VE230 Electromagnetics

Chapter 7

September 21, 2022

Exercise 7.1

Express the transformer emf induced in a stationary loop in terms of time-varying vector potential A.

Answer:

$$V = -\int_S \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s} = -\int_S \frac{\partial}{\partial t}(\bar{\nabla} \times \bar{A}) \cdot d\bar{s} = -\oint_C \frac{\partial \bar{A}}{\partial t} \cdot d\bar{l}.$$

Exercise 7.2

The circuit in Fig. 7-10 is situated in a magnetic field

$$\mathbf{B} = \mathbf{a}_z 3 \cos\left(5\pi 10^7 t - \frac{2}{3}\pi x\right)$$

Assuming $R = 15(\Omega)$, find the current i.

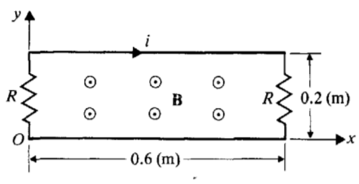


FIGURE 7–10
A circuit in a time-varying magnetic field (Problem P.7–2).

Answer:

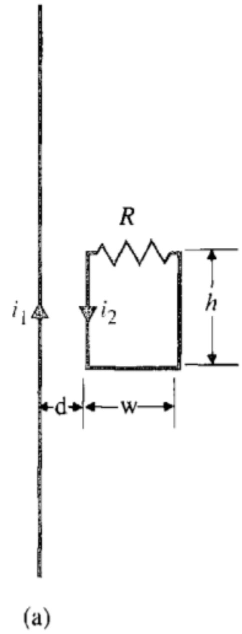
$$\begin{aligned}
&\bar{B} = \bar{a}_z 3\cos\left(5\pi 10^7 t - \frac{2}{3}\pi x\right) \cdot 10^{-6}(\pi),\\
&\int_S \bar{B} \cdot d\bar{s} = \int_0^{0.6} \bar{a}_z 3\cos\left(5\pi 10^7 t - \frac{2}{3}\pi x\right) 10^{-6} \cdot (\bar{a}_z 0.2 dx)\\
&= -\frac{0.18}{2\pi}\left[\sin\left(5\pi 10^7 t - 0.4\pi\right) - \sin 5\pi 10^7 t\right] \cdot 10^{-6}(Wb),\\
&\alpha = -\frac{d}{dt}\int \bar{B} \cdot d\bar{s} = 45\left[\cos\left(5\pi 10^7 t - 0.4\pi\right) - \cos 5\pi 10^2 t\right] (\mathrm{V}),\\
&i = \frac{V}{2R} = 1.5\left[\cos\left(5\pi 10^7 t - 0.4 \cdot \pi\right) - \cos 5\pi 10^7 t\right]\\
&= 1.76 \sin\left(5\pi 10^7 t - 0.2\pi\right) \ (\mathrm{A}).
\end{aligned}$$

Exercise 7.3

A rectangular loop of width w and height h is situated near a very long wire carrying a current i_1 as in Fig. 7-11(a). Assume i_1 to be a rectangular pulse as shown in Fig. 7-11(b).

a) Find the induced current i_2 in the rectangular loop whose self-inductance is L.

b) Find the energy dissipated in the resistance R if $T \gg L/R$.



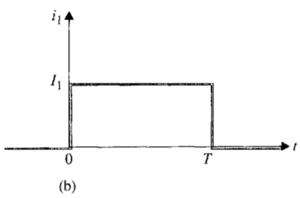


FIGURE 7–11
A rectangular loop near a long current-carrying wire (Problem P.7–3).

Answer:

In the rectangular loop with the assigned direction for i_2

$$L_{12}\frac{di_1}{dt} = L\frac{di_2}{dt} + Ri_2$$

where

$$L_{12} = \frac{\Phi_{i_2}}{i_2} = \frac{h}{i_2}\int_d^{d+w_1} B_{12}dr = \frac{h_1}{i_2}\int_d^{d+w} \frac{\mu_0 i_1}{2\pi r}dr$$

$$= \frac{\mu_{Gh}}{2\pi}\ln\left(1+\frac{\omega_v}{d}\right).$$

a) At $t = 0, i, (t) = I, L(t)$ is applied and (1) becomes

$$L\frac{di_2}{dt} + Ri_2 = L_{r/2}I, \delta(t)$$

Solution of (3):

