Chapter 3 Static Electric Fields

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3-1 Introduction

- A field is a specified distribution of a scalar or vector quantity, which may or may not be a function of time.
- In this chapter, we deal with electrostatics, which means electric charges are at rest (not moving), and electric field do not change with time.
- There is no magnetic field existed in this chapter.

3-2 Fundamental Postulates of Electrostatics in Free Space

- The simplest case:
  - Static electric charges (source) in free space ➔ electric fields
- Electric field intensity
  - If q is small enough not to disturb the charge distribution of the source,

$$\boxed{\mathbf{E} = \lim_{q \to 0} \frac{\mathbf{F}}{q} \qquad (\text{V/m}).}$$

$$\boxed{\mathbf{F} = q\mathbf{E} \qquad (\text{N}).}$$

- Fundamental postulates of electrostatics (in free space)

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}}$$

The charge is a flow source of E field.

$$\boxed{\nabla \times \mathbf{E} = 0.}$$

Integral Form

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}} \Rightarrow \int_V \nabla \cdot \mathbf{E}\, dv = \frac{1}{\epsilon_0} \int_V \rho\, dv. \Rightarrow \boxed{\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0},}$$

A form of Gauss's law: the total outward flux of the electric field intensity over any closed surface in free space is equal to the total charge enclosed in the surface divided by ε_0

Integral Form

$$\boxed{\nabla \times \mathbf{E} = 0.} \Rightarrow \qquad \Rightarrow \boxed{\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = 0.}$$

The scalar line integral of the static electric field intensity around any closed path vanishes.

KVL in circuit theory: the algebraic sum of voltage drops around any closed circuit is zero.

E is Irrotational (Conservative)

$$\boxed{\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = 0.}$$

$$\int_{C_1} \mathbf{E} \cdot d\boldsymbol{\ell} + \int_{C_2} \mathbf{E} \cdot d\boldsymbol{\ell} = 0$$

$$\underset{\text{Along } C_1}{\int_{P_1}^{P_2}} \mathbf{E} \cdot d\boldsymbol{\ell} = - \underset{\text{Along } C_2}{\int_{P_2}^{P_1}} \mathbf{E} \cdot d\boldsymbol{\ell}$$

$$\underset{\text{Along } C_1}{\int_{P_1}^{P_2}} \mathbf{E} \cdot d\boldsymbol{\ell} = \underset{\text{Along } C_2}{\int_{P_1}^{P_2}} \mathbf{E} \cdot d\boldsymbol{\ell}.$$

The scalar line integral of the irrotational **E** field is *independent of the path*; it depends only on the end points.

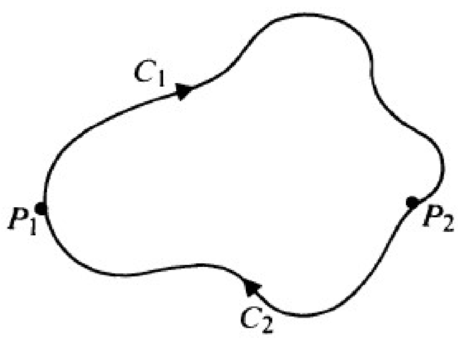


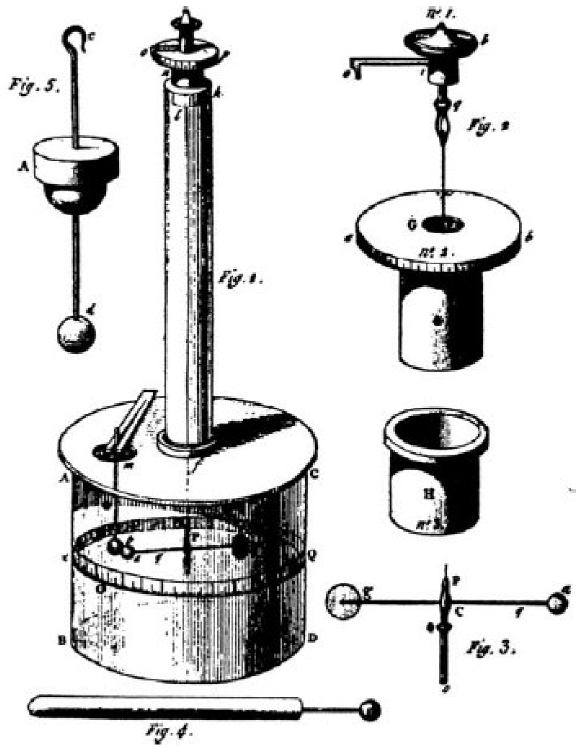
FIGURE 3–1
An arbitrary contour.

Postulates of Electrostatics in Free Space	
Differential Form	Integral Form
$\nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0}$	$\oint_S \mathbf{E} \cdot d\mathbf{s} = \dfrac{Q}{\epsilon_0}$
$\nabla \times \mathbf{E} = 0$	$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = 0$

3-3 Coulomb's Law

- The first person to quantify this law was Coulomb, whose experiment using torsion meters in 1785 demonstrated two things:

1) The force is proportional to the charges on each object
2) The force is inversely proportional to the square of distance
3) The force is a "central body" force (parallel to the position vector)



Coulomb's Law

- The simplest case: a single point charge

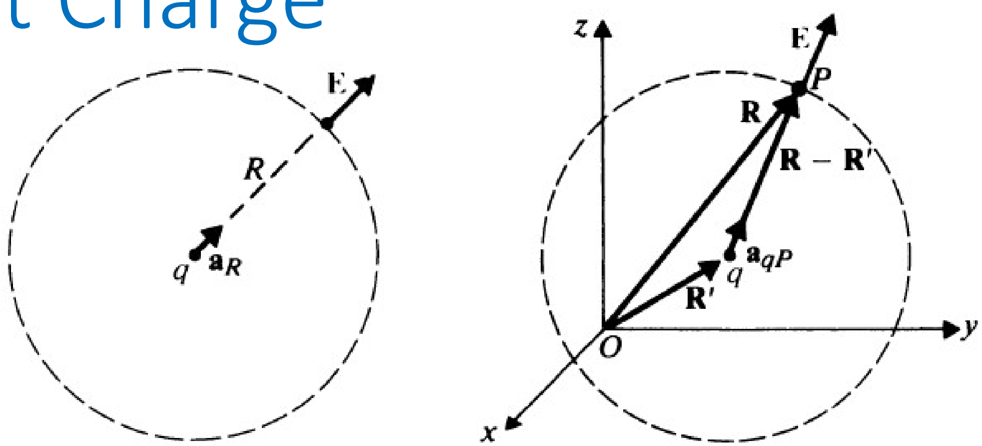
$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \oint_S (\mathbf{a}_R E_R) \cdot \mathbf{a}_R \, ds = \frac{q}{\epsilon_0}$$

$$E_R \oint_S ds = E_R(4\pi R^2) = \frac{q}{\epsilon_0}.$$

$$\boxed{\mathbf{E} = \mathbf{a}_R E_R = \mathbf{a}_R \frac{q}{4\pi\epsilon_0 R^2} \qquad \text{(V/m)}.}$$

The electric field intensity of a positive point charge is in the outward direction and has a magnitude proportional to the charge and inversely proportional to the square of the distance from the charge.

E of a Point Charge



(a) Point charge at the origin. (b) Point charge not at the origin.

FIGURE 3–2

$$\mathbf{E} = \mathbf{a}_R E_R = \mathbf{a}_R \frac{q}{4\pi\epsilon_0 R^2}$$

R* → *R*–*R’

R: field
R’: source

$$\mathbf{E}_P = \mathbf{a}_{qP} \frac{q}{4\pi\epsilon_0 |\mathbf{R} - \mathbf{R}'|^2}$$

$$\mathbf{a}_{qP} = \frac{\mathbf{R} - \mathbf{R}'}{|\mathbf{R} - \mathbf{R}'|}$$

$$\boxed{\mathbf{E}_p = \frac{q(\mathbf{R} - \mathbf{R}')}{4\pi\epsilon_0 |\mathbf{R} - \mathbf{R}'|^3} \qquad \text{(V/m).}}$$

Mathematical Form of Coulomb's Law

- Force on q_2 in an **E** field due to q_1

$$\boxed{\mathbf{F}_{12} = q_2\mathbf{E}_{12} = \mathbf{a}_R \frac{q_1 q_2}{4\pi\epsilon_0 R^2} \quad \text{(N).}}$$

3-3.1 Electric Field due to a System of Discrete Charges

- ss

$$\boxed{\mathbf{E}_p = \frac{q(\mathbf{R} - \mathbf{R}')}{4\pi\epsilon_0|\mathbf{R} - \mathbf{R}'|^3} \quad \text{(V/m).}}$$

$$\Downarrow$$

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\sum_{k=1}^{n} \frac{q_k(\mathbf{R} - \mathbf{R}'_k)}{|\mathbf{R} - \mathbf{R}'_k|^3} \quad \text{(V/m).}}$$

An Electric Dipole

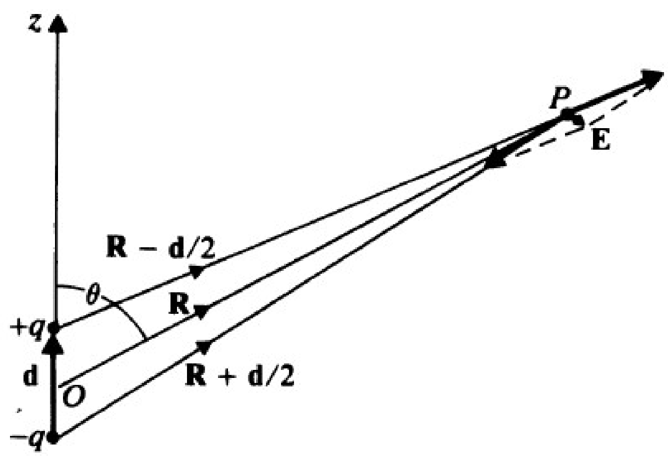


FIGURE 3–5
Electric field of a dipole.

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left\{\frac{\mathbf{R}-\frac{\mathbf{d}}{2}}{\left|\mathbf{R}-\frac{\mathbf{d}}{2}\right|^3} - \frac{\mathbf{R}+\frac{\mathbf{d}}{2}}{\left|\mathbf{R}+\frac{\mathbf{d}}{2}\right|^3}\right\}.$$

If $d << R$

$$\begin{aligned}\left|\mathbf{R}-\frac{\mathbf{d}}{2}\right|^{-3} &= \left[\left(\mathbf{R}-\frac{\mathbf{d}}{2}\right)\cdot\left(\mathbf{R}-\frac{\mathbf{d}}{2}\right)\right]^{-3/2} \\ &= \left[R^2 - \mathbf{R}\cdot\mathbf{d} + \frac{d^2}{4}\right]^{-3/2} \\ &\cong R^{-3}\left[1 - \frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right]^{-3/2} \\ &\cong R^{-3}\left[1 + \frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right],\end{aligned}$$

$$\left|\mathbf{R}-\frac{\mathbf{d}}{2}\right|^{-3} \cong R^{-3}\left[1 + \frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right]$$

$$\left|\mathbf{R}+\frac{\mathbf{d}}{2}\right|^{-3} \cong R^{-3}\left[1 - \frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right].$$

$$\mathbf{E} \cong \frac{q}{4\pi\epsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\mathbf{R} - \mathbf{d}\right].$$

Electric Dipole Moment

- Definition: The product of the charge q and the vector **d** $\mathbf{p} = q\mathbf{d}.$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{p}}{R^2}\mathbf{R} - \mathbf{p}\right]$$

$$\mathbf{p} = \mathbf{a}_z p = p(\mathbf{a}_R \cos\theta - \mathbf{a}_\theta \sin\theta),$$

$$\mathbf{R}\cdot\mathbf{p} = Rp\cos\theta,$$

$$\boxed{\mathbf{E} = \frac{p}{4\pi\epsilon_0 R^3}(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta \sin\theta) \quad \text{(V/m)}.}$$

3-3.2 Electric Field due to a Continuous Distribution of Charge

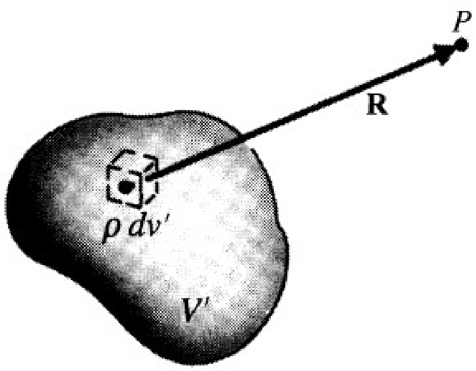


FIGURE 3–6
Electric field due to a continuous charge distribution.

$$d\mathbf{E} = \mathbf{a}_R \frac{\rho\, dv'}{4\pi\epsilon_0 R^2}.$$

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_{V'} \mathbf{a}_R \frac{\rho}{R^2}\, dv' \qquad \text{(V/m)},}$$

$$\mathbf{a}_R = \mathbf{R}/R$$

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_{V'} \rho \frac{\mathbf{R}}{R^3}\, dv' \qquad \text{(V/m)}.}$$

For a Surface or Line Charge

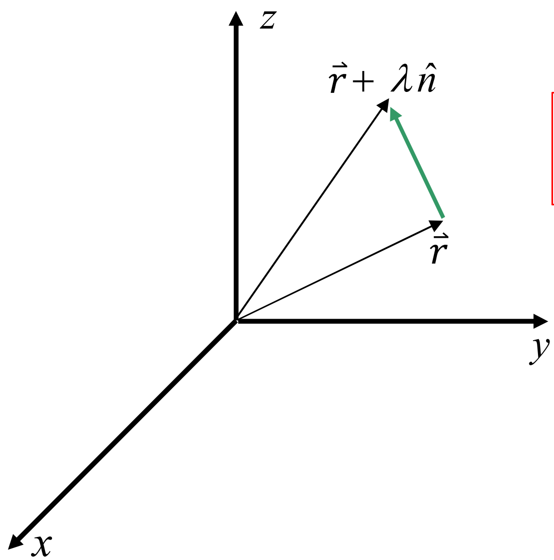
$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_{S'} \mathbf{a}_R \frac{\rho_s}{R^2}\, ds' \qquad \text{(V/m).}}$$

