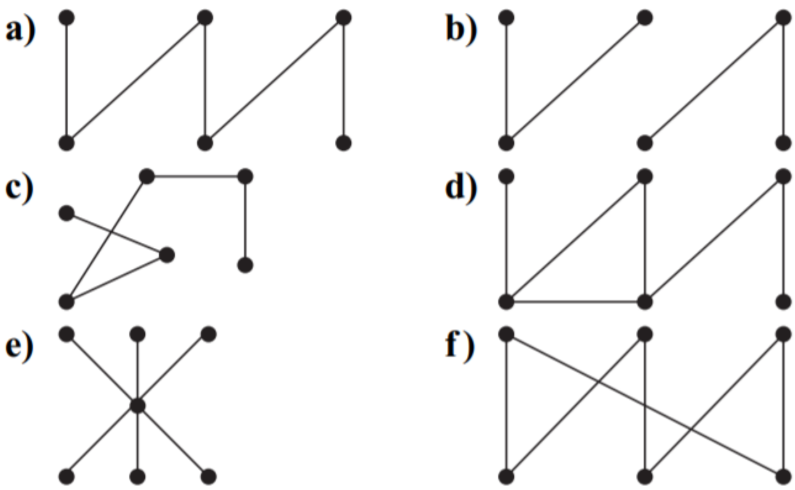
VE203 Discrete Math
Spring 2022 — Worksheet 8 Solutions

April 16, 2022

Exercise 8.1 Tree Definition

Which of these graphs are trees?



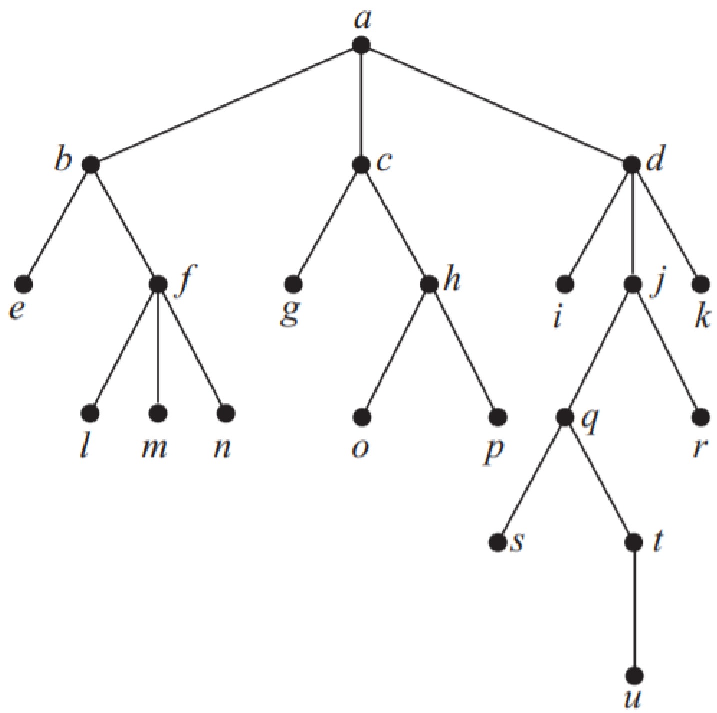
Solution:

a) This graph is connected and has no simple circuits, so it is a tree.
b) This graph is not connected, so it is not a tree.
c) This graph is connected and has no simple circuits, so it is a tree.
d) This graph has a simple circuit, so it is not a tree.
e) This graph is connected and has no simple circuits, so it is a tree.
f) This graph has a simple circuit, so it is not a tree.

Exercise 8.2 Tree Definition

Answer these questions about the rooted tree illustrated.

a) Which vertex is the root?
b) Which vertices are internal?
c) Which vertices are leaves?
d) Which vertices are children of j ?
e) Which vertex is the parent of h ?
f) Which vertices are siblings of o ?
g) Which vertices are ancestors of m ?
h) Which vertices are descendants of b ?



Solution:

a) Vertex a is the root, since it is drawn at the top.

b) The internal vertices are the vertices with children, namely a, b, c, d, f, h, j, q, and t.

c) The leaves are the vertices without children, namely $e, g, i, k, l, m, n, o, p, r, s$, and u.

d) The children of j are the vertices adjacent to j and below j, namely q and r.

e) The parent of h is the vertex adjacent to h and above h, namely c.

f) Vertex o has only one sibling, namely p, which is the other child of o's parent. h.

g) The ancestors of m are all the vertices on the unique simple path frorn m back to the root, namely f, b, and a

h) The descendants of b are all the vertices that have b as an ancestor, namely e, f, l, m, and n.

