

Dice Is the Plural; Die Is the Singular

Let X be the outcome of a single die roll.
X is a random variable.

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$$\mathbb{E}[X] = \sum_x x P(X = x)$$

$$\begin{aligned} \mathrm{Var}(X) &= \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] \\ &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \end{aligned}$$

1. What is the expectation, $E[X]$?

$$\begin{aligned} \mathbb{E}[X] &= 1(1/6) + 2(1/6) + 3(1/6) + 4(1/6) + 5(1/6) + 6(1/6) \\ &= (1/6)(1+2+3+4+5+6) = \frac{7}{2} \end{aligned}$$

2. What is the variance, $\mathrm{Var}(X)$?

Dice Is the Plural; Die Is the Singular

Let X be the outcome of a single die roll.
X is a random variable.

$$P(X = x) = \begin{cases} 1/6 & \text{if } x \in \{1,2,3,4,5,6\} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X] = \sum_x xP(X = x)$$
$$\mathrm{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$$
$$= \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

1. What is the expectation, $E[X]$?

$$\mathbb{E}[X] = 1(1/6) + 2(1/6) + 3(1/6) + 4(1/6) + 5(1/6) + 6(1/6)$$
$$= (1/6)(1 + 2 + 3 + 4 + 5 + 6) = \frac{7}{2}$$

2. What is the variance, $\mathrm{Var}(X)$?

Approach 1: Definition

$$\mathrm{Var}(X) = (1/6)\left((1 - 7/2)^2 + (2 - 7/2)^2 + (3 - 7/2)^2 + (4 - 7/2)^2 + (5 - 7/2)^2 + (6 - 7/2)^2\right)$$
$$= 35/12$$

Approach 2: Property

$$\mathbb{E}[X^2] = \sum_x x^2 P(X = x)$$

1^2*(⅙) + 2^2 * (⅙) +... 6^2*(⅙) = 91/6

$$\mathrm{Var}(X) = 91/6 - (7/2)^2 = 35/12$$

Sums of Random Variables

Functions of Multiple Random Variables

A function of a random variable is also a random variable!

If you create multiple random variables based on your sample…

…then functions of those random variables are also random variables.

For instance, if $X_1, X_2, \ldots, X_n$ are random variables, then so are all of these:

$$X_n^2 \qquad \#\{i : X_i > 10\}$$

$$\max(X_1, X_2, \ldots, X_n) \qquad \frac{1}{n}\sum_{i=1}^{n}(X_i - c)^2$$

$$\boxed{\frac{1}{n}\sum_{i=1}^{n} X_i}$$

Many functions of RVs that we care about (**counts**, **means**) involve **sums of RVs**, so we expand on properties of sums of RVs.

Equal vs. Identically Distributed vs. IID

Suppose that we have two random variables X and Y.

X and Y are **equal** if:

- $X(s) = Y(s)$ for every sample s.
- We write $X = Y$.

X and Y are **identically distributed** if:

- The distribution of X is the same as the distribution of Y
- We say "X and Y are equal in distribution."
- If $X = Y$, then X and Y are identically distributed; but the converse is not true.

X and Y are **independent and identically distributed (IID)** if:

- X and Y are identically distributed, *and*
- Knowing the outcome of X does not influence your belief of the outcome of Y, and vice versa ("X and Y are independent.")

Equal RVs

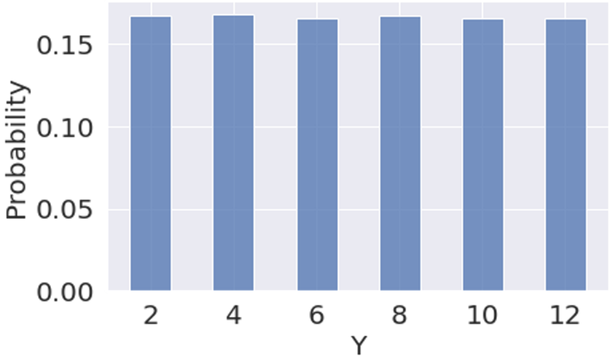
IID RVs

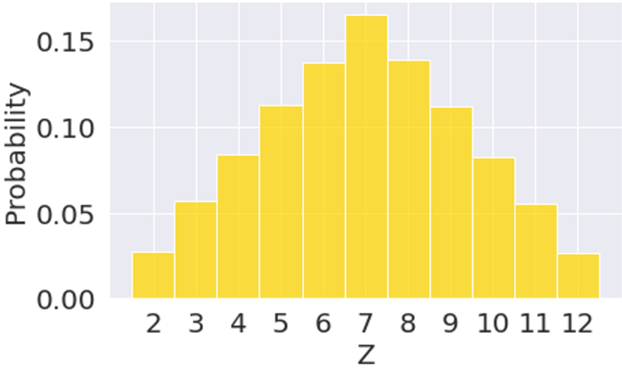
Distributions of Sums

Let X_1 and X_2 be numbers on two rolls of a die.

