Chapter 4 Solution of Electrostatic Problems

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$\rho(v) \rightarrow \vec{E}(v)$

$V(v)$

✓ partial $\rho(v)$

$V_1, V_2 \ldots$ $\Big\} \Rightarrow \vec{E}(v)$

$V(v)$

4-1 Introduction

- Electrostatic problems: $\mathbf{E}$, V, ρ
- $\rho(\boldsymbol{r})$ known exactly everywhere ➔ $\mathbf{E}(\boldsymbol{r})$, $V(\boldsymbol{r})$

- In practical problems, $\rho(\boldsymbol{r})$ is not known everywhere (e.g., only partial $\rho(\boldsymbol{r})$ known). We need techniques:
  - Method of images
  - Boundary-value problems

4-2 Poisson's and Laplace's Equations

Maxwell's 1st and 2nd equations

$$\nabla \cdot \mathbf{D} = \rho.$$
$$\nabla \times \mathbf{E} = 0.$$

$$\nabla \times \mathbf{E} = 0. \quad \Rightarrow \quad \mathbf{E} = -\nabla V.$$

$$\nabla \cdot \mathbf{D} = \rho.$$

In a linear medium

$$\mathbf{D} = \epsilon \mathbf{E},$$

$$\nabla \cdot \epsilon \mathbf{E} = \rho.$$

$$\mathbf{E} = -\nabla V.$$

$$\nabla \cdot (\epsilon \nabla V) = -\rho,$$

homogeneous

Poisson's Equations

$$\nabla \cdot (\epsilon \nabla V) = -\rho,$$

In a homogeneous medium
ε is a constant over space

Poisson's equation $$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon}.}$$

Laplacian operator: $\nabla^2 = \nabla \bullet \nabla$

Poisson's equation:
ρ may be a function of space coordinates
ε must be a constant over space

Poisson's Equation in Cartesian Coordinate

$$\nabla^2 V = \nabla \cdot \nabla V = \left(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}\right) \cdot \left(\mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z}\right);$$

$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon}.} \Rightarrow \boxed{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\epsilon} \qquad (\mathrm{V/m^2}).}$$

∇^2 in Cylindrical and Spherical Coordinates

- Cylindrical:

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}.$$

- Spherical:

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}.$$

Laplace's Equation

- In a simple medium where there is no free charge, $\rho = 0$

Laplace's Equation $\boxed{\nabla^2 V = 0,}$

- Example to use Laplace's equation: a set of conductors at different potentials

  Solve V by Laplace's equation ➔ $\mathbf{E} = -\nabla V$ ➔ $\rho_s = \varepsilon E_n$

  (see example 4-1)

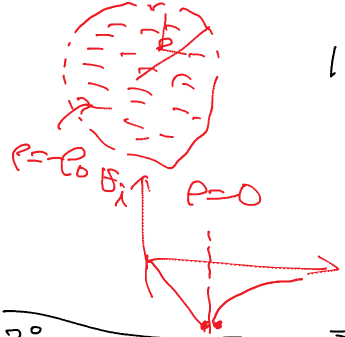
$\Rightarrow V(v)$

$\rho_s \rightarrow V$

$\vec{E}$

EXAMPLE 4–2 Determine the **E** field both inside and outside a spherical cloud of electrons with a uniform volume charge density $\rho = -\rho_0$ (where ρ_0 is a positive quantity) for $0 \le R \le b$ and $\rho = 0$ for $R > b$ by solving ~~Poisson's and~~ Laplace's equations for V.

chap. 3 : $\rho(v) \rightarrow \vec{E}(v)$



1° $R < b$, $\quad \nabla^2 V_i = \dfrac{-(-\rho_0)}{\epsilon_0} = \dfrac{\rho_0}{\epsilon_0}$

$V_i(R)$

$$\Rightarrow \frac{1}{R^2}\frac{d}{dR}\left(R^2 \frac{dV_i}{dR}\right) + 0 + 0 = \frac{\rho_0}{\epsilon_0}$$

$$\Rightarrow \frac{d}{dR}\left(R^2 \frac{dV_i}{dR}\right) = \frac{\rho_0}{\epsilon_0} R^2 \quad \text{—— (1)}$$

$$\Rightarrow \quad \cdots \quad \Rightarrow \frac{dV_i}{dR} = \frac{\rho_0}{3\epsilon_0} R + \frac{C_1}{R^2}$$

$$\vec{E}_i = -\nabla V_i = -\hat{a}_R \frac{dV_i}{dR} + 0 + 0 = -\hat{a}_R\left(\frac{\rho_0}{3\epsilon_0} R + \cancel{\frac{C_1}{R^2}}\right)$$

0, $\because E_i$ at $R = 0$ will be ∞, which is not allowed.

2° from (1), and let $\rho_0 = 0$

$$\Rightarrow \frac{d}{dR}\left(R^2 \frac{dV_o}{dR}\right) = 0 \Rightarrow \cdots \Rightarrow \frac{dV_o}{dR} = \frac{C_2}{R^2}$$

$$\vec{E}_o = -\hat{a}_R \frac{dV_o}{dR} = -\hat{a}_R \frac{C_2}{R^2}$$

$\vec{E}_{i,R=b} = \vec{E}_{o,R=b}$ because of the same material

$\rho_s = 0$, E_n is continuous

$\Downarrow$

$C_2 = \cdots$

$$\Rightarrow \vec{E}_o = -\hat{a}_R \frac{\rho_o b^3}{3\epsilon_o R^2}$$

total $Q = -\frac{4}{3}\pi b^3 \cdot \rho_o$

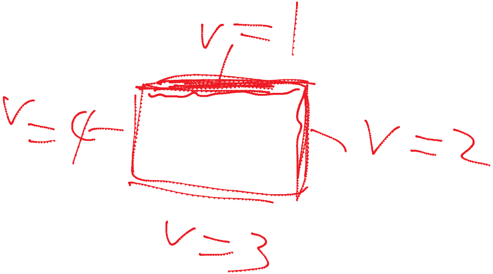
$$\Rightarrow \vec{E}_o = \hat{a}_R \frac{Q}{4\pi\epsilon_o R^2}$$

4-3 Uniqueness of Electrostatic Solutions

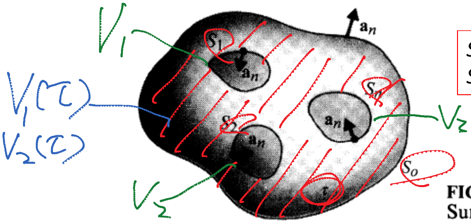
1

- Uniqueness theorem: a solution of Poisson's equation that satisfies the given boundary conditions is a unique solution.

