VE230 Electromagnetics

Chapter 3

September 11, 2022

Exercise 3.1

Refer to Fig. 3-4.

a) Find the relation between the angle of arrival, α, of the electron beam at the screen and the deflecting electric field intensity E_{d}.

b) Find the relation between w and L such that $d_1 = d_0/20$.

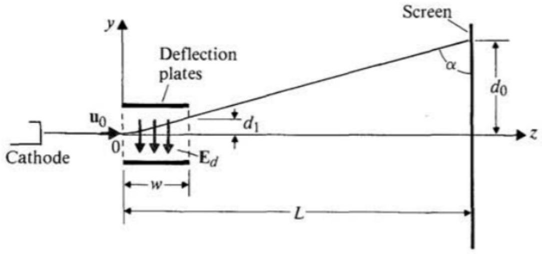


FIGURE 3–4
Electrostatic deflection system of a cathode-ray oscilloscope (Example 3–3)

Answer:

(a)

$$\alpha = \tan^{-1}\left(\frac{L-w}{d_0-d_1}\right) = \tan^{-1}\left(\frac{L-w}{d_2}\right) = \tan^{-1}\left(\frac{mu_0^2}{ewE_d}\right)$$

(b)

$$d_1 = \frac{d_0}{20} \longrightarrow \frac{eE_d}{2m}\left(\frac{w}{u_0}\right)^2 = \frac{1}{20}\frac{eE_d}{mu_0^2}w\left(L-\frac{w}{2}\right)$$
$$L/w = 10.5$$

Exercise 3.2

The cathode-ray oscilloscope (CRO) shown in Fig. 3-4 is used to measure the voltage applied to the parallel deflection plates.

a) Assuming no breakdown in insulation, what is the maximum voltage that can be measured if the distance of separation between the plates is h ?

b) What is the restriction on L if the diameter of the screen is D ?

c) What can be done with a fixed geometry to double the CRO's maximum measurable voltage?

Answer:

a) Max. voltage V max. will make $d_1 = h/2$ at $z = w$. $\frac{h}{2} = \frac{e}{2m}\left(\frac{V_{max}}{h}\right)\left(\frac{w}{u_0}\right)^2 \longrightarrow$ $V_{\max} = \frac{m}{e}\left(\frac{u_0 h}{v_v}\right)^2$.

b) At the screen, $(d_0)_{\max} = D/2$. L mast be $\leq L_{\max}$, Where

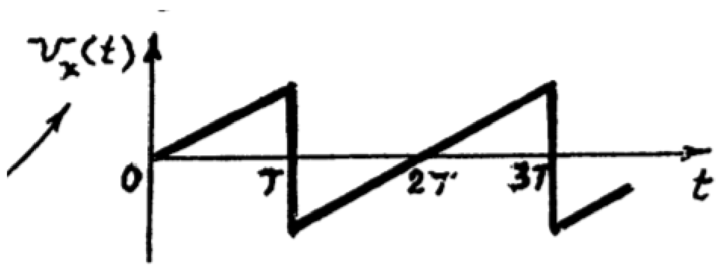
$$L_{\max} = \frac{1}{2}\left(w + \frac{mu_0^2 Dh}{ewV_{\max}}\right).$$

c) Double V_{max} by doubling u_0^2, or doubling the anode accelerating voltage.

Exercise 3.3

The deflection system of a cathode-ray oscilloscope usually consists of two pairs of parallel plates producing orthogonal electric fields. Assume the presence of another set of plates in Fig. 3-4 that establishes a uniform electric field $\mathbf{E}_x = \mathbf{a}_x E_x$ in the deflection region. Deflection voltages $v_x(t)$ and $v_y(t)$ are applied to produce $\mathbf{E}_x$ and $\mathbf{E}_y$, respectively. Determine the types of waveforms that $v_x(t)$ and $v_y(t)$ should have if the electrons are to trace the following graphs on the fluorescent screen:
a) a horizontal line,
b) a straight line having a negative unity slope,
c) a circle,
d) two cycles of a sine wave.

Answer:
a) $v_x(t) = V_0 \cos\omega t\,(\text{ or } V_0 \sin\omega t)\,, v_y = 0$.
b) $v_x(t) = V_0 \cos\omega t, v_y(t) = -V_0 \cos\omega t$.
c) $v_x(t) = V_0 \cos\omega t, v_y(t) = V_0 \sin\omega t$.
d) $v_y(t) = V_0 \sin\frac{2\pi}{T}t$,
$v_x(t)$: Periodic saw-tooth wave with period $2T$.



Exercise 3.4

Write a short article explaining the principle of operation of xerography. (Use library resources if needed.)

Answer:
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Exercise 3.5

Two point charges, Q_1 and Q_2, are located at $(1, 2, 0)$ and $(2, 0, 0)$, respectively. Find the relation between Q_1 and Q_2 such that the total force on a test charge at the point $P(-1, 1, 0)$ will have

a) no x-component,
b) no y-component.

Answer:

$$\overrightarrow{Q_1P} = -\bar{a}_x 2 - \bar{a}_y; \quad \overrightarrow{Q_2P} = -\bar{a}_x 3 + \bar{a}_y$$

$$\bar{E}_{p_1} = \frac{Q_1}{4\pi\epsilon_0(\sqrt{5})^3}\left(-\bar{a}_x 2 - \bar{a}_y\right); \bar{E}_{p_2} = \frac{Q_2}{4\pi\varepsilon_0(\sqrt{10})^3}\left(-\bar{a}_x 3 + \bar{a}_y\right)$$

a) No x-component of $\bar{E}_p$: $-\frac{2Q_1}{(\sqrt{5})^3} - \frac{3Q_2}{(\sqrt{10})^3} = 0$, or $\frac{Q_1}{Q_2} = -\frac{3}{4\sqrt{2}}$.
b) No y-component of $\bar{E}_p$: $-\frac{Q_1}{(\sqrt{5})^3} + \frac{Q_2}{(\sqrt{10})^3} = 0$, or $\frac{Q_1}{Q_2} = \frac{1}{2\sqrt{2}}$.

Exercise 3.6

Two very small conducting spheres, each of a mass $1.0 \times 10^{-4}(\text{ kg})$, are suspended at a common point by very thin nonconducting threads of a length $0.2(\text{ m})$. A charge Q is placed on each sphere. The electric force of repulsion separates the spheres, and an equilibrium is reached when the suspending threads make an angle of $10°$. Assuming a gravitational force of $9.80(\text{ N/kg})$ and a negligible mass for the threads, find Q

Answer:
At equilibrium, electric force F_e and gravitational force F_9 must add to give a resultant along the thread.

$$\frac{F_e}{F_g} = \tan 5° = 0.0875$$

$$F_g = mg = 9.80 \times 10^{-4}(\text{N})$$

$$F_e = \frac{Q^2}{4\pi\varepsilon_0\left(2 \times 0.2 \sin 5°\right)^2} = 7.41 \times 10^{-12} Q^2 (N)$$

$$\longrightarrow Q = 3.40(\text{nC})$$

Exercise 3.7

Find the force between a charged circular loop of radius b and uniform charge density ρ_ℓ and a point charge Q located on the loop axis at a distance h from the plane of the loop. What is the force when $h \gg b$, and when $h = 0$? Plot the force as a function of h.

Answer:

$$dV_Q = \frac{\rho_l b d\phi'}{4\pi E_0\left(z^2 + b^2\right)^{1/2}}.$$

$$V_Q = \frac{\rho_l b}{4\pi\varepsilon_0\left(z^2 + b^2\right)^{1/2}} \int_0^{2\pi} d\phi' = \frac{\rho_l b}{2E_0\left(z^2 + b^2\right)^{1/2}}$$

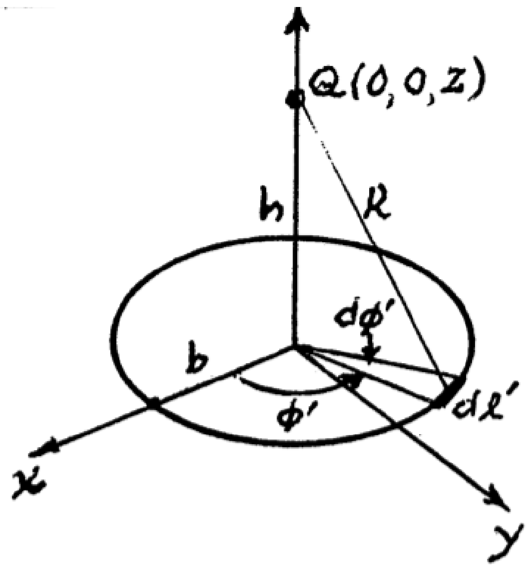
$$E_Q = -\bar{\nabla} V = -\bar{a}_z \frac{dV_a}{dz} = a_z \frac{\rho_l b z}{2\varepsilon_0\left(z^2 + b^2\right)^{3/2}}$$

$$\bar{F}_Q = Q\bar{E}_Q = \bar{a}_z \frac{Q\rho_l b z}{2E_0\left(z^2 + b^2\right)^{3/2}}.$$

(Direction depending on whether Q is above or below the loop.)

When $h \gg b$, $\left|\bar{F}_Q\right| \cong \frac{QP_l b}{2\epsilon_0 h^2}$ (inverse-square law).

When $h = 0$, $\quad \bar{F}_Q = 0$, $\quad$ Max. $\left|\bar{F}_Q\right|$ occurs at $\frac{z}{b} = \frac{1}{\sqrt{2}}$.