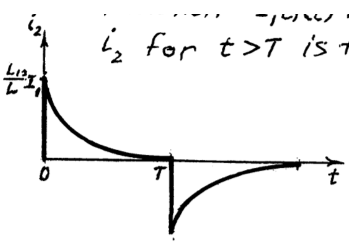
$$i_2 = \frac{L_{12}}{L} I_1 e^{-(R/L)t}, 0 < t < T$$

At $t = T, c_2 = \frac{L_{12}}{L} I_1 e^{-RT/L}$, when a negative step function $-I, L(t)$ is applied. If $T \gg L/R$, then i_2 for $t > T$ is the reverse of in far $0 < t < T$;

$$\text{i.e.,} \quad i_2 = -\frac{L_{12}}{L} I, e^{-(R/L)(t-T)}, t > T.$$

b) Energy dissipated in R,

$$\begin{aligned} W &\cong 2 \left(\frac{L_{12}}{L} I_1 \right)^2 R \int_0^\infty e^{-(2R/L)t} dt \\ &= \frac{1}{L} \left(L_{12} I_1 \right)^2. \end{aligned}$$



Exercise 7.4

A conducting equilateral triangular loop is placed near a very long straight wire, shown in Fig. 6-48, with $d = b/2$. A current $i(t) = I \sin \omega t$ flows in the straight wire.

a) Determine the voltage registered by a high-impedance rms voltmeter inserted in the loop.

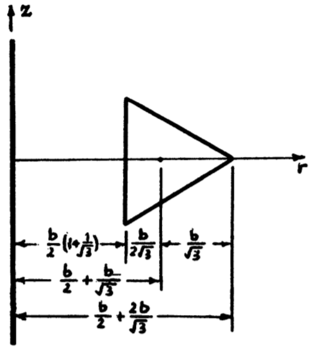
b) Determine the voltmeter reading when the triangular loop is rotated by 60° about a perpendicular axis through its center.

Answer:

$$\bar{B} = \bar{a}_\phi \frac{\mu_0 I \sin \omega t}{2\pi r}, \quad \Phi = \int_S \bar{B} \cdot d\bar{s}, \quad d\bar{s} = a_\phi 2z dr, \quad z = \frac{\sqrt{3}}{3}(r - d)$$

$$\begin{aligned} d = \frac{b}{2}, \quad V &= -\frac{\partial \Phi}{\partial t} = -\frac{\sqrt{3}\mu_0 I \omega b}{3\pi} \left[\frac{\sqrt{3}}{2} - \frac{1}{2} \ln\left(\sqrt{3} + 1\right) \right] \cos \omega t \\ &= V_m \cos \omega t \end{aligned}$$

$$\begin{aligned} V_{rms} &= \frac{\sqrt{2}}{2} |V_m| = \frac{\sqrt{6}\mu_0 I \omega b}{12\pi} [\sqrt{3} - \ln\left(\sqrt{3} + 1\right)] \\ &= 0.0472 \mu_0 I \omega b (V). \end{aligned}$$



b) $\quad z = \frac{1}{\sqrt{3}}\left[\frac{b}{2}\left(1+\frac{4}{\sqrt{3}}\right) - r\right];$

$$\begin{aligned}
\int \bar{B}\cdot d\bar{s} &= \frac{\mu_0 I \sin\omega t}{\sqrt{3}\pi}\int_{\frac{b}{2}\left(1+\frac{1}{\sqrt{3}}\right)}^{\frac{b}{2}\left(1+\frac{4}{\sqrt{3}}\right)}\left[\frac{b}{2}\left(1+\frac{4}{\sqrt{3}}\right)\frac{1}{r} - 1\right] dr \\
&= \frac{\mu_0 I \sin\omega t}{\sqrt{3}\pi}\left[\frac{b}{2}\left(1+\frac{4}{\sqrt{3}}\right)\ln\left(\frac{4+\sqrt{3}}{1+\sqrt{3}}\right) - \frac{\sqrt{3}}{2}b\right]. \\
V_{rms} &= \frac{1}{\sqrt{2}}\frac{\mu_0 I\omega}{\sqrt{3}\pi}\frac{b}{2}\left|\left(1+\frac{4}{\sqrt{3}}\right)\ln\left(\frac{4+\sqrt{3}}{1+\sqrt{3}}\right) - \sqrt{3}\right| \\
&= 0.0469\mu_0 I\omega b \quad (\mathrm{V}).
\end{aligned}$$