- X_1, X_2 are **IID**, so X_1, X_2 have the same distribution.
- But the sums $Y = X_1 + X_1 = 2X_1$ and $Z = X_1 + X_2$ have different distributions!

Let's show this through simulation:





- Same expectation…
- But $Y = 2X_1$ has larger variance!

	Y	Z
E[•]	6.984400	6.984950
Var(•)	11.669203	5.817246
SD(•)	3.416021	2.411897

How can we directly compute $E[Y]$, $\mathrm{Var}(Y)$, **without** simulating distributions?

Demo

Properties of Expectation [1/3]

Instead of simulating full distributions, we often just compute expectation and variance directly.

Recall definitions of expectation:

$$\mathbb{E}[X] = \sum_x x P(X = x)$$

$$\mathbb{E}[X] = \sum_{\substack{\text{all samples} \\ s}} X(s)P(s)$$

Properties:

1. **Expectation is linear**.
Intuition: summations are linear. Proof

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

Properties of Expectation [2/3]

Instead of simulating full distributions, we often just compute expectation and variance directly.

Recall definitions of expectation: $\mathbb{E}[X] = \sum_x xP(X = x)$ $\qquad \mathbb{E}[X] = \sum_{\substack{\text{all samples}\\ s}} X(s)P(s)$

Properties:

1. **Expectation is linear**.
Intuition: summations are linear. Proof

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

2. Expectation is linear in sums of RVs,
for any relationship between X and Y. Proof

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

Properties of Expectation [3/3]

Instead of simulating full distributions, we often just compute expectation and variance directly.

Recall definitions of expectation: $\mathbb{E}[X] = \sum_x xP(X = x)$ $\mathbb{E}[X] = \sum_{\substack{\text{all samples} \\ s}} X(s)P(s)$

Properties:

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Intuition: summations are linear. Proof

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

2. Expectation is linear in sums of RVs,
for any relationship between X and Y. Proof

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

3. If g is a non-linear function, then in general $\mathbb{E}[g(X)] \neq g(\mathbb{E}[X])$.

- Example: if X is -1 or 1 with equal probability, then $E[X] = 0$ but $E[X^2] = 1 \neq 0$.

Properties of Variance [1/2]

Recall definition of variance:

$$\mathrm{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$$

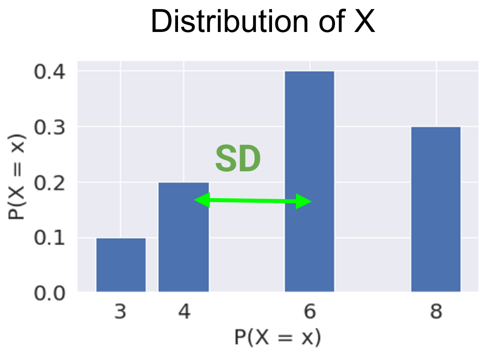
Properties:

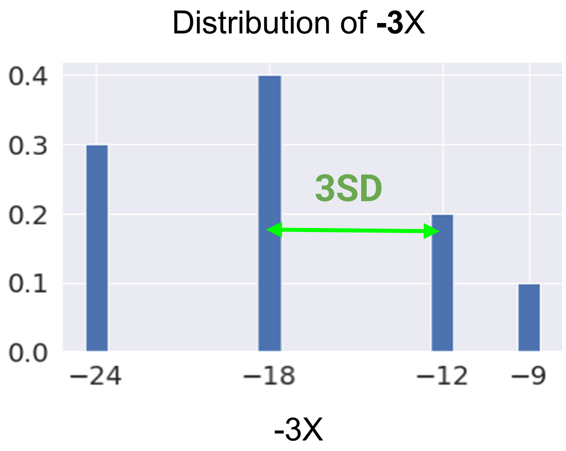
1. **Variance is non-linear**:

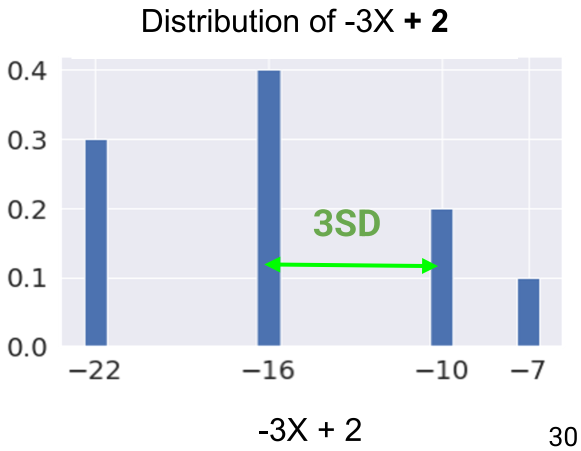
$$\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$$

Intuition (full proof): Consider the Standard Deviation for $Y = -3X+2$:

$$\mathrm{SD}(aX + b) = |a|\mathrm{SD}(X)$$







Properties of Variance [2/2]

Recall definition of variance: $\mathrm{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$

Properties:

1. **Variance is non-linear**:

$$\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$$

Intuition (full proof): Consider the Standard Deviation for $Y = -3X+2$:

$$\mathrm{SD}(aX + b) = |a|\mathrm{SD}(X)$$

2. Variance of sums of RVs is affected by the (in)dependence of the RVs (derivation):

$$\mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \underbrace{2\mathrm{Cov}(X, Y)}$$

Covariance of X and Y (next slide).
If X, Y independent,
then Cov(X, Y) = 0.

