VE203 Discrete Math RC6

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April 15, 2022

Linear Recurrence Equations

Consider the following linear homogeneous recurrence equation

$$p(A)f_n = (A - r_1)(A - r_2) \cdots (A - r_k) f_n = 0$$

with $r_1, r_2, \ldots, r_k$ distinct non-zero constants. The the general solutions is given by

$$f_n = c_1 r_1^n + c_2 r_2^n + \cdots + c_k r_k^n$$

with constants $c_1, c_2, \ldots, c_k$.

Linear Recurrence Equations

Let $k \geq 1$ and consider the recurrence equation

$$(A - r)^k f_n = 0$$

Then the general solution is given by

$$\begin{aligned} f_n &= c_1 r^n + c_2 n r^n + c_3 n^2 r^n + c_4 n^3 r^n + \cdots + c_k n^{k-1} r^n \\ &= \left(c_1 + c_2 n + c_3 n^2 + c_4 n^3 + \cdots + c_k n^{k-1} \right) r^n \end{aligned}$$

with constants $c_1, c_2, \ldots, c_k$.

Linear Recurrence Equations

Consider

$$(A-1)^5(A+1)^3(A-3)^2(A+8)(A-9)^4 f_n = 0$$

The general solution is given by

$$\begin{aligned} f_n = {} & c_1 + c_2 n + c_3 n^2 + c_4 n^3 + c_5 n^4 \\ & + \left(c_6 + c_7 n + c_8 n^2\right)(-1)^n \\ & + \left(c_9 + c_{10} n\right) 3^n \\ & + c_{11}(-8)^n \\ & + \left(c_{12} + c_{13} n + c_{14} n^2 + c_{15} n^3\right) 9^n \end{aligned}$$

with constants $c_1, c_2, \ldots, c_{15}$.

Exercise 10

Solve these recurrence relations together with the initial conditions given.

1.
$$a_n = 5a_{n-1} - 6a_{n-2} \text{ for } n \geq 2, a_0 = 1, a_1 = 0$$

2.
$$a_n = -4a_{n-1} - 4a_{n-2} \text{ for } n \geq 2, a_0 = 0, a_1 = 1$$

3. Find the solution to $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$ with $a_0 = 7, a_1 = -4$, and $a_2 = 8$.

4. Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 2n^2$.

Exercise 10 Solution

1. The characteristic equation is $r^2 - 5r + 6 = 0$, which factors as $(r-2)(r-3) = 0$, so the roots are $r = 2$ and $r = 3$. Therefore by Theorem 1 the general solution to the recurrence relation is $a_n = \alpha_1 2^n + \alpha_2 3^n$ for some constants α_1 and α_2. We plug in the initial condition to solve for the α 's. Since $a_0 = 1$ we have $1 = \alpha_1 + \alpha_2$, and since $a_1 = 0$ we have $0 = 2\alpha_1 + 3\alpha_2$. These linear equations are easily solved to yield $\alpha_1 = 3$ and $\alpha_2 = -2$. Therefore the solution is $a_n = 3 \cdot 2^n - 2 \cdot 3^n$.
2. This time the characteristic equation is $r^2 + 4r + 4 = 0$, which factors as $(r+2)^2 = 0$, so again there is only one root, $r = -2$, which occurs with multiplicity 2 . Therefore by Theorem 2 the general solution to the recurrence relation is $a_n = \alpha_1(-2)^n + \alpha_2 n(-2)^n$ for some constants α_1 and α_2. We plug in the initial conditions to solve for the α 's. Since $a_0 = 0$ we have $0 = \alpha_1$, and since $a_1 = 1$ we have $1 = -2\alpha_1 - 2\alpha_2$. These linear equations are easily solved to yield $\alpha_1 = 0$ and $\alpha_2 = -1/2$. Therefore the solution is $a_n = (-1/2)n(-2)^n = n(-2)^{n-1}$.