Exercise 7.5

A conducting circular loop of a radius 0.1(m) is situated in the neighborhood of a very long power line carrying a 60 – (Hz) current, as shown in Fig. 6-49, with $d = 0.15(\mathrm{~m})$. An a-c milliammeter inserted in the loop reads 0.3(mA). Assume the total impedance of the loop including the milliammeter to be 0.01(Ω).

a) Find the magnitude of the current in the power line.

b) To what angle about the horizontal axis should the circular loop be rotated in order to reduce the milliammeter reading to 0.2(mA) ?

Answer:

From Problem P.6-40: $\Phi_{12} = \mu_0 I(\sin\omega t)\left(d - \sqrt{d^2-b^2}\right)$.

a) $V = -\frac{d\Phi}{dt} = -\mu_0 I\omega(\cos\omega t)\left(d-\sqrt{d^2-b^2}\right) = V_m\cos\omega t,$

$$3\times 10^{-4} = \frac{|V_m|}{\sqrt{2}R} = \frac{\mu_0 I\omega\left(d-\sqrt{d^2-b^2}\right)}{\sqrt{2}R}.$$

$$I = \frac{\sqrt{2}R\times 3\times 10^{-4}}{\mu_0\omega\left(d-\sqrt{d^2-b^2}\right)} = \frac{3\sqrt{2}\times 10^{-6}}{4\pi 10^{-7}(2\pi 60)\times 0.0382} = 0.234(A).$$

b)

$$\alpha = \cos^{-1}(0.2/0.3) = 48.2^\circ.$$

Exercise 7.6

A suggested scheme for reducing eddy-current power loss in transformer cores with a circular cross section is to divide the cores into a large number of small insulated filamentary parts. As illustrated in Fig. 7-12, the section shown in part (a) is replaced by that in part (b). Assuming that $B(t) = B_0 \sin \omega t$ and that N filamentary areas fill 95% of the original cross-sectional area, find

a) the average eddy-current power loss in the section of core of height h in Fig. 7-12(a),

b) the total average eddy-current power loss in the N filamentary sections in Fig. 7-12(b).

The magnetic field due to eddy currents is assumed to be negligible. (Hint: First find the current and power dissipated in the differential circular ring section of height h and width dr at radius r.)

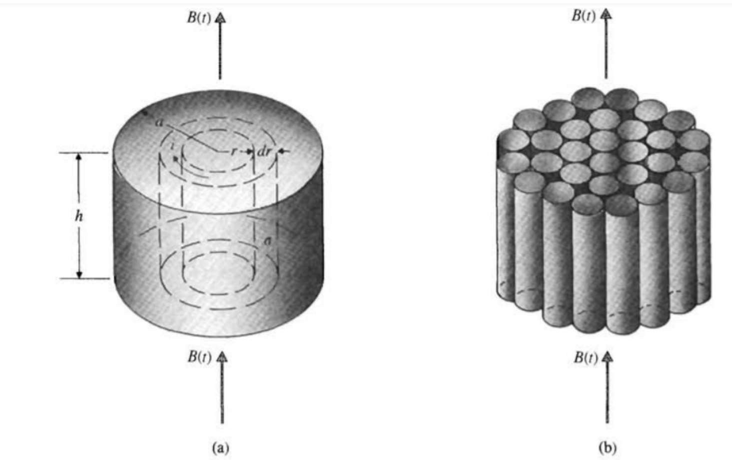


FIGURE 7–12
Suggested eddy-current power-loss reduction scheme (Problem P.7–6).

Answer:

a) Flux enclosed in the ring in Fig. 7-12(a): $\Phi = \pi r^2 B(t)$. The induced emf in the ring referring to the assigned direction for current: $V = iR_r = \frac{d\Phi}{dt} = \pi r^2 \frac{dB(t)}{dt}$.

