VE203 Discrete Math
Spring 2022 — Worksheet 4 Solutions

March 25, 2022

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Exercise 4.1 Divisibility

1. Show that if $a \mid b$ and $b \mid a$, where a and b are integers, then $a=b$ or $a=-b$.
2. Show that if a, b, c, and d are integers, where $a \neq 0$, such that $a \mid c$ and $b \mid d$, then $ab \mid cd$.
3. Show that if a, b, and c are integers, where $a \neq 0$ and $c \neq 0$, such that $ac \mid bc$, then $a \mid b$.

Solution:

1. The given conditions imply that there are integers s and t such that $a=bs$ and $b=at$. Combining these, we obtain $a=ats$; since $a \neq 0$, we conclude that $st=1$. Now the only way for this to happen is for $s=t=1$ or $s=t=-1$. Therefore either $a=b$ or $a=-b$.
2. Under the hypotheses, we have $c=as$ and $d=bt$ for some s and t. Multiplying we obtain $cd=ab(st)$, which means that $ab \mid cd$, as desired.
3. The given condition means that $bc=(ac)t$ for some integer t. Since $c \neq 0$, we can divide both sides by c to obtain $b=at$. This is the definition of $a \mid b$, as desired.

Exercise 4.2 Prime

Find the prime factorization of 10!.

Solution:

$10! = 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 = 2 \cdot 3 \cdot 2^2 \cdot 5 \cdot (2 \cdot 3) \cdot 7 \cdot 2^3 \cdot 3^2 \cdot (2 \cdot 5) = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$

Exercise 4.3 Prime

How many zeros are there at the end of 100 !?

Solution:

A 0 appears at the end of a number for every factor of $10 (= 2 \cdot 5)$ the number has. Now 100 ! certainly has more factors of 2 than it has factors of 5 , so the number of factors of 10 it has is the same as the number of factors of 5 . Each of the twenty numbers $5, 10, 15, \ldots, 100$ contributes a factor of 5 to 100 !, and in addition the four numbers $25, 50, 75$, and 100 contribute one more factor of 5 . Therefore there are 24 factors of 5 in 100 !, so 100 ! ends in exactly 240 's.

Exercise 4.4 Prime

The value of the Euler ϕ-function at the positive integer n is defined to be the number of positive integers less than or equal to n that are relatively prime to n. [Note: ϕ is the Greek letter phi.]

What is the value of $\phi\left(p^k\right)$ when p is prime and k is a positive integer?

Solution:

All the positive integers less than or equal to p^k (and there are clearly p^k of them) are less than p^k and relatively prime to p^k unless they are a multiple of p. Since the fraction $1/p$ of them are multiples of p, we have $\phi\left(p^k\right) = p^k(1-1/p) = p^k - p^{k-1}$.

Exercise 4.5 Euclidean algorithm

Use the Euclidean algorithm to find

1. $\gcd(1, 5)$.
2. $\gcd(100, 101)$.
3. $\gcd(123, 277)$.
4. $\gcd(1529, 14039)$.
5. $\gcd(1529, 14038)$.
6. $\gcd(11111, 111111)$.

Solution:

To apply the Euclidean algorithm, we divide the larger number by the smaller, replace the larger by the smaller and the smaller by the remainder of this division, and repeat this process until the remainder is 0 . At that point, the smaller number is the greatest common divisor.

1. $\gcd(1, 5) = \gcd(1, 0) = 1$
2. $\gcd(100, 101) = \gcd(100, 1) = \gcd(1, 0) = 1$
3. $\gcd(123, 277) = \gcd(123, 31) = \gcd(31, 30) = \gcd(30, 1) = \gcd(1, 0) = 1$
4. $\gcd(1529, 14039) = \gcd(1529, 278) = \gcd(278, 139) = \gcd(139, 0) = 139$
5. $\gcd(1529, 14038) = \gcd(1529, 277) = \gcd(277, 144) = \gcd(144, 133) = \gcd(133, 11) = \gcd(11, 1) = \gcd(1, 0) = 1$
6. $\gcd(11111, 111111) = \gcd(11111, 1) = \gcd(1, 0) = 1$

Exercise 4.6 Prime

Adapt the proof in the text that there are infinitely many primes to prove that there are infinitely many primes of the form $3k + 2$, where k is a nonnegative integer. [Hint: Suppose that there are only finitely many such primes $q_1, q_2, \ldots, q_n$, and consider the number $3q_1q_2 \cdots q_n - 1$.]

Solution:

Suppose by way of contradiction that $q_1, q_2, \ldots, q_n$ are the only primes of the form $3k+2$. Notice that this list necessarily includes 2 . Let $Q = 3q_1q_2 \cdots q_n - 1$. Notice that neither 3 nor any prime of the form $3k+2$ is a factor of Q. But $Q \geq 3 \cdot 2 - 1 = 5 > 1$, so it must have prime factors. Therefore all of its prime factors are of the form $3k+1$. However, the product of numbers of the form $3k+1$ is again of that form, because $(3k+1)(3l+1) = 3(3kl+k+l)+1$. Patently Q is not of that form, and we have a contradiction, which completes the proof.

Exercise 4.7 Goldbach's conjecture

Show that Goldbach's conjecture, which states that every even integer greater than 2 is the sum of two primes, is equivalent to the statement that every integer greater than 5 is the sum of three primes

Solution:

Assume that every even integer greater than 2 is the sum of two primes, and let n be an integer greater than 5 . If n is odd, then we can write $n = 3 + (n - 3)$, decompose

$n-3=p+q$ into the sum of two primes (since $n-3$ is an even integer greater than 2), and therefore have written $n=3+p+q$ as the sum of three primes. If n is even, then we can write $n=2+(n-2)$, decompose $n-2=p+q$ into the sum of two primes (since $n-2$ is an even integer greater than 2), and therefore have written $n=2+p+q$ as the sum of three primes. For the converse, assume that every integer greater than 5 is the sum of three primes, and let n be an even integer greater than 2. By our assumption we can write $n+2$ as the sum of three primes. Since $n+2$ is even, these three primes cannot all be odd, so we have $n+2=2+p+q$, where p and q are primes, whence $n=p+q$, as desired.

Reference

1. Rosen, Kenneth H., and Kamala Krithivasan. Discrete mathematics and its applications: with combinatorics and graph theory. Tata McGraw-Hill Education, 2012.

2. Fraleigh, John B. A first course in abstract algebra. Pearson Education India, 2003.