VE230 Electromagnetics

Chapter 5

September 13, 2022

Exercise 5.1

Assuming S to be the area of the electrodes in the space-charge-limited vacuum diode in Fig. 5-2, find

a) $V(y)$ and $E(y)$ within the interelectrode region,

b) the total amount of charge in the interelectrode region,

c) the total surface charge on the cathode and on the anode,

d) the transit time of an electron from the cathode to the anode with $V_0 = 200(\text{ V})$ and $d = 1(\text{ cm})$

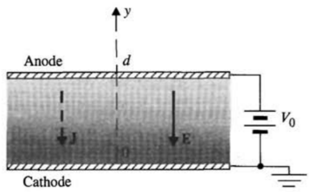

FIGURE 5-2
Space-charge-limited vacuum diode (Example 5-1).

Answer:

a) Integrating $E_g(5-16)$: $V(y) = \left[\frac{9J}{4\epsilon_0}\sqrt{\frac{m}{2e}}\right]^{2/3} y^{4/3} = V_0\left(\frac{y}{d}\right)^{4/3}$

$$E(y) = -\bar{a}_y \frac{dV(y)}{dy} = -\bar{a}_y \frac{4V_0}{3d}\left(\frac{y}{d}\right)^{1/3}$$

b) $\rho(y) = \epsilon_0 \frac{dE(y)}{dy} = -\frac{4\epsilon_0 V_0}{9d^2}\left(\frac{y}{d}\right)^{-2/3}$

$$Q = \int_0^d \rho(y) S dy = -\frac{4\epsilon_0 V_0 S}{9d^{4/3}} \int_0^d y^{-2/3} dy = -\frac{4\epsilon_0 V_0}{3d} S.$$

c) On the anode, $y = d, \rho_s = -\epsilon_0 E(d) = \frac{4\epsilon_0 V_0}{3d}$. Total surface charge on anode $= \rho_s S = \frac{4\epsilon_0 V_0}{3d} S$. Total charge on cathode $= 0$.

Exercise 5.2

Starting with Ohm's law as expressed in Eq. $(5-26)$

$$V_{12} = \left(\frac{\ell}{\sigma S}\right) I = RI$$

applied to a resistor of length ℓ, conductivity σ, and uniform cross-section S, verify the point form of Ohm's law represented by Eq. $(5-21)$

$$\mathbf{J} = \sigma \mathbf{E}\left(\mathrm{A/m^2}\right)$$

Answer:

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Exercise 5.3

A long, round wire of radius a and conductivity σ is coated with a material of conductivity 0.1σ.

a) What must be the thickness of the coating so that the resistance per unit length of the uncoated wire is reduced by 50% ?

b) Assuming a total current I in the coated wire, find $\mathbf{J}$ and $\mathbf{E}$ in both the core and the coating material.

Answer:

$R_1 =$ Resistance per unit length of core $= \frac{1}{\sigma S_1} = \frac{1}{\sigma\pi a^2}$.

$R_2 =$ Resistance per unit length of coating $= \frac{1}{0.1\sigma S_2}$.

Let $b =$ Thickness of coating. $\longrightarrow S_2 = \pi(a+b)^2 - \pi a^2 = \pi\left(2ab + b^2\right)$.

a) $R_1 = R_2 \longrightarrow b = (\sqrt{11} - 1)a = 2.32a$.

b) $I_1 = I_2 = \frac{I}{2}$. $\quad J_1 = \frac{I}{2\pi a^2}, \quad J_2 = \frac{I}{2S_2} = \frac{I}{20S_1} = \frac{I}{20\pi a^2}$.

$$E_1 = \frac{J_1}{\sigma} = \frac{I}{2\pi a^2\sigma}, E_2 = \frac{J_2}{0.1\sigma} = \frac{I}{2\pi a^2\sigma}.$$

Thus, $J_1 = 10J_2$ and $E_1 = E_2$.

Exercise 5.4

Find the current and the heat dissipated in each of the five resistors in the network shown in Fig. 5-9 if

$$R_1 = \frac{1}{3}(\Omega), \quad R_2 = 20(\Omega), \quad R_3 = 30(\Omega), \quad R_4 = 8(\Omega), \quad R_5 = 10(\Omega),$$

and if the source is an ideal d-c voltage generator of $0.7(V)$ with its positive polarity at terminal 1 . What is the total resistance seen by the source at terminal pair $1-2$?

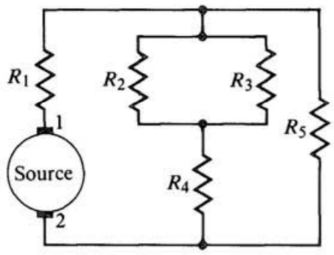


FIGURE 5–9
A network problem (Problem P.5–4).