Resistance of differential circular ring: $R_r = \frac{2\pi r}{\sigma h dr}$. Combining (2) and (3):

$$i = \frac{\pi r^2}{R_r}\frac{dB(t)}{dt} = \frac{\sigma h}{2} r dr \left(\frac{dB}{dt}\right)^2$$

$$dp = i^2 R_r = \frac{\pi \sigma h}{2} r^3 dr \left(\frac{dB}{dt}\right)^2.$$

$$p = \int dp = \frac{\pi\sigma h}{8} a^4 \left(\frac{dB}{dt}\right)^2 = \frac{\pi \sigma h}{8} a^4 \omega^2 B_0^2 \cos^2 \omega t.$$

$$P_{av} = \frac{\pi\sigma h}{16} a^4 \omega^2 B_0^2.$$

b) For N insulated filamentary parts, each with an area

$$S = \frac{0.95\pi a^2}{N} = \pi b^2 \longrightarrow b = \sqrt{\frac{S}{N}} = a\sqrt{\frac{0.95}{N}}.$$

Power loss in N filaments in Fig. $7-12(b)$ $\quad P' = N\left(\frac{\pi\sigma h}{8}\right)\left(a\sqrt{\frac{0.95}{N}}\right)^4 \omega^2 B_0^2 \cos^2 \omega t = \frac{0.95^2}{N} P.$ from (6)

$$P'_{av} = \frac{0.95^2}{N} P_{av}$$

Exercise 7.7

A conducting sliding bar oscillates over two parallel conducting rails in a sinusoidally varying magnetic field

$$\mathbf{B} = \mathbf{a}_z 5 \cos \omega t \quad (\mathrm{mT})$$

as shown in Fig. 7-13. The position of the sliding bar is given by $x = 0.35(1 - \cos \omega t)(\mathrm{m})$, and the rails are terminated in a resistance $R = 0.2(\Omega)$. Find i.

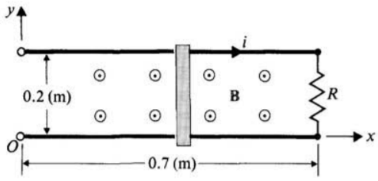


FIGURE 7–13
A conducting bar sliding over parallel rails in a time-varying magnetic field (Problem P.7–7).

Answer:

$$\begin{aligned}
&\Phi(t) = \bar{B}(t) \cdot \bar{s}(t) = -(5 \cos \omega t) \times 0.2(0.7 - x) \\
&= -0.35 \cos \omega t(1 + \cos \omega t) \quad (mT), \\
&L' = -\frac{1}{R}\frac{d\Phi}{dt} = -\frac{1}{R} 0.35\omega(\sin \omega t + \sin 2\omega t) \\
&= -1.75\omega \sin \omega t(1 + 2\cos \omega t)(mA).
\end{aligned}$$

Exercise 7.8

In the d-c motor illustrated in Fig. 6-32 we noted that a current I sent through the loop in a magnetic field $\mathbf{B}$ produces a torque that makes the loop rotate. As loop rotates, the amount of the magnetic flux linking with the loop changes, giving rise to an induced emf. Energy must be expended by an external electric source to counter this emf and establish the current in the loop. Prove that this electric energy is equal to the mechanical work done by the rotating loop. (Hint: Consider the normal of the loop at an arbitary angle α with $\mathbf{B}$, and let it rotate by an angle $\Delta\alpha$.)

Answer:

Assuming the loop to have N turns each with an area $a \times b$, the torque on the loop is $\bar{T} = \bar{m} \times \bar{B} = -\bar{a}_x NabIB \sin \alpha$.

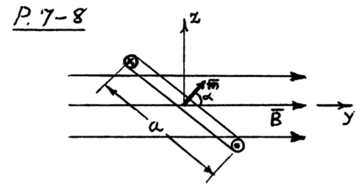
Mechanical work done by the motor in rotating through an angle $(-\Delta\alpha)$ is

$$W_m = T(-\Delta\alpha) = NabIB(\Delta\alpha) \sin \alpha.$$

Flux linking with the loop, $\Phi = NabB \cos \alpha$.