Covariance and Correlation: The Basics

Covariance is the expected product of deviations from expectation.

$$\mathrm{Cov}(X, Y) = \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right]$$

- A generalization of variance. Note $\mathrm{Cov}(X, X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathrm{Var}(X)$.
- Interpret by defining **correlation** (yes, *that* correlation!):

$$r(X, Y) = \mathbb{E}\left[\underbrace{\left(\frac{X - \mathbb{E}[X]}{\mathrm{SD}(X)}\right)}_{\textbf{standard units} \text{ of } X \text{ (link)}}\left(\frac{Y - \mathbb{E}[Y]}{\mathrm{SD}(Y)}\right)\right] = \frac{\mathrm{Cov}(X, Y)}{\mathrm{SD}(X)\mathrm{SD}(Y)}$$

Correlation (and therefore covariance) measures a linear relationship between X and Y.

Covariance and Correlation: The Basics

Covariance is the expected product of deviations from expectation.

$$\text{Cov}(X,Y) = \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right]$$

- A generalization of variance. Note $\text{Cov}(X,X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \text{Var}(X)$.
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$$r(X,Y) = \mathbb{E}\left[\underbrace{\left(\frac{X - \mathbb{E}[X]}{\text{SD}(X)}\right)}_{\text{standard units of } X \text{ (link)}}\left(\frac{Y - \mathbb{E}[Y]}{\text{SD}(Y)}\right)\right] = \frac{\text{Cov}(X,Y)}{\text{SD}(X)\text{SD}(Y)}$$

Correlation (and therefore covariance) measures a linear relationship between X and Y.

- If X and Y are correlated, then knowing X tells you something about Y.
- “X and Y are uncorrelated” is the same as “Correlation and covariance equal to 0”

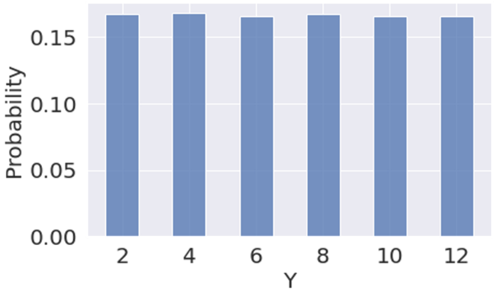
- **Independent X, Y are uncorrelated**, because knowing X tells you nothing about Y.
- The converse is not necessarily true: **X, Y could be uncorrelated but not independent**.

Dice, Our Old Friends: Expectation

Let X_1 and X_2 be numbers on two rolls of a die.

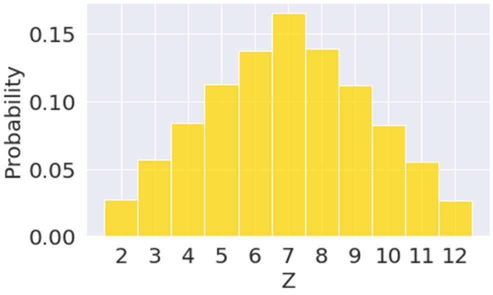
- X_1 , X_2 are **IID**, so X_1 , X_2 have the same distribution.
- Therefore $E[X_1] = E[X_2] = 7/2$ $\qquad$ $\text{Var}(X_1) = \text{Var}(X_2) = 35/12$

$$Y = 2X_1$$



$$E[Y] = E[2X_1] = 2E[X_1] = 7$$

$$Z = X_1 + X_2$$



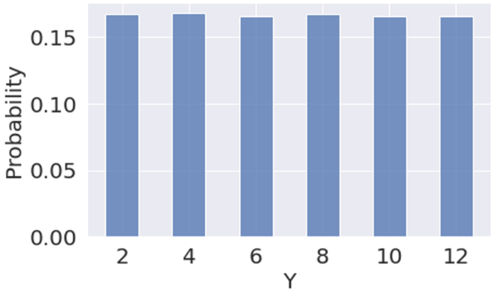
$$E[Z] \qquad = E[X_1] + E[X_2] = (7/2) + (7/2) \quad = 7$$

Dice, Our Old Friends: Variance

Let X_1 and X_2 be numbers on two rolls of a die.

