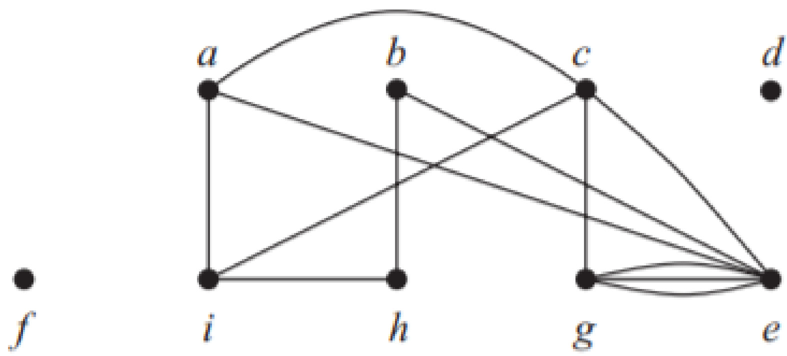
VE203 Discrete Math
Spring 2022 — Worksheet 7 Solutions

April 16, 2022

Exercise 7.1 Graph Definition

Find the number of vertices, the number of edges, and the degree of each vertex in the given undirected graph. Identify all isolated and pendant vertices.



Solution:

There are 9 vertices here, and 12 edges. The degree of each vertex is the number of edges incident to it. Thus $\deg(a) = 3, \deg(b) = 2, \deg(c) = 4, \deg(d) = 0$ (and hence d is isolated), $\deg(e) = 6, \deg(f) = 0$ (and hence f is isolated), $\deg(g) = 4, \deg(h) = 2$, and $\deg(i) = 3$. Note that the sum of the degrees is $3 + 2 + 4 + 0 + 6 + 0 + 4 + 2 + 3 = 24$, which is twice the number of edges.

Exercise 7.2 Graph Definition

How many vertices and how many edges do these graphs have?

a) K_n

b) C_n

c) W_n

d) $K_{m,n}$

e) Q_n

Solution:

a) Obviously K_n has n vertices. It has $C(n, 2) = n(n-1)/2$ edges, since each unordered pair of distinct vertices is an edge.

b) Obviously C_n has n vertices. Just as obviously it has n edges.

c) The wheel W_n is the same as C_n with an extra vertex and n extra edges incident to that vertex. Therefore it has $n + 1$ vertices and $n + n = 2n$ edges.

d) By definition $K_{m,n}$ has $m + n$ vertices. Since it has one edge for each choice of a vertex in the one part and a vertex in the other part, it has mn edges.

e) Since the vertices of Q_n are the bit strings of length n, there are 2^n vertices. Each vertex has degree n, since there are n strings that differ from any given string in exactly one bit (any one of the n different bits can be changed). Thus the sum of the degrees is $n2^n$. Since this must equal twice the number of edges (by the handshaking theorem), we know that there are $n2^n/2 = n2^{n-1}$ edges.

Exercise 7.3 K-regular Graph

A simple graph is called regular if every vertex of this graph has the same degree. A regular graph is called n-regular if every vertex in this graph has degree n.

For which values of n are these graphs regular?

a) K_n
b) C_n
c) W_n
d) Q_n
e) For which values of m and n is $K_{m,n}$ regular?

Solution:

a) The complete graph K_n is regular for all values of $n \geq 1$, since the degree of each vertex is $n - 1$.

b) The degree of each vertex of C_n is 2 for all n for which C_n is defined, namely $n \geq 3$, so C_n is regular for all these values of n.

c) The degree of the middle vertex of the wheel W_n is n, and the degree of the vertices on the "rim" is 3 . Therefore W_n is regular if and only if $n = 3$. Of course W_3 is the same as K_4.

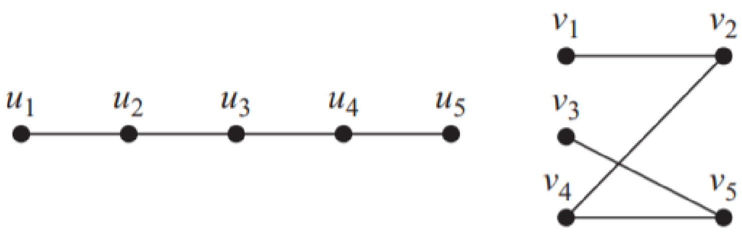
d) The cube Q_n is regular for all values of $n \geq 0$, since the degree of each vertex in Q_n is n. (Note that Q_0 is the graph with 1 vertex.)

e) Since the vertices in one part have degree m, and vertices in the other part have degree n, we conclude that $K_{m,n}$ is regular if and only if $m = n$.

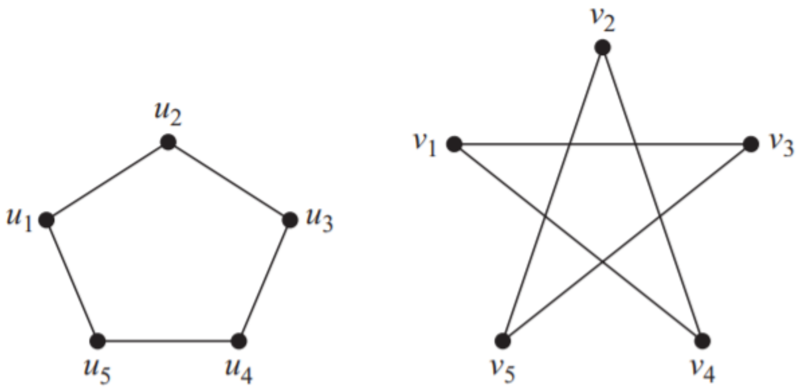
Exercise 7.4 Isomorphism

Determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.