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_{L'} \mathbf{a}_R \frac{\rho_\ell}{R^2}\, d\ell' \qquad \text{(V/m),}}$$

Vector Calculus

Directional derivative $$\frac{Du(\vec{r})}{Dn} = \lim_{\lambda \to 0} \frac{u(\vec{r} + \lambda\hat{n}) - u(\vec{r})}{\lambda}$$

Gradient $$\nabla u(\vec{r}) \cdot \hat{n} = \frac{Du(\vec{r})}{Dn} = \lim_{\lambda \to 0} \frac{u(\vec{r} + \lambda\hat{n}) - u(\vec{r})}{\lambda}$$



Derivative along a direction $\hat{n}$ with a distance λ

Very Useful Formulas and Examples!!!!

$$\left|\vec{r}+\lambda\hat{n}\right| = \sqrt{r^2 + \underline{2\lambda\hat{n}\cdot\vec{r}} + \lambda^2} = \underline{r}\sqrt{1 + 2\frac{\lambda}{\underline{r}}\underline{\hat{n}\cdot\hat{r}} + \frac{\lambda^2}{r^2}} \approx r\left(1 + \frac{\lambda}{r}\hat{n}\cdot\hat{r}\right)$$

$$when \quad \lambda << r$$

Example $\quad \nabla r = ?$

$\boxed{u(\vec{r}) = r}$

$$\nabla r\cdot\hat{n} = \lim_{\lambda\to 0}\frac{\left|\vec{r}+\lambda\hat{n}\right| - r}{\lambda} \approx \lim_{\lambda\to 0}\frac{r\left(1+\frac{\lambda}{r}\hat{n}\cdot\hat{r}\right) - r}{\lambda} = \hat{r}\cdot\hat{n}, \;\; \nabla r = \hat{r}$$

Example $\quad \nabla\frac{1}{r} = ?$

$\boxed{u(\vec{r}) = \frac{1}{r}}$

$$\nabla\frac{1}{r}\cdot\hat{n} = \lim_{\lambda\to 0}\frac{\frac{1}{\left|\vec{r}+\lambda\hat{n}\right|} - \frac{1}{r}}{\lambda} \approx \frac{1}{r}\lim_{\lambda\to 0}\frac{\frac{1}{\left(1+\frac{\lambda}{r}\hat{n}\cdot\hat{r}\right)} - 1}{\lambda} = -\frac{\hat{r}}{r^2}\cdot\hat{n},$$

$$or \quad \nabla\frac{1}{r} = -\frac{\hat{r}}{r^2} = -\frac{\vec{r}}{r^3}$$

When r is not 0 !!!

Example: Determine the Laplacian of the function 1/*r*

$$\nabla^2 \frac{1}{r} = \nabla \cdot \nabla \frac{1}{r} = -\nabla \cdot \frac{\vec{r}}{r^3} = ?$$

$$\nabla \frac{1}{r} = -\frac{\vec{r}}{r^3}$$

$$\nabla \cdot (\psi \mathbf{A}) = \psi \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla \psi$$

$$\nabla \frac{1}{r^n} = -n \frac{\hat{r}}{r^{n+1}} = -n \frac{\vec{r}}{r^{n+2}}$$

$$\nabla \cdot \frac{\vec{r}}{r^3} = \frac{1}{r^3} \nabla \cdot \vec{r} + \vec{r} \cdot \nabla \frac{1}{r^3} = \frac{3}{r^3} - 3\vec{r} \cdot \frac{\vec{r}}{r^5} = 0$$

The above is valid when r ≠ 0. What happens when r = 0???

Consider volume integral of the function over all space. This integral can be found by integrating over a spherical volume shown in the figure and letting the radius increase to infinity

$$\lim_{R\to\infty} \iiint_{V_R} \nabla \cdot \frac{\vec{r}}{r^3} dV = \lim_{R\to\infty} \oint\!\!\oint_{S_R} \frac{\vec{r} \cdot \hat{r}}{r^3} dS = \lim_{R\to\infty} \oint\!\!\oint_{S_R} \frac{1}{r^2} dS = \lim_{R\to\infty} \frac{1}{R^2} 4\pi R^2 = 4\pi$$

What we have just shown is that the Laplacian of the function $1/r$ is zero everywhere except at the origin where $r = 0$, and yet its volume integral is finite and equal to 4π.

This implies that the Laplacian of $1/r$ is actually a Dirac Delta Function, given by:

$$\nabla^2 \frac{1}{r} = -\nabla \cdot \frac{\vec{r}}{r^3} = -4\pi\delta(\vec{r})$$

Or alternatively, we can say that the function: $\varphi = \dfrac{1}{4\pi r}$

is a solution to the differential equation: $\nabla^2 \varphi = -\delta(\vec{r})$

This result is of fundamental importance in the subject of electrostatic and magnetostatic fields !!!!!

EXAMPLE 3–2 A total charge Q is put on a thin spherical shell of radius b. Determine the electric field intensity at an arbitrary point inside the shell.

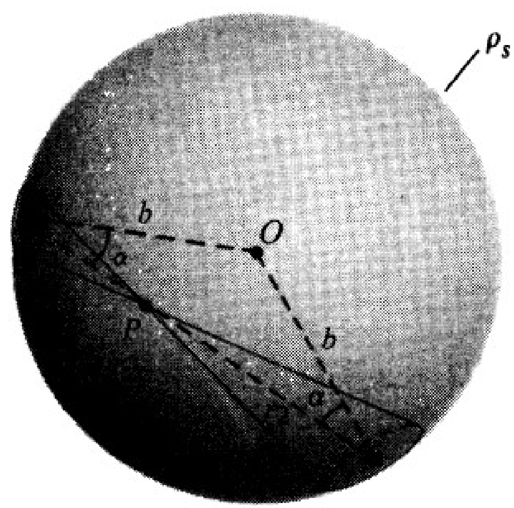


FIGURE 3–3
A charged shell (Example 3–2).

EXAMPLE 3–4 Determine the electric field intensity of an infinitely long, straight, line charge of a uniform density ρ_ℓ in air. •

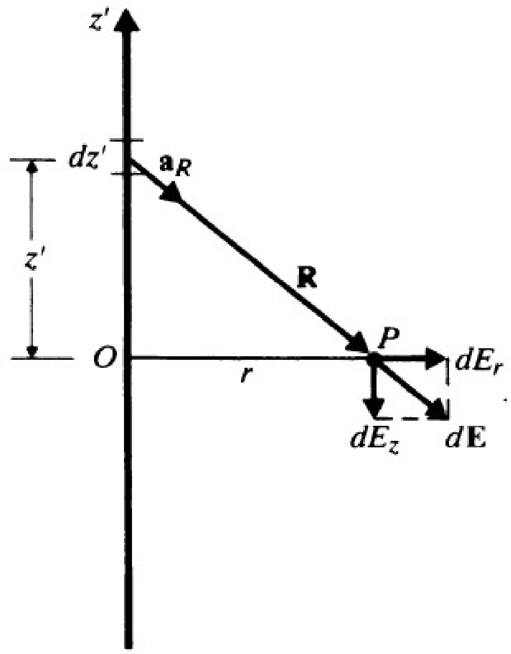


FIGURE 3–7
An infinitely long, straight, line charge.

3-4 Gauss's Law and Applications

Recap of slide#4

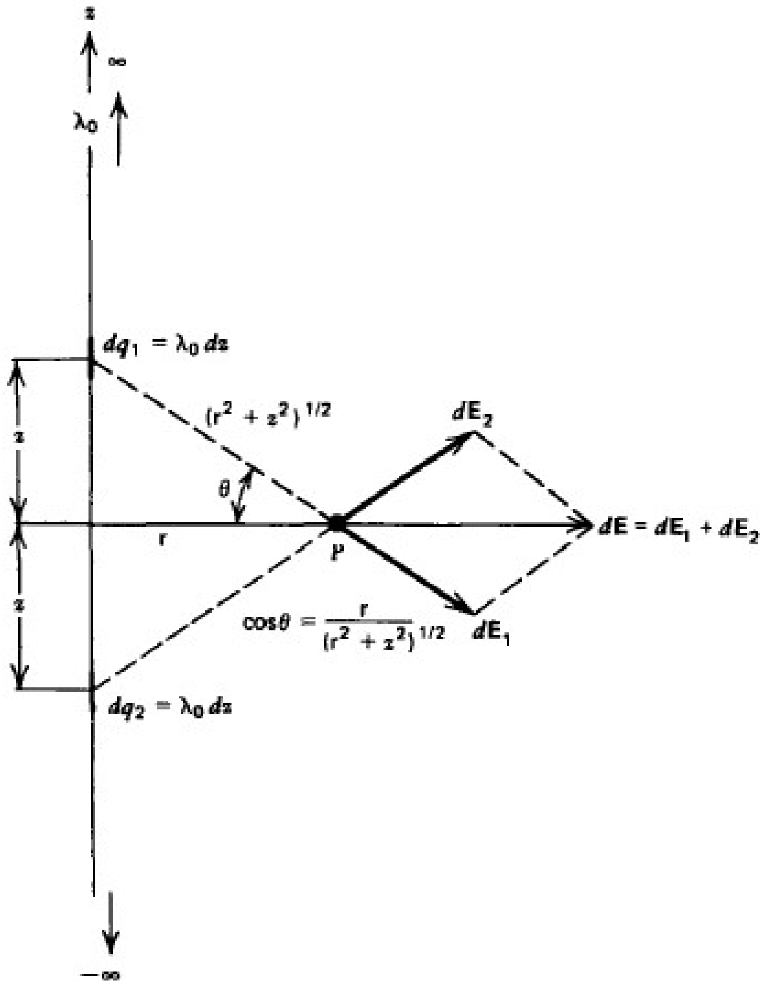
$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}} \Rightarrow \boxed{\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0}.}$$

Gauss's law: The total outward flux of the E-field over any closed surface in free space is equal to the total charge enclosed in the surface divided by ε_0

S: can be any hypothetical closed surface

Example 1

- Let us calculate the electric field of an infinitely long line charge in two ways.
- In the first way, we will directly integrate the electrical field of a line charge as a superposition of small segments, *dl*, containing total charge *λdl*.



$$\vec{E}(\vec{r}) = \int_{-\infty}^{+\infty} \frac{\lambda(\vec{r} - z_s\hat{z})}{4\pi\varepsilon_0 |\vec{r} - z_s\hat{z}|^3} dz_s$$

Let us solve this problem in cylindrical coordinates, therefore

$$\vec{E}(\vec{r}) = \frac{\lambda}{4\pi\varepsilon_0} \int_{-\infty}^{+\infty} \frac{r\hat{r} - z_s\hat{z}}{\sqrt{r^2 + z_s^2}^3} dz_s$$

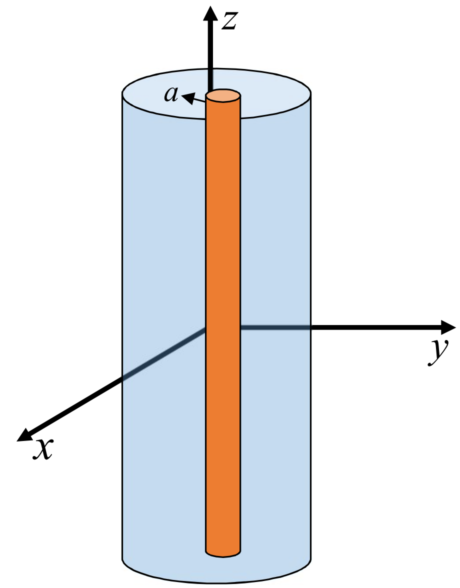
Note that due to symmetry the ***z*** component of the field cancels:

$$\vec{E}(\vec{r}) = \frac{\lambda\hat{r}}{4\pi\varepsilon_0} \int_{0}^{+\infty} \frac{2r}{\sqrt{r^2 + z_s^2}^3} dz_s$$

$$\vec{E}(\vec{r}) = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$$

Now let's solve this problem using Maxwell's equations…

Let us draw a cylindrical surface around the line charge, where it is assumed to be placed at the center of the cylinder. Due to symmetry, the field evaluated anywhere on the surface must be in the ***r*** direction and must be constant.



$$\oint\!\!\oint_S \vec{E}\cdot\hat{n}\,dS = \frac{Q}{\varepsilon_0}$$

$$\frac{Q}{\varepsilon_0} = \int_{-\infty}^{+\infty}\frac{\lambda}{\varepsilon_0}dz \qquad \oint\!\!\oint_S \vec{E}\cdot\hat{n}\,dS = \oint\!\!\oint_S E_r\hat{r}\cdot\hat{r}r\,d\phi\,dz \;= \int_{-\infty}^{+\infty} 2\pi r E_r\,dz$$

$$\therefore \int_L 2\pi r E_r\,dz = \int_L \frac{\lambda}{\varepsilon_0}dz$$

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$$

Maxwell's equations thus provide a more direct way to obtain the electric field of these highly symmetrical systems.

Where can Maxwell’s Integral Equations be Used?