Answer:

$$I_1 = 0.1(\text{ A}), P_{R1} = 3.33(\text{ mW}); \qquad I_2 = 0.02(\text{ A}), P_{R_2} = 8.00(\text{ mW});$$
$$I_3 = 0.0133(\text{ A}), P_{R3} = 5.31(\text{ mW}); \quad I_4 = 0.0333(\text{ A}), P_{R4} = 8.87(\text{ mW});$$
$$I_5 = 0.0667(\text{ A}), P_{R5} = 44.5(\text{ mW}). \qquad \sum_n P_{Rn} = V_0 I_1 = 70(\text{ mW}).$$

Exercise 5.5

Solve Problem P.5-4, assuming that the source is an ideal current generator that supplies a direct current of 0.7 (A) out of terminal 1.

Answer:

$$I_1 = 0.700(\text{ A}), P_{R1} = 0.163(\text{ W}); \quad I_2 = 0.140(\text{ A}), P_{R2} = 0.392(\text{ W});$$
$$I_3 = 0.093(\text{ A}), P_{R3} = 0.261(\text{ W}); \quad I_4 = 0.233(\text{ A}), P_{R4} = 0.436(\text{ W});$$
$$I_5 = 0.467(\text{ A}), P_{R5} = 2.178(\text{ W}).$$

Exercise 5.6

Lightning strikes a lossy dielectric sphere $-\epsilon = 1.2\epsilon_0, \sigma = 10(\text{ S/m})-$ of radius $0.1(\text{ m})$ at time $t = 0$, depositing uniformly in the sphere a total charge 1(mC). Determine, for all t,

a) the electric field intensity both inside and outside the sphere,

b) the current density in the sphere.

Answer:

$$\rho_0 = \frac{Q_0}{(4\pi/3)b^3} = \frac{10^{-3}}{(4\pi/3)(0.1)^3} = 0.239\left(\text{C/m}^3\right), \quad \rho = \rho_0\epsilon^{-(\sigma/\epsilon)t}$$

$$\text{a) } R < b : \bar{E}_i = \bar{a}_R\frac{(4\pi/3)R^3p}{4\pi\epsilon R^2} = \bar{a}_R\frac{p_0R}{3t}e^{-(\sigma/t)t} = \bar{a}_R 7.5\times 10^9 \text{R}e^{-9.42\times 10^{11}t}(\text{ V/m})$$

$$R > b : \bar{E}_0 = \bar{a}_R\frac{Q_0}{4\pi\epsilon_0 R^2} = \bar{a}_R\frac{q}{R^2}\times 10^6(\text{ V/m})$$

$$\text{b) } R < b : \bar{J}_i = \sigma\bar{E}_i = \bar{a}_R 7.5\times 10^{10} Re^{-9.42\times 10^{11}t}$$

$$R > b : \bar{J}_0 = 0$$

Exercise 5.7

Refer to Problem P.5-6.

a) Calculate the time it takes for the charge density in the sphere to diminish to 1% of its initial value.

b) Calculate the change in the electrostatic energy stored in the sphere as the charge density diminishes from the initial value to 1% of its value. What happens to this energy?

c) Determine the electrostatic energy stored in the space outside the sphere. Does this energy change with time?

Answer:

a) $e^{-(\sigma/\epsilon)t} = \frac{\rho}{\rho_0} = 0.01 \longrightarrow t = \frac{\ln 100}{(\sigma/t)} = 4.88\times 10^{-12}(s) = 4.88(ps)$.

b) $W_i = \frac{\epsilon}{2}\int_v E_i^2 dv' = \frac{2\pi P_0 b^3}{45\epsilon}e^{-2(\sigma/\epsilon)t} = (W_i)_0\left[e^{-(\sigma/\epsilon)t}\right]^2$.

$\therefore \frac{W_i}{(W_i)_0} = \left[e^{-(\sigma/\epsilon)t}\right]^2 = 0.01^2 = 10^{-4}$. Energy dissipated as heat loss.

c) Electrostatic energy $W_0 = \frac{\epsilon_0}{2}\int_b^\infty E_0^2 4\pi R^2 dR = \frac{Q_0^2}{8\pi\epsilon_0 b} = 45(kJ)$ stored outside the sphere constant.

Exercise 5.8

A d-c voltage of 6(V) applied to the ends of 1(km) of a conducting wire of 0.5(mm) radius results in a current of 1/6(A). Find

a) the conductivity of the wire,

b) the electric field intensity in the wire,

c) the power dissipated in the wire,

d) the electron drift velocity, assuming electron mobility in the wire to be 1.4×10^{-3} $(\mathrm{m}^2/\mathrm{V}\cdot\mathrm{s})$

Answer:

a) $R = \frac{l}{\sigma S} = \frac{V}{I} \longrightarrow \sigma = \frac{lI}{SV} = 3.54 \times 10^7(\ \mathrm{S/m})$.

b) $E = \frac{V}{l} = 6 \times 10^{-3}(\ \mathrm{V/m})$.

