Chapter 8 Plane Electromagnetic Waves

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8-1 Introduction

- In Chap. 7, homogeneous vector wave equations

$$\nabla^2 \mathbf{E} - \frac{1}{u^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0.$$

where $u = 1/\sqrt{\mu\epsilon}$,

- In free space the source-free wave equation

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0,$$

where $c = \dfrac{1}{\sqrt{\mu_0 \epsilon_0}} \cong 3 \times 10^8 \text{ (m/s)} = 300 \quad \text{(Mm/s)}$

Plane Wave

- Waves with one-dimensional spatial dependence
- A uniform plane wave:
  - a particular solution of Maxwell's equations
  - **E** (or **H**) with the same direction, same magnitude, and same phase in infinite planes perpendicular to the direction of propagation.
- If we are far enough away from a source, the wavefront (surface of constant phase) becomes almost spherical; and a very small portion of the surface of a giant sphere is very nearly a plane.

8-2 Plane Waves in Lossless Media

- Wave equation for source free, in free space:

Homogeneous vector Helmholtz's equation

$$\boxed{\nabla^2 \mathbf{E} + k_0^2 \mathbf{E} = 0,}$$

where k_0: free-space wavenumber

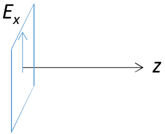
$$\boxed{k_0 = \omega\sqrt{\mu_0 \epsilon_0} = \frac{\omega}{c} \qquad \text{(rad/m).}}$$

In Cartesian coordinates

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k_0^2\right) E_x = 0.$$

1D wave equation

Consider a uniform plane wave: uniform E_x (uniform magnitude and constant phase) over plane surfaces $\perp z$



E_x uniform in x and y; $E_x(z)$ ➔ $\partial^2 E_x/\partial x^2 = 0$ and $\partial^2 E_x/\partial y^2 = 0.$

$$\frac{d^2 E_x}{dz^2} + k_0^2 E_x = 0,$$

E_x: a phasor
2^{nd}-order ODE ➔ 2 integration constants

Solution

$$\begin{aligned} E_x(z) &= E_x^+(z) + E_x^-(z) \\ &= E_0^+ e^{-jk_0 z} + E_0^- e^{jk_0 z}, \end{aligned}$$

E_0^+, E_0^-: arbitrary complex constants, to be determined by boundary conditions

$$E_x(z) = E_x^+(z) + E_x^-(z)$$
$$= E_0^+ e^{-jk_0 z} + E_0^- e^{jk_0 z},$$

Check time-dependent E_x
(Phasor → time domain)

Time domain

phasor

$$E_x^+(z, t) = \mathcal{R}e[E_x^+(z)e^{j\omega t}]$$
$$= \mathcal{R}e[E_0^+ e^{j(\omega t - k_0 z)}]$$
$$= E_0^+ \cos(\omega t - k_0 z) \qquad \text{(V/m).}$$

Assuming E_0^+ is a real constant

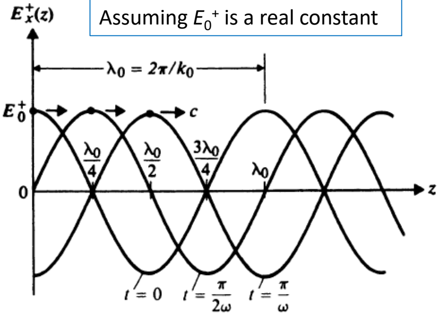
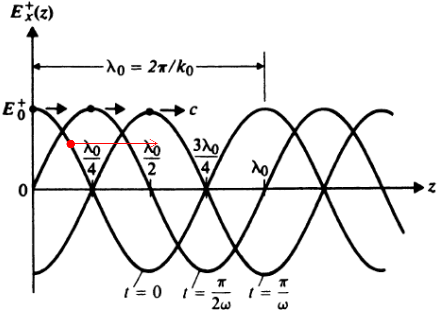


FIGURE 8–1
Wave traveling in positive z direction $E_x^+(z, t) = E_0^+ \cos(\omega t - k_0 z)$, for several values of t.

- Phase velocity: the velocity of propagation of a point of a particular phase on the wave



$$\cos(\omega t - k_0 z) = \text{a constant}$$

$$\omega t - k_0 z = \text{A constant phase,}$$

$$u_p = \frac{dz}{dt} = \frac{\omega}{k_0} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c.$$

FIGURE 8–1
Wave traveling in positive z direction $E_x^+(z, t) = E_0^+ \cos(\omega t - k_0 z)$, for several values of t.

$$u_p = \frac{dz}{dt} = \frac{\omega}{k_0} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c.$$

$$k_0 = 2\pi f / c$$

Wavenumber: The number of wavelength in a complete cycle

$$\boxed{k_0 = \frac{2\pi}{\lambda_0} \qquad \text{(rad/m)},}$$

Inverse relation

$$\boxed{\lambda_0 = \frac{2\pi}{k_0} \qquad \text{(m)}.}$$

For lossless dielectrics:

$k=2\pi/\lambda$

$\lambda=2\pi/k$

$E_0^+ e^{-jk_0 z}$ A wave traveling in the +z direction $u_p = \frac{dz}{dt} = \frac{\omega}{k_0}$ >0

$E_0^- e^{jk_0 z}$ A wave traveling in the -z direction

The Associated Magnetic Fields H

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H},$$

Consider only the wave traveling in the +z direction

$$E_x(z) = E_x^+(z)$$
$$= E_0^+ e^{-jk_0 z}$$

Only z dependence

$$\nabla \times \mathbf{E} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ E_x^+(z) & 0 & 0 \end{vmatrix} = -j\omega\mu_0(\mathbf{a}_x H_x^+ + \mathbf{a}_y H_y^+ + \mathbf{a}_z H_z^+),$$

Only E_x component

$$H_x^+ = 0,$$
$$H_y^+ = \frac{1}{-j\omega\mu_0}\frac{\partial E_x^+(z)}{\partial z},$$
$$H_z^+ = 0.$$

The only nonzero component H_y^+

$$H_y^+ = \frac{1}{-j\omega\mu_0}\frac{\partial E_x^+(z)}{\partial z},$$

$$E_x^+(z) = E_0^+ e^{-jk_0 z}$$

$$\frac{\partial E_x^+(z)}{\partial z} = \frac{\partial}{\partial z}(E_0^+ e^{-jk_0 z}) = -jk_0 E_x^+(z),$$

$$H_y^+(z) = \frac{k_0}{\omega\mu_0} E_x^+(z) = \frac{1}{\eta_0} E_x^+(z) \qquad \text{(A/m)}.$$

where

$$\boxed{\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \cong 120\pi \cong 377 \qquad (\Omega),}$$

intrinsic impedance of the free space

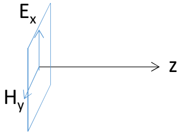
η_0 is real ➔ $H_y^+(z)$ is in phase with $E_x^+(z)$

Check time-domain **H**

$$\mathbf{H}(z, t) = \mathbf{a}_y H_y^+(z, t) = \mathbf{a}_y \mathscr{R}e[H_y^+(z)e^{j\omega t}]$$
$$= \mathbf{a}_y \frac{E_0^+}{\eta_0} \cos(\omega t - k_0 z) \qquad \text{(A/m)}.$$

For a Uniform Plane Wave

- $|\mathbf{E}|/|\mathbf{H}|=\eta_0$
- **H**, **E**, and the direction of propagation are perpendicular to each other.



EXAMPLE 8–1 A uniform plane wave with $\mathbf{E} = \mathbf{a}_x E_x$ propagates in a lossless simple medium ($\epsilon_r = 4$, $\mu_r = 1$, $\sigma = 0$) in the $+z$-direction. Assume that E_x is sinusoidal with a frequency 100 (MHz) and has a maximum value of $+10^{-4}$ (V/m) at $t = 0$ and $z = \frac{1}{8}$ (m).

a) Write the instantaneous expression for $\mathbf{E}$ for any t and z.

b) Write the instantaneous expression for $\mathbf{H}$.

c) Determine the locations where E_x is a positive maximum when $t = 10^{-8}$ (s).

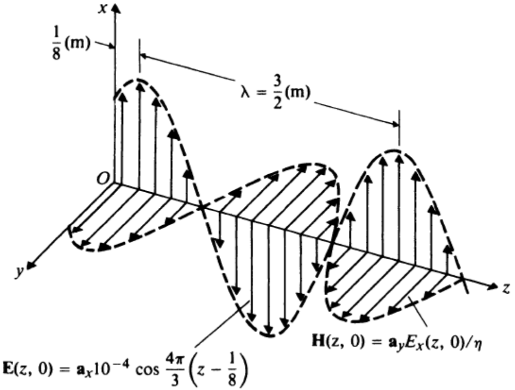


FIGURE 8–2
E and H fields of a uniform plane wave at $t = 0$ (Example 8–1).

8-2.1 Doppler Effect (excluded)

8-2.2 Transverse Electromagnetic Waves

- For a uniform plane wave, we have seen
  - $\mathbf{E} = \mathbf{a_x} E_x$; $\mathbf{H} = \mathbf{a_y} H_y$; direction of propagation in z
  - **E** and **H** are transvers to the direction of propagation, so it is called transverse electromagnetic (TEM) wave
  - Phasors **E** and **H** are functions of z only
- General case: Consider a uniform plane wave along an arbitrary direction (not necessarily coincide with a coordinate axis)

For a uniform plane wave

$$\mathbf{E}(z) = \mathbf{E}_0 e^{-jkz},$$

Propagating in the +z direction

General case $\mathbf{E}(x, y, z) = \mathbf{E}_0 e^{-jk_x x - jk_y y - jk_z z}$

Propagating in the +x, +y, +z direction

Substitution in Helmholtz's equation

$$\nabla^2 \mathbf{E} + k_0^2 \mathbf{E} = 0,$$

$$k_x^2 + k_y^2 + k_z^2 = \omega^2 \mu \epsilon.$$

$$\boxed{\mathbf{E}(x, y, z) = \mathbf{E}_0 e^{-jk_x x - jk_y y - jk_z z}.}$$

Define a wavenumber vector **k**:

$$\mathbf{k} = \mathbf{a}_x k_x + \mathbf{a}_y k_y + \mathbf{a}_z k_z = k\mathbf{a}_n$$

$$\mathbf{R} = \mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z,$$

$$\boxed{\mathbf{E}(\mathbf{R}) = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{R}} = \mathbf{E}_0 e^{-jk\mathbf{a}_n\cdot\mathbf{R}} \qquad \text{(V/m)},}$$

$\mathbf{a}_n$: unit vector of **k**; direction of propagation (explained next)

$$\mathbf{k} = \mathbf{a}_x k_x + \mathbf{a}_y k_y + \mathbf{a}_z k_z = k\mathbf{a}_n \Rightarrow \begin{aligned} k_x &= \mathbf{k}\cdot\mathbf{a}_x = k\mathbf{a}_n\cdot\mathbf{a}_x, \\ k_y &= \mathbf{k}\cdot\mathbf{a}_y = k\mathbf{a}_n\cdot\mathbf{a}_y, \\ k_z &= \mathbf{k}\cdot\mathbf{a}_z = k\mathbf{a}_n\cdot\mathbf{a}_z, \end{aligned}$$

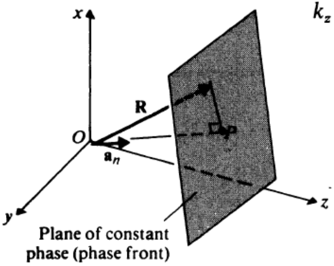


FIGURE 8–4
Radius vector and wave normal to a phase front of a uniform plane wave.

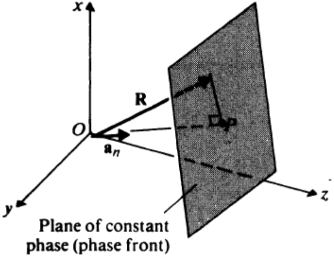


FIGURE 8–4
Radius vector and wave normal to a phase front of a uniform plane wave.

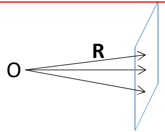
$$\mathbf{a}_n \cdot \mathbf{R} = \text{Length } \overline{OP} \quad \text{(a constant)}$$

$$\boxed{\mathbf{E}(\mathbf{R}) = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{R}} = \mathbf{E}_0 e^{-jk\mathbf{a}_n\cdot\mathbf{R}} \qquad \text{(V/m)},}$$

= phase

For constant phase, $\mathbf{a}_n \cdot \mathbf{R}$ = constant = $\overline{OP}$

R forms a constant-phase plane
$\mathbf{a}_n$ // $\hat{\boldsymbol{n}}$ of constant-phase plane // direction of propagation



In a charge-free region, $\nabla \cdot \mathbf{E} = 0$

$$\mathbf{E}(\mathbf{R}) = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{R}} = \mathbf{E}_0 e^{-jk\mathbf{a}_n\cdot\mathbf{R}}$$

$$\nabla \cdot (\mathbf{E_0}\, e^{-j\mathbf{k}\cdot\mathbf{R}}) = e^{-j\mathbf{k}\cdot\mathbf{R}}\, \nabla\cdot\mathbf{E_0} + \mathbf{E_0}\cdot\nabla(e^{-j\mathbf{k}\cdot\mathbf{R}})$$

$$\mathbf{E}_0 \cdot \nabla(e^{-jk\mathbf{a}_n\cdot\mathbf{R}}) = 0.$$

$$\begin{aligned}\nabla(e^{-jk\mathbf{a}_n\cdot\mathbf{R}}) &= \left(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}\right) e^{-j(k_x x + k_y y + k_z z)} \\ &= -j(\mathbf{a}_x k_x + \mathbf{a}_y k_y + \mathbf{a}_z k_z) e^{-j(k_x x + k_y y + k_z z)} \\ &= -jk\mathbf{a}_n e^{-jk\mathbf{a}_n\cdot\mathbf{R}},\end{aligned}$$

$$-jk(\mathbf{E}_0 \cdot \mathbf{a}_n) e^{-jk\mathbf{a}_n\cdot\mathbf{R}} = 0,$$

which requires $\mathbf{a}_n \cdot \mathbf{E}_0 = 0.$

Thus, for a plane-wave solution, $\mathbf{E_0} \perp \mathbf{a}_n$

The Associated Magnetic Fields H

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H},$$

$$\Downarrow$$

$$\mathbf{H}(\mathbf{R}) = -\frac{1}{j\omega\mu}\nabla \times \mathbf{E}(\mathbf{R})$$

$\nabla\times$**E**=?

$$\nabla \times (\psi \mathbf{A}) = \psi \nabla \times \mathbf{A} + \nabla \psi \times \mathbf{A}$$

$$\mathbf{E}(\mathbf{R}) = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{R}} = \mathbf{E}_0 e^{-jk\mathbf{a}_n\cdot\mathbf{R}}$$

$$\Downarrow$$

$$\boxed{\mathbf{H}(\mathbf{R}) = \frac{1}{\eta}\mathbf{a}_n \times \mathbf{E}(\mathbf{R}) \qquad (\mathrm{A/m}),}$$

where

$$\boxed{\eta = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\epsilon}} \qquad (\Omega)}$$

the intrinsic impedance of the medium

$$\mathbf{H}(\mathbf{R}) = \frac{1}{\eta}\,\mathbf{a}_n \times \mathbf{E}(\mathbf{R}) \qquad \text{(A/m)},$$

$$\mathbf{E}(\mathbf{R}) = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{R}} = \mathbf{E}_0 e^{-jk\mathbf{a}_n\cdot\mathbf{R}}$$

$$\mathbf{H}(\mathbf{R}) = \frac{1}{\eta}(\mathbf{a}_n \times \mathbf{E}_0)e^{-jk\mathbf{a}_n\cdot\mathbf{R}} \qquad \text{(A/m)}.$$

A uniform plane wave propagating in an arbitrary direction, $\mathbf{a}_n$:

- a TEM wave
- $\mathbf{H} \perp \mathbf{E}$; $\mathbf{H} \perp \mathbf{a}_n$; $\mathbf{E} \perp \mathbf{a}_n$

8-2.3 Polarization of Plane Waves

- Polarization of a uniform plane wave: time-varying behavior of **E** vector at a given point in space.
    - E.g., $\mathbf{E} = \mathbf{a_x}E_x$, the wave is linearly polarized in x direction

- In some cases, direction of **E** of a plane wave may change with time
  - Two linearly polarized waves in *x* and *y* direction

Phasor notation:

$$\mathbf{E}(z) = \mathbf{a}_x E_1(z) + \mathbf{a}_y E_2(z)$$
$$= \mathbf{a}_x E_{10} e^{-jkz} - \mathbf{a}_y j E_{20} e^{-jkz},$$

$\boxed{\mathbf{a}_y \text{ lags } 90°}$

E_{10}, E_{20}: real numbers, denoting amplitudes

Time-domain expression:

$$\mathbf{E}(z, t) = \mathcal{R}e\{[\mathbf{a}_x E_1(z) + \mathbf{a}_y E_2(z)]e^{j\omega t}\}$$
$$= \mathbf{a}_x E_{10} \cos(\omega t - kz) + \mathbf{a}_y E_{20} \cos\left(\omega t - kz - \frac{\pi}{2}\right).$$

  - Examine the direction change of **E** at a given point as *t* changes (z = 0 for convenience)

$$\mathbf{E}(0, t) = \mathbf{a}_x E_1(0, t) + \mathbf{a}_y E_2(0, t)$$
$$= \mathbf{a}_x E_{10} \cos \omega t + \mathbf{a}_y E_{20} \sin \omega t.$$

