Chapter 7 Time-Varying Fields and Maxwell's Equations

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7-1 Introduction

- Electrostatics:

$$\nabla \times \mathbf{E} = 0,$$
$$\nabla \cdot \mathbf{D} = \rho.$$

For linear and isotropic media $\mathbf{D} = \epsilon \mathbf{E}.$

- Magnetostatics:

$$\nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{H} = \mathbf{J}.$$

For linear and isotropic media $\mathbf{H} = \frac{1}{\mu} \mathbf{B}.$

TABLE 7–1
Fundamental Relations for Electrostatic and Magnetostatic Models

Fundamental Relations	Electrostatic Model	Magnetostatic Model
Governing equations	$\nabla \times \mathbf{E} = 0$ $\nabla \cdot \mathbf{D} = \rho$	$\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{H} = \mathbf{J}$
Constitutive relations (linear and isotropic media)	$\mathbf{D} = \epsilon \mathbf{E}$	$\mathbf{H} = \frac{1}{\mu}\mathbf{B}$

Static Case

- (**E** and **D**) and (**B** and **H**) form separate and independent pairs.
- Electromagnetostatic field:
  - In a conducting medium, static **E** → static **J** → static **B**.
  - Static electric and static magnetic fields both exist.
  - **B** is a consequence, not affecting **E**

Time-varying Case

- (**E** and **D**) and (**B** and **H**) are related.
- A changing magnetic field gives rise to an electric field, and vice versa.
- Table 7-1 must be modified.

7-2 Faraday's Law of Electromagnetic Induction

- Faraday's law: the quantitative relationship between the induced emf and the rate of change of flux linkage, based on experimental observation (emf = $-d\Phi/dt$).
- Fundamental postulate for electromagnetic induction:

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.}$$

  - Applies whether it be in free space or in a material medium
  - The electric field intensity in a region of time-varying magnetic flux density is therefore nonconservative and cannot be expressed as the gradient of a scalar potential

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.}$$

$\Downarrow$ Surface integral over an open surface

Integral form $$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}.$$

Several Cases

$\boxed{emf = -d\Phi/dt}$

- A stationary circuit in a time-varying magnetic field (transformer emf)
- A moving conductor in a static magnetic field (motional emf)
- A moving circuit in a time-varying magnetic field (combined)

$-\frac{d\phi}{dt}$

7-2.1 A Stationary Circuit in a Time-Varying Magnetic Field

- For a stationary circuit with a contour *C* and surface *S*

$$\boxed{\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}.}$$

$\Rightarrow$ $d\Phi$

Stationary *S* (i.e., *S* not a function of time)

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{s}.$$

- Define

$$\mathcal{V} = \oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = \text{emf induced in circuit with contour } C \qquad \text{(V)}$$

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \text{magnetic flux crossing surface } S \qquad \text{(Wb)},$$

C may or may not be a physical circuit

- Then,

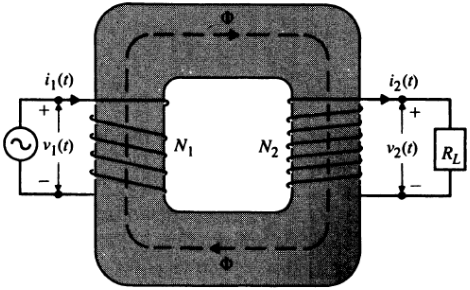
$$\boxed{\mathscr{V} = -\frac{d\Phi}{dt} \qquad \text{(V)}.}$$

The induced emf will cause a current to flow in the closed loop in such a direction as to oppose the change in the linking magnetic flux. (Lenz's law)

Faraday's law of electromagnetic induction: The emf induced in a stationary closed circuit is equal to the negative rate of increase of the magnetic flux linking the circuit. (Transformer emf)

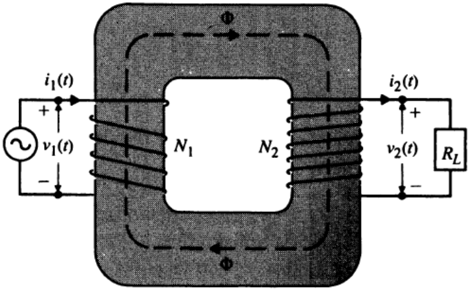
7-2.2 Transformers

- A transformer: two or more coils coupled magnetically through a common ferromagnetic core.



(a) Schematic diagram of a transformer.

Primary circuit



Secondary circuit

(a) Schematic diagram of a transformer.

KVL for magnetic circuit: $$N_1 i_1 - N_2 i_2 = \mathscr{R}\Phi,$$

By Lenz's law, the induced mmf, $N_2 i_2$, opposes flux Φ created by the mmf in the primary circuit, $N_1 i_1$.

$$\mathscr{R} = \frac{\ell}{\mu S}.$$

$$N_1 i_1 - N_2 i_2 = \frac{\ell}{\mu S}\Phi.$$

(a) Ideal transformer

- Assume $\mu \rightarrow \infty$,

$$\boxed{N_1 i_1 - N_2 i_2 = \frac{\ell}{\mu S}\Phi.}$$

$$\Downarrow$$

$$\boxed{\frac{i_1}{i_2} = \frac{N_2}{N_1}.}$$

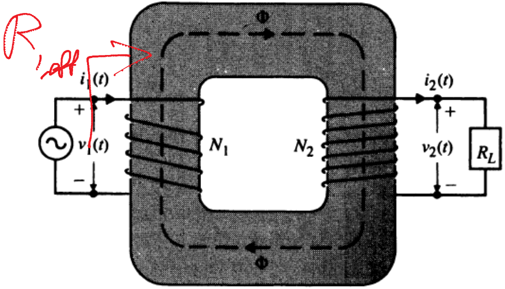
The ratio of the currents in the primary and secondary windings of an ideal transformer is equal to the inverse ratio of the numbers of turns.

From Faraday's law: $$v_1 = N_1 \frac{d\Phi}{dt}$$

$$v_2 = N_2 \frac{d\Phi}{dt},$$

$$\boxed{\frac{v_1}{v_2} = \frac{N_1}{N_2}.}$$

The ratio of the voltages across the primary and secondary windings of an ideal transformer is equal to the turns ratio.



(a) Schematic diagram of a transformer.

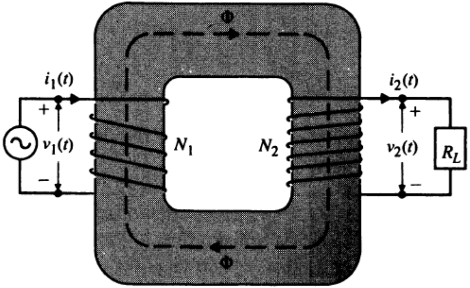
When the secondary winding is terminated in a load resistance R_L, the effective load seen by the source

$$(R_1)_{\text{eff}} = \frac{v_1}{i_1} = \frac{(N_1/N_2)v_2}{(N_2/N_1)i_2},$$

$$\boxed{\frac{i_1}{i_2} = \frac{N_2}{N_1}.}$$

$$\boxed{\frac{v_1}{v_2} = \frac{N_1}{N_2}.}$$

$$\boxed{(R_1)_{\text{eff}} = \left(\frac{N_1}{N_2}\right)^2 R_L,}$$



(a) Schematic diagram of a transformer.

For a sinusoidal source $v_1(t)$ and a load impedance Z_L, the effect load seen by the source

$$\boxed{(Z_1)_{\text{eff}} = \left(\frac{N_1}{N_2}\right)^2 Z_L.}$$

$$\boxed{N_1 i_1 - N_2 i_2 = \frac{\ell}{\mu S}\Phi.}$$

Replace Φ

Total flux

$$\Lambda_1 = N_1\Phi = \frac{\mu S}{\ell}(N_1^2 i_1 - N_1 N_2 i_2),$$

$$\Lambda_2 = N_2\Phi = \frac{\mu S}{\ell}(N_1 N_2 i_1 - N_2^2 i_2).$$

Substitution of Λ_1 Λ_2 into

$$v_1 = N_1\frac{d\Phi}{dt} \qquad v_2 = N_2\frac{d\Phi}{dt},$$

$$v_1 = L_1\frac{di_1}{dt} - L_{12}\frac{di_2}{dt},$$

$$v_2 = L_{12}\frac{di_1}{dt} - L_2\frac{di_2}{dt},$$

circuit

where $L_1 = \frac{\mu S}{\ell}N_1^2,$ Self-inductance of the primary winding

$L_2 = \frac{\mu S}{\ell}N_2^2,$ Self-inductance of the secondary winding

$L_{12} = \frac{\mu S}{\ell}N_1 N_2$ Mutual inductance

See Ex. 6-15 (Eq. 6-135)

(b) Real transformer

- For an ideal transformer:
  - $L_1 = \frac{\mu S}{\ell} N_1^2, \quad L_2 = \frac{\mu S}{\ell} N_2^2, \quad L_{12} = \frac{\mu S}{\ell} N_1 N_2 \quad \Rightarrow \quad L_{12} = \sqrt{L_1 L_2}.$
  - Infinite μ → infinite L
- For a real transformer: $L_{12} = k\sqrt{L_1 L_2}, \quad k < 1,$

  k: coefficient of coupling

7-2.3 A Moving Conductor in a Static Magnetic Field

$\vec{u} \times \vec{B} \Rightarrow \vec{E}_m \Rightarrow$ charges moving

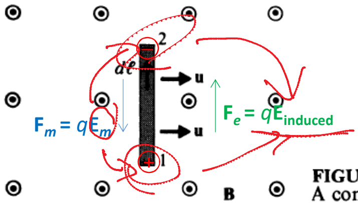


FIGURE 7–2
A conducting bar moving in a magnetic field.

