VE203 Discrete Math
Spring 2022 — Worksheet 3 Solutions

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Exercise 3.1 Poset
Which of these are posets?

a) $(\mathbf{Z}, =)$

b) $(\mathbf{Z}, \neq)$

c) $(\mathbf{Z}, \geq)$

d) $(\mathbf{Z}, \nmid)$

e) $(\mathbf{R}, =)$

f) $(\mathbf{R}, <)$

g) $(\mathbf{R}, \leq)$

h) $(\mathbf{R}, \neq)$

Solution:
The question in each case is whether the relation is reflexive, antisymmetric, and transitive.

a) The equality relation on any set satisfies all three conditions and is therefore a partial partial ordering. (It. is the smallest partial partial ordering; reflexivity insures that every partial order contains at least all the pairs (a, a).)

b) This is not a poset, since the relation is not reflexive, not antisymmetric, and not transitive (the absence of one of these properties would have been enough to give a negative answer).

c) This is a poset

d) This is not a poset. The relation is not reflexive, since it is not true, for instance, that $2X2$. (It also is not antisymmetric and not transitive.)

e) The equality relation on any set satisfies all three conditions and is therefore a partial order. (It is the smallest partial order; reflexivity insures that every partial order contains at least all the pairs (a, a).)

f) This is not a poset, since the relation is not reflexive, although it is antisymmetric and transitive. Any relation of this sort can be turned into a partial ordering by adding in all the pairs (a, a).

g) This is a poset

h) This is not a poset, since the relation is not reflexive, not antisymmetric, and not transitive (the absence of one of these properties would have been enough to give a negative answer).

Exercise 3.2 Partial Order
Answer these questions for the poset $(\{2,4,6,9,12,\ 18,27,36,48,60,72\},|)$.

a) Find the maximal elements.

b) Find the minimal elements.

c) Is there a greatest element?

d) Is there a least element?

e) Find all upper bounds of $\{2,9\}$.

f) Find the least upper bound of $\{2,9\}$, if it exists.

g) Find all lower bounds of $\{60,72\}$.

h) Find the greatest lower bound of $\{60,72\}$, if it exists.

Solution:
The reader should draw the Hasse diagram to aid in answering these questions.

a) Clearly the numbers $27,48,60$, and 72 are maximal, since each divides no number in the list other than itself. All of the other numbers divide 72 , however, so they are not maximal.

b) Only 2 and 9 are minimal. Every other element is divisible by either 2 or 9 .

c) There is no greatest element, since, for example, there is no number in the set that both 60 and 72 divide.

d) There is no least element, since there is no number in the set that divides both 2 and 9.

e) We need to find numbers in the list that are multiples of both 2 and 9 . Clearly 18,36 , and 72 are the numbers we are looking for.

f) Of the numbers we found in the previous part, 18 satisfies the definition of the least upper bound, since it divides the other two upper bounds.

g) We need to find numbers in the list that are divisors of both 60 and 72 . Clearly $2,4,6$, and 12 are the numbers we are looking for.

h) Of the numbers we found in the previous part, 12 satisfies the definition of the greatest lower bound, since the other three lower bounds divide it.

Exercise 3.3 Lattice
Determine whether these posets are lattices.

a) $(\{1,3,6,9,12\},1)$

b) $(\{1,5,25,125\},|)$

c) $(\mathbf{Z},\geq)$

d) $(P(S),\supseteq)$, where $P(S)$ is the power set of a set S

Solution:
In each case, we need to decide whether every pair of elements has a least upper bound and a greatest lower bound.

a) This is not a lattice, since the elements 6 and 9 have no upper bound (no element in our set is a multiple of both of them).

b) This is a lattice; in fact it is a linear order, since each element in the list divides the next one. The least upper bound of two numbers in the list is the larger, and the greatest lower bound is the smaller.

c) Again, this is a lattice because it is a linear order. The least upper bound of two numbers in the list is the smaller number (since here "greater" really means "less"!), and the greatest lower bound is the larger of the two numbers.

d) Yes. Here the g.l.b. of two subsets A and B is $A \cup B$, and their l.u.b. is $A \cap B$.

Exercise 3.4 Chain and Antichain

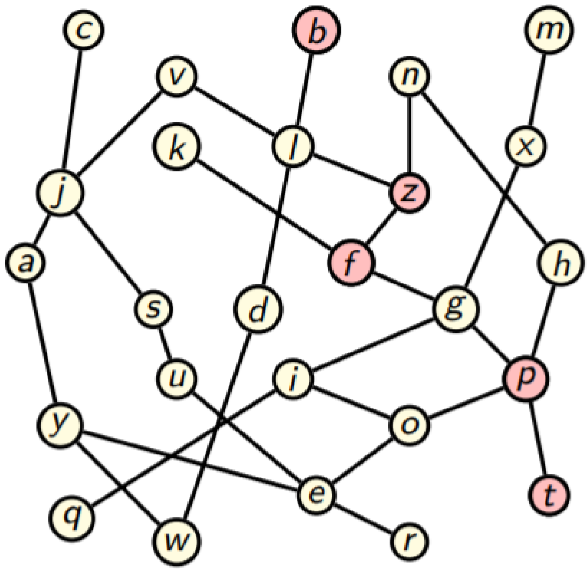

Figure 1: problem 3.4

a) Find a chain

b) Find an antichain

c) Find a maximal chain

d) Find a maximal antichain

Solution:

a) $\{t, p, f, z, b\}$ is a chain

b) $\{j, d, g, h\}$ is an antichain

c) $\{r, e, o, i, g, f, z, l, v\}$ is a maximal chain

d) $\{a, s, d, k, z, x, h\}$ is a maximal antichain

Exercise 3.5 Cardinality
Give an example of two uncountable sets A and B such that $A \cap B$ is
a) finite.
b) countably infinite.
c) uncountable.

Solution:
In each case, we can make the intersection what we want it to be, and then put additional elements into A and into B (with no overlap) to make them uncountable.
a) The simplest solution would be to make $A \cap B = \varnothing$. So, for example, take A to be the interval $(1, 2)$ of real numbers, and take B to be the interval $(3, 4)$.
b) Take the example from part (a) and adjoin the positive integers. Thus, let $A = (1, 2) \cup \mathbf{Z}^+$and let $B = (3, 4) \cup \mathbf{Z}^+$.
c) Let $A = (1, 3)$ and $B = (2, 4)$.

Exercise 3.6 Cardinality
Show that there is no infinite set A such that $|A| < |\mathbf{Z}^+| = \aleph_0$.

Solution:
Because $|A| < |\mathbf{Z}^+|$, there is a one-to-one function $f : A \to \mathbf{Z}^+$. We are also given that A is infinite, so the range of f has to be infinite. We will construct a bijection g from $\mathbf{Z}^+$to A. For each $n \in \mathbf{Z}^+$, let m be the n^{th} smallest element in the range of f. Then $g(n) = f^{-1}(m)$. The existence of g contradicts the definition of $|A| < |\mathbf{Z}^+|$, and our proof is complete.

Reference

1. Rosen, Kenneth H., and Kamala Krithivasan. Discrete mathematics and its applications: with combinatorics and graph theory. Tata McGraw-Hill Education, 2012.

2. https://services.math.duke.edu/ lpereira/Teaching/ApCombSlides/3012_Lecture_18_handout.p