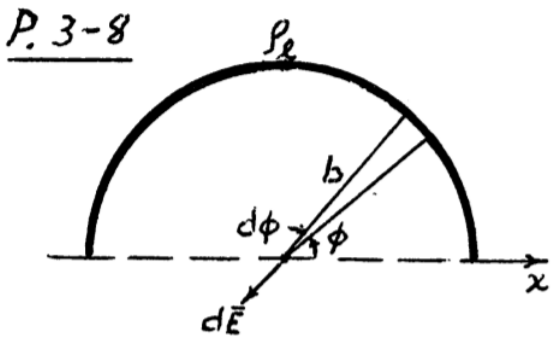


Exercise 3.8

A line charge of uniform density ρ_ℓ in free space forms a semicircle of radius b. Determine the magnitude and direction of the electric field intensity at the center of the semicircle.

Answer:

$$\begin{aligned} dE_y &= -\frac{\rho_l(bd\phi)}{4\cdot\pi\epsilon_0 b^2}\sin\phi_1 \\ E = \bar{a}_y E_y &= -\bar{a}_y\frac{\rho_l}{4\pi\epsilon_0 b}\int_0^\pi \sin\phi d\phi \\ &= -\bar{a}_y\frac{\rho_l}{2\pi\epsilon_0 b} \end{aligned}$$



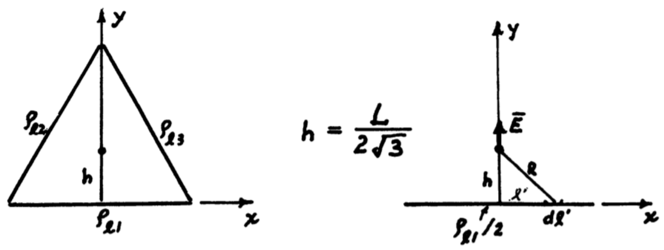
Exercise 3.9

Three uniform line charges $-\rho_{\ell 1}, \rho_{\ell 2}$, and $\rho_{\ell 3}$, each of length $L-$ form an equilateral triangle. Assuming that $\rho_{\ell 1}=2\rho_{\ell 2}=2\rho_{\ell 3}$, determine the electric field intensity at the center of the triangle.

Answer:

E at the center of the triangle would be zero it all three line charges were of the same charge density. The problem is equivalent to that of a single line charge of density $\mathrm{P}_l, 1/2$. By symmetry, there will only be a y-component

$$\bar{E}=\bar{a}_y E_y=\bar{a}_y \int_{-L/2}^{L/2} \frac{\left(P_{l_1}/2\right) dl'}{4\pi\epsilon_0 R^3}\left(\frac{h}{R}\right)=\bar{a}_y \int_{-L/2}^{L/2} \frac{\rho_{l1} h dl}{8\pi\epsilon_0\left(h^2+l'^2\right)^{3/2}}$$
$$=\bar{a}_y \frac{3\rho_{\ell_1}}{4\pi\epsilon_0 L}=\bar{a}_y \frac{3\rho_{\ell_2}}{2\pi\epsilon_0 L}.$$



Exercise 3.10

Assuming that the electric field intensity is $\mathbf{E}=\mathbf{a}_x 100x(\mathrm{~V/m})$, find the total electric charge contained inside

a) a cubical volume 100(mm) on a side centered symmetrically at the origin,

b) a cylindrical volume around the z-axis having a radius 50(mm) and a height 100(mm) centered at the origin.

Answer:

Use Gauss's law: $\oint_s \bar{E}\cdot d\bar{s}=Q/\epsilon_0$.

a) E is normal to the two faces at $x=\pm 0.05(\mathrm{~m})$, where $\bar{E}=\pm\bar{a}_x 5$ and $\bar{a}_n=\pm\bar{a}_x$ respectively.

$$Q=2\epsilon_0\left(5\times 0.1^2\right)=0.1\epsilon_0=8.84\times 10^{-13}(C)$$

b) $E=\bar{a}_r(100x)\cos\phi-\bar{a}_\phi(100x)\sin\phi=\bar{a}_r\left(100r\cos^2\phi\right)-\bar{a}_\phi\left(100r\sin^2\phi\right)$.

$$\oint_s E\cdot\bar{a}_n ds=\int_0^{2\pi}\left(1\,\mathrm{co}\times 0.05\cos^2\phi\right)(0.1\times 0.05)d\phi=0.025\pi.$$
$$Q=0.025\pi\epsilon_0=6.44\times 10^{-13}(C).$$

Exercise 3.11

A spherical distribution of charge $\rho=\rho_0\left[1-\left(R^2/b^2\right)\right]$ exists in the region $0\leq R\leq b$. This charge distribution is concentrically surrounded by a conducting shell with inner radius $R_i(>b)$ and outer radius R_0. Determine $\mathbf{E}$ everywhere.

Answer:
Spherical symmetry: $\bar{E} = \bar{a}_R E_R$. Apply Gauss's law.
1) $0 \leqslant R \leqslant b$. $\quad 4\pi R^2 E_{R1} = \frac{\rho_0}{\epsilon_0} \int_0^R \left(1 - \frac{R^2}{b^2}\right) + \pi R^2 dR = \frac{4\pi\rho_0}{\epsilon_0} \left(\frac{R^3}{3} - \frac{R^5}{5b^2}\right)$.

$$E_{R1} = \frac{\rho_0}{\epsilon_0} R \left(\frac{1}{3} - \frac{R^2}{5b^2}\right)$$

2) $b \geqslant R < R_i$. $\quad 4\pi R^2 = \frac{\rho_0}{\epsilon_0} \int_0^b \left(1 - \frac{R^2}{b^2}\right) 4\pi R^2 dR = \frac{5\pi\rho_0}{15\epsilon_0} b^3$,

$$E_{R_2} = \frac{2\rho_0 b^3}{15 \cdot \epsilon_0 R^2}$$

3)

$$R_i < R < R_0. \quad E_{R3} = 0$$

4) $R > R_0$.

$$E_{R4} = \frac{2\rho_0 b^3}{15\epsilon_0 R^2}$$

Exercise 3.12

Two infinitely long coaxial cylindrical surfaces, $r = a$ and $r = b(b > a)$, carry surface charge densities ρ_{sa} and ρ_{sb} , respectively.
a) Determine $\mathbf{E}$ everywhere.
b) What must be the relation between a and b in order that $\mathbf{E}$ vanishes for $r > b$?

Answer:
Cylindrical symmetry: $\bar{E} = \bar{a}_r E_r$. Apply Gauss's law.
a)

$$r < a, E_r = 0$$

$$a < r < b, E_r = a\rho_{sa}/\epsilon_0 r$$

$$r > b, E_r = \frac{a\rho_{sa} + b\rho_{sb}}{\epsilon_0 r}$$

b)

$$\frac{b}{a} = -\frac{\rho_{sa}}{\rho_{sb}}$$

Exercise 3.13

Determine the work done in carrying a $-2(\mu\text{C})$ charge from $P_1(2, 1, -1)$ to $P_2(8, 2, -1)$ in the field $\mathbf{E} = \mathbf{a}_x y + \mathbf{a}_y x$
a) along the parabola $x = 2y^2$
b) along the straight line joining P_1 and P_2.

Answer:
$W = -q \int \bar{E} \cdot d\bar{l} = -q \int (ydx + xdy)$.
a) Along the parabola $x = 2y^2$, $dx = 4ydy$.

$$W = -q \int_1^2 6y^2 dy = -14q = 28(\mu J).$$

b) Along the straight line $x = 6y - 4$, $dx = 6dy$.

$$W = -q \int_1^2 (12y - 4)dy = -14q = 28(\mu J).$$

Exercise 3.14

At what values of θ does the electric field intensity of a z-directed dipole have no z-component?

Answer:
Refer to

$$\begin{aligned}\mathbf{E} &= -\nabla V = -\mathbf{a}_R \frac{\partial V}{\partial R} - \mathbf{a}_\theta \frac{\partial V}{R\partial\theta} \\ &= \frac{p}{4\pi\epsilon_0 R^3}\left(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta \sin\theta\right)\end{aligned}$$

E will have no z - component if $E_R \cos\theta = E_\theta \sin\theta$, or $2\cos^2\theta = \sin^2\theta$. $\longrightarrow \theta = 54.7°$ and $125.3°$.

Exercise 3.15

Three charges ($+q, -2q$, and $+q$) are arranged along the z-axis at $z = d/2, z = 0$, and $z = -d/2$, respectively.
a) Determine V and $\mathbf{E}$ at a distant point $P(R, \theta, \phi)$.
b) Find the equations for equipotential surfaces and streamlines.
c) Sketch a family of equipotential lines and streamlines.
(Such an arrangement of three charges is called a linear electrostatic quadrupole.)