E(0, *t*): the sum of two linearly polarized waves in both space quadrature (**a**$_x$ and **a**$_y$) and time quadrature (cos ωt and sin ωt)

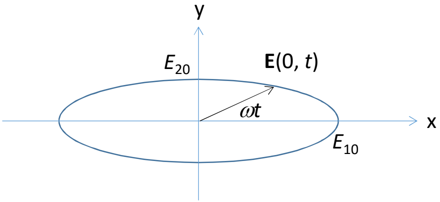
$$\mathbf{E}(0, t) = \mathbf{a}_x E_1(0, t) + \mathbf{a}_y E_2(0, t)$$
$$= \underline{\mathbf{a}_x E_{10} \cos \omega t} + \underline{\mathbf{a}_y E_{20} \sin \omega t}.$$

$$\cos \omega t = \frac{E_1(0, t)}{E_{10}}$$

$$\sin \omega t = \frac{E_2(0, t)}{E_{20}}$$
$$= \sqrt{1 - \cos^2 \omega t} = \sqrt{1 - \left[\frac{E_1(0, t)}{E_{10}}\right]^2},$$

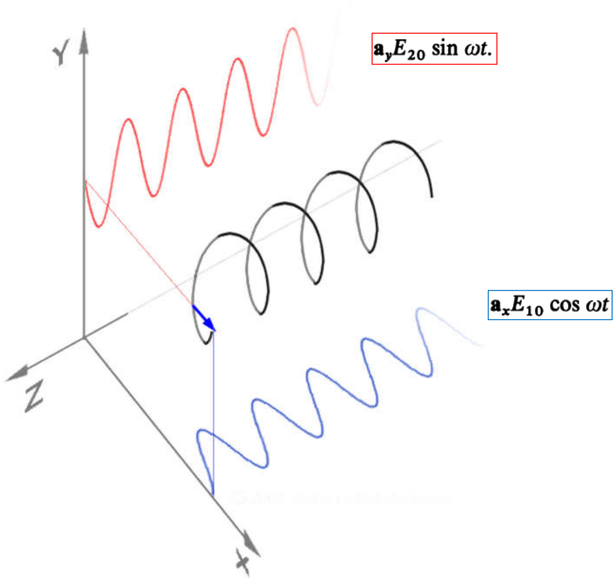
$$\left[\frac{E_2(0, t)}{E_{20}}\right]^2 + \left[\frac{E_1(0, t)}{E_{10}}\right]^2 = 1.$$

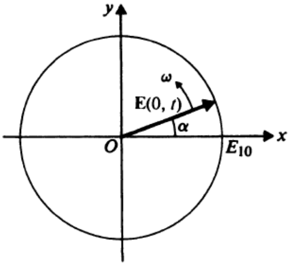
As ωt increases, **E**(0, *t*) will traverse an elliptical locus in the counterclockwise direction



If $E_{20} \neq E_{10}$, elliptical polarization

$$\mathbf{E}(0, t) = \mathbf{a}_x E_1(0, t) + \mathbf{a}_y E_2(0, t)$$
$$= \mathbf{a}_x E_{10} \cos \omega t + \mathbf{a}_y E_{20} \sin \omega t.$$





If $E_{20} = E_{10}$, circular polarization

$$\cos \omega t = \frac{E_1(0, t)}{E_{10}}$$

$$\sin \omega t = \frac{E_2(0, t)}{E_{20}}$$

And the angle $\alpha = \tan^{-1} \dfrac{E_2(0, t)}{E_1(0, t)} = \omega t,$

E(0, *t*) rotates counterclockwise

Right-hand (positive) circularly polarized wave:

- finger: direction of rotation of **E**
- thumb: direction of propagation (+z)

For $\mathbf{a}_y$ lags 90°

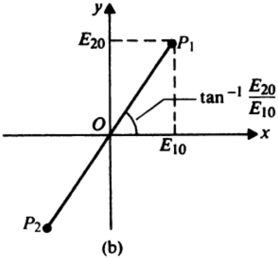
Left-hand (negative) circularly polarized wave:

For $\mathbf{a}_y$ leads 90°

FIGURE 8–5
Polarization diagrams for sum of two linearly polarized waves in space quadrature at $z = 0$: (a) circular polarization, $\mathbf{E}(0, t) = E_{10}(\mathbf{a}_x \cos \omega t + \mathbf{a}_y \sin \omega t)$;

E(0,t): the sum of two linearly polarized waves in space quadrature ($\mathbf{a}_x$ and $\mathbf{a}_y$) but in time phase

$$\mathbf{E}(0, t) = (\mathbf{a}_x E_{10} + \mathbf{a}_y E_{20}) \cos \omega t.$$



tip at P_1 when $t = 0$;
tip at origin O when $\omega t = \pi/2$;
tip at P_2 when $\omega t = \pi$
➔ Linear polarization

FIGURE 8–5
Polarization diagrams for sum of two linearly polarized waves in space quadrature at $z = 0$:
(b) linear polarization,
$\mathbf{E}(0, t) = (\mathbf{a}_x E_{10} + \mathbf{a}_y E_{20}) \cos \omega t.$

Polarizations

- AM broadcast station: $\updownarrow$
- Television signals: $\longleftrightarrow$
- FM broadcast stations: $\bigcirc$
- Receiving antennas should have similar orientation to get the best signals

8-3 Plane Waves in Lossy Media

- Wave equation for source free and in lossy media:

  $k \rightarrow k_c$ ⇨ $$\nabla^2 \mathbf{E} + k_c^2 \mathbf{E} = 0,$$

  $$k_c = \omega\sqrt{\mu\epsilon_c}$$ A complex number

  - Conventional notation in transmission-line theory: propagation constant γ

$$\boxed{\gamma = jk_c = j\omega\sqrt{\mu\epsilon_c} \qquad (\mathrm{m}^{-1}).}$$

γ is complex

$$\begin{aligned} k_c &= \omega\sqrt{\mu\epsilon_c} \\ &= \omega\sqrt{\mu(\epsilon' - j\epsilon'')} \end{aligned}$$

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon'}\left(1 - j\frac{\epsilon''}{\epsilon'}\right)^{1/2},$$

A special case:
$\varepsilon'' = \sigma/\omega$

$$\epsilon_c = \epsilon - j\frac{\sigma}{\omega}$$

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon}\left(1 + \frac{\sigma}{j\omega\epsilon}\right)^{1/2},$$

- For a lossless medium, σ=0
  - σ=0

$$\boxed{\gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon}\left(1 + \frac{\sigma}{j\omega\epsilon}\right)^{1/2},}$$

  - α=0,

$$\gamma = j\beta$$

$$\beta = k = \omega\sqrt{\mu\epsilon}$$

- Wave equation in lossy media expressed by γ:

$$\nabla^2 \mathbf{E} + k_c^2 \mathbf{E} = 0, \quad \Rightarrow \quad \nabla^2 \mathbf{E} - \gamma^2 \mathbf{E} = 0.$$

$$\boxed{\gamma = jk_c}$$

  - Solution of a uniform plane wave
    - Propagating in +*z*
    - Linearly polarized in *x*

$$\mathbf{E} = \mathbf{a}_x E_x = \mathbf{a}_x E_0 e^{-\gamma z},$$

$$\Downarrow \quad \boxed{\gamma = \alpha + j\beta}$$

$$E_x = E_0 \underbrace{e^{-\alpha z}}_{\text{Attenuation along +z}} \underbrace{e^{-j\beta z}}_{\text{Propagation along +z}}.$$

  - $\alpha>0$, $\beta>0$
    - $e^{-\alpha z}$ decreases as *z* increases, so it is called attenuation factor (α: attenuation constant)
    - $e^{-j\beta z}$ determines the phase, so it is called phase factor (β: phase constant)

8-3.1 Low-Loss Dielectrics

- Low-loss dielectrics = good but imperfect insulator
  - Low $\sigma \rightarrow$ small current $\rightarrow$ low loss
  - $\varepsilon'' << \varepsilon'$ (or $\sigma/\omega\varepsilon<<1$)

$$\boxed{\gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon'}\left(1 - j\frac{\epsilon''}{\epsilon'}\right)^{1/2},}$$

Binomial expansion
$\varepsilon''/\varepsilon' << 1 \Rightarrow$ neglect H.O.T.

$$\gamma = \alpha + j\beta \cong j\omega\sqrt{\mu\epsilon'}\left[1 - j\frac{\epsilon''}{2\epsilon'} + \frac{1}{8}\left(\frac{\epsilon''}{\epsilon'}\right)^2\right],$$

$$\gamma = \alpha + j\beta \cong \underline{j\omega\sqrt{\mu\epsilon'}}\left[1 - j\underline{\frac{\epsilon''}{2\epsilon'}} + \frac{1}{8}\left(\frac{\epsilon''}{\epsilon'}\right)^2\right],$$

Attenuation constant $$\alpha \cong \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} \quad \text{(Np/m)}$$

Phase constant $$\beta \cong \omega\sqrt{\mu\epsilon'}\left[1 + \frac{1}{8}\left(\frac{\epsilon''}{\epsilon'}\right)^2\right] \quad \text{(rad/m)}.$$

- α >0, α is proportional to ω
- When $\varepsilon''/\varepsilon'$ → 0, β reduces to the case of lossless dielectrics

The intrinsic impedance $\eta_c = (\mu/\varepsilon)^{1/2} = (\mu/(\varepsilon' - j\varepsilon''))^{1/2}$

$$\eta_c = \sqrt{\frac{\mu}{\epsilon'}}\left(1 - j\frac{\epsilon''}{\epsilon'}\right)^{-1/2}$$

$$\cong \sqrt{\frac{\mu}{\epsilon'}}\left(1 + j\frac{\epsilon''}{2\epsilon'}\right) \quad (\Omega).$$

For $\varepsilon'' << \varepsilon'$

For a uniform plane wave $\eta_c = E_x/H_y$

- In lossless case, η is real, E_x and H_y are in time phase
- In low-loss case, η_c is complex, E_x/H_y are out of phase

The phase velocity $u_p = \omega/\beta$

$$u_p = \frac{\omega}{\beta} \cong \frac{1}{\sqrt{\mu\epsilon'}}\left[1 - \frac{1}{8}\left(\frac{\epsilon''}{\epsilon'}\right)^2\right] \quad \text{(m/s)}.$$

$$\boxed{\beta \cong \omega\sqrt{\mu\epsilon'}\left[1 + \frac{1}{8}\left(\frac{\epsilon''}{\epsilon'}\right)^2\right]}$$

8-3.2 Good Conductors

- A good conductor
  - $\sigma/\omega\varepsilon >> 1$

$$\boxed{\gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon}\left(1 + \frac{\sigma}{j\omega\epsilon}\right)^{1/2},}$$

$\sigma/\omega\varepsilon >> 1$

$$\sqrt{j} = (e^{j\pi/2})^{1/2} = e^{j\pi/4} = (1 + j)/\sqrt{2}$$

$$\gamma \cong j\omega\sqrt{\mu\epsilon}\sqrt{\frac{\sigma}{j\omega\epsilon}} = \sqrt{j}\sqrt{\omega\mu\sigma} = \frac{1+j}{\sqrt{2}}\sqrt{\omega\mu\sigma}$$

$$\omega = 2\pi f.$$

$$\gamma = \alpha + j\beta \cong (1 + j)\sqrt{\pi f\mu\sigma},$$

$$\boxed{\gamma = \alpha + j\beta \cong (1+j)\sqrt{\pi f \mu \sigma},}$$

$$\boxed{\alpha = \beta = \sqrt{\pi f \mu \sigma}.}$$

α and β are approximately equal

$$\alpha, \beta \sim f^{1/2}, \mu^{1/2}, \sigma^{1/2}$$

The intrinsic impedance $\eta_c = (\mu/\varepsilon)^{1/2} = (\mu/\varepsilon_c)^{1/2}$

$$\eta_c = \sqrt{\frac{\mu}{\epsilon_c}} \cong \sqrt{\frac{j\omega\mu}{\sigma}} = (1+j)\sqrt{\frac{\pi f \mu}{\sigma}} = (1+j)\frac{\alpha}{\sigma} \qquad (\Omega),$$

$\varepsilon_c \cong -j\sigma/\omega$ for a good conductor ($\sigma/\omega >> \varepsilon'$)

For a uniform plane wave $\eta_c = E_x/H_y$

- For a good conductor ($\angle\eta_c = 45°$), H_y lags E_x by 45°

The phase velocity $u_p = \omega/\beta$

$$u_p = \frac{\omega}{\beta} \cong \sqrt{\frac{2\omega}{\mu\sigma}} \qquad \text{(m/s)},$$

The wavelength of a plane wave

$$\lambda = \frac{2\pi}{\beta} = \frac{u_p}{f} = 2\sqrt{\frac{\pi}{f\mu\sigma}} \qquad \text{(m)}.$$

Example: copper, f = 3MHz

1. u_p

$$\sigma = 5.80 \times 10^7 \quad \text{(S/m)},$$
$$\mu = 4\pi \times 10^{-7} \quad \text{(H/m)},$$

$$\boxed{u_p = \frac{\omega}{\beta} \cong \sqrt{\frac{2\omega}{\mu\sigma}} \quad \text{(m/s)},}$$

$$u_p = 720 \text{ (m/s)} \quad \text{at} \quad 3 \text{ (MHz)},$$

Much slower than the velocity of light in air

2. λ

$$\boxed{\lambda = u_p/f}$$

$$\lambda = 0.24 \text{ (mm)}.$$

Much shorter than electromagnetic wave in air (λ = 100m)

3. α

$$\boxed{\alpha = \beta = \sqrt{\pi f \mu \sigma}.}$$

$$\alpha = \sqrt{\pi(3 \times 10^6)(4\pi \times 10^{-7})(5.80 \times 10^7)} = 2.62 \times 10^4 \quad \text{(Np/m)}.$$

Very large attenuation

Skin Depth (Depth of Penetration)

- Attenuation: $e^{-\alpha z}$,
- When z = 1/α, the intensity reduces to e^{-1}
  ➔ The amplitude of a wave will be attenuated by a factor of e^{-1} (= 0.368) when it travels a distance δ = 1/α (skin depth)

$$\boxed{\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f \mu \sigma}} \qquad \text{(m).}}$$

  - For copper, f = 3MHz: δ = 1/α = 1/(2.62×10^4) (m) = 0.038 (mm)
  - For copper, f = 10GHz: δ = 0.66 (μm) (very small distance)

- Thus, a high-frequency electromagnetic wave is attenuated very rapidly as it propagates in a good conductor. $\delta \sim \frac{1}{\sqrt{f}}$
- At microwave frequencies, the fields and currents can be considered confined in a very thin layer (i.e., in the skin) of the conductor surface.
- For a good conductor, $\alpha = \beta$, $\delta = 1/\beta$

$$\boxed{\delta = \frac{1}{\beta} = \frac{\lambda}{2\pi} \qquad \text{(m)}.}$$

TABLE 8–1
Skin Depths, δ in (mm), of Various Materials

Material	σ (S/m)	$f = 60$ (Hz)	1 (MHz)	1 (GHz)
Silver	6.17×10^7	8.27 (mm)	0.064 (mm)	0.0020 (mm)
Copper	5.80×10^7	8.53	0.066	0.0021
Gold	4.10×10^7	10.14	0.079	0.0025
Aluminum	3.54×10^7	10.92	0.084	0.0027
Iron ($\mu_r \cong 10^3$)	1.00×10^7	0.65	0.005	0.00016
Seawater	4	32 (m)	0.25 (m)	†

EXAMPLE 8–4 The electric field intensity of a linearly polarized uniform plane wave propagating in the $+z$-direction in seawater is $\mathbf{E} = \mathbf{a}_x 100 \cos(10^7 \pi t)$ (V/m) at $z = 0$. The constitutive parameters of seawater are $\epsilon_r = 72$, $\mu_r = 1$, and $\sigma = 4$ (S/m). (a) Determine the attenuation constant, phase constant, intrinsic impedance, phase velocity, wavelength, and skin depth. (b) Find the distance at which the amplitude of $\mathbf{E}$ is 1% of its value at $z = 0$. (c) Write the expressions for $\mathbf{E}(z, t)$ and $\mathbf{H}(z, t)$ at $z = 0.8$ (m) as functions of t.

8-3.3 Ionized Gases (excluded)

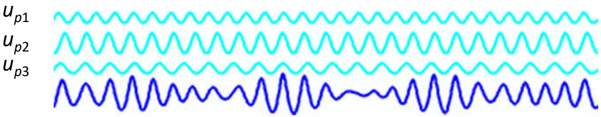
8-4 Group Velocity

- Phase velocity: velocity of propagation of an equiphase wavefront
  - For plane waves in lossless media:
    β is a linear function of ω $\beta = \omega\sqrt{\mu\epsilon}$
    ➔ $u_p = 1/\sqrt{\mu\epsilon}$ a constant, independent of ω

$$\boxed{u_p = \frac{\omega}{\beta} \qquad \text{(m/s).}}$$

  - In some cases (e.g., in lossy dielectrics):
    β is not a linear function of ω
    ➔ $u_p(\omega)$

- Dispersion: signal distortion due to $u_p(\omega)$
  - Waves of the component frequencies travel with different phase velocities ➔ a distortion in the signal wave shape
  - A lossy dielectric is a dispersive medium



a distortion in waveform

Through a non-dispersive medium

Through a dispersive medium

Waveform is changing with time when they propagate (i.e., distortion)

Group Velocity

- An information bearing signal has a small spread of frequencies (Δf) around a high carrier frequency (f_c).
- Group velocity u_g: the velocity of propagation of the wave-packet envelope of a group of frequencies

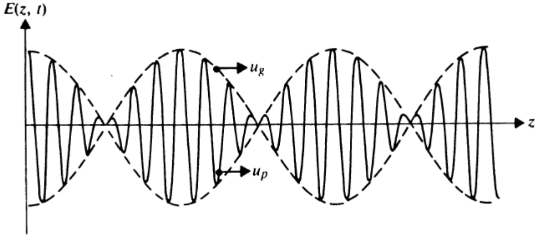


FIGURE 8–6
Sum of two time-harmonic traveling waves of equal amplitude and slightly different frequencies at a given t.