c) $P = VI = 1(W)$.

d) $\rho_e = -\frac{\sigma}{\mu_e}$. The given electron mobility $1.4 \times 10^{-3}\,(\ \mathrm{m}^2 \cdot \mathrm{V/s})$ is that of a good conductor.

$$u = \left|\frac{J}{\rho_e}\right| = \left|\frac{\mu_e J}{\sigma}\right| = |\mu_e E| = 1.4 \times 10^{-3} \times \left(6 \times 10^{-3}\right)$$
$$= 8.4 \times 10^{-6}(\ \mathrm{m/s}).$$

Exercise 5.9

Two lossy dielectric media with permittivities and conductivities (ϵ_1, σ_1) and (ϵ_2, σ_2) are in contact. An electric field with a magnitude E_1 is incident from medium 1 upon the interface at an angle α_1 measured from the common normal, as in Fig. 5-10.

a) Find the magnitude and direction of $\mathbf{E}_2$ in medium 2.

b) Find the surface charge density at the interface.

c) Compare the results in parts (a) and (b) with the case in which both media are perfect dielectrics.

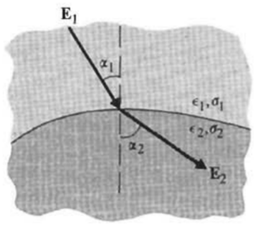


FIGURE 5–10
Boundary between two lossy dielectric media (Problem P.5–9).

Answer:

a) $E_q(3-118): E_{1t} = E_{2t} \longrightarrow E_2 \sin\alpha_2 = E_1 \sin\alpha_1$.

$E_{q.}(5-58): J_{1n} = J_{2n} \longrightarrow \sigma_1 E_{1n} = \sigma_2 E_{2n} \longrightarrow \sigma_2 E_2 \cos\alpha_2 = \sigma_1 E_1 \cos\alpha_1$.

$\therefore E_2 = E_i\sqrt{\sin^2\alpha_1 + \left(\frac{\sigma_i}{\sigma_2}\cos\alpha_1\right)^2}$. $\tan\alpha_2 = \frac{\sigma_2}{\sigma_1}\tan\alpha_1 \longrightarrow \alpha_2 = \tan^{-1}\left(\frac{\sigma_2}{\sigma_1}\tan\alpha_1\right)$.

b) Eq. (3-121b): $D_{2n} - D_{1n} = \rho_s \longrightarrow \epsilon_2 E_{2n} - \epsilon_1 E_{1n} = \rho_s$.

$$\rho_s = \left(\frac{\sigma_1}{\sigma_2}\epsilon_2 - \epsilon_1\right) E_{1n} = \left(\frac{\sigma_1}{\sigma_2}\epsilon_2 - \epsilon_1\right) E_1 \cos\cdot\alpha_1.$$

c) If both media are perfect dielectrics, $\sigma_1 = \sigma_2 = 0, E_q$. (1) and (2) revert to Eqs. $(3-130)$ and $(3-129)$ respectively and $\rho_s = 0$.

Exercise 5.10

The space between two parallel conducting plates each having an area S is filled with an inhomogeneous ohmic medium whose conductivity varies linearly from σ_1 at one plate ($y = 0$) to σ_2 at the other plate ($y = d$). A d-c voltage V_0 is applied across the plates as in Fig. 5-11. Determine

a) the total resistance between the plates,
b) the surface charge densities on the plates,
c) the volume charge density and the total amount of charge between the plates.

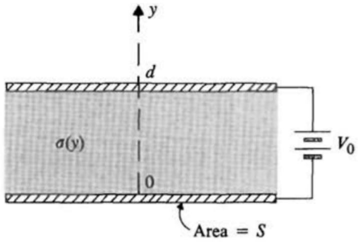


FIGURE 5–11
Inhomogeneous ohmic medium with conductivity $\sigma(y)$ (Problem P.5–10).

Answer:

$$\sigma(y) = \sigma_1 + (\sigma_2 - \sigma_1)\frac{y}{d}$$

a) Neglecting fringing effect and assuming a current density

$$\bar{J} = -\bar{a}_y J_0 \rightarrow \bar{E} = \frac{\bar{J}}{\sigma} = -\bar{a}_y \frac{J_0}{\sigma(y)}.$$

$$V_0 = -\int_0^d \bar{E}\cdot\bar{a}_y dy = \int_0^d \frac{J_0 dy}{\sigma_1 + (\sigma_2-\sigma_1)\frac{y}{d}} = \frac{J_0 d}{\sigma_2-\sigma_1}\ln\frac{\sigma_2}{\sigma_1}.$$

$$R = \frac{V_0}{I} = \frac{V_0}{J_0 S} = \frac{d}{(\sigma_2-\sigma_1)S}\ln\frac{\sigma_2}{\sigma_1}.$$