A magnetic force $\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$

Moving velocity of a conductor

Static magnetic field

$\mathbf{F}_m$ (magnetic force) ➔ charge separation ➔ $\mathbf{E}_{induced}$ ➔ $\mathbf{F}_e$ (electric force)

At equilibrium, the net force ($\mathbf{F}_m + \mathbf{F}_e$) on the free charges in the moving conductor is zero.

$$-\int^{2} \vec{E}_{induced} \cdot d\vec{l}$$

An induced electric field acting along the conductor and producing a voltage

$$-\mathbf{E}_{\text{induced}} = \mathbf{E}_m = \mathbf{u} \times \mathbf{B}$$

$$V_{21} = \int_1^2 (\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell}.$$

If the moving conductor is a part of a closed circuit *C*, the emf

$$\boxed{\mathcal{V}' = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell} \qquad \text{(V)}.}$$

Called flux cutting emf or motional emf

For **u**//**B** (no flux is cut), emf $V'=0$

motional emf

= flux-cutting emf

EXAMPLE 7–2 A metal bar slides over a pair of conducting rails in a uniform magnetic field $\mathbf{B} = \mathbf{a}_z B_0$ with a constant velocity $\mathbf{u}$, as shown in Fig. 7–3.

a) Determine the open-circuit voltage V_0 that appears across terminals 1 and 2.

b) Assuming that a resistance R is connected between the terminals, find the electric power dissipated in R.

c) Show that this electric power is equal to the mechanical power required to move the sliding bar with a velocity $\mathbf{u}$. Neglect the electric resistance of the metal bar and of the conducting rails. Neglect also the mechanical friction at the contact points.

$$\frac{V_0^2}{R}$$

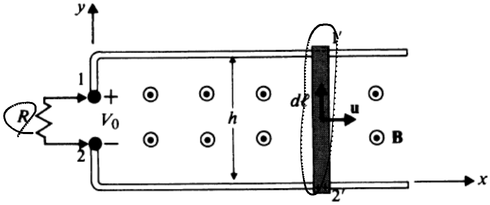
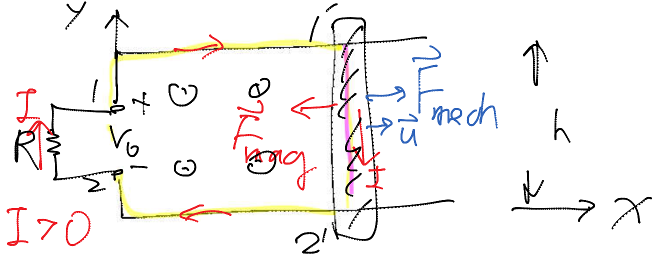


FIGURE 7–3
A metal bar sliding over conducting rails (Example 7–2).



(a) open-circuit voltage V_0

(b) P_e dissipated by R

(c) $P_e = P_{mech}$

$I = -\frac{V_0}{R}$ $\quad \vec{B} = \hat{a}_z B_0$

P_e : electric power

P_{mech} : mechanical power

$$\boxed{\text{(b)}\quad P_e = \frac{V_0^2}{R} = \frac{(uB_0h)^2}{R} \quad \#}$$

(a) $$V_0 = V_1 - V_2 = \oint_C (\vec{u}\times\vec{B})\cdot d\vec{l}$$

$$= \int_{2'\,(\text{ref})}^{1'} (\hat{a}_x u \times \hat{a}_z B_0)\cdot \hat{a}_y\, dl = -uB_0h \text{ (V)} \quad \#$$

(c) $$P_{mech} = \vec{F}_{mech} \cdot \vec{u}$$

$$\vec{F}_{mech} = -\vec{F}_{mag}$$

$$\vec{F}_{mag} = I \int_{1'}^{2'} d\vec{l} \times \vec{B} = I \left(\hat{a}_y(-h) \times \hat{a}_z B_0 \right)$$

$$= \hat{a}_x \left(-I B_0 h \right)$$

$$\Rightarrow P_{mech} = \hat{a}_x I B_0 h \cdot \hat{a}_x u = I B_0 h u = \frac{(B_0 h u)^2}{R} = P_e$$

$$-\frac{V_0}{R} = I$$

7-2.4 A Moving Circuit in a Time-Varying Magnetic Field

- Transformer emf + motional emf
- Lorentz's force equation:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}).$$

- The effective electric field **E'** on *q*:

$$\mathbf{E}' = \underbrace{\mathbf{E}}_{\text{Due to time-varying magnetic field (transformer emf)}} + \underbrace{\mathbf{u} \times \mathbf{B}}_{\text{Due to a moving circuit (motional emf)}}$$

Considering a conducting circuit with contour C and surface S moves with a velocity **u** in a field (**E**,**B**):

$$\mathbf{E}' = \mathbf{E} + \mathbf{u} \times \mathbf{B}$$

$\Downarrow$ Integral along C on both sides

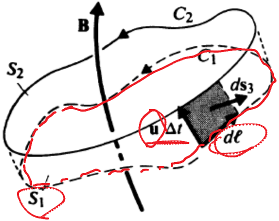
and use $\boxed{\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}.}$

General form of Faraday's law for a moving circuit in a time-varying magnetic field.

$$\boxed{\oint_C \mathbf{E}' \cdot d\boldsymbol{\ell} = \underbrace{-\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}}_{\text{Transformer emf}} + \underbrace{\oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell}}_{\text{Motional emf}} \qquad (\mathrm{V}).} \qquad (34)$$

$$\boxed{\mathcal{V} = -\frac{d\Phi}{dt} \qquad (\mathrm{V}).} \qquad \boxed{\mathcal{V}' = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell} \qquad (\mathrm{V}).}$$

A Moving Circuit



Moving velocity: **u**

FIGURE 7–5
A moving circuit in a time-varying magnetic field.

- The contour C moves from C_1 at time t to C_2 at time $t+\Delta t$
- The motion can be translation, rotation, and distortion in an arbitrary manner.

B(t)

$$\frac{d\Phi}{dt} = \frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{s}$$

ds(t)

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_{S_2} \mathbf{B}(t+\Delta t) \cdot d\mathbf{s}_2 - \int_{S_1} \mathbf{B}(t) \cdot d\mathbf{s}_1\right].$$

t+Δt

t

$$\frac{d\Phi}{dt} = \frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s}$$

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{S_2}\mathbf{B}(t+\Delta t)\cdot d\mathbf{s}_2 - \underbrace{\int_{S_1}\mathbf{B}(t)\cdot d\mathbf{s}_1}_{(1)}\right].$$

Expand this term **B**(t+Δt) as a Taylor's series

$$\mathbf{B}(t+\Delta t) = \underbrace{\mathbf{B}(t)}_{(2)} + \underbrace{\frac{\partial \mathbf{B}(t)}{\partial t}\Delta t}_{(3)} + \underbrace{\text{H.O.T.}}_{(4)},$$

$\Delta t^2, \Delta t^3 \ldots$

$$\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} = \underbrace{\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s}}_{(3)} + \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\underbrace{\int_{S_2}\mathbf{B}\cdot d\mathbf{s}_2}_{(2)} - \underbrace{\int_{S_1}\mathbf{B}\cdot d\mathbf{s}_1}_{(1)} + \underbrace{\text{H.O.T.}}_{(4)}\right], \qquad (37)$$

$\Delta t^2, \Delta t^3 \ldots$

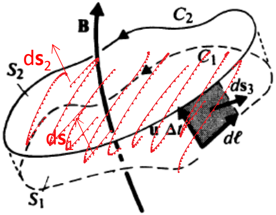


FIGURE 7–5
A moving circuit in a time-varying magnetic field.

