Chapter 3 Static Electric Fields

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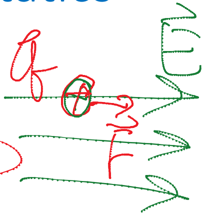
3-1 Introduction

- A field is a specified distribution of a scalar or vector quantity, which may or may not be a function of time.
- In this chapter, we deal with electrostatics, which means electric charges are at rest (not moving), and electric field do not change with time.
- There is no magnetic field existed in this chapter.

3-2 Fundamental Postulates of Electrostatics in Free Space

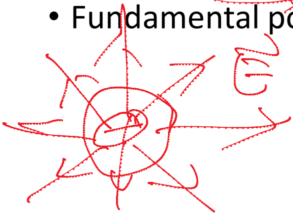


- The simplest case:
  - Static electric charges (source) in free space ➔ electric fields
- Electric field intensity
  - If q is small enough not to disturb the charge distribution of the source,

$$\boxed{\mathbf{E} = \lim_{q \to 0} \frac{\mathbf{F}}{q} \quad (\mathrm{V/m}).}$$

$$\boxed{\mathbf{F} = q\mathbf{E} \quad (\mathrm{N}).}$$

- Fundamental postulates of electrostatics (in free space)



$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}}$$

The charge is a flow source of E field.

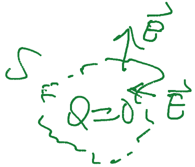
$$\boxed{\nabla \times \mathbf{E} = 0.}$$

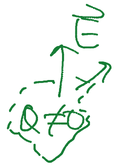
Integral Form

total flux of $\vec{E}$

$\int dv$

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}} \Rightarrow \int_V \nabla \cdot \mathbf{E}\, dv = \frac{1}{\epsilon_0}\int_V \rho\, dv \Rightarrow \boxed{\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0},}$$





A form of Gauss's law: the total outward flux of the electric field intensity over any closed surface in free space is equal to the total charge enclosed in the surface divided by ε_0

Integral Form

$$\nabla \times \mathbf{E} = 0.$$

$\int \hat{n} \, ds$

$E = -\nabla V$

$-\nabla V$

$$\oint_C \mathbf{E} \cdot d\ell = 0.$$

$= V(r_f) - V(r_i)$

$\shortparallel$

0

The scalar line integral of the static electric field intensity around any closed path vanishes.

KVL in circuit theory: the algebraic sum of voltage drops around any closed circuit is zero.

E is Irrotational (Conservative)

$$\boxed{\oint_C \mathbf{E} \cdot d\ell = 0.}$$

$$\int_{C_1} \mathbf{E} \cdot d\ell + \int_{C_2} \mathbf{E} \cdot d\ell = 0$$

$$\underset{\text{Along } C_1}{\int_{P_1}^{P_2}} \mathbf{E} \cdot d\ell = - \underset{\text{Along } C_2}{\int_{P_2}^{P_1}} \mathbf{E} \cdot d\ell$$

$$\underset{\text{Along } C_1}{\int_{P_1}^{P_2}} \mathbf{E} \cdot d\ell = \underset{\text{Along } C_2}{\int_{P_1}^{P_2}} \mathbf{E} \cdot d\ell.$$

The scalar line integral of the irrotational **E** field is *independent of the path*; it depends only on the end points.

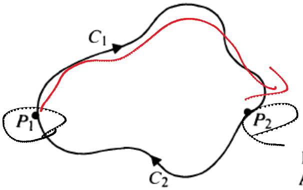


FIGURE 3–1
An arbitrary contour.

Postulates of Electrostatics in Free Space	
Differential Form	Integral Form
$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$	$\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0}$
$\nabla \times \mathbf{E} = 0$	$\oint_C \mathbf{E} \cdot d\ell = 0$

3-3 Coulomb’s Law

- The first person to quantify this law was Coulomb, whose experiment using torsion meters in 1785 demonstrated two things:

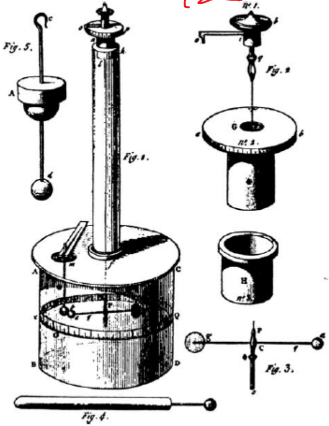
1) The force is proportional to the charges on each object

$|F| \sim q$

2) The force is inversely proportional to the square of distance

$|F| \sim \frac{1}{r^2}$

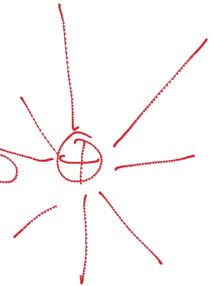
3) The force is a “central body” force (parallel to the position vector)



Coulomb's Law

$R^2 \sin\theta\, d\theta\, d\phi$ $\quad \hat{a}_R \cdot \hat{a}_\theta = 0$

$\hat{a}_R \cdot \hat{a}_\phi = 0$



- The simplest case: a single point charge

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \oint_S (\mathbf{a}_R E_R) \cdot \mathbf{a}_R \, ds = \frac{q}{\epsilon_0}$$

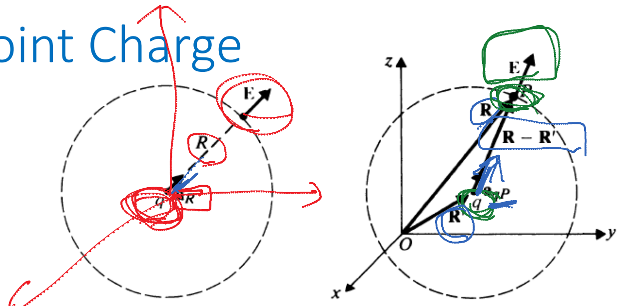
$$E_R \oint_S ds = E_R(4\pi R^2) = \frac{q}{\epsilon_0}.$$

$$\boxed{\mathbf{E} = \mathbf{a}_R E_R = \mathbf{a}_R \frac{q}{4\pi\epsilon_0 R^2} \quad \text{(V/m)}.}$$

The electric field intensity of a positive point charge is in the outward direction and has a magnitude proportional to the charge and inversely proportional to the square of the distance from the charge.

E of a Point Charge

O' : source
O : field



(a) Point charge at the origin. (b) Point charge not at the origin.