- Computing electric field by the direct charge integration method will always lead to the correct field, however it can involved tedious integrals and possibly require numerical calculations.
- The integral form of Maxwell’s equations provides a more direct method to obtaining the electrical field of simple charge distributions, however it is only applicable when there is a high degree of symmetry in the charge distribution or in the electrical field (i.e., spherically symmetric, planar, line charge, etc.
- Let’s look at a few examples where the integral form of Maxwell’s equations cannot be used.

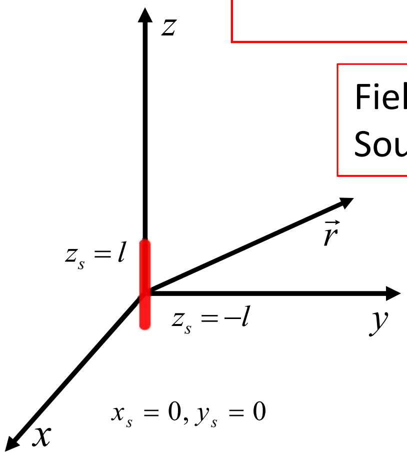
Example 2: Field of a Line charge of Finite Length

$$V(\vec{r}) = \int_L \frac{\lambda(\vec{r_s})}{4\pi\varepsilon_0|\vec{r}-\vec{r_s}|} dl_s$$

Field: (x, y, z)
Source: $(0, 0, z_s)$

$$V(x,y,z) = \int_{-l}^{l} \frac{\lambda}{4\pi\varepsilon_0\sqrt{x^2+y^2+(z-z_s)^2}} dz_s$$

$$V(x,y,z) = \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{z-l+\sqrt{x^2+y^2+(z-l)^2}}{z+l+\sqrt{x^2+y^2+(z+l)^2}} \right)$$



The electric field can then be computed from the negative gradient of this potential.

$$\vec{E} = -\nabla \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{z-l+\sqrt{x^2+y^2+(z-l)^2}}{z+l+\sqrt{x^2+y^2+(z+l)^2}} \right)$$

Example 3: Field of a Charged Ring

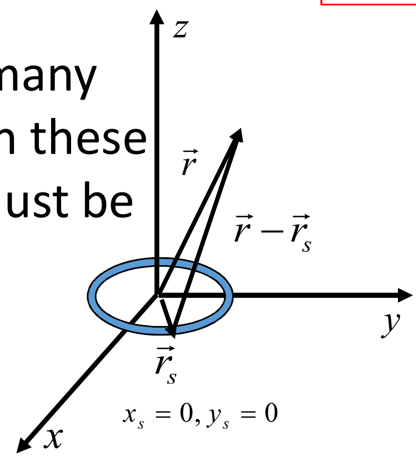
Consider a ring with radius *a* and uniform charge density with a center at the origin and located in the xy- plane.

$$V(x,y,z)=\frac{\lambda a}{4\pi\varepsilon_0}\oint\frac{d\theta}{\sqrt{(x-a\cos\theta)^2+(y-a\sin\theta)^2+z^2}}$$

$$\boxed{V(\vec{r}) = \int_L \frac{\lambda(\vec{r_s})}{4\pi\varepsilon_0|\vec{r}-\vec{r_s}|}dl_s}$$

Field: (*x*, *y*, *z*)
Source: (*a*cos*θ*, *a*sin*θ*, 0)

This integral is more difficult to solve, and in many cases they don't have an analytical solution. In these and other examples, the potential and field must be computed numerically on a computer:



$$\vec{E}=-\nabla\frac{\lambda a}{4\pi\varepsilon_0}\oint\frac{d\theta}{\sqrt{(x-a\cos\theta)^2+(y-a\sin\theta)^2+z^2}}$$

EXAMPLE 3–7 Determine the **E** field caused by a spherical cloud of electrons with a volume charge density $\rho = -\rho_o$ for $0 \le R \le b$ (both ρ_o and b are positive) and $\rho = 0$ for $R > b$.

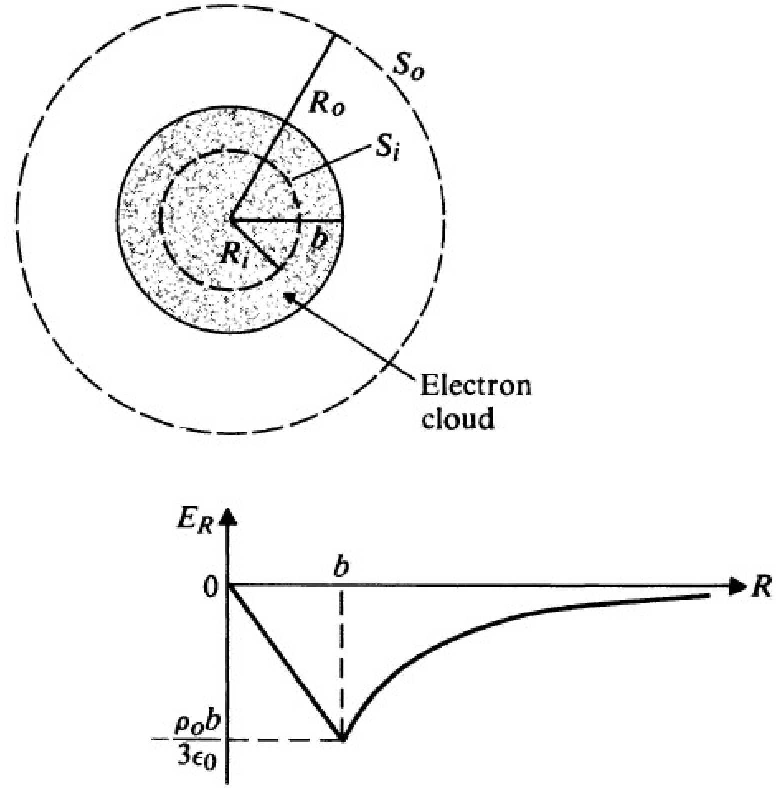


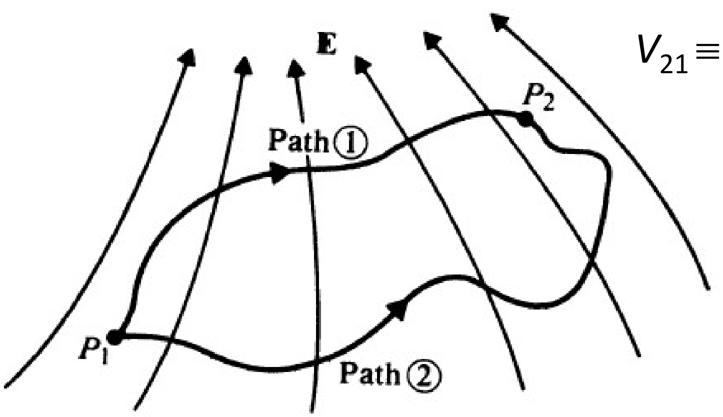
FIGURE 3–10
Electric field intensity of a spherical electron cloud (Example 3–7).

3-5 Electric Potential

$$\nabla \times \mathbf{E} = 0 \quad \Rightarrow \quad \boxed{\mathbf{E} = -\nabla V}$$

Scalar quantities are easier to handle than vector quantities

Physical meaning: Work done in carrying a charge from one point to another



$$V_{21} \equiv \frac{W}{q} = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\boldsymbol{\ell} \qquad \text{(J/C or V).}$$

$\boxed{\mathbf{E} = \mathbf{F}/q}$

Work is done against the field

FIGURE 3–11
Two paths leading from P_1 to P_2 in an electric field.

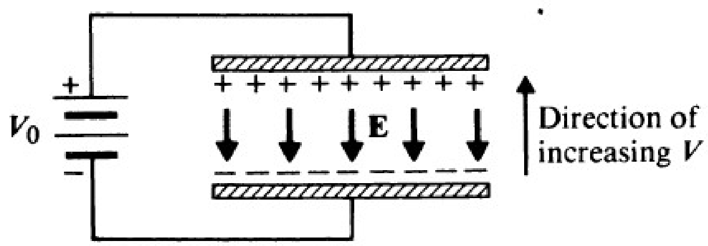
Electrical Potential Difference

$$V_{21} = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\ell = \int_{P_1}^{P_2} (\nabla V)\cdot(\mathbf{a}_\ell\, d\ell)$$
$$= \int_{P_1}^{P_2} dV = V_2 - V_1.$$

Recap: $\nabla V \cdot \mathbf{a}_\ell = dV/d\ell$

$$\boxed{V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\ell \qquad \text{(V).}}$$

Usually, the zero-potential point is taken at infinity (P_1).



E and increasing V are in opposite direction

FIGURE 3–12
Relative directions of **E** and increasing V.

In going against the **E** field, the electric potential V increases.

$$\boxed{\mathbf{E} = -\nabla V}$$

$\downarrow$

$\mathbf{E} \perp$ constant-V surfaces

Field lines
Streamlines

Equipotential lines
Equipotential surfaces

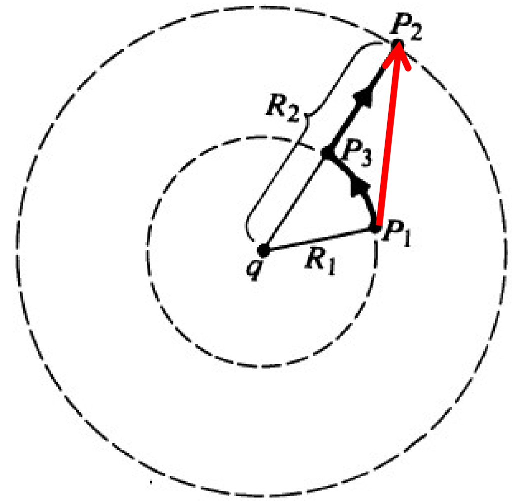
3-5.1 Electric Potential due to a Charge Distribution

- $V(R)$ of a point charge at origin

$$V = -\int_{\infty}^{R} \left(\mathbf{a}_R \frac{q}{4\pi\epsilon_0 R^2} \right) \cdot (\mathbf{a}_R \, dR),$$

With reference point at infinity

$$\boxed{V = \frac{q}{4\pi\epsilon_0 R} \qquad \text{(V).}}$$



- Potential difference between any two points

$$V_{21} = V_{P_2} - V_{P_1} = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{R_2} - \frac{1}{R_1} \right).$$

A charge is moved from P_1 to P_2.

$V_{21} = V_{31} + V_{23}$

V due to n Discrete Point Charges

$$V = \frac{q}{4\pi\epsilon_0 R} \quad \text{(V)}.$$

Reference point at infinity

R ➔ ***R* – *R'*** (Charges located at ***R'***)

$\sum$

$$V = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \frac{q_k}{|\mathbf{R} - \mathbf{R}'_k|} \quad \text{(V)}.$$

V of an Electric Dipole

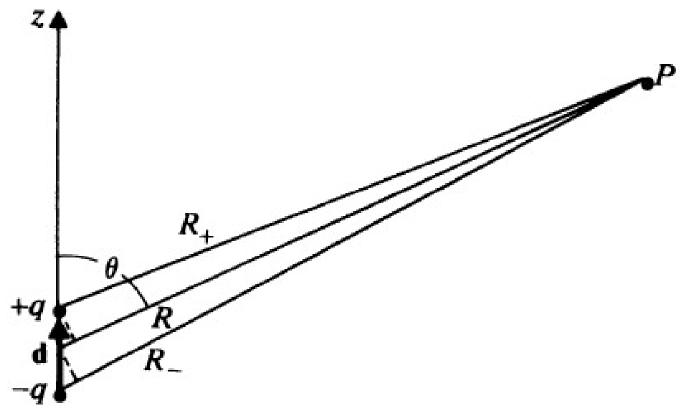


FIGURE 3–14
An electric dipole.

$$V = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right).$$

If $d \ll R$

$$\frac{1}{R_+} \cong \left(R - \frac{d}{2}\cos\theta\right)^{-1} \cong R^{-1}\left(1 + \frac{d}{2R}\cos\theta\right)$$

$$\frac{1}{R_-} \cong \left(R + \frac{d}{2}\cos\theta\right)^{-1} \cong R^{-1}\left(1 - \frac{d}{2R}\cos\theta\right).$$

$$V = \frac{qd\cos\theta}{4\pi\epsilon_0 R^2}$$

or

$$\boxed{V = \frac{\mathbf{p}\cdot\mathbf{a}_R}{4\pi\epsilon_0 R^2} \qquad \text{(V)},}$$

where $\mathbf{p} = q\mathbf{d}$.

$$\mathbf{E} = -\nabla V = -\mathbf{a}_R\frac{\partial V}{\partial R} - \mathbf{a}_\theta\frac{\partial V}{R\,\partial\theta}$$

$$= \frac{p}{4\pi\epsilon_0 R^3}(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta\sin\theta).$$

A much simpler approach (vs. slide#15)

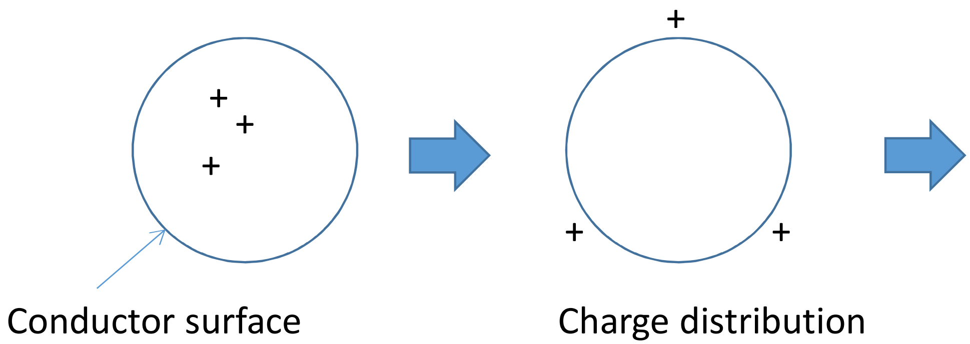
3-6 Conductors in Static Electric Field

- 3 types: conductors, semiconductors, insulators (or dielectrics)
- Conductors: Orbiting electrons are loosely held by an atom and migrate easily from one atom to another.
- Insulators: Electrons confined to their orbits
- Semiconductors: A small number of freely moveable charges (between conductors and insulators)

Band theory

- Crucial to the conduction process is whether or not there are electrons in the conduction band

E and ρ inside a Conductor



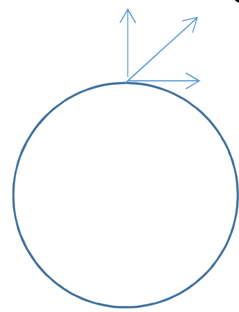
$\rho_{inside} = 0$

Inside a Conductor (*Under Static Conditions*)
$\rho = 0$
$\mathbf{E} = 0$

By Gauss's law

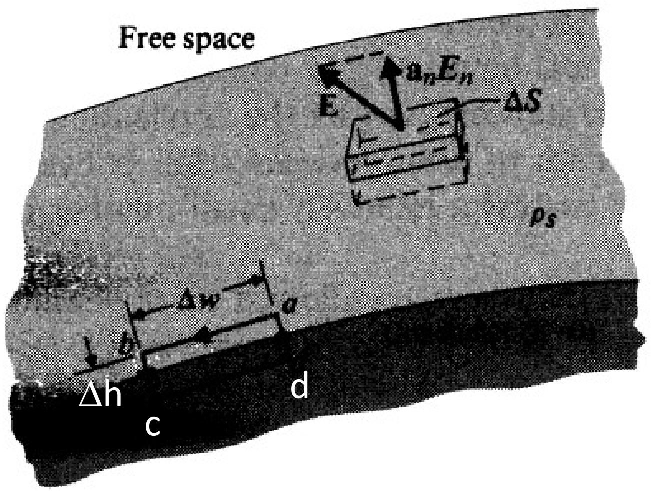
Equilibrium

- At a state of equilibrium (static charges), tangential **E** = 0. Otherwise, charges move.