Exercise 8.3 Spanning Tree

Find a spanning tree for each of these graphs.

a) K_5

b) $K_{4,4}$

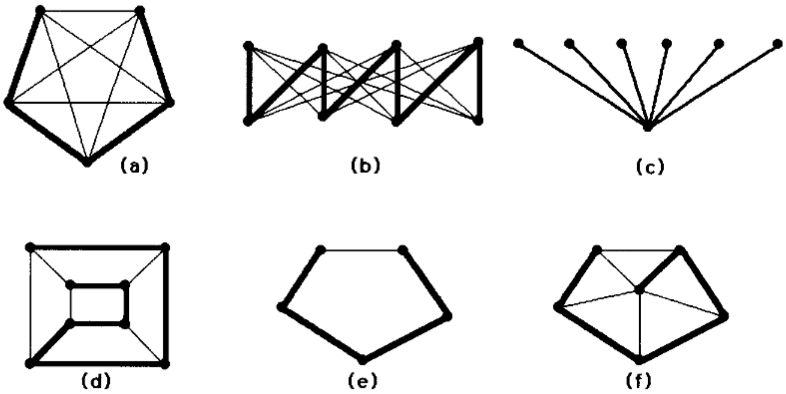
c) $K_{1,6}$

d) Q_3

e) C_5

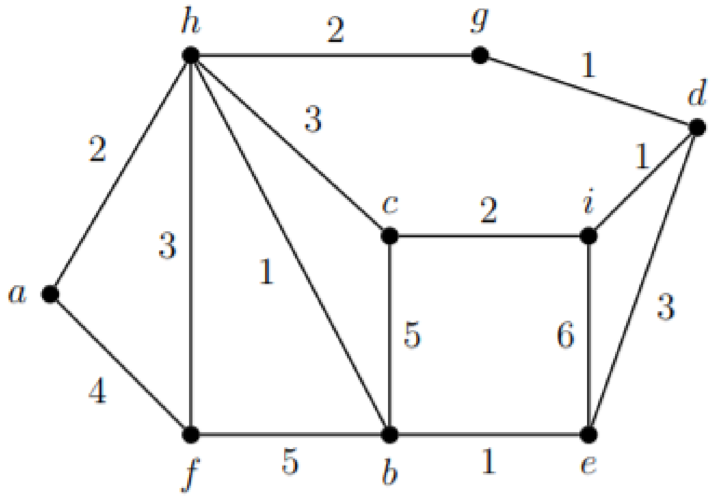
f) W_5

Solution:



Exercise 8.4 Kruskal's algorithm

Find a minimum-weight spanning tree via Kruskal's algorithm. List the edges chosen in order and sketch the tree.



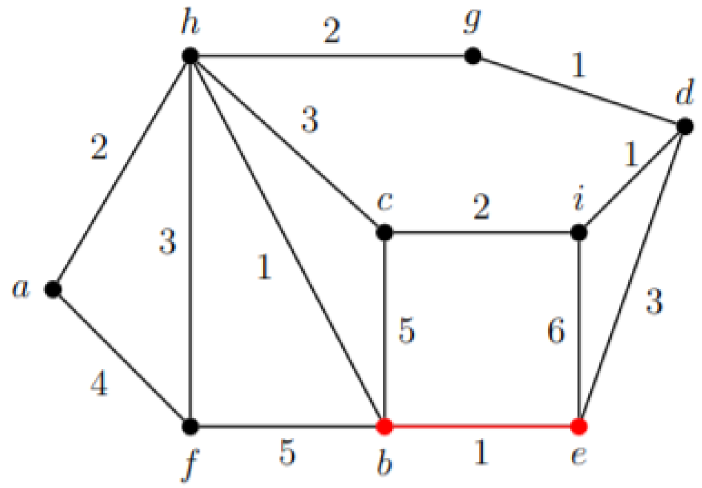
Solution:

For Kruskal's algorithm, we first list the edges in the increasing order of its weighted length, which is given by