FIGURE 3–2

$$\mathbf{E} = \mathbf{a}_R E_R = \mathbf{a}_R \frac{q}{4\pi\epsilon_0 R^2}$$

R* → *R*–*R'

R: field
R': source

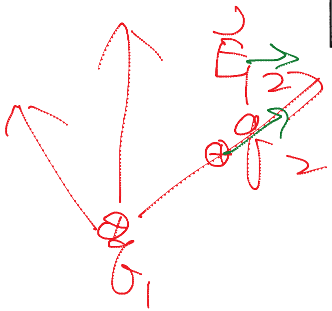
$$\mathbf{E}_P = \mathbf{a}_{qP} \frac{q}{4\pi\epsilon_0 |\mathbf{R} - \mathbf{R}'|^2}$$

$$\mathbf{a}_{qP} = \frac{\mathbf{R} - \mathbf{R}'}{|\mathbf{R} - \mathbf{R}'|}$$

$$\boxed{\mathbf{E}_P = \frac{q(\mathbf{R} - \mathbf{R}')}{4\pi\epsilon_0 |\mathbf{R} - \mathbf{R}'|^3} \qquad \text{(V/m).}}$$

Mathematical Form of Coulomb's Law

- Force on q_2 in an **E** field due to q_1



$$\mathbf{F}_{12} = q_2\mathbf{E}_{12} = \mathbf{a}_R \frac{q_1 q_2}{4\pi\epsilon_0 R^2} \quad \text{(N)}.$$

R_{12}

3-3.1 Electric Field due to a System of Discrete Charges

$$\mathbf{E}_p = \frac{q(\mathbf{R} - \mathbf{R}')}{4\pi\epsilon_0 |\mathbf{R} - \mathbf{R}'|^3} \quad \text{(V/m)}.$$

$$\Downarrow$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \frac{q_k(\mathbf{R} - \mathbf{R}'_k)}{|\mathbf{R} - \mathbf{R}'_k|^3} \quad \text{(V/m)}.$$

An Electric Dipole

far-field : R >> d

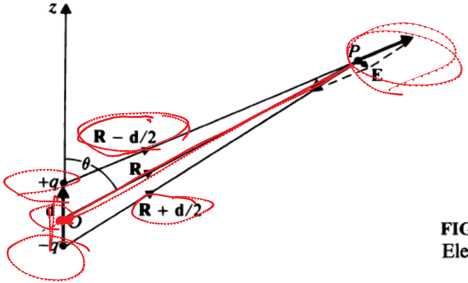


FIGURE 3–5
Electric field of a dipole.

$+q$ $-q$

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left\{\frac{\mathbf{R}-\frac{\mathbf{d}}{2}}{\left|\mathbf{R}-\frac{\mathbf{d}}{2}\right|^3} - \frac{\mathbf{R}+\frac{\mathbf{d}}{2}}{\left|\mathbf{R}+\frac{\mathbf{d}}{2}\right|^3}\right\}.$$

$A^2 = \vec{A}\cdot\vec{A}$

If $d << R$

$$\left|\mathbf{R}-\frac{\mathbf{d}}{2}\right|^{-3} = \left[\left(\mathbf{R}-\frac{\mathbf{d}}{2}\right)\cdot\left(\mathbf{R}-\frac{\mathbf{d}}{2}\right)\right]^{-3/2}$$

$$= \left[R^2 - \mathbf{R}\cdot\mathbf{d} + \frac{d^2}{4}\right]^{-3/2}$$

$\frac{1}{R^2}$

$$\cong R^{-3}\left[1-\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right]^{-3/2}$$

$$\cong R^{-3}\left[1+\frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right],$$

$(1+x)^n \cong 1+nx$

if $x \ll 1$

$$\left|\mathbf{R}-\frac{\mathbf{d}}{2}\right|^{-3} \cong R^{-3}\left[1+\frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right]$$

$$\left|\mathbf{R}+\frac{\mathbf{d}}{2}\right|^{-3} \cong R^{-3}\left[1-\frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right].$$

$$\mathbf{E} \cong \frac{q}{4\pi\epsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\mathbf{R}-\mathbf{d}\right].$$

Electric Dipole Moment

- Definition: The product of the charge q and the vector **d** $\qquad \mathbf{p} = q\mathbf{d}.$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{p}}{R^2}\mathbf{R} - \mathbf{p}\right]$$

$$\mathbf{p} = \mathbf{a}_z p = p(\mathbf{a}_R \cos\theta - \mathbf{a}_\theta \sin\theta),$$

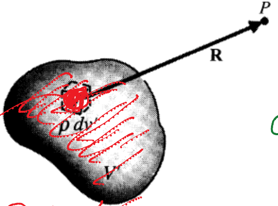
$$\mathbf{R}\cdot\mathbf{p} = Rp\cos\theta,$$

$$\boxed{\mathbf{E} = \frac{p}{4\pi\epsilon_0 R^3}(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta \sin\theta) \quad \text{(V/m)}.}$$

$\oplus$: $\vec{E} \sim \frac{1}{R^2}$

$\oplus$ $\ominus$: $\vec{E} \sim \frac{1}{R^3}$ decays faster than $\vec{E}$ of a single charge. why?

3-3.2 Electric Field due to a Continuous Distribution of Charge



$d\vec{E} = \hat{a}_R \frac{dq}{4\pi\epsilon_0 R^2}$

$dq = \rho\, dv'$

C $C/m^3 \cdot m^3$

FIGURE 3–6
Electric field due to a continuous charge distribution.

$$d\mathbf{E} = \mathbf{a}_R \frac{\rho\, dv'}{4\pi\epsilon_0 R^2}.$$

$\int$ dv'

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_{V'} \mathbf{a}_R \frac{\rho}{R^2}\, dv' \qquad \text{(V/m)},}$$

if ρ in uniform

$\mathbf{a}_R = \mathbf{R}/R$

$\rho(v')$

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_{V'} \rho \frac{\mathbf{R}}{R^3}\, dv' \qquad \text{(V/m)}.}$$

$\Rightarrow \int_{v'} \rho\, dv' = Q$ (total charge)

For a Surface or Line Charge

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 R^2}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{S'} \mathbf{a}_R \frac{\rho_s}{R^2}\, ds' \quad \text{(V/m)}.$$

C

$\rho_s(r')$

$R(r')$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{L'} \mathbf{a}_R \frac{\rho_\ell}{R^2}\, d\ell' \quad \text{(V/m)},$$

C

Vector Calculus

$$\frac{dy}{dx} = \lim_{dx\to 0} \frac{Y(2)-Y(1)}{X}$$

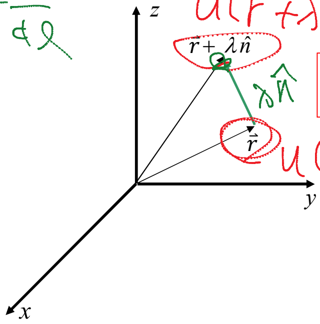
Directional derivative $$\frac{Du(\vec{r})}{Dn} = \lim_{\lambda\to 0}\frac{u(\vec{r}+\lambda\hat{n})-u(\vec{r})}{\lambda}$$

Gradient $$\nabla u(\vec{r})\cdot\hat{n} = \frac{Du(\vec{r})}{Dn} = \lim_{\lambda\to 0}\frac{u(\vec{r}+\lambda\hat{n})-u(\vec{r})}{\lambda}$$

$$\nabla V \cdot d\vec{l} = \frac{dV}{dl}$$

$u(\vec{r}+\lambda n)$

u: scalar fields



$u(\vec{r})$

Derivative along a direction $\hat{n}$ with a distance λ

Very Useful Formulas and Examples!!!!