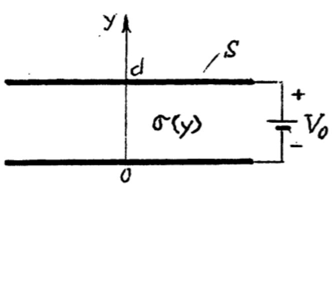
b)

$$(\rho_s)_u = \epsilon_0 E_y(d) = \frac{\epsilon_0 J_0}{\sigma_2} = \frac{\epsilon_0(\sigma_2-\sigma_1)V_0}{\sigma_2 d\ln(\sigma_2/\sigma_1)} \text{on upper plate}$$

$$(\rho_s)_l = -\epsilon_0 E_y(0) = -\frac{\epsilon_0 J_0}{\sigma_1} = -\frac{\epsilon_0(\sigma_2-\sigma_1)V_0}{\sigma_1 d\ln(\sigma_2/\sigma_1)} \text{on lower plate}$$

c)

$$\rho = \bar{\bar{\nabla}}\cdot\bar{D} = \frac{d}{dy}[\epsilon_0 E] = -\epsilon_0 J_0 \frac{d}{dy}\left[\frac{1}{\sigma_1+(\sigma_2-\sigma_1)y/d}\right] = \epsilon_0 J_0 \frac{(\sigma_2-\sigma_1)/d}{[\sigma_1+(\sigma_2-\sigma_1)y/d]^{-2}}$$



Exercise 5.11

Refer to Example 5-4.

a) Draw the equivalent circuit of the two-layer, parallel-plate capacitor with lossy dielectrics, and identify the magnitude of each component.

b) Determine the power dissipated in the capacitor.

EXAMPLE 5-4 An emf $\mathscr{V}$ is applied across a parallel-plate capacitor of area S. The space between the conducting plates is filled with two different lossy dielectrics of thicknesses d_1 and d_2, permittivities ϵ_1 and ϵ_2, and conductivities σ_1 and σ_2, respectively. Determine (a) the current density between the plates, (b) the electric field intensities in both dielectrics, and (c) the surface charge densities on the plates and at the interface.

Answer:

a)

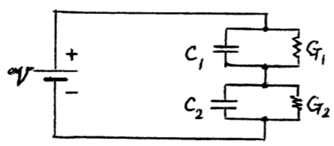
$$C_1 = \frac{\epsilon_1 S}{d_1}, \quad G_1 = \frac{\sigma_1 S}{d_1}$$

$$C_2 = \frac{\epsilon_2 S}{d_2}, \quad G_2 = \frac{\sigma_2 S}{d_2}$$

b)

$$P = V^2 G = V^2 \frac{G_1 G_2}{G_1 + G_2}$$

$$= V^2 s \frac{\sigma_1 \sigma_2}{\sigma_1 d_2 + \sigma_2 d_1}$$



Exercise 5.12

Refer again to Example 5-4. Assuming that a voltage V_0 is applied across the parallel-plate capacitor with the two layers of different lossy dielectrics at $t = 0$,

a) express the surface charge density ρ_{si} at the dielectric interface as a function of t,

b) express the electric field intensities $\mathbf{E}_1$ and $\mathbf{E}_2$ as functions of t.

Answer:

Refer to Fig.5-6. In the transient state, the equation of continuity must be satisfied at the interface. Now

$$-\frac{\partial \rho_{si}}{\partial t} = J_2 - J_1 = \sigma_2 E_2 - \sigma_1 E_1$$

$$E_1 d_1 + E_2 d_2 = v$$

$$\epsilon_2 E_2 - \epsilon_1 E_1 = \rho_{si}$$

Solving (2) and (3) for E_1 and E_2 in terms of V and ρ_{si} :

$$E_1 = \frac{\epsilon_2 V - d_2 \rho_{si}}{\epsilon_2 d_1 + \epsilon_1 d_2} (4)$$

$$E_2 = \frac{\epsilon_1 V + d_1 \rho_{si}}{\epsilon_2 d_1 + \epsilon_1 d_2} (5)$$

a) Substituting (4) and (5) in (1):

$$-\frac{\partial \rho_{si}}{\partial t} = \frac{\sigma_2 d_1 + \sigma_1 d_2}{\epsilon_2 d_1 + \epsilon_1 d_2} \rho_{si} + \frac{\epsilon_1 \sigma_2 - \epsilon_2 \sigma_1}{\epsilon_2 d_1 + \epsilon_1 d_2} V$$

Solution of (6):

$$\rho_{si} = \left(\frac{\epsilon_2 \sigma_1 - \epsilon_1 \sigma_2}{\sigma_2 d_1 + \sigma_1 d_2} \right) V \left[1 - e^{-t/\tau} \right],$$

where $\tau =$ Relaxation time $= \frac{\epsilon_2 d_1 + \epsilon_1 d_2}{\sigma_2 d_1 + \sigma_1 d_2}$.