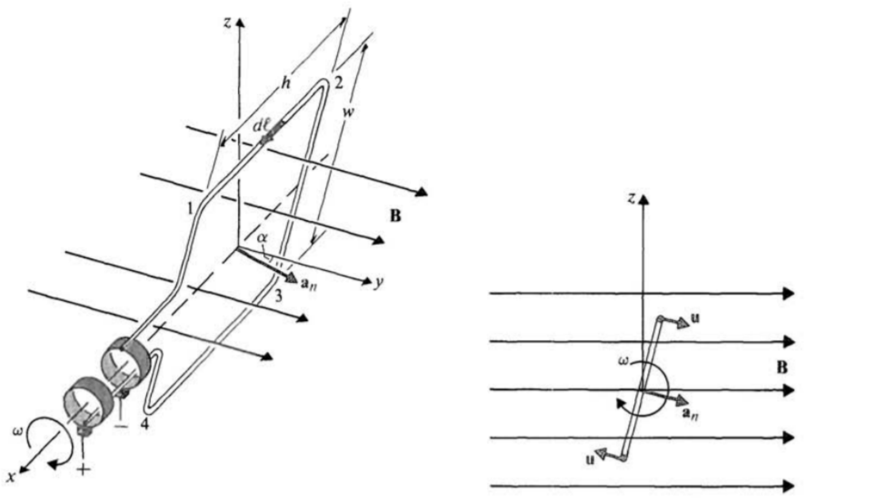
Emf induced in the loop, $\mathscr{V} = -\frac{d\Phi}{dt} = NabB \left(\frac{\Delta\alpha}{\Delta t}\right) \sin \alpha$.

Electric energy required to send current I against this emf in time $\Delta t, W_e = \mathscr{V} I(\Delta t) = NabB(\Delta\alpha) \sin \alpha = W_m$.



Exercise 7.9

Assuming that a resistance R is connected across the slip rings of the rectangular conducting loop that rotates in a constant magnetic field $\mathbf{B} = \mathbf{a}_y B_0$, shown in Fig. 7-6, prove that the power dissipated in R is equal to the power required to rotate the loop at an angular frequency ω.



(a) Perspective view. (b) View from $+x$ direction.

FIGURE 7–6
A rectangular conducting loop rotating in a changing magnetic field (Example 7–4).

Answer:

$$\begin{aligned} i &= \frac{\mathscr{V}}{R} = -\frac{1}{R}\frac{d}{dt}\int_s \bar{B}\cdot d\bar{s} = -\frac{1}{R}\frac{d}{dt}\left(B_0 h\omega \cos\omega t\right) \\ &= \frac{\omega B_0 h w}{R}\sin\omega t, \\ P_d = i^2 R &= \frac{\left(\omega B_0 h\omega\right)^2}{R}\sin^2\omega t \quad \text{(Power dissipated in } R\text{)} \end{aligned}$$

On the other hand,
for side 1-2: $\bar{F}_{12} = \bar{a}_z i h B_0, \bar{u}_{12} = \frac{\omega w}{2}\left(\bar{a}_y \cos\omega t - \bar{a}_z \sin\omega t\right)$;
for side 4-3: $\bar{F}_{43} = -\bar{a}_z i h B_0, \bar{u}_{43} = \frac{\omega w}{2}\left(-\bar{a}_y \cos\omega t + \bar{a}_z \sin\omega t\right)$.
Mechanical power required to rotate coil:
$p_m = -\left(\bar{F}_{12}\cdot\bar{u}_{12} + \bar{F}_{43}\cdot\bar{u}_{43}\right) = \omega B_0 h w i \sin\omega t = p_d$.
(Alternatively, $\bar{F}_m = \bar{T}\cdot\bar{\omega}$; where $\bar{T} = -\bar{a}_x\left(ihB_0\right)\omega\sin\omega t$, and $\bar{\omega} = -\bar{a}_x\omega$)

Exercise 7.10

A hollow cylindrical magnet with inner radius a and outer radius b rotates about its axis at an angular frequency ω. The magnet has a uniform axial magnetization $\mathbf{M} = \mathbf{a}_z M_0$. Sliding brush contacts are provided at the inner and outer surfaces as shown in Fig. 7-14. Assuming that $\mu_r = 5000$ and $\sigma = 10^7$(S/m) for the magnet, find

a) $\mathbf{H}$ and $\mathbf{B}$ in the magnet,

b) open-circuit voltage V_0,

c) short-circuit current.

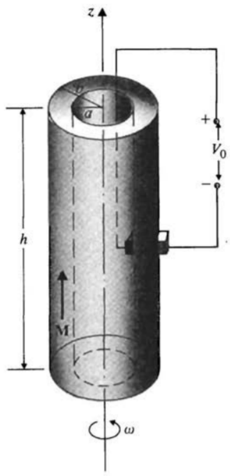


FIGURE 7–14
A rotating hollow cylindrical magnet (Problem P.7–10).