2



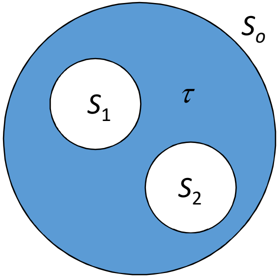
Proof of Uniqueness Theorem



S_o: outer surface enclosing the volume τ
$S_1, S_2, \ldots, S_n$: surfaces of **conducting bodies**

FIGURE 4–2
Surface S_o enclosing volume τ with conducting bodies.

- Volume τ is bounded (enclosed) by a surface S_o and surfaces S_1, S_2, … S_n
- Inside S_o, many charged **conducting bodies** with surfaces S_1, S_2,…, S_n **at specified potentials**.



$S_{1,}$ S_2: Surface of a conducting body
τ: Volume bounded by S_o, S_1, and S_2

Proof of Uniqueness Theorem

- Uniqueness theorem: There is only one solution of potential V in τ
- To prove uniqueness theorem, we assume two solutions V_1 and V_2 in τ:

$$\nabla^2 V_1 = -\frac{\rho}{\epsilon},$$

$$\nabla^2 V_2 = -\frac{\rho}{\epsilon}.$$

Also assume that V_1 and V_2 satisfy the same boundary conditions on S_1, S_2, … S_n and S_o

Proof of Uniqueness Theorem

$V_d(\tau)$

- (i) Define a potential difference V_d: $V_d = V_1 - V_2$.

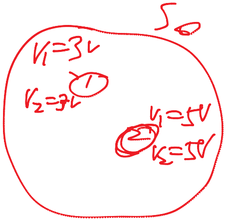
$$\nabla^2 V_1 = -\frac{\rho}{\epsilon}, \quad - \quad \nabla^2 V_2 = -\frac{\rho}{\epsilon}. \quad \Rightarrow \quad \nabla^2 V_d = 0. \quad \text{in } \tau$$

∇^2

- (ii) On conducting boundaries, **the potentials are specified** → $V_d = 0$

on the conductor surfaces

2



Uniqueness theorem: a solution of Poisson's equation that **satisfies the given boundary conditions (i.e., potentials are specified on surfaces of conducting bodies)** is a unique solution.

Proof of Uniqueness Theorem

$$\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f;$$

letting $f = V_d$ and $\mathbf{A} = \nabla V_d$;

$$\nabla \cdot (V_d \nabla V_d) = V_d \nabla^2 V_d + |\nabla V_d|^2,$$

Integration over τ

$\boxed{\nabla^2 V_d = 0. \quad \text{in } \tau}$ Using (i)

$$\oint_S (V_d \nabla V_d) \cdot \mathbf{a}_n \, ds = \int_\tau |\nabla V_d|^2 \, dv,$$

$\boxed{S: S_1, S_2, \ldots S_n, \text{ and } S_o}$

Proof of Uniqueness Theorem

$$\oint_{S} (V_d \nabla V_d) \cdot \mathbf{a}_n \, ds = \int_{\tau} |\nabla V_d|^2 \, dv,$$

$S_1 - S_n$

S_o

$\int_{S_o}$

1. For S_1, S_2, … S_n , $V_d = 0$ Using (ii)
2. For S_o

    Consider the surface of a sphere with radius $R \rightarrow \infty$

    $V_d \sim 1/R$

    $\nabla V_d \sim 1/R^2$

    $s \sim R^2$

    Thus, the integration ~ $1/R$

    As $R \rightarrow \infty$, Left side $\rightarrow$ 0

$\Rightarrow$ $$\int_{\tau} |\nabla V_d|^2 \, dv = 0.$$

Proof of Uniqueness Theorem

$\nabla V_d = 0$ in τ

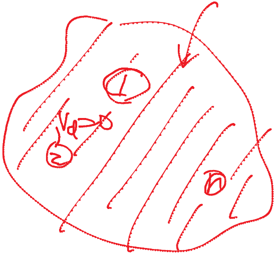
$\Downarrow$

$V_d = \text{const.}$ in τ

$$\int_\tau |\nabla V_d|^2 \, dv = 0.$$

$\int A^2 dv = 0$

$\rightarrow$ ~~|A|~~ $A = 0$



$|\nabla V_d|^2$ is nonnegative everywhere,

$$|\nabla V_d| = 0. \quad \text{everywhere in } \tau$$

V_d is a constant everywhere in τ

➔ "V_d in τ" = "V_d on surfaces" by the continuity of V_d

$\Downarrow$

We know $V_d = 0$ on surfaces S_1, S_2,… S_n

$V_d = 0$ in τ

$\Downarrow$

$V_d = 0$ everywhere in τ

That is, $V_1 = V_2$ everywhere in τ, and there is only one possible solution!

$V_1 = V_2$ in τ

Two Cases for the Uniqueness Theorem

- Known **potentials** of conducting bodies, which is just proved.
- Known **charge distributions** of conducting bodies:

ρ_S is known on conducting bodies

$$(\rho_s = \epsilon E_n = -\epsilon\, \partial V/\partial n),$$

$\nabla V_d = 0.$ on conducting body surfaces

Substitute $\nabla V_d = 0$. on $S_1, S_2,\ldots S_n$ into $\oint_S (V_d \nabla V_d) \cdot \mathbf{a}_n \, ds = \int_\tau |\nabla V_d|^2 \, dv,$

$$\Downarrow$$

$$\left.\begin{array}{l} \oint_S (V_d \nabla V_d) \cdot \mathbf{a}_n \, ds \\ \qquad = 0 \text{ over } S_1, S_2,\ldots S_n. \\ \text{Also, } \oint_{S_o} (V_d \nabla V_d) \cdot \mathbf{a}_n \, ds = 0 \text{ as } R \rightarrow \infty \end{array}\right\} \text{LHS} \rightarrow 0$$

$$\Downarrow$$

$$\int_\tau |\nabla V_d|^2 \, dv = 0.$$

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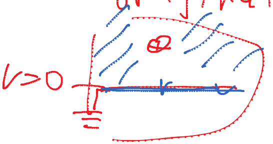
$\Rightarrow$ The same conclusion can be obtained!