Exercise 10 Solution

3. This is a third degree recurrence relation. The characteristic equation is $r^3 - 2r^2 - 5r + 6 = 0$. By the rational root test, the possible rational roots are $\pm 1, \pm 2, \pm 3. \pm 6$. We find that $r = 1$ is a root. Dividing $r - 1$ into $r^3 - 2r^2 - 5r + 6$, we find that $r^3 - 2r^2 - 5r + 6 = (r-1)\left(r^2 - r - 6\right)$. By inspection we factor the rest, obtaining $r^3 - 2r^2 - 5r + 6 = (r-1)(r-3)(r+2)$. Hence the roots are 1,3 , and -2, so the general solution is $a_n = \alpha_1 1^n + \alpha_2 3^n + \alpha_3(-2)^n$, or more simply $a_n = \alpha_1 + \alpha_2 3^n + \alpha_3(-2)^n$. To find these coefficients, we plug in the initial conditions:

$$7 = a_0 = \alpha_1 + \alpha_2 + \alpha_3$$

$$-4 = a_1 = \alpha_1 + 3\alpha_2 - 2\alpha_3$$

$$8 = a_2 = \alpha_1 + 9\alpha_2 + 4\alpha_3$$

Solving this system of equations, we get $\alpha_1 = 5, \alpha_2 = -1$, and $\alpha_3 = 3$. Therefore the specific solution is $a_n = 5 - 3^n + 3(-2)^n$

Exercise 10 Solution

4. The associated homogeneous recurrence relation is $a_n = 2a_{n-1}$. We easily solve it to obtain $a_n^{(h)} = \alpha 2^n$. Next we need a particular solution to the given recurrence relation. By Theorem 6 we want to look for a function of the form $a_n = p_2 n^2 + p_1 n + p_0$. (Note that $s = 1$ here, and 1 is not a root of the characteristic polynomial.) We plug this into our recurrence relation and obtain $p_2 n^2 + p_1 n + p_0 = 2\left(p_2(n-1)^2 + p_1(n-1) + p_0\right) + 2n^2$. We rewrite this by grouping terms with equal powers of n, obtaining $(-p_2 - 2)\, n^2 + (4p_2 - p_1)\, n + (-2p_2 + \ 2p_1 - p_0) = 0$. In order for this equation to be true for all n, we must have $p_2 = -2, 4p_2 = p_1$, and $-2p_2 + 2p_1 - p_0 = 0$. This tells us that $p_1 = -8$ and $p_0 = -12$. Therefore the particular solution we seek is $a_n^{(p)} = -2n^2 - 8n - 12$. So the general solution is the sum of the homogeneous solution and this particular solution, namely $a_n = \alpha 2^n - 2n^2 - 8n - 12$.

Exercise 11

Find the general solution r_n to the the following recurrence equations.
$r_n = r_{n-1} + 3r_{n-2} + 2^{n+1} - n^2, n \geq 0.$

Exercise 11 Solution

From the following recurrence equation

$$r_n = r_{n-1} + 3r_{n-2} + 2^{n+1} - n^2, \quad n \geq 0,$$

with the shift operator, we have

$$A^2 r_n = A r_n + 3r_n + 2^{n+3} - (n+2)^2, \quad n \geq -2$$

where we have implicitly performed a change of variable $n \to n+2$. By the linearity of the operator, we have

$$\left(A^2 - A - 3\right) r_n = \left(A - \frac{1+\sqrt{13}}{2}\right)\left(A - \frac{1-\sqrt{13}}{2}\right) r_n = 2^{n+3} - (n+2)^2$$

We first solve the homogeneous equation, which is

$$\left(A - \frac{1+\sqrt{13}}{2}\right)\left(A - \frac{1-\sqrt{13}}{2}\right) r_n^{(h)} = 0, \quad n \geq -2,$$

Exercise 11 Solution

and by the theorem, we have

$$r_n^{(h)} = c_1\left(\frac{1+\sqrt{13}}{2}\right)^n + c_2\left(\frac{1-\sqrt{13}}{2}\right)^n, \quad c_i \in \mathbb{R}, i = 1, 2.$$

Then we consider the inhomogeneous equation, which is

$$\left(A - \frac{1+\sqrt{13}}{2}\right)\left(A - \frac{1-\sqrt{13}}{2}\right) r_n^{(p)} = 2^{n+3} - (n+2)^2, \quad n \geq -2.$$

