VE203 Discrete Math
Spring 2022 — Worksheet 2 Solutions

February 14, 2022

Exercise 2.1 Relation
Determine whether the relation R on the set of all real numbers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if
a) $x + y = 0$.
b) $x - y$ is a rational number.
c) $xy \geq 0$.

Solution:
a) Since $1 + 1 \neq 0$, this relation is not reflexive. Since $x + y = y + x$, it follows that $x + y = 0$ if and only if $y + x = 0$, so the relation is symmetric. Since $(1, -1)$ and $(-1, 1)$ are both in R, the relation is not antisymmetric. The relation is not transitive; for example, $(1, -1) \in R$ and $(-1, 1) \in R$, but $(1, 1) \notin R$.
b) The relation is reflexive, since $x - x = 0$ is a rational number. The relation is symmetric, because if $x - y$ is rational, then so is $-(x - y) = y - x$. Since $(1, -1)$ and $(-1, 1)$ are both in R, the relation is not antisymmetric. To see that the relation is transitive, note that if $(x, y) \in R$ and $(y, z) \in R$, then $x - y$ and $y - z$ are rational numbers. Therefore their sum $x - z$ is rational, and that means that $(x, z) \in R$.
c) This relation is reflexive since squares are always nonnegative. It is clearly symmetric (the roles of x and y in the statement are interchangeable). It is not antisymmetric, since $(2, 3)$ and $(3, 2)$ are both in R. It is not transitive; for example, $(1, 0) \in R$ and $(0, -2) \in R$, but $(1, -2) \notin R$.

Exercise 2.2 Relation
How many transitive relations are there on a set with n elements if
a) $n = 1$?
b) $n = 2$?
c) $n = 3$?

Solution:
a) There are two relations on a set with only one element, and they are both transitive.
b) There are 16 relations on a set with two elements, and we saw that 13 of them are transitive.
*1. $\varnothing$
*2. $\{(0, 0)\}$
*3. $\{(0, 1)\}$
*4. $\{(1, 0)\}$
*5. $\{(1, 1)\}$
*6. $\{(0, 0), (0, 1)\}$
*7. $\{(0, 0), (1, 0)\}$
*8. $\{(0, 0), (1, 1)\}$
9. $\{(0, 1), (1, 0)\}$
*10. $\{(0, 1), (1, 1)\}$
*11. $\{(1, 0), (1, 1)\}$

12. $\{(0,0),(0,1),(1,0)\}$
*13. $\{(0,0),(0,1),(1,1)\}$
*14. $\{(0,0),(1,0),(1,1)\}$
15. $\{(0,1),(1,0),(1,1)\}$
*16. $\{(0,0),(0,1),(1,0),(1,1)\}$
c) (optional) For $n = 3$ there are $2^{3^2} = 512$ relations. One way to find out how many of them are transitive is to use a computer to generate them all and check each one for transitivity. If we do this, then we find that 171 of them are transitive. Doing this by hand is not pleasant, since there are many cases to consider.

Exercise 2.3 Equinumerosity
Prove: Let A be an arbitrary set. Then A and $\mathcal{P}(A)$ are not equinumerous.

Solution:
We will show that there is no one-to-one function from A onto $\mathcal{P}(A)$; in fact, no function from A onto $\mathcal{P}(A)$ exist, whether or not we require it to be one-to-one. To see this, let $f : A \to \mathcal{P}(A)$ be an arbitrary function, and consider the subset C of A defined as

$$C = \{x \in A : x \notin f(x)\}$$

Then there is no $y \in A$ for which $f(y) = C$. Indeed, for an arbitrary $y \in A$, if $y \in f(y)$ then $y \notin C$ by the definition of C, and if $y \notin f(y)$ then $y \in C$. This shows that $f(y)$ and C do not have the same elements (y is an element of exactly one of these two sets), so $f(y) \neq C$, as claimed.

Exercise 2.4 Equinumerosity
Prove: The set $\mathbb{R}$ and the interval $(-1,1)$ are equinumerous.

Solution:
The function $f : (-1,1) \to \mathbb{R}$ such that

$$f(x) = \frac{x}{x^2 - 1} \qquad \text{for} \qquad x \in (-1,1)$$

is one-to-one and onto $\mathbb{R}$. Indeed, let $y \in \mathbb{R}$ be arbitrary. We need to show that there is exactly one $x \in (-1,1)$ such that $f(x) = y$. If $y = 0$ then we have $f(x) = y$ only for $x = 0$. Assume now that $y \neq 0$. Then the equation $f(x) = y$ can be equivalently written as $y\,(x^2 - 1) = x$.

Observe that this latter equation makes sense for $x = \pm 1$ while the equation $f(x) = y$ does not. The important point, however, is that $x = \pm 1$ does not satisfy the latter equation, since for this choice of x the left-hand side is 0 while the right-hand side is ± 1. That is, the exceptional case of $x = \pm 1$ does not affect the equivalence of the two equations.