Boundary Conditions at a Conductor/Free Space Interface
$E_t = 0$ $E_n = \dfrac{\rho_s}{\epsilon_0}$

- Under static conditions,
  - **E** field on a conductor surface is everywhere normal to the surface.
  - The surface of a conductor is an equipotential surface.



Conductor surface

Inside conductor

FIGURE 3–18
A conductor–free space interface.

$$\oint_{abcda} \mathbf{E} \cdot d\ell = E_t \Delta w = 0$$

(1) Let Δh → 0 ➔ Integrals along bc, da = 0
(2) E_{inside} = 0 ➔ Integral along cd = 0

$$E_t = 0,$$

The tangential component of the E field on a conductor surface is zero.

Normal Component of E

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = E_n \Delta S = \frac{\rho_s \Delta S}{\epsilon_0}$$

6 faces:
(1) Let $\Delta h \rightarrow 0 \Rightarrow$ Integrals over 4 side surfaces = 0
(2) $E_{inside}=0 \Rightarrow$ Integral over the bottom surface = 0

$$E_n = \frac{\rho_s}{\epsilon_0}.$$

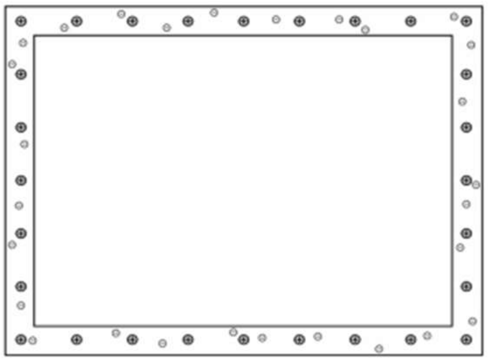
The normal component of the E field at a conductor/free space boundary is equal to the surface charge density on the conductor divided by the permittivity of free space.

An Uncharged Conductor in a Static E Field

- $E_{external}$ ➔ Electrons moving ➔ $E_{induced}$
- $E_{induced}$ cancels $E_{external}$ both inside the conductor and tangent to its surface

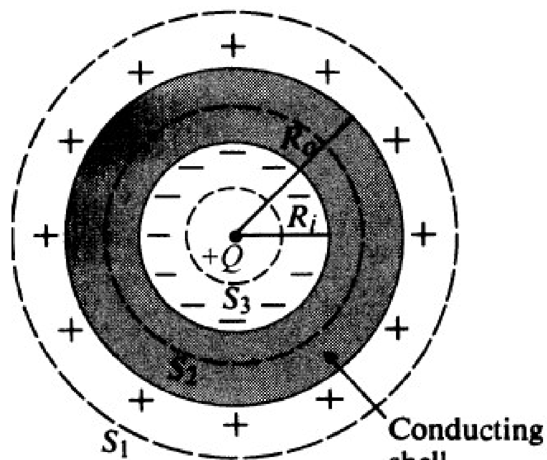
$E_{external}$

Faraday Cage



$E_{induced}$

EXAMPLE 3–11 A positive point charge Q is at the center of a spherical conducting shell of an inner radius R_i and an outer radius R_o. Determine $\mathbf{E}$ and V as functions of the radial distance R.



(a)

3-7 Dielectrics in Static Electric Field

- Dielectrics: Bound charges
- A dielectric material placed under $\mathbf{E}_{external}$
  ➔ polarize a dielectric material and create electric dipoles, which is $\mathbf{E}_{induced}$
  ➔ modify **E** both inside and outside the dielectric material

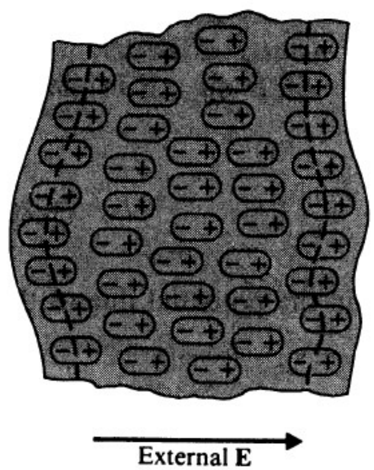


FIGURE 3–20
A cross section of a polarized dielectric medium.

Permanent Dipole Moments

- Some materials have non-zero dipole moments in the absence of $\mathbf{E}_{external}$ field
  - E.g., H_2O (polar molecule)
  - Macroscopic viewpoint of H_2O
    - ❖Without $\mathbf{E}_{external}$: No net dipole moment
    - ❖With $\mathbf{E}_{external}$: Molecules aligned due to $\mathbf{E}_{external}$ ➔ nonzero net dipole moment

3-7.1 Equivalent Charge Distributions of Polarized Dielectrics

- Macroscopic effect
- Polarization vector:

$N = n\Delta v$,
where N is the total # in a volume (Δv);
n is the number density

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \mathbf{p}_k}{\Delta v} \quad (\mathrm{C/m^2}),$$

P: Volume density of electric dipole moment **p**

$$d\mathbf{p} = \mathbf{P}\, dv'$$

- Derivation for dielectrics:

$$\boxed{V = \frac{\mathbf{p} \cdot \mathbf{a}_R}{4\pi\epsilon_0 R^2} \quad (\mathrm{V}),} \Rightarrow dV = \frac{\mathbf{P} \cdot \mathbf{a}_R}{4\pi\epsilon_0 R^2}\, dv'.$$

$$V = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\mathbf{P} \cdot \mathbf{a}_R}{R^2}\, dv',$$

Primed is the coordinate of source

$$R^2 = (x - x')^2 + (y - y')^2 + (z - z')^2,$$

$$\underline{\nabla'}\left(\frac{1}{R}\right) = \frac{\mathbf{a}_R}{R^2}.$$

Gradient w.r.t the primed coordinate. Thus, no “–” sign at the right side (see slide#20).

$$V = \frac{1}{4\pi\epsilon_0} \int_{V'} \mathbf{P} \cdot \nabla'\left(\frac{1}{R}\right) dv'.$$

$$\nabla' \cdot (f\mathbf{A}) = f\nabla' \cdot \mathbf{A} + \mathbf{A} \cdot \nabla' f, \qquad \text{letting } \mathbf{A} = \mathbf{P} \text{ and } f = 1/R,$$

$$V = \frac{1}{4\pi\epsilon_0} \left[\int_{V'} \nabla' \cdot \left(\frac{\mathbf{P}}{R}\right) dv' - \int_{V'} \frac{\nabla' \cdot \mathbf{P}}{R}\, dv' \right].$$

$$V = \frac{1}{4\pi\epsilon_0} \oint_{S'} \frac{\mathbf{P} \cdot \mathbf{a}'_n}{R} \, ds' + \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{(-\nabla' \cdot \mathbf{P})}{R} \, dv',$$

By divergence theorem

Comparison with

$$\boxed{V = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho}{R} \, dv' \qquad (\mathrm{V}).}$$

$$\boxed{V = \frac{1}{4\pi\epsilon_0} \int_{S'} \frac{\rho_s}{R} \, ds' \qquad (\mathrm{V});}$$

V = contribution of surface charge distribution

\+

contribution of volume charge distribution

$$\boxed{\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n}$$

$$\boxed{\rho_p = -\nabla \cdot \mathbf{P}.}$$

- ′ has been dropped for simplicity
- Polarization charge densities or bound-charge densities

$$V = \frac{1}{4\pi\epsilon_0} \oint_{S'} \frac{\rho_{ps}}{R} \, ds' + \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho_p}{R} \, dv'.$$

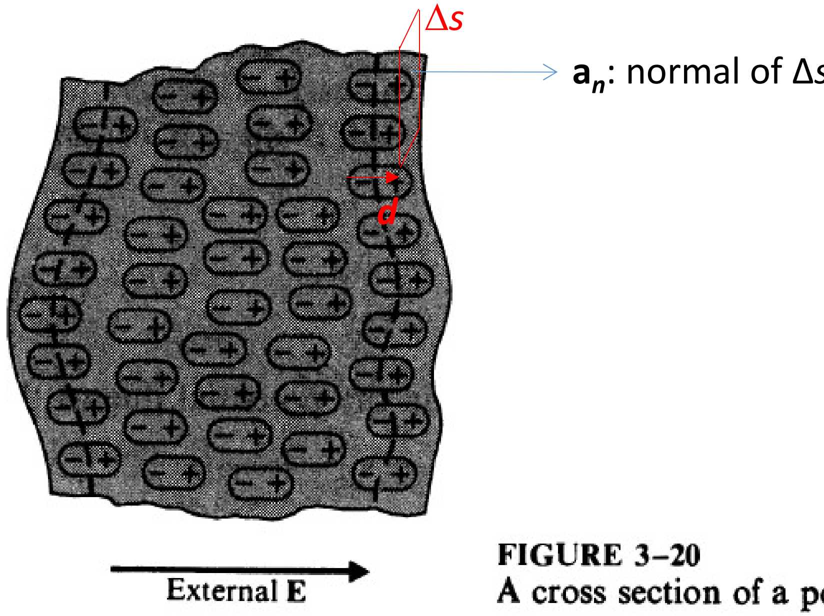


FIGURE 3–20
A cross section of a polarized dielectric medium.

External **E**

➔ Causes a separation *d* of bound charges; *q* displaced by *d* along the direction of the external **E**

➔ Total charge crossing the surface Δs: $nq\,d(\Delta s)$, for $\boldsymbol{d} \,//\, \mathbf{a}_n$

$$\Delta Q = nq(\mathbf{d} \cdot \mathbf{a}_n)(\Delta s). \quad \text{for } \boldsymbol{d} \not\parallel \mathbf{a}_n$$

$\boxed{\begin{array}{l}\mathbf{p}=q\mathbf{d}\\ \mathbf{P}=n\mathbf{p}=nq\mathbf{d}\end{array}}$

$$\Delta Q = \mathbf{P}\cdot\mathbf{a}_n(\Delta s) \qquad \rho_{ps} = \frac{\Delta Q}{\Delta s} = \mathbf{P}\cdot\mathbf{a}_n,$$

The net charge remaining within the volume V is the negative of the integral

$$Q = -\oint_S \mathbf{P}\cdot\mathbf{a}_n\,ds$$
$$= \int_V (-\nabla\cdot\mathbf{P})\,dv = \int_V \rho_p\,dv,$$

$$\boxed{\rho_p = -\nabla\cdot\mathbf{P}.}$$

Since starting with an electrically neutral dielectric body, the total charge of the body after polarization must remain zero.

$$\text{Total charge} = \oint_S \rho_{ps}\,ds + \int_V \rho_p\,dv$$
$$= \oint_S \mathbf{P}\cdot\mathbf{a}_n\,ds - \int_V \nabla\cdot\mathbf{P}\,dv = 0,$$

3-8 Electric Flux Density and Dielectric Constant

- In 3-7, polarization **P** or bound volume charge density ρ_p
  ➔ produces **E** field due to ρ_p

- Modification of divergence postulates:

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}} \Rightarrow \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}(\rho + \rho_p).$$

$$\boxed{\rho_p = -\nabla \cdot \mathbf{P}.}$$

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho.$$

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \qquad (\mathrm{C/m^2}).}$$

Where **D**: electric flux density, electric displacement

$$\boxed{\nabla \cdot \mathbf{D} = \rho \qquad (\mathrm{C/m^3}),}$$

$$\int_V \nabla \cdot \mathbf{D}\, dv = \int_V \rho\, dv$$

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \qquad (\mathrm{C}).}$$

Integral form

Another from of Gauss's law: The total outward flux of the electric displacement (or, simply, the total outward electric flux) over any closed surface is equal to the total ***free*** charge enclosed in the surface.