4-4 Methods of Images

- Methods of images: replacing boundaries by appropriate **image charges** in lieu of a formal solution of Poisson's or Laplace's equation
  - Condition on boundaries unchanged
  - $V(\boldsymbol{R})$ can be determined easily

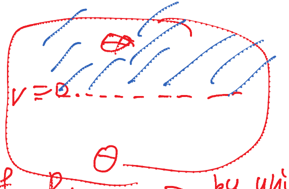
original case



$V_1(\tau)$

τ: upper space

new case: easy to compute



$V_2(\tau)$

① $V_1(\tau)$, $V_2(\tau)$ are sol. of P's eq.

② $V_1(\tau)$, $V_2(\tau)$ both satisfy B.C.

by uni theorem $\Longrightarrow V_1(\tau) = V_2(\tau)$

Point Charge and Grounded Plane Conductor (by Laplace Eq.)

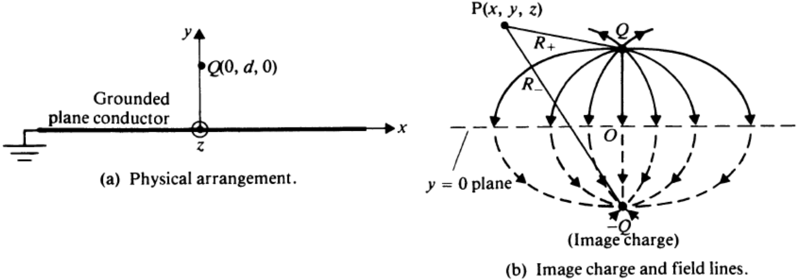
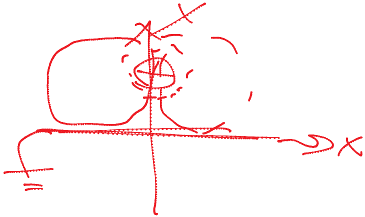


FIGURE 4–3
Point charge and grounded plane conductor.

Why Not Laplace's Eq.



- Solved by Laplace eq.:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

Hold for $y > 0$ except at the point charge

- **4 Conditions** should be satisfied:

1. $V(x, 0, z) = 0.$
2. for points very close to Q

$$V \to \frac{Q}{4\pi\epsilon_0 R}, \text{ as } R \to 0,$$

3. $V \rightarrow 0$ for points very far from Q

$$x \to \pm\infty,\ y \to +\infty, \text{ or } z \to \pm\infty$$

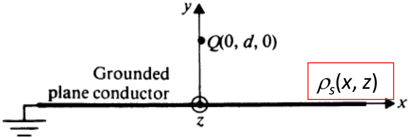
4. Even functions w.r.t. x and z coordinates

$$V(x, y, z) = V(-x, y, z) \qquad V(x, y, z) = V(x, y, -z).$$

Difficult to solve…

- $+Q \rightarrow$ induce ρ_s on conducting plane

$$V(x, y, z) = \frac{Q}{4\pi\epsilon_0\sqrt{x^2 + (y-d)^2 + z^2}} + \frac{1}{4\pi\epsilon_0}\int_S \frac{\rho_s}{R_1}\,ds,$$

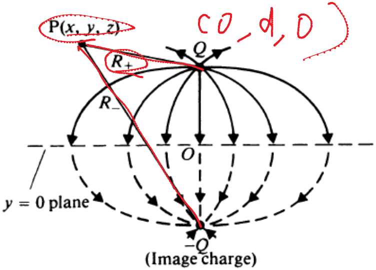


(a) Physical arrangement.

  - $\rho_s(x, z)$ not easy to determine.
  - Besides, 2nd term is difficult to evaluate.

4-4.1 Point Charge and Conducting Planes

- Image methods:
  - Remove the conductor
  - Replace with an image point charge $-Q$ at $y = -d$



(b) Image charge and field lines.

$$V(x, y, z) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right),$$

$$R_+ = [x^2 + (y-d)^2 + z^2]^{1/2},$$
$$R_- = [x^2 + (y+d)^2 + z^2]^{1/2}.$$

Solution $$V(x, y, z) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right),$$

- By direct substitution, we can verify:
  - Laplace's Eq. is satisfied
  - **4 conditions** are satisfied
- In view of uniqueness theorem, the solution is the only solution.

Uniqueness theorem: a solution of Poisson's equation that **satisfies the given boundary conditions** is a unique solution.

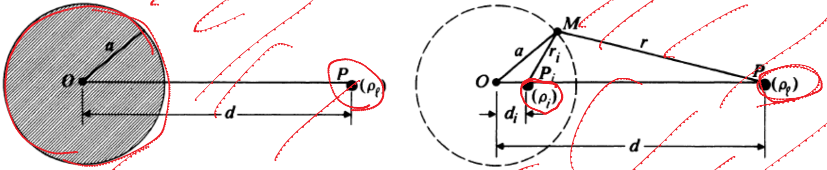
- $\mathbf{E} = -\nabla V$

Extremely simple!

Notes of Image Methods

- The solution **cannot** be used to calculate V or $\mathbf{E}$ **in the y<0 region**.
- For $y < 0$ region, $\mathbf{E} = 0$, $V = 0$.

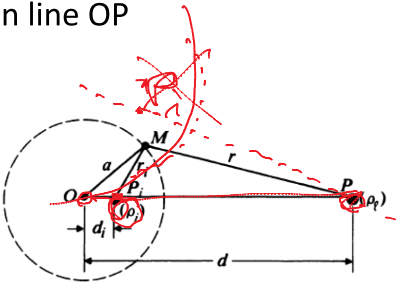
4-4.2 Line Charge and Parallel Conducting Cylinder



(a) Line charge and parallel conducting cylinder. (b) Line charge and its image.