Answer:

$$V = \frac{q}{4\pi\epsilon_0 R}\left(\frac{R}{R_1} + \frac{R}{R_2} - 2\right).$$

$$R_1^2 = R^2 + \left(\frac{d}{2}\right)^2 - Rd\cos\theta$$

$$\begin{aligned}\frac{R}{R_1} &\cong \left[1 + \left(\frac{d}{2R}\right)^2 - \frac{d}{R}\cos\theta\right]^{-1/2} \\ &\cong 1 + \frac{d}{2R}\cos\theta + \frac{d^2}{4R^2}\frac{3\cos^2\theta - 1}{2}, \\ \frac{R}{R_2} &\cong 1 - \frac{d}{2R}\cos\theta + \frac{d^2}{4R^2}\frac{3\cos^2\theta - 1}{2}\end{aligned}$$

$$V = \frac{gd^2}{16\pi\epsilon_0 R^3}\left(3\cos^2\theta - 1\right), \quad R^3 \gg d^3.$$

$$\begin{aligned}\text{a) } \bar{E} &= -\bar{\nabla} V = -\bar{a}_R \frac{\partial V}{\partial R} - \overline{a_\theta} \cdot \frac{\partial V}{R\theta} \\ &= \frac{3qd^2}{16\pi\epsilon_0 R^4}\left[\bar{a}_R\left(3\cos^2\theta - 1\right) + \bar{a}_\theta \sin 2\theta\right].\end{aligned}$$

b) Equation for equipotential surfaces: $R^3 = c_V\left(3\cos^2\theta - 1\right)$.
Equation for streamlines:

$$\frac{dR}{E_R} = \frac{Rd\theta}{E_\theta}$$

or

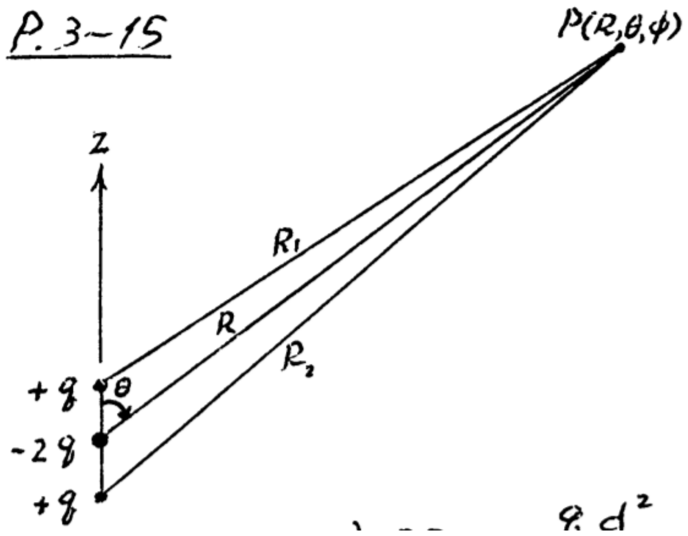
$$\frac{dR}{3\cos^2\theta - 1} = \frac{Rd\theta}{\sin 2\theta}$$

$$\frac{dR}{R} = \frac{3}{2}\frac{d(\sin\theta)}{\sin\theta} - \frac{d\theta}{\sin 2\theta}$$

$$\longrightarrow R = c'_E \frac{\sin^{3/2}\theta}{\sqrt{|\tan\theta|}}$$

or

$$R^2 = c_E \sin^2\theta\cos\theta$$



Exercise 3.16

A finite line charge of length L carrying uniform line charge density ρ_ℓ is coincident with the x-axis.

a) Determine V in the plane bisecting the line charge.

b) Determine $\mathbf{E}$ from ρ_ℓ directly by applying Coulomb's law.

c) Check the answer in part (b) with $-\nabla V$.

Answer:

a)

$$\begin{aligned} V =& 2\int_0^{L/2} \frac{\rho_l dx}{4\pi\epsilon_0 R} \\ =& \frac{\rho_l}{2\pi\epsilon_0}\int_0^{L/2} \frac{dx}{\sqrt{x^2+y^2}} \\ =& \frac{\rho_l}{2\pi\epsilon_0}\{\ln[\sqrt{(\frac{L}{2})^2 + y^2} + \frac{L}{2}] - \ln y\}. \end{aligned}$$

b) From Coulomb's law:

$$\bar{E} = \bar{a}_y E_y = 2\int_0^{L/2} \bar{\alpha}_y \frac{\rho_l y dx}{4\pi\epsilon_0 R^3} := \bar{a}_y \frac{\rho_l}{2\pi\epsilon_0 y}\left[\frac{L/2}{\sqrt{(L/2)^2 + y^2}}\right]$$

c) $\bar{E} = -\bar{\nabla} V$ gives the same answer as in b).

Exercise 3.17

In Example 3-5 we obtained the electric field intensity around an infinitely long line charge of a uniform charge density in a very simple manner by applying Gauss's law. Since $|\mathbf{E}|$ is a function of r only, any coaxial cylinder around the infinite line charge is an equipotential surface. In practice, all conductors are of finite length. A finite line charge carrying a constant charge density ρ_ℓ along the axis, however, does not produce a constant potential on a concentric cylindrical surface. Given the finite line charge ρ_ℓ of length L in Fig. 3-40, find the potential on the cylindrical surface of radius b as a function of x and plot it.

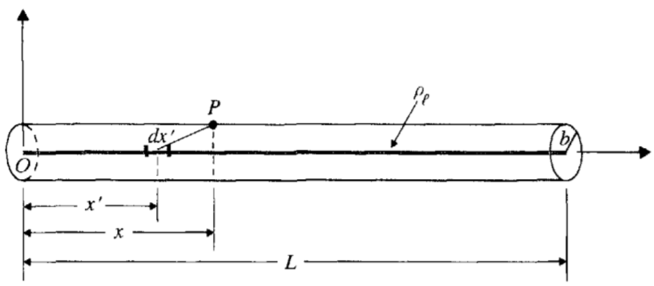


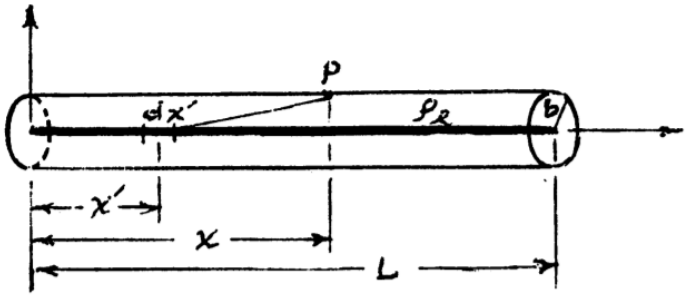
FIGURE 3–40
A finite line charge (Problem P.3–17).

Answer:

$$dV_P(x) = \frac{\rho_l dx'}{4\pi\epsilon_0\sqrt{(x-x')^2 + b^2}}$$

$$V_P(x) = \frac{\rho_l}{4\pi\varepsilon_0}\int_0^L \frac{dx}{\sqrt{(x-x')^2 + b^2}}$$

$$= \frac{\rho_l}{4\pi\epsilon_0}\left[\sinh^{-1}\left(\frac{L-x}{b}\right) + \sinh^{-1}\left(\frac{x}{b}\right)\right]$$



Exercise 3.18

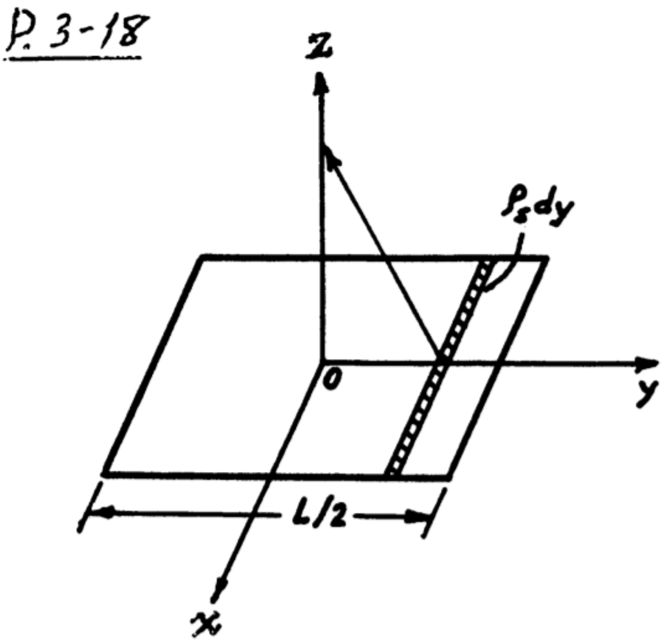
A charge Q is distributed uniformly over an $L \times L$ square plate. Determine V and $\mathbf{E}$ at a point on the axis perpendicular to the plate and through its center.

Answer:

$$V = 2 \cdot \frac{\rho_s}{2\pi\epsilon_0} \int_0^{L/2} \left\{ \ln \left[\sqrt{\left(\frac{L}{2}\right)^2 + y^2 + z^2} + \left(\frac{L}{2}\right) \right] - \ln \sqrt{y^2 + z^2} \right\} dy$$

$$= \frac{Q}{\pi\epsilon_0 L^2} \left\{ \frac{L}{2} \ln \left[\frac{\sqrt{2\left(\frac{L}{2}\right)^2 + z^2} + \frac{L}{2}}{\sqrt{2\left(\frac{L}{2}\right)^2 + z^2} - \frac{L}{2}} \right] - z \tan^{-1} \left[\frac{\left(\frac{L}{2}\right)^2}{z\sqrt{2\left(\frac{L}{2}\right)^2 + z^2}} \right] \right\}$$

$$\bar{E} = -\bar{\nabla} V = \bar{a}_z \frac{Q}{\pi\epsilon_0 L^2} \tan^{-1} \left[\frac{\left(\frac{L}{2}\right)^2}{2\sqrt{2\left(\frac{L}{2}\right)^2 + z^2}} \right]$$



Exercise 3.19

A charge Q is distributed uniformly over the wall of a circular tube of radius b and height h. Determine V and $\mathbf{E}$ on its axis

a) at a point outside the tube, then

b) at a point inside the tube.