Answer:

a) $\mu_n = 1 + \chi_{m,}, \quad \chi_m = 5000 - 1 = 4999,$

$\bar{H} = \frac{\bar{M}}{x_m} = \bar{a}_z \frac{M_0}{4999}; \quad \bar{B} = \mu_0 \mu_r \bar{H} = \bar{a}_z \frac{5000}{4999} \mu_0 M_0.$

b)

$$V_0 = \oint \bar{u} \times \bar{B} \cdot d\bar{l} = \int_b^\alpha (\bar{a}_\phi \omega r) \times (\bar{a}_z B) \cdot \bar{a}_r dr$$

$$= -\frac{\omega B}{2} \left(b^2 - a^2\right) = -\frac{2500}{4999} \mu_0 M_0 \omega \left(b^2 - a^2\right).$$

c) $\bar{E}_r = \bar{E}'_r - \bar{u} \times \bar{B} = \frac{\bar{J}_r}{\sigma} - (\bar{a}_\phi u) \times (\bar{a}_z B) = \bar{a}_r \left(\frac{i}{2\pi r h \sigma} - \omega r B\right)$

Induced voltage $V = \int_a^b E_r dr = \frac{i}{2\pi n \sigma} \ln \frac{b}{a} - \frac{\omega B}{2} (b^2 - a^2).$

$$= iR + V_0.$$

Short circuit: $V_0 = 0, i_{sc} = \frac{\omega B}{2R} (b^2 - a^2)$, where $R = \frac{\ln(b/a)}{2\pi h \sigma}$.

Exercise 7.11

Derive the two divergence equations, Eqs. (7-53c) and (7-53d), from the two curl equations, Eqs. (7-53a) and (7-53b), and the equation of continuity, Eq. (7-48).

Answer:

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Exercise 7.12

Prove that the Lorentz condition for potentials as expressed in Eq. $(7-62)$ is consistent with the equation of continuity.

Answer:
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Exercise 7.13

The vector magnetic potential $\mathbf{A}$ and scalar electric potential V defined in Section 7-4 are not unique in that it is possible to add to $\mathbf{A}$ the gradient of a scalar $\psi, \nabla\psi$, with no change in $\mathbf{B}$ from Eq. (7-55).

$$\mathbf{A}' = \mathbf{A} + \nabla\psi.$$

In order not to change $\mathbf{E}$ in using Eq. (7-57), V must be modified to V'.

a) Find the relation between V' and V.

b) Discuss the condition that ψ must satisfy so that the new potentials $\mathbf{A}'$ and V' remain governed by the uncoupled wave equations $(7-63)$ and (7-65).

Answer:
a)

$$\bar{E} = -\bar{\nabla}V - \frac{\partial \bar{A}}{\partial t} = -\bar{\nabla}\left(V - \frac{\partial\psi}{\partial t}\right) - \frac{\partial \bar{A}'}{\partial t} \longrightarrow V' = V - \frac{\partial\psi}{\partial t}.$$

b) Eq. $(7-62)$: $\bar{\nabla}\cdot\overline{A'} + \mu\epsilon\frac{\partial V'}{\partial t} = 0.$

$$\longrightarrow \bar{\nabla}\cdot(\bar{A} + \bar{\nabla}\psi) + \mu\epsilon\frac{\partial}{\partial t}\left(V - \frac{\partial\psi}{\partial t}\right) = 0.$$

$$\longrightarrow \bar{\nabla}^2\psi - \mu\epsilon\frac{\partial^2\psi}{\partial t^2} = 0.$$

Exercise 7.14

Substitute Eqs. (7-55) and (7-57) in Maxwell's equations to obtain wave equations for scalar potential V and vector potential $\mathbf{A}$ for a linear, isotropic but inhomogeneous medium. Show that these wave equations reduce to Eqs. (7-65) and (7-63) for simple media. (Hint: Use the following gauge condition for potentials in an inhomogeneous medium:

$$\boldsymbol{\nabla}\cdot(\epsilon\mathbf{A}) + \mu\epsilon^2\frac{\partial V}{\partial t} = 0.)$$