- In going from C_1 to C_2, the circuit covers a region bounded by S_1, S_2, and S_3.
- S_3: side surface, the area swept out by the contour in time Δt. An element of S_3

$$d\mathbf{s}_3 = d\boldsymbol{\ell} \times \mathbf{u}\,\Delta t.$$

$d\vec{s}_1$

- Apply the divergence theorem for **B** at time t

$$\int_V \nabla \cdot \mathbf{B}\, dv = \int_{S_2} \mathbf{B} \cdot d\mathbf{s}_2 - \int_{S_1} \mathbf{B} \cdot d\mathbf{s}_1 + \int_{S_3} \mathbf{B} \cdot d\mathbf{s}_3,$$

Because outward normal must be used

$$\parallel \quad \oint_S \vec{B}\cdot d\vec{S} =$$

$S: S_1, S_2, S_3$

$$\int_V \nabla \cdot \mathbf{B}\, dv = \int_{S_2} \mathbf{B} \cdot d\mathbf{s}_2 - \int_{S_1} \mathbf{B} \cdot d\mathbf{s}_1 + \int_{S_3} \mathbf{B} \cdot d\mathbf{s}_3,$$

0

$$\nabla \cdot \mathbf{B} = 0,$$

$$d\mathbf{s}_3 = d\boldsymbol{\ell} \times \mathbf{u}\, \Delta t.$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})$$

$$\int_{S_2} \mathbf{B} \cdot d\mathbf{s}_2 - \int_{S_1} \mathbf{B} \cdot d\mathbf{s}_1 = -\Delta t \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell}. \qquad (40)$$

Combine Eqs. (37) and (40)

$$\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s} + \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{S_2}\mathbf{B}\cdot d\mathbf{s}_2 - \int_{S_1}\mathbf{B}\cdot d\mathbf{s}_1 + \text{H.O.T.}\right], \tag{37}$$

$$\int_{S_2}\mathbf{B}\cdot d\mathbf{s}_2 - \int_{S_1}\mathbf{B}\cdot d\mathbf{s}_1 = -\Delta t\oint_C (\mathbf{u}\times\mathbf{B})\cdot d\boldsymbol{\ell}. \tag{40}$$

H.O.T is neglected as $\Delta t \rightarrow 0$

$$\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s} - \oint_C (\mathbf{u}\times\mathbf{B})\cdot d\boldsymbol{\ell},$$

Compared with (34)

$$\boxed{\oint_C \mathbf{E}'\cdot d\boldsymbol{\ell} = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s} + \oint_C (\mathbf{u}\times\mathbf{B})\cdot d\boldsymbol{\ell} \qquad \text{(V)}.}$$

(34)

$$-\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} = \oint_C \mathbf{E}'\cdot d\boldsymbol{\ell}$$

$$-\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} = \oint_C \mathbf{E}'\cdot d\boldsymbol{\ell}$$

By designating

$$\mathcal{V}' = \oint_C \mathbf{E}'\cdot d\boldsymbol{\ell}$$

$$= \text{emf induced in circuit } C \text{ measured in the moving frame,}$$

$$\boxed{\begin{aligned}\mathcal{V}' &= -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s}\\ &= -\frac{d\Phi}{dt} \quad \text{(V)},\end{aligned}} \tag{43}$$

Comparison of Eqs. (43) and (6)

$$\mathcal{V}' = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{s} = -\frac{d\Phi}{dt} \quad \text{(V)}, \qquad (43)$$

motional emf

$d\vec{s}(t)$

transformer emf

$$\mathcal{V} = -\frac{d\Phi}{dt} \quad \text{(V)}. \qquad (6)$$

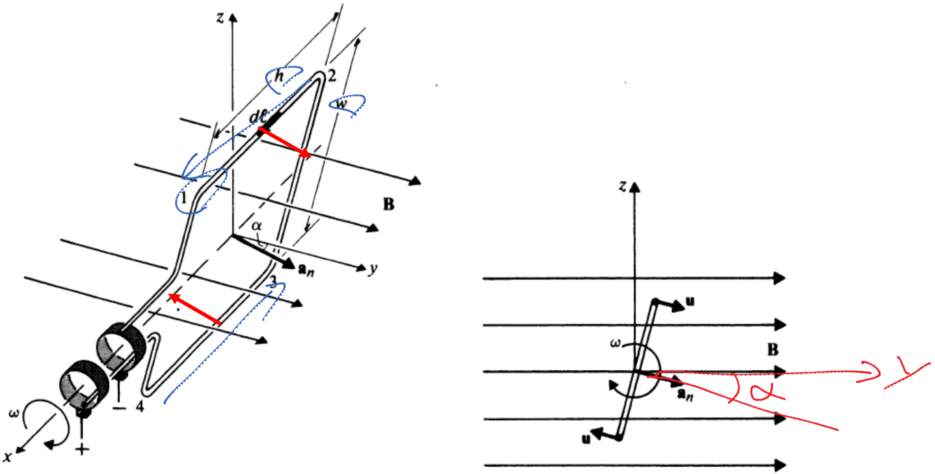
$\vec{u} = 0$

- They are exactly the same.
- *V*′ is for circuits in motion; *V* is for circuits not in motion

Faraday's law that the emf induced in a closed circuit equals the negative time-rate of increase of the magnetic flux linking a circuit applies to a stationary circuit as well as a moving one.

Also see example 7-4

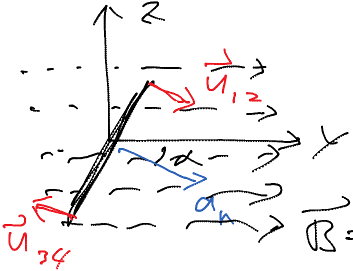
EXAMPLE 7–4 An h by w rectangular conducting loop is situated in a changing magnetic field $\mathbf{B} = \mathbf{a}_y B_0 \sin \omega t$. The normal of the loop initially makes an angle α with $\mathbf{a}_y$, as shown in Fig. 7–6. Find the induced emf in the loop: (a) when the loop is at rest, and (b) when the loop rotates with an angular velocity ω about the x-axis.



(a) Perspective view.

(b) View from $+x$ direction.

FIGURE 7–6
A rectangular conducting loop rotating in a changing magnetic field (Example 7–4).



$\vec{B} = \hat{a}_y B_0 \sin\omega t$

(a)

$$V_a = -\frac{d\Phi}{dt}$$

$$\Phi = \int \vec{B} \cdot d\vec{S}$$

$$= \oint (\hat{a}_y B_0 \sin\omega t) \cdot (\hat{a}_n h w)$$

$$= B_0 \underline{hw} \sin\omega t \cos\alpha$$

$$V_a = -B_0 S \omega \cos\omega t \cos\alpha$$

(b) Method 1

$$V_t = \underline{V_a} + \underline{V_b}$$

motional

$$V_b = \oint_C (\vec{u} \times \vec{B}) \cdot d\vec{l}$$

sections 12 and 34 contributes to V_b

$$= \int_1^2 (\hat{a}_n \frac{w}{2}\omega \times \hat{a}_y B_0 \sin\omega t) \cdot (\hat{a}_x dl)$$

$$+ \int_4^3 (-\hat{a}_n \frac{w}{2}\omega \times \hat{a}_y B_0 \sin\omega t) \cdot (\hat{a}_x dl)$$

$$= \cdots$$

$$= S \omega B_0 \sin\omega t \sin\alpha$$

$\Rightarrow V_t = V_a + V_b$

if $\alpha = \omega t$ $= - - - -$

$=$

$= -B_0 S \omega \cos 2\omega t$

$\boxed{\text{Method 2}}$: $V_t = -\dfrac{d\Phi}{dt}$ $\vec{n} \neq 0$

$\Phi = \vec{B}(t) \cdot d\vec{S}(t)$

$= \hat{a}_y B_0 \sin\omega t \cdot \hat{a}_n(t) S$

$= \cos\omega t \, B_0 \sin\omega t \, S$

$= \frac{1}{2} B_0 S \sin 2\omega t$

$V_t = -\dfrac{d\Phi}{dt} = -B_0 S \omega \cos 2\omega t$

7-3 Maxwell's Equations

$V = -\frac{d\Phi}{dt} \Longleftrightarrow \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

- Electromagnetic induction: a time-varying magnetic field gives rise to an electric field.

