上 海 交 通 大 学 试 卷

2021 – 2022 Academic Year (Fall Term)

Ve203 Discrete Mathematics
First Midterm Exam

Exercise 1 (20 points)
In the following questions, write down the corresponding labels (ABCD) of the true statements in the table below, and **fill the answer card** (with your name and student number). In each case, it is possible that none of the statements are true or that more than one statement is true.

(i)	(ii)	(iii)	(iv)	(v)
AB	BC	CD	AB	A

(i) Given a sequence $a = (0, 8, 4, 12, 2, 10, 6, 14, 1, 9, 5, 13, 3, 11, 7, 15)$. Let

$$S := \{s \mid s \text{ is a longest increasing subsequence of } a\}$$
$$T := \{t \mid t \text{ is a longest decreasing subsequence of } a\}$$

Which of the following statements are true?

A. If $s \in S$, then s contains 15.
B. If $t \in T$, then length of t is at most 5.
C. $|S| \cdot |T| \geq 16$.
D. If $s \in S \cup T$, then s contains 6.

(ii) Given a relation R on a non-empty finite set A, then
A. $R \cup R^{-1}$ is an equivalence relation.
B. If R is transitive and asymmetric, then R is irreflexive.
C. R is total iff $R \cup R^{-1} = A^2$.
D. $R^{\circ k}$ is transitive for $k > |A|$, where $R^{\circ k} := R \circ R \circ \cdots \circ R$ is the composition of R with itself k times.

(iii) Given relations R and T on a non-empty set A, $R \circ T$ is a function, then
A. R is a function.
B. T is a function.
C. If $R \circ T$ is surjective, then R is a surjective function.
D. If $R \circ T$ is injective, then T is an injective function.

(iv) Given logical variables p and q, which of the following are tautologies?
A. $p \vee \neg p$
B. $((p \to q) \wedge \neg q) \to \neg p$
C. $p \vee q \to p$
D. $p \to (p \wedge q)$

(v) Which of the following statements are correct?
A. The set $\mathbb{N} \times \mathbb{Q}$ is countable.
B. The set $\mathbb{N}^{\mathbb{Q}}$ is countable.
C. If there exists an injective function $f : \mathbb{N} \to S$, then S is countable.
D. If there exists a surjective function $f : S \to \mathbb{N}$, then S is countable.

Exercise 2 (10 points)
Let M be a set and let $X, Y, Z \subset M$. We define the *symmetric difference* as

$$X \triangle Y := (X - Y) \cup (Y - X)$$

(Note that $X^c := M - X$ represents the usual set complement.)

(i) (5 points) Show that $X \triangle Y = X^c \triangle Y^c$.

Solution: We know that for two sets A and B, $A = B$ iff $A \triangle B = \varnothing$. Now note that the symmetric difference is associative and commutative, then

$$(X \triangle Y) \triangle (X^c \triangle Y^c) = (X \triangle X^c) \triangle (Y \triangle Y^c) = M \triangle M = \varnothing \tag{1}$$

hence result follows.

(ii) (5 points) Show that $X \triangle Y \triangle Z = (X^c \triangle Y^c \triangle Z^c)^c$.

Solution: Again, consider the following symmetric difference

$$\begin{aligned}(X \triangle Y \triangle Z) \triangle (X^c \triangle Y^c \triangle Z^c)^c &= (X \triangle Y \triangle Z)^c \triangle (X^c \triangle Y^c \triangle Z^c) && (2)\\ &= ((X \triangle Y \triangle Z)^c \triangle X^c) \triangle (Y^c \triangle Z^c) && (3)\\ &= X \triangle Y \triangle Z \triangle X \triangle Y \triangle Z && (4)\\ &= \varnothing && (5)\end{aligned}$$

and the result follows.

Exercise 3 (15 points)
Given the logical proposition $\varphi = p \to (q \wedge r)$

(i) (5 points) Write the truth table for φ.