Answer:
Eq. (7-53b): $\bar{\nabla}\times\bar{H} = \bar{J} + \frac{\partial\bar{D}}{\partial t} \longrightarrow \bar{\nabla}\times\left(\frac{\bar{B}}{\mu}\right) = \bar{J} + \epsilon\frac{\partial\bar{E}}{\partial t}$
Eqs. $(7-55)\&(7-57)$: $\bar{B} = \bar{\nabla}\times\bar{A}$, $\bar{E} = -\bar{\nabla}V - \frac{\partial\bar{A}}{\partial t}$
Substituting (2)& (3) in (1):

$$\mu\bar{\nabla}\times\left(\frac{1}{\mu}\bar{\nabla}\times\bar{A}\right) = \mu\bar{J} - \mu\in\frac{\partial^2\bar{A}}{\partial t^2} - \mu\in\bar{\nabla}\left(\frac{\partial V}{\partial t}\right)$$

Using gauge condition for potentials in an inhomogeneous medium:

$$\bar{\nabla}\cdot(\epsilon\bar{A}) + \mu\epsilon^2\frac{\partial V}{\partial t} = 0$$

We obtain the wave equation for vector potential:

$$-\bar{\nabla}\times\left(\frac{1}{\mu}\bar{\nabla}\times\bar{A}\right)+\epsilon\bar{\nabla}\left[\frac{1}{\mu\epsilon^2}\bar{\nabla}\cdot(\epsilon\bar{A})\right]-\epsilon\frac{\partial^2\bar{A}}{\partial t^2}=-\bar{J}$$

Eq. (7-53c):

$$\bar{\nabla}\cdot\bar{D}=\rho\longrightarrow\bar{\nabla}\cdot(\epsilon\bar{\nabla}V)+\frac{\partial}{\partial t}\bar{\nabla}\cdot(\epsilon\bar{A})=-\rho$$

Wave equation for scalar potentials

$$\frac{1}{\epsilon}\bar{\nabla}\cdot(\epsilon\bar{\nabla}V)-\mu\epsilon\frac{\partial^2 V}{\partial t^2}=-\frac{\rho}{\epsilon}$$

Exercise 7.15

Write the set of four Maxwell's equations, Eqs. (7-53a, b, c and d), as eight scalar equations

a) in Cartesian coordinates,
b) in cylindrical coordinates,
c) in spherical coordinates.

Answer:
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Exercise 7.16

Supply the detailed steps for the derivation of the electromagnetic boundary conditions, Eqs. (7-66a, b, c, and d).

Answer:
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Exercise 7.17

Discuss the relations

a) between the boundary conditions for the tangential components of **E** and those for the normal components of **B**,
b) between the boundary conditions for the normal components of **D** and those for the tangential components of **H**.

Answer:
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Exercise 7.18

In Eqs. $(3-88)$ and $(3-89)$ it was shown that for field calculations a polarized dielectric may be replaced by an equivalent polarization surface charge density ρ_{ps} and an equivalent polarization volume charge density ρ_p. Find the boundary conditions at the interface of two different media for

a) the normal component of **P**

b) the normal components of **E**
in terms of free and equivalent polarization surface charge densities ρ_s and ρ_{ps}.

Answer:
a) Eq. (3-89):

$$\bar{\nabla}\cdot\bar{p} = -\rho_p \longrightarrow \bar{a}_{n2}\cdot(\bar{p}_1-\bar{p}_2) = -\rho_{ps}$$

.

$$\text{b) } \epsilon_0\bar{E} = \bar{D}-\bar{P}\begin{cases} \bar{\nabla}\cdot\bar{D} = \rho_f \longrightarrow \bar{a}_{n2}\cdot\left(\overline{D_1}-\bar{D}_2\right) = \rho_{fs} \\ -\bar{\nabla}\cdot\bar{p} = \rho_p \longrightarrow \bar{a}_{n2}\cdot(P_1-P_2) = -\rho_{ps} \end{cases}$$

$$\xrightarrow[\text{(1)\&(2)}]{\text{Combining}} \bar{a}_{n2}\cdot\left(\overline{E_1}-\bar{E}_2\right) = \tfrac{1}{\epsilon_0}\left(\rho_{fs}+\rho_{ps}\right).$$

Subscript f signifies free charge.