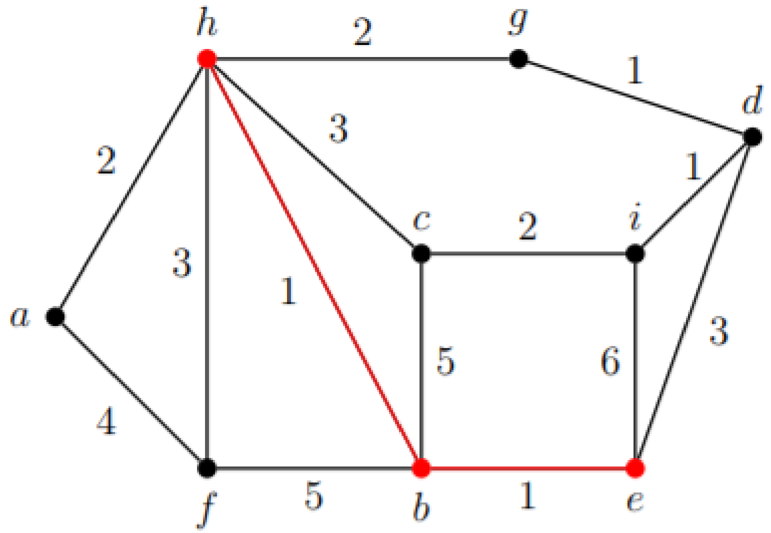
$$1, 1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 5, 5, 6$$

The ties don't matter for the result(or one can say for different order of tied edges will give a specific minimum-weight spanning tree). We randomly choose the following order of edge and list as $be, bh, id, dg, ah, hg, ci, hf, hc, ed, af, fb, bc, ie$.

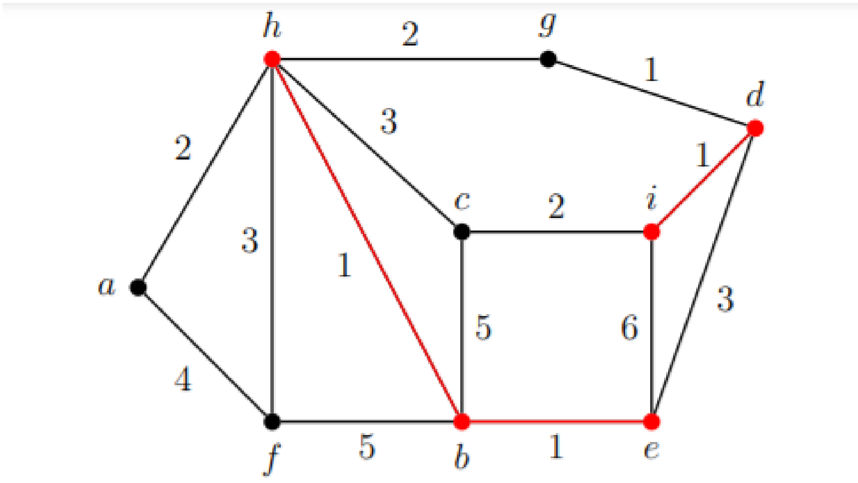
Now, we draw the following spanning tree by Kruskal's algorithm. First, we add *be* in our graph



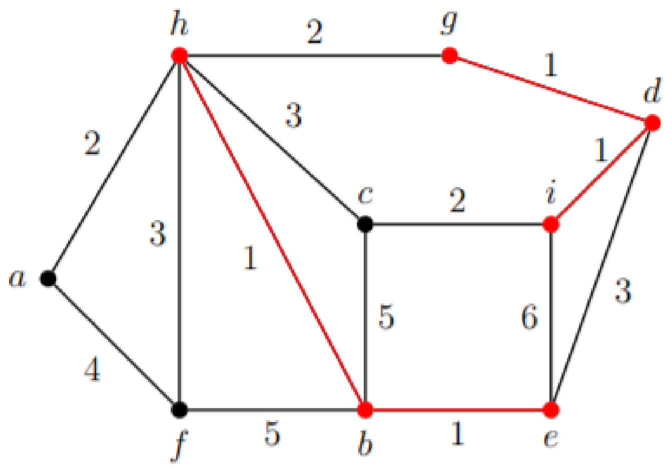
Next, we see that adding *bh* will not introduce a cycle, hence we add it in our graph