Solution:

p	q	r	$p \to (q \wedge r)$
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

(ii) (5 points) Express φ in disjunctive normal form (i.e., sum of products) φ_{dnf}.

Solution: By the truth table, we can write

$$\varphi_{\text{dnf}} = (\neg p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge q \wedge r)$$

(iii) (5 points) Express φ in conjunctive normal form (i.e., product of sums) φ_{cnf}.

Solution: Based on the truth table, let

$$\neg\varphi_{\text{cnf}} = (p \wedge q \wedge \neg r) \wedge (p \wedge \neg q \wedge r) \wedge (p \wedge \neg q \wedge \neg r)$$

then

$$\begin{aligned}\varphi_{\text{cnf}} &= \neg[(p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r)] \\ &= \neg(p \wedge q \wedge \neg r) \wedge \neg(p \wedge \neg q \wedge r) \wedge \neg(p \wedge \neg q \wedge \neg r) \\ &= (\neg p \vee \neg q \vee r) \wedge (\neg p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r)\end{aligned}$$

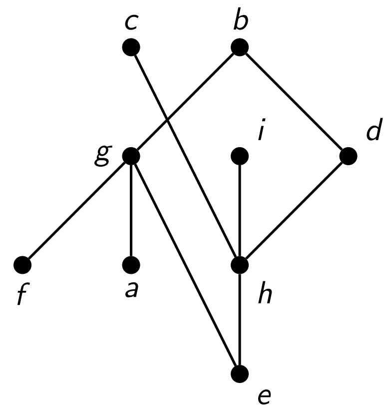
Exercise 4 (10 points)
Given an uncountably infinite set A, show that A is not equinumerous to its power set 2^A.

Solution: Consider $f : A \to 2^A$, and define $B \coloneqq \{x \in A \mid x \notin f(x)\} \in 2^A$.
Suppose that f is bijective, then f is onto, so there exists $b \in A$ such that $f(b) = B$, yet

- If $b \in B$, then by definition $b \notin f(b) = B$.
- If $b \notin B$, then by definition $b \in f(b) = B$.

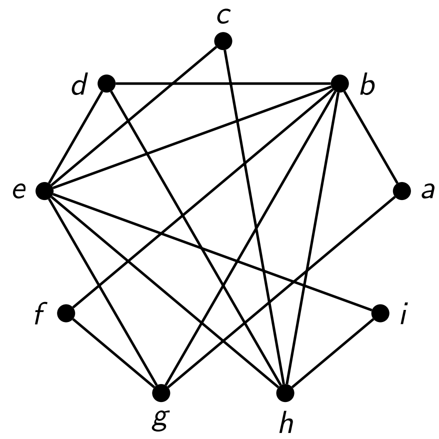
Contradiction.

Exercise 5 (15 points)
Given a poset with the Hasse diagram below

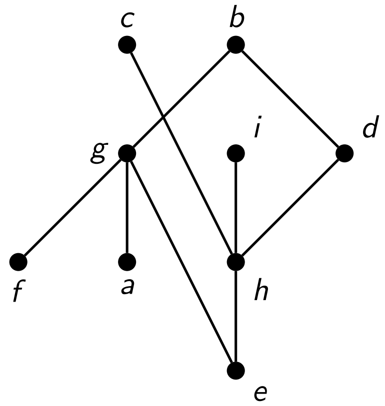


(i) (3 points) Sketch the comparability graph of the poset.

Solution:

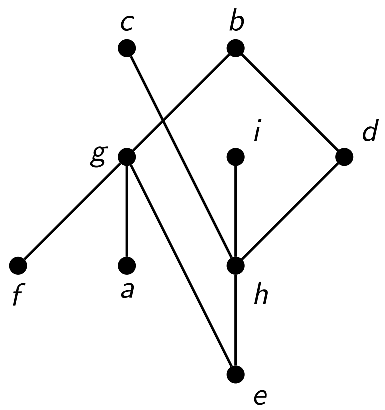


(ii) (4 points) Find an antichain of maximum size. Write down the set explicitly as well as indicate it on the following diagram.



