VE203 Discrete Math

Spring 2022 — Worksheet 1 Solutions

February 13, 2022

Exercise 1.1 Set

How many elements does each of these sets have where a and b are distinct elements?

a) $\mathcal{P}(\{a, b, \{a, b\}\})$

b) $\mathcal{P}(\{\emptyset, a, \{a\}, \{\{a\}\}\})$

c) $\mathcal{P}(\mathcal{P}(\emptyset))$

Solution:

a) Since the set we are working with has 3 elements, the power set has $2^3 = 8$ elements.

b) Since the set we are working with has 4 elements, the power set has $2^4 = 16$ elements.

c) The power set of the empty set has $2^0 = 1$ element. The power set of this set therefore has $2^1 = 2$ elements.

In particular, it is $\{\varnothing, \{\varnothing\}\}$.

Exercise 1.2 Set

Let $A = \{a, b, c\}$, $B = \{x, y\}$, and $C = \{0, 1\}$. Find

a) $A \times B \times C$.

b) $C \times B \times A$.

c) $C \times A \times B$.

d) $B \times B \times B$.

Solution:

In each case the answer is a set of 3-tuples.

a) $\{(a, x, 0), (a, x, 1), (a, y, 0), (a, y, 1), (b, x, 0), (b, x, 1), (b, y, 0), (b, y, 1),$
$(c, x, 0), (c, x, 1), (c, y, 0), (c, y, 1)\}$

b) $\{(0, x, a), (0, x, b), (0, x, c), (0, y, a), (0, y, b), (0, y, c), (1, x, a), (1, x, b),$
$(1, x, c), (1, y, a), (1, y, b), (1, y, c)\}$

c) $\{(0, a, x), (0, a, y), (0, b, x), (0, b, y), (0, c, x), (0, c, y), (1, a, x), (1, a, y),$
$(1, b, x), (1, b, y), (1, c, x), (1, c, y)\}$

d) $\{(x, x, x), (x, x, y), (x, y, x), (x, y, y), (y, x, x), (y, x, y), (y, y, x), (y, y, y)\}$

Exercise 1.3 Set

Let A, B, and C be sets. Show that

a) $(A - B) - C \subseteq A - C$.

b) $(A - C) \cap (C - B) = \emptyset$.

Solution:

a) Suppose that $x \in (A - B) - C$. Then x is in $A - B$ but not in C. Since $x \in A - B$, we know that $x \in A$ (we also know that $x \notin B$, but that won't be used here). Since we have established that $x \in A$ but $x \notin C$, we have proved that $x \in A - C$.

b) To show that the set given on the left-hand side is empty, it suffices to assume that x is some element in that set and derive a contradiction, thereby showing that no such x exists. So suppose that $x \in (A - C) \cap (C - B)$. Then $x \in A - C$ and $x \in C - B$. The first of these statements implies by definition that $x \notin C$, while the second implies that $x \in C$. This is impossible, so our proof by contradiction is complete.

Exercise 1.4 Set
Show that if A is an infinite set, then whenever B is a set, $A \cup B$ is also an infinite set.

Solution:
We give a proof by contradiction. If $A \cup B$ is finite, then it has n elements for some natural number n. But A already has more than n elements, because it is infinite, and $A \cup B$ has all the elements that A has, so $A \cup B$ has more than n elements. This contradiction shows that $A \cup B$ must be infinite.

Exercise 1.5 Logic
Determine whether each of these conditional statements is true or false.
a) If $1 + 1 = 3$, then unicorns exist.
b) If $1 + 1 = 3$, then dogs can fly.
c) If $1 + 1 = 2$, then dogs can fly.
d) If $2 + 2 = 4$, then $1 + 2 = 3$.

Solution:
(a) The conditional statement, if $1 + 1 = 3$ then unicorns exists - is true.
(b) The conditional statement, if $1 + 1 = 3$ then dogs can fly is true.
(c) The conditional statement, if $1 + 1 = 2$ then dogs can fly is false.
(d) The conditional statement, if $2 + 2 = 4$ then $1 + 2 = 3$ is true.