χ_e and ε_r

- Electric susceptibility
  - For linear and isotropic medium,

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E},$$

χ_e dimensionless quantity called ***electric susceptibility***

- Relative permittivity (dielectric constant)

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \qquad (\mathrm{C/m^2}).}$$

$$\Rightarrow$$

$$\boxed{\begin{aligned} \mathbf{D} &= \epsilon_0 (1 + \chi_e) \mathbf{E} \\ &= \epsilon_0 \epsilon_r \mathbf{E} = \epsilon \mathbf{E} \qquad (\mathrm{C/m^2}), \end{aligned}}$$

$$\epsilon_r = 1 + \chi_e = \frac{\epsilon}{\epsilon_0}$$

ε: absolute permittivity (or simply permittivity)

	$\varepsilon_r = \varepsilon/\varepsilon_0$
Carbon Tetrachloride[a]	2.2
Ethanol[a]	24
Methanol[a]	33
n-Hexane[a]	1.9
Nitrobenzene[a]	35
Pure Water[a]	80
Barium Titanate[b] (with 20% Strontium Titanate)	>2100
Borosilicate Glass[b]	4.0
Ruby Mica (Muscovite)[b]	5.4
Polyethylene[b]	2.2
Polyvinyl Chloride[b]	6.1
Teflon[b] (Polytetrafluorethylene)	2.1
Plexiglas[b]	3.4
Paraffin Wax[b]	2.2

[a] *From* Lange's Handbook of Chemistry, *10th ed., McGraw-Hill, New York, 1961, pp. 1234–37.*

[b] *From A. R. von Hippel (Ed.) Dielectric Materials and Applications, M.I.T., Cambridge, Mass., 1966, pp. 301–370*

A Simple Medium

- Linear: χ_e is dependent of **E** only (not $|\mathbf{E}|^2$, $|\mathbf{E}|^3$…)
- Homogeneous: χ_e is independent of space
- Isotropic: χ_e is a scalar, not a tensor ➔ **P//E**

- A simple medium: linear, homogeneous, and isotropic

- ε_r in a simple medium is a constant

Anisotropic Medium

- The ε_r is different for different directions of the electric field
  - **D** and **E** vectors generally have different directions (i.e., not parallel)
  - $\overline{\overline{\varepsilon}}$ is a tensor

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}.$$

- For crystals, choosing a proper coordinate system, $\overline{\overline{\varepsilon}}$ can be simplified

Anisotropic: Biaxial and Uniaxial

- Biaxial: $\varepsilon_1 \neq \varepsilon_2 \neq \varepsilon_3$

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}.$$

$$D_x = \epsilon_1 E_x,$$
$$D_y = \epsilon_2 E_y,$$
$$D_z = \epsilon_3 E_z.$$

- Uniaxial: $\varepsilon_1 = \varepsilon_2 \neq \varepsilon_3$

- In this book, only deal with isotropic media

3-8.1 Dielectric Strength

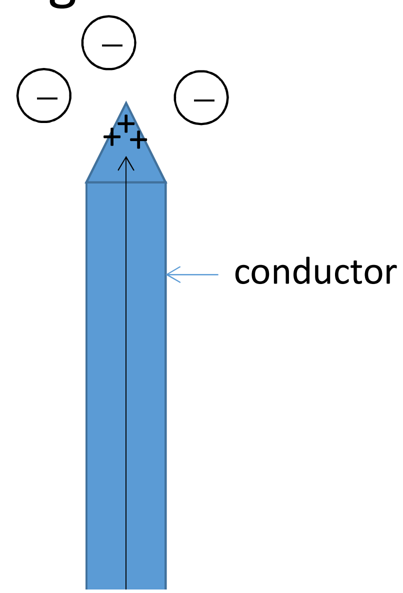
- External **E** ➔ Displacement of bound charges ➔ Polarization **P**
- Dielectric breakdown: If very strong external **E** causes permanent dislocation of electrons and damage in the material, avalanche effect of ionization due to collisions may occur. The material becomes conducting and may result in large currents.
- Dielectric strength: The maximum **E** intensity that a dielectric material can withstand without breakdown

TABLE 3–1
Dielectric Constants and Dielectric Strengths of Some Common Materials

Material	Dielectric Constant	Dielectric Strength (V/m)
Air (atmospheric pressure)	1.0	3×10^6
Mineral oil	2.3	15×10^6
Paper	2–4	15×10^6
Polystyrene	2.6	20×10^6
Rubber	2.3–4.0	25×10^6
Glass	4–10	30×10^6
Mica	6.0	200×10^6

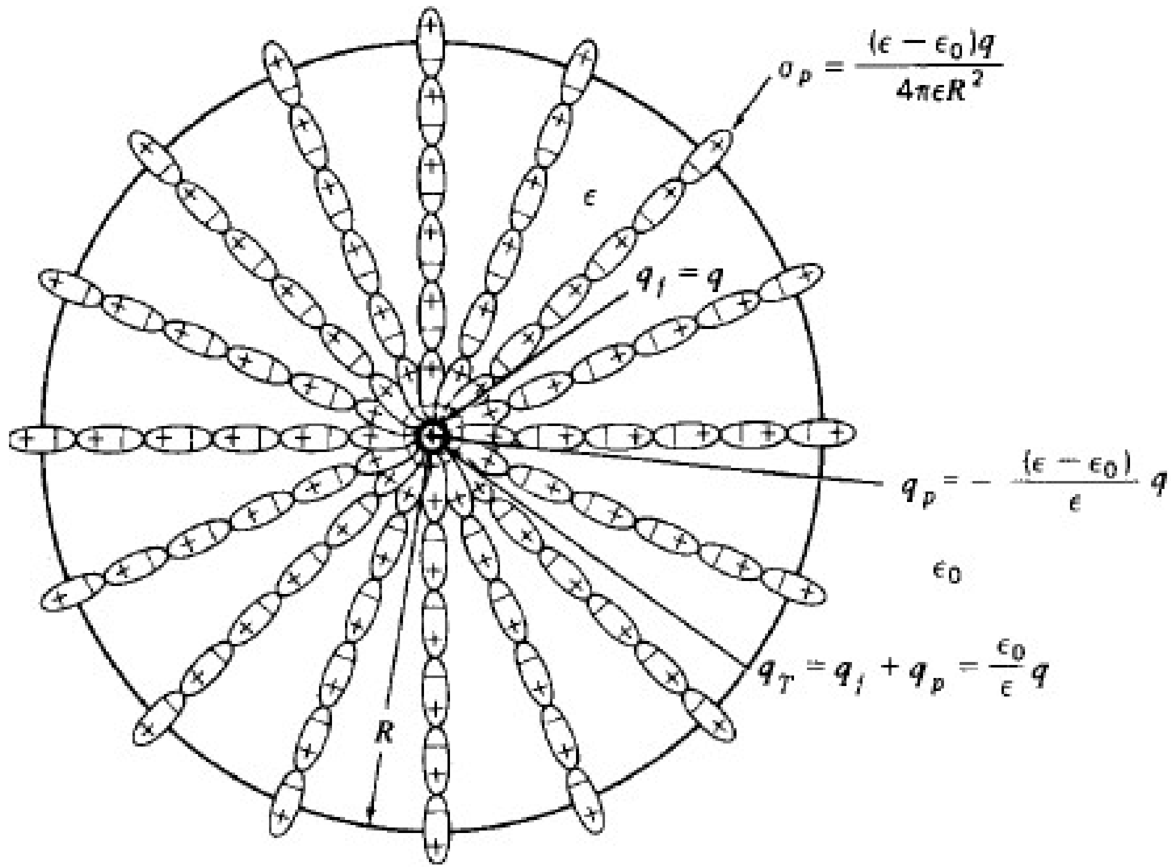
A Lighting Arrester with a Sharp Rod

- Example 3-13: The electric field intensities are inversely proportional to the radii. That is, **E** is higher at the surface with a larger curvature.
- A cloud containing an abundance of electric charges
  ➔ charges of opposite sign are attracted from the ground to the tip
  ➔ **E** is very strong at the tip (sharp points)
  ➔ When **E** at tip > $\mathbf{E}_{\text{breakdown,wet air}}$
  ➔ Air ionized, becomes conducting
  ➔ ⊖ in the cloud are discharged safely to the ground



Example: Point Charge Embedded in a Dielectric Sphere

Let's use Maxwell's equations to solve the fields inside materials.



Point Charge Embedded in a Dielectric Sphere

$$\oiint \vec{D}\cdot\hat{n}\,dS = \iiint_V \rho\, dV_s$$

$$D_r 4\pi r^2 = Q$$

$$\therefore \vec{D} = \frac{Q}{4\pi r^2}\hat{r}$$

$$\vec{E} = \frac{Q}{4\pi}\begin{cases} \dfrac{1}{\varepsilon r^2}\hat{r} & r<R \\ \dfrac{1}{\varepsilon_0 r^2}\hat{r} & r>R \end{cases}$$

$$\vec{P} = \vec{D} - \varepsilon_0\vec{E} =$$

$$\vec{P} = \frac{Q}{4\pi r^2}\begin{cases} \left(\dfrac{\varepsilon-\varepsilon_0}{\varepsilon}\right)\hat{r} & r<R \\ 0 & r>R \end{cases}$$

$$\vec{E} = -\nabla V$$

$$V(\vec{r}) = \frac{Q}{4\pi}\begin{cases} \dfrac{1}{\varepsilon r} + \dfrac{1}{\varepsilon R}\left(\dfrac{\varepsilon-\varepsilon_0}{\varepsilon_0}\right) & r<R \\ \dfrac{1}{\varepsilon_0 r} & r>R \end{cases}$$

Polarization Induced Charge on Sphere Surface

At the outer surface of the sphere $\boldsymbol{P}(\boldsymbol{R}) \cdot \hat{\boldsymbol{n}} = P_r(\boldsymbol{R}) = \rho_{ps} = \dfrac{Q(\varepsilon - \varepsilon_0)}{4\pi\varepsilon \ R^2}$

In order to maintain charge neutrality, we must have an equal and opposite polarization induced point charge at the center of the sphere. The total polarization charge must sum to zero.

At the center: $q_p = -Q\left(\dfrac{\varepsilon - \varepsilon_0}{\varepsilon}\right)$

$\boxed{q_p + q_{ps} = 0}$

Thus the sphere can be modeled as the combination of the true and polarization charge at the center along with the polarization charge on the outer surface.

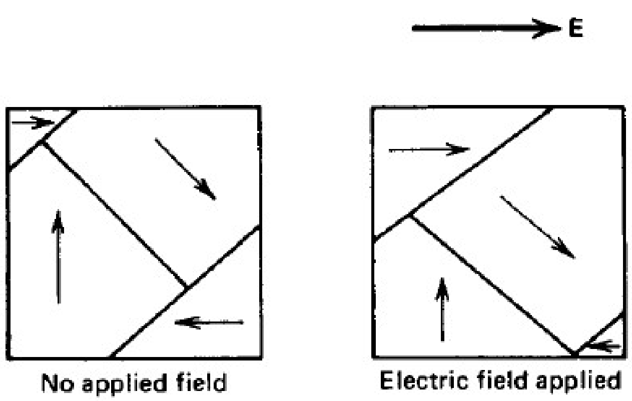
At the center: $q_T = q + q_p = Q\dfrac{\varepsilon_0}{\varepsilon} = \dfrac{Q}{\varepsilon_r}$

$\boxed{<Q}$

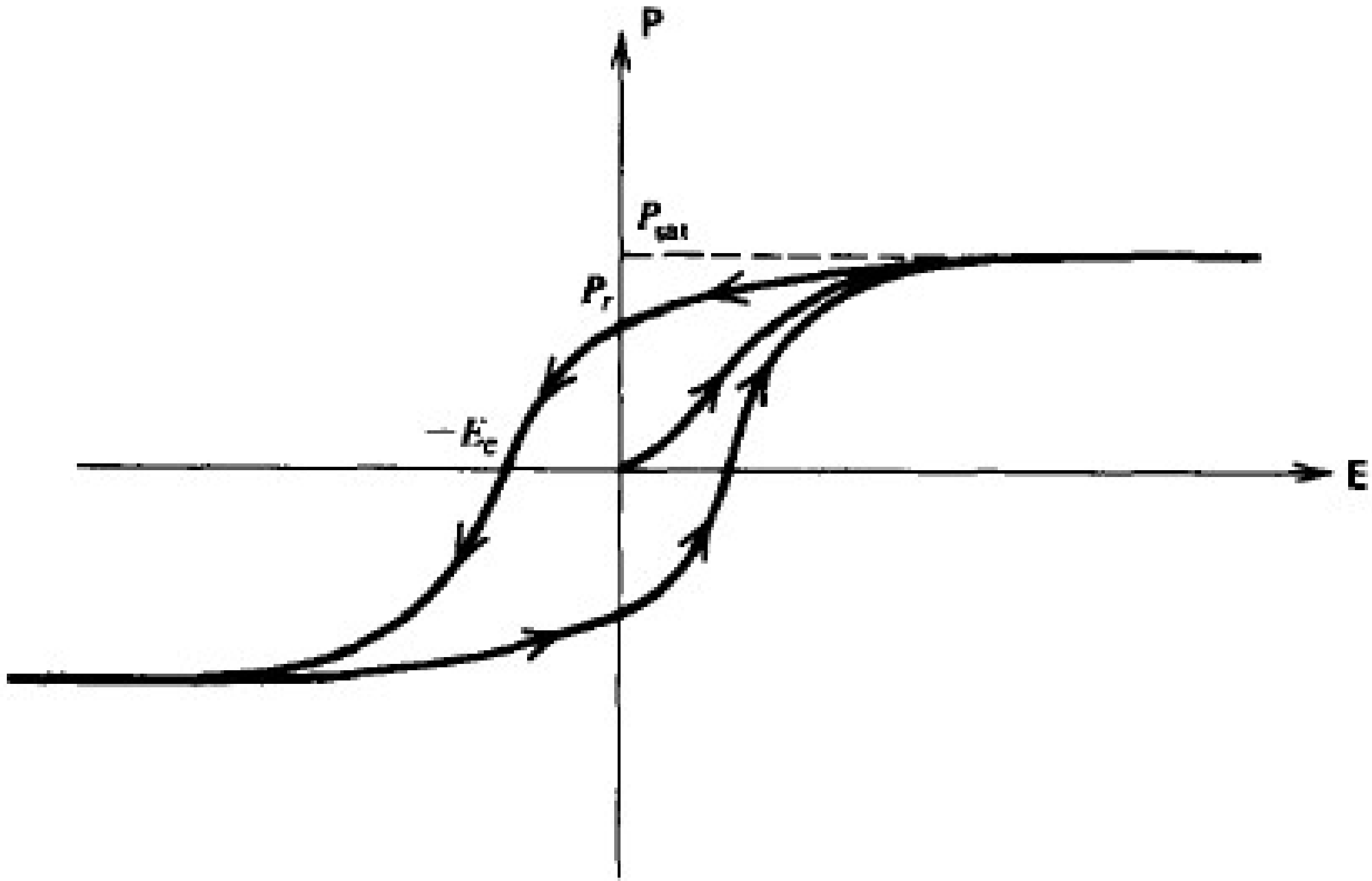
The effect of the dielectric is therefore ***the displacement of some of the original charge to its outer surface***.