- Consider the simplest case of a wave packet
  - Two travelling waves with equal amplitude and slightly different angular frequencies $\omega_0 + \Delta\omega \quad \omega_0 - \Delta\omega \quad (\Delta\omega \ll \omega_0)$

    ➔ the corresponding phase constants

$$\beta_0 + \Delta\beta \qquad \beta_0 - \Delta\beta.$$

$$\Downarrow$$

$$\begin{aligned} E(z,t) &= E_0 \cos\left[(\omega_0 + \Delta\omega)t - (\beta_0 + \Delta\beta)z\right] \\ &\quad + E_0 \cos\left[(\omega_0 - \Delta\omega)t - (\beta_0 - \Delta\beta)z\right] \\ &= 2E_0 \cos(t\,\Delta\omega - z\,\Delta\beta)\cos(\omega_0 t - \beta_0 z). \end{aligned}$$

$$E(z, t) = E_0 \cos[(\omega_0 + \Delta\omega)t - (\beta_0 + \Delta\beta)z] + E_0 \cos[(\omega_0 - \Delta\omega)t - (\beta_0 - \Delta\beta)z] = 2E_0 \underbrace{\cos(t\,\Delta\omega - z\,\Delta\beta)}_{\text{A slowly-varying envelope } (\Delta\omega)} \underbrace{\cos(\omega_0 t - \beta_0 z)}_{\text{A rapidly oscillating wave } (\omega_0)}.$$

$\Delta\omega \ll \omega_0$

A slowly-varying envelope ($\Delta\omega$)

A rapidly oscillating wave (ω_0)

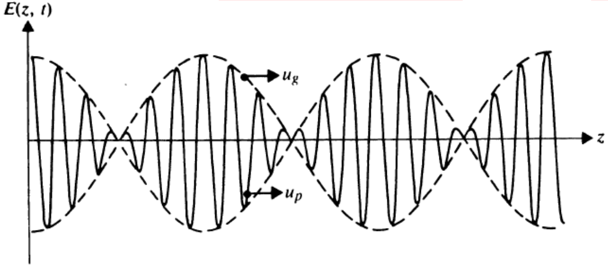
$\omega_0 t - \beta_0 z = \text{Constant}$

group velocity $u_g = \dfrac{dz}{dt} = \dfrac{\Delta\omega}{\Delta\beta} = \dfrac{1}{\Delta\beta/\Delta\omega}$.

Velocity of the envelope

$u_p = \dfrac{dz}{dt} = \dfrac{\omega_0}{\beta_0}$.

Velocity of the carrier



Group velocity: the velocity of a point on the envelope of the wave packet

FIGURE 8–6
Sum of two time-harmonic traveling waves of equal amplitude and slightly different frequencies at a given t.

$$u_g = \frac{dz}{dt} = \frac{\Delta\omega}{\Delta\beta} = \frac{1}{\Delta\beta/\Delta\omega}.$$

In the limit $\Delta\omega \to 0$ (narrow-band signal)

$$\boxed{u_g = \frac{1}{d\beta/d\omega} \qquad \text{(m/s).}}$$

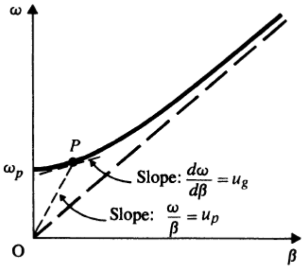


FIGURE 8–7
ω–β graph for ionized gas.

u_p: the slope drawn from the origin to a point

u_g: local slope

- In an ionized medium:

8-3.3

$$\boxed{\gamma = j\omega\sqrt{\mu\epsilon_0}\sqrt{1-\left(\frac{f_p}{f}\right)^2},}$$

$$\beta = \omega\sqrt{\mu\epsilon_0}\sqrt{1-\left(\frac{f_p}{f}\right)^2}$$
$$= \frac{\omega}{c}\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}.$$

  - At $\omega = \omega_p$, $\beta = 0$.
  - At $\omega > \omega_p$, wave propagation is possible

$$\boxed{u_g = \frac{1}{d\beta/d\omega}}$$

$$u_p = \frac{\omega}{\beta} = \frac{c}{\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}}. \qquad u_g = c\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}.$$

$$\boxed{u_p \geq c \text{ and } u_g \leq c;\ u_p u_g = c^2}$$

Similar situation exists in waveguides!

- A general relation between u_g and u_p:

$$\boxed{u_p = \frac{\omega}{\beta}} \qquad \boxed{u_g = \frac{1}{d\beta/d\omega}}$$

$$\frac{d\beta}{d\omega} = \frac{d}{d\omega}\left(\frac{\omega}{u_p}\right) = \frac{1}{u_p} - \frac{\omega}{u_p^2}\frac{du_p}{d\omega}.$$

$$\boxed{u_g = \frac{1}{d\beta/d\omega}}$$

$$u_g = \frac{u_p}{1 - \dfrac{\omega}{u_p}\dfrac{du_p}{d\omega}}.$$

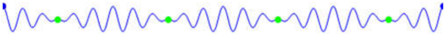
a) No dispersion: $\dfrac{du_p}{d\omega} = 0$ $\quad u_g = u_p.$ $\quad u_p$ independent of ω

b) Normal dispersion: $\dfrac{du_p}{d\omega} < 0$ $\quad u_g < u_p.$ $\quad u_p$ decreasing with ω

c) Anomalous dispersion: $\dfrac{du_p}{d\omega} > 0$ $\quad u_g > u_p.$ $\quad u_p$ increasing with ω

Example of a material with normal dispersion?

No dispersion: $\dfrac{du_p}{d\omega} = 0$ $u_g = u_p$. u_p independent of ω



Normal dispersion: $\dfrac{du_p}{d\omega} < 0$ $u_g < u_p$. u_p decreasing with ω

Red dot: u_p; Green dot: u_g

EXAMPLE 8–6 A narrow-band signal propagates in a lossy dielectric medium which has a loss tangent 0.2 at 550 (kHz), the carrier frequency of the signal. The dielectric constant of the medium is 2.5. (a) Determine α and β. (b) Determine u_p and u_g. Is the medium dispersive?

8-5 Flow of Electromagnetic Power and the Poynting Vector

- Electromagnetic waves carry with them electromagnetic power.
- Energy is transported through space to distant receiving points by electromagnetic waves.

Vector identity

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}).$$

$$\boxed{\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \end{aligned}}$$

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{E} \cdot \mathbf{J}.$$

In a simple medium, whose ε, μ, and σ do not change with time

Product rule

$$\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = \mathbf{H} \cdot \frac{\partial (\mu \mathbf{H})}{\partial t} = \frac{1}{2} \frac{\partial (\mu \mathbf{H} \cdot \mathbf{H})}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right),$$

$$\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = \mathbf{E} \cdot \frac{\partial (\epsilon \mathbf{E})}{\partial t} = \frac{1}{2} \frac{\partial (\epsilon \mathbf{E} \cdot \mathbf{E})}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon E^2\right),$$

$$\mathbf{E} \cdot \mathbf{J} = \mathbf{E} \cdot (\sigma \mathbf{E}) = \sigma E^2.$$

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right) - \sigma E^2,$$

A point-function relationship

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right) - \sigma E^2,$$

Integration over the volume of concern & Divergence theorem

An integral form

$$\oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s} = \underbrace{-\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right) dv}_{\text{The time-rate of change of the energy stored in the electric and magnetic fields, respectively}} - \underbrace{\int_V \sigma E^2\, dv}_{\text{Ohmic power dissipation (due to conduction current)}},$$

Law of conservation of energy

- Right side: the rate of decrease of the electric and magnetic energies stored, subtracted by the ohmic power dissipated as heat in the volume V
- Left side: power (rate of energy) leaving the volume through its surface

Power flow per unit area $\boxed{\mathscr{P} = \mathbf{E} \times \mathbf{H} \qquad (\mathrm{W/m^2}).}$

Known as Poynting vector, a power density vector associated with electromagnetic field

- Poynting theorem: the surface integral of ***P*** (=**E**×**H**) over a closed surface = power leaving the enclosed volume

$$\boxed{\oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s} = -\frac{\partial}{\partial t} \int_V \left(\frac{1}{2} \epsilon E^2 + \frac{1}{2} \mu H^2\right) dv - \int_V \sigma E^2 \, dv,}$$

- Another form

$$-\oint_S \mathcal{P} \cdot d\mathbf{s} = \frac{\partial}{\partial t} \int_V (w_e + w_m) \, dv + \int_V p_\sigma \, dv,$$

where
$$w_e = \tfrac{1}{2}\epsilon E^2 = \tfrac{1}{2}\epsilon \mathbf{E} \cdot \mathbf{E}^* = \text{Electric energy density,}$$
$$w_m = \tfrac{1}{2}\mu H^2 = \tfrac{1}{2}\mu \mathbf{H} \cdot \mathbf{H}^* = \text{Magnetic energy density,}$$
$$p_\sigma = \sigma E^2 = J^2/\sigma = \sigma \mathbf{E} \cdot \mathbf{E}^* = \mathbf{J} \cdot \mathbf{J}^*/\sigma = \text{Ohmic power density.}$$

  - The total power flowing into a closed surface at any instant = the sum of rates of increase of the stored electric and magnetic energies and the ohmic power dissipated within the enclosed volume

$$-\oint_S \mathscr{P} \cdot d\mathbf{s} = \frac{\partial}{\partial t} \int_V (w_e + w_m)\, dv + \int_V p_\sigma\, dv,$$

- ***P*** (=**E**×**H**)
  - ***P***⊥**E**, ***P***⊥**H**
- Lossless case: σ=0
  - Right side: only the rate of increase of the stored electric and magnetic energies
- Static case: $\partial/\partial t$ = 0
  - Right side: only the ohmic power dissipated in the enclosed volume

8-5.1 Instantaneous and Average Power Densities

- Phasor: $\mathbf{E}(z) = \mathbf{a}_x E_x(z) = \mathbf{a}_x E_0 e^{-(\alpha + j\beta)z}$,

  Instantaneous expression: $$\mathbf{E}(z, t) = \underline{\mathcal{Re}}[\mathbf{E}(z)\underline{e^{j\omega t}}] = \mathbf{a}_x E_0 e^{-\alpha z}\, \mathcal{Re}[e^{j(\omega t - \beta z)}]$$ $$= \mathbf{a}_x E_0 e^{-\alpha z} \cos(\omega t - \beta z).$$

- For a uniform plane wave propagating in a lossy medium in the +z direction, the **H** field:

  Phasor $$\mathbf{H}(z) = \mathbf{a}_y H_y(z) = \mathbf{a}_y \frac{E_0}{|\eta|} e^{-\alpha z} e^{\underline{-j(\beta z} \underline{+\theta_\eta)}},$$

  Due to lossy media, $\exp(-j\vartheta_\eta)$

  The intrinsic impedance of the medium $\eta = |\eta| e^{j\theta_\eta}$

  +z propagation

  Instantaneous expression $$\mathbf{H}(z, t) = \mathcal{Re}[\mathbf{H}(z)e^{j\omega t}] = \mathbf{a}_y \frac{E_0}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta).$$

$$\mathcal{R}e[\mathbf{E}(z)e^{j\omega t}] \times \mathcal{R}e[\mathbf{H}(z)e^{j\omega t}] \neq \mathcal{R}e[\mathbf{E}(z) \times \mathbf{H}(z)e^{j\omega t}].$$

Or a·cos ωt × b·cos $\omega t \neq$ ab cos ωt

To get time-domain Poynting vector $\mathcal{P}(z, t)$, one cannot do the simple cross product of **E** and **H** in phasor domain and then change it to time-domain expression!

Method 1: check in time domain directly

Thus, for the instantaneous expression for Poynting vector:

$$\mathbf{E}(z, t) = \mathbf{a}_x E_0 e^{-\alpha z} \cos(\omega t - \beta z).$$

$$\mathbf{H}(z, t) = \mathbf{a}_y \frac{E_0}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta).$$

$$\begin{aligned}\mathcal{P}(z, t) &= \mathbf{E}(z, t) \times \mathbf{H}(z, t) = \underline{\mathcal{R}e[\mathbf{E}(z)e^{j\omega t}] \times \mathcal{R}e[\mathbf{H}(z)e^{j\omega t}]} \\ &= \mathbf{a}_z \frac{E_0^2}{|\eta|} e^{-2\alpha z} \cos(\omega t - \beta z) \cos(\omega t - \beta z - \theta_\eta) \\ &= \mathbf{a}_z \frac{E_0^2}{2|\eta|} e^{-2\alpha z}[\cos\theta_\eta + \cos(2\omega t - 2\beta z - \theta_\eta)].\end{aligned}$$

E, **H** in phasors

Correct expression for instantaneous $\mathcal{P}$

Obviously, not equal to

$$\mathcal{R}e[\mathbf{E}(z) \times \mathbf{H}(z)e^{j\omega t}] = \mathbf{a}_z \frac{E_0^2}{|\eta|} e^{-2\alpha z} \cos(\omega t - 2\beta z - \theta_\eta),$$

Time-average Poynting vector, $\boldsymbol{P}_{\text{av}}(z)$:

$$\mathscr{P}(z, t) = \mathbf{a}_z \frac{E_0^2}{2|\eta|} e^{-2\alpha z}[\underbrace{\cos \theta_\eta}_{\text{Independent of } t} + \underbrace{\cos (2\omega t - 2\beta z - \theta_\eta)}_{\text{Average = 0 over } T/2}].$$

Integration over a period T

$$\mathscr{P}_{\text{av}}(z) = \frac{1}{T} \int_0^T \mathscr{P}(z, t)\, dt = \mathbf{a}_z \frac{E_0^2}{2|\eta|} e^{-2\alpha z} \cos \theta_\eta \qquad (\text{W/m}^2),$$

where $T = 2\pi/\omega$

As far as the power transmitted by an electromagnetic wave is concerned, its average value is a more significant quantity than its instantaneous value.

Method 2: check in time domain with phasor expression

Consider two general complex vectors **A** and **B**:

*: complex conjugate of

$$\mathscr{Re}(\mathbf{A}) = \tfrac{1}{2}(\mathbf{A} + \mathbf{A}^*) \quad \text{and} \quad \mathscr{Re}(\mathbf{B}) = \tfrac{1}{2}(\mathbf{B} + \mathbf{B}^*),$$

$$\begin{aligned}\mathscr{Re}(\mathbf{A}) \times \mathscr{Re}(\mathbf{B}) &= \tfrac{1}{2}(\mathbf{A} + \mathbf{A}^*) \times \tfrac{1}{2}(\mathbf{B} + \mathbf{B}^*)\\ &= \tfrac{1}{4}[(\mathbf{A} \times \mathbf{B}^* + \mathbf{A}^* \times \mathbf{B}) + (\mathbf{A} \times \mathbf{B} + \mathbf{A}^* \times \mathbf{B}^*)]\\ &= \tfrac{1}{2}\mathscr{Re}(\underline{\mathbf{A} \times \mathbf{B}^* + \mathbf{A} \times \mathbf{B}}).\end{aligned}$$

(C+C*)=2Re[C]

Express the instantaneous Poynting vector (in phasors):

$$\begin{aligned}\mathscr{P}(z,t) &= \mathscr{Re}[\mathbf{E}(z)e^{j\omega t}] \times \mathscr{Re}[\mathbf{H}(z)e^{j\omega t}]\\ &= \tfrac{1}{2}\mathscr{Re}[\underline{\mathbf{E}(z) \times \mathbf{H}^*(z) + \mathbf{E}(z) \times \mathbf{H}(z)e^{j2\omega t}}].\end{aligned}$$

Let **A** = **E**$e^{j\omega t}$; **B** = **H**$e^{j\omega t}$

Independent of t

Integrating $\mathscr{P}(z, t)$ over a fundamental period T

Average of the last term ($e^{j2\omega t}$) vanishes

Time-average Poynting vector, $\mathscr{P}_{av}(z)$:

$$\mathscr{P}_{av}(z) = \tfrac{1}{2}\mathscr{Re}[\mathbf{E}(z) \times \mathbf{H}^*(z)].$$

The general formula for computing the average power density in a propagating wave $\mathcal{P}_{av}$:

$$\boxed{\mathcal{P}_{av}(z) = \tfrac{1}{2}\mathcal{R}e[\mathbf{E}(z) \times \mathbf{H}^*(z)].}$$

Not necessarily propagating in *z* direction

General expression

$$\boxed{\mathcal{P}_{av} = \tfrac{1}{2}\mathcal{R}e(\mathbf{E} \times \mathbf{H}^*) \qquad (\mathrm{W/m^2}),}$$

Point form:

$\mathcal{P}$: power density vector

Recall in circuits: P_{av} = ½ Re(VI*)

P: Power

Analogy between electromagnetics and circuits:

	Electromagnetics	Circuits
	E	*V*
	H	*I*
Impedance	η = E/H	*Z* = *V*/*I*

- Power density vector, $\boldsymbol{\mathcal{P}}$:
  - W/m^2
  - vector (energy propagation direction)
  - a point value
- Power, P:
  - Watt
  - scalar
  - a value for a certain volume

EXAMPLE 8–7 Find the Poynting vector on the surface of a long, straight conducting wire (of radius b and conductivity σ) that carries a direct current I. Verify Poynting's theorem.