FIGURE 4–5
Cross section of line charge and its image in a parallel, conducting, circular cylinder.

1. Cylinder surface is an equi-potential surface → image must be a parallel line charge (ρ_i) inside the cylinder
2. By symmetry of line OP, ρ_i should be on line OP



(b) Line charge and its image.

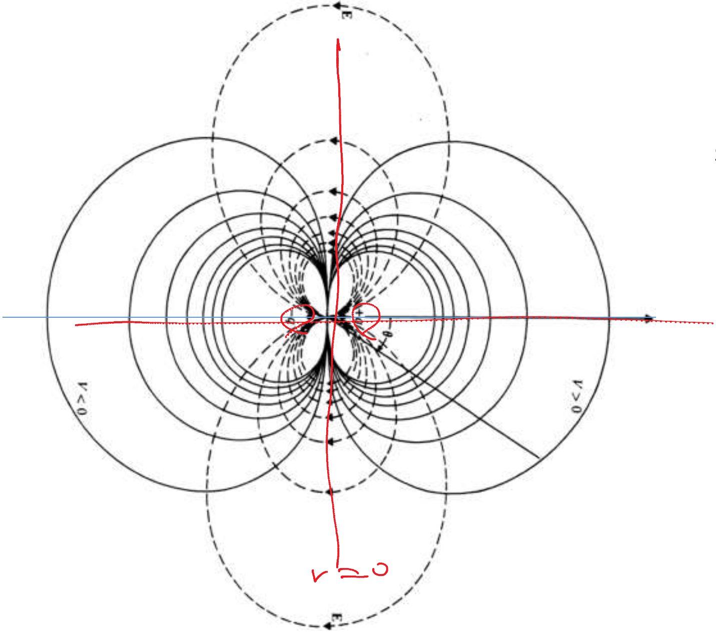


FIGURE 3–15
Equipotential and electric field lines of an electric dipole (Example 3–8).

Solid line: equi-potential surface

Assume $\boxed{\rho_i = -\rho_\ell.}$ (intelligent guess, also see figure 3-15)

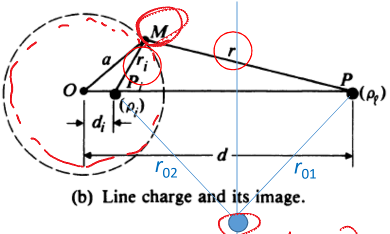
- Voltage due to ρ_ℓ

$$V = -\int_{r_0}^{r} E_r\,dr = -\frac{\rho_\ell}{2\pi\epsilon_0}\int_{r_0}^{r}\frac{1}{r}\,dr$$

$$= \frac{\rho_\ell}{2\pi\epsilon_0}\ln\frac{r_0}{r}.$$

Reference point, $V = 0$

Point of interest



(b) Line charge and its image.

- Voltage due to ρ_ℓ and ρ_i on cylindrical surface

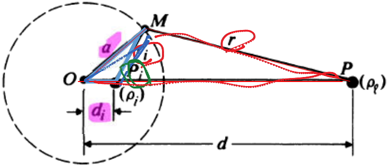
$$V_M = \frac{\rho_\ell}{2\pi\epsilon_0}\ln\frac{r_{01}}{r} - \frac{\rho_\ell}{2\pi\epsilon_0}\ln\frac{r_{02}}{r_i}$$

$$= \frac{\rho_\ell}{2\pi\epsilon_0}\ln\frac{r_i}{r}.$$

Choosing the same reference point with equidistance from ρ_ℓ and ρ_i so that $r_{01} = r_{02}$.

- To make V_M = constant

$$\frac{r_i}{r} = \text{Constant.}$$



(b) Line charge and its image.

- To make M coincide with the cylindrical surface ($OM = a$), P_i should be chosen to make the two triangles OMP_i and OPM similar. (Otherwise, r_i/r = constant over the cylindrical surface cannot be satisfied.)

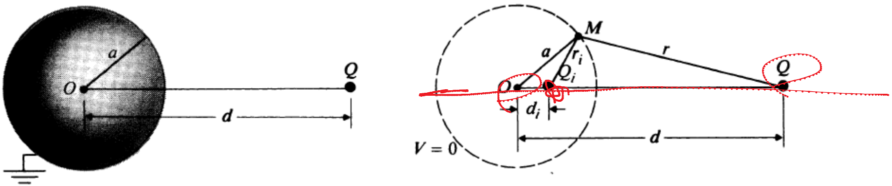
$$\frac{\overline{P_iM}}{\overline{PM}} = \frac{\overline{OP_i}}{\overline{OM}} = \frac{\overline{OM}}{\overline{OP}}$$

$$\frac{r_i}{r} = \frac{d_i}{a} = \frac{a}{d} = \text{Constant.}$$

$$\boxed{d_i = \frac{a^2}{d}}$$

P_i is called the **inverse point** of P

4-4.3 Point Charge and Conducting Sphere



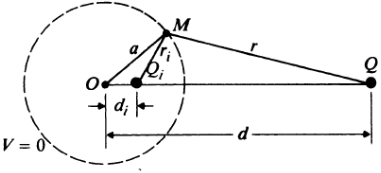
(a) Point charge and grounded conducting sphere.

(b) Point charge and its image.

FIGURE 4–11
Point charge and its image in a grounded sphere.

Intelligent Guess

- By symmetry, Q_i
  - Negative
  - Inside the sphere
  - On line OQ



(b) Point charge and its image.

