VE203 Discrete Math
Spring 2022 — Worksheet 5 Solutions

March 25, 2022

Exercise 5.1 Group

In Exercises 1 through 6, determine whether the binary operation $*$ gives a group structure on the given set. If no group results, give the first axiom in the order $\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3$ from Definition that does not hold.

1. Let $*$ be defined on $\mathbb{Z}$ by letting $a * b = ab$.
2. Let $*$ be defined on $2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\}$ by letting $a * b = a + b$.
3. Let $*$ be defined on $\mathbb{R}^+$by letting $a * b = \sqrt{ab}$.
4. Let $*$ be defined on $\mathbb{Q}$ by letting $a * b = ab$.
5. Let $*$ be defined on the set $\mathbb{R}^*$ of nonzero real numbers by letting $a * b = a/b$.
6. Let $*$ be defined on $\mathbb{C}$ by letting $a * b = |ab|$.

Solution:

1. No. G_3 fails.
2. Yes
3. No. G_1 fails.
4. No. G_3 fails.
5. No. G_1 fails.
6. No. G_2 fails.

Exercise 5.2 Group

Let S be the set of all real numbers except -1. Define $*$ on S by

$$a * b = a + b + ab.$$

a. Show that $*$ gives a binary operation on S.
b. Show that $\langle S, * \rangle$ is a group.
c. Find the solution of the equation $2 * x * 3 = 7$ in S.

Solution:

a. We must show that S is closed under $*$, that is, that $a + b + ab \neq -1$ for $a, b \in S$. Now $a + b + ab = -1$ if and only if $0 = ab + a + b + 1 = (a + 1)(b + 1)$. This is the case if and only if either $a = -1$ or $b = -1$, which is not the case for $a, b \in S$.

b. Associative: We have $a * (b * c) = a * (b + c + bc) = a + (b + c + bc) + a(b + c + bc) = a + b + c + ab + ac + bc + abc$ and $(a * b) * c = (a + b + ab) * c = (a + b + ab) + c + (a + b + ab)c = a + b + c + ab + ac + bc + abc$. Identity: 0 acts as identity element for $*$, for $0 * a = a * 0 = a$.

Inverses: $\frac{-a}{a+1}$ acts as inverse of a, for

$$a * \frac{-a}{a+1} = a + \frac{-a}{a+1} + a\frac{-a}{a+1} = \frac{a(a+1) - a - a^2}{a+1} = \frac{0}{a+1} = 0$$

c. Because the operation is commutative, $2 * x * 3 = 2 * 3 * x = 11 * x$. Now the inverse of 11 is $-11/12$ by Part(**b**). From $11 * x = 7$, we obtain

$$x = \frac{-11}{12} * 7 = \frac{-11}{12} + 7 + \frac{-11}{12}7 = \frac{-11 + 84 - 77}{12} = \frac{-4}{12} = -\frac{1}{3}$$

Exercise 5.3 Subgroup

In Exercises 1 through 6, determine whether the given subset of the complex numbers is a subgroup of the group $\mathbb{C}$ of complex numbers under addition.

1. $\mathbb{R}$
2. $\mathbb{Q}^+$
3. $7\mathbb{Z}$
4. The set $i\mathbb{R}$ of pure imaginary numbers including 0
5. The set $\pi\mathbb{Q}$ of rational multiples of π
6. The set $\{\pi^n \mid n \in \mathbb{Z}\}$

Solution:

1. Yes
2. No, there is no identity element.
3. Yes
4. Yes
5. Yes
6. No, the set is not closed under addition.

Exercise 5.4 Cyclic Group

In Exercises 1 through 5, find all orders of subgroups of the given group.

1. $\mathbb{Z}_6$
2. $\mathbb{Z}_8$
3. $\mathbb{Z}_{12}$
4. $\mathbb{Z}_{20}$
5. $\mathbb{Z}_{17}$

Solution:

1. $1, 2, 3, 6$
2. $1, 2, 4, 8$
3. $1, 2, 3, 4, 6, 12$
4. $1, 2, 4, 5, 10, 20$
5. 1,17

Exercise 5.5 Permutation Group

In Exercises 1 through 5 , compute the indicated product involving the following permutations in S_6 : $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}$, $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$, $\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 4 & 3 & 1 & 6 \end{pmatrix}$

1. $\tau\sigma$
2. $\tau^2\sigma$
3. $\mu\sigma^2$
4. $\sigma^{-2}\tau$
5. $\sigma^{-1}\tau\sigma$

Solution:

1. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 6 & 5 & 4 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 5 & 6 & 3 \end{pmatrix}$

3. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 6 & 2 & 5 \end{pmatrix}$

4. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 6 & 2 & 4 & 3 \end{pmatrix}$

5. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 1 & 5 & 4 & 3 \end{pmatrix}$

Exercise 5.6 Permutation Group

In Exercises 1 through 4, compute the expressions shown for the permutations σ, τ and μ defined prior to Exercise 5.5 .