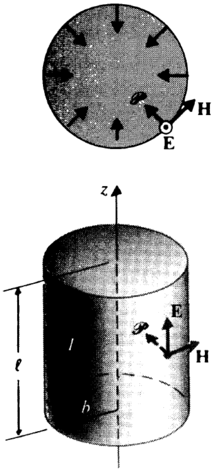


FIGURE 8–8
Illustrating Poynting's theorem (Example 8–7).

8-6 Normal Incidence at a Plane Conducting Boundary

- When an electromagnetic wave traveling in one medium impinges on another medium with a different intrinsic impedance, it experiences a reflection.
  - A plane conducting boundaries (8-6, 8-7)
  - An interface between two dielectric media (8-8, ~~8-9~~, 8-10)

Normal Incidence

- Assume
  - The incident wave ($\mathbf{E_i}$, $\mathbf{H_i}$) travels in a lossless medium ($\sigma_1 = 0$)
  - The boundary is an interface with a perfect conductor ($\sigma_2 = \infty$)

The Analogy between EM Waves and Transmission Lines

- EM waves:
  - Incoming wave with a certain frequency
  - Terminated by **a perfect conductor (η = 0)**
  - Waves are totally reflected.

- Transmission lines:
  - Voltage applied with a certain frequency
  - Terminated by **a short circuit (*Z* = 0)**
  - Voltage signals are totally reflected.

Medium 1 (lossless medium)

Incident waves

$$\mathbf{E}_i(z) = \mathbf{a}_x E_{i0} e^{\underline{-j\beta_1 z}},$$

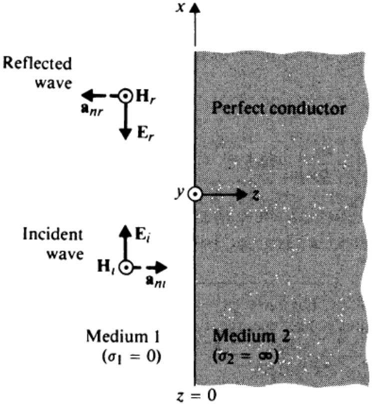
$$\mathbf{H}_i(z) = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z},$$

Travels in the +z direction

E_{i0}: magnitude of $\mathbf{E}_i$

β_1: phase constant of medium 1

η_1: intrinsic impedance of medium 1



Poynting vector:

$$\mathscr{P}_i(z) = \mathbf{E}_i(z) \times \mathbf{H}_i(z),$$

In $\mathbf{a}_z$ direction (direction of energy propagation)

FIGURE 8–9
Plane wave incident normally on a plane conducting boundary.

Medium 2 (perfect conductor)

$\mathbf{E_2} = 0$, $\mathbf{H_2} = 0$

➔ No wave is transmitted

Incident wave is reflected ($\mathbf{E}_r$, $\mathbf{H}_r$)

$$\mathbf{E}_r(z) = \mathbf{a}_x E_{r0} e^{+j\beta_1 z},$$

Travels in the −z direction

To be determined by B.C.

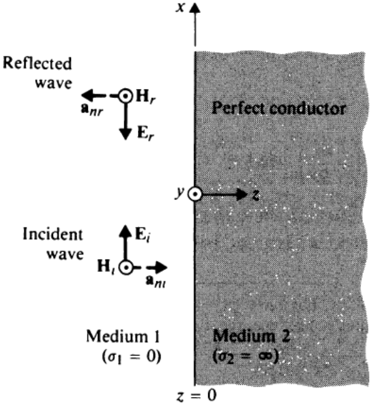


FIGURE 8–9
Plane wave incident normally on a plane conducting boundary.

Boundary

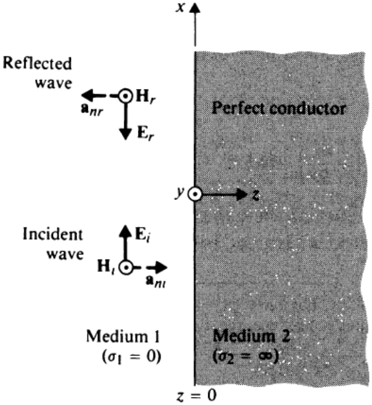
Total field in medium 1 = $\mathbf{E}_i + \mathbf{E}_r$

$$\mathbf{E}_1(z) = \mathbf{E}_i(z) + \mathbf{E}_r(z) = \mathbf{a}_x(E_{i0}e^{-j\beta_1 z} + E_{r0}e^{+j\beta_1 z}).$$

B.C.: $E_{1t} = E_{2t}$ at $z = 0$

$$\mathbf{E}_1(0) = \mathbf{a}_x(E_{i0} + E_{r0}) = \mathbf{E}_2(0) = 0,$$

Perfect conductor of medium 2



$$E_{r0} = -E_{i0}.$$

Thus, if $\mathbf{E}_i$ is along $\mathbf{a}_x$, $\mathbf{E}_r$ should be along $-\mathbf{a}_x$, as shown in the figure.

FIGURE 8–9
Plane wave incident normally on a plane conducting boundary.

$\boxed{\mathbf{E}_1}$

$$\boxed{\mathbf{E}_1(z) = \mathbf{E}_i(z) + \mathbf{E}_r(z) = \mathbf{a}_x(E_{i0}e^{-j\beta_1 z} + E_{r0}e^{+j\beta_1 z}).}$$

$$\boxed{E_{r0} = -E_{i0}.}$$

$$\begin{aligned}\mathbf{E}_1(z) &= \mathbf{a}_x E_{i0}(e^{-j\beta_1 z} - e^{+j\beta_1 z}) \\ &= -\mathbf{a}_x j2E_{i0} \sin \beta_1 z.\end{aligned}$$

$\boxed{\mathbf{H}_1}$

$$\boxed{\mathbf{E}_r(z) = \mathbf{a}_x E_{r0} e^{+j\beta_1 z},}$$

$$\boxed{\mathbf{H}(\mathbf{R}) = \frac{1}{\eta} \mathbf{a}_n \times \mathbf{E}(\mathbf{R})} \quad (8\text{-}29)$$

$$\mathbf{H}_r(z) = \frac{1}{\eta_1} \mathbf{a}_{nr} \times \mathbf{E}_r(z) = \frac{1}{\eta_1} (-\mathbf{a}_z) \times \mathbf{E}_r(z)$$

$$= -\mathbf{a}_y \frac{1}{\eta_1} E_{r0} e^{+j\beta_1 z} = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{+j\beta_1 z}.$$

$$\mathbf{H}_1(z) = \mathbf{H}_i(z) + \mathbf{H}_r(z)$$

$$\boxed{\mathbf{H}_i(z) = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z},}$$

H$_r$ and **H**$_i$ are both along **a**$_y$, as shown in the figure

$$\mathbf{H}_1(z) = \mathbf{H}_i(z) + \mathbf{H}_r(z) = \mathbf{a}_y 2 \frac{E_{i0}}{\eta_1} \cos \beta_1 z.$$

$$\boxed{\begin{aligned}\mathbf{E}_1(z) &= \mathbf{a}_x E_{i0}(e^{-j\beta_1 z} - e^{+j\beta_1 z}) \\ &= -\mathbf{a}_x j2E_{i0} \sin \beta_1 z.\end{aligned}}$$

$$\boxed{\mathbf{H}_1(z) = \mathbf{H}_i(z) + \mathbf{H}_r(z) = \mathbf{a}_y 2 \frac{E_{i0}}{\eta_1} \cos \beta_1 z.}$$

$$\boxed{\mathscr{P}_{av} = \frac{1}{2}\mathscr{Re}(\mathbf{E} \times \mathbf{H}^*)} \qquad \text{(8-96)}$$

- $\mathbf{E}_1$ and $\mathbf{H}_1$ are in phase quadrature (i.e., in time quadrature)

- No average power is associated with the total electromagnetic wave in medium 1

Time-domain behavior

$$\boxed{\begin{aligned}\mathbf{E}_1(z) &= \mathbf{a}_x E_{i0}(e^{-j\beta_1 z} - e^{+j\beta_1 z})\\ &= -\mathbf{a}_x j2E_{i0} \sin \beta_1 z.\end{aligned}}$$

$$\boxed{\mathbf{H}_1(z) = \mathbf{H}_i(z) + \mathbf{H}_r(z) = \mathbf{a}_y 2 \frac{E_{i0}}{\eta_1} \cos \beta_1 z.}$$

Total field in time domain

$$\mathbf{E}_1(z, t) = \mathscr{R}e[\mathbf{E}_1(z)e^{j\omega t}] = \mathbf{a}_x 2E_{i0} \sin \beta_1 z \underline{\sin \omega t},$$

$\boxed{=\cos(\omega t - \pi/2)}$

$$\mathbf{H}_1(z, t) = \mathscr{R}e[\mathbf{H}_1(z)e^{j\omega t}] = \mathbf{a}_y 2 \frac{E_{i0}}{\eta_1} \cos \beta_1 z \underline{\cos \omega t}.$$

Zeros and maxima along *z* for all *t*

$$\left.\begin{array}{r}\text{Zeros of } \mathbf{E}_1(z, t)\\ \text{Maxima of } \mathbf{H}_1(z, t)\end{array}\right\} \text{occur at } \beta_1 z = -n\pi, \quad \text{or } z = -n\frac{\lambda}{2}, \qquad n = 0, 1, 2, \ldots$$

$$\left.\begin{array}{r}\text{Maxima of } \mathbf{E}_1(z, t)\\ \text{Zeros of } \mathbf{H}_1(z, t)\end{array}\right\} \text{occur at } \beta_1 z = -(2n+1)\frac{\pi}{2}, \quad \text{or } z = -(2n+1)\frac{\lambda}{4},$$

$$n = 0, 1, 2, \ldots$$

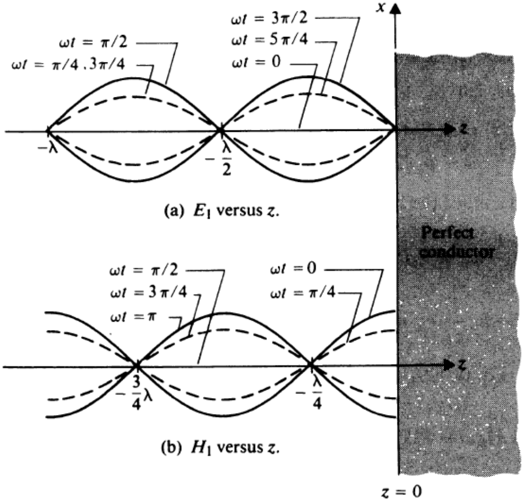


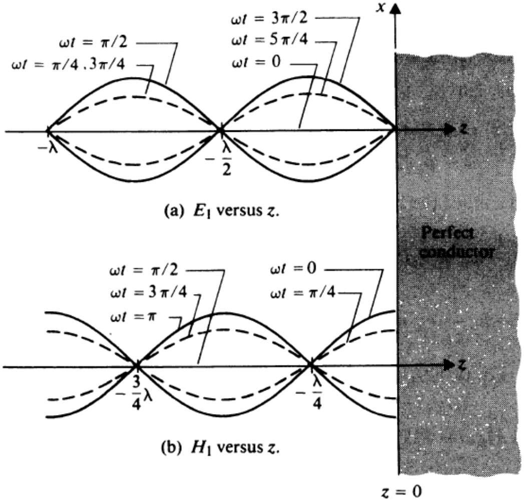
FIGURE 8–10
Standing waves of $\mathbf{E}_1 = \mathbf{a}_x E_1$ and $\mathbf{H}_1 = \mathbf{a}_y H_1$ for several values of ωt.

The total wave in medium 1 is a standing wave (not a traveling wave).

Space-time behavior:

1) $\mathbf{E}_1$ vanishes on the conducting boundary ($E_{r0} = -E_{i0}$) and at $-n\lambda/2$
2) $\mathbf{H}_1$ is a maximum on the conducting boundary ($H_{r0} = H_{i0} = E_{i0}/\eta_1$) and at $-n\lambda/2$
3) $\mathbf{E}_1$ and $\mathbf{H}_1$ are in time quadrature ($\pi/2$) and are shifted in space by $\lambda/4$

Recall:
$\mathbf{E}_i$: $\mathbf{a}_x$ and $\mathbf{E}_r$: $-\mathbf{a}_x$
$\mathbf{H}_i$: $\mathbf{a}_y$ and $\mathbf{H}_r$: $\mathbf{a}_y$



For a given t, both $\mathbf{E}_1$ and $\mathbf{H}_1$ vary sinusoidally with z

$$\mathbf{E}_1(z, t) = \mathbf{a}_x 2E_{i0} \sin \beta_1 z \sin \omega t,$$

$$\mathbf{H}_1(z, t) = \mathbf{a}_y 2 \frac{E_{i0}}{\eta_1} \cos \beta_1 z \cos \omega t.$$

FIGURE 8–10
Standing waves of $\mathbf{E}_1 = \mathbf{a}_x E_1$ and $\mathbf{H}_1 = \mathbf{a}_y H_1$ for several values of ωt.

EXAMPLE 8–8 The far field of a short vertical current element $I\,d\ell$ located at the origin of a spherical coordinate system in free space is

$$\mathbf{E}(R, \theta) = \mathbf{a}_\theta E_\theta(R, \theta) = \mathbf{a}_\theta \left(j \frac{60\pi I\,d\ell}{\lambda R} \sin\theta \right) e^{-j\beta R} \qquad \text{(V/m)}$$

and

$$\mathbf{H}(R, \theta) = \mathbf{a}_\phi \frac{E_\theta(R, \theta)}{\eta_0} = \mathbf{a}_\phi \left(j \frac{I\,d\ell}{2\lambda R} \sin\theta \right) e^{-j\beta R} \qquad \text{(A/m)},$$

where $\lambda = 2\pi/\beta$ is the wavelength.

a) Write the expression for instantaneous Poynting vector.

b) Find the total average power radiated by the current element.

EXAMPLE 8–9 A y-polarized uniform plane wave $(\mathbf{E}_i, \mathbf{H}_i)$ with a frequency 100 (MHz) propagates in air in the $+x$ direction and impinges normally on a perfectly conducting plane at $x = 0$. Assuming the amplitude of $\mathbf{E}_i$ to be 6 (mV/m), write the phasor and instantaneous expressions for (a) $\mathbf{E}_i$ and $\mathbf{H}_i$ of the incident wave; (b) $\mathbf{E}_r$ and $\mathbf{H}_r$ of the reflected wave; and (c) $\mathbf{E}_1$ and $\mathbf{H}_1$ of the total wave in air. (d) Determine the location nearest to the conducting plane where E_1 is zero.

8-7 Oblique Incidence at a Plane Conducting Boundary

- The behavior of the reflected wave depends on the polarization of the incident wave in oblique incidence.
- Plane of incidence: the plane containing the direction of propagation (of the incident wave) and the normal of the boundary surface.
- Consider the two cases separately
  - $\mathbf{E}_i \perp$ plane of incidence
  - $\mathbf{E}_i$ // plane of incidence

8-7.1 Perpendicular Polarization

The incident wave

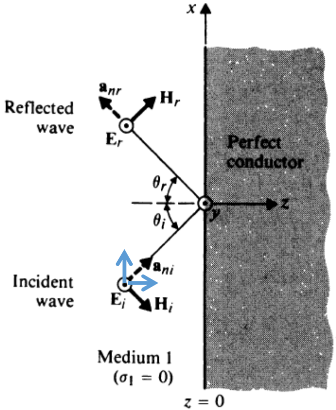
$$\mathbf{a}_{ni} = \mathbf{a}_x \sin\theta_i + \mathbf{a}_z \cos\theta_i,$$

$\mathbf{k} = \beta_1 \mathbf{a}_{ni}$
$\mathbf{R} = \mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z$

$$\mathbf{E}(\mathbf{R}) = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{R}} = \mathbf{E}_0 e^{-jk\mathbf{a}_n\cdot\mathbf{R}}$$

$$\mathbf{H}(\mathbf{R}) = \frac{1}{\eta}\mathbf{a}_n \times \mathbf{E}(\mathbf{R})$$

θ_i: angle of incidence



$$\mathbf{E}_i(x, z) = \mathbf{a}_y E_{i0} e^{-j\beta_1 \mathbf{a}_{ni}\cdot\mathbf{R}} = \mathbf{a}_y E_{i0} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)},$$

$$\mathbf{H}_i(x, z) = \frac{1}{\eta_1}[\mathbf{a}_{ni} \times \mathbf{E}_i(x, z)]$$

$$= \frac{E_{i0}}{\eta_1}(-\mathbf{a}_x \cos\theta_i + \mathbf{a}_z \sin\theta_i) e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}.$$

Different polarization

Same direction of propagation

Perpendicular polarization:
also called horizontal polarization, E-polarization, s-polarization

FIGURE 8–11
Plane wave incident obliquely on a plane conducting boundary (perpendicular polarization).