Answer:

Assume the circular tube sits on the xy-plane with its axis coinciding with the z-axis. The surface charge on the tube wall is $\rho_s = Q/2\pi bh$. First find the potential along the axis at z due to a circular line. charge of density situated at z'.

$$V = \oint \frac{\rho_l dl}{4\pi\epsilon_0 R} = \int_0^{2\pi} \frac{\rho_l b d\phi'}{4\pi\epsilon_0 \sqrt{b^2 + (z - z')^2}} = \frac{\rho_l b}{2\epsilon_0 \sqrt{b^2 + (z - z')^2}}.$$

a) The expression above is the contribution dV due to a circular line charge of density $\rho_l = \rho_s dz^{\text{in}}$ at $z^{'}$

$$dV = \frac{\rho_s b dz'}{2\epsilon_0 \sqrt{b^2 + (z - z')^2}}.$$

At a point outside the tube, $z > h$:

$$V_0 = \int_{z'=0}^{z'=h} dV = \frac{b\rho_s}{2\epsilon_0} \ln \frac{z + \sqrt{b^2 + z^2}}{(z - h) + \sqrt{b^2 + (z - h)^2}}.$$

$$\bar{E}_0 = -\bar{a}_z \frac{dV}{dz} = \bar{a}_z \frac{b\rho_s}{2\epsilon_0} \left[\frac{1}{\sqrt{b^2 + (z - h)^2}} - \frac{1}{\sqrt{b^2 + z^2}} \right].$$

b) At a point inside the tube, $z < h$:

$$V_i = \frac{\rho_s b}{2\epsilon_0} \left[\int_0^z \frac{dz'}{\sqrt{b^2 + (z - z')^2}} + \int_z^h \frac{dz'}{\sqrt{b^2 + (z' - z)^2}} \right]$$
$$= \frac{\rho_s b}{2\epsilon_0} \ln \frac{1}{b^2} \left(z + \sqrt{b^2 + z^2} \right) \left[(h - z) + \sqrt{b^2 + (h - z)^2} \right].$$

$\bar{E}_i$ has the-same expression as $\bar{E}_0$.

Exercise 3.20

An early model of the atomic structure of a chemical element was that the atom was a spherical cloud of uniformly distributed positive charge Ne, where N is the atomic number and e is the magnitude of electronic charge. Electrons, each carrying a negative charge $-e$, were considered to be imbedded in the cloud. Assuming the spherical charge cloud to have a radius R_0 and neglecting collision effects,

a) find the force experienced by an imbedded electron at a distance r from the center;

b) describe the motion of the electron;

c) explain why this atomic model is unsatisfactory.

Answer:

Positive charge Ne uniformly distributed over a sphere of radius $R_0 : \rho = \frac{Ne}{\frac{4}{3}\pi R_0^3} = \frac{3Ne}{4\pi R_0^3}$.
Inside the sphere (applying Gauss's law): $\bar{E} = \bar{a}_R E_R = \bar{a}_R \frac{Ner}{4\pi\epsilon_0 R_0^3}$.
a) Force experienced by an electron $-e : \bar{F} = -\overline{a_R} \frac{Ne^2 r}{4\pi\epsilon_0 R_0^3}$.
b) Equation of motion for an electron with mass m :

$$m\frac{d^2r}{dt^2} = -\frac{Ne^2 r}{4\pi\epsilon_0 R_0^3},$$

or

$$\frac{d^2r}{dt^2} + \left(\frac{Ne^2}{4\pi\epsilon_0 m R_0^3}\right) r = 0$$

or

$$\frac{d^2r}{dt^2} + \omega_e^2 r = 0$$

where

$$\omega_e = \sqrt{\frac{Ne^2}{4\pi\epsilon_0 m R_0^3}}$$

Hence the electrons would undergo a simple harmonic motion with an angular frequency ω_e.

c) The oscillating electrons would lase power through radiation and lead to an unstable atomic model.

Exercise 3.21

A simple classical model of an atom consists of a nucleus of a positive charge Ne surrounded by a spherical electron cloud of the same total negative charge. (N is the atomic number and e is the magnitude of electronic charge.) An external electric field $\mathbf{E}_o$ will cause the nucleus to be displaced a distance r_o from the center of the electron cloud, thus polarizing the atom. Assuming a uniform charge distribution within the electron cloud of radius b, find r_o

Answer:
Applied E_0 causes a displacement r_0.
Force of separation: qE_0
Restoring force(attraction): qE_x
E_x at q due to spherical volume of electrons- of radius r_0 is (by Gauss's law)

$$E_x = \frac{\rho r_0}{3\epsilon_0} = -\frac{r_0}{3\epsilon_0}|\rho|$$

where

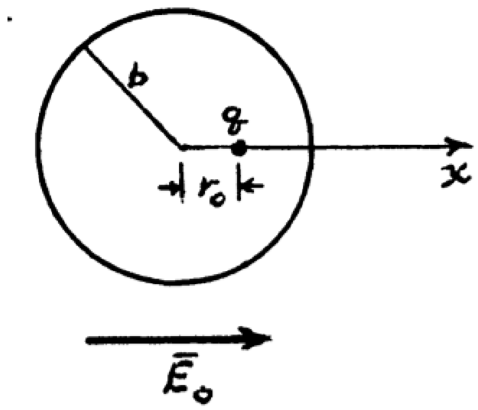
$$|\rho| = \frac{q}{\frac{4}{3}\pi b^3} = \frac{3\text{Ne}}{4\pi b^3}$$

At equilibrium:

$$E_0 = |E_x| = \frac{r_0 Ne}{4\pi\epsilon_0 b^3}$$

or

$$r_0 = \frac{4\pi\epsilon_0 b^3}{Ne} E_0$$



Exercise 3.22

The polarization in a dielectric cube of side L centered at the origin is given by $\mathbf{P} = P_o(\mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z)$

a) Determine the surface and volume bound-charge densities.

b) Show that the total bound charge is zero.

Answer:

a) $\rho_{ps} = \bar{p} \cdot \bar{a}_n = P_0 \frac{L}{2}$ on all six faces of the-cube.

$\rho_p = -\bar{\nabla} \cdot \bar{p} = -3P_0$.

b) $Q_{s'} = (6L^2)\,\rho_{ps} = 3P_0L^3, \quad Q_v = (L^3)\,\rho_{p,} = -3P_0L^3$.

Total bound charge $= Q_s + Q_v = 0$.

Exercise 3.23

Determine the electric field intensity at the center of a small spherical cavity cut out of a large block of dielectric in which a polarization $\mathbf{P}$ exists.

Answer:

Assume $\bar{p} = \bar{a}_z P$. Surface charge density

$$\rho_{ps} = \bar{p} \cdot \bar{a}_n = (\bar{a}_z p) \cdot (-\bar{a}_R)$$

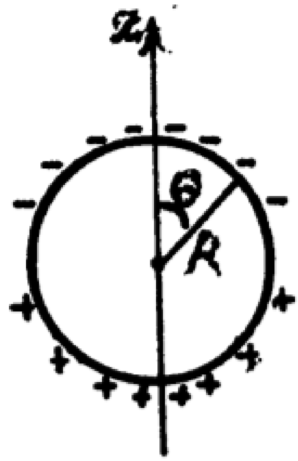
$$= -P\cos\theta.$$

The z-component of the electric field intensity due to a ring of ρ_{ps} contained in width $Rd\theta$ at θ is

$$dE_z = \frac{P\cos\theta}{4\pi\varepsilon_0 R^2}(2\pi R\sin\theta)(Rd\theta)\cos\theta$$
$$= \frac{P}{2\varepsilon_0}\cos^2\theta\sin\theta d\theta.$$

At the center of the cavity: $\bar{E} = \bar{a}_z E_z = \bar{a}_z \frac{\rho}{2\epsilon_0}\int_0^\pi \cos^2\theta\sin\theta d\theta$

$$= \bar{P}/3\epsilon_0.$$



Exercise 3.24

Solve the following problems:

a) Find the breakdown voltage of a parallel-plate capacitor, assuming that conducting plates are 50(mm) apart and the medium between them is air.

b) Find the breakdown voltage if the entire space between the conducting plates is filled with plexiglass, which has a dielectric constant 3 and a dielectric strength 20(kV/mm)

c) If a $10-$(mm) thick plexiglass is inserted between the plates, what is the maximum voltage that can be applied to the plates without a breakdown?

Answer:

a: air
p: plexigiass
b : breakdown
a) $V_b = E_{ba}d_a = 3 \times 50 = 150(\mathrm{kV})$
b) $V_b = E_{bp}d_p = 20 \times 50 = 1,000(kV)$
c) $V_b = E_a d_a + E_p d_p = E_a(50 - d_p) + E_p d_p$
Now, $D_a = D_p \rightarrow \epsilon_0 E_a = \epsilon_0 \epsilon_{rp} E_p \longrightarrow E_p = \frac{E_a}{\epsilon_{rp}} = \frac{E_u}{3} < E_a$
$E_{ba} < E_{bp} \longrightarrow$ Breakdown occurs in air-region first

$$\therefore V_b = E_{ba}(50-10) + \frac{E_{ba}}{3} \times 10 = 3\left(40 + \frac{1}{3} \times 10\right) = 130(\mathrm{kV}).$$

Exercise 3.25

Assume that the $z = 0$ plane separates two lossless dielectric regions with $\epsilon_{r1} = 2$ and $\epsilon_{r2} = 3$. If we know that $\mathbf{E}_1$ in region 1 is $\mathbf{a}_x 2y - \mathbf{a}_y 3x + \mathbf{a}_z(5+z)$, what do we also know about $\mathbf{E}_2$ and $\mathbf{D}_2$ in region 2? Can we determine $\mathbf{E}_2$ and $\mathbf{D}_2$ at any point in region 2? Explain.