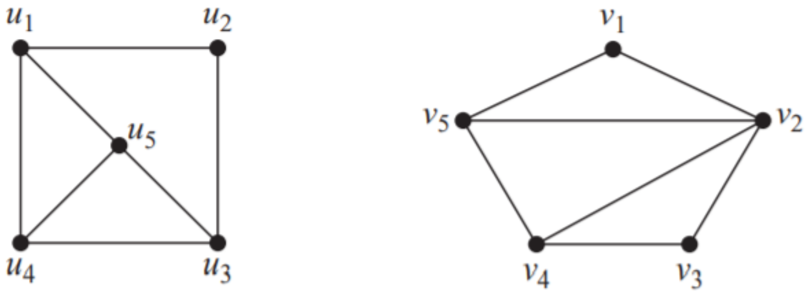
a)



b)



c)



Solution:

a) These graphs are isomorphic, since each is a path with five vertices. One isomorphism is $f(u_1) = v_1, f(u_2) = v_2, f(u_3) = v_4, f(u_4) = v_5$, and $f(u_5) = v_3$.

b) These graphs are isomorphic, since each is the 5-cycle. One isomorphism is $f(u_1) = v_1, f(u_2) = v_3, f(u_3) = v_5, f(u_4) = v_2$, and $f(u_5) = v_4$.

c) These graphs are not isomorphic. The second has a vertex of degree 4, whereas the first does not.

Exercise 7.5 Handshaking Theorem

Show that the sum, over the set of people at a party, of the number of people a person has shaken hands with, is even. Assume that no one shakes his or her own hand.

Solution:

Model this problem by letting the vertices of a graph be the people at the party, with an edge between two people if they shake hands. Then the degree of each vertex is the number of people the person that vertex represents shakes hands with. By Theorem 1 the sum of the degrees is even (it is $2e$).

Exercise 7.6 Hall's Theorem

Suppose that there are five young women and six young men on an island. Each woman is willing to marry some of the men on the island and each man is willing to marry any woman who is willing to marry him. Suppose that Anna is willing to marry Jason, Larry, and Matt; Barbara is willing to marry Kevin and Larry; Carol is willing to marry Jason, Nick, and Oscar; Diane is willing to marry Jason, Larry, Nick, and Oscar; and Elizabeth is willing to marry Jason and Matt.

a) Model the possible marriages on the island using a bipartite graph.

b) Find a matching of the young women and the young men on the island such that each young woman is matched with a young man whom she is willing to marry.

c) Is the matching you found in part (b) a complete matching? Is it a maximum matching?

Solution:

a) The partite sets are the set of women ({Anna, Barbara, Carol, Diane, Elizabeth }) and the set of men ({Jason, Kevin, Larry, Matt, Nick, Oscar }). We will use first letters for convenience. The given information tells us to have edges AJ, AL, AM, BK, BL, CJ, CN, CO, DJ, DL, DN, DO, EJ, and EM in our graph. We do not put an edge between a woman and a man she is not willing to marry.

b) By trial and error we easily find a matching (it's not unique), such as AL, BK, CJ, DN, and EM.

c) This is a complete matching from the women to the men (as well as from the men to the women). A complete matching is always a maximum matching.

Exercise 7.7 Subgraph

1. How many subgraphs with at least one vertex does K_2 have?
2. How many subgraphs with at least one vertex does K_3 have?
3. How many subgraphs with at least one vertex does W_3 have?

Solution:

1. We list the subgraphs: the subgraph consisting of K_2 itself, the subgraph consisting of two vertices and no edges, and two subgraphs with 1 vertex each. Therefore the answer is 4 .

2. We will count the subgraphs in terms of the number of vertices they contain. There are clearly just 3 subgraphs consisting of just one vertex. If a subgraph is to have two vertices, then there are $C(3, 2) = 3$ ways to choose the vertices, and then 2 ways in each case to decide whether or not to include the edge joining them. This gives us $3 \cdot 2 = 6$ subgraphs with two vertices. If a subgraph is to have all three vertices, then there are $2^3 = 8$ ways to decide whether or not to include each of the edges. Thus our answer is $3 + 6 + 8 = 17$.

3. We need to count this in an organized manner. First note that W_3 is the same as K_4, and it will be easier if we think of it as K_4. We will count the subgraphs in terms of the number of vertices they contain. There are clearly just 4 subgraphs consisting of just one vertex. If a subgraph is to have two vertices, then there are $C(4, 2) = 6$ ways to choose the vertices, and then 2 ways in each case to decide whether or not to include the edge joining them. This gives us $6 \cdot 2 = 12$ subgraphs with two vertices. If a subgraph is to have three vertices, then there are $C(4, 3) = 4$ ways to choose the vertices, and then $2^3 = 8$ ways in each case to decide whether or not to include each of the edges joining pairs of them. This gives us $4 \cdot 8 = 32$ subgraphs with three vertices. Finally, there are the subgraphs containing all four vertices. Here there are $2^6 = 64$ ways to decide which edges to include. Thus our answer is $4 + 12 + 32 + 64 = 112$.

Exercise 7.8 Bipartition

Show that if G is a bipartite simple graph with v vertices and e edges, then $e \leq v^2/4$.

Solution:

Suppose the parts are of sizes k and $v - k$. Then the maximum number of edges the graph may have is $k(v - k)$ (an edge between each pair of vertices in different parts). By algebra or calculus, we know that the function $f(k) = k(v-k)$ achieves its maximum when $k = v/2$, giving $f(k) = v^2/4$. Thus there are at most $v^2/4$ edges.

Reference

1. Rosen, Kenneth H., and Kamala Krithivasan. Discrete mathematics and its applications: with combinatorics and graph theory. Tata McGraw-Hill Education, 2012.