The reflected wave

$$\mathbf{a}_{nr} = \mathbf{a}_x \sin\theta_r - \mathbf{a}_z \cos\theta_r,$$

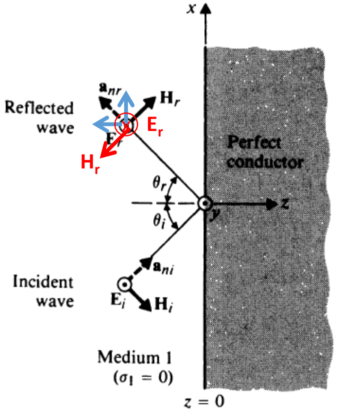
$$\boxed{\mathbf{E}_i(x, z) = \mathbf{a}_y E_{i0} e^{-j\beta_1 (x \sin\theta_i + z \cos\theta_i)},}$$

$\mathbf{k} = \beta_1 \mathbf{a}_{nr}$
$\mathbf{R} = \mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z$

Assuming polarization in $\mathbf{a}_y$ ($\mathbf{a}_y$ or $-\mathbf{a}_y$ can be confirmed later)

$$\mathbf{E}_r(x, z) = \mathbf{a}_y E_{r0} e^{-j\beta_1 (x \sin\theta_r - z \cos\theta_r)}.$$

θ_r: angle of reflection



B.C.: $E_{1t} = E_{2t} = 0$ at $z = 0$
For perpendicular polarization, only E tangential component
➔ Total **E** at boundary = 0

$$\mathbf{E}_1(x, 0) = \mathbf{E}_i(x, 0) + \mathbf{E}_r(x, 0)$$
$$= \mathbf{a}_y \left(E_{i0} e^{-j\beta_1 x \sin\theta_i} + E_{r0} e^{-j\beta_1 x \sin\theta_r}\right) = 0.$$

$$E_{r0} = -E_{i0} \qquad \theta_r = \theta_i$$

Snell's law of reflection

Thus, if $\mathbf{E}_i$ is in $\mathbf{a}_y$, $\mathbf{E}_r$ should be along $-\mathbf{a}_y$, which is different from the figure.

FIGURE 8–11
Plane wave incident obliquely on a plane conducting boundary (perpendicular polarization).

$$\boxed{\mathbf{E}_r(x, z) = \mathbf{a}_y E_{r0} e^{-j\beta_1(x \sin\theta_r - z\cos\theta_r)}.}$$

$$E_{r0} = -E_{i0} \qquad \theta_r = \theta_i$$

$\boxed{\mathbf{E_r}}$

$$\mathbf{E}_r(x, z) = -\mathbf{a}_y E_{i0} e^{-j\beta_1(x \sin\theta_i - z\cos\theta_i)}.$$

$$\mathbf{H}(\mathbf{R}) = \frac{1}{\eta}\mathbf{a}_n \times \mathbf{E}(\mathbf{R})$$

$\boxed{\mathbf{H_r}}$

$$\begin{aligned}\mathbf{H}_r(x, z) &= \frac{1}{\eta_1}[\mathbf{a}_{nr} \times \mathbf{E}_r(x, z)] \\ &= \frac{E_{i0}}{\eta_1}(-\mathbf{a}_x \cos\theta_i - \mathbf{a}_z \sin\theta_i) e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)}.\end{aligned}$$

The total field

$$\mathbf{E}_i(x, z) = \mathbf{a}_y E_{i0} e^{-j\beta_1(x \sin\theta_i + z\cos\theta_i)},$$

$$\mathbf{E}_r(x, z) = -\mathbf{a}_y E_{i0} e^{-j\beta_1(x \sin\theta_i - z\cos\theta_i)}.$$

$\mathbf{E}_1 = \mathbf{E}_i + \mathbf{E}_r$

$$\begin{aligned}\mathbf{E}_1(x, z) &= \mathbf{E}_i(x, z) + \mathbf{E}_r(x, z)\\ &= \mathbf{a}_y E_{i0}(e^{-j\beta_1 z\cos\theta_i} - e^{j\beta_1 z\cos\theta_i})e^{-j\beta_1 x\sin\theta_i}\\ &= -\mathbf{a}_y j2E_{i0}\sin(\beta_1 z\cos\theta_i)e^{-j\beta_1 x\sin\theta_i}.\end{aligned}$$

$$\mathbf{H}_i(x, z) = \frac{E_{i0}}{\eta_1}(-\mathbf{a}_x\cos\theta_i + \mathbf{a}_z\sin\theta_i)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}.$$

$$\mathbf{H}_r(x, z) = \frac{E_{i0}}{\eta_1}(-\mathbf{a}_x\cos\theta_i - \mathbf{a}_z\sin\theta_i)e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)}.$$

$\mathbf{H}_1 = \mathbf{H}_i + \mathbf{H}_r$

$$\begin{aligned}\mathbf{H}_1(x, z) = -2\frac{E_{i0}}{\eta_1}[&\mathbf{a}_x\cos\theta_i\cos(\beta_1 z\cos\theta_i)e^{-j\beta_1 x\sin\theta_i}\\ &+ \mathbf{a}_z j\sin\theta_i\sin(\beta_1 z\cos\theta_i)e^{-j\beta_1 x\sin\theta_i}].\end{aligned}$$

The total field

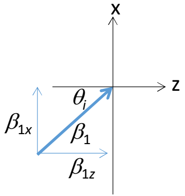
$$\mathbf{E}_1(x, z) = -\mathbf{a}_y j2E_{i0} \sin(\beta_1 z \cos\theta_i) e^{-j\beta_1 x \sin\theta_i}.$$

$$\mathbf{H}_1(x, z) = -2\frac{E_{i0}}{\eta_1}\left[\mathbf{a}_x \cos\theta_i \cos(\beta_1 z \cos\theta_i) e^{-j\beta_1 x \sin\theta_i} + \mathbf{a}_z j \sin\theta_i \sin(\beta_1 z \cos\theta_i) e^{-j\beta_1 x \sin\theta_i}\right].$$

E_{1y}

H_{1x} H_{1z}

- 1. Power along the *z* direction ($\perp$ to boundary)
  - E_{1y}, H_{1x}
  - E_{1y}, H_{1x} maintain standing-wave patterns:
    - ❖ $E_{1y} \sim \sin(\beta_{1z} z)$, $H_{1x} \sim \cos(\beta_{1z} z)$, where $\beta_{1z} = \beta_1 \cos\theta_i$
  - No average power in +*z* direction
    - ❖ $\mathcal{P}_z = 1/2\mathrm{Re}[E_{1y} \times H_{1x}^*]$
    - ❖ $E_{1y} \sim -j \times \exp(-j\beta_{1x} x)$; $H_{1x}^* \sim \exp(+j\beta_{1x} x)$ ➔ $\mathcal{P}_z = 0$



- 2. Power along the x direction (// to boundary)
  - E_{1y}, H_{1z}
  - Propagation in x direction
    - ❖ $\mathcal{P}_x = 1/2\mathrm{Re}[E_{1y} \times H_{1z}^*] \neq 0$
    - ❖ E_{1y} and H_{1z} are in phase for both time and space (time: $-j$ and $-j$ ($\theta = -90$ degree); space: $\sin(\beta_{1z} z)$)

$$\mathbf{E}_1(x, z) = -\mathbf{a}_y j2E_{i0} \sin(\beta_1 z \cos\theta_i) e^{-j\beta_{1x} x \sin\theta_i}.$$

$$\mathbf{H}_1(x, z) = -2\frac{E_{i0}}{\eta_1}\left[\mathbf{a}_x \cos\theta_i \cos(\beta_1 z \cos\theta_i) e^{-j\beta_{1x} x \sin\theta_i} + \mathbf{a}_z j \sin\theta_i \sin(\beta_1 z \cos\theta_i) e^{-j\beta_{1x} x \sin\theta_i}\right].$$

E_{1y}

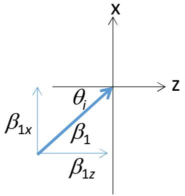
H_{1x} H_{1z}

- Phase velocity in *x* direction $u_p = \omega/\beta_{1x}$ (faster than u_1)

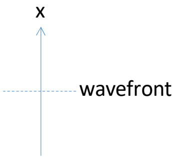
$$u_{1x} = \frac{\omega}{\beta_{1x}} = \frac{\omega}{\beta_1 \sin\theta_i} = \frac{u_1}{\sin\theta_i}.$$

- Wavelength in *x* direction $\lambda = 2\pi/\beta_{1x}$ (longer than λ_1)

$$\lambda_{1x} = \frac{2\pi}{\beta_{1x}} = \frac{\lambda_1}{\sin\theta_i}.$$



- 3. A nonuniform plane wave for the propagating wave in *x* direction
  - E_{1y} (or H_{1z}) ~ $\sin(\beta_{1z} z)$ ➔ Amplitude varies with z



$$\mathbf{E}_1(x, z) = -\mathbf{a}_y j2E_{i0} \sin(\beta_1 z \cos\theta_i) e^{-j\beta_1 x \sin\theta_i}.$$

$$\mathbf{H}_1(x, z) = -2\frac{E_{i0}}{\eta_1}[\mathbf{a}_x \cos\theta_i \cos(\beta_1 z \cos\theta_i) e^{-j\beta_1 x \sin\theta_i} + \mathbf{a}_z j \sin\theta_i \sin(\beta_1 z \cos\theta_i) e^{-j\beta_1 x \sin\theta_i}].$$

E_{1y}

H_{1x} H_{1z}

- 4. $\mathbf{E_1}$ = 0 for all x when $\sin(\beta_{1z}z)=0$

$$\beta_1 z \cos\theta_i = \frac{2\pi}{\lambda_1} z \cos\theta_i = -m\pi, \qquad m = 1, 2, 3, \ldots,$$

  - That is, a conducting plate could be inserted at

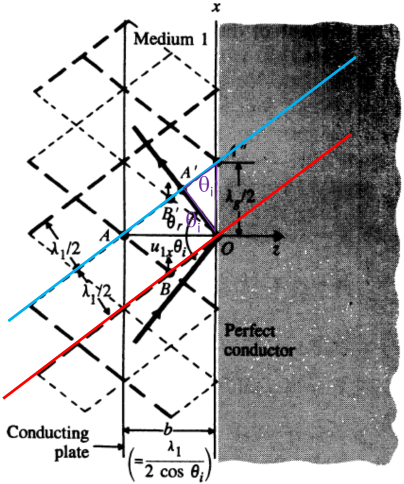
$$z = -\frac{m\lambda_1}{2\cos\theta_i}, \qquad m = 1, 2, 3, \ldots,$$

without changing $\mathbf{E_1}$ between the conducting plate and the conducting boundary
➔ A transverse electric (TE) wave (**E** transverse to plan of propagation (xz plane) ➔ only $E_{1y} \neq 0$) would bounce back and forth (i.e., a waveguide).

Long (thick) dashed line: plane wave crests (peaks)

Short (thin) dashed line: plane wave troughs (valleys)

- Conducting surfaces: $\mathbf{E}_i$ and $\mathbf{E}_r$ are 180°out of phase ➔ $\mathbf{E}(z = 0) = 0$, such as O, A, and A"
- Intersections of two crests: $\mathbf{E}$ = max, along $\mathbf{a}_y$ (e.g., B)
  Intersections of two troughs: $\mathbf{E}$ = min, along $-\mathbf{a}_y$ (e.g., B')
- OA': reflected wave from a crest (red) to a trough (blue), = $\lambda_1/2$ ➔ $\overline{OA'} = \frac{\lambda_1}{2} = \frac{\pi}{\beta_1}$,



- OA: length from the inserted plate to the boundary, b ➔

$$\overline{OA} = b = \frac{\lambda_1}{2\cos\theta_i}.$$

Also recall $\boxed{z = -\frac{m\lambda_1}{2\cos\theta_i}, \qquad m = 1, 2, 3, \ldots,}$

- Wavelength in x direction (traveling wave in the parallel-plate waveguide), λ :
  OA"= $\lambda_g/2$

$$\lambda_g = 2\overline{OA''} = 2\frac{\overline{OA'}}{\sin\theta_i}$$

$$= \frac{\lambda_1}{\sin\theta_i} > \lambda_1. \qquad \text{(longer)}$$

- At $\theta_i = 0$ (normal incidence) ➔ no propagating wave in x direction

FIGURE 8–12
Illustrating bouncing waves and interference patterns of oblique incidence at a plane conducting boundary (perpendicular polarization).

8-7.2 Parallel Polarization

The incident wave

$$\mathbf{a}_{ni} = \mathbf{a}_x \sin\theta_i + \mathbf{a}_z \cos\theta_i,$$

$\mathbf{E}_i$, $\mathbf{E}_r$: $\mathbf{a}_x$, $\mathbf{a}_z$ components
$\mathbf{H}_i$, $\mathbf{H}_r$: $\mathbf{a}_y$ component

$\mathbf{k} = \beta_1 \mathbf{a}_{ni}$
$\mathbf{R} = \mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z$

$$\mathbf{E}(\mathbf{R}) = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{R}} = \mathbf{E}_0 e^{-jk\mathbf{a}_n\cdot\mathbf{R}}$$

$$\mathbf{H}(\mathbf{R}) = \frac{1}{\eta}\mathbf{a}_n \times \mathbf{E}(\mathbf{R})$$

or

$$\mathbf{H}(\mathbf{R}) = \mathbf{H}_0 e^{-jk\mathbf{a}_n\cdot\mathbf{R}},$$

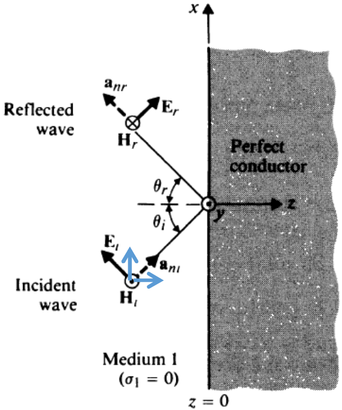
$$\mathbf{E}(\mathbf{R}) = -\eta\mathbf{a}_n \times \mathbf{H}(\mathbf{R})$$

$$\mathbf{E}_i(x,z) = E_{i0}(\mathbf{a}_x\cos\theta_i - \mathbf{a}_z\sin\theta_i)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)},$$

$$\mathbf{H}_i(x,z) = \mathbf{a}_y\frac{E_{i0}}{\eta_1}e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}.$$

Same direction of propagation

Different polarization



Parallel polarization:
also called vertical polarization, H-polarization, p-polarization

FIGURE 8–13
Plane wave incident obliquely on a plane conducting boundary (parallel polarization).

The reflected wave

$$\mathbf{a}_{nr} = \mathbf{a}_x \sin\theta_r - \mathbf{a}_z \cos\theta_r,$$

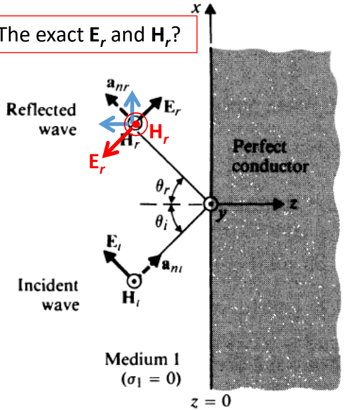
$$\mathbf{E}_i(x, z) = E_{i0}(\mathbf{a}_x \cos\theta_i - \mathbf{a}_z \sin\theta_i)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)},$$

$$\mathbf{H}_i(x, z) = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}.$$

1. Propagation direction in phase term
2. Denote polarization

θ_r: angle of reflection

The exact $\mathbf{E}_r$ and $\mathbf{H}_r$?



$$\mathbf{E}_r(x, z) = E_{r0}(\mathbf{a}_x \cos\theta_r + \mathbf{a}_z \sin\theta_r)e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)},$$

$$\mathbf{H}_r(x, z) = -\mathbf{a}_y \frac{E_{r0}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}.$$

Polarization

Propagation (same direction)

B.C.: $E_{1t} = E_{2t} = 0$

➔ Total E_{1x} at boundary = 0

$$E_{ix}(x, 0) + E_{rx}(x, 0) = 0.$$

$$(E_{i0}\cos\theta_i)e^{-j\beta_1 x\sin\theta_i} + (E_{r0}\cos\theta_r)e^{-j\beta_1 x\sin\theta_r} = 0,$$

Should be satisfied for all x

$$E_{r0} = -E_{i0} \qquad \theta_r = \theta_i$$

Snell's law of reflection

FIGURE 8–13
Plane wave incident obliquely on a plane conducting boundary (parallel polarization).