Answer:

At the $z = 0$ plane: $\bar{E}_1 = \bar{a}_{x^2}{}^2y - \bar{a}_y \cdot 3x + \bar{a}_y 5$.

$$\bar{E}_{1t}(z=0) = \bar{E}_{2t}(z=0) = \bar{a}_x 2y - \bar{a}_y 3x.$$
$$\bar{D}_{1n}(z=0) = \bar{D}_{2n}(z=0) \longrightarrow 2\bar{E}_{1n}(z=0) = 3\bar{E}_{2n}(z=0),$$
$$\therefore E_{2n}(z=0) = \frac{2}{3}(\bar{a}_z 5) = \bar{a}_z \frac{10}{3}.$$
$$\bar{E}_2(z=0) = \bar{a}_x 2y - \bar{a}_y 3x + \bar{a}_z \frac{10}{3}.$$
$$\bar{D}_2(z=0) = \left(\bar{a}_x 2y - \bar{a}_y 3x + \bar{a}_z \frac{10}{3}\right) 3\epsilon_0.$$

Exercise 3.26

Determine the boundary conditions for the tangential and the normal components of **P** at an interface between two perfect dielectric media with dielectric constants ϵ_{r1} and ϵ_{r2}.

Answer:

$$\bar{P}_t = \epsilon_0 (\epsilon_r - 1) \bar{E}_t. \quad \bar{E}_{t1} = \bar{E}_{t2} \quad \longrightarrow \frac{1}{\epsilon_{r1} - 1} p_{t1} = \frac{1}{\epsilon_{r2} - 1} p_{t2}.$$
$$\epsilon_{r1} \bar{E}_{n1} = \epsilon_{r2} \bar{E}_{n_2} \longrightarrow \frac{\epsilon_{r1}}{\epsilon_{r1} - 1} p_{n1} = \frac{\epsilon_{r2}}{\epsilon_{r_2} - 1} p_{n_2}.$$

Exercise 3.27

What are the boundary conditions that must be satisfied by the electric potential at an interface between two perfect dielectrics with dielectric constants ϵ_{r1} and ϵ_{r2} ?

Answer:

$$\epsilon_1 \frac{\partial V_1}{\partial n} = \epsilon_2 \frac{\partial V_2}{\partial n} \qquad \text{and } V_1 = V_2$$

Exercise 3.28

Dielectric lenses can be used to collimate electromagnetic fields. In Fig. 3-41 the left surface of the lens is that of a circular cylinder, and the right surface is a plane. If $\mathbf{E}_1$ at point $P(r_o, 45°, z)$ in region 1 is $\mathbf{a}_r 5 - \mathbf{a}_\phi 3$, what must be the dielectric constant of the lens in order that $\mathbf{E}_3$ in region 3 is parallel to the x-axis?

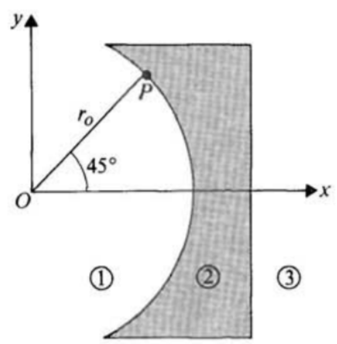


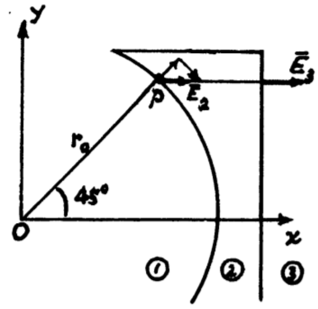
FIGURE 3–41
A dielectric lens (Problem P.3–28).

Answer:
Assume $\bar{E}_2 = \bar{a}_r E_{2r} + \bar{a}_\phi E_{2\phi}$.
Boundary condition: $\bar{a}_n \times \bar{E}_1 = \bar{a}_n \times \bar{E}_2 \longrightarrow E_{2\phi} = -3$.
For E_3, and hence E_2, to be parallel to the x-axis, $E_{2\phi} = -E_{2r}$

$$\to E_{2r} = 3.$$

Boundary condition: $\bar{a}_n \cdot \bar{D}_1 = \bar{a}_n \cdot \bar{D}_2$

$$\longrightarrow \epsilon_1 E_{r1} = \epsilon_2 E_2 \longrightarrow \epsilon_0 5 = \epsilon_0 \epsilon_{r2} 3$$
$$\epsilon_{r2} = \frac{5}{3} = 1.667$$



Exercise 3.29

Refer to Example 3-16. Assuming the same r_i and r_o and requiring the maximum electric field intensities in the insulating materials not to exceed 25% of their dielectric strengths, determine the voltage rating of the coaxial cable
a) if $r_p = 1.75 r_i$
b) if $r_p = 1.35 r_i$
c) Plot the variations of E_r and V versus r for both part (a) and part (b).

Answer:
Given: $r_i = 0.4(\text{ cm}) = 0.004(\text{ m}), r_0 = 0.832(\text{ cm}) = 2.08 r_i$.
25% of the dielectric strength of rubber- ($\epsilon_{rr} = 3.2$) is $0.25 \times 25 \times 10^6 = 6.25 \times 10^6$(v/m).
25% of the dielectric strength of polystyrene ($\epsilon_{r-p} = 2.6$) is $0.25 \times 20 \times 10^6 = 5 \times 10^6$(v/m).
a) $r_p = 1.75 r_i, r_0 = 1.189 r_p$.

$$\text{Max. } E_r = \frac{P_l}{2\pi\epsilon_0}\left(\frac{1}{3.2 r_i}\right) \longrightarrow \frac{\rho_l}{2\pi\epsilon_b} = 6.25 \times 10^6 \times 3.2 r_i = 20 \times 10^6 r_i,$$
$$\text{Max. } E_p = \frac{P_l}{2\pi\epsilon_0}\left(\frac{1}{2.6 r_p}\right) = \frac{\rho_l}{2\pi\epsilon_0}\left(\frac{1}{4.55 r_i}\right) \to \frac{P_l}{2\pi t_0} = 5 \times 10^6 \times 4.55 r_i = 22.75 \times 10^6 r_i.$$
$$\frac{\text{Max. } E_r}{\text{Max. } E_p} = \frac{4.55}{3.2} > \frac{6.25}{5} \longrightarrow \text{Max. } E_r \text{ determines the allowable } P_l/2\pi\epsilon_0.$$

$$\therefore \frac{\rho_l}{2\pi\epsilon_0} = 20 \times 10^6 \times 0.004 = 8 \times 10^4.$$

Voltage rating $V_{\text{max.}} = 8 \times 10^4 \left(\frac{1}{2.6} \ln \frac{r_0}{r_p} + \frac{1}{3.2} \ln \frac{r_p}{r_i}\right)$

$$= 8 \times 10^4 \left(\frac{1}{2.6} \ln 1.189 + \frac{1}{3.2} \ln 1.75\right) = 19.3(kV).$$

b) $r_p = 1.35 r_i, r_0 = 1.54 r_p$

Max, $r_p = \frac{\rho_l}{2\pi\epsilon_0}\left(\frac{1}{2.6 r_p}\right) = \frac{\rho}{2\pi\epsilon_0}\left(\frac{1}{3.51 r_i}\right)$, which determines the allowable $\frac{\rho_l}{2\pi\epsilon_0} = 5 \times 10^6 \times 3.51 r_i = 1.76 \times 10^6 r_i = 7 \times 10^3$.

$$V_{\text{max.}} = 7 \times 10^3 \left(\frac{1}{2.6} \ln 1.54 + \frac{1}{3.2} \ln 1.35\right) = 1.82(\text{ kV }).$$

Exercise 3.30

The space between a parallel-plate capacitor of area S is filled with a dielectric whose permittivity varies linearly from ϵ_1 at one plate ($y = 0$) to ϵ_2 at the other plate ($y = d$). Neglecting fringing effect, find the capacitance.

Answer:

$$\epsilon = \frac{\epsilon_2 - \epsilon_1}{d} y + \epsilon_1$$

Assume Q on plate at $y = d$. $\quad \bar{E} = -\bar{a}_y \frac{\rho_s}{\epsilon} = \frac{\rho_s}{s\left(\frac{\epsilon_2 - \epsilon_1}{d} y + t_1\right)}$

$$V = -\int_{y=0}^{y=d} \bar{E} \cdot d\bar{l} = \frac{Qd \ln\left(\epsilon_2/\epsilon_1\right)}{S\left(\epsilon_2 - \epsilon_1\right)}$$

$$C = \frac{Q}{V} = \frac{S\left(\epsilon_2 - \epsilon_1\right)}{d \ln\left(\epsilon_2/\epsilon_1\right)}$$

Exercise 3.31

Assume that the outer conductor of the cylindrical capacitor in Example $3 - 18$ is grounded and that the inner conductor is maintained at a potential V_0.

a) Find the electric field intensity, $\mathbf{E}(a)$, at the surface of the inner conductor.

b) With the inner radius, b, of the outer conductor fixed, find a so that $E(a)$ is minimized.

c) Find this minimum $E(a)$.

d) Determine the capacitance under the conditions of part (b).