$|\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}}$

$$\left|\vec{r}+\lambda\hat{n}\right| = \sqrt{r^2+2\lambda\hat{n}\cdot\vec{r}+\lambda^2} = r\sqrt{1+2\frac{\lambda}{r}\hat{n}\cdot\hat{r}+\frac{\lambda^2}{r^2}} \approx r\left(1+\frac{\lambda}{r}\hat{n}\cdot\hat{r}\right)$$

when $\lambda \ll r$

$(1+x)^n \approx 1+nx$ if $x \ll 1$

Example $\nabla r = ?$

$\boxed{u(\vec{r}) = r}$

$$\nabla r\cdot\hat{n} = \lim_{\lambda\to 0}\frac{\left|\vec{r}+\lambda\hat{n}\right|-r}{\lambda} \approx \lim_{\lambda\to 0}\frac{r\left(1+\frac{\lambda}{r}\hat{n}\cdot\hat{r}\right)-r}{\lambda} = \hat{r}\cdot\hat{n}, \quad \nabla r = \hat{r}$$

Example $\nabla\frac{1}{r} = ?$

$\boxed{u(\vec{r}) = \frac{1}{r}}$

$$\nabla\frac{1}{r}\cdot\hat{n} = \lim_{\lambda\to 0}\frac{\frac{1}{\left|\vec{r}+\lambda\hat{n}\right|}-\frac{1}{r}}{\lambda} \approx \frac{1}{r}\lim_{\lambda\to 0}\frac{\frac{1}{\left(1+\frac{\lambda}{r}\hat{n}\cdot\hat{r}\right)}-1}{\lambda} = -\frac{\hat{r}}{r^2}\cdot\hat{n},$$

or $\nabla\frac{1}{r} = -\frac{\hat{r}}{r^2} = -\frac{\vec{r}}{r^3}$ $\quad \hat{r} = \frac{\vec{r}}{r}$

When r is not 0 !!!

Example: Determine the Laplacian of the function $1/r$

$$\nabla^2 \frac{1}{r} = \nabla \cdot \nabla \frac{1}{r} = -\nabla \cdot \frac{\vec{r}}{r^3} = ?$$

A, ψ

$$\nabla \cdot \frac{\vec{r}}{r^3} = \frac{1}{r^3} \nabla \cdot \vec{r} + \vec{r} \cdot \nabla \frac{1}{r^3} = \frac{3}{r^3} - 3\vec{r} \cdot \frac{\vec{r}}{r^5} = 0$$

$$\nabla \frac{1}{r} = -\frac{\vec{r}}{r^3}$$

$$\nabla \cdot (\psi \mathbf{A}) = \psi \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla \psi$$

$$\nabla \frac{1}{r^n} = -n \frac{\hat{r}}{r^{n+1}} = -n \frac{\vec{r}}{r^{n+2}}$$

The above is valid when r ≠ 0. What happens when r = 0???

① $\nabla V(r) = \begin{cases} 0 & \text{when } r \neq 0 \\ ? & \text{when } r = 0 \end{cases}$

Consider volume integral of the function over all space. This integral can be found by integrating over a spherical volume shown in the figure and letting the radius increase to infinity

② $\int V \, dV = 4\pi$

$$\lim_{R\to\infty} \iiint_{V_R} \nabla \cdot \frac{\vec{r}}{r^3} dV = \lim_{R\to\infty} \oiint_{S_R} \frac{\vec{r} \cdot \hat{r}}{r^3} dS = \lim_{R\to\infty} \oiint_{S_R} \frac{1}{r^2} dS = \lim_{R\to\infty} \frac{1}{R^2} 4\pi R^2 = 4\pi$$

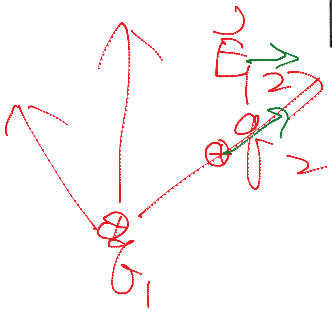
$\Downarrow$

$V(r=0) \neq 0$

Mathematical Form of Coulomb's Law

- Force on q_2 in an **E** field due to q_1

$$\boxed{\mathbf{F}_{12} = q_2\mathbf{E}_{12} = \mathbf{a}_R \frac{q_1 q_2}{4\pi\epsilon_0 R^2} \quad \text{(N)}.}$$



R

3-3.1 Electric Field due to a System of Discrete Charges

$$\boxed{\mathbf{E}_p = \frac{q(\mathbf{R} - \mathbf{R}')}{4\pi\epsilon_0 |\mathbf{R} - \mathbf{R}'|^3} \qquad \text{(V/m).}}$$

$$\Downarrow$$

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \frac{q_k(\mathbf{R} - \mathbf{R}'_k)}{|\mathbf{R} - \mathbf{R}'_k|^3} \qquad \text{(V/m).}}$$

An Electric Dipole

far-field : R >> d

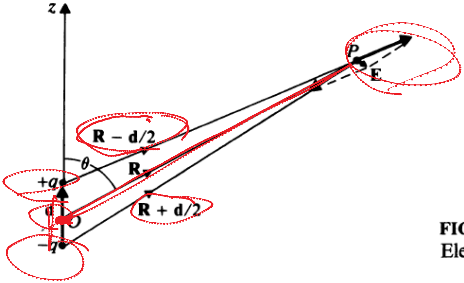


FIGURE 3–5
Electric field of a dipole.

$$A^2 = \vec{A} \cdot \vec{A}$$

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left\{\frac{\mathbf{R} - \frac{\mathbf{d}}{2}}{\left|\mathbf{R} - \frac{\mathbf{d}}{2}\right|^3} - \frac{\mathbf{R} + \frac{\mathbf{d}}{2}}{\left|\mathbf{R} + \frac{\mathbf{d}}{2}\right|^3}\right\}.$$

$+q$ $\quad -q$

If $d << R$

$$\left|\mathbf{R} - \frac{\mathbf{d}}{2}\right|^{-3} = \left[\left(\mathbf{R} - \frac{\mathbf{d}}{2}\right)\cdot\left(\mathbf{R} - \frac{\mathbf{d}}{2}\right)\right]^{-3/2}$$

$$= \left[R^2 - \mathbf{R}\cdot\mathbf{d} + \frac{d^2}{4}\right]^{-3/2}$$

$$\frac{1}{R^2}$$

$$\cong R^{-3}\left[1 - \frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right]^{-3/2}$$

$$\cong R^{-3}\left[1 + \frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right],$$

