VE203
Discrete Math
RC2

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1 Sort (Optional)

2 Relations

3 Equivalence Classes

4 Cardinality and Equinumerosity

5 Pigeonhole Principle

6 Q&A

Heap Sort

See Blackboard

Sorting Visualization

https://www.bilibili.com/video/BV1yt411V7Bz
https://www.bilibili.com/video/BV1bg4y1q7es

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Relations

Definition

A subset $R \subset A \times B$ is called a (binary) relation from A to B. If $A = B$, we say that R is a relation on A.

Notation

1. $(x, y) \in R$
2. xRy
3. via predefined symbols, e.g.,

$x \preceq y$, i.e., $(x, y) \in \preceq \subset A \times B$;
$x \sim y$, i.e., $(x, y) \in \sim \subset A \times B$.

Relations

$R = \varnothing$, the empty relation, with domain $(\varnothing) = \text{range}(\varnothing) = \varnothing$
When $A = B$, we have the identity relation,

$$\text{id}_A = \{(a, a) \mid a \in A\}$$

The identity relation relates every element to itself. Note that domain $(\mathrm{id}_A) = \text{range}\,(\mathrm{id}_A) = A$.
The relation $A \times B$ itself. This relation relates every element of A to every element of B. Note that domain $(A \times B) = A$ and range $(A \times B) = B$.

Functions

Definition

A function is a relation F such that

$$\forall x \in \text{dom } F(\exists! y(xFy))$$

Definition (Easy to understand)

Let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$

Relations and Functions

The inverse of F is the set

$$F^\top = F^{-1} = \{(y, x) \mid xFy\}$$

The composition of F and G is the set

$$F \circ G = \{(x, z) \mid \exists y \in A(xFy \wedge yGz)\}$$

The restriction of F to A is the set

$$F \mid A = \{(x, y) \in F \mid x \in A\}$$

The image of A under F is the set

$$F(A) = \mathrm{ran}(F \mid A) = \{y \mid (\exists x \in A)xFy\}$$

If F is a function, then $F(A) = \{F(x) \mid x \in A\}$.

Injection and Surjection

Given a function $F : A \to B$, with dom $F = A$ and $\text{ran}(F) \subset B$, then
F is injective or one-to-one if $\forall x, y \in A(F(x) = F(y) \Rightarrow x = y)$;
F is surjective or onto if $\text{ran}(F) = B$.
F is bijective if it is both injective and surjective.
The above function F is also called an injection, surjection, or bijection, respectively.

Injection and Surjection

Let $f : A \to B, g : B \to C$
If $g \circ f$ is injective, then f is injective.
If $g \circ f$ is surjective, then g is surjective.

Relations

Given a relation $R \subset A \times B$, the associated boolean function is given by

$$\phi_R : A \times B \to \{\top, \bot\}$$
$$(x, y) \mapsto \begin{cases} \top, & xRy \\ \bot, & \text{otherwise} \end{cases}$$

Properties of Relations

1. reflexive if $aRa \Rightarrow \top$.
2. irreflexive if $aRa \Rightarrow \perp$.
3. total if $aRb \vee bRa \Rightarrow \top$.
4. transitive if $aRb \wedge bRc \Rightarrow aRc$.
5. symmetric if $aRb \Leftrightarrow bRa$.
6. anti-symmetric if $aRb \wedge bRa \Rightarrow a = b$.
7. asymmetric if $aRb \wedge bRa \Rightarrow \perp$.

Properties of Relations

1. reflexive if $aRa \Rightarrow \top$.

Example: Consider the following relations on $\{1, 2, 3, 4\}$:

$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$

$R_2 = \{(1,1),(1,2),(2,1)\}$

$R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$

$R_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$

$R_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$

$R_6 = \{(3,4)\}$

Which of these relations are reflexive?

Properties of Relations

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$R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$

$R_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$

$R_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$

$R_6 = \{(3,4)\}$

Which of these relations are reflexive?

Properties of Relations

Solution: The relations R_3 and R_5 are reflexive because they both contain all pairs of the form (a, a), namely, $(1, 1), (2, 2), (3, 3)$, and $(4, 4)$. The other relations are not reflexive because they do not contain all of these ordered pairs. In particular, R_1, R_2, R_4, and R_6 are not reflexive because $(3, 3)$ is not in any of these relations.