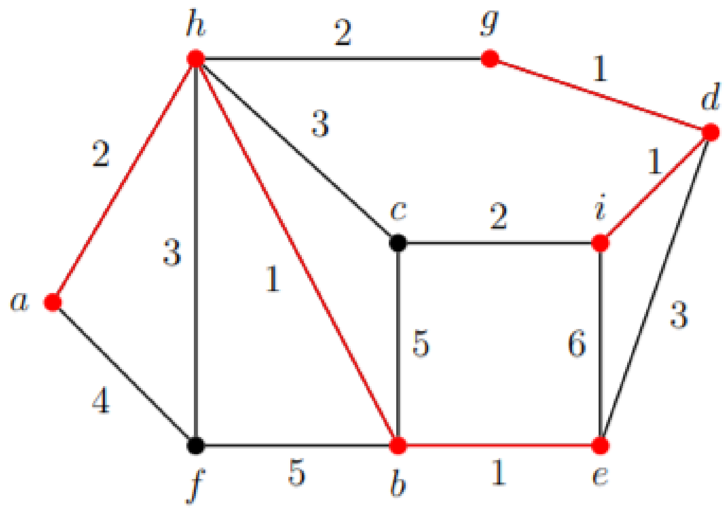
Next, we see that adding id will not introduce a cycle, hence we add it in our graph



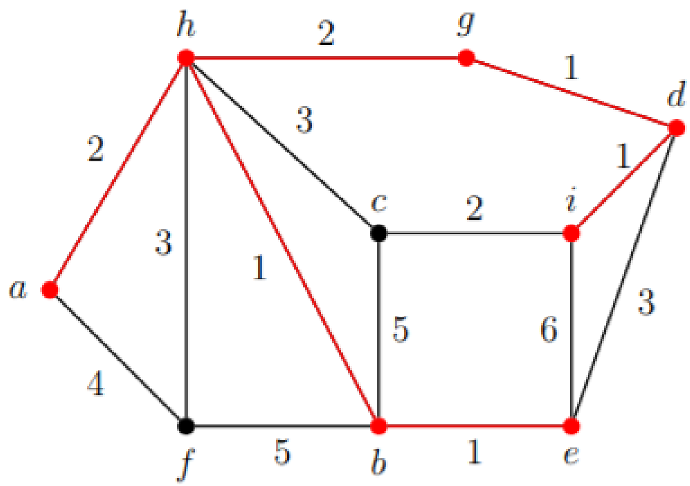
Next, we see that adding dg will not introduce a cycle, hence we add it in our graph



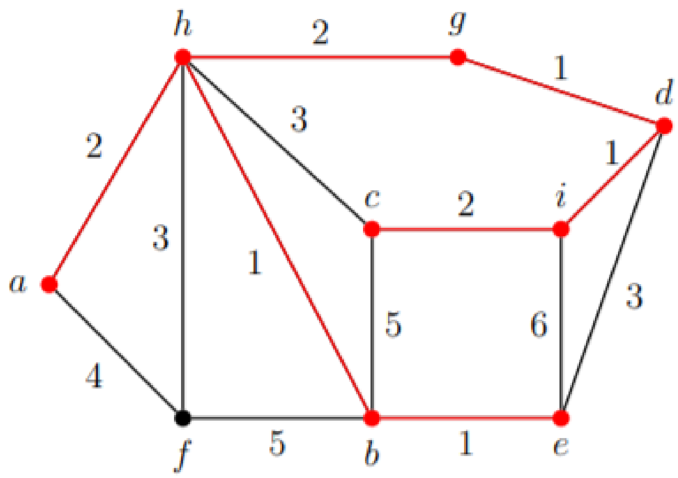
Next, we see that adding ah will not introduce a cycle, hence we add it in our graph