$(1+x)^n \cong 1 + nx$

if $x \ll 1$

$$\left|\mathbf{R} - \frac{\mathbf{d}}{2}\right|^{-3} \cong R^{-3}\left[1 + \frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right]$$

$$\left|\mathbf{R} + \frac{\mathbf{d}}{2}\right|^{-3} \cong R^{-3}\left[1 - \frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right].$$

$$\mathbf{E} \cong \frac{q}{4\pi\epsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\mathbf{R} - \mathbf{d}\right].$$

Electric Dipole Moment

- Definition: The product of the charge q and the vector $\mathbf{d}$ $\qquad \mathbf{p} = q\mathbf{d}.$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{p}}{R^2}\mathbf{R} - \mathbf{p}\right]$$

$$\mathbf{p} = \mathbf{a}_z p = p(\mathbf{a}_R \cos\theta - \mathbf{a}_\theta \sin\theta),$$

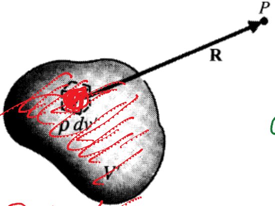
$$\mathbf{R}\cdot\mathbf{p} = Rp\cos\theta,$$

$$\boxed{\mathbf{E} = \frac{p}{4\pi\epsilon_0 R^3}(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta \sin\theta) \quad (\text{V/m}).}$$

$\oplus$: $\vec{E} \sim \frac{1}{R^2}$

$\oplus\ominus$: $\vec{E} \sim \frac{1}{R^3}$ decays faster than $\vec{E}$ of a single charge. why?

3-3.2 Electric Field due to a Continuous Distribution of Charge



$d\vec{E} = \hat{a}_R \dfrac{dQ}{4\pi\epsilon_0 R^2}$

$dq = \rho\, dv'$

C $\quad$ $C/m^3 \cdot m^3$

FIGURE 3–6
Electric field due to a continuous charge distribution.

$$d\mathbf{E} = \mathbf{a}_R \frac{\rho\, dv'}{4\pi\epsilon_0 R^2}.$$

$\int$ $\quad$ dv'

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{V'} \mathbf{a}_R \frac{\rho}{R^2}\, dv' \qquad \text{(V/m)},}$$

$\mathbf{a}_R = \mathbf{R}/R$

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{V'} \rho \frac{\mathbf{R}}{R^3}\, dv' \qquad \text{(V/m)}.}$$

$\rho(v')$

if ρ in uniform

$\Rightarrow \int_{v'} \rho\, dv' = Q$ (total charge)

For a Surface or Line Charge

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 R^2}$$

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{S'} \mathbf{a}_R \frac{\rho_s}{R^2}\, ds' \quad \text{(V/m).}}$$

C

$\rho_s(r)$

$R(r)$

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{L'} \mathbf{a}_R \frac{\rho_\ell}{R^2}\, d\ell' \quad \text{(V/m),}}$$

C

Vector Calculus

$$\frac{dy}{dx} = \lim_{dx\to 0}\frac{y(2)-y(1)}{x}$$

Directional derivative

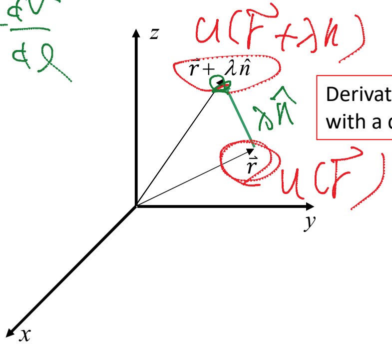
$$\frac{Du(\vec{r})}{Dn} = \lim_{\lambda\to 0}\frac{u(\vec{r}+\lambda\hat{n})-u(\vec{r})}{\lambda}$$

Gradient

$$\nabla u(\vec{r})\cdot\hat{n} = \frac{Du(\vec{r})}{Dn} = \lim_{\lambda\to 0}\frac{u(\vec{r}+\lambda\hat{n})-u(\vec{r})}{\lambda}$$

$$\nabla V\cdot d\vec{l} = \frac{dV}{dl}$$

u : scalar fields



Derivative along a direction $\hat{n}$ with a distance λ

Very Useful Formulas and Examples!!!!

$|\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}}$

$$|\vec{r}+\lambda\hat{n}| = \sqrt{r^2+2\lambda\hat{n}\cdot\vec{r}+\lambda^2} = r\sqrt{1+2\frac{\lambda}{r}\hat{n}\cdot\hat{r}+\frac{\lambda^2}{r^2}} \approx r\left(1+\frac{\lambda}{r}\hat{n}\cdot\hat{r}\right)$$

$(1+x)^n \approx 1+nx$ if $x \ll 1$

when $\lambda \ll r$

Example $\quad \nabla r = ?$ $\qquad \boxed{u(\vec{r}) = r}$

$$\nabla r\cdot\hat{n} = \lim_{\lambda\to 0}\frac{|\vec{r}+\lambda\hat{n}|-r}{\lambda} \approx \lim_{\lambda\to 0}\frac{r\left(1+\frac{\lambda}{r}\hat{n}\cdot\hat{r}\right)-r}{\lambda} = \hat{r}\cdot\hat{n}, \quad \nabla r = \hat{r}$$

Example $\quad \nabla\frac{1}{r} = ?$ $\qquad \boxed{u(\vec{r}) = \frac{1}{r}}$

$$\nabla\frac{1}{r}\cdot\hat{n} = \lim_{\lambda\to 0}\frac{\frac{1}{|\vec{r}+\lambda\hat{n}|}-\frac{1}{r}}{\lambda} \approx \frac{1}{r}\lim_{\lambda\to 0}\frac{\frac{1}{\left(1+\frac{\lambda}{r}\hat{n}\cdot\hat{r}\right)}-1}{\lambda} = -\frac{\hat{r}}{r^2}\cdot\hat{n},$$

or $\quad \nabla\frac{1}{r} = -\frac{\hat{r}}{r^2} = -\frac{\vec{r}}{r^3}$ $\qquad \hat{r} = \frac{\vec{r}}{r}$ $\qquad$ When r is not 0 !!!

Example: Determine the Laplacian of the function $1/r$

$$\nabla^2 \frac{1}{r} = \nabla \cdot \nabla \frac{1}{r} = -\nabla \cdot \frac{\vec{r}}{r^3} = ?$$

A

$$\nabla \cdot \frac{\vec{r}}{r^3} = \frac{1}{r^3}\nabla \cdot \vec{r} + \vec{r} \cdot \nabla \frac{1}{r^3} = \frac{3}{r^3} - 3\vec{r} \cdot \frac{\vec{r}}{r^5} = 0$$

ψ

$$\nabla \frac{1}{r} = -\frac{\vec{r}}{r^3}$$

$$\nabla \cdot (\psi \mathbf{A}) = \psi \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla \psi$$

$$\nabla \frac{1}{r^n} = -n \frac{\hat{r}}{r^{n+1}} = -n \frac{\vec{r}}{r^{n+2}}$$

The above is valid when r ≠ 0. What happens when r = 0???