- $Q_i \neq -Q$; otherwise the equi-potential surface of **V = 0** is a plane

- Thus, both Q_i and d_i should be solved

$$V_M = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{Q_i}{r_i}\right) = 0,$$

$$\frac{r_i}{r} = -\frac{Q_i}{Q} = \text{Constant}.$$

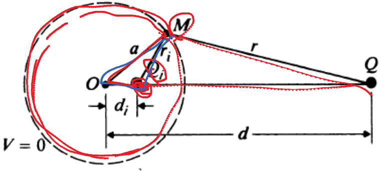
Similar to the case in 4-4.2

$$\frac{r_i}{r} = \frac{d_i}{a} = \frac{a}{d} = \text{Constant}.$$

$$-\frac{Q_i}{Q} = \frac{a}{d}$$

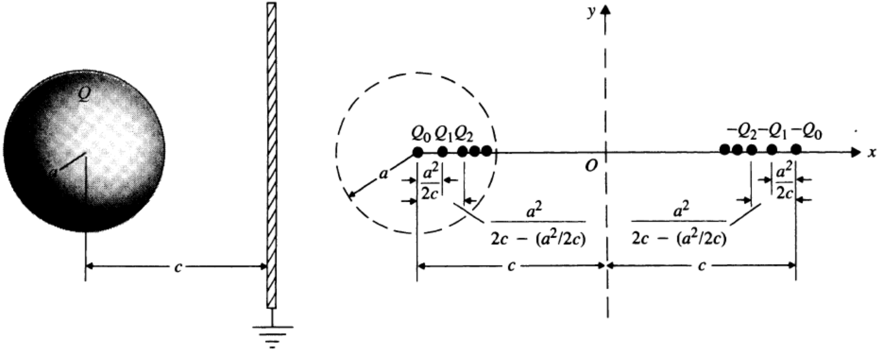
$$\boxed{Q_i = -\frac{a}{d}Q} \qquad \boxed{d_i = \frac{a^2}{d}.}$$

Q_i is called the **inverse point** of Q



(b) Point charge and its image.

4-4.4 Charged Sphere and Grounded Plane



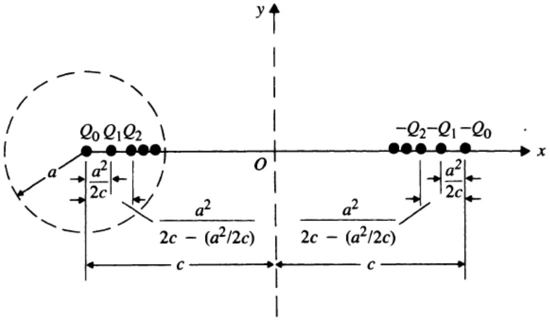
(a) Physical arrangement.

(b) Two groups of image point charges.

FIGURE 4–13
Charged sphere and grounded conducting plane.

• The sphere and the plane must be equi-potential surfaces.

• Method of images: The charged sphere and grounded plane can be replaced by charges.



(b) Two groups of image point charges.

Q_0 at $(-c, 0)$ ➔ the sphere is an equi-potential plane

(−*c*, 0) is the origin of the sphere.

To make *yz* plane equi-potential ($V = 0$)

$-Q_0$ at $(c, 0)$

Destroy equi-potential of the sphere
To make the sphere surface equi-potential

Q_1 inside the sphere

Destroy equi-potential of yz plane ($V = 0$)
To make yz plane equi-potential

$-Q_1$

.
.
.

Q_0,

$(-Q_0, Q_1)$

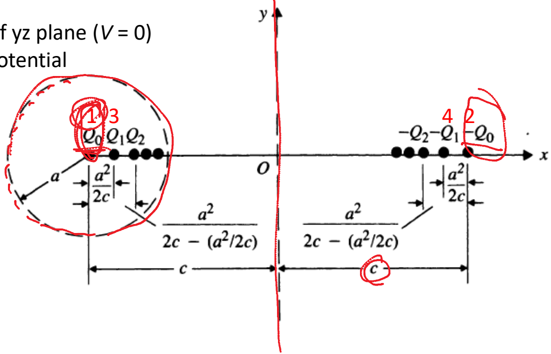
original image
(3rd example)

$Q_1 = -\frac{a}{2c}(-Q_0)$

$= \frac{a}{2c} Q_0$

$V=0$, Q_0, $-Q_0$

1 2



(b) Two groups of image point charges.

$$Q_1 = \left(\frac{a}{2c}\right) Q_0 = \alpha Q_0,$$

$$Q_2 = \frac{a}{\left(2c - \frac{a^2}{2c}\right)} Q_1 = \frac{\alpha^2}{1-\alpha^2} Q_0, \qquad \alpha = \frac{a}{2c}.$$

$$Q_3 = \frac{a}{2c - \dfrac{a^2}{\left(2c - \frac{a^2}{2c}\right)}} Q_2 = \frac{\alpha^3}{(1-\alpha^2)\left(1 - \dfrac{\alpha^3}{1-\alpha^3}\right)} Q_0,$$

$$\vdots$$

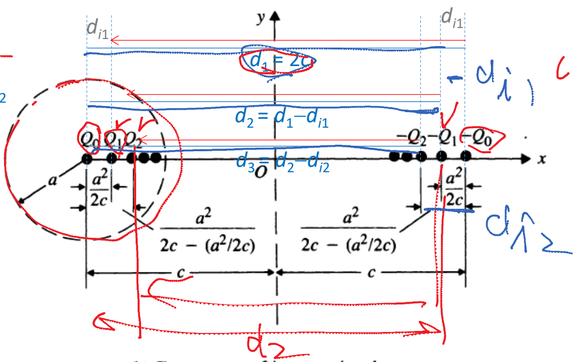
$$\boxed{Q_i = -\frac{a}{d} Q} \qquad \boxed{d_i = \frac{a^2}{d}.}$$

d: from the charge to the sphere center

Q_0 to Q_1 ➔ $Q_1 = (a/d_1)Q_0$; $d_{i1} = a^2/d_1$

Q_1 to Q_2 ➔ $Q_2 = (a/d_2)Q_1$; $d_{i2} = a^2/d_2$

Q_2 to Q_3 …



(b) Two groups of image point charges.