- X_1, X_2 are **IID**, so X_1, X_2 have the same distribution.
- Therefore $E[X_1] = E[X_2] = 7/2$ $\qquad$ $\text{Var}(X_1) = \text{Var}(X_2) = 35/12$

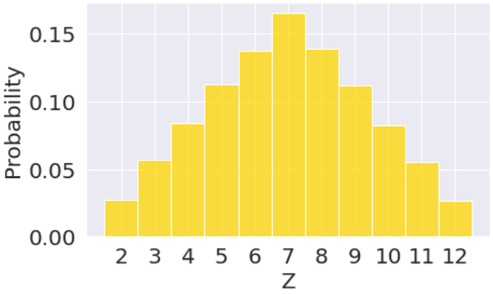
$$\boxed{Y = 2X_1}$$



$E[Y] = E[2X_1] = 2E[X_1] = \mathbf{7}$

$$\begin{aligned}\text{Var}(Y) &= \text{Var}(2X_1) = 4\text{Var}(X_1)\\ &= 4(35/12)\\ &\approx \mathbf{11.67}\end{aligned}$$

$$\boxed{Z = X_1 + X_2}$$



$E[Z] \quad = E[X_1] + E[X_2] = (7/2) + (7/2) = \mathbf{7}$

$$\begin{aligned}\text{Var}(Z) &= \text{Var}(X_1) + \text{Var}(X_2) + \overset{0}{\cancel{2\text{Cov}(X1, X2)}}\\ &= (35/12) + (35/12) + 0\\ &\approx \mathbf{5.83}\end{aligned}$$

X_1, X_2 independent

[Summary] Expectation and Variance for Linear Functions of Random Variables

Let X be a random variable with distribution $P(X = x)$.

$$\mathbb{E}[X] = \sum_x x P(X = x)$$

$$\text{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] \quad \text{(definition)}$$

$$= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \quad \text{(easier computation)}$$

Let a and b be scalar values.

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

$$\text{Var}(aX + b) = a^2\text{Var}(X)$$

Let Y be another random variable.

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

$$\text{Var}(X_1 + X_2) = \text{Var}(X_1) + \text{Var}(X_2) + \underbrace{2\text{Cov}(X, Y)}_{\text{Zero if X, Y independent.}}$$

Bernoulli and Binomial Random Variables

Common Random Variables

Bernoulli(p)

- Takes on value 1 with probability p, and 0 with probability 1 - p
- AKA the "indicator" random variable.

Binomial(n, p)

- Number of 1s in n independent Bernoulli(p) trials
- Probabilities given by the binomial formula (Lecture 2)

Uniform on a finite set of values

- Probability of each value is 1 / (size of set)
- For example, a standard die

Uniform on the unit interval(0, 1)

- Density is flat on (0, 1) and 0 elsewhere

Normal(μ, σ^2)

We'll now revisit these to solidify our understanding of expectation/variance.

Properties of Bernoulli Random Variables

$$\mathbb{E}[X] = \sum_x x P(X = x)$$

$$\begin{aligned} \mathrm{Var}(X) &= \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] \\ &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \end{aligned}$$

Definitions

Let X be a **Bernoulli**(p) random variable.

- Takes on value 1 with probability p, and 0 with probability 1 - p
- AKA the "indicator" random variable.

Expectation:

$$\mathbb{E}[X] = 1 \cdot p + 0 \cdot (1 - p) = p$$

We will get an average value of p across many, many samples

Variance:

$$\mathbb{E}[X^2] = 1^2 \cdot p + 0 \cdot (1 - p) = p$$

$$\begin{aligned} \mathrm{Var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= p - p^2 = p(1 - p) \end{aligned}$$

Lower Var: p = 0.1 or 0.9
Higher Var: p close to 0.5

Properties of Binomial Random Variables

Let Y be a **Binomial**(n, p) random variable.

- Y is the number (i.e., count) of 1s in n independent Bernoulli(p) trials.
- Distribution of Y given by the binomial formula (Lecture 2).

We can write: $$Y = \sum_{i=1}^{n} X_i$$

A count is a **sum** of 0's and 1's.

- X_i is the indicator of success on trial *i*. X_i = 1 if trial i is a success, else 0.
- All X_i s are **IID** (independent and identically distributed) and **Bernoulli**(p).

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Expectation: $$\mathbb{E}[Y] = \sum_{i=1}^{n} \mathbb{E}[X_i] = np$$

Variance: Because all X_i s are independent, $\mathrm{Cov}(X_i, X_j)$ = 0 for all i, j.

$$\mathrm{Var}(Y) = \sum_{i=1}^{n} \mathrm{Var}(X_i) = np(1-p)$$

Interlude

Suppose you win cash based on the number of heads you get in a series of 20 coin flips.

Let $X_i = 1$ if the i-th coin is heads, 0 otherwise

Which payout strategy would you choose?

Hint: Compare expectations and variances.

A. $Y_A = 10 \cdot X_1 + 10 \cdot X_2$

B. $Y_B = \left(\sum_{i=1}^{20} X_i \right)$

C. $Y_C = 20 \cdot X_1$

Which would you pick?

Expectation of a linear function is linear

$$\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$$

$$\mathrm{Var}(X_1+X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + 2\mathrm{Cov}(X,Y)$$

Suppose you win cash based on the number of heads you get in a series of 20 coin flips.

Let $X_1, X_2, \ldots, X_{20}$ be 20 **IID** Bernoulli(0.5) random variables.