b) Using (4) and (5):

$$E_1 = \frac{\sigma_2 V}{\sigma_2 d_1 + \sigma_1 d_2} \left(1 - e^{-t/\tau} \right) + \frac{\epsilon_2 V}{\epsilon_2 d_1 + \epsilon_1 d_2} e^{-t/\tau};$$
$$E_2 = \frac{\sigma_1 V}{\sigma_2 d_1 + \sigma_1 d_2} \left(1 - e^{-t/\tau} \right) + \frac{\epsilon_1 V}{\epsilon_2 d_1 + \epsilon_1 d_2} e^{-t/\tau}.$$

Exercise 5.13

A d-c voltage V_0 is applied across a cylindrical capacitor of length L. The radii of the inner and outer conductors are a and b, respectively. The space between the conductors is filled with two different lossy dielectrics having, respectively, permittivity ϵ_1 and conductivity σ_1 in the region $a < r < c$, and permittivity ϵ_2 and conductivity σ_2 in the region $c < r < b$. Determine

a) the current density in each region,

b) the surface charge densities on the inner and outer conductors and at the interface between the two dielectrics.

Answer:

$$\text{a) } G_1 = \frac{2\pi\sigma_1 L}{\ln(c/a)}, \quad G_2 = \frac{2\pi\sigma_2 L}{\ln(b/c)}.$$
$$I = V_0 G = V_0 \frac{G_1 G_2}{G_1 + G_2} = \frac{2\pi\sigma_1\sigma_2 L V_0}{\sigma_1 \ln(b/c) + \sigma_2 \ln(c/a)}.$$
$$J_1 = J_2 = \frac{I}{2\pi r L} = \frac{\sigma_1 \sigma_2 V_0}{r\left[\sigma_1 \ln(b/c) + \sigma_2 \ln(c/a)\right]}.$$
$$\text{b) } \rho_{sa} = \epsilon_1 E_1|_{r=a} = \frac{\epsilon_1 \sigma_2 V_0}{a\left[\sigma_1 \ln(b/c) + \sigma_2 \ln(c/a)\right]}.$$
$$\rho_{sb} = -\ \epsilon_2 E_2|_{r=b} = -\frac{E_2 \sigma_1 V_0}{b\left[\sigma_1 \ln(b/c) + \sigma_2 \ln(c/a)\right]}.$$
$$\rho_{sc} = -\ (\epsilon_1 E_1 - \epsilon_2 E_2)|_{r=c} = \frac{(\epsilon_2\sigma_1 - \epsilon_1\sigma_2) V_0}{c\left[\sigma_1 \ln(b/c) + \sigma_2 \ln(c/a)\right]}.$$

Exercise 5.14

Refer to the flat conducting quarter-circular washer in Example 5-6 and Fig. 5-8. Find the resistance between the curved sides.

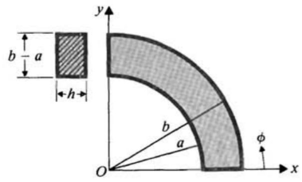


FIGURE 5–8
A quarter of a flat conducting circular washer (Example 5–6).

Answer:
$\bar{\nabla}^2 V = 0 \longrightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0.$
Solution: $T(r) = c_1 \ln r + c_2$.
Boundary conditions: $V(a) = V_0; \quad V(b) = 0.$

$$\begin{aligned}
\longrightarrow & V(r) = V_0\frac{\ln(b/r)}{\ln(b/a)}.\\
& \bar{E}(r) = -\bar{a}_r\frac{\partial V}{\partial r} = \bar{a}_r\frac{V_0}{r\ln(b/a)}.\\
& \bar{J}(r) = \sigma\bar{E}(r).\\
& I = \int_S \bar{J}\cdot d\bar{s} = \int_0^{\pi/2}\bar{J}\cdot(\bar{a}_r hrd\phi) = \frac{\pi\sigma h V_0}{2\ln(b/a)}.\\
& R = \frac{V_0}{I} = \frac{2\ln(b/a)}{\pi\sigma h}.
\end{aligned}$$

Exercise 5.15

Find the resistance between two concentric spherical surfaces of radii R_1 and R_2 ($R_1 < R_2$) if the space between the surfaces is filled with a homogeneous and isotropic material having a conductivity σ.

Answer:
Assume a potential difference V_0 between the inner and outer spheres.

$$\bar{\nabla}^2 V = 0 \to \frac{1}{R^2}\frac{d}{dR}\left(R^2 V\right) = 0 \to V = \frac{K}{R} \to E_R = \frac{K}{R^2}.$$

$$V_0 = -\int_{R_2}^{R_1} E_R dR = -K\int_{R_2}^{R_1}\frac{1}{R^2}dR = K\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

$$\longrightarrow K = \frac{V_0}{\frac{1}{R_1} - \frac{1}{R_2}}. \quad J_R = \sigma E_R = \frac{\sigma V_0}{\frac{1}{R_1} - \frac{1}{R_2}}.$$

$$I = \int_0^{2\pi}\int_0^{\pi} J_R R^2 \sin\theta d\theta d\phi = \frac{4\pi\sigma V_0}{\frac{1}{R_1} - \frac{1}{R_2}}.$$

$R = \frac{V_0}{I} = \frac{1}{4\pi\sigma}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$, which can be obtained by combining Eqs. $(3-140)$ and $(5-81)$.

