VE203 Discrete Math
Spring 2022 — Worksheet 5

March 25, 2022

Exercise 5.1 Group

In Exercises 1 through 6, determine whether the binary operation $*$ gives a group structure on the given set. If no group results, give the first axiom in the order $\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3$ from Definition that does not hold.

1. Let $*$ be defined on $\mathbb{Z}$ by letting $a * b = ab$.
2. Let $*$ be defined on $2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\}$ by letting $a * b = a + b$.
3. Let $*$ be defined on $\mathbb{R}^+$by letting $a * b = \sqrt{ab}$.
4. Let $*$ be defined on $\mathbb{Q}$ by letting $a * b = ab$.
5. Let $*$ be defined on the set $\mathbb{R}^*$ of nonzero real numbers by letting $a * b = a/b$.
6. Let $*$ be defined on $\mathbb{C}$ by letting $a * b = |ab|$.

Exercise 5.2 Group

Let S be the set of all real numbers except -1. Define $*$ on S by

$$a * b = a + b + ab.$$

a. Show that $*$ gives a binary operation on S.
b. Show that $\langle S, * \rangle$ is a group.
c. Find the solution of the equation $2 * x * 3 = 7$ in S.

Exercise 5.3 Subgroup

In Exercises 1 through 6, determine whether the given subset of the complex numbers is a subgroup of the group $\mathbb{C}$ of complex numbers under addition.

1. $\mathbb{R}$
2. $\mathbb{Q}^+$
3. $7\mathbb{Z}$
4. The set $i\mathbb{R}$ of pure imaginary numbers including 0
5. The set $\pi\mathbb{Q}$ of rational multiples of π
6. The set $\{\pi^n \mid n \in \mathbb{Z}\}$

Exercise 5.4 Cyclic Group

In Exercises 1 through 5, find all orders of subgroups of the given group.

1. $\mathbb{Z}_6$
2. $\mathbb{Z}_8$
3. $\mathbb{Z}_{12}$
4. $\mathbb{Z}_{20}$
5. $\mathbb{Z}_{17}$

Exercise 5.5 Permutation Group

In Exercises 1 through 5 , compute the indicated product involving the following permutations in S_6 : $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}$, $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$, $\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 4 & 3 & 1 & 6 \end{pmatrix}$

1. $\tau\sigma$
2. $\tau^2\sigma$
3. $\mu\sigma^2$
4. $\sigma^{-2}\tau$
5. $\sigma^{-1}\tau\sigma$

Exercise 5.6 Permutation Group

In Exercises 1 through 4, compute the expressions shown for the permutations σ, τ and μ defined prior to Exercise 5.5 .

1. $|\langle\sigma\rangle|$
2. $|\langle\tau^2\rangle|$
3. σ^{100}
4. μ^{100}

Exercise 5.7 Homomorphism

Determine whether the given map ϕ is a homomorphism.

1. Let $\phi : \mathbb{Z} \to \mathbb{R}$ under addition be given by $\phi(n) = n$.
2. Let $\phi : \mathbb{R} \to \mathbb{Z}$ under addition be given by $\phi(x) =$ the greatest integer $\leq x$.
3. Let $\phi : \mathbb{R}^* \to \mathbb{R}^*$ under multiplication be given by $\phi(x) = |x|$.

Exercise 5.8 Coset and Lagrange Theorem

1. Find all cosets of the subgroup $4\mathbb{Z}$ of $2\mathbb{Z}$.
2. Find all cosets of the subgroup $\langle 2 \rangle$ of $\mathbb{Z}_{12}$.
3. Find all cosets of the subgroup $\langle 4 \rangle$ of $\mathbb{Z}_{12}$.
4. Find all cosets of the subgroup $\langle 18 \rangle$ of $\mathbb{Z}_{36}$.
5. Find the index of $\langle 3 \rangle$ in the group $\mathbb{Z}_{24}$.
6. Let $\sigma = (1, 2, 5, 4)(2, 3)$ in S_5. Find the index of $\langle \sigma \rangle$ in S_5.
7. Let $\mu = (1, 2, 4, 5)(3, 6)$ in S_6. Find the index of $\langle \mu \rangle$ in S_6.

Reference

1. Rosen, Kenneth H., and Kamala Krithivasan. Discrete mathematics and its applications: with combinatorics and graph theory. Tata McGraw-Hill Education, 2012.

2. Fraleigh, John B. A first course in abstract algebra. Pearson Education India, 2003.