Properties of Relations

Is the “divides” relation on the set of positive integers reflexive?

Solution: Because $a \mid a$ whenever a is a positive integer, the "divides" relation is reflexive. (Note that if we replace the set of positive integers with the set of all integers the relation is not reflexive because by definition 0 does not divide 0 .)

Symmetric and anti-Symmetric

A relation R on a set A is called symmetric if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$. A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called antisymmetric.

Symmetric and anti-Symmetric

Consider these relations on the set of integers:

$$R_1 = \{(a, b) \mid a \leq b\}$$
$$R_2 = \{(a, b) \mid a > b\}$$
$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$$
$$R_4 = \{(a, b) \mid a = b\}$$
$$R_5 = \{(a, b) \mid a = b + 1\}$$
$$R_6 = \{(a, b) \mid a + b \leq 3\}$$

Which of the relations from above are symmetric and which are antisymmetric?

Symmetric and anti-Symmetric

Solution: The relations R_3, R_4, and R_6 are symmetric. R_3 is symmetric, for if $a = b$ or $a = -b$, then $b = a$ or $b = -a. R_4$ is symmetric because $a = b$ implies that $b = a. R_6$ is symmetric because $a + b \leq 3$ implies that $b + a \leq 3$.

The relations R_1, R_2, R_4, and R_5 are antisymmetric. R_1 is antisymmetric because the inequalities $a \leq b$ and $b \leq a$ imply that $a = b. R_2$ is antisymmetric because it is impossible that $a > b$ and $b > a. R_4$ is antisymmetric, because two elements are related with respect to R_4 if and only if they are equal. R_5 is antisymmetric because it is impossible that $a = b + 1$ and $b = a + 1$.

Symmetric and anti-Symmetric

Is the “divides” relation on the set of positive integers symmetric? Is it antisymmetric?
Solution: This relation is not symmetric because $1 \mid 2$, but $2 \nmid 1$. It is antisymmetric, for if a and b are positive integers with $a \mid b$ and $b \mid a$, then $a = b$

Properties of Relations

4. transitive if $aRb \wedge bRc \Rightarrow aRc$.

Consider these relations on the set of integers:

$$R_1 = \{(a, b) \mid a \leq b\}$$
$$R_2 = \{(a, b) \mid a > b\}$$
$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$$
$$R_4 = \{(a, b) \mid a = b\}$$
$$R_5 = \{(a, b) \mid a = b + 1\}$$
$$R_6 = \{(a, b) \mid a + b \leq 3\}$$

Which of the relations from above are transitive?

Properties of Relations

Solution: The relations R_1, R_2, R_3, and R_4 are transitive. R_1 is transitive because $a \leq b$ and $b \leq c$ imply that $a \leq c. R_2$ is transitive because $a > b$ and $b > c$ imply that $a > c. R_3$ is transitive because $a = \pm b$ and $b = \pm c$ imply that $a = \pm c. R_4$ is clearly transitive, as the reader should verify. R_5 is not transitive because $(2, 1)$ and $(1, 0)$ belong to R_5, but $(2, 0)$ does not. R_6 is not transitive because $(2, 1)$ and $(1, 2)$ belong to R_6, but $(2, 2)$ does not.

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Equivalence Relation

Definition

An equivalence relation on a set A is a relation that is
reflexive
symmetric
transitive

Example

On $\mathbb{Z}$ (or $\mathbb{R}$ etc.): $a = b$
On $\mathbb{Z}$: $a \equiv b(\text{mod} 12)$
On 2^S for given S : $A \equiv B$ iff $|A| = |B|$
On square matrices: $A \cong B$ iff $A = PBP^{-1}$

Equivalence Class

Definition

Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the equivalence class of a. The equivalence class of a with respect to R is denoted by $[a]_R$. When only one relation is under consideration, we can delete the subscript R and write $[a]$ for this equivalence class.

Equivalence Class

What are the equivalence classes of 0 and 1 for congruence modulo 4?