Static case

$$\nabla \times \mathbf{E} = 0$$

Time-varying case

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$
$$\nabla \times \mathbf{H} = \mathbf{J},$$
$$\nabla \cdot \mathbf{D} = \rho,$$
$$\nabla \cdot \mathbf{B} = 0.$$

Modification of ∇×H=J in a Time-varying Case

- Charge conservation (or the equation of continuity) must be satisfied at all times.

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

- Check if $\nabla \times \mathbf{H} = \mathbf{J}$ is consistent with the requirement of charge conservation in a time-varying situation

$$\nabla \times \mathbf{H} = \mathbf{J},$$

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J},$$

By null identity

not consistent

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}.$$

$$0 = \nabla \cdot \mathbf{J},$$

Not consistent!

Since $\nabla \cdot \mathbf{J} = 0$ does not vanish in a time-varying situation (ρ is changing in a time-varying situation), $\nabla \cdot \mathbf{J} = 0$ is in general not true.

➔ $\nabla \times \mathbf{H} = \mathbf{J}$ should be modified in a time-varying situation

In order to satisfy $\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}.}$

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J}, \quad \rightarrow \quad \nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t}.$$

$$\boxed{\nabla \cdot \mathbf{D} = \rho,}$$

$$\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right)$$

$$\nabla \cdot \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.}$$

A/m²

- Thus, a time-varying electric field will give rise to a magnetic field, even in the absence of a current flow.

- A recap:

$$\nabla \times \mathbf{H} = \mathbf{J}, \quad \Longrightarrow \quad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

To satisfy charge conservation

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}.$$

$\partial \mathbf{D}/\partial t$ Displacement current density (Introduced by James Clerk Maxwell)

A/m^2

Maxwell's Equation

$$\boxed{\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \\ \nabla \cdot \mathbf{D} &= \rho, \\ \nabla \cdot \mathbf{B} &= 0. \end{aligned}}$$

ρ: free charge
J: free currents (including convection current (ρ**u**) and conduction current (σ**E**))

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}.$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}).$$

The above 6 equations form the foundation of electromagnetic theory!

Electromagnetic Problem

- 4 unknowns: **E, D, B, H**
- 4 independent equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \qquad \text{(1) and (2)}$$

$$\nabla \cdot \mathbf{D} = \rho,$$

$$\nabla \cdot \mathbf{B} = 0.$$

$$\mathbf{D} = \epsilon \mathbf{E} \qquad (3)$$

$$\mathbf{H} = \mathbf{B}/\mu, \qquad (4)$$

7-3.1 Integral Form of Maxwell's Equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t},$$

$$\nabla \cdot \mathbf{D} = \rho,$$

$$\nabla \cdot \mathbf{B} = 0.$$

Surface integral
Stoke's theorem

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

$$\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{s}.$$

Volume integral
Divergence theorem

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = \int_V \rho \, dv$$

$$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0.$$

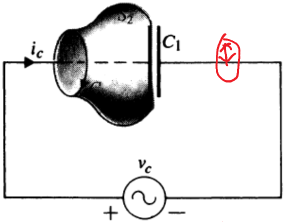
TABLE 7–2
Maxwell's Equations

Differential Form	Integral Form	Significance
$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\dfrac{d\Phi}{dt}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I + \int_S \dfrac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}$	Ampère's circuital law
$\nabla \cdot \mathbf{D} = \rho$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$	No isolated magnetic charge

A/m^2

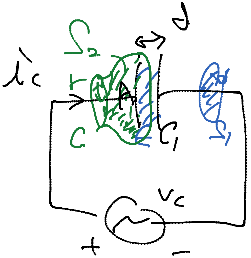
$$i_D = \int \frac{\partial D}{\partial t} \cdot dS \quad (A)$$

EXAMPLE 7–5 An a-c voltage source of amplitude V_0 and angular frequency ω, $v_c = V_0 \sin \omega t$, is connected across a parallel-plate capacitor C_1, as shown in Fig. 7–7. (a) Verify that the displacement current in the capacitor is the same as the conduction current in the wires. (b) Determine the magnetic field intensity at a distance r from the wire.



$$i_c = C \frac{dV}{dt}$$

FIGURE 7–7
A parallel-plate capacitor connected to an a-c voltage source (Example 7–5).



(a) conduction current: $i_c = C_1 \frac{dV}{dt}$

$=$

$= C_1 V_0 \omega \cos\omega t$

$C_1 = \frac{\epsilon A}{d}$

$E = \frac{V_c}{d}$

$D = \epsilon E = \epsilon \frac{V_c}{d} = \epsilon \frac{V_0 \sin\omega t}{d}$

$i_D = \int \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}$

$= \frac{\partial D}{\partial t} \cdot A$

$= \epsilon \frac{V_0}{d} \omega \cos\omega t\, A = C_1 V_0 \omega \cos\omega t$

$= i_c$

(b) $\oint \vec{H} \cdot d\vec{l} = i_c + i_D$

S_1: $H_\phi \cdot 2\pi r = i_c + 0$

(b) $\oint \vec{H} \cdot d\vec{l} = i_c + i_D$

S_2 $H_\phi \cdot 2\pi r = 0 + i_D$

7-4 Potential Functions

$$\nabla \cdot \mathbf{B} = 0$$

divergenceless

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A} \qquad \text{(T).}}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A})$$

or $$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0.$$

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0.$$

curl free

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V,$$

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \quad \text{(V/m).}}$$

$-\nabla V$ Due to **charge** distribution

$\rho \rightarrow V$ $\quad V = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho}{R}\, dv',$

$-\frac{\partial \mathbf{A}}{\partial t}$ Due to time-varying **current**

$\mathbf{J} \rightarrow \mathbf{A}$ $\quad \mathbf{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J}}{R}\, dv'.$

V and **A** here are solutions of Poisson's equations

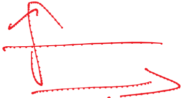
Quasi-static Fields

- The two equations were obtained under static conditions

$$V = \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{\rho}{R}\,dv', \qquad \mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\mathbf{J}}{R}\,dv'.$$

source source

- They can be time dependent: $\rho(t)$, $\mathbf{J}(t)$ ➔ $V(t)$, $\mathbf{A}(t)$
- If ρ and **J** **vary slowly with time** and the range of interest R is small in comparison with the wavelength (**low frequency, long wavelength**), it is allowable to use the 2 equations to find **quasi-static fields**.

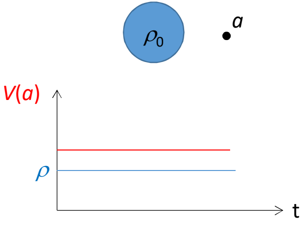


static

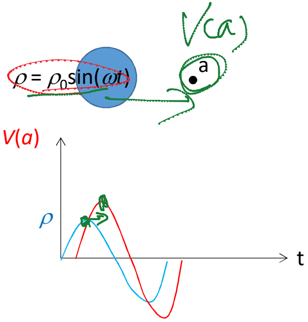
quasi-static

Time-retardation Effects

- Quasi-static fields are approximations.
- When the source frequency is high, quasi-static solutions will not suffice. Time-retardation effects must be included. (Discussed in 7-6)



Static



Time-retardation effects for high-frequency sources

As the source changes in time, it takes time to change the potential at a certain distance from the source!

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}}$$

$\nabla \times \mu$

$\nabla \times (\mu H) = \mu \vec{J} + \mu (\vec{J} + \frac{\partial \vec{D}}{\partial t})$

$\nabla \times (\nabla \times \vec{A}) = \mu \vec{J} +$

$\mu \frac{\partial}{\partial t} (\varepsilon)(-\nabla V - \frac{\partial \vec{A}}{\partial t})$

LHS RHS

$$\mathbf{H} = \mathbf{B}/\mu$$

$$\mathbf{D} = \epsilon \mathbf{E}.$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

Assume a homogeneous medium

$$\nabla \times \nabla \times \mathbf{A} = \mu \mathbf{J} + \mu\epsilon \frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \mathbf{A}}{\partial t}\right),$$

Vector identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu \mathbf{J} - \nabla\left(\mu\epsilon \frac{\partial V}{\partial t}\right) - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

or $$\nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} + \nabla\left(\nabla \cdot \mathbf{A} + \mu\epsilon \frac{\partial V}{\partial t}\right).$$

- A vector requires the specification of both its curl and its divergence.
  - Curl has been specified $\mathbf{B} = \nabla \times \mathbf{A}$
  - How to choose divergence!?

$$\nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} + \nabla\left(\nabla \cdot \mathbf{A} + \mu\epsilon \frac{\partial V}{\partial t}\right).$$

We let

$$\boxed{\nabla \cdot \mathbf{A} + \mu\epsilon \frac{\partial V}{\partial t} = 0,}$$

- Lorentz condition (or Lorentz gauge) for potentials
- Also, the condition is consistent with equation of continuity (see P7-12)

$$\boxed{\nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}.}$$

Nonhomogeneous wave equation for vector potential A

- Reduced to Poisson's equation for static cases
- Its solutions represent waves traveling with a velocity $1/\sqrt{\mu\epsilon}$. (Discussed more in 7-6)

source free:

$$\nabla^2 \vec{A} - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = 0$$

homo. wave eq.