1. $|\langle\sigma\rangle|$
2. $|\langle\tau^2\rangle|$
3. σ^{100}
4. μ^{100}

Solution:

1. Starting with 1 and applying σ repeatedly, we see that σ takes 1 to 3 to 4 to 5 to 6 to 2 to 1 , so σ^6 is the smallest possible power of σ that is the identity permutation. It is easily checked that σ^6 carries $2,3,4,5$ and 6 to themselves also, so σ^6 is indeed the identity and $|\langle\sigma\rangle| = 6$.

2. $\tau^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 5 & 6 \end{pmatrix}$ and it is clear that $\left(\tau^2\right)^2$ is the identity. Thus we have $|\langle\tau^2\rangle| = 2$.

3. Because σ^6 is the identity permutation (see Exercise 6), we have

$$\sigma^{100} = \left(\sigma^6\right)^{16} \sigma^4 = \sigma^4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 1 & 3 & 4 \end{pmatrix}$$

4. We find that μ^2 is the identity permutation, so $\mu^{100} = \left(\mu^2\right)^{50}$ is also the identity permutation.

Exercise 5.7 Homomorphism

Determine whether the given map ϕ is a homomorphism.

1. Let $\phi : \mathbb{Z} \to \mathbb{R}$ under addition be given by $\phi(n) = n$.
2. Let $\phi : \mathbb{R} \to \mathbb{Z}$ under addition be given by $\phi(x) =$ the greatest integer $\leq x$.
3. Let $\phi : \mathbb{R}^* \to \mathbb{R}^*$ under multiplication be given by $\phi(x) = |x|$.

Solution:

1. It is a homomorphism, because $\phi(m+n) = m+n = \phi(m) + \phi(n)$.

2. It is not a homomorphism, because $\phi(2.6 + 1.6) = \phi(4.2) = 4$ but $\phi(2.6) + \phi(1.6) = 2 + 1 = 3$.

3. It is a homomorphism, because $\phi(xy) = |xy| = |x||y| = \phi(x)\phi(y)$ for $x, y \in \mathbb{R}^*$.

Exercise 5.8 Coset and Lagrange Theorem

1. Find all cosets of the subgroup $4\mathbb{Z}$ of $2\mathbb{Z}$.
2. Find all cosets of the subgroup $\langle 2\rangle$ of $\mathbb{Z}_{12}$.
3. Find all cosets of the subgroup $\langle 4\rangle$ of $\mathbb{Z}_{12}$.
4. Find all cosets of the subgroup $\langle 18\rangle$ of $\mathbb{Z}_{36}$.
5. Find the index of $\langle 3\rangle$ in the group $\mathbb{Z}_{24}$.
6. Let $\sigma = (1,2,5,4)(2,3)$ in S_5. Find the index of $\langle\sigma\rangle$ in S_5.
7. Let $\mu = (1,2,4,5)(3,6)$ in S_6. Find the index of $\langle\mu\rangle$ in S_6.

Solution:

1. $4\mathbb{Z} = \{\cdots,-8,-4,0,4,8,\cdots\}, 2+4\mathbb{Z} = \{\cdots,-6,-2,2,6,10,\cdots\}$
2. $\langle 2\rangle = \{0,2,4,6,8,10\}, 1+\langle 2\rangle = \{1,3,5,7,9,11\}$
3. $\langle 4\rangle = \{0,4,8\}, \quad 1+\langle 4\rangle = \{1,5,9\}, \quad 2+\langle 4\rangle = \{2,6,10\}, \quad 3+\langle 4\rangle = \{3,7,11\}$
4. $\langle 18\rangle = \{0,18\}, \quad 1+\langle 18\rangle = \{1,19\}, \quad 2+\langle 18\rangle = \{2,20\}, \quad \cdots, \quad 17+\langle 18\rangle = \{17,35\}$
5. $\langle 3\rangle = \{1,3,6,9,12,15,18,21\}$ has 8 elements, so its index (the number of cosets) is $24/8 = 3$.
6. $\sigma = (1,2,5,4)(2,3) = (1,2,3,5,4)$ generates a cyclic subgroup of S_5 of order 5 , so its index (the number of left cosets) is $5!/5 = 4! = 24$.
7. $\mu = (1,2,4,5)(3,6)$ generates a cyclic subgroup of S_6 of order 4 , (the cycles are disjoint) so its index (the number of left cosets) is $6!/4 = 720/4 = 180$.

Reference

1. Rosen, Kenneth H., and Kamala Krithivasan. Discrete mathematics and its applications: with combinatorics and graph theory. Tata McGraw-Hill Education, 2012.

2. Fraleigh, John B. A first course in abstract algebra. Pearson Education India, 2003.