The total field

$$\boxed{\mathbf{E}_i(x, z) = E_{i0}(\mathbf{a}_x \cos\theta_i - \mathbf{a}_z \sin\theta_i)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)},}$$

$$\boxed{\mathbf{E}_r(x, z) = E_{r0}(\mathbf{a}_x \cos\theta_r + \mathbf{a}_z \sin\theta_r)e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)},}$$

$$\boxed{E_{r0} = -E_{i0}} \qquad \boxed{\theta_r = \theta_i.}$$

$\mathbf{E}_1 = \mathbf{E}_i + \mathbf{E}_r$

$$\begin{aligned}\mathbf{E}_1(x, z) &= \mathbf{E}_i(x, z) + \mathbf{E}_r(x, z)\\ &= \mathbf{a}_x E_{i0}\cos\theta_i(e^{-j\beta_1 z\cos\theta_i} - e^{j\beta_1 z\cos\theta_i})e^{-j\beta_1 x\sin\theta_i}\\ &\quad - \mathbf{a}_z E_{i0}\sin\theta_i(e^{-j\beta_1 z\cos\theta_i} + e^{j\beta_1 z\cos\theta_i})e^{-j\beta_1 x\sin\theta_i}\end{aligned}$$

or
$$\begin{aligned}\mathbf{E}_1(x, z) = -2E_{i0}[&\mathbf{a}_x j\cos\theta_i \sin(\beta_1 z\cos\theta_i)\\ &+ \mathbf{a}_z \sin\theta_i \cos(\beta_1 z\cos\theta_i)]e^{-j\beta_1 x\sin\theta_i}.\end{aligned}$$

$$\boxed{\mathbf{H}_i(x, z) = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}.}$$

$$\boxed{\mathbf{H}_r(x, z) = -\mathbf{a}_y \frac{E_{r0}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}.}$$

$\mathbf{H}_1 = \mathbf{H}_i + \mathbf{H}_r$

$$\begin{aligned}\mathbf{H}_1(x, z) &= \mathbf{H}_i(x, z) + \mathbf{H}_r(x, z)\\ &= \mathbf{a}_y 2\frac{E_{i0}}{\eta_1}\cos(\beta_1 z\cos\theta_i)e^{-j\beta_1 x\sin\theta_i}.\end{aligned}$$

Similar notes as in perpendicular polarization

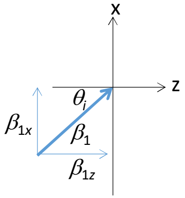
$$\mathbf{E}_1(x,z) = -2E_{i0}[\mathbf{a}_x j\cos\theta_i \sin(\beta_1 z\cos\theta_i) + \mathbf{a}_z \sin\theta_i \cos(\beta_1 z\cos\theta_i)]e^{-j\beta_1 x\sin\theta_i}.$$

$$\mathbf{H}_1(x,z) = \mathbf{a}_y 2\frac{E_{i0}}{\eta_1}\cos(\beta_1 z\cos\theta_i)e^{-j\beta_1 x\sin\theta_i}.$$

E_{1x} E_{1z}

H_{1y}

- 1. Power along the z direction (⊥ to boundary)
  - E_{1x}, H_{1y}
  - E_{1x}, H_{1y} maintain standing-wave patterns:
    - ❖ $E_{1x} \sim \sin(\beta_{1z}z)$, $H_{1y} \sim \cos(\beta_{1z}z)$, where $\beta_{1z} = \beta_1\cos\theta_i$
  - No average power in +z direction
    - ❖ $\mathcal{P}_z = 1/2\mathrm{Re}[E_{1x}\times H_{1y}^*]$
    - ❖ $E_{1x} \sim -j\times\exp(-j\beta_{1x}x)$; $H_{1y}(t)^* \sim \exp(+j\beta_{1x}x)$ ➔ $\mathcal{P}_z = 0$



- 2. Power along the x direction (// to boundary)
  - E_{1z}, H_{1y}
  - Propagation in *x* direction (E_{1z}, H_{1y})
    - ❖ $\mathcal{P}_x = 1/2\mathrm{Re}[E_{1z}\times H_{1y}^*] \neq 0$
    - ❖ E_{1z} and H_{1y} are in phase in both time and space (time: θ = 0; space: $\cos(\beta_{1z}z)$)

$$\mathbf{E}_1(x, z) = -2E_{i0}[\mathbf{a}_x j \cos\theta_i \sin(\beta_1 z \cos\theta_i) + \mathbf{a}_z \sin\theta_i \cos(\beta_1 z \cos\theta_i)]e^{-j\beta_1 x \sin\theta_i}.$$

$$\mathbf{H}_1(x, z) = \mathbf{a}_y 2\frac{E_{i0}}{\eta_1}\cos(\beta_1 z \cos\theta_i)e^{-j\beta_1 x \sin\theta_i}.$$

E_{1x} E_{1z}

H_{1y}

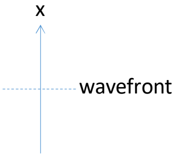
- Phase velocity in x direction $u_p = \omega/\beta_{1x}$ (faster than u_1) $u_{1x} = u_1/\sin\theta_i$
- Wavelength in x direction $\lambda = 2\pi/\beta_{1x}$ (longer than λ_1)

$$\lambda_{1x} = \frac{2\pi}{\beta_{1x}} = \frac{\lambda_1}{\sin\theta_i}.$$

Same as perpendicular polarization

- 3. A nonuniform plane wave for the propagating wave in x direction
  - H_{1y} (or E_{1z}) ~ $\cos(\beta_{1z}z)$ ➔ Amplitude varies with z

Same as perpendicular polarization



$$\mathbf{E}_1(x, z) = -2E_{i0}[\mathbf{a}_x j \cos\theta_i \sin(\beta_1 z \cos\theta_i) + \mathbf{a}_z \sin\theta_i \cos(\beta_1 z \cos\theta_i)]e^{-j\beta_1 x \sin\theta_i}.$$

$$\mathbf{H}_1(x, z) = \mathbf{a}_y 2 \frac{E_{i0}}{\eta_1} \cos(\beta_1 z \cos\theta_i) e^{-j\beta_1 x \sin\theta_i}.$$

E_{1x} E_{1z}

H_{1y}

- 4. $\mathbf{E}_{1x}$ = 0 for all x when $\sin(\beta_{1x} z)=0$

$$\beta_1 z \cos\theta_i = \frac{2\pi}{\lambda_1} z \cos\theta_i = -m\pi, \qquad m = 1, 2, 3, \ldots,$$

  - That is, a conducting plate could be inserted at

$$z = -\frac{m\lambda_1}{2\cos\theta_i}, \qquad m = 1, 2, 3, \ldots,$$

without changing $\mathbf{E}_{1x}$ (tangential component) between the conducting plate and the conducting boundary
➔ A transverse magnetic (TM) wave (H transverse to plan of propagation (xz plane) ➔ only $H_{1y} \neq 0$) would bounce back and forth (i.e., a waveguide).

A Summary of Oblique Incidence

- Perpendicular polarization

$$\mathbf{E}_1(x, z) = -\mathbf{a}_y j2E_{i0} \sin(\beta_1 z \cos\theta_i) e^{-j\beta_1 x \sin\theta_i}.$$

$$\mathbf{H}_1(x, z) = -2\frac{E_{i0}}{\eta_1}[\mathbf{a}_x \cos\theta_i \cos(\beta_1 z \cos\theta_i) e^{-j\beta_1 x \sin\theta_i} + \mathbf{a}_z j \sin\theta_i \sin(\beta_1 z \cos\theta_i) e^{-j\beta_1 x \sin\theta_i}].$$

E_{1y}

H_{1x} H_{1z}

- Parallel polarization

$$\mathbf{E}_1(x, z) = -2E_{i0}[\mathbf{a}_x j \cos\theta_i \sin(\beta_1 z \cos\theta_i) + \mathbf{a}_z \sin\theta_i \cos(\beta_1 z \cos\theta_i)] e^{-j\beta_1 x \sin\theta_i}.$$

$$\mathbf{H}_1(x, z) = \mathbf{a}_y 2\frac{E_{i0}}{\eta_1} \cos(\beta_1 z \cos\theta_i) e^{-j\beta_1 x \sin\theta_i}.$$

E_{1x} E_{1z}

H_{1y}

8-8 Normal Incidence at a Plane Dielectric Boundary

- When an electromagnetic wave is incident on the surface of a dielectric medium that has an intrinsic impedance different from that of the medium in which the wave is originated, part of incident power is reflected and part is transmitted.
- For a normal plane wave incident on a plane dielectric medium:
  - Dissipationless media ($\sigma_1 = \sigma_2 = 0$)
  - Normal incidence (8-8); Oblique incidence (8-10)

The incident wave

$$\mathbf{E}_i(z) = \mathbf{a}_x E_{i0} e^{-j\beta_1 z},$$

$$\mathbf{H}_i(z) = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z}.$$

Propagating in +z

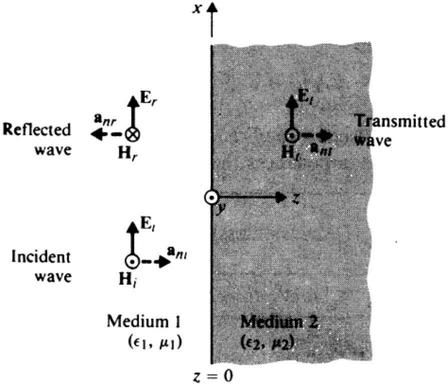


FIGURE 8–14
Plane wave incident normally on a plane dielectric boundary.

The reflected wave

$$\mathbf{E}_r(z) = \mathbf{a}_x E_{r0} e^{j\beta_1 z},$$

$$\mathbf{H}_r(z) = (-\mathbf{a}_z) \times \frac{1}{\eta_1} \mathbf{E}_r(z) = -\mathbf{a}_y \frac{E_{r0}}{\eta_1} e^{j\beta_1 z}.$$

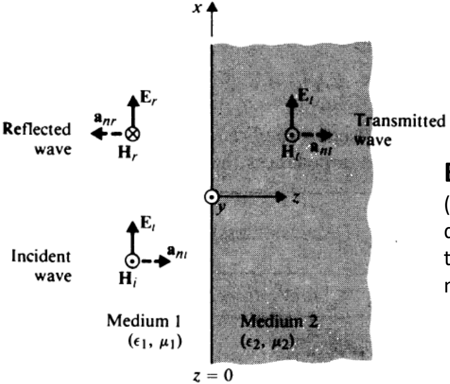
Propagating in −z

The transmitted wave

$$\mathbf{E}_t(z) = \mathbf{a}_x E_{t0} e^{-j\beta_2 z},$$

$$\mathbf{H}_t(z) = \mathbf{a}_z \times \frac{1}{\eta_2} \mathbf{E}_t(z) = \mathbf{a}_y \frac{E_{t0}}{\eta_2} e^{-j\beta_2 z},$$

Propagating in +z



E_{t0}: magnitude of $\mathbf{E}_t$
β_2: phase constant of medium 2
η_2: intrinsic impedance of medium 2

$\mathbf{E}_r$ and $\mathbf{E}_t$ are drawn arbitrarily
(E_{r0} and E_{t0} may be positive or negative, depending on the relative magnitudes of the constitutive parameters of the two media.)

FIGURE 8–14
Plane wave incident normally on a plane dielectric boundary.

$$\mathbf{E}_i(z) = \mathbf{a}_x E_{i0} e^{-j\beta_1 z},$$

$$\mathbf{H}_i(z) = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z}.$$

$$\mathbf{E}_r(z) = \mathbf{a}_x E_{r0} e^{j\beta_1 z},$$

$$\mathbf{H}_r(z) = (-\mathbf{a}_z) \times \frac{1}{\eta_1} \mathbf{E}_r(z) = -\mathbf{a}_y \frac{E_{r0}}{\eta_1} e^{j\beta_1 z}.$$

$$\mathbf{E}_t(z) = \mathbf{a}_x E_{t0} e^{-j\beta_2 z},$$

$$\mathbf{H}_t(z) = \mathbf{a}_z \times \frac{1}{\eta_2} \mathbf{E}_t(z) = \mathbf{a}_y \frac{E_{t0}}{\eta_2} e^{-j\beta_2 z},$$

Two unknowns: E_{r0} and E_{t0}

Two B.C. equations: $E_{1t} = E_{2t}$; $H_{1t} = H_{2t}$ (J_s =0)

$$\mathbf{E}_i(0) + \mathbf{E}_r(0) = \mathbf{E}_t(0) \quad \text{or} \quad E_{i0} + E_{r0} = E_{t0}$$

$$\mathbf{H}_i(0) + \mathbf{H}_r(0) = \mathbf{H}_t(0) \quad \text{or} \quad \frac{1}{\eta_1}(E_{i0} - E_{r0}) = \frac{E_{t0}}{\eta_2}.$$

$$E_{r0} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E_{i0},$$

$$E_{t0} = \frac{2\eta_2}{\eta_2 + \eta_1} E_{i0}.$$

$$E_{r0} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E_{i0},$$

$$E_{t0} = \frac{2\eta_2}{\eta_2 + \eta_1} E_{i0}.$$

Reflection coefficient
= E_{r0}/E_{i0}

$$\boxed{\Gamma = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \qquad \text{(Dimensionless)}}$$

Transmission coefficient
= E_{t0}/E_{i0}

$$\boxed{\tau = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2}{\eta_2 + \eta_1} \qquad \text{(Dimensionless).}}$$

Γ can be positive or negative
τ is always positive

$$\boxed{1 + \Gamma = \tau \qquad \text{(Dimensionless).}}$$

For dissipative media (η_1 and η_2 are complex), Γ and τ equations still apply*.
➔ Γ and τ are complex in the general case
➔ a phase shift is introduced at the interface upon reflection (or transmission)

*: the previous equations can be derived by considering complex η_c for **lossy media**, and complex $\mathbf{k}_c$ for **wave with attenuation** (they are all connected).

$$\Gamma = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \qquad \text{(Dimensionless)}$$

$$\tau = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2}{\eta_2 + \eta_1} \qquad \text{(Dimensionless).}$$

If medium 2 is a perfect conductor ($\eta_2 = 0$)

➔ $\Gamma = -1$, $\tau = 0$

➔ $E_{r0} = -E_{i0}$, $E_{t0} = 0$

➔ The incident wave is totally reflected (as discussed in Section 8-6)

If medium 2 is NOT a perfect conductor

➔ Partial reflection, partial transmission

➔ Total field in medium 1

E_{r0} replaced

Propagation $-z$

$$\begin{aligned}\mathbf{E}_1(z) &= \mathbf{E}_i(z) + \mathbf{E}_r(z) = \mathbf{a}_x E_{i0}(e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z}) \\ &= \mathbf{a}_x E_{i0}[(1+\Gamma)e^{-j\beta_1 z} + \Gamma(e^{j\beta_1 z} - e^{-j\beta_1 z})] \\ &= \mathbf{a}_x E_{i0}[(1+\Gamma)e^{-j\beta_1 z} + \Gamma(j2\sin\beta_1 z)]\end{aligned}$$

$1 + \Gamma = \tau$

$$\mathbf{E}_1(z) = \mathbf{a}_x E_{i0}[\tau e^{-j\beta_1 z} + \Gamma(j2\sin\beta_1 z)].$$

$$\mathbf{E}_1(z) = \mathbf{a}_x E_{i0}[\tau e^{-j\beta_1 z} + \Gamma(j2\sin\beta_1 z)].$$

1^{st} term: a traveling wave with an amplitude τE_{i0}
2^{nd} term: a standing wave with an amplitude $2\Gamma E_{i0}$

How to know?
Check in time domain:
traveling wave: $\cos(\omega t - \beta_1 z)$
standing wave: $\sin(\beta_1 z)$ $(-\sin(\omega t))$ ⇨ **$|\mathbf{E}_1|$** has locations of max. and min. values?

$$\mathbf{E}_1(z) = \mathbf{a}_x E_{i0}(e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z})$$

$$\Downarrow$$

$$\mathbf{E}_1(z) = \mathbf{a}_x E_{i0} e^{-j\beta_1 z}(1 + \Gamma e^{j2\beta_1 z}).$$

$$\boxed{\Gamma = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}}$$

$$\boxed{\tau = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2}{\eta_2 + \eta_1}}$$

For dissipationless media, η_1 and η_2 are real

➔ Γ and τ are real; Γ can be positive or negative

Check amplitude $|\mathbf{E}_1(z)|$

(1) $\Gamma > 0$ ($\eta_2 > \eta_1$)

max. of $|\mathbf{E}_1(z)|$: $E_{i0}(1+1\Gamma)$, when $2\beta_1 z_{\max} = -2n\pi \ (n = 0, 1, 2, \ldots)$,

$$\text{or} \qquad z_{\max} = -\frac{n\pi}{\beta_1} = -\frac{n\lambda_1}{2}, \qquad n = 0, 1, 2, \ldots$$

min. of $|\mathbf{E}_1(z)|$: $E_{i0}(1-1\Gamma)$, when $2\beta_1 z_{\min} = -(2n+1)\pi$

$$\text{or} \quad z_{\min} = -\frac{(2n+1)\pi}{2\beta_1} = -\frac{(2n+1)\lambda_1}{4}, \qquad n = 0, 1, 2, \ldots$$

(2) $\Gamma < 0$ ($\eta_2 < \eta_1$)

max. of $|\mathbf{E}_1(z)|$: $E_{i0}(1-1\Gamma)$,

min. of $|\mathbf{E}_1(z)|$: $E_{i0}(1+1\Gamma)$,

In other words, the location for $|E_1(z)|_{\max}$ and $|E_1(z)|_{\min}$ when $\Gamma > 0$ are interchanged when $\Gamma > 0$.

Standing-wave ratio (SWR): ratio of maximum value to the minimum value of $|\mathbf{E}|$ of a standing wave

$$\boxed{S = \frac{|E|_{\max}}{|E|_{\min}} = \frac{1+|\Gamma|}{1-|\Gamma|} \qquad \text{(Dimensionless).}}$$

$$\Downarrow$$

Inverse relation of Γ and S

$$\boxed{|\Gamma| = \frac{S-1}{S+1} \qquad \text{(Dimensionless).}}$$

Range of Γ: -1 to 1
Range of S: 1 to ∞

The magnetic field in medium 1:

$$\mathbf{H}_i(z) = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z}.$$

$$\mathbf{H}_r(z) = -\mathbf{a}_y \frac{E_{r0}}{\eta_1} e^{j\beta_1 z}.$$

$$\mathbf{H}_1(z) = \mathbf{a}_y \frac{E_{i0}}{\eta_1}(e^{-j\beta_1 z} - \Gamma e^{j\beta_1 z})$$

$$= \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z}\underline{(1 - \Gamma e^{j2\beta_1 z})}.$$

Compared with

$$\mathbf{E}_1(z) = \mathbf{a}_x E_{i0} e^{-j\beta_1 z}\underline{(1 + \Gamma e^{j2\beta_1 z})}.$$

Compared with $\mathbf{E}_1(z)$: In a dissipationless medium, Γ is real.
$|\mathbf{H}_1(z)|$ is max. at locations where $|\mathbf{E}_1(z)|$ is min.
$|\mathbf{H}_1(z)|$ is min. at locations where $|\mathbf{E}_1(z)|$ is max.

