VE203 Discrete Math
Spring 2022 — Worksheet 9 Solutions

April 16, 2022

Exercise 9.1 Twelvefold Way

Consider the functions $f : B \to U$, count the number of functions and fill in the blanks below. Express the results in binomial coefficients, factorials, or powers (AVOID double bracket notation for first two rows).

Domain	Codomain	Any	Injective	Surjective
distinguishable	distinguishable			
indistinguishable	distinguishable			
distinguishable	indistinguishable			
indistinguishable	indistinguishable			

where
(i) $B = \{1, 2, 3\}$ and $U = \{1, 2, 3, 4, 5\}$.
(ii) $B = \{1, 2, 3, 4, 5\}$ and $U = \{1, 2, 3\}$.
Solution:
i) For

$$f : B \to U, \quad B = \{1, 2, 3\}, U = \{1, 2, 3, 4, 5\}$$

we have

Domain	Codomin	Any	Injectve	surjective
distinguishable	distinguishable	5^3	$\begin{pmatrix} 5 \\ 3 \end{pmatrix} \cdot 3!$	0
indistinguishable	distinguishable	$\begin{pmatrix} 7 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 5 \\ 3 \end{pmatrix}$	0
distinguishable	indistinguishable	5	1	0
indistinguishable	indistinguishable	3	1	0

ii) For

$$f : B \to U, \quad B = \{1,2,3,4,5\}, U = \{1,2,3\}$$

we have

Domain	Codomain	Any	Injective	Surjective
distinguishable	distinguishable	3^5	0	$30 \times 3!$
indistinguishable	distinguishable	$\left(\begin{array}{c}7\\5\end{array}\right)$	0	$\left(\begin{array}{c}4\\2\end{array}\right)$
distinguishable	indistinguishable	41	0	25
indistinguishable	indistinguishable	5	0	2

Exercise 9.2 Multichoosing

Consider

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \leq 100$$

What are the number of integer solutions if

(i) $x_i > 0$ and $=$ holds;
(ii) $x_i \geq 0$ and $=$ holds;
(iii) $x_i > 0$ and $<$ holds;
(iv) $x_i \geq 0$ and $<$ holds;
(v) $x_i \geq 0$.

AVOID double bracket notation in the final solution.

Solution:

i) If $x_i > 0$ and the equality holds. Namely, we have

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 100$$

with $x_i > 0$, or equivalently, $x_i \geq 1$. Then we can divide 100 into

$$\underbrace{1 + 1 + \cdots + 1}_{100\text{terms}} = 100$$

Since we have 7 variables, then the number of integer solutions is given by

$$\left(\begin{array}{c}99\\6\end{array}\right)$$

which stands for choosing 6 of $+$ sign out of $99+$ sign to form a $7-$ partition of 100 .

ii) If $x_i \geq 0$ and the equality holds. Namely, we have

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 100$$

with $x_i \geq 0$, or equivalently, we have

$$x_1' + x_2' + x_3' + x_4' + x_5' + x_6' + x_7' = 107$$

with $x_i' > 0$, or equivalently, $x_i' \geq 1$ since $x_i' := x_i + 1$. Then by the same argument from (i), we know that the number of integer solutions is given by

$$\left(\begin{array}{c}106\\6\end{array}\right)$$

iii) If $x_i > 0$ and the equality does not hold. We introduce another variable $x_8 > 0$ such that

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 100$$

which is equivalent to the original one. Then by the same argument from (i), we know that the number of integer solutions is given by

$$\binom{99}{7}$$

iv) If $x_i \geq 0$ and the equality does not hold. We introduce another variable $x_8 > 0$ such that

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 100,$$

which is equivalent to the original one. By the same transformation of variables as in (ii), we have

$$x_1' + x_2' + x_3' + x_4' + x_5' + x_6' + x_7' + x_8 = 107,$$

then now all the variables are strictly greater than 0 . By the same argument as in (i), we know that the number of integer solution is given by

$$\binom{106}{7}$$

v) If $x_i \geq 0$. Then the number of integer solution is given by the results in (ii) and (iv) combined. Since we know that the solution sets in the case of

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 100$$

and the case of

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 < 100$$

are mutually disjoint, hence the number of solution can be combined, namely

$$\binom{106}{6} + \binom{106}{7} = \binom{107}{7}$$

by the recursive identity for binomial coefficient.

Exercise 9.3 Inclusion-Exclusion Principle

Find the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 = 17$, where $x_i, i = 1, 2, 3, 4$, are nonnegative integers such that $x_1 \leq 3, x_2 \leq 4, x_3 \leq 5$, and $x_4 \leq 8$.

Solution:

$C(4+17-1,17) - C(4+13-1,13) - C(4+12-1,12) - C(4+11-1,11) - C(4+8-1,8) + C(4+8-1,8) + C(4+7-1,7) + C(4+4-1,4) + C(4+6-1,6) + C(4+3-1,3) + C(4+2-1,2) - C(4+2-1,2) = 20$

Exercise 9.4 Derangement

What is the probability that none of 10 people receives the correct hat if a hatcheck person hands their hats back randomly?