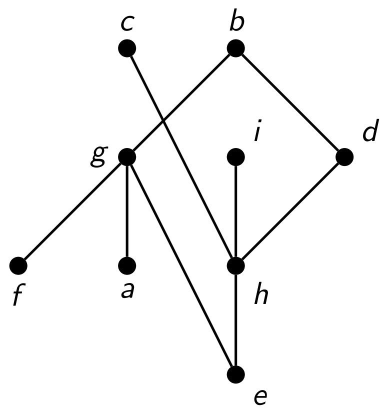
Solution: e.g., $\{f, a, i, d, c\}$

(iii) (4 points) Find a chain partition of minimum size. Write down the partition explicitly as well as indicate it on the following diagram.



Solution: e.g., $\{\{f, g, b\}, \{a\}, \{e, h, i\}, \{d\}, \{c\}\}$

(iv) (4 points) Find an antichain partition of minimum size. Write down the partition explicitly as well as indicate it on the following diagram.



Solution: e.g., $\{\{a, e, f\}, \{g, h\}, \{c, d, i\}, \{b\}\}$

Exercise 6 (30 points)

(i) (10 points) Given a poset $(P, \leq)$. Use induction to show that every finite non-empty set $Q \subset P$ admits a minimal element with respect to $\leq$.

Solution: `(Note:` $\not<$ `is not equivalent to` $\geq$`.)` `Also the following version of the proof deliberately avoids the symbol` $<$ `(i.e.,` $\leq$ `but not` $=$`), and use proof by contradiction to better exploit the properties of poset.`

Induction on the size of the poset Q. Since $Q \neq \varnothing$, we start with the base case $|P| = 1$.
Base case. Let $|Q| = 1$, then $Q \subset P$ contains only one element, say, x. Obviously x is a minimal element, since $\nexists y \in Q$ such that $y \leq x$ and $y \neq x$ (or $\forall y \in Q\ y \not< x$).
Inductive case. Assume the IH that all subset of P with size n admits a minimal element. Let $Q \subset P$ with $|Q| = n+1$, and $q \in Q$ an arbitrary element. Let $Q' = Q \backslash \{q\}$, then $|Q'| = n$. By IH Q' admits a minimal element $q' \in Q'$.

- If $q \leq q'$, we show that q is a minimal element of Q. Indeed, suppose not, then $\exists x \in Q$ such that $x \leq q$ and $x \neq q$, hence $x \in Q'$. By transitivity, $x \leq q$ and $q \leq q'$ implies $x \leq q'$. But q' is minimal in Q', hence $q' \leq x$, then by antisymmetry, $x = q'$, hence $q' = x \leq q$. Again by antisymmetry, $q = q'$, contradicting the fact that $q \notin Q'$.

- If $q \not\leq q'$, we show that q' is a minimal element of Q. Indeed, suppose not, then $\exists x \in Q$ such that $x \leq q'$ and $x \neq q'$. Since q' is minimal in Q', hence $x \notin Q'$, therefore $x = q$. But this implies $q = x \leq q'$, contradicting $q \not\leq q'$.

(ii) (10 points) Given a poset $(P, \leq)$ on a finite set P. Use induction to show that there exists a total order T on P such that $\leq \subset T$.

Solution: Induction on the size of P.
Base case. Let $|P| = 0$, i.e., $P = \varnothing$, thus $\leq = \varnothing$, the statement is vacuously true.