The magnetic field in medium 2 (expressed in terms of E_{i0} and τ):

$$\mathbf{H}_t(z) = \mathbf{a}_y \frac{E_{t0}}{\eta_2} e^{-j\beta_2 z},$$

$$\mathbf{H}_t(z) = \mathbf{a}_y \frac{\tau}{\eta_2} E_{i0} e^{-j\beta_2 z}.$$

$$\tau = \frac{E_{t0}}{E_{i0}}$$

$$\mathbf{E}_t(z) = \mathbf{a}_x E_{t0} e^{-j\beta_2 z},$$

$$\mathbf{E}_t(z) = \mathbf{a}_x \tau E_{i0} e^{-j\beta_2 z}.$$

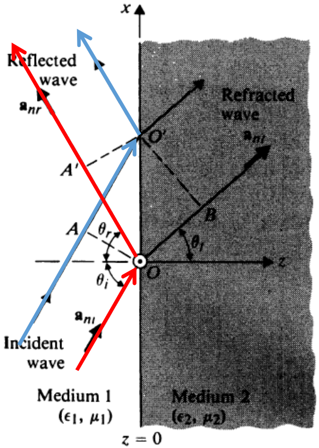
EXAMPLE 8–11 A uniform plane wave in a lossless medium with intrinsic impedance η_1 is incident normally onto another lossless medium with intrinsic impedance η_2 through a plane boundary. Obtain the expressions for the time-average power densities in both media.

8-9 Normal Incidence at Multiple Dielectric Interfaces (excluded)

8-10 Oblique Incidence at a Plane Dielectric Boundary

- Oblique incidence on a plane interface between two dielectric media.
  - Lossless media assumed

Intersection of wavefronts (surfaces of constant) with the plane of incidence
AO: incident waves
O'A': reflected waves
O'B: refracted waves



In medium 1, incident and reflected waves propagate with the same u_{p1}

$OA' = AO'$

$$\overline{OO'} \sin \theta_r = \overline{OO'} \sin \theta_i$$

or $\boxed{\theta_r = \theta_i.}$ Snell's law of reflection

FIGURE 8–16
Uniform plane wave incident obliquely on a plane dielectric boundary.

In medium 1, incident waves propagate with u_{p1}
In medium 2, refracted waves propagate with u_{p2}
The same time is taken for OB and AO'

$$OB/u_{p2} = AO'/u_{p1}$$

$$\frac{\overline{OB}}{\overline{AO'}} = \frac{\overline{OO'}\sin\theta_t}{\overline{OO'}\sin\theta_i} = \frac{u_{p2}}{u_{p1}},$$

$$\boxed{\frac{\sin\theta_t}{\sin\theta_i} = \frac{u_{p2}}{u_{p1}} = \frac{\beta_1}{\beta_2} = \frac{n_1}{n_2},}$$

Snell's law of refraction

n: the index of refraction
By definition, *n* is the ratio of the speed of light in free space to that in the medium ➔ $n = c/u_p$

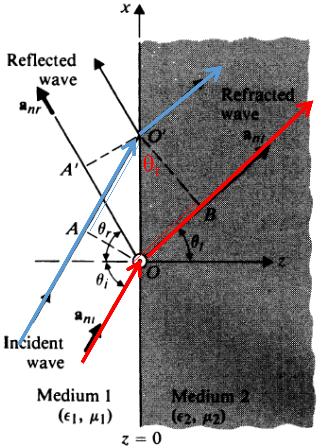


FIGURE 8–16
Uniform plane wave incident obliquely on a plane dielectric boundary.

Snell's law of refraction: at an interface between two dielectric media, the ratio of the sine of the angle of refraction (transmission) in medium 2 to the sine of the angle of incidence in medium 1 is equal to the inverse ratio of indices of refraction n_1/n_2

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{n_1}{n_2}$$

For nonmagnetic material, $\mu_1 = \mu_2 = \mu_0$

$$\boxed{\frac{\sin\theta_t}{\sin\theta_i} = \sqrt{\frac{\epsilon_1}{\epsilon_2}} = \sqrt{\frac{\epsilon_{r1}}{\epsilon_{r2}}} = \frac{n_1}{n_2} = \frac{\eta_2}{\eta_1},}$$

$$\boxed{\frac{\sin\theta_t}{\sin\theta_i} = \frac{u_{p2}}{u_{p1}} \qquad u_p = \frac{1}{\sqrt{\epsilon\mu}}}$$

$$\boxed{\eta = \sqrt{\frac{\mu}{\epsilon}}}$$

- $n, \sqrt{\epsilon}, \beta$
- $\sin\theta, u_p, \eta$

For nonmagnetic material, $\mu_1 = \mu_2 = \mu_0$
And if medium 1 is free space: $\varepsilon_{r1} = 1$, $n_1 = 1$, $\eta_1 = 120\pi$

$$\boxed{\frac{\sin\theta_t}{\sin\theta_i} = \frac{1}{\sqrt{\epsilon_{r2}}} = \frac{1}{n_2} = \frac{\eta_2}{120\pi}.}$$

$n_2 > 1 \rightarrow \theta_t < \theta_i$
$\rightarrow$ Wave will be bent toward normal (for oblique incidence to a denser medium)

- In these derivation, no indications of the wave polarizations have been made.

- Snell's law of reflection and Snell's law of refraction are independent of wave polarization.

8-10.1 Total Reflection

- For $\varepsilon_1 > \varepsilon_2$:
  - wave in medium 1 is incident on a less dense medium 2
  - $\theta_t > \theta_i$
  - θ_t increases with θ_i; When $\theta_t = \pi/2$, the refracted wave will glaze along the interface.
  - A further increase in θ_i ➔ no refracted wave, and the incident wave is totally reflected.
  - Critical angle θ_c: the angle of θ_i corresponds to $\theta_t = \pi/2$ (threshold of total reflection)

- n, $\sqrt{\epsilon}$, β
- $\sin\theta$, u_p, η

Unit vectors for propagation direction
$\mathbf{a}_{ni}$: direction of incident waves
$\mathbf{a}_{nr}$: direction of reflected waves
$\mathbf{a}_{nt}$: direction of transmitted waves

$$\frac{\sin\theta_t}{\sin\theta_i} = \sqrt{\frac{\epsilon_1}{\epsilon_2}} = \frac{n_1}{n_2}$$

$\theta_i = \theta_c$, $\theta_t = \pi/2$

$$\sin\theta_c = \sqrt{\frac{\epsilon_2}{\epsilon_1}}$$

or

$$\boxed{\theta_c = \sin^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}} = \sin^{-1}\left(\frac{n_2}{n_1}\right).}$$

- n, $\sqrt{\epsilon}$, β
- $\sin\theta$, u_p, η

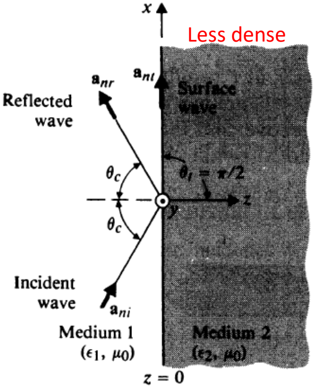


FIGURE 8–17
Plane wave incident at critical angle, $\epsilon_1 > \epsilon_2$.

What happens mathematically if $\theta_i > \theta_c$?

$$\boxed{\frac{\sin\theta_t}{\sin\theta_i} = \sqrt{\frac{\epsilon_1}{\epsilon_2}} = \frac{n_1}{n_2}}$$

$$\sin\theta_i > \sin\theta_c = \sqrt{\epsilon_2/\epsilon_1}$$

$$\sin\theta_t = \sqrt{\frac{\epsilon_1}{\epsilon_2}}\sin\theta_i > 1,$$

θ_t: not real

$\sin\theta_t$: still real

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \pm j\sqrt{\frac{\epsilon_1}{\epsilon_2}\sin^2\theta_i - 1}.$$

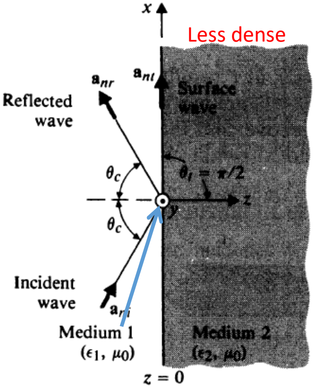


FIGURE 8–17
Plane wave incident at critical angle, $\epsilon_1 > \epsilon_2$.

$$\mathbf{a}_{nt} = \mathbf{a}_x \sin\theta_t + \mathbf{a}_z \cos\theta_t.$$

$\mathbf{E}_t, \mathbf{H}_t \sim \exp(-j\,\beta_2\, \mathbf{a}_{nt}\cdot\mathbf{R})$

$$e^{-j\beta_2 \mathbf{a}_{nt}\cdot\mathbf{R}} = e^{-j\beta_2(x\,\underline{\sin\theta_t} + z\,\underline{\cos\theta_t})},$$

$$\sin\theta_t = \sqrt{\frac{\epsilon_1}{\epsilon_2}}\sin\theta_i \qquad \text{real}$$

$$\cos\theta_t = \pm j\sqrt{\frac{\epsilon_1}{\epsilon_2}\sin^2\theta_i - 1}. \qquad \text{imaginary}$$

$(-j)\times(-j) = -1$

$$e^{-\alpha_2 z} e^{-j\beta_{2x} x},$$

where $\alpha_2 = \beta_2\sqrt{(\epsilon_1/\epsilon_2)\sin^2\theta_i - 1}$

$$\beta_{2x} = \beta_2\sqrt{\epsilon_1/\epsilon_2}\,\sin\theta_i.$$

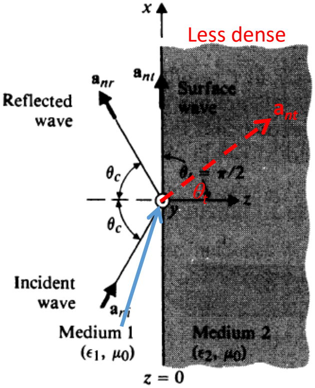
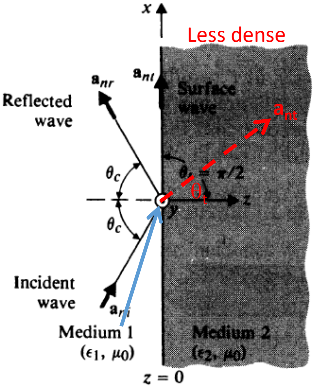


FIGURE 8–17
Plane wave incident at critical angle, $\epsilon_1 > \epsilon_2$.

Evanescent wave exists along the interface (in the x direction) $e^{-\alpha_2 z} e^{-j\beta_{2x} x}$,

$e^{-\alpha_2 z}$ The evanescent wave attenuated exponentially (rapidly) in medium 2 in the normal direction (z direction);
No power is transmitted into medium 2

$e^{-j\beta_{2x} x}$ The wave is tightly bound to the interface and is called a surface wave (Not a uniform plane wave due to exp($-\alpha_2 z$))



Evanescent wave:
Attenuation in z direction
Propagation along x direction

FIGURE 8–17
Plane wave incident at critical angle, $\epsilon_1 > \epsilon_2$.

8-10.2 Perpendicular Polarization

- **E** $\perp$ the plane of incidence
- Also called s-polarization (German origin: s = senkrecht = perpendicular)
- TE

The incident fields

$$\mathbf{E}_i(x, z) = \mathbf{a}_y E_{i0} e^{-j\beta_1(x \sin\theta_i + z\cos\theta_i)}$$

$$\mathbf{H}_i(x, z) = \frac{E_{i0}}{\eta_1}(-\mathbf{a}_x \cos\theta_i + \mathbf{a}_z \sin\theta_i) e^{-j\beta_1(x \sin\theta_i + z\cos\theta_i)}.$$

polarization

propagation

$$\mathbf{H}(\mathbf{R}) = \frac{1}{\eta}\mathbf{a}_n \times \mathbf{E}(\mathbf{R})$$

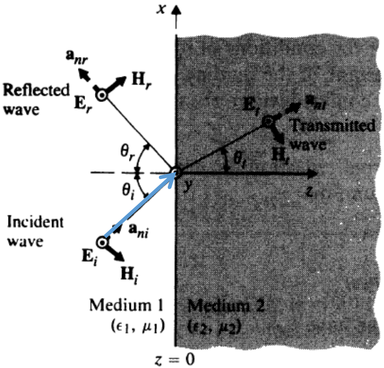


FIGURE 8–20
Plane wave incident on a plane dielectric boundary (perpendicular polarization).

The reflected fields

$$\mathbf{E}_r(x, z) = \underline{\mathbf{a}_y} E_{r0} e^{\underline{-j\beta_1(x \sin\theta_r - z\cos\theta_r)}}$$

$$\mathbf{H}_r(x, z) = \frac{E_{r0}}{\eta_1}\underline{(\mathbf{a}_x \cos\theta_r + \mathbf{a}_z \sin\theta_r)}e^{\underline{-j\beta_1(x \sin\theta_r - z\cos\theta_r)}}.$$

polarization

propagation

$$\mathbf{H}(\mathbf{R}) = \frac{1}{\eta}\mathbf{a}_n \times \mathbf{E}(\mathbf{R})$$

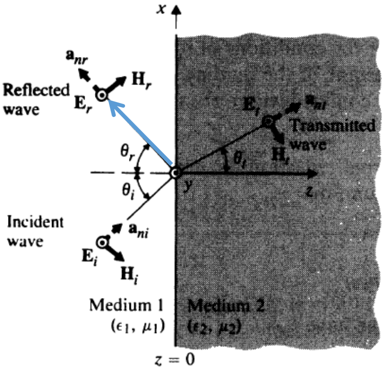


FIGURE 8–20
Plane wave incident on a plane dielectric boundary (perpendicular polarization).

The transmitted fields

$$\mathbf{E}_t(x, z) = \mathbf{a}_y E_{t0} e^{-j\beta_2(x \sin\theta_t + z\cos\theta_t)}$$

$$\mathbf{H}_t(x, z) = \frac{E_{t0}}{\eta_2}(-\mathbf{a}_x \cos\theta_t + \mathbf{a}_z \sin\theta_t) e^{-j\beta_2(x \sin\theta_t + z\cos\theta_t)}.$$

polarization

propagation

$$\mathbf{H}(\mathbf{R}) = \frac{1}{\eta}\mathbf{a}_n \times \mathbf{E}(\mathbf{R})$$

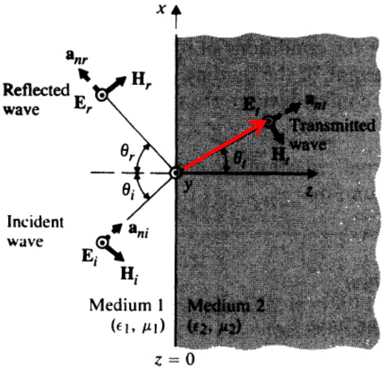


FIGURE 8–20
Plane wave incident on a plane dielectric boundary (perpendicular polarization).