EXAMPLE 3-18 A cylindrical capacitor consists of an inner conductor of radius a and an outer conductor whose inner radius is b. The space between the conductors is filled with a dielectric of permittivity ϵ, and the length of the capacitor is L. Determine the capacitance of this capacitor.

Solution We use cylindrical coordinates for this problem. First we assume charges $+Q$ and $-Q$ on the surface of the inner conductor and the inner surface of the outer conductor, respectively. The $\mathbf{E}$ field in the dielectric can be obtained by applying Gauss's law to a cylindrical Gaussian surface within the dielectric $a < r < b$. (Note that Eq. (3-122) gives only the normal component of the $\mathbf{E}$ field at a conductor surface. Since the conductor surfaces are not planes here, the $\mathbf{E}$ field is not constant in the dielectric and Eq. (3-122) cannot be used to find $\mathbf{E}$ in the $a < r < b$ region.) Referring to Fig. 3-29 and applying Gauss's law, we have

$$\mathbf{E} = \mathbf{a}_r E_r = \mathbf{a}_r \frac{Q}{2\pi\epsilon L r}$$

Again we neglect the fringing effect of the field near the edges of the conductors. The potential difference between the inner and outer conductors is

$$\begin{aligned} V_{ab} &= -\int_{r=b}^{r=a} \mathbf{E} \cdot d\ell = -\int_b^a \left(\mathrm{a}_r \frac{Q}{2\pi\epsilon L r} \right) \cdot (\mathbf{a}_r dr) \\ &= \frac{Q}{2\pi\epsilon L} \ln\left(\frac{b}{a}\right) \end{aligned}$$

Therefore, for a cylindrical capacitor,

$$C = \frac{Q}{V_{ab}} = \frac{2\pi\epsilon L}{\ln\left(\frac{b}{a}\right)}$$

We could not solve this problem from an assumed V_{ab} because the electric field is not uniform between the inner and outer conductors. Thus we would not know how to express $\mathbf{E}$ and Q in terms of V_{ab} until we learned how to solve such a boundary value problem.

Answer:

Let ρ_l be the-lineal charge density on the inner conductor. $E = \bar{a}_r \frac{\rho_R}{2\pi\in r}$.

$$V_0 = -\int_b^a E \cdot d\bar{r} = \frac{\rho_l}{2\pi\epsilon} \ln\left(\frac{b}{a}\right) \longrightarrow \rho_l = \frac{2\pi\epsilon V_0}{\ln(b/a)}.$$

a) $E(a) = \bar{a}_r \frac{V_0}{a \ln(b/a)}$.

b) For a fixed b, the function to be minimized is. $(x = b/a)$ $f(x) = \frac{V_0 x}{b \cdot \ln x}$. Setting $\frac{df(x)}{dx} = 0$ yields $\ln x = 1$, or $x = \frac{b}{a} = e = 2.718$.

c) $\min E(a) = eV_0/b$.

d) $(C' = \frac{\rho_l}{V_0} = \frac{2\pi\epsilon}{\ln(b/a)} = 2\pi\epsilon(F/m)$

Exercise 3.32

The radius of the core and the inner radius of the outer conductor of a very long coaxial transmission line are r_i and r_o, respectively. The space between the conductors is filled with two coaxial layers of dielectrics. The dielectric constants of the dielectrics are ϵ_{r1} for $r_i < r < b$ and ϵ_{r2} for $b < r < r_o$. Determine its capacitance per unit length.

Answer:

$$\bar{D} = \bar{a}_r \frac{\rho_l}{2\pi r} \cdot \bar{E}_i = \bar{a}_r \frac{\rho_l}{2\pi\epsilon_0\epsilon_{r1} r}, \quad r_i < r < b$$

$$\bar{E}_2 = \bar{a}_r \frac{\rho_l}{2\pi\epsilon_0\epsilon_{r2} r}, \quad b < r < r_c$$

$$\begin{aligned} V &= -\int_{r_0}^{r_i} \bar{E} \cdot d\bar{r} = \frac{\rho_l}{2 \cdot \pi\epsilon_0} \left[\frac{1}{\epsilon_{r1}} \ln\left(\frac{b}{r_i}\right) + \frac{1}{\epsilon_{r2}} \ln\left(\frac{r_0}{b}\right) \right] \\ C' &= \frac{\rho_l}{V} = \frac{2\pi\epsilon_0}{\frac{1}{\epsilon_{r1}} \ln\left(\frac{b}{r_i}\right) + \frac{1}{\epsilon_{r2}} \ln\left(\frac{r_c}{b}\right)} \quad \text{(F/m)}. \end{aligned}$$

Exercise 3.33

A cylindrical capacitor of length L consists of coaxial conducting surfaces of radii r_i and r_o. Two dielectric media of different dielectric constants ϵ_{r1} and ϵ_{r2} fill the space between the conducting surfaces as shown in Fig. 3-42. Determine its capacitance.

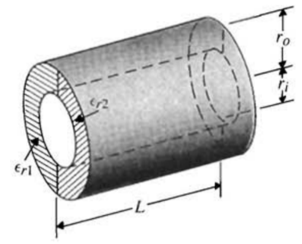


FIGURE 3–42
A cylindrical capacitor with two dielectric media (Problem P.3–33).

Answer:

$$\begin{aligned}
&\text{Gauss's law: } \oint_s \bar{D} \cdot d\bar{s} = \rho_l L \\
&\bar{E}_1 = \bar{E}_2 = \bar{a}_r E_r, \quad \pi r L \left(\epsilon_0 \epsilon_{r1} + \epsilon_0 \epsilon_{r_2}\right) E_r = \rho_l L \\
&\longrightarrow E_r = \frac{\rho_l}{\pi r \epsilon_0 \left(\epsilon_{r1} + \epsilon_{r_2}\right)}; V = -\int_{r_o}^{r_i} E_r dr = \frac{\rho_l}{\epsilon_0 \left(\epsilon_{r1} + \epsilon_{r2}\right)} \ln\left(\frac{r_o}{r_i}\right) \\
&\therefore C = \frac{\rho_l L}{V} = \frac{\pi \epsilon_0 \left(\epsilon_{r1} + \epsilon_{r2}\right) L}{\ln\left(r_o / r_i\right)}.
\end{aligned}$$

Exercise 3.34

A capacitor consists of two coaxial metallic cylindrical surfaces of a length 30(mm) and radii 5(mm) and 7(mm). The dielectric material between the surfaces has a relative permittivity $\epsilon_r = 2 + (4/r)$, where r is measured in mm. Determine the capacitance of the capacitor.

Answer:

$$\begin{aligned}
\bar{E} &= \bar{a}_r \frac{\rho_l}{2\pi \epsilon r} = \bar{a}_r \frac{\rho_l}{2\pi \epsilon_0 \left(2 + \frac{4}{r}\right) r} = \bar{a}_r \frac{\rho_l}{4\pi \epsilon_0 (r+2)} \\
V &= -\int_{r_0}^{r_i} \bar{E} \cdot d\bar{r} = \frac{\rho_l}{4\pi\epsilon_0} \ln(r+2) \Big|_5^7 = \frac{\rho_l}{4\pi\epsilon_0} \ln\left(\frac{9}{7}\right) \\
C &= \frac{\rho_l L}{V} = \frac{4\pi\epsilon_0 L}{\ln(9/7)} = 1500\epsilon_0 = 13.26(\mu F).
\end{aligned}$$

Exercise 3.35

Assuming the earth to be a large conducting sphere (radius $= 6.37 \times 10^3$ km) surrounded by air, find

a) the capacitance of the earth;

b) the maximum charge that can exist on the earth before the air breaks down.

Answer:

a) $C = 4\pi\epsilon_0 R = \frac{1}{9} \times 10^{-9} \times \left(6.37 \times 10^6\right) = 7.08 \times 10^{-4}(\text{ F})$

b) $E_b = 3 \times 10^6 = \frac{Q_{\max}}{4\pi\epsilon_0 R^2} \longrightarrow Q_{\max} = 1.35 \times 10^{10}(C)$

Exercise 3.36

Determine the capacitance of an isolated conducting sphere of radius b that is coated with a dielectric layer of uniform thickness d. The dielectric has an electric susceptibility χ_e.

Answer:
Assume charge Q on conducting sphere,

$$
\begin{aligned}
&b < R < b + d: \quad \bar{E}_1 = \bar{a}_R \frac{Q}{4\pi\epsilon_0 (1+\chi_e) R^2} \\
&R > b + d \quad \bar{E}_2 = \bar{a}_R \frac{Q}{4\pi\epsilon_0 R^2}. \\
V =& -\int_\infty^b \bar{E} \cdot d\bar{l} = -\int_\infty^{b+d} E_2 dR - \int_{b+d}^b E_1 dR = \frac{Q}{4\pi\epsilon_0 (1+\chi_e)} \left(\frac{\chi_e}{b+d} + \frac{1}{b} \right) \\
C =& \frac{Q}{V} = \frac{4\pi\epsilon_0 (1+\chi_e)}{\frac{\chi_e}{b+d} + \frac{1}{b}}.
\end{aligned}
$$

Exercise 3.37

A capacitor consists of two concentric spherical shells of radii R_i and R_o. The space between them is filled with a dielectric of relative permittivity ϵ_r from R_i to $b\,(R_i < b < R_o)$ and another dielectric of relative permittivity $2\epsilon_r$ from b to R_o.
a) Determine **E** and **D** everywhere in terms of an applied voltage V.
b) Determine the capacitance.