Inductive case. Assume that IH that for every partial order on a set of size n, there exists a total order as its superset. Now consider $(P, \leq)$ with $|P| = n+1$, then there exists a minimal element $m \in P$. Let $Q = P \setminus \{m\}$ and $\leq_Q = \leq \cap (Q \times Q)$, we claim that $(Q, \leq_Q)$ is a partial order. Indeed, we verify that $\leq_Q$ is (For this step, no verification is needed on recognizing $(Q, \leq_Q)$ is an **(induced) subposet**)

- reflexive. Let $x \in Q$. Since $\leq$ is reflexive, $(x,x) \in \leq$, hence $(x,x) \in \leq \cap (Q \times Q) = \leq_Q$.
- transitive. Suppose $(x,y) \in \leq_Q$, $(y,z) \in \leq_Q$, $x,y,z \in Q$. Since $\leq$ is transitive. thus $(x,z) \in \leq$, therefore $(x,z) \in \leq \cap (Q \times Q) = \leq_Q$
- antisymmetric. Suppose $(x,y) \in \leq_Q$ and $(y,x) \in \leq_Q$, then $(x,y) \in \leq$ and $(y,x) \in \leq$. Since $\leq$ is antisymmetric, thus $x = y$.

By IH, there exists a total order T_Q on Q such that $\leq_Q \subset T_Q$. Let $T = T_Q \cup (\{m\} \times P)$, then we can show that T is a total order on P and $\leq \subset T$. Indeed, to see that T is a total order, we verify that T is

- reflexive. Let $x \in P$,
  - If $x = m$, then $(x,x) = (m,m) \in \{m\} \times P \subset T$.
  - If $x \neq m$, then $x \in Q$. Since $\leq_Q$ is reflexive, $(x,x) \in \leq_Q \subset T_Q \subset T$.
- transitive. Suppose $(x,y) \in T$ and $(y,z) \in T$, $x,y,z \in P$.
  - If $x = m$, then $(x,z) = (m,z) \in \{m\} \times P \subset T$.
  - If $x \neq m$, then $(x,y) \notin \{m\} \times P$, but $(x,y) \in T = T_Q \cup (\{m\} \times P)$, hence $(x,y) \in T_Q$. Note that $T_Q \subset Q \times Q$, hence $y \neq m$, similarly $(y,z) \in T_Q$. Since T_Q is transitive, then $(x,z) \in T_Q \subset T$.
- antisymmetric. Suppose $(x,y) \in T$ and $(y,x) \in T$,
  - If at least one of x,y coincides with m, w.l.o.g., say $x = m$, then $(x,y) \notin T_Q$, hence $(y,x) \in \{m\} \times P$, so $y = x$.
  - If $x \neq m$ and $y \neq m$, then similarly as in verifying transitivity, $(x,y) \in T_Q$ and $(y,x) \in T_Q$. By antisymmetry of T_Q, it follows that $x = y$.
- total. Suppose $x,y \in P$.
  - If $x = m$, then $(x,y) \in \{m\} \times P \subset T$.
  - If $y = m$, then $(y,x) \in \{m\} \times P \subset T$
  - If $x \neq m$ and $y \neq m$, then $x,y \in Q$. Since T_Q is a total order, then either $(x,y) \in T_Q \subset T$, or $(y,x) \in T_Q \subset T$.

To see that $\leq \subset T$, suppose $(x,y) \in \leq$.

- If $x = m$, then $(x,y) \in \{m\} \times P \subset T$.
- If $x \neq m$, then also $y \neq m$ (otherwise $(x,y) \in \leq$, i.e., $x \leq m$, and $x \neq m$, contradicting that m is minimal in P). Thus $(x,y) \in \leq \cap (Q \times Q) = \leq_Q \subset T_Q \subset T$.

(iii) Given a poset $(P, \leq)$, the dimension of of $(P, \leq)$ is given by

$$\dim(P, \leq) := \min\left\{ n \in \mathbb{N} \;\middle|\; \begin{array}{c} \leq \, = \cap_{k=1}^{n} T_k, \\ (P, T_k) \text{ is a total order } \forall k = 1, \ldots, n \end{array} \right\}$$

(a) (5 points) What is the dimension of a chain of size 4? Explain.

Solution: The dimension is 1. A chain is a total order.

(b) (5 points) What is the dimension of an antichain of size 4? Explain.

Solution: The dimension is 2. Consider the union of a chain and its dual.