Keeping in mind that we assumed that $y \neq 0$, the latter equation can also be written as

$$x^2 - \frac{1}{y}x - 1 = 0$$

This is a quadratic equation for x. We can solve this equation for x as

$$x = \frac{\frac{1}{y} \pm \sqrt{\frac{1}{y^2} + 4}}{2}$$

Given that the discriminant (the expression under the square root) of this equation is positive, this equation has two distinct real solutions; call them x_1 and x_2. The product of these two solutions is the constant term of the equation; that is, $x_1 x_2 = -1$.

Therefore $|x_1| |x_2| = 1$. Given that $|x_1|, |x_2| \neq 1$, as we remarked above, one of x_1 and x_2 must be inside the interval $(-1, 1)$ and the other one must be outside. That is, there is exactly one $x \in (-1, 1)$ for which $f(x) = y$, as we wanted to show.

Exercise 2.5 Pigeonhole Principle
How many numbers must be selected from the set $\{1, 2, 3, 4, 5, 6\}$ to guarantee that at least one pair of these numbers add up to 7 ?

Solution:
We can apply the pigeonhole principle by grouping the numbers cleverly into pairs (subsets) that add up to 7 , namely $\{1, 6\}, \{2, 5\}$, and $\{3, 4\}$. If we select four numbers from the set $\{1, 2, 3, 4, 5, 6\}$, then at least two of them must fall within the same subset, since there are only three subsets. Two numbers in the same subset are the desired pair that add up to 7. We also need to point out that choosing three numbers is not enough, since we could choose $\{1, 2, 3\}$, and no pair of them add up to more than 5.

Exercise 2.6 Pigeonhole Principle
In the 17th century, there were more than 800,000 inhabitants of Paris. At the time, it was believed that no one had more than 200,000 hairs on their head. Assuming these numbers are correct and that everyone has at least one hair on their head (that is, no one is completely bald), use the pigeonhole principle to show, as the French writer Pierre Nicole did, that there had to be two Parisians with the same number of hairs on their heads. Then use the generalized pigeonhole principle to show that there had to be at least five Parisians at that time with the same number of hairs on their heads.

Solution:
The numbers from 1 to 200,000 are the pigeonholes, and the inhabitants of Paris are the pigeons, which number at least 800,001 . Therefore by Theorem 1 there are at least two Parisians with the same number of hairs on their heads; and by Theorem 2 there are at least $\lceil 800,001/200,000 \rceil = 5$ Parisians with the same number of hairs on their heads.

Exercise 2.7 Pigeonhole Principle
a) Assume that $i_k \leq n$ for $k = 1, 2, \ldots, n^2 + 1$. Use the generalized pigeonhole principle to show that there are $n + 1$ terms $a_{k_1}, a_{k_2}, \ldots, a_{k_{n+1}}$ with $i_{k_1} = i_{k_2} = \cdots = i_{k_{n+1}}$, where $1 \leq k_1 < k_2 < \cdots < k_{n+1}$.
b) Show that $a_{k_j} > a_{k_{j+1}}$ for $j = 1, 2, \ldots, n$. [Hint: Assume that $a_{k_j} < a_{k_{j+1}}$, and show that this implies that $i_{k_j} > i_{k_{j+1}}$, which is a contradiction.]
c) Use parts (a) and (b) to show that if there is no increasing subsequence of length $n + 1$, then there must be a decreasing subsequence of this length.

Solution:
a) Assuming that each $i_k \leq n$, there are only n pigeonholes (namely $1, 2, \ldots, n$) for the $n^2 + 1$ numbers $i_1, i_2, \ldots, i_{n^2+1}$. Hence, by the generalized pigeonhole principle at least $\lceil (n^2 + 1) / n \rceil = n + 1$ of the numbers are in the sarne pigeonhole, i.e., equal.
b) If $a_{k_J} < a_{k_{J+1}}$, then the subsequence consisting of $a_{k,}$ followed by a maximal increasing subsequence of length $i_{k_{j+1}}$ starting at $a_{k_{J+1}}$ contradicts the fact that $i_{k_J} = i_{k_{J+1}}$. Hence

$a_{k_j} > a_{k_{3+1}}$.

c) If there is no increasing subsequence of length greater than n, then parts (a) and (b) apply. Therefore we have $a_{k_{n+1}} > a_{k_n} > \cdots > a_{k_2} > a_{k_1}$, a decreasing subsequence of length $n + 1$.

Reference

1. Rosen, Kenneth H., and Kamala Krithivasan. Discrete mathematics and its applications: with combinatorics and graph theory. Tata McGraw-Hill Education, 2012.

2. http://www.sci.brooklyn.cuny.edu/ mate/misc/cardinalities.pdf