Non-Linear Materials: Hysteresis

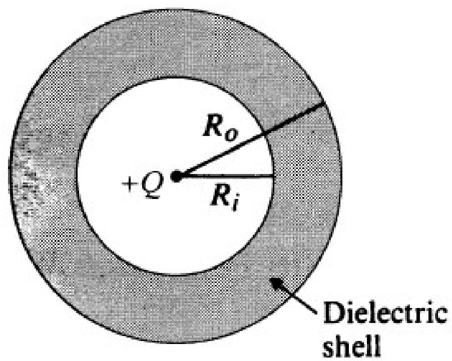
- Ferroelectric materials: such materials exists ***spontaneous polarization*** even when there is no external field. Ferromagnetic materials (permanent magnets) also display similar behavior.
- Domains: There is a whole field for describing how the material changes its polarization when an external field is applied. It is also well know that there are "domains" inside material that have constant polarization and well defined boundaries. The application of an electric field amounts to the ***shrinking or growing*** of these domains.



Ferroelectric Hysteresis Curve



EXAMPLE 3–12 A positive point charge Q is at the center of a spherical dielectric shell of an inner radius R_i and an outer radius R_o. The dielectric constant of the shell is ϵ_r. Determine $\mathbf{E}$, V, $\mathbf{D}$, and $\mathbf{P}$ as functions of the radial distance R.



(a)

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \qquad \text{(C).}}$$

Gauss's law: E, D → V and P

EXAMPLE 3–13 Consider two spherical conductors with radii b_1 and b_2 ($b_2 > b_1$) that are connected by a conducting wire. The distance of separation between the conductors is assumed to be very large in comparison to b_2 so that the charges on the spherical conductors may be considered as uniformly distributed. A total charge Q is deposited on the spheres. Find (a) the charges on the two spheres, and (b) the electric field intensities at the sphere surfaces.

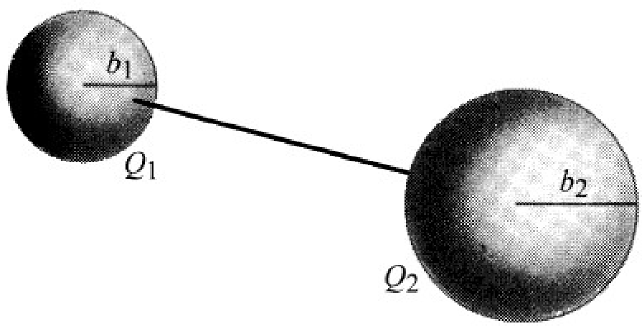


FIGURE 3–22
Two connected conducting spheres (Example 3–13).

3-9 Boundary Conditions for Electrostatic Fields

- Knowledge of the relations of the ***field quantities at an interface*** between two media is of importance for electromagnetic problems.

B.C.: Tangential Component

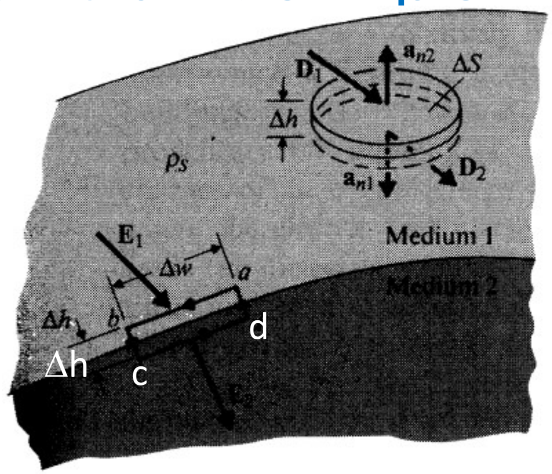


FIGURE 3–23
An interface between two media.

let sides $bc = da = \Delta h$ approach zero

$$\oint_{abcda} \mathbf{E} \cdot d\ell = \mathbf{E}_1 \cdot \Delta\mathbf{w} + \mathbf{E}_2 \cdot (-\Delta\mathbf{w}) = E_{1t}\Delta w - E_{2t}\Delta w = 0.$$

$$\boxed{E_{1t} = E_{2t} \qquad \text{(V/m)},}$$

The tangential component of an **E** field is continuous across an interface.

B.C.: Tangential Component

- A conductor/free space interface:

$$E_{2t,\text{conductor}} = 0 \Rightarrow E_{1t,\text{free space}} = 0$$

- Two dielectrics:

$$\boxed{E_{1t} = E_{2t} \qquad \text{(V/m)},} \Rightarrow \frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2}.$$

B.C.: Normal Component

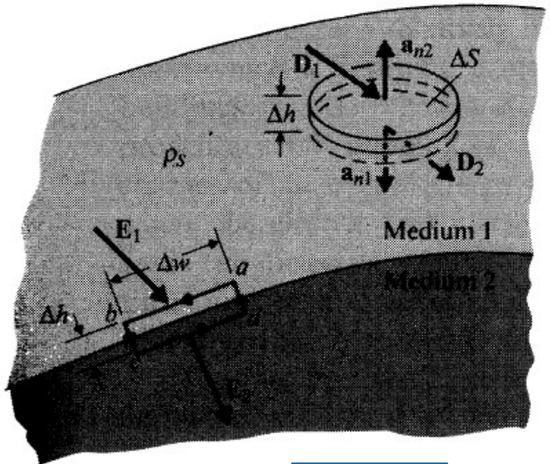


FIGURE 3–23
An interface between two media.

Δh → 0

Gauss's law: $\oint_S \mathbf{D}\cdot d\mathbf{s} = (\mathbf{D}_1 \cdot \mathbf{a}_{n2} + \mathbf{D}_2 \cdot \mathbf{a}_{n1})\Delta S$

$$= \mathbf{a}_{n2}\cdot(\mathbf{D}_1 - \mathbf{D}_2)\Delta S$$

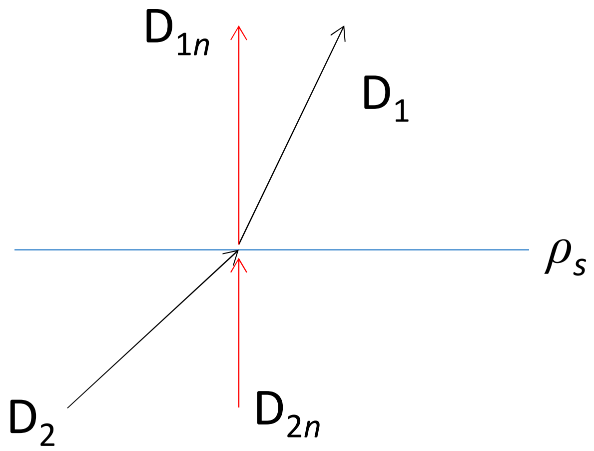
$$= \rho_s \Delta S,$$

$\boxed{\mathbf{a}_{n2} = -\mathbf{a}_{n1}}$

$\boxed{\mathbf{a}_{n2}\cdot(\mathbf{D}_1 - \mathbf{D}_2) = \rho_s}$ $\boxed{D_{1n} - D_{2n} = \rho_s \qquad (\mathrm{C/m^2}),}$ Ref.: $\mathbf{a}_{n2}$

The normal of **D** field is ***discontinuous*** across an interface where ***a surface charge*** exists—the amount of discontinuity being equal to the surface charge density.

Which one is correct?



$D_{1n} - D_{2n} = \rho_s$

$D_{2n} - D_{1n} = \rho_s$

B.C.: Normal Component

- For a dielectric (Medium 1)/conductor (Medium 2) interface:

$$\mathbf{D}_2 = 0 \quad \Rightarrow \quad D_{1n} = \epsilon_1 E_{1n} = \rho_s,$$

- For no charge existing at the interface

$$\rho_s = 0, \quad \Rightarrow \quad \begin{aligned} D_{1n} &= D_{2n} \\ \epsilon_1 E_{1n} &= \epsilon_2 E_{2n}. \end{aligned}$$

Continuity of D_n and E_n

$$(\mathbf{D_1} - \mathbf{D_2}) \cdot \hat{\boldsymbol{n}} = (\mathbf{P_1} + \varepsilon_0 \mathbf{E_1} - \mathbf{P_2} - \varepsilon_0 \mathbf{E_2}) \cdot \hat{\boldsymbol{n}} = \rho_s$$

$$(\varepsilon_0 \mathbf{E_1} - \varepsilon_0 \mathbf{E_2}) \cdot \hat{\boldsymbol{n}} = \rho_s - \underline{(\mathbf{P_1} - \mathbf{P_2}) \cdot \hat{\boldsymbol{n}}} = \rho_s - \underline{\rho_{ps}}$$

$$(\mathbf{E_1} - \mathbf{E_2}) \cdot \hat{\boldsymbol{n}} = \frac{\rho_s - \rho_{ps}}{\varepsilon_0}$$

- The normal component of the **D** field is discontinuous by the amount of TRUE charge on the surface.
- The normal component of the **E** field is discontinuous by the amount of TOTAL charge (true plus polarization charge).
- Therefore, the D_n field may be continuous (when $\rho_s = 0$).

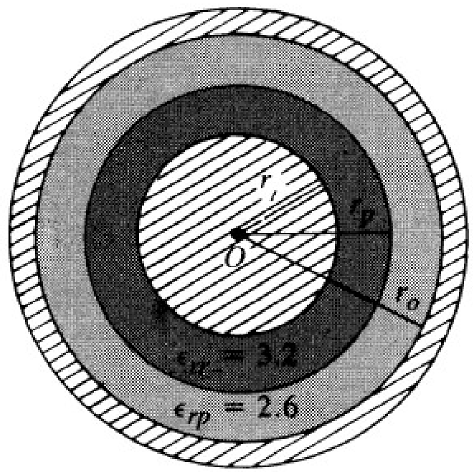
Continuity of D$_t$ and E$_t$

$$(\mathbf{E_1} - \mathbf{E_2}) \times \hat{\boldsymbol{n}} = \left(\frac{\mathbf{D_1} - \mathbf{P_1}}{\varepsilon_0} - \frac{\mathbf{D_2} - \mathbf{P_2}}{\varepsilon_0}\right) \times \hat{\boldsymbol{n}} = \mathbf{0}$$

$$(\mathbf{D_1} - \mathbf{D_2}) \times \hat{\boldsymbol{n}} = (\mathbf{P_1} - \mathbf{P_2}) \times \hat{\boldsymbol{n}} = \varepsilon_0 \underline{(\Delta\chi_e)}\mathbf{E} \times \hat{\mathbf{n}}$$

- The tangential component of the **E** field is continuous across EVERY interface.
- The tangential component of the **D** field is discontinuous by the amount of ***susceptibility difference*** of the interface.

EXAMPLE 3–16 When a coaxial cable is used to carry electric power, the radius of the inner conductor is determined by the load current, and the overall size by the voltage and the type of insulating material used. Assume that the radius of the inner conductor is 0.4 (cm) and that concentric layers of rubber ($\epsilon_{rr} = 3.2$) and polystyrene ($\epsilon_{rp} = 2.6$) are used as insulating materials. Design a cable that is to work at a voltage rating of 20 (kV). In order to avoid breakdown due to voltage surges caused by lightning and other abnormal external conditions, the maximum electric field intensities in the insulating materials are not to exceed 25% of their dielectric strengths.