- Since X_i s are independent: $\mathrm{Cov}(X_i, X_j) = 0$ for all i, j.
- Since X_i is Bernoulli(p = 0.5): $E[X_i] = p = 0.5,\ \mathrm{Var}(X_i) = p(1-p) = 0.25$.

Which payout strategy would you choose?

	A. $Y_A = 10 \cdot X_1 + 10 \cdot X_2$	**B.** $Y_B = \left(\sum_{i=1}^{20} X_i\right)$	**C.** $Y_C = 20 \cdot X_1$
Expectation			
Variance			
Std. Deviation			

Which would you pick?

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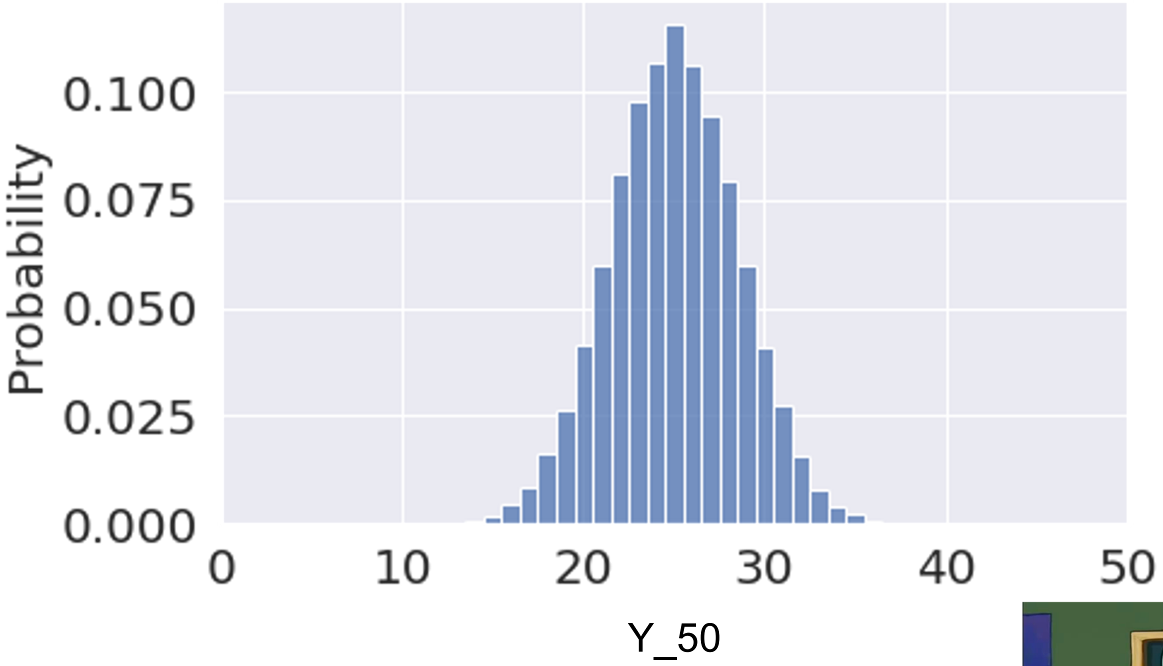
Which payout strategy would you choose?

	A. $Y_A = \$10 \cdot X_1 + \$10 \cdot X_2$	**B.** $Y_B = \$\left(\sum_{i=1}^{20} X_i\right)$	**C.** $Y_C = \$20 \cdot X_1$
Expectation	$E[Y_A] = 10(0.5) + 10(0.5) = 10$	$E[Y_B] = 0.5 + \ldots + 0.5 = 10$	$E[Y_C] = 20(0.5) = 10$
Variance	$\mathrm{Var}(Y_A) = 10^2(0.25) + 10^2(0.25)$ $= 50$	$\mathrm{Var}(Y_B) = 0.25 + \ldots + 0.25$ $= 20(0.25)$ $= 5$	$\mathrm{Var}(Y_C) = 20^2(0.25)$ $= 100$
Std. Deviation	$\mathrm{SD}(Y_A) \approx \mathbf{7.07}$	$\mathrm{SD}(Y_B) \approx \mathbf{2.24}$	$\mathrm{SD}(Y_C) = \mathbf{10}$

Binomial(n, p) for large n

For p = 0.5, n = 50 (i.e. number of heads in 50 fair coin flips):



Sample Statistics

From Populations to Samples

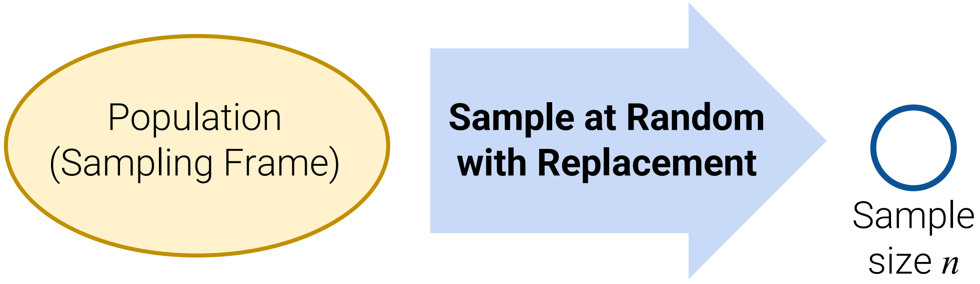
Today, we've talked extensively about **populations**:

- If we know the **distribution of a random variable**, we can reliably compute expectation, variance, functions of the random variable, etc.