Exercise 1.6 Logic
Is the assertion ”This statement is false” a proposition?

Solution:
Consider the assertion ”This statement is false”

Definition: A proposition is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

In view of this definition, consider the assertion again.

”This statement is false”

Observe that there is no statement preceding this. So, the statement may be true or false which cannot be confirmed by this statement.

If at all there is a statement, then it can be checked for its validity and confirmed whether it is true or false. But, in the present case, there is no statement which has a confirmation like ”this statement is false”.

So, even though this statement looks like a declarative sentence, it is not declaring the truth or falacity of a particular text. Therefore, this is not a proposition.

Exercise 1.7 Logic

1. Find the negation of $\forall x \forall y \exists z\ A(x, y, z) \Rightarrow B(x, y, z)$
2. Show that $(\exists x(P(x) \Rightarrow Q(x))) \Leftrightarrow ((\forall x P(x)) \Rightarrow (\exists x Q(x)))$ is a tautology.

Solution:

1. $\neg(\forall x \forall y \exists z\ A(x,y,z) \Rightarrow B(x,y,z))$
$\equiv \exists x \neg(\forall y \exists z\ A(x,y,z) \Rightarrow B(x,y,z))$
$\equiv \exists x \exists y \neg(\exists z\ A(x,y,z) \Rightarrow B(x,y,z))$
$\equiv \exists x \exists y \forall z(\neg(\neg A(x,y,z) \vee B(x,y,z)))$
$\equiv \exists x \exists y \forall z(A(x,y,z) \wedge \neg B(x,y,z))$

2. $\exists x(P(x) \Rightarrow Q(x))$
$\Leftrightarrow \exists x(\neg P(x) \vee Q(x))$
$\Leftrightarrow (\exists x \neg P(x)) \vee (\exists x Q(x))$
$\Leftrightarrow \neg(\forall x P(x)) \vee (\exists x Q(x))$
$\Leftrightarrow (\forall x P(x)) \Rightarrow (\exists x Q(x))$

Exercise 1.8 Induction Prove that for every positive integer n,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > 2(\sqrt{n+1} - 1).$$

Solution:
This exercise involves some messy algebra, but the logic is the usual logic for proofs using the principle of mathematical induction. The basis step ($n = 1$) is true, since 1 is greater than $2(\sqrt{2} - 1) \approx 0.83$. We assume that

$$1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > 2(\sqrt{n+1} - 1)$$

and try to derive the corresponding statement for $n + 1$:

$$1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} > 2(\sqrt{n+2} - 1)$$

Since by the inductive hypothesis we know that

$$1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} > 2(\sqrt{n+1} - 1) + \frac{1}{\sqrt{n+1}}$$

we will be finished if we can show that the inequality

$$2(\sqrt{n+1} - 1) + \frac{1}{\sqrt{n+1}} > 2(\sqrt{n+2} - 1)$$

holds. By canceling the -2 from both sides and rearranging, we obtain the equivalent inequality

$$2(\sqrt{n+2} - \sqrt{n+1}) < \frac{1}{\sqrt{n+1}}$$

which in turn is equivalent to

$$2(\sqrt{n+2} - \sqrt{n+1})(\sqrt{n+2} + \sqrt{n+1}) < \frac{\sqrt{n+1}}{\sqrt{n+1}} + \frac{\sqrt{n+2}}{\sqrt{n+1}}.$$

This last inequality simplifies to

$$2 < 1 + \frac{\sqrt{n+2}}{\sqrt{n+1}}$$

which is clearly true. Therefore the original inequality is true, and our proof is complete.

Reference

1. Rosen, Kenneth H., and Kamala Krithivasan. Discrete mathematics and its applications: with combinatorics and graph theory. Tata McGraw-Hill Education, 2012.

2. Chengjun Peng, Worksheet 1 for VE203