(a)

$E \leq 0.25\times E_{max,rubber}$
$E \leq 0.25\times E_{max,ps}$
The E field from an infinitely long conductor = $\rho_l/(2\pi\varepsilon)$
V by taking integration of E field

3-10 Capacitance and Capacitors

- Deposit charges Q on a conductor ➔ V
- kQ ➔ $k\rho_s$ ➔ kV

$$\boxed{V = \frac{1}{4\pi\epsilon_0}\int_{S'} \frac{\rho_s}{R}\, ds' \qquad \text{(V);}}$$

- The ratio Q/V unchanged

$$\boxed{Q = CV,}$$

C: capacitance (C/V, or Farad)

Capacitor

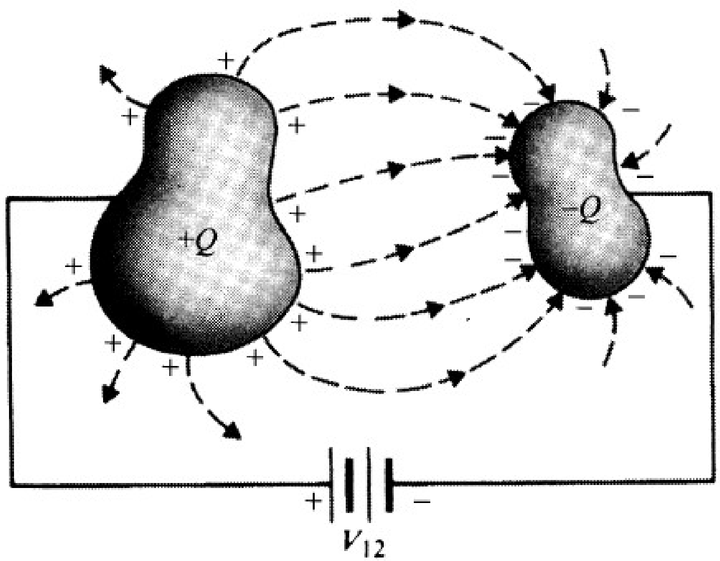


FIGURE 3–27
A two-conductor capacitor.

E $\perp$ conductor surfaces (equipotential surfaces)

$$\boxed{C = \frac{Q}{V_{12}} \qquad \text{(F).}}$$

Capacitance

- C depends on
  - The geometry of the conductors
  - The permittivity of the medium between conductors
  - ***Independent of Q and V***
- Measurement of C
  - Method 1: V_{12} known, determine Q (Chap.4)
  - Method 2: Q known, determine V_{12}
    - ❖ 1. Choose a proper coordinate system
    - ❖ 2. Assume $+Q$, $-Q$ on the conductors
    - ❖ 3. Find **E** from Q (Gauss's law, etc.); Find V_{12} by $V_{12} = -\int_2^1 \mathbf{E} \cdot d\boldsymbol{\ell}$
    - ❖ 4. $C = Q/V_{12}$

    See examples 3-17, 3-18, 3-19

3-10.1 Series and Parallel Connections of Capacitors

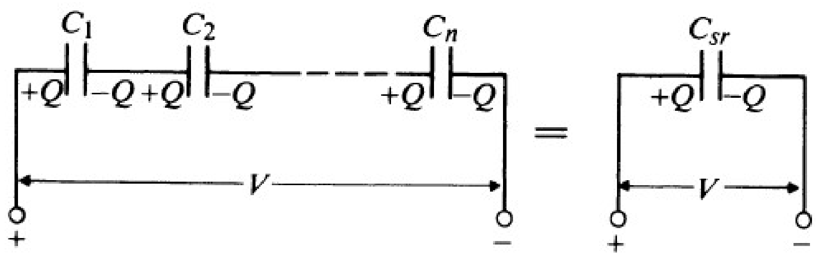


FIGURE 3–31
Series connection of capacitors.

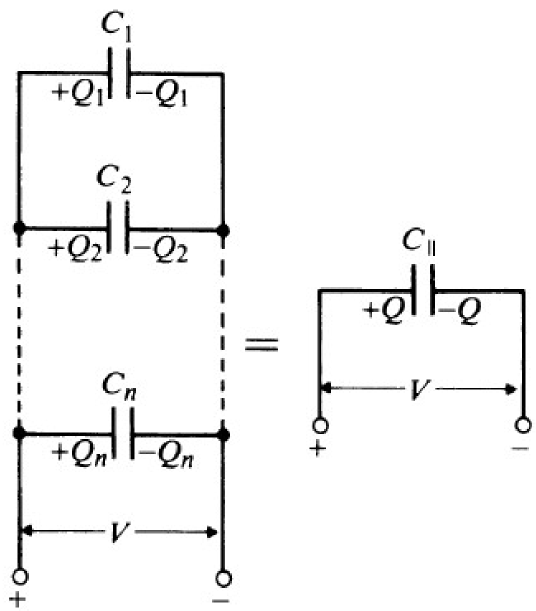


FIGURE 3–32
Parallel connection of capacitors.

Series

- V

  ➔ $+Q$ and $-Q$ on two external terminals

  ➔ $+Q$ and $-Q$ also induced internally

➔ $$V = \frac{Q}{C_{sr}} = \frac{Q}{C_1} + \frac{Q}{C_2} + \cdots + \frac{Q}{C_n},$$

$$\boxed{\frac{1}{C_{sr}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}.}$$

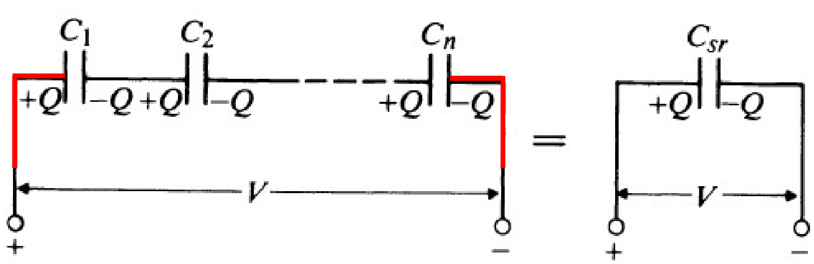
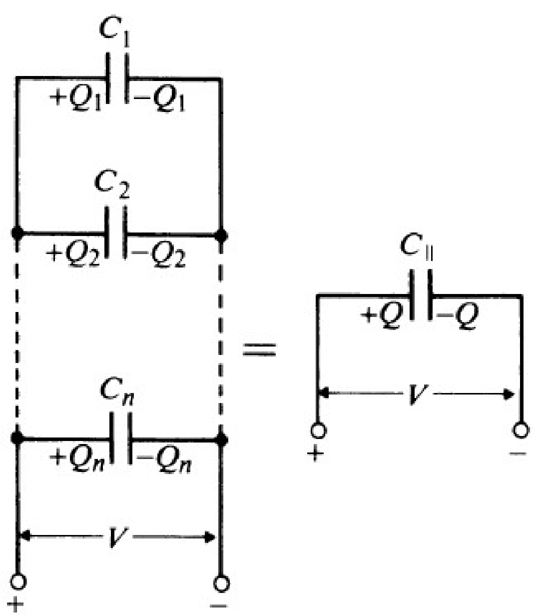


FIGURE 3–31
Series connection of capacitors.

Parallel

- V
  ➔ Q_1, Q_2, Q_3, … on each capacitor



$$Q = Q_1 + Q_2 + \cdots + Q_n$$
$$= C_1V + C_2V + \cdots + C_nV = C_{||}V$$

$$\boxed{C_{||} = C_1 + C_2 + \cdots + C_n.}$$

FIGURE 3–32
Parallel connection of capacitors.

3-10.2 Capacitances in Multiconductor Systems

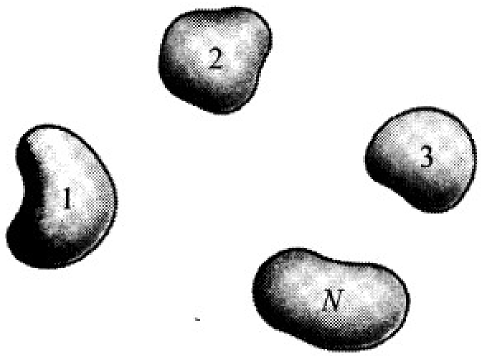


FIGURE 3–34
A multiconductor system.

Presence of a charge on any one of the conductors affects potential of all the other conductors

$$
\begin{aligned}
V_1 &= p_{11}Q_1 + p_{12}Q_2 + \cdots + p_{1N}Q_N, \\
V_2 &= p_{21}Q_1 + p_{22}Q_2 + \cdots + p_{2N}Q_N, \\
&\vdots \\
V_N &= p_{N1}Q_1 + p_{N2}Q_2 + \cdots + p_{NN}Q_N.
\end{aligned}
$$

p $_{ij}$: **coefficients of potential**; depends on
1. Shape and position of the conductor
2. Permittivity of surroundings

For an isolated system $Q_1 + Q_2 + Q_3 + \cdots + Q_N = 0.$

$$Q_1 = c_{11}V_1 + c_{12}V_2 + \cdots + c_{1N}V_N,$$
$$Q_2 = c_{21}V_1 + c_{22}V_2 + \cdots + c_{2N}V_N,$$
$$\vdots$$
$$Q_N = c_{N1}V_1 + c_{N2}V_2 + \cdots + c_{NN}V_N,$$

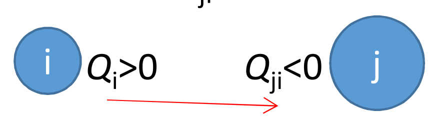
c_{ii}: coefficients of capacitance
c_{ij}: coefficients of induction ($i \neq j$)

c_{ii}: ground all other conductors, then $c_{ii} = Q_i/V_i$

c_{ji}: Induced charge $Q_{ji} = c_{ji}V_i$

If Q_i on ith conductor and $Q_i>0$, then $V_i>0$ and induced $Q_{ji}<0$

Thus, $c_{ii}>0$; $c_{ji}<0$



By reciprocity, $p_{ij} = p_{ji}$ and $c_{ij} = c_{ji}$

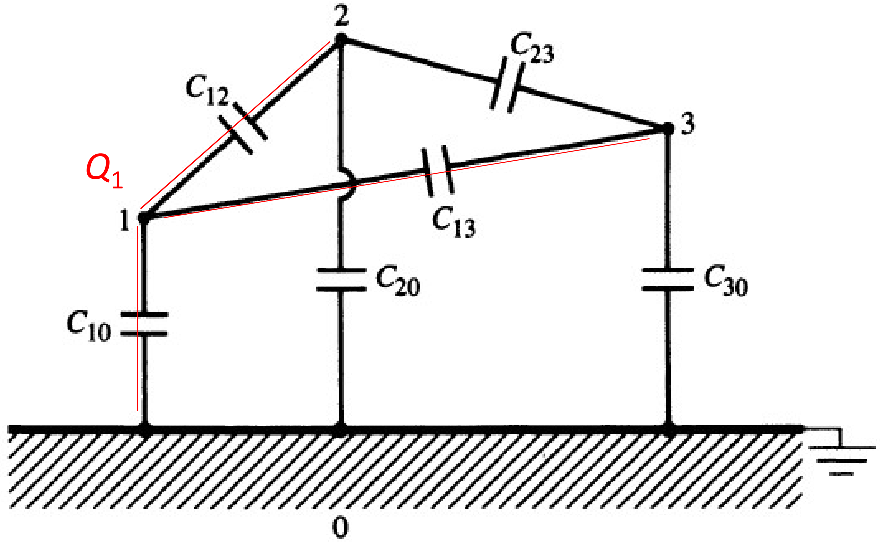
A Four-conductor System

$$
\begin{aligned}
Q_1 &= c_{11}V_1 + c_{12}V_2 + \cdots + c_{1N}V_N,\\
Q_2 &= c_{21}V_1 + c_{22}V_2 + \cdots + c_{2N}V_N,\\
&\vdots\\
Q_N &= c_{N1}V_1 + c_{N2}V_2 + \cdots + c_{NN}V_N,
\end{aligned}
$$

Conductors 0,1,2,3. Let conductor 0 be grounded (i.e., $V_0 = 0$).

$$
\begin{aligned}
Q_1 &= c_{11}V_1 + c_{12}V_2 + c_{13}V_3,\\
Q_2 &= c_{12}V_1 + c_{22}V_2 + c_{23}V_3,\\
Q_3 &= c_{13}V_1 + c_{23}V_2 + c_{33}V_3,
\end{aligned}
$$

A Four-conductor System



c: Coefficient of capacitance
C: Capacitance

FIGURE 3–35
Schematic diagram of three conductors and the ground.

Rewrite the Q ~ V relation (by definition of capacitance)

$$Q_1 = C_{10}V_1 + C_{12}(V_1 - V_2) + C_{13}(V_1 - V_3),$$
$$Q_2 = C_{20}V_2 + C_{12}(V_2 - V_1) + C_{23}(V_2 - V_3),$$
$$Q_3 = C_{30}V_3 + C_{13}(V_3 - V_1) + C_{23}(V_3 - V_2),$$

C_{10}, C_{20}, C_{30}: self-partial capacitance
C_{ij} $(i \neq j)$: mutual partial capacitance

Compare:

$$Q_1 = c_{11}V_1 + c_{12}V_2 + c_{13}V_3, \qquad Q_1 = C_{10}V_1 + C_{12}(V_1 - V_2) + C_{13}(V_1 - V_3),$$
$$Q_2 = c_{12}V_1 + c_{22}V_2 + c_{23}V_3, \qquad Q_2 = C_{20}V_2 + C_{12}(V_2 - V_1) + C_{23}(V_2 - V_3),$$
$$Q_3 = c_{13}V_1 + c_{23}V_2 + c_{33}V_3, \qquad Q_3 = C_{30}V_3 + C_{13}(V_3 - V_1) + C_{23}(V_3 - V_2),$$

$$\Downarrow$$

Coefficient of capacitance:

$$c_{11} = C_{10} + C_{12} + C_{13},$$
$$c_{22} = C_{20} + C_{12} + C_{23},$$
$$c_{33} = C_{30} + C_{13} + C_{23},$$

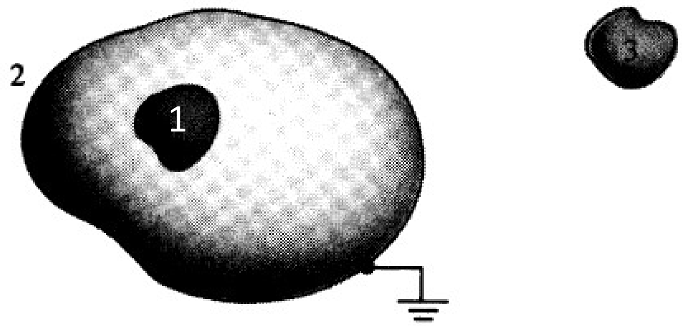
Coefficient of inductance:

$$c_{12} = -C_{12},$$
$$c_{23} = -C_{23},$$
$$c_{13} = -C_{13}.$$

$$\Rightarrow$$

$$C_{10} = c_{11} + c_{12} + c_{13},$$
$$C_{20} = c_{22} + c_{12} + c_{23},$$
$$C_{30} = c_{33} + c_{13} + c_{23}.$$

3-10.3 Electrostatic Shielding



$$Q_1 = C_{10}V_1 + C_{12}(V_1 - V_2) + C_{13}(V_1 - V_3),$$
$$Q_2 = C_{20}V_2 + C_{12}(V_2 - V_1) + C_{23}(V_2 - V_3),$$
$$Q_3 = C_{30}V_3 + C_{13}(V_3 - V_1) + C_{23}(V_3 - V_2),$$

FIGURE 3–37
Illustrating electrostatic shielding.