- Total charge on the sphere

$$Q = Q_0 + Q_1 + Q_2 + \cdots$$
$$= Q_0\left(1 + \alpha + \frac{\alpha^2}{1-\alpha^2} + \cdots\right).$$

- The V on the sphere

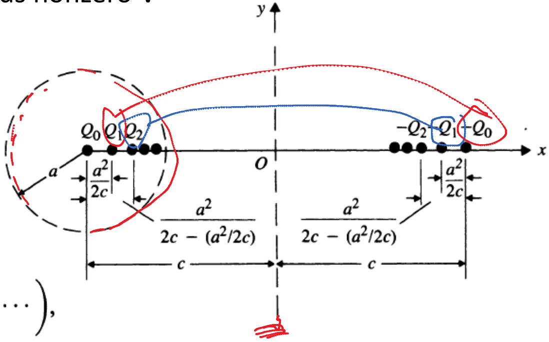
Pairs $(-Q_0, Q_1), (-Q_1, Q_2), \ldots$ yield zero potential on the sphere

Only Q_0 at the center yields nonzero V

$$V_0 = \frac{Q_0}{4\pi\epsilon_0 a}.$$

- The C between the sphere and the conducting plane

$$C = \frac{Q}{V_0} = 4\pi\epsilon_0 a\left(1 + \alpha + \frac{\alpha^2}{1-\alpha^2} + \cdots\right),$$



(b) Two groups of image point charges.

EXAMPLE 4–4 Determine the capacitance per unit length between two long, parallel, circular conducting wires of radius a. The axes of the wires are separated by a distance D.

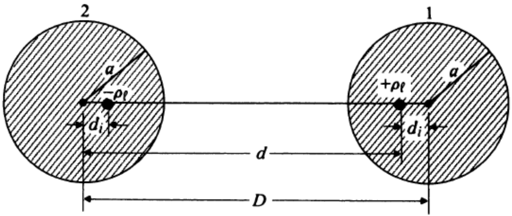
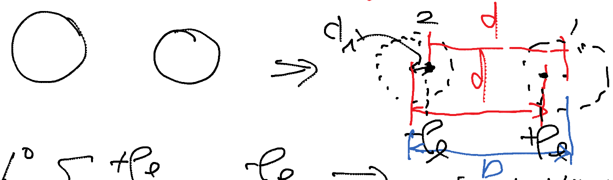


FIGURE 4–6
Cross section of two-wire transmission line and equivalent line charges (Example 4–4).

$C_e = \frac{\rho_e}{V}$

2nd example



1° $\begin{cases} \underset{\text{original}}{+\rho_e}, \ \underset{\text{image}}{-\rho_e} \Rightarrow \text{equi-potential for conductor 2} \\ -\rho_e, \ +\rho_e \Rightarrow \quad 1 \end{cases}$

$\Rightarrow V_2 = \frac{\rho_e}{2\pi\epsilon_0}\ln\left(\frac{a}{d}\right)$

$\Rightarrow V_1 = \frac{-\rho_e}{2\pi\epsilon_0}\ln\left(\frac{a}{d}\right)$

2° $C_e = \frac{\rho_e}{V_1 - V_2} = \frac{2\pi\epsilon_0}{\ln\left(\frac{d}{a}\right)}$ —— (1)

$d = D - d_1 = D - \frac{a^2}{d} \Rightarrow d = \ldots a, D$

4-5 Boundary-Value Problems in Cartesian Coordinates

methods of Image

- Method of images: useful for the case with isolated free charges (see 4-4.1 to 4-4.4)
- Laplace's equation: can be used to solve the case **w/o isolated free charges** (see Example 4-1: **charges on conductors (= non-isolated)**)
  - Known boundary values (potential or its normal derivative specified)

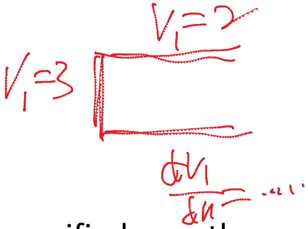
BVP

Three Types of Boundary Conditions

- Dirichlet: *V* is specified on boundaries
- Neumann: *dV/dn* is specified on boundaries

normal

- Mixed: *V* specified on some boundaries; *dV/dn* specified over the remaining boundaries.



Separation of Variables

Laplace's equation
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

By separation of variables
$$V(x, y, z) = X(x)Y(y)Z(z),$$

$$Y(y)Z(z)\frac{d^2X(x)}{dx^2} + X(x)Z(z)\frac{d^2Y(y)}{dy^2} + X(x)Y(y)\frac{d^2Z(z)}{dz^2} = 0,$$

$\times 1/X(x)Y(y)Z(z)$

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = 0.$$

$$\boxed{f(x) + f(y) + f(z) = 0}$$

$2x$ $\quad 2 \quad -1 \quad -1 \quad y$

$2x + y = 0$
not always true
for arbitrary x, y

$f(x)+f(y)+f(z) = 0$ to be satisfied for all values of x, y, z

$f(x)$: function of x only

$f(y)$: function of y only

$f(z)$: function of z only

$f(x)$, $f(y)$, $f(z)$ must be a constant

$df(x)/dx=0$, $df(y)/dy=0$, $df(z)/dz=0$

$df(x)/dx = 0$

$$\frac{d}{dx}\left[\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}\right] = 0,$$

$f(x)$ is constant

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -k_x^2,$$

$$\frac{d^2X(x)}{dx^2} + k_x^2 X(x) = 0.$$

$$X'' = -k_x^2 X$$

Possible Solutions of $X''(x) + k_x^2 X(x) = 0$

k_x^2	k_x	$X(x)$	Exponential forms[†] of $X(x)$
0	0	$A_0 x + B_0$	
+	k	$A_1 \sin kx + B_1 \cos kx$	$C_1 e^{jkx} + D_1 e^{-jkx}$
−	jk	$A_2 \sinh kx + B_2 \cosh kx$	$C_2 e^{kx} + D_2 e^{-kx}$

$\boxed{\text{k is real}}$

$$\frac{d^2X(x)}{dx^2} + k_x^2X(x) = 0.$$

$$\frac{d^2Y(y)}{dy^2} + k_y^2Y(y) = 0$$

$$\frac{d^2Z(z)}{dz^2} + k_z^2Z(z) = 0,$$

$$\boxed{\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = 0.}$$

$f(x)$	$+$	$f(y)$	$+$	$f(z)$	$=0$
$-k_x^2$		$-k_y^2$		$-k_z^2$	$=0$

$k_x^2 + k_y^2 + k_z^2 = 0.$, which should be satisfied.