Nonhomogeneous wave equation for scalar potential V

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho,$$

$$\nabla \cdot (\varepsilon E) = \rho$$

$$-\nabla \cdot \epsilon \left(\nabla V + \frac{\partial \mathbf{A}}{\partial t} \right) = \rho,$$

Assume a constant ε

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\epsilon}.$$

$$\nabla \cdot \mathbf{A} + \mu\epsilon \frac{\partial V}{\partial t} = 0,$$

Lorentz condition uncouples the wave equations for **A** and V

$$\boxed{\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon},}$$

- Reduced to Poisson's equation in static cases
- Its solutions represent waves traveling with a velocity

Solution of Wave Equations for **A** and V

Poisson's equations (static cases)

$$\nabla^2 V = -\frac{\rho}{\epsilon},$$

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J}.$$

Solutions

$$V = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho}{R} dv',$$

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J}}{R} dv'.$$

Wave equations (time-varying cases)

$$\boxed{\nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}.}$$

$$\boxed{\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon},}$$

Solutions ?

$V = ?$

$\vec{A} = ?$

Different equations ➔ solutions must be modified! (Discussed more in 7-6)

7-5 Electromagnetic Boundary Conditions

- In general, the application of the integral form of a curl equation to a flat closed path at a boundary with top and bottom sides in the two touching media yields the boundary condition for the tangential components
- The application of the integral form of a divergence equation to a shallow pillbox at an interface with top and bottom faces in the two contiguous media gives the boundary condition for the normal components

$\oint_c \vec{E} \cdot d\vec{l} = 0$

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi}{dt}$$

$$\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}$$

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$$

$$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$$

$$\boxed{E_{1t} = E_{2t} \qquad (\text{V/m});}$$

$$\boxed{\mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \qquad (\text{A/m}).}$$

$$\boxed{\mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s \qquad (\text{C/m}^2);}$$

$$\boxed{B_{1n} = B_{2n} \qquad (\text{T}).}$$

- For curl equations:

Let the height of the flat closed path approach zero (area → 0)

→ The surface integral of $\partial \mathbf{B}/\partial t$ and $\partial \mathbf{D}/\partial t$ vanishes

→ Same equations as static cases

→ Same boundary conditions as static cases

$$\boxed{E_{1t} = E_{2t} \qquad (\text{V/m});}$$

$$\boxed{\mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \qquad (\text{A/m}).} \tag{2}$$

$$\boxed{\mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s \qquad (\text{C/m}^2);} \tag{3}$$

$$\boxed{B_{1n} = B_{2n} \qquad (\text{T}).}$$

1. The tangential component of an E field is continuous across an interface.
2. The tangential component of an H field is discontinuous across an interface where a surface current exists, the amount of discontinuity being determined by Eq. (2).
3. The normal component of a D field is discontinuous across an interface where a surface charge exists, the amount of discontinuity being determined by Eq. (3).
4. The normal component of a B field is continuous across an interface.

Due to the dependence of Maxwell's equations, divergence equations can be derived from curl equations and equation of continuity.

$$\boxed{E_{1t} = E_{2t} \qquad \text{(V/m);}}$$

$$\boxed{\mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \qquad \text{(A/m).}}$$

$$\boxed{\mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s \qquad \text{(C/m}^2\text{);}}$$

$$\boxed{B_{1n} = B_{2n} \qquad \text{(T).}}$$

Equivalent BC

Equivalent BC

7-5.1 Interface between Two Lossless Linear Media

- A lossless linear media: $\sigma=0$

  J = 0 ➔ power dissipation = 0 ➔ lossless

$$P = \int_V \mathbf{E} \cdot \mathbf{J}\, dv \qquad \text{(W)}.$$

- Usually no free charges and no surface currents at the interface of two lossless media. ($\rho_s = 0$, $\mathbf{J}_s = 0$)

TABLE 7–3
Boundary Conditions between Two Lossless Media

$E_{1t} = E_{2t} \rightarrow \dfrac{D_{1t}}{D_{2t}} = \dfrac{\epsilon_1}{\epsilon_2}$
$H_{1t} = H_{2t} \rightarrow \dfrac{B_{1t}}{B_{2t}} = \dfrac{\mu_1}{\mu_2}$
$D_{1n} = D_{2n} \rightarrow \epsilon_1 E_{1n} = \epsilon_2 E_{2n}$
$B_{1n} = B_{2n} \rightarrow \mu_1 H_{1n} = \mu_2 H_{2n}$

7-5.2 Interface between a Dielectric and a Perfect Conductor

dielectric

- Conductors
  - Good conductors: $\sigma \sim 10^7$ (S/m)
  - Superconductors: $\sigma \sim 10^{20}$ (S/m) PEC

PEC

- In order to simplify the analytical solution of field problems, good conductors are often considered perfect conductors in regard to boundary conditions.

Perfect Conductors

- $\sigma \rightarrow \infty$
- $\mathbf{E}_{\mathbf{inside}}$ = 0 (otherwise, infinite **J** inside)
- Charges only reside on the surface
- In a time-varying situation, (**E**, **D**) and (**B**, **H**) in the interior of a conductor are zero.

E = 0 → D = 0

$$\mathbf{D} = \epsilon \mathbf{E}$$

E = 0 → B(t) = 0 ⇨ B=0 → H=0

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = 0 \text{ inside a conductor} \qquad \mathbf{H} = \mathbf{B}/\mu,$$

In a time-varying situation, B should be time varying (i.e., cannot be a nonzero constant)!

in the static case, B and H may not be zero!

if B is a nonzero constant, it contradicts the "time-varying" ~~sit~~ condition.

In medium 2 (a perfect conductor), $\mathbf{E_2} = 0$, $\mathbf{H_2} = 0$, $\mathbf{D_2} = 0$, $\mathbf{B_2} = 0$

TABLE 7–4
Boundary Conditions between a Dielectric (Medium 1) and a Perfect Conductor (Medium 2) (Time-Varying Case)

$J_s \neq 0$

On the Side of Medium 1	On the Side of Medium 2
$E_{1t} = 0$	$E_{2t} = 0$
$\mathbf{a}_{n2} \times \mathbf{H}_1 = \mathbf{J}_s$	$H_{2t} = 0$
$\mathbf{a}_{n2} \cdot \mathbf{D}_1 = \rho_s$	$D_{2n} = 0$
$B_{1n} = 0$	$B_{2n} = 0$

Q: How about if medium 2 is a conductor with finite conductivity?

$J_s = 0$

A: As mentioned in Section 6-10, currents in media with finite conductivities are expressed in terms of volume current densities **J**, and surface current densities $\mathbf{J_s}$ for currents flowing through an infinitesimal thickness (τ) is zero.

