# Ve203 Discrete Mathematics (Spring 2022)

## Assignment 1

**Date Due: 21:00 PM, Tuesday, Mar. 01, 2022**
This assignment has a total of **(31 points)**.

**Exercise 1.1**

(i) `(1 point)` Let $a, b$ be statements. Write out the truth tables to prove *de Morgan's rules*:

$$\neg(a \wedge b) \Leftrightarrow \neg a \vee \neg b, \qquad \neg(a \vee b) \Leftrightarrow \neg a \wedge \neg b.$$

(ii) `(1 point)` Let $M$ be a set and $A, B \subset M$. Prove the following equalities by writing out the sets in terms of predicates and applying de Morgan's rules.

$$M - (A \cap B) = (M - A) \cup (M - B), \qquad M - (A \cup B) = (M - A) \cap (M - B).$$

**(2 points)**

**Exercise 1.2** Given $\varphi = (A \to (B \to C)) \to (B \to (A \to C))$,

(i) `(2 points)` Write the truth table for $\varphi$.

(ii) `(2 points)` Write $\varphi$ in disjunctive normal form.

(iii) `(2 points)` Write $\varphi$ in conjunctive normal form.

**(6 points)**

**Exercise 1.3** The following shows the truth table for all $2^{2^2} = 16$ different binary logical operators $\varphi_i$, $i = 0, \ldots, 15$.

| $p$ | $q$ | $\varphi_0$ | $\varphi_1$ | $\varphi_2$ | $\varphi_3$ | $\varphi_4$ | $\varphi_5$ | $\varphi_6$ | $\varphi_7$ | $\varphi_8$ | $\varphi_9$ | $\varphi_{10}$ | $\varphi_{11}$ | $\varphi_{12}$ | $\varphi_{13}$ | $\varphi_{14}$ | $\varphi_{15}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |

Using infix notation, for example, $\varphi_{13}$ can be represented as $\varphi_{13} = \to(p, q) = p \to q$.

A set $S$ of logical operators is called *functionally complete* if every compound proposition is logically equivalent to a compound proposition involving only these logical operators in $S$. In this exercise, in order to show $S$ is a functionally complete set, it suffices to verify that for all $i = 0, \ldots, 15$, $\varphi_i$ over logical variables $p$ and $q$ can be represented using only operators in $S$.

(i) `(1 point)` Show that $\{\wedge, \vee, \neg\}$ is functionally complete.

(ii) `(1 point)` Show that $\{\wedge, \neg\}$ is functionally complete.

(iii) `(1 point)` Show that $\{\vee, \neg\}$ is functionally complete.

(iv) `(1 point)` Show that $\{\vee, \wedge\}$ is *not* functionally complete.

**(4 points)**

**Exercise 1.4** In computer design, the logical operations NAND and NOR play an important role.[1] In logic, NAND is represented by the *Scheffer stroke* $|$ while NOR is represented by the *Peirce arrow* $\downarrow$. They are defined as

$$A \mid B :\!= \neg(A \wedge B), \qquad A \downarrow B :\!= \neg(A \vee B).$$

(i) `(1 point)` Give the truth tables for $A \mid B$ and $A \downarrow B$.

(ii) `(2 points)` Prove that $A \downarrow A \Leftrightarrow \neg A$ and $(A \downarrow B) \downarrow (A \downarrow B) \Leftrightarrow A \vee B$.

(iii) `(1 point)` Deduce that $\{\downarrow\}$ is functionally complete.

(iv) `(1 point)` Represent the exclusive or $\oplus$ solely through $\downarrow$.

[1] According to `https://en.wikipedia.org/wiki/Logical_NOR`, "The computer used in the spacecraft that first carried humans to the moon, the Apollo Guidance Computer, was constructed entirely using NOR gates with three inputs." A reference for this claim is is given in that article. See also `https://en.wikipedia.org/wiki/Flash_memory` for a discussion of NAND and NOR flash memory.

(v) `(1 point)` Prove that $\{|\}$ is functionally complete.

(vi) `(1 point)` Is the Scheffer stroke $|$ acting on logical statements is associative? That is, is it correct that $(A|B)|C \Leftrightarrow A \,|\, (B \,|\, C)$?

**(7 points)**

**Exercise 1.5** For any sets $A$ and $B$, show that

(i) `(1 point)` $2^A \cap 2^B = 2^{A \cap B}$.

(ii) `(1 point)` $(2^A \cup 2^B) \subset 2^{A \cup B}$.

**(2 points)**

**Exercise 1.6** Let $M$ be a set and let $X, Y, Z, W \subset M$. We define the *symmetric difference*:

$$X \triangle Y := (X - Y) \cup (Y - X)$$

(i) `(1 point)` Prove that $X \triangle Y = (X \cup Y) - (X \cap Y)$.

(ii) `(1 point)` Prove that $(M - X) \triangle (M - Y) = X \triangle Y$.

(iii) `(1 point)` Show that the symmetric difference is associative, i.e., $(X \triangle Y) \triangle Z = X \triangle (Y \triangle Z)$.

(iv) `(1 point)` Prove that $X \cap (Y \triangle Z) = (X \cap Y) \triangle (X \cap Z)$.

(v) `(1 point)` Show that $X \triangle Y = Z \triangle W$ iff $X \triangle Z = Y \triangle W$.

(vi) `(1 point)` Indicate the region of $X \triangle Y \triangle Z$ in a Venn diagram.

**(6 points)**

**Exercise 1.7** Let $X$ be a finite set, define the distance/metric $\varrho(A, B)$ of two sets $A, B \in 2^X$ by

$$\varrho(A, B) := |A \triangle B|.$$

Show that $(2^X, \varrho)$ is a *metric space* by verifying that for all $A, B, C \in 2^X$,

(i) `(1 point)` $\varrho(A, B) = 0$ iff $A = B$;

(ii) `(1 point)` $\varrho(A, B) = \varrho(B, A)$;

(iii) `(2 points)` $\varrho(A, C) \leq \varrho(A, B) + \varrho(B, C)$.

**(4 points)**