However, in Data Science, we often collect **samples**.

- We don't know the distribution of our population.
- We'd like to use the distribution of your sample to estimate/infer properties of the population.

The **big assumption** we make in modeling/inference:



The Sample is a Set of IID Random Variables

Population
(Sampling Frame)

Sample at Random
with Replacement

Sample
size n

x	P(X = x)
3	0.1
4	0.2
6	0.4
8	0.3

or

	X(s)
0	3
1	4
2	4
3	6
4	8
...	...
79995	6
79996	6
79997	4
79998	6
79999	6
...	...

Population
(really large N)

df.sample(n, replace=True)
[documentation]

	x
0	6
1	8
2	6
3	6
4	3
...	...
95	8
96	6
97	6
98	3
99	8

Each observation in our sample is a **Random Variable** drawn **IID** from our population distribution.

Sample
(n << N)

$\boxed{X_1, X_2, \ldots, X_n}$

The Sample is a Set of IID Random Variables

Population (Sampling Frame)

x	P(X = x)
3	0.1
4	0.2
6	0.4
8	0.3

or

	X(s)
0	3
1	4
2	4
3	6
4	8
...	...
79995	6
79996	6
79997	4
79998	6
79999	6
...	...

$E[X] = 5.9$

Population Mean
A **number**,
i.e., fixed value

Sample at Random with Replacement

df.sample(n, replace=True)
[documentation]

Sample size n

	X
0	6
1	8
2	6
3	6
4	3
...	...
95	8
96	6
97	6
98	3
99	8

Sample Mean
A **random variable**!
Depends on our randomly drawn sample!!

np.mean(...) = 5.71

Sample $X_1, X_2, \ldots, X_n$

[Terminology] Sample Mean

Consider an IID sample X_1 , X_2 , ..., X_n drawn from a numerical population with mean μ and SD σ.

Define the **sample mean**:

$$\boxed{\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i}$$

Expectation:

$$\begin{aligned}\mathbb{E}[\bar{X}_n] &= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i] \\ &= \frac{1}{n}(n\mu) = \boxed{\mu}\end{aligned}$$

Variance/Standard Deviation:

$$\begin{aligned}\mathrm{Var}(\bar{X}_n) &= \frac{1}{n^2}\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\underbrace{\left(\sum_{i=1}^{n}\mathrm{Var}(X_i)\right)}_{\text{IID} \to \mathrm{Cov}(X_i, X_j) = 0} \\ &= \frac{1}{n^2}\left(n\sigma^2\right) = \boxed{\frac{\sigma^2}{n}}\end{aligned}$$

$$\mathrm{SD}(\bar{X}_n) = \boxed{\frac{\sigma}{\sqrt{n}}}$$

Distribution?

$\bar{X}_n$ is **normally distributed** by the **Central Limit Theorem**.

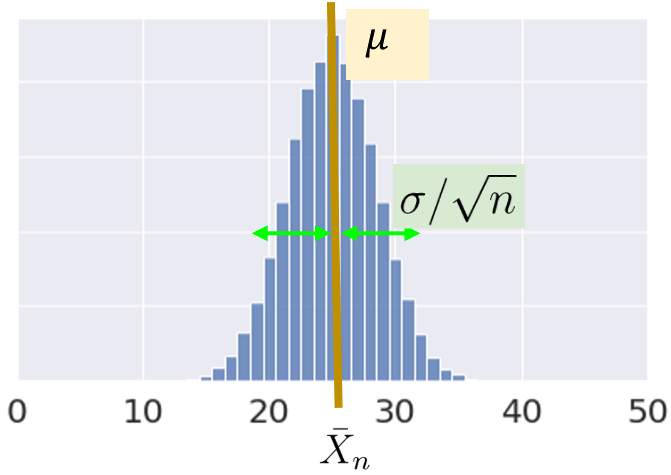
Central Limit Theorem

No matter what population you are drawing from:

If an IID sample of size n is large,

the probability distribution of the **sample mean**

is **roughly normal** with mean μ and SD $\sigma/\sqrt{n}$.



Any theorem that provides the rough distribution of a statistic
and **doesn't need the distribution of the population** is valuable to data scientists.

- Because we rarely know a lot about the population!

How Large Is "Large"?