Possible Solutions of $X''(x) + k_x^2 X(x) = 0$

k_x^2	k_x	$X(x)$	Exponential forms† of $X(x)$
0	0	$A_0 x + B_0$	
+	k	$A_1 \sin kx + B_1 \cos kx$	$C_1 e^{jkx} + D_1 e^{-jkx}$
−	jk	$A_2 \sinh kx + B_2 \cosh kx$	$C_2 e^{kx} + D_2 e^{-kx}$

$e^{\pm jkx} = \cos kx \pm j \sin kx, \quad \cos kx = \frac{1}{2}(e^{jkx} + e^{-jkx}), \quad \sin kx = \frac{1}{2j}(e^{jkx} - e^{-jkx});$

$e^{\pm kx} = \cosh kx \pm \sinh kx, \quad \cosh kx = \frac{1}{2}(e^{kx} + e^{-kx}), \quad \sinh kx = \frac{1}{2}(e^{kx} - e^{-kx}).$ See Fig. 4-14

A, B (or C, D) will be determined by boundary conditions

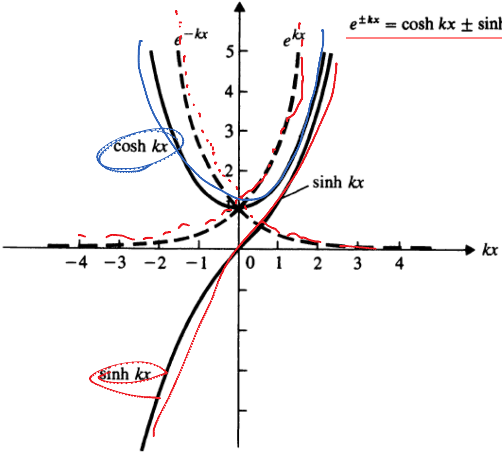


FIGURE 4–14
Hyperbolic and exponential functions.

EXAMPLE 4–6 Two grounded, semi-infinite, parallel-plane electrodes are separated by a distance b. A third electrode perpendicular to and insulated from both is maintained at a constant potential V_0 (see Fig. 4–15). Determine the potential distribution in the region enclosed by the electrodes.

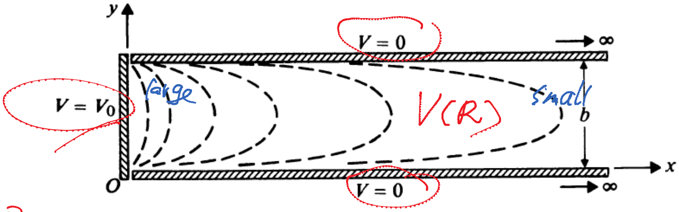


FIGURE 4–15 Cross-sectional figure for Example 4–6. The plane electrodes are infinite in z-direction.

1° $V(x,y,z) = X(x)Y(y)Z(z)$

2° z: $V(x,y,z) = V(x,y)$

x: $V(0,y) = V_0$ — ①

$V(\infty,y) = 0$ — ②

y: $V(x,0) = 0$ — ③

$V(x,b) = 0$ — ④

3° $Z(z) = B \Rightarrow k_z = 0 \Rightarrow k_y^2 = -k_x^2 = k^2$

$\Rightarrow$ exponential $\Rightarrow k_x^2 < 0$

$\Rightarrow$ sin, cos $\Rightarrow k_y^2 > 0$

4° $X''+k_x^2X=0 \Rightarrow X''-k^2X=0 \Rightarrow X(x)=D_2e^{-kx}$

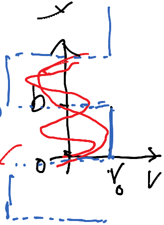
$\pm k$, choose $-k$ only

(2)

$Y''+k_y^2Y=0 \Rightarrow Y''+k^2Y=0 \Rightarrow Y(y)=A_1\sin ky$

(3)

$\Rightarrow V_n(x,y,z)=(B_0D_2A_1)e^{-kx}\cancel{\sin ky} = C_n e^{-kx}\cancel{\sin ky}$



5° to satisfy ④ $V_n(x,b)=0 \Rightarrow \sin kb=0 \Rightarrow k=\frac{n\pi}{b},\ n=1,2,3\ldots$

$\Rightarrow V_n(x,y)$

6° to satisfy ① $V(0,y)=V_0 \Rightarrow \sum_{n=1}^{\infty} V_n(0,y)=\sum_{n=1}^{\infty} C_n\sin\frac{n\pi}{b}y = V_0$

$n=1 \Rightarrow \sin\frac{\pi}{b}y$

$n=0 \Rightarrow \sin kb=0 \Rightarrow \sin 0y=0$ ✗

$C_n=\begin{cases}\frac{4V_0}{n\pi} & \text{if } n \text{ is odd}\\ 0 & \text{even}\end{cases}$

4-6 Boundary-Value Problems in Cylindrical Coordinates

Laplace's equation $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{\partial^2 V}{\partial z^2}=0.$$

General solution: Bessel functions

Assuming z independent

$\partial^2 V/\partial z^2 = 0$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2}=0.$$

By separation of variables

$V(r, \phi) = R(r)\Phi(\phi),$

$$\frac{r}{R(r)}\frac{d}{dr}\left[r\frac{dR(r)}{dr}\right]+\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2}=0.$$

To hold for all values of r and ϕ

$$\boxed{\frac{r}{R(r)}\frac{d}{dr}\left[r\frac{dR(r)}{dr}\right]+\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2}=0.}$$

$$\frac{r}{R(r)}\frac{d}{dr}\left[r\frac{dR(r)}{dr}\right]=k^2$$

$$\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2}=-k^2, \quad \xrightarrow[\text{rewrite}]{} \quad \frac{d^2\Phi(\phi)}{d\phi^2}+k^2\Phi(\phi)=0.$$

$$\boxed{V(r, \phi) = R(r)\underline{\Phi(\phi)},}$$

$$\frac{d^2\Phi(\phi)}{d\phi^2} + k^2\Phi(\phi) = 0.$$

For circular configurations, if ϕ is unrestricted, $\Phi(\phi)$ is **periodic over every 2π** (same values at a certain ϕ)

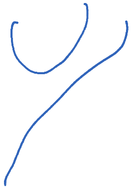
- k must be an integer → n
- sinh, cosh are not periodic!