Again, by linearity, we can further divide the question into two part, namely we can find the particular solution $r_n^{(p)}$ for this equation by first found

$$\left(A - \frac{1+\sqrt{13}}{2}\right)\left(A - \frac{1-\sqrt{13}}{2}\right) r_n^{(p_1)} = 2^{n+3}, \quad n \geq -2$$

Exercise 11 Solution

and then found

$$\left(A - \frac{1+\sqrt{13}}{2}\right)\left(A - \frac{1-\sqrt{13}}{2}\right) r_n^{(p_2)} = -(n+2)^2, \quad n \geq -2,$$

then we have

$$r^{(p)} = r^{(p_1)} + r^{(p_2)}.$$

For the first one, we can set up the usual ansatz as

$$r_n^{(p_1)} := an^2 + bn + c$$

and plug it into the equation, we get

$$\begin{aligned} A^2 - A - 3)\, r_n^{(p_1)} &= -(n+2)^2 = -n^2 - 4n - 4 \\ \Rightarrow n^2(-3a) + n(2a-3b) + (3a+b-3c) &= -n^2 - 4n - 4 \end{aligned}$$

Exercise 11 Solution

which implies

$$\begin{cases} a = \frac{1}{3} \\ b = \frac{14}{9} \\ c = \frac{59}{27} \end{cases}$$

Hence, we have

$$r_n^{(p_1)} = \frac{1}{3}n^2 + \frac{14}{9}n + \frac{59}{27}.$$

Exercise 11 Solution

For the second one, we can set up the usual ansatz as

$$r_n^{(p_2)} := d \cdot 2^n$$

and plug it into the equation, we get

$$\begin{aligned}
\left(A^2 - A - 3\right) r_n^{(p_2)} &= 2^{n+3} = 3 \cdot 2^n \\
\Rightarrow & c \cdot 2^{n+2} - d \cdot 2^{n+1} - 3d \cdot 2^n = 8 \cdot 2^n \\
\Rightarrow & 4d - 2d - 3d = 8 \\
\Rightarrow & d = -8.
\end{aligned}$$

Hence, we have

$$r_n^{(p_2)} = -8 \cdot 2^n.$$

Exercise 11 Solution

In all, we have the general solution r_n given by

$$r_n = r_n^{(h)} + r_n^{(p_1)} + r_n^{(p_2)}$$

$$= c_1 \left(\frac{1+\sqrt{13}}{2} \right)^n + c_2 \left(\frac{1-\sqrt{13}}{2} \right)^n + \frac{1}{3}n^2 + \frac{14}{9}n + \frac{59}{27} - 8 \cdot 2^n$$

where $c_i \in \mathbb{R}, i = 1, 2$ are arbitrary constants.

Asymptotic Notations

A way to compare "sizes" of functions

$$O \approx \leq$$

$$\Omega \approx \geq$$

$$\Theta \approx =$$

In addition,

$$o \approx <$$

$$\omega \approx >$$

O Notation

A function $g(n)$ is an asymptotic upper bound for $f(n)$, denoted by

$$f(n) = O(g(n))$$

if there exist positive constants c and n_0 such that

$$0 \leq f(n) \leq cg(n) \text{ for all } n \geq n_0$$

i.e.,

$$\limsup_{n\to\infty} \frac{f(n)}{g(n)} < \infty$$

Ω Notation

A function $g(n)$ is an asymptotic lower bound for $f(n)$, denoted by

$$f(n) = \Omega(g(n))$$

if there exist positive constants c and n_0 such that

$$0 \leq cg(n) \leq f(n) \text{ for all } n \geq n_0$$

i.e.,

$$\liminf_{n\to\infty} \frac{f(n)}{g(n)} > 0$$

Θ Notation

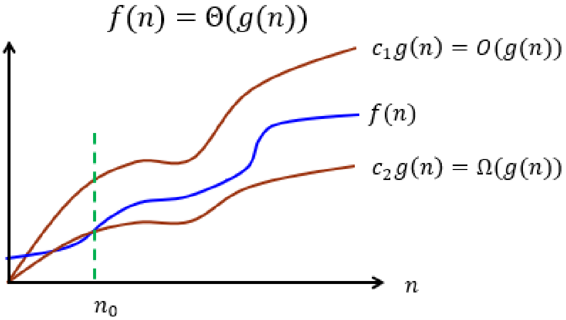
A function $g(n)$ is an asymptotic tight bound for $f(n)$, denoted by

$$f(n) = \Theta(g(n))$$

if there exist constants c_1, c_2, and n_0 such that

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \text{ for all } n \geq n_0$$