Answer:
Assume charge Q in inner shall and $-Q$ on inner- shell. $R_i < R < R_0 : \bar{D} = \bar{a}_R \frac{Q}{4\pi R^2}$.

$$
\begin{aligned}
&R_i < R < b : E_1 = \frac{\bar{D}}{\epsilon_0 \cdot \epsilon_r}; \quad b < R < R_0 : \bar{E}_2 = \frac{\bar{D}}{2\epsilon_0\epsilon_r} \\
&V = -\int_{R_0}^{R_i} \bar{E} \cdot d\bar{R} = -\int_b^{R_i} E_1 dR - \int_{R_0}^b E_2 dR = \frac{Q}{4\pi\epsilon_0\epsilon_r} \left(\frac{1}{R_i} - \frac{1}{2b} - \frac{1}{2R_0} \right)
\end{aligned}
$$

a) $\bar{D} = \bar{a}_R \frac{\epsilon_0\epsilon_r V}{R^2\left(\frac{1}{R_i} - \frac{1}{2b} - \frac{1}{2R_0}\right)}, R_i < R_0$. $\quad \bar{D} = 0, \bar{E} = 0$ for. $R < R_i$ and $R > R_0$
$\bar{E}_1 = \bar{a}_R \frac{V}{R^2\left(\frac{1}{R_i} - \frac{1}{2b} - \frac{1}{2R_0}\right)}; \quad \bar{E}_2 = \bar{a}_R \frac{V}{R^2\left(\frac{2}{R_i} - \frac{1}{b} - \frac{1}{R_0}\right)}$.
b) $C = \frac{4\pi\epsilon_0\epsilon_r}{\frac{1}{R_i} - \frac{1}{2b} - \frac{1}{2R_0}}$

Exercise 3.38

The two parallel conducting wires of a power transmission line have a radius a and are spaced at a distance d apart. The wires are at a height h above the ground. Assuming the ground to be perfectly conducting and both d and h to be much larger than a, find the expressions for the mutual and self-partial capacitances per unit length.

Answer:

Image wires $1'$ and $2'$.
The potential differences referring to the ground are:

$$V_{10} = \frac{1}{2} V_{11'} = \frac{\rho_{l1}}{2\pi\epsilon_0} \ln \frac{2h}{a} + \frac{\rho_{l2}}{2\pi\epsilon_0} \ln \frac{D}{d}$$

$$V_{20} = \frac{1}{2} V_{22'} = \frac{\rho_{l1}}{2\pi\epsilon_0} \ln \frac{D}{d} + \frac{\rho_{l_2}}{2\pi\epsilon_0} \ln \frac{2h}{d}$$

where $D = (4h^2 + d^2)^{1/2}$
Eqs. (1)and (2) can be solved for ρ_{l1} and ρ_{l2} in terms of V_{10} and V_{20}

$$\rho_{l1} = \Delta_0 \left(V_{10} \ln \frac{2h}{a} - V_{20} \ln \frac{D}{a} \right) = c_{11} V_{10} + c_{12} V_{20}$$

$$\rho_{l2} = \Delta_0 \left(-V_{10} \ln \frac{D}{a} + V_{10} \ln \frac{2h}{a} \right) = c_{21} V_{10} + c_{22} V_{20}$$

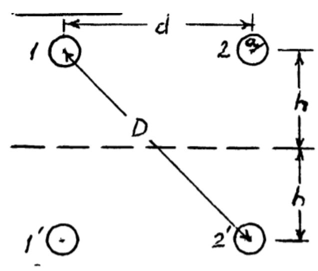
where

$$\Delta_0 = \frac{2\pi\epsilon_0}{\left(\ln \frac{2h}{a}\right)^2 - \left(\ln \frac{D}{a}\right)^2}$$

From Eqs. (3-157a, b, c):

$$C_{12} = -c_{12} = \Delta_0 \ln \frac{D}{a} = \frac{2\pi\epsilon_0 \ln(D/a)}{\left(\ln \frac{2h}{a}\right)^2 - \left(\ln \frac{D}{a}\right)^2}.$$

$$C_{10} = C_{20} = c_{11} + c_{12} = \frac{2\pi\epsilon_0}{\ln(2h/a) + \ln(D/a)}.$$



Exercise 3.39

An isolated system consists of three very long parallel conducting wires. The axes of all three wires lie in a plane. The two outside wires are of a radius b and both are at a distance $d = 500b$ from a center wire of a radius $2b$. Determine the partial capacitances per unit length.

Answer:

Assuming line charge densities we have (see Example 3-21)

$$\begin{aligned}
2\pi\epsilon_0 V_{10} &= \rho_{l0}\ln\frac{b}{d}+\rho_{l1}\ln\frac{d}{2b}+\rho_{l2}\ln\frac{2d}{d} \\
&= \rho_{l0}\ln\frac{1}{500}+\rho_{l1}\ln 250+\rho_{l2}\ln 2 \\
2\pi\epsilon_0 V_{20} &= \rho_{l0}\ln\frac{b}{2d}+\rho_{l1}\ln\frac{d}{d}+\rho_{l2}\ln\frac{2d}{b} \\
&= \rho_{l0}\ln\frac{1}{1000}+0+\rho_{l_2}\ln 1000
\end{aligned}$$

For an isolated system:

$$\rho_{l0}=-\left(\rho_{l1}+\rho_{l2}\right)$$

Eqs. (1) and (2)become

$2\pi\epsilon_0 V_{10}=\rho_{l1}\left(\ln 250+\ln 500\right)+\rho_{l2}(\ln 2+\ln 500)$

$=\rho_{l1}\ln 125\times 10^3+\rho_{l2}\ln 10^3$

$2\pi\epsilon_0 V_{20}=\rho_{l_1}\ln 1000+\rho_{\ell_2}(\ln 1000+\ln 1000)=\rho_{\ell 1}\ln 10^3+\rho_{\ell 2}\ln 10^6$

Solving (3) and (4):

$\rho_{l1}=\Delta_0\left(V_{10}\ln 10^6-V_{20}\ln 10^3\right),$

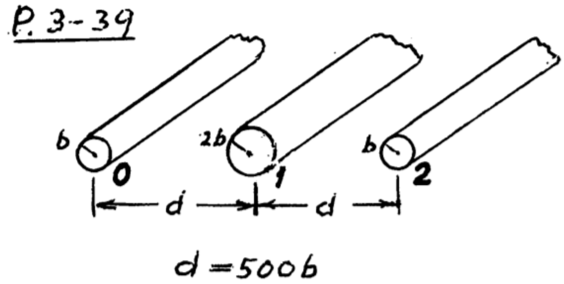
$\rho_{l2}=\Delta_0\left(-V_{10}\ln 10^3+V_{20}\ln 125\times 10^3\right),$

where $\Delta_0=\dfrac{2\pi\epsilon_0}{\left(\ln 10^6\right)\left(\ln 125\times 10^3\right)-\left(\ln 10^3\right)^2}.$

$C_{12}=-c_{12}=\Delta_0\ln 10^3=3.36(\mathrm{pF/m}),$

$C_{10}=c_{11}+c_{12}=\Delta_0\left(\ln 10^6-\ln 10^3\right)=3.36(\mathrm{pF/m}),$

$C_{20}=c_{22}+c_{12}=\Delta_0\left(\ln 125\times 10^3-\ln 10^3\right)=2.35(\mathrm{pF/\ m}).$



Exercise 3.40

Calculate the amount of electrostatic energy of a uniform sphere of charge with radius b and volume charge density ρ stored in the following regions:

a) inside the sphere,

b) outside the sphere.

Check your results with those in Example 3-22.

Answer:

$$\bar{D} = \bar{a}_R \frac{R}{3}\rho, R < b; \quad \bar{D} = \overline{a_R} \frac{b^3\rho}{3R^2}, R > b; \quad \bar{E} = \frac{\bar{D}}{\epsilon_0}$$

a)

$$W_i = \frac{1}{2}\int_V \bar{D} \cdot \bar{E} dv = \frac{1}{2}\int_0^b \frac{1}{\epsilon_0}\left(\frac{R}{3}\rho\right)^2 4\pi R^2 dR = \frac{2\pi b^5 \rho^2}{45\epsilon_0}$$

b)

$$W_o = \frac{1}{2}\int_b^\infty \frac{1}{\epsilon_0}\left(\frac{b^3\rho}{3R^2}\right)^2 4\pi R^2 dR = \frac{2\pi b^5 p^2}{9\epsilon_0}$$

Total

$$W = W_i + W_o = \frac{4\pi b^5 p^2}{15\epsilon_0}$$

Exercise 3.41

Einstein's theory of relativity stipulates that the work required to assemble a charge is stored as energy in the mass and is equal to mc^2, where m is the mass and c $\cong 3\times 10^8$(m/s) is the velocity of light. Assuming the electron to be a perfect sphere, find its radius from its charge and mass (9.1×10^{-31} kg).

Answer:
From Eq. $(3 - 169)$

$$W = \frac{3Q^2}{20\pi\epsilon_0 b}$$

$W = \frac{3e^2}{20\pi\epsilon_0 b} = mc^2$

$$b = \frac{3e^2}{20\pi\epsilon_0 mc^2} = 1.69 \times 10^{-15}(\text{ m}).$$

Exercise 3.42

Find the electrostatic energy stored in the region of space $R > b$ around an electric dipole of moment p.