No matter what population you are drawing from:

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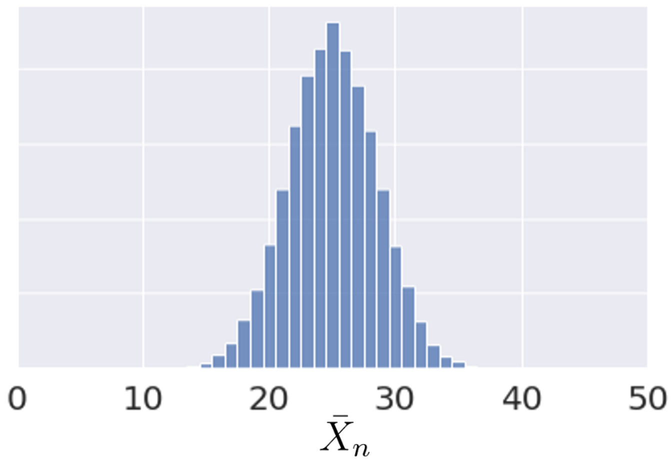


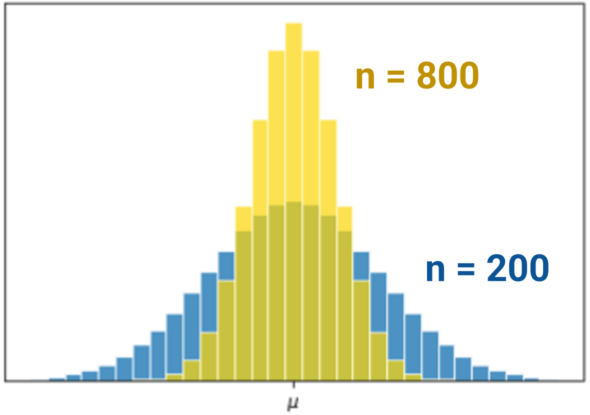
How large does n have to be for the normal approximation to be good?

- …It depends on the shape of the distribution of the population…
- If population is **roughly symmetric and unimodal**/uniform, could need as few as **n = 20**.
  If population is very skewed, you will need bigger n.
- If in doubt, you can bootstrap the sample mean and see if
  the bootstrapped distribution is bell-shaped.

Accuracy and Spread of the Sample Mean

Our goal is often to **estimate** some characteristic of a population.

- Example: average height of Cal undergraduates.
- We typically can collect a **single sample**. It has just one average.
- Since that sample was random, it *could have* come out differently.



We should consider the **average value and spread** of all possible sample means, and what this means for how big n should be.

$$\mathbb{E}[\bar{X}_n] = \mu$$

$$\mathrm{SD}(\bar{X}_n) = \frac{\sigma}{\sqrt{n}}$$

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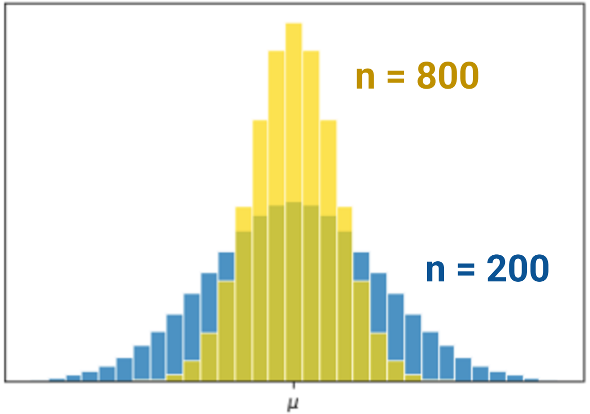

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For every sample size, the expected value of the sample mean is the population mean.

We call the sample mean an **unbiased estimator** of the population mean.

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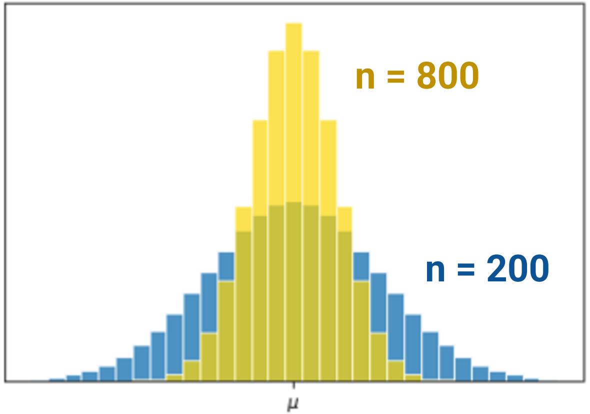

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For every sample size, the expected value of the sample mean is the population mean.

We call the sample mean an **unbiased estimator** of the population mean. (more in next lecture)

Square root law: If you increase the sample size by a factor, the SD decreases by the square root of the factor.

The sample mean is more likely to be close to the population mean if we have a larger sample size.

[Extra Slides] Derivations

Random Variables and Distributions

- Expectation and Variance
- Equality vs Identically Distributed
- Common RVs: Bernoulli, Binomial

Functions of Random Variables

- Distributions through Simulation, I.I.D.
- Properties of Expectation and Variance
- Covariance, Correlation
- Standard Units

Sample Statistics

- Sample Mean
- Central Limit Theorem

Standardization of random variables

Jump back: link

X in **standard units** is the random variable $X_{su} = \dfrac{X - \mathbb{E}(X)}{\mathbb{SD}(X)}$.