Exercise 5.16

Determine the resistance between two concentric spherical surfaces of radii R_1 and R_2 ($R_1 < R_2$), assuming that a material of conductivity $\sigma = \sigma_0(1 + k/R)$ fills the space between them. (Note: Laplace's equation for V does not apply here.)

Answer:
Assume a current I between the spherical surfaces.

$$\begin{aligned}
\bar{J} &= \overline{a_R}\frac{I}{4\pi R^2} = \sigma\bar{E} \\
V_0 &= -\int_{R_2}^{R_1} \bar{E}\cdot d\bar{R} = \int_{R_1}^{R_2} \frac{IdR}{4\pi\sigma R^2} = \frac{I}{4\pi\sigma_0}\int_{R_1}^{R_2} \frac{dR}{R^2(1+k/R)} \\
&= \frac{I}{4\pi\sigma_0}\cdot\int_{R_1}^{R_2} \frac{1}{k}\left(\frac{1}{R}-\frac{1}{R+k}\right)dR = \frac{I}{4\pi\sigma_0 k}\ln\frac{R_2\left(R_1+k\right)}{R_1\left(R_2+k\right)} \\
R &= \frac{V_0}{I} = \frac{1}{4\pi\sigma_0 k}\ln\frac{R_2\left(R_1+k\right)}{R_1\left(R_2+k\right)}
\end{aligned}$$

Exercise 5.17

A homogeneous material of uniform conductivity σ is shaped like a truncated conical block and defined in spherical coordinates by

$$R_1 \le R \le R_2 \quad \text{and} \quad 0 \le \theta \le \theta_0.$$

Determine the resistance between the $R = R_1$ and $R = R_2$ surfaces.

Answer:
Assume I. $\bar{J}(R) = \bar{a}_R\frac{I}{S(R)}$ $S(R) = \int_0^{2\pi}\int_0^{\theta_0} R^2\sin\theta d\theta d\phi = 2\pi R^2\left(1-\cos\theta_0\right)$

$$\bar{E}(R) = \frac{1}{\sigma}\bar{J}(R) = \bar{a}_R\frac{I}{2\pi\sigma R^2\left(1-\cos\theta_0\right)}$$

$$\begin{aligned}
V_0 &= -\int_{R_2}^{R_1} E(R)dR = \frac{I\left(R_2-R_1\right)}{2\pi\sigma R_1R_2\left(1-\cos\theta_0\right)} \\
R &= \frac{V_0}{I} = \frac{1}{2\pi\left(1-\cos\theta_0\right)}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)
\end{aligned}$$

Exercise 5.18

Redo Problem P.5-17, assuming that the truncated conical block is composed of an inhomogeneous material with a nonuniform conductivity $\sigma(R) = \sigma_0 R_1/R$, where $R_1 \le R \le R_2$

Answer:

$$\begin{aligned}
&\bar{\nabla}\cdot\bar{J} = 0 = \bar{\nabla}\cdot\left(\sigma\bar{E}\right) = \sigma\bar{\nabla}\cdot\bar{E} - \left(\bar{\nabla}\sigma\right)\cdot\bar{E} = 0 \\
&\bar{E} = \bar{a}_R E, \bar{\nabla}\cdot\bar{E} = \frac{1}{R^2}\frac{d}{dR}\left(R^2E\right); \quad \bar{\nabla}\sigma = \bar{a}_R\frac{d\sigma}{dR} = -\bar{a}_R\frac{\sigma_0 R_1}{R^2}
\end{aligned}$$

Substituting back: $R\dfrac{dE}{dR} = E \longrightarrow \bar{E} = \bar{a}_R \dfrac{c}{R}$

$$V = -\int_{R_2}^{R_1} \bar{E} \cdot d\bar{R} = c \ln \frac{R_2}{R_1} \longrightarrow c = \frac{V_0}{\ln(R_2/R_1)}, \bar{E} = \bar{a}_R \frac{V_0}{R \ln(R_2/R_1)}.$$

$$I = \int_S \bar{J} \cdot d\bar{s} = \int_S \sigma \bar{E} \cdot d\bar{s} = \int_0^{2\pi} \int_0^{\theta_0} \left(\frac{\sigma_0 R_1}{R} \right) \left[\frac{V_0}{R \ln(R_2/R_1)} \right] R^2 \sin\theta d\theta d\phi$$

$$= \frac{2\pi\sigma_0 R_1 V_0 (1 - \cos\theta_0)}{\ln(R_2/R_1)}.$$

$$R = \frac{V_0}{I} = \frac{\ln(R_2/R_1)}{2\pi\sigma_0 R_1 (1 - \cos\theta_0)}.$$

Exercise 5.19

Two conducting spheres of radii b_1 and b_2 that have a very high conductivity are immersed in a poorly conducting medium (for example, they are buried very deep in the ground) of conductivity σ and permittivity ϵ. The distance, d, between the spheres is very large in comparison with the radii. Determine the resistance between the conducting spheres. (Hint: Find the capacitance between the spheres by following the procedure in Section 3-10 and using Eq. (5-81).)