① $\nabla V(r) = \begin{cases} 0 & \text{when } r \neq 0 \\ ? & \text{when } r = 0 \end{cases}$

Consider volume integral of the function over all space. This integral can be found by integrating over a spherical volume shown in the figure and letting the radius increase to infinity

② $\int_V dV = 4\pi$

$$\lim_{R\to\infty} \iiint_{V_R} \nabla \cdot \frac{\vec{r}}{r^3} dV = \lim_{R\to\infty} \oiint_{S_R} \frac{\vec{r} \cdot \hat{r}}{r^3} dS = \lim_{R\to\infty} \oiint_{S_R} \frac{1}{r^2} dS = \lim_{R\to\infty} \frac{1}{R^2} 4\pi R^2 = 4\pi$$

$\Downarrow$

$V(r=0) \neq 0$

What we have just shown is that the Laplacian of the function $1/r$ is zero everywhere except at the origin where $r = 0$, and yet its volume integral is finite and equal to 4π.

This implies that the Laplacian of $1/r$ is actually a Dirac Delta Function, given by:

$$\nabla^2 \frac{1}{r} = -\nabla \cdot \frac{\vec{r}}{r^3} = -4\pi\delta(\vec{r})$$

$\int \nabla^2 \frac{1}{r} dV$

$= \int -4\pi\delta(\vec{r})\, dV$

$= -4\pi$

Or alternatively, we can say that the function: $\varphi = \dfrac{1}{4\pi r}$

is a solution to the differential equation: $\nabla^2 \varphi = -\delta(\vec{r})$

This result is of fundamental importance in the subject of electrostatic and magnetostatic fields !!!!!

Field

Source

EXAMPLE 3–2 A total charge Q is put on a thin spherical shell of radius b. Determine the electric field intensity at an arbitrary point inside the shell.

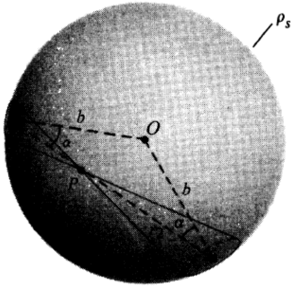


FIGURE 3–3
A charged shell (Example 3–2).

EXAMPLE 3–4 Determine the electric field intensity of an infinitely long, straight, line charge of a uniform density ρ_ℓ in air.

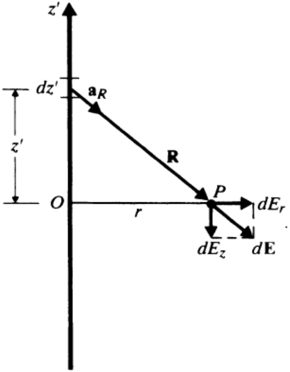


FIGURE 3–7
An infinitely long, straight, line charge.

3-4 Gauss's Law and Applications

Recap of slide#4

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}} \Rightarrow \boxed{\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0}.}$$

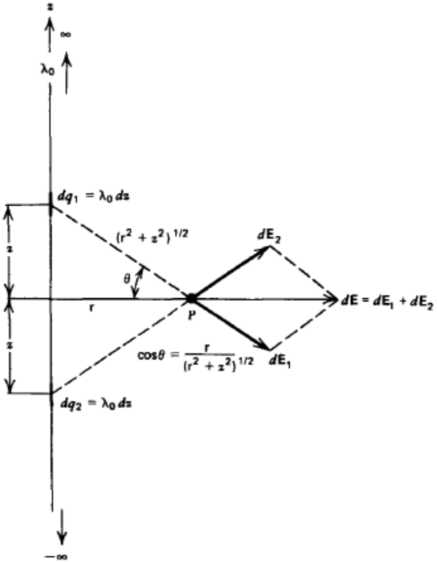
Gauss's law: The total outward flux of the E-field over any closed surface in free space is equal to the total charge enclosed in the surface divided by ε_0

S: can be any hypothetical closed surface

Example 1

- Let us calculate the electric field of an infinitely long line charge in two ways.
- In the first way, we will directly integrate the electrical field of a line charge as a superposition of small segments, *dl*, containing total charge *λdl*.

$$\vec{E}(\vec{r}) = \int_{-\infty}^{+\infty} \frac{\lambda(\vec{r} - z_s\hat{z})}{4\pi\varepsilon_0 \left|\vec{r} - z_s\hat{z}\right|^3} dz_s$$



Let us solve this problem in cylindrical coordinates, therefore

$$\vec{E}(\vec{r}) = \frac{\lambda}{4\pi\varepsilon_0} \int_{-\infty}^{+\infty} \frac{r\hat{r} - z_s\hat{z}}{\sqrt{r^2 + z_s^2}^{\,3}} dz_s$$

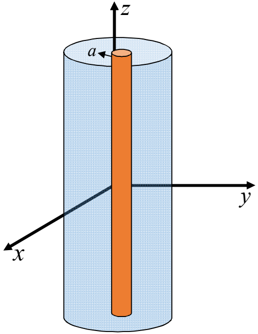
Note that due to symmetry the ***z*** component of the field cancels:

$$\vec{E}(\vec{r}) = \frac{\lambda\hat{r}}{4\pi\varepsilon_0} \int_{0}^{+\infty} \frac{2r}{\sqrt{r^2 + z_s^2}^{\,3}} dz_s$$

$$\vec{E}(\vec{r}) = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$$

Now let's solve this problem using Maxwell's equations…

Let us draw a cylindrical surface around the line charge, where it is assumed to be placed at the center of the cylinder. Due to symmetry, the field evaluated anywhere on the surface must be in the ***r*** direction and must be constant.



$$\oiint_S \vec{E}\cdot\hat{n}\,dS = \frac{Q}{\varepsilon_0}$$

$$\frac{Q}{\varepsilon_0} = \int_{-\infty}^{+\infty}\frac{\lambda}{\varepsilon_0}dz \qquad \oiint_S \vec{E}\cdot\hat{n}\,dS = \oiint_S E_r\hat{r}\cdot\hat{r}r\,d\phi\,dz \;=\int_{-\infty}^{+\infty} 2\pi r E_r\,dz$$

$$\therefore \int_L 2\pi r E_r\,dz = \int_L \frac{\lambda}{\varepsilon_0}dz$$

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$$

Maxwell's equations thus provide a more direct way to obtain the electric field of these highly symmetrical systems.

Where can Maxwell’s Integral Equations be Used?

- Computing electric field by the direct charge integration method will always lead to the correct field, however it can involved tedious integrals and possibly require numerical calculations.
- The integral form of Maxwell’s equations provides a more direct method to obtaining the electrical field of simple charge distributions, however it is only applicable when there is a high degree of symmetry in the charge distribution or in the electrical field (i.e., spherically symmetric, planar, line charge, etc.
- Let’s look at a few examples where the integral form of Maxwell’s equations cannot be used.