4 unknowns: E_{r0}, E_{t0}, θ_r, θ_t

B.C.: tangential **E** and **H** should be continuous

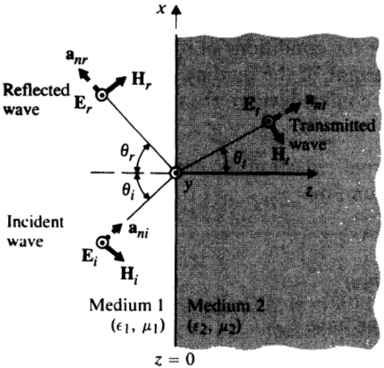


FIGURE 8–20
Plane wave incident on a plane dielectric boundary (perpendicular polarization).

$$\mathbf{E}_i(x, z) = \mathbf{a}_y E_{i0} e^{-j\beta_1(x \sin\theta_i + z\cos\theta_i)}$$

$$\mathbf{H}_i(x, z) = \frac{E_{i0}}{\eta_1}(-\mathbf{a}_x \cos\theta_i + \mathbf{a}_z \sin\theta_i) e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}.$$

$$\mathbf{E}_r(x, z) = \mathbf{a}_y E_{r0} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\mathbf{H}_r(x, z) = \frac{E_{r0}}{\eta_1}(\mathbf{a}_x \cos\theta_r + \mathbf{a}_z \sin\theta_r) e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}.$$

$$\mathbf{E}_t(x, z) = \mathbf{a}_y E_{t0} e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$\mathbf{H}_t(x, z) = \frac{E_{t0}}{\eta_2}(-\mathbf{a}_x \cos\theta_t + \mathbf{a}_z \sin\theta_t) e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}.$$

Tangential **E** and **H** should be continuous

E_{1y}

$$E_{iy}(x, 0) + E_{ry}(x, 0) = E_{ty}(x, 0)$$

$$\Downarrow$$

$$E_{i0} e^{-j\beta_1 x\sin\theta_i} + E_{r0} e^{-j\beta_1 x\sin\theta_r} = E_{t0} e^{-j\beta_2 x\sin\theta_t}.$$

H_{1x}

$$H_{ix}(x, 0) + H_{rx}(x, 0) = H_{tx}(x, 0)$$

$$\Downarrow$$

$$\frac{1}{\eta_1}(-E_{i0}\cos\theta_i e^{-j\beta_1 x\sin\theta_i} + E_{r0}\cos\theta_r e^{-j\beta_1 x\sin\theta_r}) = -\frac{E_{t0}}{\eta_2}\cos\theta_t e^{-j\beta_2 x\sin\theta_t}.$$

$$E_{i0}e^{-j\beta_1 x\sin\theta_i} + E_{r0}e^{-j\beta_1 x\sin\theta_r} = E_{t0}e^{-j\beta_2 x\sin\theta_t}. \quad (8\text{-}202)$$

$$\frac{1}{\eta_1}(-E_{i0}\cos\theta_i e^{-j\beta_1 x\sin\theta_i} + E_{r0}\cos\theta_r e^{-j\beta_1 x\sin\theta_r}) = -\frac{E_{t0}}{\eta_2}\cos\theta_t e^{-j\beta_2 x\sin\theta_t}. \quad (8\text{-}203)$$

The 2 equations are to be satisfied for all *x* (boundary)
➔ Exponential terms that are functions of *x* (phase terms) must be equal (phase matching)

$$\beta_1 x\sin\theta_i = \beta_1 x\sin\theta_r = \beta_2 x\sin\theta_t,$$

$$\theta_r = \theta_i \qquad \sin\theta_t/\sin\theta_i = \beta_1/\beta_2 = n_1/n_2$$

Snell's law of reflection
Snell's law of refraction

Substitute in Eqs. (8-202) and (8-203)

$$E_{i0} + E_{r0} = E_{t0}$$

$$\frac{1}{\eta_1}(E_{i0} - E_{r0})\cos\theta_i = \frac{E_{t0}}{\eta_2}\cos\theta_t,$$

$$E_{i0} + E_{r0} = E_{t0}$$

$$\frac{1}{\eta_1}(E_{i0} - E_{r0}) \cos \theta_i = \frac{E_{t0}}{\eta_2} \cos \theta_t,$$

Derivation

Express E_{r0} and E_{t0}

$$\Gamma_\perp = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_t}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t} = \frac{(\eta_2/\cos \theta_t) - (\eta_1/\cos \theta_i)}{(\eta_2/\cos \theta_t) + (\eta_1/\cos \theta_i)}$$

$$\tau_\perp = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t} = \frac{2(\eta_2/\cos \theta_t)}{(\eta_2/\cos \theta_t) + (\eta_1/\cos \theta_i)}.$$

Fresnel's equations

Comparison with normal incidence

Normal incidence

$$\Gamma = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

$$\tau = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2}{\eta_2 + \eta_1}$$

$$1 + \Gamma = \tau$$

$\eta_1 \rightarrow (\eta_1/\cos\theta_i)$

$\eta_2 \rightarrow (\eta_2/\cos\theta_t)$

Oblique incidence

$$\Gamma_\perp = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} = \frac{(\eta_2/\cos\theta_t) - (\eta_1/\cos\theta_i)}{(\eta_2/\cos\theta_t) + (\eta_1/\cos\theta_i)}$$

$$\tau_\perp = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} = \frac{2(\eta_2/\cos\theta_t)}{(\eta_2/\cos\theta_t) + (\eta_1/\cos\theta_i)}.$$

$$\boxed{1 + \Gamma_\perp = \tau_\perp,}$$

When $\theta_i = 0$, $\theta_r = \theta_t = 0$

➔ reduce to normal incidence

If medium 2 is a perfect conductor, η_2=0

$$\Gamma_\perp = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} = \frac{(\eta_2/\cos\theta_t) - (\eta_1/\cos\theta_i)}{(\eta_2/\cos\theta_t) + (\eta_1/\cos\theta_i)}$$

$$\tau_\perp = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} = \frac{2(\eta_2/\cos\theta_t)}{(\eta_2/\cos\theta_t) + (\eta_1/\cos\theta_i)}.$$

η_2=0 $\Rightarrow$

$$\Gamma_\perp = -1 \; (E_{r0} = -E_{i0})$$

$$\tau_\perp = 0 \; (E_{t0} = 0)$$

E tangential on the surface of conductor = 0. No energy is transmitted across a perfectly conducting boundary (as was noted).

When reflection = 0 ?

$$\Gamma_\perp = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

$$= \frac{(\eta_2/\cos\theta_t) - (\eta_1/\cos\theta_i)}{(\eta_2/\cos\theta_t) + (\eta_1/\cos\theta_i)}$$

$$\Gamma_\perp = 0$$

Denote the $\theta_i = \theta_{B\perp}$ for no reflection

$$\eta_2 \cos\theta_{B\perp} = \eta_1 \cos\theta_t.$$

Derivation

$$\cos\theta_t = \sqrt{1 - \sin^2\theta_t} = \sqrt{1 - \frac{n_1^2}{n_2^2}\sin^2\theta_i}$$

By Snell's law of refraction

$$\sin^2\theta_{B\perp} = \frac{1 - \mu_1\epsilon_2/\mu_2\epsilon_1}{1 - (\mu_1/\mu_2)^2}.$$

$\theta_{B\perp}$: Brewster angle of no reflection of s-polarization

For nonmagnetic material, $\mu_1 = \mu_2 = \mu_0$, $\theta_{B\perp}$ does not exist.
For materials $\varepsilon_1 = \varepsilon_2$ and $\mu_1 \neq \mu_2$ (**very rare situation**), $\theta_{B\perp}$ exists: $\sin\theta_{B\perp} = \dfrac{1}{\sqrt{1 + (\mu_1/\mu_2)}}$,

8-10.3 Parallel Polarization

- **E** // the plane of incidence
- p-polarization
- TM

The incident fields

$$\mathbf{E}_i(x, z) = E_{i0}(\mathbf{a}_x \cos\theta_i - \mathbf{a}_z \sin\theta_i)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\mathbf{H}_i(x, z) = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

polarization

propagation

$$\mathbf{H}(\mathbf{R}) = \frac{1}{\eta}\mathbf{a}_n \times \mathbf{E}(\mathbf{R})$$

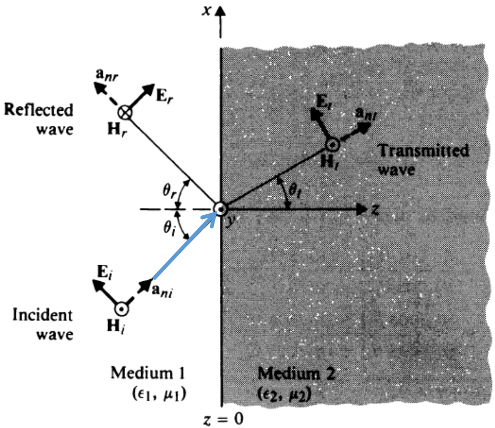


FIGURE 8–21
Plane wave incident obliquely on a plane dielectric boundary (parallel polarization).

The reflected fields

$$\mathbf{E}_r(x, z) = E_{r0}(\mathbf{a}_x \cos\theta_r + \mathbf{a}_z \sin\theta_r)e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)},$$

$$\mathbf{H}_r(x, z) = -\mathbf{a}_y \frac{E_{r0}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}.$$

polarization

propagation

$$\mathbf{H}(\mathbf{R}) = \frac{1}{\eta}\mathbf{a}_n \times \mathbf{E}(\mathbf{R})$$

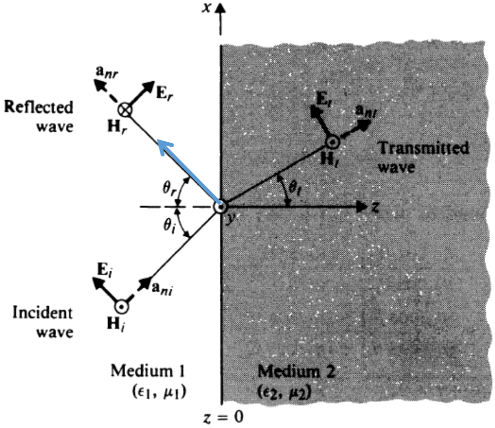


FIGURE 8–21
Plane wave incident obliquely on a plane dielectric boundary (parallel polarization).

The transmitted fields

$$\mathbf{E}_t(x, z) = E_{t0}(\mathbf{a}_x \cos\theta_t - \mathbf{a}_z \sin\theta_t)e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)},$$

$$\mathbf{H}_t(x, z) = \mathbf{a}_y \frac{E_{t0}}{\eta_2} e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}.$$

polarization

propagation

$$\mathbf{H}(\mathbf{R}) = \frac{1}{\eta}\mathbf{a}_n \times \mathbf{E}(\mathbf{R})$$

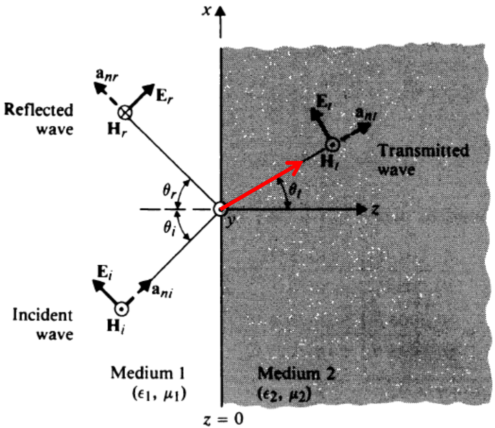


FIGURE 8–21
Plane wave incident obliquely on a plane dielectric boundary (parallel polarization).

$$\mathbf{E}_i(x, z) = E_{i0}(\mathbf{a}_x \cos\theta_i - \mathbf{a}_z \sin\theta_i)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\mathbf{H}_i(x, z) = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\mathbf{E}_r(x, z) = E_{r0}(\mathbf{a}_x \cos\theta_r + \mathbf{a}_z \sin\theta_r)e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)},$$

$$\mathbf{H}_r(x, z) = -\mathbf{a}_y \frac{E_{r0}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}.$$

$$\mathbf{E}_t(x, z) = E_{t0}(\mathbf{a}_x \cos\theta_t - \mathbf{a}_z \sin\theta_t)e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)},$$

$$\mathbf{H}_t(x, z) = \mathbf{a}_y \frac{E_{t0}}{\eta_2} e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}.$$

Tangential **E** and **H** should be continuous at *z* = 0

.

.

.

$$(E_{i0} + E_{r0})\cos\theta_i = E_{t0}\cos\theta_t,$$

$$\frac{1}{\eta_1}(E_{i0} - E_{r0}) = \frac{1}{\eta_2}E_{t0}.$$

$$(E_{i0} + E_{r0}) \cos \theta_i = E_{t0} \cos \theta_t,$$

$$\frac{1}{\eta_1}(E_{i0} - E_{r0}) = \frac{1}{\eta_2} E_{t0}.$$

Express E_{r0} and E_{t0}

$$\Gamma_{||} = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 \cos \theta_t - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i}$$

$$\tau_{||} = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i}.$$

Fresnel's equations

$$\boxed{1 + \Gamma_{||} = \tau_{||}\left(\frac{\cos \theta_t}{\cos \theta_i}\right).}$$

Different from the case in s-polarization

$$\boxed{1 + \Gamma_{\perp} = \tau_{\perp},}$$

When $\theta_i = 0$, $\theta_r = \theta_t = 0$

➔ reduce to normal incidence

If medium 2 is a perfect conductor, η_2=0

$$\boxed{\Gamma_{||} = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}}$$

$$\boxed{\tau_{||} = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}.}$$

$$\xrightarrow{\eta_2=0}$$

$$\Gamma_{||} = -1$$

$$\tau_{||} = 0$$

E tangential on the surface of conductor = 0.
No energy is transmitted across a perfectly conducting boundary (as was noted).

When is reflection = 0 ?

$$\Gamma_{||} = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

$\Gamma_{||} = 0$

Denote the $\theta_i = \theta_{B||}$ for no reflection

$$\eta_2 \cos\theta_t = \eta_1 \cos\theta_{B||},$$

Derivation

$$\cos\theta_t = \sqrt{1 - \sin^2\theta_t} = \sqrt{1 - \frac{n_1^2}{n_2^2}\sin^2\theta_i}$$

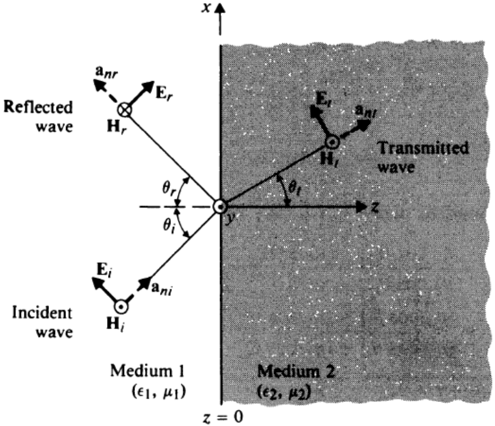
By Snell's law of refraction

$$\boxed{\sin^2\theta_{B||} = \frac{1 - \mu_2\epsilon_1/\mu_1\epsilon_2}{1 - (\epsilon_1/\epsilon_2)^2}.}$$

$\theta_{B||}$: Brewster angle of no reflection of p-polarization

Directions of $\mathbf{E}_r$, $\mathbf{H}_r$ in figures 8-11, 8-13, 8-20, and 8-21 are chosen arbitrarily.
The actual directions depend on the sign of the expression.

- In Figs. 8-11 and 8-13, actual directions of $\mathbf{E}_r$ $\mathbf{H}_r$ are opposite to those chosen because $E_{r0} = -E_{i0}$
- In Figs. 8-20 and 8-21, actual directions of $\mathbf{E}_r$ $\mathbf{H}_r$ depends on the sign of $\Gamma_\perp$ and $\Gamma_{||}$, respectively



$$\mathbf{E}_r(x, z) = E_{r0}(\mathbf{a}_x \cos\theta_r + \mathbf{a}_z \sin\theta_r)e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)},$$

$$\mathbf{H}_r(x, z) = -\mathbf{a}_y \frac{E_{r0}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}.$$

$$\Gamma_{||} = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

- If $\Gamma_{||} > 0$, $\mathbf{H}_r$ is in $-\mathbf{a}_y$ direction (same as shown in figure)
- If $\Gamma_{||} < 0$, $\mathbf{H}_r$ is in $+\mathbf{a}_y$ direction (opposite to that shown in figure)

FIGURE 8–21
Plane wave incident obliquely on a plane dielectric boundary (parallel polarization).

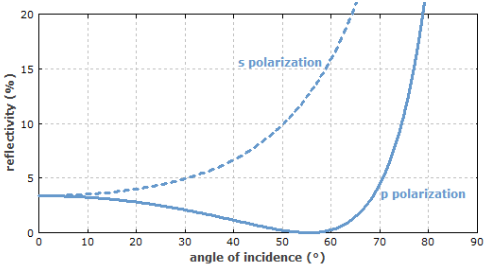
Comparison of $|\Gamma_{\perp}|^2$ and $|\Gamma_{||}|^2$

$$\Gamma_{\perp} = \frac{E_{r0}}{E_{i0}} = \frac{(\eta_2/\cos\theta_t) - (\eta_1/\cos\theta_i)}{(\eta_2/\cos\theta_t) + (\eta_1/\cos\theta_i)}$$

$$\Gamma_{||} = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

$|\Gamma_{\perp}|^2$ is always larger than $|\Gamma_{||}|^2$

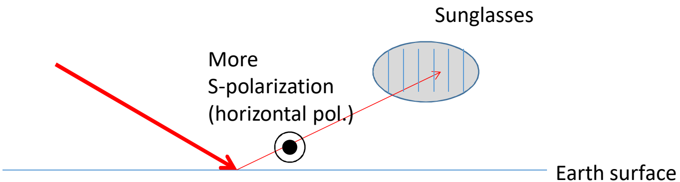
➔ When an unpolarized light strikes a plane dielectric interface, the reflected wave will contain more power in s-polarization than p-polarization.



Power reflectivity of the interface for s and p polarization, if a beam is incident from air onto a medium with refractive index 1.45 (e.g., silica at 1064 nm).

http://www.rp-photonics.com/fresnel_equations.html

Polaroid Sunglasses



The light reaching the eye is predominately s-polarization
(i,e, **E** $\perp$ plane of reflection, or **E** field is parallel to the earth surface)

Polaroid sunglasses (a polarizer) are designed to filter out this component $\mathbf{E}_{\perp}$
As a result, a dim light of $\mathbf{E}_{//}$ will penetrate into the sunglasses.

For materials $\mu_1 = \mu_2$, $\theta_{B||}$:

$$\boxed{\sin\theta_{B||} = \frac{1}{\sqrt{1+(\epsilon_1/\epsilon_2)}}.} \qquad (\mu_1 = \mu_2)$$

or

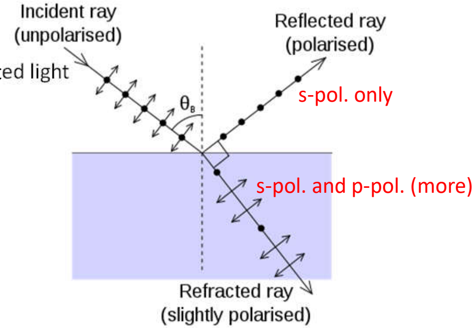
$$\boxed{\theta_{B||} = \tan^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}} = \tan^{-1}\left(\frac{n_2}{n_1}\right).} \qquad (\mu_1 = \mu_2)$$

Because of different formulas for Brewster angles for s- and p-polarization, it is possible to separate these 2 types of polarizations from an unpolarized light.

E.g.,

Light is incident at angle $\theta_{B||}$

➔ no p-polarization component in the reflected light
or only s-polarization in the reflected light



EXAMPLE 8–15 The dielectric constant of pure water is 80. (a) Determine the Brewster angle for parallel polarization, $\theta_{B\|}$, and the corresponding angle of transmission. (b) A plane wave with perpendicular polarization is incident from air on water surface at $\theta_i = \theta_{B\|}$. Find the reflection and transmission coefficients.