➔ $\mathbf{J_s} = \mathbf{J}*\tau = 0$ as $\tau \to 0$

➔ H_t continuous

Boundary Conditions between a Dielectric (Medium 1) and a Perfect Conductor (Medium 2) (Time-Varying Case)

On the Side of Medium 1	On the Side of Medium 2
$E_{1t} = 0$	$E_{2t} = 0$
$\boxed{\mathbf{a}_{n2} \times \mathbf{H}_1 = \mathbf{J}_s}$	$H_{2t} = 0$
$\mathbf{a}_{n2} \cdot \mathbf{D}_1 = \rho_s$	$D_{2n} = 0$
$B_{1n} = 0$	$B_{2n} = 0$

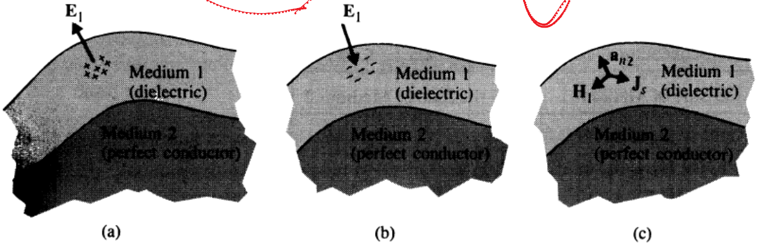


FIGURE 7–8
Boundary conditions at an interface between a dielectric (medium 1) and a perfect conductor (medium 2).

$$|\mathbf{E}_1| = E_{1n} = \frac{\rho_s}{\epsilon_1}.$$

$$|\mathbf{H}_1| = |\mathbf{H}_{1t}| = |\mathbf{J}_s|.$$

$E_{1t} = 0$ ➔ E is normal to the points away from (into) the conductor surface when the surface charges are positive (negative)

Importance of Boundary Conditions

- Maxwell’s equations are partial differential equations. Their solutions will contain integration constants that are determined from the additional information supplied by boundary conditions so that each solution will be unique for each given problem.

7-6 Wave Equations and Their Solutions

- Importance of Maxwell's equations
  - Give a complete description of the relation between electromagnetic fields and charge and current distributions (sources).
  - Their solutions provide the answers to all electromagnetic problems.
- For given charge and current distributions

$$\rho, \mathbf{J} \longrightarrow V, \mathbf{A} \longrightarrow \mathbf{E}, \mathbf{B}$$

$$V = \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{\rho}{R}\, dv',$$

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\mathbf{J}}{R}\, dv'.$$

(Quasi-static)

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

7-6.1 Solution of Wave Equations for Potentials

- Nonhomogeneous wave equation

$$\boxed{\nabla^2 V - \mu\epsilon\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon},}$$

$V = \rho$?

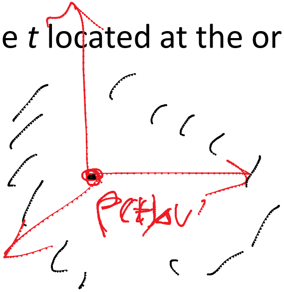
- Finding V for an elemental point charge at time t located at the origin $\rho(t)\,\Delta v'$

  Spherical symmetry → $V(\boldsymbol{R},t)$ is only function of R

  Except at origin, the wave equation is:

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) - \mu\epsilon\frac{\partial^2 V}{\partial t^2} = 0.$$

$\cancel{\frac{\partial V}{\partial \theta}}$ $\cancel{\frac{\partial V}{\partial \phi}}$



$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) - \mu\epsilon\frac{\partial^2 V}{\partial t^2} = 0. \qquad (7\text{-}71)$$

Introduce a new variable U

$$V(R, t) = \frac{1}{R}U(R, t),$$

$$\frac{\partial^2 U}{\partial R^2} - \mu\epsilon\frac{\partial^2 U}{\partial t^2} = 0.$$

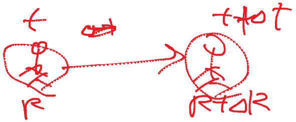
A 1D homogeneous wave eq.

$$U(R, t) = f(t - R\sqrt{\mu\epsilon}).$$

$$\cancel{U(R, t) = f(t + R\sqrt{\mu\epsilon}).}$$

Solution, which can be verified by direct substitution (7-74)

"+" solution doesn't satisfy causality and thus is neglected (discussed later).



$$U(R, t) = f(t - R\sqrt{\mu\epsilon}).$$

$$V(R, t) = \frac{1}{R} f(t - R/u).$$

Check the function *U* at *R*+Δ*R* at a later time *t*+Δ*t*

$$U(R + \Delta R, t + \Delta t) = f[t + \Delta t - (R + \Delta R)\sqrt{\mu\epsilon}] = f(t - R\sqrt{\mu\epsilon}). = U(R, t)$$

$\Delta t - \Delta R\sqrt{\mu\epsilon} = 0$

The function retains its form if $\Delta t = \Delta R\sqrt{\mu\epsilon} = \Delta R/u$, where $u = 1/\sqrt{\mu\epsilon}$

$\frac{\Delta R}{\Delta t}$

Thus, the function *U*(*R*,*t*) represents a wave traveling in the positive *R* direction with a velocity $u = \Delta R/\Delta t = 1/\sqrt{\mu\epsilon}$

$$\frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c \cong 3 \times 10^8$$

Next, to determine the specific function $f(t - R/u)$

A static point charge $\rho(t)\Delta v'$ at origin

$$V = \frac{q}{4\pi\epsilon_0 R} \quad \Longrightarrow \quad \Delta V(R) = \frac{\rho(t)\,\Delta v'}{4\pi\epsilon R}.$$

$\rho(t)$

$\Delta V(R, t)$

Comparison with

$$V(R, t) = \frac{1}{R} f(t - R/u).$$

$$R\,\Delta V(R) = \Delta f(t - R/u) = \frac{\rho(t - R/u)\,\Delta v'}{4\pi\epsilon}.$$

Incorporate the retardation effect !

$\rho(t - \frac{R}{u})$

Potential due to a charge distribution (integration)

$$\boxed{V(R, t) = \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho(t - R/u)}{R}\, dv' \qquad \text{(V).}}$$

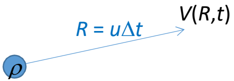
$$\boxed{V(R,t) = \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho(t - R/u)}{R}\, dv' \qquad \text{(V).}}$$

[cause] The value of ρ at an earlier time ($t-R/u$)

➔ [effect] $V(R, t)$ at a distance R from the source at time t

It takes time R/u for the effect of ρ to be felt at distance R.

That is, there is time retardation ($\Delta t = R/u$) from ρ to V



Q: can you explain now why "+" cannot be a solution?

$$U(R,t) = f(t + R\sqrt{\mu\epsilon}).$$

A: it would lead to the impossible situation that the effect of ρ would be felt at a distant point before it occurs at the source. That is, "+" solution doesn't satisfy causality.

Wave equation

$$\boxed{\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon},}$$

$$\boxed{V(R, t) = \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho(t - R/u)}{R} \, dv' \qquad \text{(V).}}$$

Retarded *V*

Wave equation

$$\boxed{\nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu\mathbf{J}.}$$

Following exactly the same way as that for *V*

$$\boxed{\mathbf{A}(R, t) = \frac{\mu}{4\pi} \int_{V'} \frac{\mathbf{J}(t - R/u)}{R} \, dv' \qquad \text{(Wb/m).}}$$

Retarded **A**

- **E** or **B** obtained from *V* and ***A*** will also be functions of $(t - R/u)$ and therefore retarded in time.
- It **takes time** for electromagnetic waves to travel and for the effects of time-varying charges and currents to be felt at distant points.
- By contrast, in the quasi-static approximation we ignore this time-retardation effect and assume instant response.

$$\begin{matrix}\rho \\ J\end{matrix}\left(t - \frac{R}{u}\right) \Rightarrow \begin{matrix} V(t) \\ \vec{A}(t) \\ \vec{E}(t) \\ \vec{B}(t) \end{matrix}$$

7-6.2 Source-Free Wave Equations

- Source free: $\rho = 0$, $\mathbf{J} = 0$
- Often interested not so much in how an electromagnetic wave is originated, but in how it propagates.
- Assuming a simple nonconducting media characterized by ε and μ (σ = 0),

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \times \mathbf{H} = \cancel{\mathbf{J}} + \frac{\partial \mathbf{D}}{\partial t},$$

$$\nabla \cdot \mathbf{D} = \cancel{\rho},$$

$$\nabla \cdot \mathbf{B} = 0.$$

D = ε**E**

B = μ**H**

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t},$$

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t},$$

$$\nabla \cdot \mathbf{E} = 0,$$

$$\nabla \cdot \mathbf{H} = 0.$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t},$$
$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t},$$
$$\nabla \cdot \mathbf{E} = 0,$$
$$\nabla \cdot \mathbf{H} = 0.$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t},$$

Curl on both sides

substitute $\boxed{\nabla \times \mathbf{H} = \epsilon \dfrac{\partial \mathbf{E}}{\partial t},}$

$$\nabla \times \nabla \times \mathbf{E} = -\mu \frac{\partial}{\partial t}(\nabla \times \mathbf{H}) = -\mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\underbrace{\nabla \cdot \mathbf{E}}_{=0}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E}$$

$$\nabla^2 \mathbf{E} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0;$$

$$u = 1/\sqrt{\mu\epsilon},$$

Homogeneous vector wave equations

$$\boxed{\nabla^2 \mathbf{E} - \frac{1}{u^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0.}$$

In an entirely similar way,

.