Ve216:

$e^{jk\phi} \neq e^{jk(\phi+2\pi)}$ $= e^{jk\phi} e^{jk2\pi}$

$e^{jn\phi} = e^{jn(\phi+2\pi)}$ $= e^{jn\phi} e$

$$\Phi(\phi) = A_\phi \sin n\phi + B_\phi \cos n\phi,$$



$$\boxed{V(r, \phi) = \underline{R(r)}\Phi(\phi),}$$

$$\underbrace{\frac{r}{R(r)} \frac{d}{dr}\left[r \frac{dR(r)}{dr}\right]} = k^2$$

k → n

Product rule in calculus

$$r^2 \frac{d^2R(r)}{dr^2} + r\frac{dR(r)}{dr} - n^2R(r) = 0,$$

Solution: $\underline{R(r) = A_r r^n + B_r r^{-n}}.$

$$\Phi(\phi) = A_\phi \sin n\phi + B_\phi \cos n\phi,$$

$$R(r) = A_r r^n + B_r r^{-n}.$$

Combine the two solutions

$$\boxed{V(r, \phi) = R(r)\Phi(\phi),}$$

$$V_n(r, \phi) = r^n(A_n \sin n\phi + B_n \cos n\phi) + r^{-n}(A'_n \sin n\phi + B'_n \cos n\phi), \qquad n \neq 0.$$

where $A_n = A_r A_\phi$; $B_n = A_r B_\phi$; $A'_n = B_r A_\phi$; $B'_n = B_r B_\phi$;

If region of interest (ROI) includes $r = 0$, 2nd term cannot exist.

If ROI includes $r = \infty$, 1st term cannot exist

A Special Case: $k = 0$

$$\boxed{\frac{d^2\Phi(\phi)}{d\phi^2} + k^2\Phi(\phi) = 0.}$$

$$\boxed{\frac{r}{R(r)}\frac{d}{dr}\left[r\frac{dR(r)}{dr}\right] = k^2}$$

for k=0

$$\frac{d^2\Phi(\phi)}{d\phi^2} = 0.$$

$$\frac{d}{dr}\left[r\frac{dR(r)}{dr}\right] = 0,$$

$$\Phi(\phi) = A_0\phi + B_0$$

$$R(r) = C_0 \ln r + D_0,$$

EXAMPLE 4–8 Consider a very long coaxial cable. The inner conductor has a radius a and is maintained at a potential V_0. The outer conductor has an inner radius b and is grounded. Determine the potential distribution in the space between the conductors.

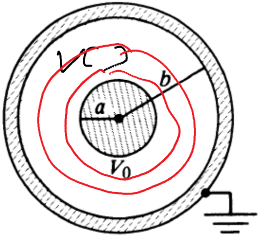


FIGURE 4–18
Cross section of a coaxial cable (Example 4–8).

$V(r,\phi)=R(r)\Phi(\phi)$

$= \frac{V_0}{\ln(a/b)}\ln\left(\frac{r}{b}\right)$

1° $V(r,\phi)=R(r)\Phi(\phi)$

2° $V(a,\phi)=V_0$ — ①

$V(b,\phi)=0$ — ②

V is ϕ independent $\Rightarrow \Phi(\phi)=B_0$, $k=0$ $\Rightarrow R(r)=C_0\ln r+D_0 \Rightarrow \cdots$

$C_0=$

$D_0=$

4-7 Boundary-Value Problems in Spherical Coordinates

Laplace's equation

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial V}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \cancel{\frac{1}{R^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}} = 0$$

Assuming ϕ independent

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial V}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) = 0.$$

By separation of variables

$$V(R, \theta) = \Gamma(R)\Theta(\theta).$$

$$\frac{1}{\Gamma(R)}\frac{d}{dR}\left[R^2 \frac{d\Gamma(R)}{dR}\right] + \frac{1}{\Theta(\theta)\sin\theta}\frac{d}{d\theta}\left[\sin\theta \frac{d\Theta(\theta)}{d\theta}\right] = 0.$$

To hold for all values of R and θ

$$\frac{1}{\Gamma(R)}\frac{d}{dR}\left[R^2\frac{d\Gamma(R)}{dR}\right] = k^2$$

$$\frac{1}{\Theta(\theta)\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta(\theta)}{d\theta}\right] = -k^2,$$

$$\boxed{V(R, \theta) = \Gamma(R)\Theta(\theta).}$$

$$\boxed{\frac{1}{\Gamma(R)}\frac{d}{dR}\left[R^2\frac{d\Gamma(R)}{dR}\right] = k^2}$$

$$\Downarrow$$

$$R^2\frac{d^2\Gamma(R)}{dR^2} + 2R\frac{d\Gamma(R)}{dR} - k^2\Gamma(R) = 0,$$

Solution: $\Gamma_n(R) = A_n R^n + B_n R^{-(n+1)}.$

where $n(n+1) = k^2,$

n = 0,1,2… is a positive integer

(Verified by direct substitution)

$$\boxed{V(R, \theta) = \Gamma(R)\Theta(\theta).}$$

$$\boxed{\frac{1}{\Theta(\theta)\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta(\theta)}{d\theta}\right] = -k^2,}$$

$$\boxed{n(n+1) = k^2,}$$

$$\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta(\theta)}{d\theta}\right] + n(n+1)\Theta(\theta)\sin\theta = 0,$$ Legendre's equation

Solution: $\Theta_n(\theta) = P_n(\cos\theta).$ Legendre's functions

if involving full range of θ = [0, π]

For integer values of n,

Several Legendre Polynomials

n	$P_n(\cos\theta)$
0	1
1	$\cos\theta$
2	$\frac{1}{2}(3\cos^2\theta - 1)$
3	$\frac{1}{2}(5\cos^3\theta - 3\cos\theta)$

$$\int_0^{2\pi} P_n \cdot P_m \, d\theta = 0$$

$n \neq m$

The Legendre polynomials are **orthogonal**