Answer:

Assume charges $+q$ and $-q$ to concentrate at the centers of spheres I and respectively. $d \gg b_1, d \gg b_2$.

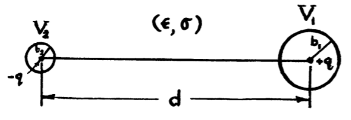
$$V_1 \cong \frac{q}{4\pi\epsilon} \left(\frac{1}{b_1} - \frac{1}{d - b_1} \right)$$

$$V_2 \cong \frac{q}{4\pi\epsilon} \left(\frac{1}{d - b_2} - \frac{1}{b_2} \right).$$

$$C = \frac{q}{V_1 - V_2} = \frac{4\pi\epsilon}{\frac{1}{b_1} + \frac{1}{b_2} - \frac{1}{d-b_1} - \frac{1}{d-b_2}}$$

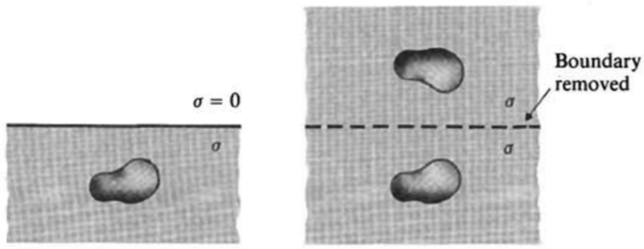
$$= G\frac{\epsilon}{\sigma} = \frac{\epsilon}{R\sigma}.$$

$$R = \frac{1}{4\pi\sigma} \left(\frac{1}{b_1} + \frac{1}{b_2} - \frac{1}{d - b_1} - \frac{1}{d - b_2} \right) \cong \frac{1}{4\pi\sigma} \left(\frac{1}{b_1} + \frac{1}{b_2} - \frac{2}{d} \right)$$



Exercise 5.20

Justify the statement that the steady-current problem associated with a conductor buried in a poorly conducting medium near a plane boundary with air, as shown in Fig. 5-12(a), can be replaced by that of the conductor and its image, both immersed in the poorly conducting medium as shown in Fig. 5-12(b).



(a) Conductor in a poorly conducting medium near a plane boundary.

(b) Image conductor in conducting medium replacing the plane boundary.

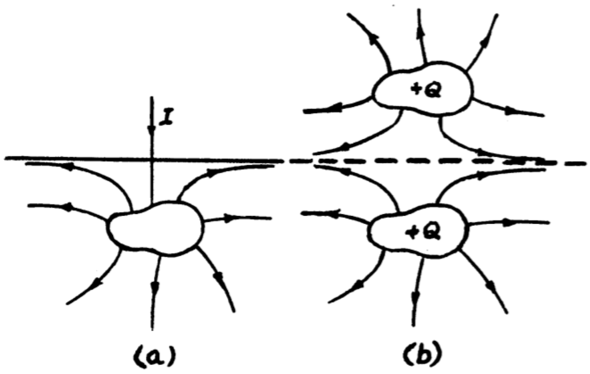
FIGURE 5–12
Steady-current problem with a plane boundary (Problem P.5–20).

Answer:

The curved flow pattern of the lower half of Fig (b), if both the conductor and its image are fed with the same current, is exactly the same as that of Fig. (a). All boundary conditions are satisfied.

$$\bar{\nabla} \times \left(\frac{\bar{J}}{\sigma}\right) = 0 \rightarrow \overline{\overline{V}} \times \bar{J} = 0.$$

We can write $\bar{J} = -\bar{\nabla}\psi$ where ψ and electrostatic potential V are simply related. The streamlines are similar to the- $\bar{E}$-lines of a conductor and its image, both carrying a charge $+Q$ in the electrostatic case.



Exercise 5.21

A ground connection is made by burying a hemispherical conductor of radius 25(mm) in the earth with its base up, as shown in Fig. 5-13. Assuming the earth conductivity to be 10^{-6} S/m, find the resistance of the conductor to far-away points in the ground. (Hint: Use the image method in P.5-20.)

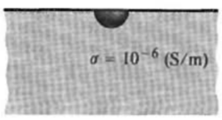


FIGURE 5–13
Hemispherical conductor in ground (Problem P.5–21).