.

.

$$\boxed{\nabla^2 \mathbf{H} - \frac{1}{u^2}\frac{\partial^2 \mathbf{H}}{\partial t^2} = 0.}$$

Homogeneous vector wave equations

$$\nabla^2\mathbf{E} - \frac{1}{u^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 0.$$

$$\nabla^2\mathbf{H} - \frac{1}{u^2}\frac{\partial^2\mathbf{H}}{\partial t^2} = 0.$$

In Cartesian coordinates, the above equations can be decomposed into three 1D wave equations, just like the equation (7-73) solved before

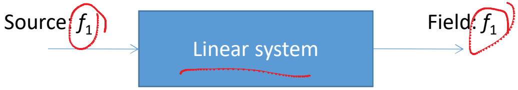
$$\frac{\partial^2 U}{\partial R^2} - \mu\epsilon\frac{\partial^2 U}{\partial t^2} = 0.$$

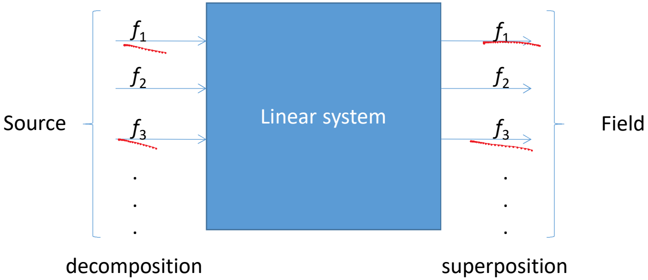
Thus, each component of **E** and **H** also represents waves, just like *U*.

7-7 Time-Harmonic Fields

- Since Maxwell's equations are linear differential equations, sinusoidal time variations of source functions of a given frequency will produce sinusoidal variations of **E** and **H** with the same frequency in the steady state.
- For source functions with an arbitrary time dependence, electrodynamic fields can be determined in terms of those caused by the various frequency components of the source functions. The applications of superposition will give us the total fields.

$\boxed{\text{analyze various frequency component} \rightarrow \text{use superposition to get the total field}}$


Source: f_1
Linear system
Field: f_1


Source
f_1
f_2
f_3
Linear system
f_1
f_2
f_3
Field
decomposition
superposition

7-7.1 The Use of Phasors—A Review

- Choose either a cosine or sine function as the reference
- Specify 3 parameters: amplitude, frequency, and phase

$$i(t) = I \cos(\omega t + \phi),$$

Example

Time domain

The loop equation for a series RLC circuit. Determine *i*(*t*)?

Applied voltage $e(t) = E \cos \omega t$

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = e(t).$$

$$i(t) = I \cos(\omega t + \phi),$$

$$I\left[-\omega L \sin(\omega t + \phi) + R\cos(\omega t + \phi) + \frac{1}{\omega C}\sin(\omega t + \phi)\right] = E\cos \omega t.$$

Complicated mathematical manipulations are required to determine *I* and ϕ

Example

Relation between time-domain and phasor expression

$\boxed{s(t) = \text{Re}[Se^{j\omega t}]}$

Phasor domain

The loop equation for a series RLC circuit. Determine $i(t)$?

Applied voltage $e(t) = E \cos \omega t$

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = e(t).$$

$$i(t) = I \cos(\omega t + \phi),$$

$\frac{d}{dt} \to j\omega$

$\int dt \to \frac{1}{j\omega}$

$i(t) \to \tilde{I}$

$e(t) \to \tilde{E}$

I. Change to phasor expressions

$$e(t) = E \cos \omega t = \mathcal{Re}[(Ee^{j0})e^{j\omega t}]$$
$$= \mathcal{Re}(E_s e^{j\omega t})$$

Phasors

$$E_s = Ee^{j0} = E$$
$$I_s = Ie^{j\phi}$$

$$i(t) = \mathcal{Re}[(Ie^{j\phi})e^{j\omega t}]$$
$$= \mathcal{Re}(I_s e^{j\omega t}),$$

Phasors contain amplitude and phase information but are independent of t

II. Differentiation and integration

$$\frac{di}{dt} = \mathcal{Re}(j\omega I_s e^{j\omega t}), \qquad \int i\,dt = \mathcal{Re}\left(\frac{I_s}{j\omega}e^{j\omega t}\right).$$

III. Equation in phasor domain

$$\left[R + j\left(\omega L - \frac{1}{\omega C}\right)\right]I_s = E_s,$$

$\boxed{I_s \text{ can be solved easily.}}$

7-7.2 Time-Harmonic Electromagnetics

/ frequency

- Vector phasors: e.g., a time-harmonic E field

$$\mathbf{E}(x, y, z, t) = \mathscr{Re}[\mathbf{E}(x, y, z)e^{j\omega t}]$$

direction, magnitude, and phase

complex

$v(t) \leftrightarrow \tilde{V}$

- Differentiation and integration

$$\partial \mathbf{E}(x, y, z, t)/\partial t \quad \Rightarrow \quad j\omega \mathbf{E}(x, y, z)$$

$$\int \mathbf{E}(x, y, z, t)\, dt \quad \Rightarrow \quad \mathbf{E}(x, y, z)/j\omega$$

$$\boxed{\partial/\partial t \rightarrow j\omega}$$

- Maxwell's equations in terms of vector field phasors (**E**, **H**) and source phasors (ρ, **J**) in a simple (linear, isotropic, and homogeneous) medium

$$
\begin{aligned}
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\
\nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \\
\nabla \cdot \mathbf{D} &= \rho, \\
\nabla \cdot \mathbf{B} &= 0.
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
\nabla \times \tilde{\mathbf{E}} &= -j\omega\mu\tilde{\mathbf{H}}, \\
\nabla \times \tilde{\mathbf{H}} &= \tilde{\mathbf{J}} + j\omega\epsilon\tilde{\mathbf{E}}, \\
\nabla \cdot \tilde{\mathbf{E}} &= \tilde{\rho}/\epsilon, \\
\nabla \cdot \tilde{\mathbf{H}} &= 0.
\end{aligned}
$$

ϵ, μ · const.

- Time-dependent quantities and phasors have the same notations for simplicity.
- In the rest of this book, we deal with phasors unless otherwise specified. (Useful note: any quantity containing *j* must necessarily be a phasor. Any quantities with *t* must be time-dependent quantities.)
- Phasor quantities are not functions of *t*.

- Time-harmonic wave equations

$(j\omega)^2 = -\omega^2$ $\quad + \omega^2\mu\epsilon = k^2$

$$\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}, \qquad \nabla^2 \tilde{V} + k^2 \tilde{V} = -\frac{\tilde{\rho}}{\epsilon}$$

$$\nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}. \qquad \nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J},$$

$2\pi f$

where $k = \omega\sqrt{\mu\epsilon} = \frac{\omega}{u} = 2\pi/\lambda$ (k: the wavenumber)

- The Lorentz condition

$$\nabla \cdot \mathbf{A} + \mu\epsilon \frac{\partial V}{\partial t} = 0, \qquad \nabla \cdot \mathbf{A} + j\omega\mu\epsilon V = 0.$$

$u = f\lambda$

- The phasor solutions for wave equations

$$V(R, t) = \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho(t - R/u)}{R}\, dv' \qquad \boxed{\tilde{V}(R) = \frac{1}{4\pi\epsilon} \int_{V'} \frac{\tilde{\rho} e^{-jkR}}{R}\, dv' \quad \text{(V)},}$$

$$\mathbf{A}(R, t) = \frac{\mu}{4\pi} \int_{V'} \frac{\mathbf{J}(t - R/u)}{R}\, dv' \qquad \boxed{\tilde{\mathbf{A}}(R) = \frac{\mu}{4\pi} \int_{V'} \frac{\tilde{\mathbf{J}} e^{-jkR}}{R}\, dv' \quad \text{(Wb/m).}}$$

$$e^{j\omega(t-R/u)} = e^{j\omega t} \times e^{-j\omega R/u} = e^{j\omega t} \times e^{-jkR}$$

Time delay (time domain) → additional phase term (phasor domain)

Taylor series expansion of the additional phase term e^{-jkR}

$$e^{-jkR} = 1 - jkR + \frac{k^2R^2}{2} + \cdots,$$

$$k = \frac{2\pi f}{u} = \frac{2\pi}{\lambda}.$$

$$kR = 2\pi\frac{R}{\lambda} \ll 1,$$

When $R \ll \lambda$ (or slow variation), $e^{-jkR} \to 1$

The solutions for V and $\mathbf{A}$ simplify to the static expressions.