Solution:

$$\frac{D_{10}}{10!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{1}{10!} = \frac{1334961}{3628800} = \frac{16481}{44800} \approx 0.3678794643,$$

which is almost exactly $e^{-1} \approx 0.3678794412\ldots$

Exercise 9.5 Derangement

A machine that inserts letters into envelopes goes haywire and inserts letters randomly into envelopes. What is the probability that in a group of 100 letters

a) no letter is put into the correct envelope?

b) exactly one letter is put into the correct envelope?

c) exactly 98 letters are put into the correct envelopes?

d) exactly 99 letters are put into the correct envelopes?

e) all letters are put into the correct envelopes?

Solution:

a) An arrangement in which no letter is put into the correct envelope is a derangement. There are by definition D_{100} derangements. Since there are $P(100,100) = 100\,!$ equally likely permutations altogether, the probability of a derangement is $D_{100}/100\,!$. Numerically, this is almost exactly equal to $1/e$, which is about 0.368.

b) We need to count the number of ways to put exactly one letter into the correct envelope. First, there are $C(100,1) = 100$ ways to choose the letter that is to be correctly stuffed. Then there are D_{99} ways to insert the remaining 99 letters so that none of them go into their correct envelopes. By the product rule, there are $100D_{99}$ such arrangements. As in part (a) the denoninator is $P(100,100) = 100\,!$. Therefore the answer is $100D_{99}/100! = D_{99}/99!$. Again this is almost exactly $1/e \approx 0.368$.

c) This time, to count the number of ways that exactly 98 letters can be put into their correct envelopes, we need simply to choose the two letters that are to be misplaced, since there is only one way to misplace them. There are of course $C(100,2) = 4950$ ways to do this. As in part (a) the denominator is $P(100,100) = 100\,!$. Therefore the answer is $4950/100\,!$. This is substantially less than 10^{-100}, so for all practical purposes, the answer is 0 .

d) There is no way that exactly 99 letters can be inserted into their correct envelopes, since as soon as 99 letters have been correctly inserted, there is only one envelope left for the remaining letter, and it is the correct one. Therefore the answer is exactly 0 . (The probability of an event that cannot happen is 0 .)

e) Only one of the $100\,!$ permutations is the correct stuffing, so the answer is $1/100!$. As in part (c) this is 0 for all practical purposes.

Exercise 9.6 Master Method

Find the O or Θ bound of $T(n)$ for the following recurrence relation.

(i) $T(n) = 4T(n/4) + 5n$.

(ii) $T(n) = 4T(n/5) + 5n$.

(iii) $T(n) = 5T(n/4) + 4n$.

(iv) $T(n) = 4T(\sqrt{n}) + \log^5 n$

(v) $T(n) = 4T(\sqrt{n}) + \log^2 n$

Solution:

i) From the Master Theorem, we see that the recurrence relation is in the form of

$$T(n) = aT(n/b) + O\left(n^d\right)$$

for constants $a \geq 1, b > 1, d \geq 0$. Specifically, we see that

$$a = 4, \quad b = 4, \quad d = 1$$

since $5n \in O(n)$. Then, we find out

$$\log_b a = \log_4 4 = 1 = d,$$

hence from Master Theorem we conclude that

$$T(n) = O\left(n^d \log n\right) = O(n \log n).$$

ii) From the Master Theorem, we see that the recurrence relation is in the form of

$$T(n) = aT(n/b) + O\left(n^d\right)$$

for constants $a \geq 1, b > 1, d \geq 0$. Specifically, we see that

$$a = 4, \quad b = 5, \quad d = 1$$

since $5n \in O(n)$. Then, we find out

$$\log_b a = \log_5 4 < 1 = d,$$

hence from Master Theorem we conclude that

$$T(n) = O\left(n^d\right) = O(n).$$

iii) From the Master Theorem, we see that the recurrence relation is in the form of

$$T(n) = aT(n/b) + O\left(n^d\right)$$

for constants $a \geq 1, b > 1, d \geq 0$. Specifically, we see that

$$a = 5, \quad b = 4, \quad d = 1$$

since $4n \in O(n)$. Then, we find out

$$\log_b a = \log_4 5 > 1 = d,$$

hence from Master Theorem we conclude that

$$T(n) = O\left(n^{\log_b a}\right) = O\left(n^{\log_5 4}\right).$$

iv)

$$\begin{aligned}
&\text{Let } n = 2^m \text{ and } S(m) = T\left(2^m\right) \\
&\Rightarrow S(m) = 4S\left(\frac{m}{2}\right) + m^5 \\
&a = 4, b = 2, d = 5, \Rightarrow \log_b a = 2 < d \\
&\Rightarrow S(m) = O\left(m^5\right) \\
&\Rightarrow T(n) = S(\log m) = O\left((\log n)^5\right)
\end{aligned}$$

v)

$$\begin{aligned}
&\text{Let } n = 2^m \text{ and } S(m) = T\left(2^m\right) \\
&\Rightarrow S(m) = 4S\left(\frac{m}{2}\right) + m^2 \\
&a = 4, b = 2, d = 2, \Rightarrow \log_b a = 2 = d \\
&\Rightarrow S(m) = O\left(m^2 \log m\right) \\
&\Rightarrow T(n) = S(\log m) = O\left((\log n)^2 \log \log n\right)
\end{aligned}$$

Reference

1. Rosen, Kenneth H., and Kamala Krithivasan. Discrete mathematics and its applications: with combinatorics and graph theory. Tata McGraw-Hill Education, 2012.