Answer:

According to problem P.5-20, the current flow pattern would be the same as that not a whole sphere in an unbounded earth medium. Hence the current lines would be radial. Assume a current I.

$$\bar{J} = \bar{a}_R \frac{I}{2\pi R^2}, \quad \bar{E} = \bar{a}_R \frac{I}{2\pi R^2}$$

$$V_0 = -\int_{\infty}^{b} E dR = -\frac{I}{2\pi\sigma}\int_{\infty}^{b}\frac{dR}{R^2} = \frac{I}{2\pi\sigma b}$$

$$R = \frac{V_0}{I} = \frac{1}{2\pi\sigma b} = \frac{1}{2\pi\left(10^{-6}\right)\left(25\times 10^{-3}\right)} = 6.36\times 10^{6}(\Omega)$$

Exercise 5.22

Assume a rectangular conducting sheet of conductivity σ, width a, and height b.A potential difference V_0 is applied to the side edges, as shown in Fig. 5-14. Find

a) the potential distribution,

b) the current density everywhere within the sheet. (Hint: Solve Laplace's equation in Cartesian coordinates subject to appropriate boundary conditions.)

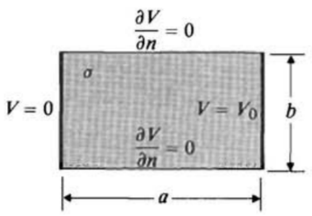


FIGURE 5–14
A conducting sheet (Problem P.5–22).

Answer:

Specified boundary conditions can be satisfied by Solutions of Laplace's equation with zero separation constants: $k_x = k_y = 0$. $X(x) = A_0 x + B_0; Y(y) = C_0 y + D_0$. $B_0 = C_0 = 0$ $V(x) = A_0 D_0 x$

a) At $x = a, V(a) = V_0 = A_0 D_0 a \longrightarrow A_0 D_0 = \frac{V_0}{a} \therefore V = \frac{V_0}{a}x$

b) $\bar{E} = -\bar{\nabla} V = -\bar{a}_x \frac{V_0}{a} \longrightarrow \bar{J} = \sigma\bar{E} = -\bar{a}_x \frac{\sigma V_0}{a}$

Exercise 5.23

A uniform current density $\mathbf{J} = \mathbf{a}_x J_0$ flows in a very large rectangular block of homogeneous material of a uniform thickness having a conductivity σ. A hole of radius b is drilled in the material. Find the new current density $\mathbf{J}'$ in the conducting material. (Hint: Solve Laplace's equation in cylindrical coordinates and note that V approaches $-\left(J_0 r/\sigma\right)\cos\phi$ as $r \to \infty$, where ϕ is the angle measured from the x-axis.)

Answer:

$$V(r,\phi)=\sum_{n=0}^{\infty}\left(A_n r^n + B_n r^{-n}\right)\left(C_n\cos n\phi + D_n\sin n\phi\right)$$

$$V(r,\phi)=V(r,-\phi)\longrightarrow D_n=0.$$

$$r\to\infty, V=-\frac{J_0}{\sigma}r\cos\phi\longrightarrow A_n=C_n=0 \text{ for } n\neq 1.$$

Write $V(r,\phi)=\left(K_1 r+\dfrac{K_2}{r}\right)\cos\phi.\quad K_1=A_1c_1=-\dfrac{J_0}{\sigma}, K_2=B_1c_1.$

B.C.: $\left.\dfrac{\partial V}{\partial r}\right|_{r=b}=0\longrightarrow K_1-\dfrac{K_2}{b^2}=0,\quad K_2=b^2K_1=-\dfrac{J_0}{\sigma}b^2.$

$$\therefore V(r,\phi)=-\frac{J_0}{\sigma}\left(r+\frac{b^2}{r}\right)\cos\phi.$$

$$\bar{J}=-\sigma\bar{\nabla}V=-\sigma\left(\bar{a}_r\frac{\partial V}{\partial r}+\bar{a}_\phi\frac{\partial V}{r\partial\phi}\right)$$

$$=\bar{a}_rJ_0\left(1-\frac{b^2}{r^2}\right)\cos\phi-\bar{a}_\phi J_0\left(1+\frac{b^2}{r^2}\right)\sin\phi$$

$$=J_0\left(\bar{a}_r\cos\phi-\bar{a}_\phi\sin\phi\right)-\frac{J_0b^2}{r^2}\left(\bar{a}_r\cos\phi+\bar{a}_\phi\sin\phi\right)$$

$$=\bar{a}_xJ_0-\frac{J_0b^2}{r^2}\left(\bar{a}_r\cos\phi+\bar{a}_\phi\sin\phi\right),\quad r>b;$$

$$J=0,\quad r<b.$$

Reference

1. Cheng, David Keun. Field and wave electromagnetics. Pearson Education India, 1989.