$$\boxed{V(R) = \frac{1}{4\pi\epsilon}\int_{V'} \frac{\rho e^{-jkR}}{R}\,dv' \qquad \text{(V)},}$$

$$\boxed{\mathbf{A}(R) = \frac{\mu}{4\pi}\int_{V'} \frac{\mathbf{J} e^{-jkR}}{R}\,dv' \qquad \text{(Wb/m)}.}$$

Procedure for Determining **E** and **H** due to Time-harmonic ρ and J

- 1. Find V and **A** $$V(R) = \frac{1}{4\pi\epsilon}\int_{V'} \frac{\rho e^{-jkR}}{R}\,dv' \qquad \mathbf{A}(R) = \frac{\mu}{4\pi}\int_{V'} \frac{\mathbf{J} e^{-jkR}}{R}\,dv'$$
- 2. Find **E** and **B** $$\mathbf{E}(R) = -\nabla V - j\omega \mathbf{A} \qquad \mathbf{B}(R) = \nabla \times \mathbf{A}.$$
- 3. Find instantaneous **E**(t) and **B**(t) $$\mathbf{E}(R,t) = \mathcal{R}e[\mathbf{E}(R)e^{j\omega t}] \qquad \mathbf{B}(R,t) = \mathcal{R}e[\mathbf{B}(R)e^{j\omega t}]$$

7-7.3 Source-Free Fields in Simple Media

- In a simple, nonconducting source-free medium: $\rho = 0$, $\mathbf{J} = 0$, $\sigma = 0$

Method 1

$$\boxed{\begin{aligned} \nabla \times \mathbf{E} &= -j\omega\mu\mathbf{H}, \\ \nabla \times \mathbf{H} &= \mathbf{J} + j\omega\epsilon\mathbf{E}, \\ \nabla \cdot \mathbf{E} &= \rho/\epsilon, \\ \nabla \cdot \mathbf{H} &= 0. \end{aligned}}$$

$$\begin{aligned} \nabla \times \tilde{\mathbf{E}} &= -j\omega\mu\tilde{\mathbf{H}}, \\ \nabla \times \tilde{\mathbf{H}} &= j\omega\epsilon\tilde{\mathbf{E}}, \\ \nabla \cdot \tilde{\mathbf{E}} &= 0, \\ \nabla \cdot \tilde{\mathbf{H}} &= 0. \end{aligned}$$

Method 2

$$\nabla^2\mathbf{E} - \frac{1}{u^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 0.$$

$$\nabla^2\mathbf{H} - \frac{1}{u^2}\frac{\partial^2\mathbf{H}}{\partial t^2} = 0.$$

$$\boxed{\nabla^2\tilde{\mathbf{E}} + k^2\tilde{\mathbf{E}} = 0}$$

$$\boxed{\nabla^2\tilde{\mathbf{H}} + k^2\tilde{\mathbf{H}} = 0,}$$

Homogeneous vector Helmholtz's equations

phasor wave eqs. for $\tilde{\vec{E}}$, $\tilde{\vec{H}}$

- If the medium is conducting ($\sigma \neq 0$), $\mathbf{J} = \sigma\mathbf{E} \neq 0$, Equation with **J** should be changed.

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H},$$
$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E}, + J$$
$$\nabla \cdot \mathbf{E} = 0,$$
$$\nabla \cdot \mathbf{H} = 0.$$

$$\nabla \times \mathbf{H} = (\sigma + j\omega\epsilon)\mathbf{E}$$

$$\nabla \times \mathbf{H} = (\sigma + j\omega\epsilon)\mathbf{E} = j\omega\left(\epsilon + \frac{\sigma}{j\omega}\right)\mathbf{E}$$
$$= j\omega\epsilon_c\mathbf{E}$$

where $\epsilon_c = \epsilon - j\dfrac{\sigma}{\omega}$ (F/m).

Complex permittivity

$$\boxed{\nabla \times \mathbf{H} = j\omega\epsilon_c\mathbf{E}}$$

If complex permittivity ε_c is used, all the previous equations for nonconducting media can be applied to conducting media.

Loss

- Damping loss: due to out-of-phase polarization
  - **E** is too quick, **P** is out of phase to **E**
- Ohmic loss: due to free charge carries
- The damping and ohmic losses can be characterized in the imaginary part of a complex permittivity ε_c (Chap. 8):
  - For an appreciable amount of free charge carriers, ohmic losses dominate and damping losses are very small and already neglected

$$\epsilon_c = \epsilon' - j\epsilon'' \quad \text{(F/m)},$$

Comparing $\boxed{\epsilon_c = \epsilon - j\frac{\sigma}{\omega} \quad \text{(F/m)}.}$

$$\sigma = \omega\epsilon'' \quad \text{(S/m)}.$$

ϵ' : real part

ϵ'' : Imag. part $\begin{cases} \text{damping} : \times \\ \text{ohmic} : \underline{\epsilon'' = \frac{\sigma}{\omega}} \end{cases}$

- Low-loss or lossless media: $\varepsilon_c = \varepsilon'$
- Lossy media: $\varepsilon_c = \varepsilon' - j\varepsilon''$

$$k_c = \omega\sqrt{\mu\epsilon_c} = \omega\sqrt{\mu(\epsilon' - j\epsilon'')}$$

The real wavenumber k should be changed to a complex wavenumber k_c in a lossy dielectric medium

- Loss tangent: a measure of power loss

$$\tan\delta_c = \frac{\epsilon''}{\epsilon'} \cong \frac{\sigma}{\omega\epsilon}.$$

if ohmic loss dominates

δ_c : loss angle

A good conductor and a good insulator

- A good conductor: σ>>ωε $\varepsilon'' \gg \varepsilon'$
- A good insulator: ωε>>σ $\varepsilon' \gg \varepsilon''$
- Thus, a material may be a good conductor at low frequencies but may have the properties of a lossy dielectric at very high frequencies.

  E.g., moist ground is a relatively good conductor at low frequency and behaves more like an insulator at high frequency.

$$\boxed{\epsilon_c = \epsilon - j\frac{\sigma}{\omega} \qquad \text{(F/m)}.}$$

7-7.4 The Electromagnetic Spectrum

- Maxwell's equations, and therefore the wave and Helmholtz's equations, impose no limit on the frequency of the waves.
- All electromagnetic waves in whatever frequency range propagate in a medium with the same velocity: $u = 1/\sqrt{\mu\epsilon}$ ($c \cong 3 \times 10^8$ m/s in air).

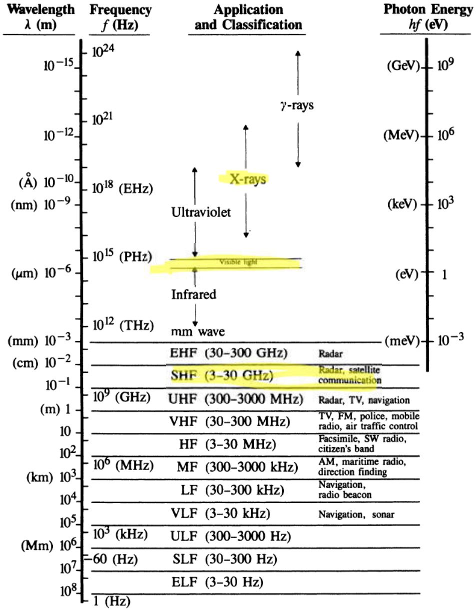


FIGURE 7–9
Spectrum of electromagnetic waves.

TABLE 7–5
Band Designations for Microwave Frequency Ranges

Old†	New	Frequency Ranges (GHz)
Ka	K	26.5–40
K	K	20–26.5
K	J	18–20
Ku	J	12.4–18
X	J	10–12.4
X	I	8–10
C	H	6–8
C	G	4–6
S	F	3–4
S	E	2–3
L	D	1–2
UHF	C	0.5–1

EXAMPLE 7–7 Show that if (**E**, **H**) are solutions of source-free Maxwell's equations in a simple medium characterized by ϵ and μ, then so also are (**E**′, **H**′), where

$$\mathbf{E}' = \eta \mathbf{H} \tag{7–107a}$$

$$\mathbf{H}' = -\frac{\mathbf{E}}{\eta}. \tag{7–107b}$$

In the above equations, $\eta = \sqrt{\mu/\epsilon}$ is called the ***intrinsic impedance*** of the medium.