Example 2: Field of a Line charge of Finite Length

$\vec{E}$ $\quad$ V

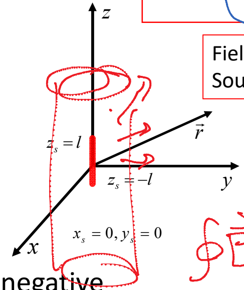
$\vec{E} = -\nabla V$

$$V(\vec{r}) = \int_L \frac{\lambda(\vec{r_s})}{4\pi\varepsilon_0 |\vec{r} - \vec{r_s}|} dl_s$$

Field: (x, y, z)
Source: $(0, 0, z_s)$

$$V(x,y,z) = \int_{-l}^{l} \frac{\lambda}{4\pi\varepsilon_0 \sqrt{x^2 + y^2 + (z - z_s)^2}} dz_s$$

$$V(x,y,z) = \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{z - l + \sqrt{x^2 + y^2 + (z-l)^2}}{z + l + \sqrt{x^2 + y^2 + (z+l)^2}}\right)$$



$$V = \frac{Q}{4\pi\varepsilon_0 r}$$

$\oint \vec{E} \cdot d\vec{S}$

The electric field can then be computed from the negative gradient of this potential.

$$V = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho_l \, dl}{r}$$

$$\vec{E} = -\nabla \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{z - l + \sqrt{x^2 + y^2 + (z-l)^2}}{z + l + \sqrt{x^2 + y^2 + (z+l)^2}}\right)$$

$\vec{E}$ is not constant through a surface

$\Rightarrow$ calculation is complicated.

Example 3: Field of a Charged Ring

Consider a ring with radius *a* and uniform charge density with a center at the origin and located in the xy- plane.

$$V(x,y,z)=\frac{\lambda a}{4\pi\varepsilon_0}\oint\frac{d\theta}{\sqrt{(x-a\cos\theta)^2+(y-a\sin\theta)^2+z^2}}$$

$$\boxed{V(\vec{r}) = \int_L \frac{\lambda(\vec{r_s})}{4\pi\varepsilon_0|\vec{r}-\vec{r_s}|}dl_s}$$

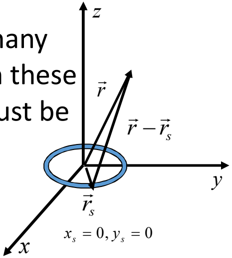
Field: (*x*, *y*, *z*)
Source: (*a*cos θ, *a*sin θ, 0)

V

$\Downarrow$

$\vec{E}$

This integral is more difficult to solve, and in many cases they don't have an analytical solution. In these and other examples, the potential and field must be computed numerically on a computer:

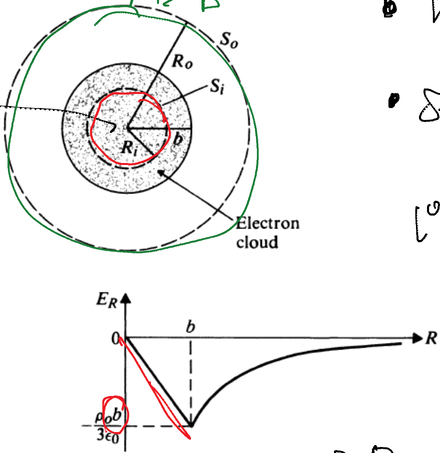


$$\vec{E}=-\nabla\frac{\lambda a}{4\pi\varepsilon_0}\oint\frac{d\theta}{\sqrt{(x-a\cos\theta)^2+(y-a\sin\theta)^2+z^2}}$$

EXAMPLE 3–7 Determine the **E** field caused by a spherical cloud of electrons with a volume charge density $\rho = -\rho_o$ for $0 \le R \le b$ (both ρ_o and b are positive) and $\rho = 0$ for $R > b$.

$R \geq b$

$\rho = -\rho_o$



- highly symmetric $\Rightarrow$ Gauss's law
- spherical symmetry $\Rightarrow$ spherical coor.

1° $0 \le R \le b$

E_R only

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q}{\epsilon_0}$$

$$\oint E_R \hat{a}_R \cdot \hat{a}_R R^2 \sin\theta \, d\theta \, d\phi = \frac{\left(\frac{4}{3}\pi R^3\right)(-\rho_o)}{\epsilon_0}$$

$$\Rightarrow E_R \cdot 4\pi R^2 =$$

$$\Rightarrow E_R = -\frac{\rho_o}{3\epsilon_0} R$$

FIGURE 3–10
Electric field intensity of a spherical electron cloud (Example 3–7).

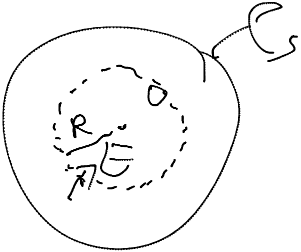
2° $R > b$

$$E_R 4\pi R^2 = \frac{\left(\frac{4}{3}\pi b^3\right)(-\rho_0)}{\epsilon_0}$$

$\Rightarrow \ldots$

$$\to E_R = \frac{-\rho_0 b^3}{3\epsilon_0 R^2} \propto \frac{1}{R^2}$$

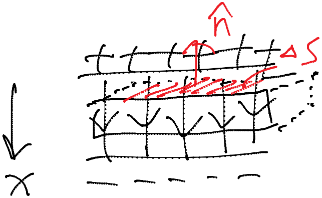
Ex3-2



$E_R(R)$ only

the same direction (either inward or outward)

$4\pi R^2 \cdot E_R\big|_R = 0 \Rightarrow E_R\big|_R = 0$



$\oint \vec{E} \cdot d\vec{S} = \frac{Q}{\epsilon_0}$

$-E_x \cdot \Delta S + E_x \Delta S = 0$

$\Delta S \neq 0$ and $E_x \neq 0$

3-5 Electric Potential

$$\nabla \times \mathbf{E} = 0 \quad \Rightarrow \quad \boxed{\mathbf{E} = -\nabla V}$$

Scalar quantities are easier to handle than vector quantities

Physical meaning: Work done in carrying a charge from one point to another

$$V_{21} \equiv \frac{W}{q} = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\boldsymbol{\ell} \qquad \text{(J/C or V).}$$

$\int \vec{F} \cdot d\vec{\ell}$

$\boxed{\mathbf{E} = \mathbf{F}/q}$

Work is done against the field

$V = \frac{W}{q}$

$gh = \frac{W}{m}$

$\int \nabla V \cdot d\ell$

path independent

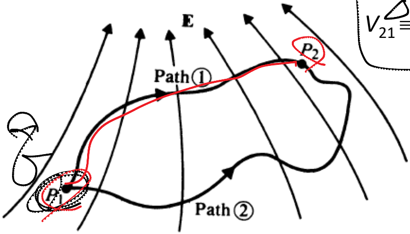


FIGURE 3–11
Two paths leading from P_1 to P_2 in an electric field.