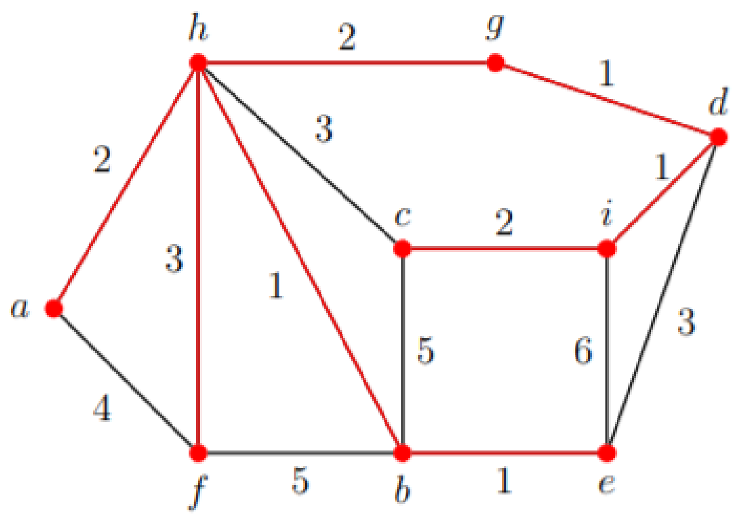
Next, we see that adding hg will not introduce a cycle, hence we add it in our graph



Next, we see that adding ci will not introduce a cycle, hence we add it in our graph



Next, we see that adding hf will not introduce a cycle, hence we add it in our graph

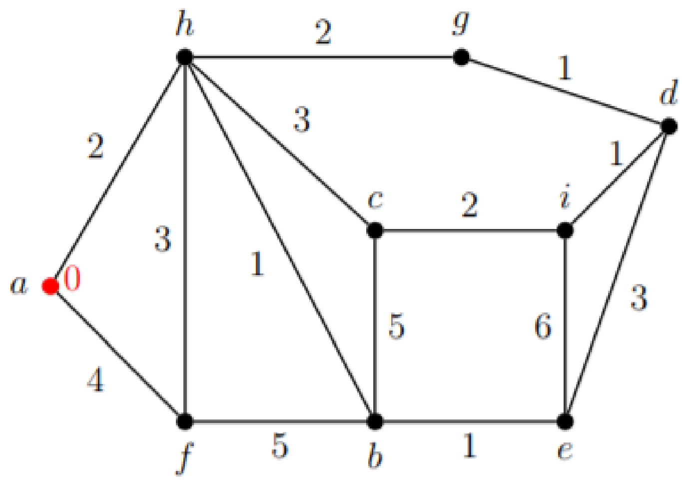


Next, we see that adding hc will introduce a cycle, hence we omit it. Also, adding ed, af, fb, bc or ie will all introduce a cycle, hence we terminate the algorithm, resulting a desired minimum-weight spanning tree.

Exercise 7.5 Dijkstra's algorithm

Given the root vertex a, find a shortest-path spanning tree via Dijkstra's algorithm. List the edges chosen in order, list the shortest path distance (from root vertex) to each vertex. Sketch the tree.

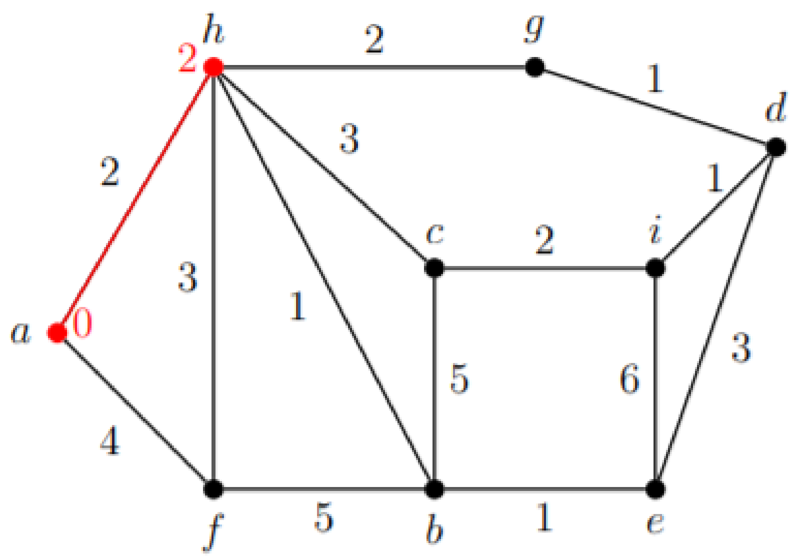
Solution:



We first find the neighbors of a and the temporary cost of them. The neighbors are h and f, with the temporary cost

$$\text{Cost}_{a\to h} = 2,$$
$$\text{Cost}_{a\to f} = 4.$$

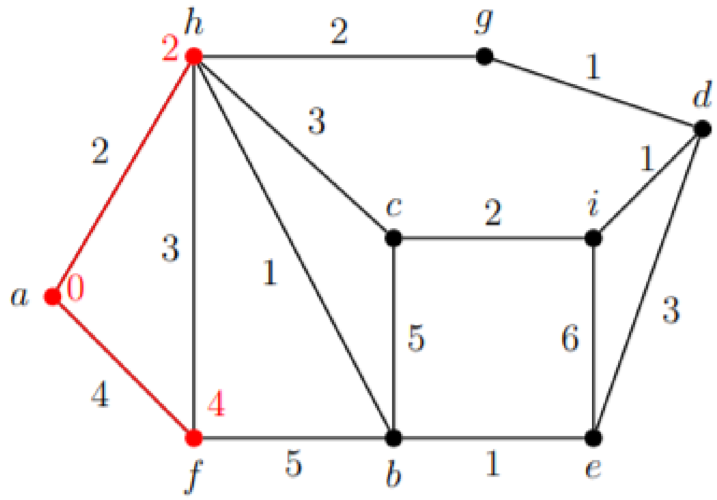
We choose the minimal costed edge, which is ah and add it in the graph where we specify the determined cost from the root a to the node beside it in red.



We first find the neighbors of a, h and the temporary cost of them. The neighbors are f, g and c, with the temporary cost

$$\begin{aligned}
&\text{Cost}_{\ a \to f} = 4, \\
&\text{Cost}_{\ h \to c} = 2 + 3 = 5, \\
&\text{Cost}_{\ h \to g} = 2 + 2 = 4.
\end{aligned}$$

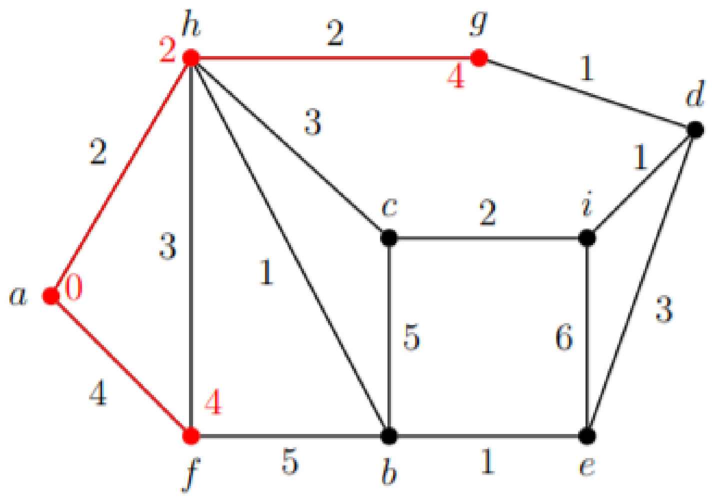
There is a tie, and we randomly choose one of which. In this case, we choose af to be added in the graph



Next, we consider the neighbors of h, f and the temporary cost of them. The neighbors are c, b and g, with the temporary cost

$$\begin{aligned}\text{Cost}_{h\to c} &= 2+3=5,\\ \text{Cost}_{h\to g} &= 2+2=4,\\ \text{Cost}_{f\to b} &= 4+5=9.\end{aligned}$$

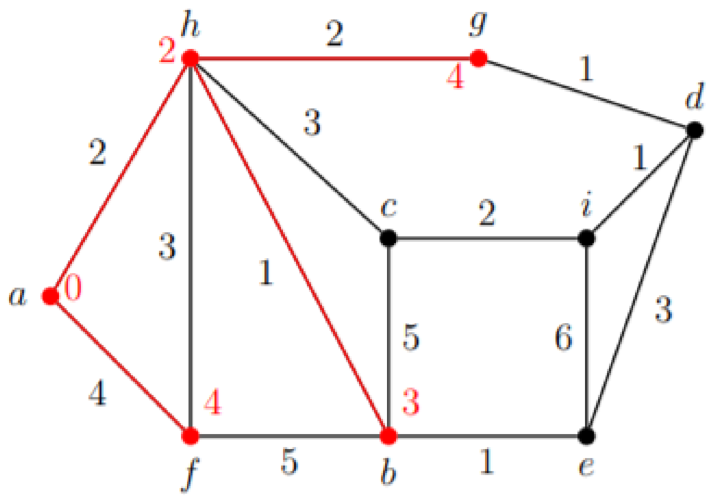
We choose the minimal costed edge, which is hg and add it in the graph