X_{su} measures X on the scale **"number of SDs from expectation**."

- It is a linear transformation of X. By the linear transformation rules for expectation and variance:

$$\mathbb{E}(X_{su}) = 0, \quad \mathbb{SD}(X_{su}) = 1$$

- Since X_{su} is centered (has expectation 0):

$$\mathbb{E}(X_{su}^2) = \mathbb{V}ar(X_{su}) = 1$$

You should prove these facts yourself.

Variance: An Alternate Derivation

There's a more convenient form of variance for use in calculations.

$$\mathbb{V}ar(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

To derive this, we make repeated use of the linearity of expectation.

$$\begin{aligned} Var(X) &= E((X - E(X))^2) \\ &= E\big(X^2 - 2XE(X) + (E(X)^2)\big) \\ &= E(X^2) - 2E(X)E(X) + (E(X))^2 \\ &= E(X^2) - (E(X))^2 \end{aligned}$$

Properties of Expectation #1

Jump back: link

Recall definition of expectation: $\mathbb{E}[X] = \sum_x xP(X = x)$ $\qquad \mathbb{E}[X] = \sum_{\substack{\text{all samples} \\ s}} X(s)P(s)$

1. **Expectation is linear**:
(intuition: summations are linear)

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

Proof:

$$\begin{aligned}
\mathbb{E}[aX + b] &= \sum_x (ax + b)P(X = x) = \sum_x \big(axP(X = x) + bP(X = x)\big) \\
&= a\sum_x xP(X = x) + b\sum_x P(X = x) \\
&= a\mathbb{E}[X] + b \cdot 1
\end{aligned}$$

Properties of Expectation #2

Jump back: link

Recall definitions of expectation:

$$\mathbb{E}[X] = \sum_{x} x P(X = x)$$

$$\mathbb{E}[X] = \sum_{\substack{\text{all samples} \\ s}} X(s)P(s)$$

3. **Expectation is linear in sums of RVs**:

For any relationship between X and Y.

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

Proof:

$$\mathbb{E}[X+Y] = \sum_{s} (X+Y)(s)P(s) = \sum_{s} \big(X(s) + Y(s)\big)P(s)$$

$$= \sum_{s} \big(X(s)P(s) + Y(s)P(s)\big)$$

$$= \sum_{s} X(s)\,P(s) + \sum_{s} Y(s)P(s)$$

$$= \mathbb{E}[X] + \mathbb{E}[Y]$$

Properties of Variance #1

Jump back: link

We know that $\mathbb{E}(aX+b) = a\mathbb{E}(X) + b$

In order to compute Var(aX + b), consider:

- A shift by **b** units **does not** affect spread. Thus, Var(aX + b) = Var(aX).
- The multiplication by **a does** affect spread!

Then,

$$\begin{aligned} Var(aX+b) = Var(aX) &= E((aX)^2) - (E(aX))^2 \\ &= E(a^2X^2) - (aE(X))^2 \\ &= a^2\left(E(X^2) - (E(X))^2\right) \\ &= a^2 Var(X) \end{aligned}$$

In summary:

$$\mathbb{E}[g(X)] \neq g(\mathbb{E}[X])$$

$$\mathbb{V}ar(aX+b) = a^2\mathbb{V}ar(X)$$
$$\mathbb{SD}(aX+b) = |a|\mathbb{SD}(X)$$

Don't forget the absolute values and squares!

Properties of Variance #2

Jump back: link

The variance of a sum is affected by the dependence between the two random variables that are being added. Let's expand out the definition of Var(X + Y) to see what's going on.

Let $\mu_x = E[X], \mu_y = E[Y]$

We see

$$\begin{aligned} Var(X+Y) &= E\left[(X+Y-E(X+Y))^2\right] \\ &= E\left[((X-\mu_x)+(Y-\mu_y))^2\right] \\ &= E\left[(X-\mu_x)^2 + 2(X-\mu_x)(Y-\mu_y) + (Y-\mu_y)^2\right] \\ &= E\left[(X-\mu_x)^2\right] + E\left[(Y-\mu_y)^2\right] + 2E\left[(X-\mu_x)(Y-\mu_y)\right] \\ &= Var(X) + Var(Y) + 2E\left[(X-E(X))(Y-E(Y))\right] \end{aligned}$$

By the linearity of expectation, and the substitution.

Addition rule for variance

If X and Y are **uncorrelated** (in particular, if they are **independent**), then

$$\mathbb{V}ar(X+Y) \;=\; \mathbb{V}ar(X) + \mathbb{V}ar(Y)$$

Therefore, under the same conditions,

$$\mathbb{SD}(X+Y) \;=\; \sqrt{\mathbb{V}ar(X) + \mathbb{V}ar(Y)} \;=\; \sqrt{(\mathbb{SD}(X))^2 + (\mathbb{SD}(Y))^2}$$

- Think of this as “Pythagorean theorem” for random variables.
- Uncorrelated random variables are like orthogonal vectors.