Θ Notation



1. Combinatorial mathematics
   - Linear Recurrence Equations
   - Asymptotic Notations
   - Master Method
2. Q&A

Master Method

Recurrence:

$$T(n) \leq aT\left(\frac{n}{b}\right) + O\left(n^d\right)$$

Claim: $$T(n) = \begin{cases} O\left(n^d \log n\right) & \text{if } a = b^d \\ O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^{\log_b a}\right) & \text{if } a > b^d \end{cases}$$

Exercise 12

Find the O or Θ bound of $T(n)$ for the following recurrence relation.

(i) $T(n) = 4T(n/4) + 5n$.
(ii) $T(n) = 4T(n/5) + 5n$.
(iii) $T(n) = 5T(n/4) + 4n$.
(iv) $T(n) = 4T(\sqrt{n}) + \log^5 n$
(v) $T(n) = 4T(\sqrt{n}) + \log^2 n$

Exercise 12 Solution

i) From the Master Theorem, we see that the recurrence relation is in the form of

$$T(n) = aT(n/b) + O\left(n^d\right)$$

for constants $a \geq 1, b > 1, d \geq 0$. Specifically, we see that

$$a = 4, \quad b = 4, \quad d = 1$$

since $5n \in O(n)$. Then, we find out

$$\log_b a = \log_4 4 = 1 = d,$$

hence from Master Theorem we conclude that

$$T(n) = O\left(n^d \log n\right) = O(n \log n).$$

Exercise 12 Solution

ii) From the Master Theorem, we see that the recurrence relation is in the form of

$$T(n) = aT(n/b) + O\left(n^d\right)$$

for constants $a \geq 1, b > 1, d \geq 0$. Specifically, we see that

$$a = 4, \quad b = 5, \quad d = 1$$

since $5n \in O(n)$. Then, we find out

$$\log_b a = \log_5 4 < 1 = d,$$

hence from Master Theorem we conclude that

$$T(n) = O\left(n^d\right) = O(n).$$

Exercise 12 Solution

iii) From the Master Theorem, we see that the recurrence relation is in the form of

$$T(n) = aT(n/b) + O\left(n^d\right)$$

for constants $a \geq 1, b > 1, d \geq 0$. Specifically, we see that

$$a = 5, \quad b = 4, \quad d = 1$$

since $4n \in O(n)$. Then, we find out

$$\log_b a = \log_4 5 > 1 = d,$$

hence from Master Theorem we conclude that

$$T(n) = O\left(n^{\log_b a}\right) = O\left(n^{\log_5 4}\right).$$

Exercise 12 Solution

iv)

$$\text{Let } n = 2^m \text{ and } S(m) = T\left(2^m\right)$$

$$\Rightarrow S(m) = 4S\left(\frac{m}{2}\right) + m^5$$

$$a = 4, b = 2, d = 5, \Rightarrow \log_b a = 2 < d$$

$$\Rightarrow S(m) = O\left(m^5\right)$$

$$\Rightarrow T(n) = S(\log m) = O\left((\log n)^5\right)$$

Exercise 12 Solution

v)

$$
\begin{aligned}
&\text{Let } n = 2^m \text{ and } S(m) = T\left(2^m\right) \\
&\Rightarrow S(m) = 4S\left(\frac{m}{2}\right) + m^2 \\
&a = 4, b = 2, d = 2, \Rightarrow \log_b a = 2 = d \\
&\Rightarrow S(m) = O\left(m^2 \log m\right) \\
&\Rightarrow T(n) = S(\log m) = O\left((\log n)^2 \log\log n\right)
\end{aligned}
$$

Q&A

Q&A