Next, we consider the neighbors of h, f and g and the temporary cost of them. The neighbors are b, c and d, with the temporary cost

$$\begin{aligned} \text{Cost}_{\ h\to b} &= 2+1=3, \\ \text{Cost}_{\ f\to b} &= 4+5=9, \\ \text{Cost}_{\ h\to c} &= 2+3=5, \\ \text{Cost}_{\ g\to d} &= 4+1=5. \end{aligned}$$

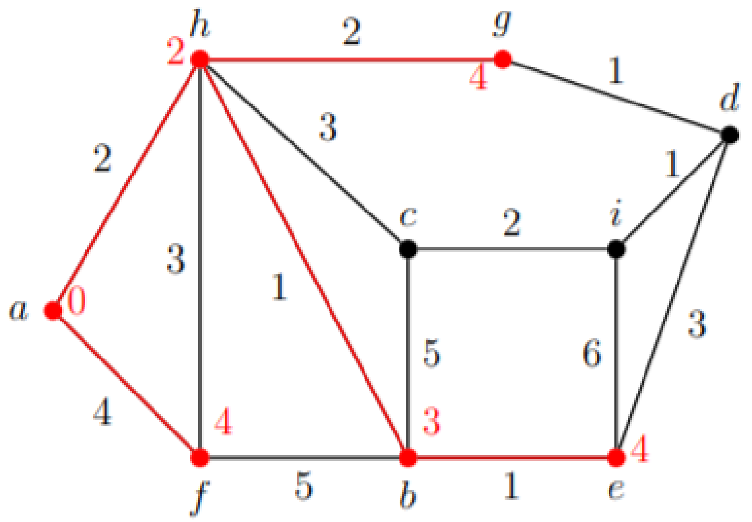
We choose the minimal costed edge, which is hb and add it in the graph



Next, we consider the neighbors of h, b and g and the temporary cost of them. The neighbors are c, e and d, with the temporary cost

$$\begin{aligned}
\text{Cost }_{h\to c} &= 2+3=5\\
\text{Cost }_{b\to c} &= 3+5=8\\
\text{Cost }_{b\to e} &= 3+1=4\\
\text{Cost }_{g\to d} &= 4+1=5.
\end{aligned}$$

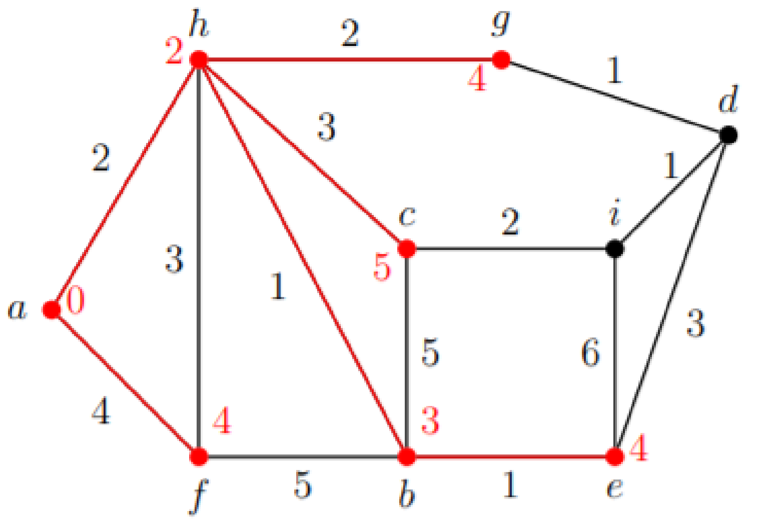
Notice that we do not need to consider f anymore, since any edges expanded from which will introduce a cycle. We choose the minimal costed edge, which is be and add it in the graph



Next, we consider the neighbors of h, b, e and g and the temporary cost of them. The neighbors are c, i and d, with the temporary cost

$$\begin{aligned}
\text{Cost }_{h\to c} &= 2+3=5\\
\text{Cost }_{b\to c} &= 3+5=8\\
\text{Cost }_{e\to i} &= 4+6=10\\
\text{Cost }_{g\to d} &= 4+1=5\\
\text{Cost }_{e\to d} &= 4+3=7
\end{aligned}$$