Following the previous system, we ground the conductor #2. (i.e., V_2=0)

➔ $Q_1 = C_{10}V_1 + C_{12}V_1 + C_{13}(V_1 - V_3).$

When $Q_1 = 0$ ➔ **E** inside #2 = 0 ➔ $V_1 = V_2 = 0$ ➔ $0 = -C_{13}V_3$ ➔ $C_{13}=0$

Gauss's law

A change of V_3 will not affect Q_1.

EXAMPLE 3–18 A cylindrical capacitor consists of an inner conductor of radius a and an outer conductor whose inner radius is b. The space between the conductors is filled with a dielectric of permittivity ϵ, and the length of the capacitor is L. Determine the capacitance of this capacitor.

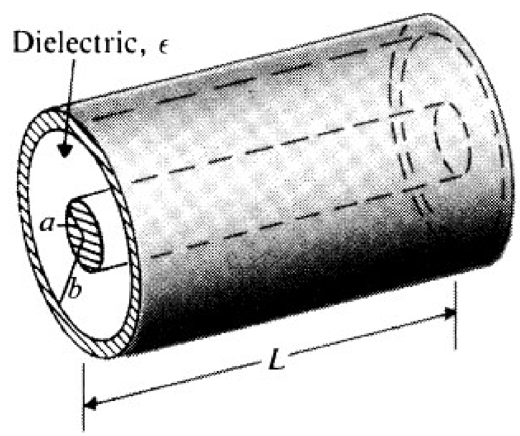


FIGURE 3–29
A cylindrical capacitor (Example 3–18).

EXAMPLE 3–21 Three horizontal parallel conducting wires, each of radius a and isolated from the ground, are separated from one another as shown in Fig. 3–36. Assuming $d \gg a$, determine the partial capacitances per unit length between the wires.

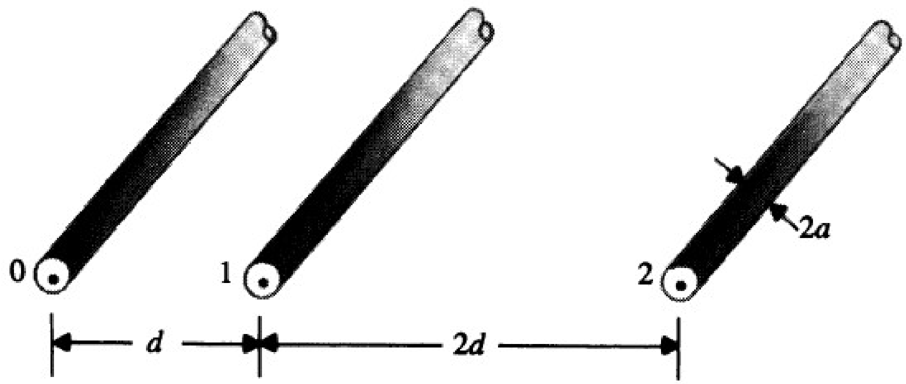


FIGURE 3–36
Three parallel wires (Example 3–21).

3-11 Electrostatic Energy and Forces

- From Eq. 3-44 $V_{21} \equiv \frac{W}{q} = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\boldsymbol{\ell}$ (J/C or V).
  - Work required to bring a charge q from P_1 to P_2
    $W = qV_{21}$
- A charge Q_1 in free space. Work required to bring a **second** charge Q_2 from infinity to a distance R_{12}: $W = Q_2 V_{2\infty} = Q_2 V_2$

Against **E** field of charge Q_1
(V_2 is due to charge Q_1)

$$W_2 = Q_2 V_2 = Q_2 \frac{Q_1}{4\pi\epsilon_0 R_{12}}$$

$$\Downarrow$$

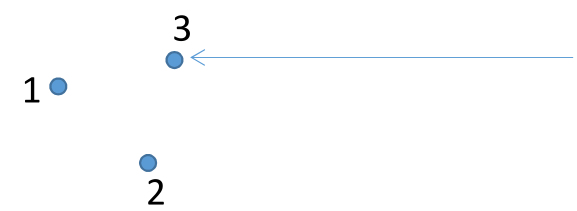
$$W_2 = Q_1 \frac{Q_2}{4\pi\epsilon_0 R_{12}} = Q_1 V_1.$$

$$\Downarrow$$

$$Q_1V_1 = Q_2V_2;\ Q_1V_1 + Q_2V_2 = 2W_2$$

$$\Downarrow$$

$$W_2 = \tfrac{1}{2}(Q_1V_1 + Q_2V_2).$$



- Another charge Q_3. Work required to bring a **third** charge Q_3 from infinity to a distance R_{13} from Q_1 and R_{23} from Q_2: $\Delta W = Q_3 V_{3\infty}$

$$\Delta W = Q_3 V_3 = Q_3\left(\frac{Q_1}{4\pi\epsilon_0 R_{13}} + \frac{Q_2}{4\pi\epsilon_0 R_{23}}\right).$$

Against **E** field of charge Q_1 and **E** field of charge Q_2 (V_3 is due to charges Q_1 and Q_2)

- Total work to assemble the 3 charges Q_1, Q_2, and Q_3: $W_3 = W_2 + \Delta W$

$$W_3 = W_2 + \Delta W = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_1Q_2}{R_{12}} + \frac{Q_1Q_3}{R_{13}} + \frac{Q_2Q_3}{R_{23}}\right).$$

Rewrite: 3 terms divided into 6 terms

$$W_3 = \frac{1}{2}\left[Q_1\left(\frac{Q_2}{4\pi\epsilon_0 R_{12}} + \frac{Q_3}{4\pi\epsilon_0 R_{13}}\right) + Q_2\left(\frac{Q_1}{4\pi\epsilon_0 R_{12}} + \frac{Q_3}{4\pi\epsilon_0 R_{23}}\right) + Q_3\left(\frac{Q_1}{4\pi\epsilon_0 R_{13}} + \frac{Q_2}{4\pi\epsilon_0 R_{23}}\right)\right]$$
$$= \tfrac{1}{2}(Q_1V_1 + Q_2V_2 + Q_3V_3).$$

- Potential V_1 is caused by charges Q_2 and Q_3
- **Different from the previous V_1 due to Q_2 only**

General expression

Self energy: work required to assemble the individual point charges

Initially, Q_1 in space

Introduce Q_2 $\Delta W = Q_2 V_{2\infty}$

Introduce Q_3 $\Delta W = Q_3 V_{3\infty}$

+

Mutual energy: the interacting energy

$$W_2 = \tfrac{1}{2}(Q_1 V_1 + Q_2 V_2).$$

$$W_3 = \tfrac{1}{2}(Q_1 V_1 + Q_2 V_2 + Q_3 V_3).$$

$$\vdots$$

$$\boxed{W_e = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k \qquad \text{(J)},}$$

Potential V_k is caused by all the other charges

$$V_k = \frac{1}{4\pi\epsilon_0}\sum_{\substack{j=1 \\ (j\neq k)}}^{N} \frac{Q_j}{R_{jk}}.$$

3-11.1 Electrostatic Energy in terms of Field Quantities

- For a continuous charge distribution of density ρ

$$\boxed{W_e = \frac{1}{2}\int_{V'} \rho V\, dv \qquad \text{(J).}}$$

Volume

Electrical potential

$$W_e = \frac{1}{2}\int_{V'} (\nabla \cdot \mathbf{D}) V\, dv.$$

$$\nabla \cdot (V\mathbf{D}) = V\nabla \cdot \mathbf{D} + \mathbf{D} \cdot \nabla V,$$

$$\begin{aligned} W_e &= \frac{1}{2}\int_{V'} \nabla \cdot (V\mathbf{D})\, dv - \frac{1}{2}\int_{V'} \mathbf{D} \cdot \nabla V\, dv \\ &= \frac{1}{2}\oint_{S'} V\mathbf{D} \cdot \mathbf{a}_n\, ds + \frac{1}{2}\int_{V'} \mathbf{D} \cdot \mathbf{E}\, dv, \end{aligned}$$

- V' can be any volume
- Choose its radius $R \rightarrow \infty$ ➔ 1st term disappears because of $V_\infty = 0$

$$\boxed{W_e = \frac{1}{2}\int_{V'} \mathbf{D} \cdot \mathbf{E}\, dv \qquad \text{(J).}}$$

Electrostatic Energy Density w_e

$$\boxed{W_e = \frac{1}{2}\int_{V'} \mathbf{D}\cdot\mathbf{E}\, dv \qquad \text{(J).}}$$

$\mathbf{D} = \epsilon \mathbf{E}$ For a linear medium

$$\boxed{W_e = \frac{1}{2}\int_{V'} \epsilon E^2\, dv \qquad \text{(J)}} \qquad \boxed{W_e = \frac{1}{2}\int_{V'} \frac{D^2}{\epsilon}\, dv \qquad \text{(J).}}$$

$$W_e = \int_{V'} w_e\, dv.$$

$$w_e = \tfrac{1}{2}\mathbf{D}\cdot\mathbf{E} \qquad (\text{J/m}^3) \qquad\qquad w_e = \tfrac{1}{2}\epsilon E^2 \qquad (\text{J/m}^3) \qquad\qquad w_e = \frac{D^2}{2\epsilon} \qquad (\text{J/m}^3).$$

Definition of density form is artificial.
Volume integral form can be verified.

3-11.2 Electrostatic Forces

- Using Coulomb's law to determine the force on one body that is caused by the charges on other bodies would be very tedious.
- Thus, a simple method of ***principle of virtual displacement*** is introduced.
  - System of bodies with ***fixed charges***
  - System of conducting bodies with ***fixed potentials***

System of Bodies with Fixed Charges

- An isolated system consisting of charged conductor and dielectric bodies.
- Condition: Charges are constant.
- Electric force displaces one of the bodies by $d\ell$(a virtual displacement)
  - Mechanical **work** done by the system: $dW = \mathbf{F}_Q \cdot d\boldsymbol{\ell},$

    $\mathbf{F}_Q$: total electric force acting on the body
  - In other words, **reduced stored electrostatic energy** produces the mechanical **work**

$dW + dW_e = 0$ ⇨ $dW = -dW_e = \mathbf{F}_Q \cdot d\boldsymbol{\ell}.$

$$dW_e = (\nabla W_e) \cdot d\boldsymbol{\ell}$$

⇩

$$\boxed{\mathbf{F}_Q = -\nabla W_e \qquad \text{(N)}.}$$

A very simple formula for the calculation of $\mathbf{F}_Q$ from **the electrostatic energy** of the system

- Electric torque rotates one of the bodies by $d\phi$ (a virtual rotation) about an axis (e.g., z axis)
  - Work done by the system:

$$dW = (T_Q)_z \, d\phi$$

$$\vdots$$

$$\boxed{(T_Q)_z = -\frac{\partial W_e}{\partial \phi} \qquad (\mathrm{N \cdot m}).}$$

System of Conducting Bodies with Fixed Potentials

- Condition: potentials are fixed.
- System connected to **external sources** to maintain fixed potentials
- A displacement $d\ell$ ➔ dW_e, dQ_k
  to maintain fixed potentials V_k
  - 1. Work done by **the external sources**: $dW_s = \sum_k V_k \, dQ_k$

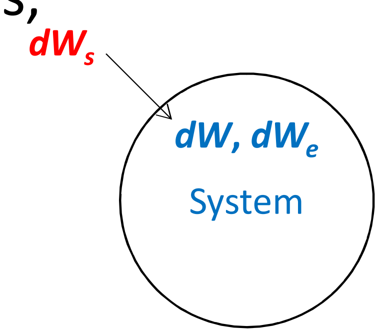
    ***dQ**$_k$*: Due to charge transfer between external sources and the system

  - 2. Produced mechanical **work**: $dW = \mathbf{F}_V \cdot d\boldsymbol{\ell}$

    F$_V$: Electric force acting on the body

- 3. Change of **electrostatic energy** due to dQ_k: $dW_e = \frac{1}{2}\sum_k V_k\, dQ_k = \frac{1}{2}\, dW_s$

- Thus,



$$dW + dW_e = dW_s.$$

$$dW = dW_e$$

$$\mathbf{F}_V \cdot d\boldsymbol{\ell} = dW_e = (\nabla W_e) \cdot d\boldsymbol{\ell}$$

$$\boxed{\mathbf{F}_V = \nabla W_e \qquad \text{(N)}.}$$

- Similarly, a displacement $d\phi \rightarrow dW_e$, dQ_k

$$\boxed{(T_V)_z = \frac{\partial W_e}{\partial \phi} \qquad (\mathrm{N \cdot m}),}$$

The difference in formulas for fixed potentials and for fixed charges is only a sign change.

EXAMPLE 3–26 Determine the force on the conducting plates of a charged parallel-plate capacitor. The plates have an area S and are separated in air by a distance x.