Electrical Potential Difference

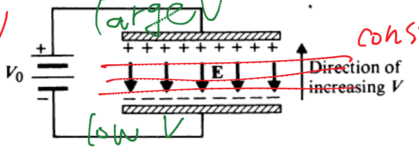
$$V_2 - V_1 = V_{21} = -\int_{P_1}^{P_2} \mathbf{E}\cdot d\ell = \int_{P_1}^{P_2} (\nabla V)\cdot(\mathbf{a}_\ell\, d\ell)$$
$$= \int_{P_1}^{P_2} dV = V_2 - V_1.$$

Recap: $\nabla V \cdot \mathbf{a}_\ell = dV/d\ell$

$$\boxed{V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E}\cdot d\ell \qquad \text{(V).}}$$

Usually, the zero-potential point is taken at infinity (P_1).

$\vec{E} = -\nabla V$



constant V

E and increasing V are in opposite direction

FIGURE 3–12
Relative directions of E and increasing V.

In going against the **E** field, the electric potential V increases.

$$\boxed{\mathbf{E} = -\nabla V}$$

$\Downarrow$

E $\perp$ constant-V surfaces

Field lines
Streamlines

Equipotential lines
Equipotential surfaces

3-5.1 Electric Potential due to a Charge Distribution

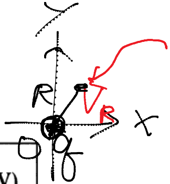
- $V(R)$ of a point charge at origin

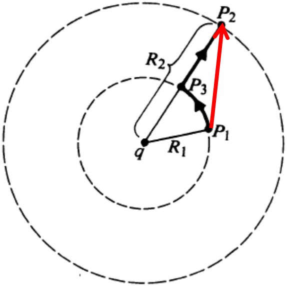
$\Phi(\infty)$ $V_\infty = 0$

$$V_R - V_\infty = V = -\int_{\infty}^{R}\left(\mathbf{a}_R \frac{q}{4\pi\epsilon_0 R^2}\right)\cdot(\mathbf{a}_R\, dR), \qquad \vec{E}\cdot d\vec{l}$$

With reference point at infinity

$$\boxed{V = \frac{q}{4\pi\epsilon_0 R} \quad \text{(V)}.}$$





- Potential difference between any two points

$$V_{21} = V_{P_2} - V_{P_1} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R_2} - \frac{1}{R_1}\right).$$

A charge is moved from P_1 to P_2.

$$V_{21} = V_{31} + V_{23}$$

$\parallel$
0

V due to n Discrete Point Charges

$$V = \frac{q}{4\pi\epsilon_0 R} \quad \text{(V)}.$$

Reference point at infinity

R ➔ ***R* – *R'*** (Charges located at ***R'***)

∑

$$V = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \frac{q_k}{|\mathbf{R} - \mathbf{R}'_k|} \quad \text{(V)}.$$

V of an Electric Dipole

V

$\Downarrow$

$\vec{E}$

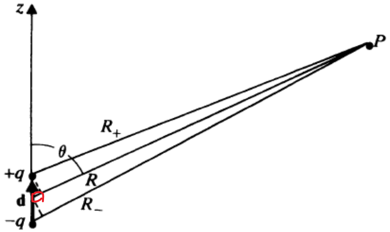


FIGURE 3–14
An electric dipole.

If $d \ll R$

$$V = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right).$$

$$\frac{1}{R_+} \cong \left(R - \frac{d}{2}\cos\theta\right)^{-1} \cong R^{-1}\left(1 + \frac{d}{2R}\cos\theta\right)$$

$$\frac{1}{R_-} \cong \left(R + \frac{d}{2}\cos\theta\right)^{-1} \cong R^{-1}\left(1 - \frac{d}{2R}\cos\theta\right).$$

$$V = \frac{qd \cos \theta}{4\pi\epsilon_0 R^2}$$

or

$$\boxed{V = \frac{\mathbf{p} \cdot \mathbf{a}_R}{4\pi\epsilon_0 R^2} \quad \text{(V)},}$$

where $\mathbf{p} = q\mathbf{d}$.

$$\mathbf{E} = -\nabla V = -\mathbf{a}_R \frac{\partial V}{\partial R} - \mathbf{a}_\theta \frac{\partial V}{R\,\partial \theta}$$

$$= \frac{p}{4\pi\epsilon_0 R^3}(\mathbf{a}_R 2 \cos \theta + \mathbf{a}_\theta \sin \theta).$$

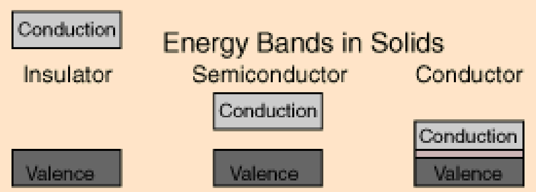
A much simpler approach (vs. slide#15)

3-6 Conductors in Static Electric Field

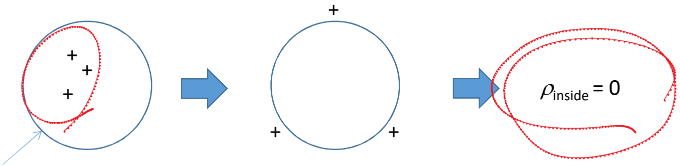
- 3 types: conductors, semiconductors, insulators (or dielectrics)
- Conductors: Orbiting electrons are loosely held by an atom and migrate easily from one atom to another.
- Insulators: Electrons confined to their orbits
- Semiconductors: A small number of freely moveable charges (between conductors and insulators)

Band theory

- Crucial to the conduction process is whether or not there are electrons in the conduction band

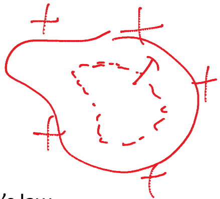


E and ρ inside a Conductor



Inside a Conductor (Under Static Conditions)
$\rho = 0$
$\mathbf{E} = 0$

By Gauss's law

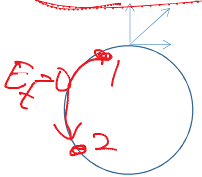


applies to any shape of conductors.

Equilibrium

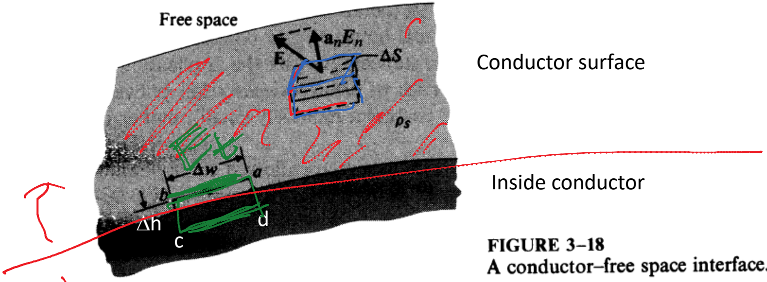
- At a state of equilibrium (static charges), tangential **E** = 0. Otherwise, charges move.

$$V_{21} = -\int \vec{E} \cdot d\vec{l} = 0$$



Boundary Conditions at a Conductor/Free Space Interface
$E_t = 0$
$E_n = \dfrac{\rho_s}{\epsilon_0}$

- Under static conditions,
  - **E** field on a conductor surface is everywhere normal to the surface.
  - The surface of a conductor is an equipotential surface.