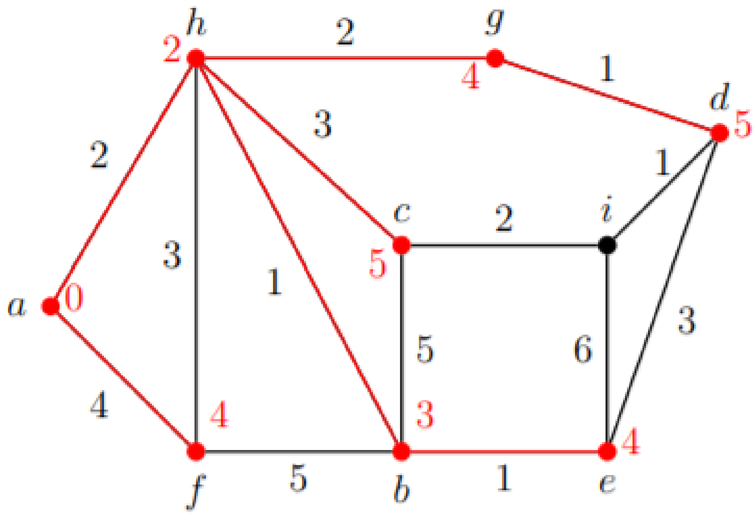
There is a tie, and we randomly choose one of which. In this case, we choose hc to be added in the graph



Next, we consider the neighbors of c, e and g and the temporary cost of them. The neighbors are i and d, with the temporary cost

$$\begin{aligned}
&\text{Cost }_{c\to i} = 5+2 = 7,\\
&\text{Cost }_{e\to i} = 4+6 = 10,\\
&\text{Cost }_{g\to d} = 4+1 = 5,\\
&\text{Cost }_{e\to d} = 4+3 = 7.
\end{aligned}$$

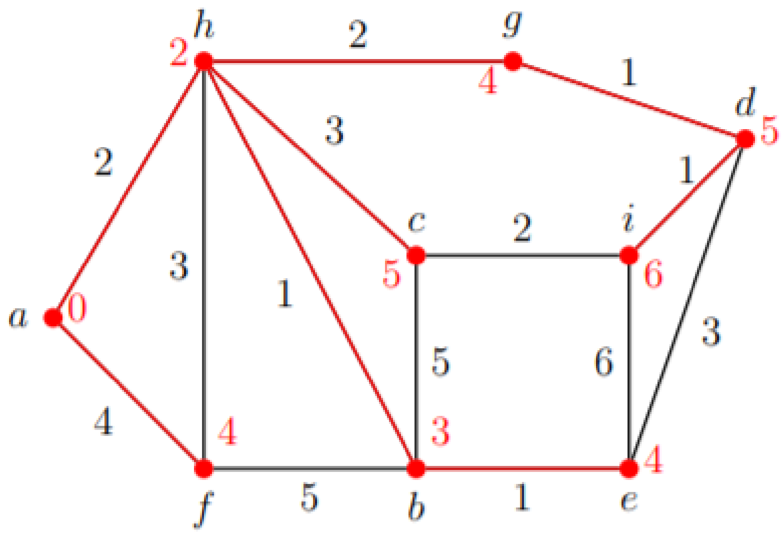
Notice that we do not need to consider b anymore, since any edges expanded from which will introduce a cycle. We choose the minimal costed edge, which is gd and add it in the graph



Finally, we consider the neighbors of c, e and d and the temporary cost of them. The only neighbor left is i, with the temporary cost

$$\begin{aligned} \text{Cost}_{c \to i} &= 5 + 2 = 7 \\ \text{Cost}_{d \to i} &= 5 + 1 = 6, \\ \text{Cost}_{e \to i} &= 4 + 3 = 7 \end{aligned}$$

We choose the minimal costed edge, which is di and add it in the graph



Follow the procedure, one can see the order to choose the vertex adding in the graph, and also the red value labeled beside each vertex indicates the shortest path distance from root vertex a to it.

Reference

1. Rosen, Kenneth H., and Kamala Krithivasan. Discrete mathematics and its applications: with combinatorics and graph theory. Tata McGraw-Hill Education, 2012.