Solution: The equivalence class of 0 contains all integers a such that $a \equiv 0(\bmod 4)$. The integers in this class are those divisible by 4 . Hence, the equivalence class of 0 for this relation is

$$[0]=\{\ldots,-8,-4,0,4,8,\ldots\}$$

The equivalence class of 1 contains all the integers a such that $a \equiv 1(\bmod 4)$. The integers in this class are those that have a remainder of 1 when divided by 4 . Hence, the equivalence class of 1 for this relation is

$$[1]=\{\ldots,-7,-3,1,5,9,\ldots\}$$

Partition

We are now in a position to show how an equivalence relation partitions a set. Let R be an equivalence relation on a set A. The union of the equivalence classes of R is all of A, because an element a of A is in its own equivalence class, namely, $[a]_R$. In other words,

$$\bigcup_{a \in A} [a]_R = A$$

Quotient set

Given an equivalence relation R on A, then the set $\{[x]_R \mid x \in A\}$ of all equivalence classes is a partition of A.
Given an equivalence relation R on A, then the quotient set is given by

$$A/R := \{[x]_R \mid x \in A\}$$

where A/R is read " A modulo R ".
Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, and define the relation $\sim$ on $\mathbb{N}$ by

$$m \sim n \quad \Leftrightarrow \quad m - n \text{ divisible by } 6$$

then $\mathbb{N}/\sim = \{[0], [1], [2], [3], [4], [5]\}$

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Cardinality and Equinumerosity

A set A is equinumerous to a set B (written $A \approx B$) if there is a bijection from A to B.

The sets A and B have the same cardinality if and only if there is a one-to-one correspondence from A to B. When A and B have the same cardinality, we write $|A| = |B|$.

Countable Set

A set that is either finite or has the same cardinality as the set of positive integers is called countable. A set that is not countable is called uncountable. When an infinite set S is countable, we denote the cardinality of S by $\aleph_0$ (where $\aleph$ is aleph, the first letter of the Hebrew alphabet). We write $|S| = \aleph_0$ and say that S has cardinality "aleph null."

Equinumerosity

Several important equinumerosity:

$\mathbb{Z} \approx \mathbb{N}$

$\mathbb{Q} \approx \mathbb{N}$

$\mathbb{R} \not\approx \mathbb{N}$

For every set A, $A \not\approx \mathcal{P}(A)$

Hilbert's Grand Hotel

See Blackboard

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Pigeonhole Principle

Definition

If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Pigeonhole Principle

Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet.

How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

Solution: There are 101 possible scores on the final. The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score.

The Generalized Pigeonhole Principle

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

The Generalized Pigeonhole Principle

Solution: The minimum number of students needed to ensure that at least six students receive the same grade is the smallest integer N such that $\lceil N/5 \rceil = 6$. The smallest such integer is $N = 5 \cdot 5 + 1 = 26$. If you have only 25 students, it is possible for there to be five who have received each grade so that no six students have received the same grade. Thus, 26 is the minimum number of students needed to ensure that at least six students will receive the same grade.

Some Applications of the Pigeonhole Principle

Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

Some Applications of the Pigeonhole Principle

Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group.

Solution: Let A be one of the six people. Of the five other people in the group, there are either three or more who are friends of A, or three or more who are enemies of A. This follows from the generalized pigeonhole principle, because when five objects are divided into two sets, one of the sets has at least $\lceil 5/2 \rceil = 3$ elements. In the former case, suppose that B, C, and D are friends of A. If any two of these three individuals are friends, then these two and A form a group of three mutual friends. Otherwise, B, C, and D form a set of three mutual enemies. The proof in the latter case, when there are three or more enemies of A, proceeds in a similar manner.

Ramsey number

The Ramsey number $R(m, n)$, where m and n are positive integers greater than or equal to 2 , denotes the minimum number of people at a party such that there are either m mutual friends or n mutual enemies, assuming that every pair of people at the party are friends or enemies.

Ramsey number

Values / known bounding ranges for Ramsey numbers

$m\&n$	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2		2	3	4	5	6	7	8
3			6	9	14	18	23	28
4				18	25	36 – 41	49 – 61	59 – 84
5					43 – 48	58 – 87	80 – 143	101 – 216
6						102 – 165	115 – 298	134 – 495
7							205 – 540	217 – 1031
8								282 – 1870

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