Conductor surface

Inside conductor

FIGURE 3–18
A conductor–free space interface.

$\nabla \times \vec{E} = 0 \Rightarrow$

$$\oint_{abcda} \mathbf{E} \cdot d\boldsymbol{\ell} = E_t \Delta w = 0$$

$$\int_{ab} \vec{E} \cdot d\vec{\ell} + \int_{bc} \vec{E} \cdot d\vec{\ell} + \int_{cd} \vec{E} \cdot d\vec{\ell} + \int_{da} \vec{E} \cdot d\vec{\ell}$$

0   0   0

$E_{inside} = 0$ $\Delta h = 0$

(1) Let Δh → 0 ➔ Integrals along bc, da = 0
(2) E_{inside} = 0 ➔ Integral along cd = 0

$$E_t = 0,$$

The tangential component of the E field on a conductor surface is zero.

Normal Component of E

6 faces

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = E_n \Delta S = \frac{\rho_s \Delta S}{\epsilon_0}$$

Q

6 faces:
(1) Let $\Delta h \rightarrow 0$ ➔ Integrals over 4 side surfaces = 0
(2) $E_{inside}=0$ ➔ Integral over the bottom surface = 0

$$E_n = \frac{\rho_s}{\epsilon_0}.$$

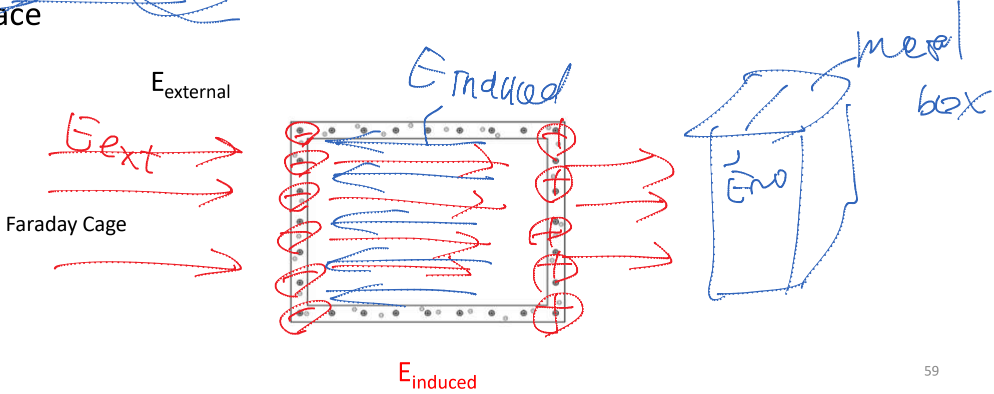
$\rho_s \uparrow \Rightarrow E_n \uparrow$

$\rho_s \downarrow$

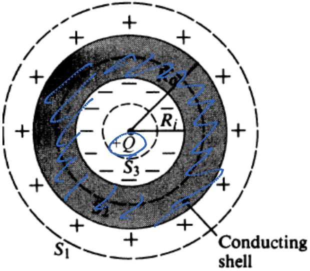
The normal component of the E field at a conductor/free space boundary is equal to the surface charge density on the conductor divided by the permittivity of free space.

An Uncharged Conductor in a Static **E** Field

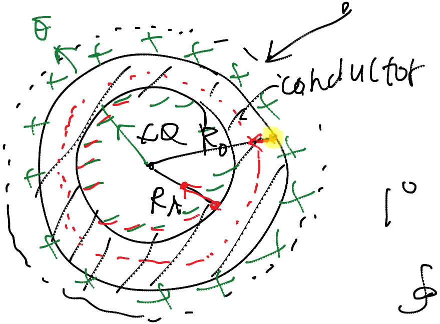
- $E_{external}$ ➔ Electrons moving ➔ $E_{induced}$
- $E_{induced}$ cancels $E_{external}$ both inside the conductor and tangent to its surface



EXAMPLE 3–11 A positive point charge Q is at the center of a spherical conducting shell of an inner radius R_i and an outer radius R_o. Determine $\mathbf{E}$ and V as functions of the radial distance R.



(a)



$\vec{E}$? V?

guess: $E_R(R)$ only

1° $R > R_o$

$\oint \vec{E}\cdot d\vec{s} = \frac{Q}{\epsilon_0}$

$E_{R1}\cdot 4\pi R^2 = \frac{Q}{\epsilon_0}$

$\Rightarrow E_{R1} = \frac{Q}{4\pi\epsilon_0 R^2}$

$V_{R\infty} = -\int_\infty^R E_R\cdot dR = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R}\right)\Big|_\infty^R$

$\hat{a}_R\cdot\hat{a}_R = 1$

$= \frac{Q}{4\pi\epsilon_0 R}$

2° $R_i < R < R_o$ $\quad E_{R2} = 0$

$V_{R_2R_o} = V_{R_2} - V_{R_o} = -\int E_{R2}\cdot dR = 0$ $\quad V_{R2} = V_{R_o} = V_{R_o} = \frac{Q}{4\pi\epsilon_0 R_o}$

3° $R < R_i$ $\quad E_{R3} = \frac{Q}{4\pi\epsilon_0 R^2}$

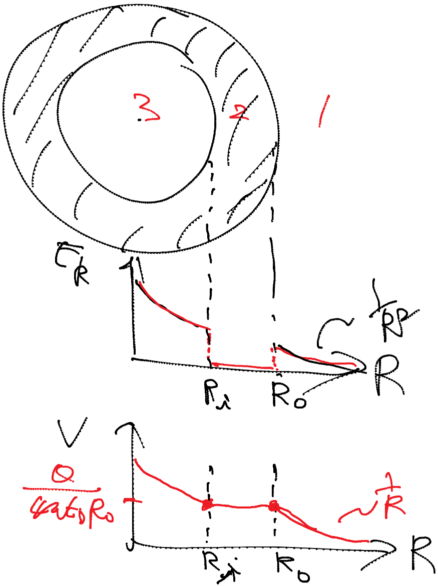
$V_{R_3R_2@R_i} = V_{R_3} - V_{R_2@R_i}$

$= -\int_{R_i}^{R} E_{R_3}\cdot dR = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R}\right)\Big|_{R_i}^{R}$

$= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R} - \frac{1}{R_i}\right)$

$V_{R_3} = V_{R_2@R_i} + \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R} - \frac{1}{R_i}\right)$

$= \frac{Q}{4\pi\epsilon_0 R_o} + \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R} - \frac{1}{R_i}\right)$ ※



$$E_{R_1} = R_3 = \frac{Q}{4\pi\varepsilon_0 R^2} \sim \frac{1}{R^2}$$

$$\vec{E} = -\nabla V$$

if V is discontinuous, $\nabla V \to \infty$, which is not allowed.