Ve203 Discrete Mathematics (Spring 2022)

Assignment 7

Exercise 7.1
Consider the functions $f : B \to U$, count the number of functions and fill in the blanks below. Express the results in binomial coefficients, factorials, or powers (**AVOID** double bracket notation).

Elements of Domain	Elements of Codomain	Any f	Injective f	Surjective f
distinguishable	distinguishable			
indistinguishable	distinguishable			

where

1. $B = \{1, 2, 3\}$ and $U = \{1, 2, 3, 4, 5\}$.
2. $B = \{1, 2, 3, 4, 5\}$ and $U = \{1, 2, 3\}$.

Exercise 7.2
Derive the following formula for the Euler's totient function φ

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

by applying the inclusion-exclusion principle to the set $\{1, 2, \ldots, n\}$.

Exercise 7.3
Consider

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \leq 100$$

What are the number of integer solutions if

(i) $x_i > 0$ and $=$ holds;
(ii) $x_i \geq 0$ and $=$ holds;
(iii) $x_i > 0$ and $<$ holds;
(iv) $x_i \geq 0$ and $<$ holds;
(v) $x_i \geq 0$.

AVOID double bracket notation in the final solution.

Exercise 7.4
Find the Θ bound of $T(n)$ for the following recurrence relation.

(i) $T(n) = 4T(n/4) + 5n$.
(ii) $T(n) = 4T(n/5) + 5n$.
(iii) $T(n) = 5T(n/4) + 4n$.
(iv) $T(n) = 4T(\sqrt{n}) + \log^5 n$
(v) $T(n) = 4T(\sqrt{n}) + \log^2 n$

Exercise 7.5
The purpose of this problem is to prove that the number of spanning trees of the complete graph K_n, $n \geq 2$, is n^{n-2}, a formula due to Cayley (1889).[1]

(i) Let $T(n; d_1, \ldots, d_n)$ be the number of trees with $n \geq 2$ vertices $v_1, \ldots, v_n$, and degrees $d(v_1) = d_1$, $d(v_2) = d_2$, $\ldots$, $d(v_n) = d_n$, with $d_i \geq 1$. Show that

$$T(n; d_1, \ldots, d_n) = \binom{n-2}{d_1 - 1, d_2 - 1, \ldots, d_n - 1}$$

(ii) Show that $d_1, \ldots, d_n$, with $d_i \geq 1$, are degrees of a tree with n vertices iff

$$\sum_{i=1}^{n} d_i = 2(n-1)$$

(iii) Use (i) and (ii) prove that the number of spanning trees of K_n is n^{n-2}.

[1] For hints, see Gallier, p. 254