Answer:

$$\bar{E} = \frac{p}{4\pi\epsilon_0 R^3}\left(\bar{a}_R 2\cos\theta + \bar{a}_\theta \sin\theta\right)$$

$$W = \frac{1}{2}\epsilon_0 \int_V E^2 dV = \frac{\epsilon_0}{2}\left(\frac{p}{4\pi\epsilon_0}\right)^2 \int_0^{2\pi} d\phi \int_0^\pi d\theta \int_b^\infty \frac{1}{R^6}\left(4\cos^2\theta + \sin^2\theta\right) R^2 \sin\theta dR$$

$$= \frac{p^2}{12\pi\epsilon_0 b^3}$$

Exercise 3.43

Prove that Eqs. $(3 - 180)$

$$W_e = \frac{1}{2}CV^2$$

for stored electrostatic energy hold true for any two-conductor capacitor.

Answer:
Two conductors at potentials V_1 and V_2 carrying charges Q and $-Q$:

$$\begin{aligned} W_e &= \frac{1}{2}V_1\int_{s_1}\rho_{s1}ds + \frac{1}{2}V_2\int_{s_2}\rho_{s2}ds = \frac{1}{2}Q\left(V_1 - V_2\right) \\ &= \frac{1}{2}CV^2, \quad V = V_1 - V_2. \end{aligned}$$

Exercise 3.44

A parallel-plate capacitor of width w, length L, and separation d is partially filled with a dielectric medium of dielectric constant ϵ_r, as shown in Fig. 3-43. A battery of V_0 volts is connected between the plates.

a) Find $\mathbf{D}, \mathbf{E}$, and ρ_s in each region.

b) Find distance x such that the electrostatic energy stored in each region is the same.

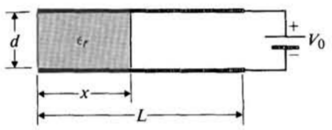


FIGURE 3–43
A parallel-plate capacitor (Problem P.3–44).

Answer:
Region 1-dielectric; region 2 -air.
a)

$$\bar{E}_1 = -\bar{a}_y\frac{V_0}{d}, \bar{D}_1 = -\bar{a}_y\epsilon_0\epsilon_r\frac{V_0}{d}, \rho_{s1} = \epsilon_0\epsilon_r\frac{V_0}{d}(\text{top plate})$$

$$\bar{E}_2 = -\bar{a}_y\frac{V_0}{d}, \bar{D}_2 = -\bar{a}_y\epsilon_0\frac{V_0}{d}, \rho_{s2} = \epsilon_0\frac{V_0}{d}(\text{top plate})$$

(
b)

$$\frac{W_{e1}}{W_{e2}} = \frac{\epsilon_r x}{L - x} = 1 \longrightarrow x = \frac{L}{\epsilon_r + 1}$$

Exercise 3.45

Using the principle of virtual displacement, derive an expression for the force between two point charges $+Q$ and $-Q$ separated by a distance x in free space.

Answer:
From Eqs. $(3-165)$

$$W_e = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k$$

and $(3-166)$

$$V_k = \frac{1}{4\pi\epsilon_0}\sum_{\substack{j=1\\(j\neq k)}}^{N}\frac{Q_j}{R_{jk}}.$$

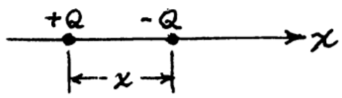
$$W_e = -\frac{Q^2}{4\pi\epsilon_0 x}, \quad x = x_- - x_+.$$

Force on

$$+Q : \bar{F}_{Q+} = -\bar{\nabla}_+ W_e = -\bar{a}_x \frac{\partial W_e}{\partial x_+} = \bar{a}_x \frac{Q^2}{4\pi\epsilon_0 x^2}$$

Force on

$$-Q : \bar{F}_{Q-} = -\bar{\nabla}_- W_e = -\bar{a}_x \frac{\partial W_e}{\partial x_-} = -\bar{a}_x \frac{Q^2}{4\pi\epsilon_0 x^2} = -\bar{F}_{a+}$$



Exercise 3.46

A constant voltage V_0 is applied to a partially filled parallel-plate capacitor shown in Fig. 3-44. The permittivity of the dielectric is ϵ, and the area of the plates is S. Find the force on the upper plate.

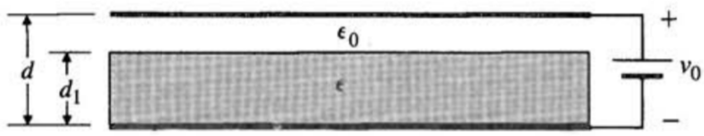


FIGURE 3-44
A parallel-plate capacitor (Problem P.3-46).

Answer:
Denoting the plate separation by y and using Eq $(3-192)$

$$dW + dW_e = dW_s$$

$$\bar{F}_v = \bar{a}_y \frac{\partial W_e}{\partial y} = \bar{a}_y \frac{\partial}{\partial y}\left(\frac{1}{2}CV_0^2\right) = \bar{a}_y \frac{V_0^2}{2}\frac{dC}{dy},$$

where C is the capacitance of the series connection of C_d and $C_a : C_d = \frac{\epsilon S}{d_1}; \quad C_a = \frac{\epsilon_0 S'}{y-d_1}$.

$$C = \frac{c_d c_a}{c_d + c_a} = \frac{\epsilon\epsilon_0 S}{\epsilon\left(y-d_1\right)+\epsilon_0 d_1}$$

$$\therefore \bar{F}_V = -\bar{a}_y \frac{\epsilon^2\epsilon_0 S V_0}{2\left[\epsilon\left(d-d_1\right)+\epsilon_0 d_1\right]^2} \text{ (attractive force).}$$

Exercise 3.47

The conductors of an isolated two-wire transmission line, each of radius b, are spaced at a distance D apart. Assuming $D \gg b$ and a voltage V_0 between the lines, find the force per unit length on the lines.

Answer:

Assume line charge densities ρ_l and $-\rho_l$ on conductors 1 and 2 respectively. At point P,

$$\bar{E}_p = \bar{E}_1 + \bar{E}_2 = \bar{a}_x \left[\frac{\rho_l}{2\pi\epsilon_0 x} + \frac{\rho_l}{2\pi\epsilon_0 (D-x)} \right]$$

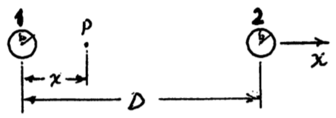
$$V_0 = V_1 - V_2 = \int_b^{D-b} \bar{E}_p \cdot d\bar{x} = \frac{\rho_l}{2\pi\epsilon_0} \int_b^{D-b} \left(\frac{1}{x} + \frac{1}{D-x} \right) dx$$

$$= \frac{\rho_l}{2\pi\epsilon_0} \left(\ln \frac{D-b}{b} - \ln \frac{b}{D-b} \right) = \frac{\rho_l}{\pi\epsilon_0} \ln \frac{D-b}{b} \cong \frac{\rho_l}{\pi\epsilon_0} \ln \frac{D}{b}$$

$$C' = \frac{\rho_l}{V_0} = \frac{\pi\epsilon_0}{\ln(D/b)} \quad (F/m).$$

$$\bar{F}' = \bar{\nabla} W_e = \bar{a}_x \frac{V_0}{2} \frac{\partial C'}{\partial D} = -\bar{a}_x \frac{\pi\epsilon_0 V_0^2}{2D[\ln(D/b)]^2}$$

in the direction of decreasing D.



Exercise 3.48

A parallel-plate capacitor of width w, length L, and separation d has a solid dielectric slab of permittivity ϵ in the space between the plates. The capacitor is charged to a voltage V_0 by a battery, as indicated in Fig. 3-45. Assuming that the dielectric slab is withdrawn to the position shown, determine the force acting on the slab

a) with the switch closed,

b) after the switch is first opened.

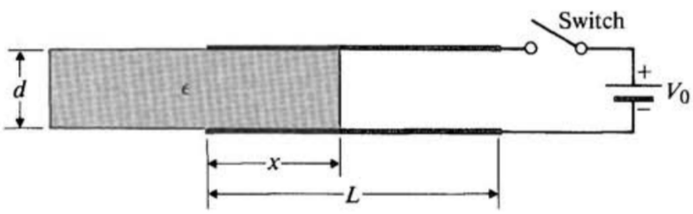


FIGURE 3–45
A partially filled parallel-plate capacitor (Problem P.3–48).

Answer:

a) Switch closed $\longrightarrow V = V_0 =$ constant.

$$W_e = \frac{1}{2} C V_0^2, \quad C = \frac{w}{d}\left[\epsilon x + \epsilon_0 (L - x)\right].$$
$$\bar{F}_v = \bar{\nabla} W_e = \bar{a}_x \frac{V_0^2}{2} \frac{\partial C}{\partial x} = \bar{a}_x \frac{V_0 w}{2d} \left(\epsilon - \epsilon_0\right).$$

b) Switch open $\longrightarrow Q =$ constant $= C V_0$.

$$W_e = \frac{Q^2}{2C},$$
$$\bar{F}_Q = -\bar{\nabla} W_e = -\bar{a}_x \frac{Q^2}{2} \frac{\partial}{\partial x}\left(\frac{1}{C}\right)$$
$$= \bar{a}_x \cdot \frac{Q^2 d}{2w} \cdot \frac{\epsilon - \epsilon_0}{\left[\epsilon x + \epsilon_0 (L - x)\right]^2} = \bar{a}_x \frac{V_0 w}{2d} \left(\epsilon - \epsilon_0\right).$$

Reference

1. Cheng, David Keun. Field and wave electromagnetics. Pearson Education India, 1989.