Exercise 7.19

Write the boundary conditions that exist at the interface of free space and a magnetic material of infinite (an approximation) permeability.

Answer:
Medium 1: Free space.
Medium 2: $\mu_2 \to \infty$. H_2 must be zero so that B_2 is not infinite.
Boundary Conditions $\bar{a}_{n2}\times\bar{H}_1 = \bar{J}_s, \quad B_{1n} = B_{2n}.$

$$E_{1t} = E_{2t}, \quad \bar{a}_{n_2}\cdot\left(\bar{D}_1-\bar{D}_2\right) = \rho_s.$$

Exercise 7.20

Prove by direct substitution that any twice differentiable function of $(t-R\sqrt{\mu\epsilon})$ or of $(t+R\sqrt{\mu\epsilon})$ is a solution of the homogeneous wave equation, Eq. (7-73).

Answer:
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Exercise 7.21

Prove that the retarded potential in Eq. (7-77) satisfies the nonhomogeneous wave equation, Eq. (7-65).

Answer:
We wish to prove

$$\bar{\nabla}^2 V - \frac{1}{u^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}$$

where

$$V(R,t) = \frac{1}{4\pi\epsilon}\int_{V'}\frac{\rho(t-R/u)}{R}dv'$$

We need:

$$\bar{\nabla}^2\left(\frac{\rho}{R}\right) = \frac{1}{R}\bar{\nabla}^2\rho + \rho\bar{\nabla}^2\left(\frac{1}{R}\right) + 2(\bar{\nabla}\rho)\cdot\bar{\nabla}\left(\frac{1}{R}\right)$$

Formula:

$$\bar{\nabla}^2(fg) = \bar{\nabla}\cdot\bar{\nabla}(fg) = g\bar{\nabla}^2 f + f\bar{\nabla}^2 g + 2(\bar{\nabla} f)\cdot(\bar{\nabla} g)$$

Let

$$\xi = t - R/u, \quad \bar{\nabla}^2\rho(\xi) = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial\rho}{\partial R}\right) = \frac{1}{u^2}\frac{d^2\rho}{d^2\xi} - \frac{2}{2R}\frac{d\rho}{d\xi}$$

$$\bar{\nabla}^2\left(\frac{1}{R}\right) = -4\cdot\pi\delta(R)$$

$$(\bar{\nabla}\rho)\cdot\left(\bar{\nabla}\frac{1}{R}\right) = \frac{\partial\rho}{\partial R}\left(-\frac{1}{R^2}\right) = \frac{1}{uR^2}\frac{d\rho}{d\xi}$$

Substituting (4), (5)& (6) in (3):

$$\bar{\nabla}^2\left(\frac{\rho}{R}\right) = \frac{1}{u^2R}\frac{d^2\rho}{d\xi^2} - 4\pi\rho\delta(R)$$

From (2):

$$\bar{\nabla}^2 V = \frac{1}{4\pi\epsilon}\bar{\nabla}^2\int_{v'}\frac{\rho}{R}dv' = \frac{1}{4\pi\epsilon}\int_{v'}\left[\frac{1}{u^2R}\frac{d^2\rho}{d\rho^2} - 4\pi\rho\delta(R)\right]dv'$$

$$\frac{\partial^2 V}{\partial t^2} = \frac{1}{4\pi\epsilon}\int_{v'}\frac{1}{R}\cdot\frac{d^2\rho}{d\xi^2}dv'$$

$$\therefore \quad \bar{\nabla}^2 V - \frac{1}{u^2}\frac{\partial^2 v}{\partial t^2} = \frac{1}{4\pi\epsilon}\int_{v'}\left[\frac{1}{u^2R}\frac{d^2\rho}{d\xi^2} - 4\pi\rho\delta(R) - \frac{1}{u^2R}\frac{d^2\rho}{d\xi^2}\right]dv'$$

$$= -\frac{\rho}{\epsilon} \quad \text{Q.E.D.}$$

Exercise 7.22

For the assumed $f(t)$ at $R = 0$ in Fig. $7 - 15$, sketch
a) $f(t - R/u)$ versus t,
b) $f(t - R/u)$ versus R for $t > T$.

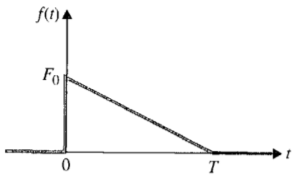
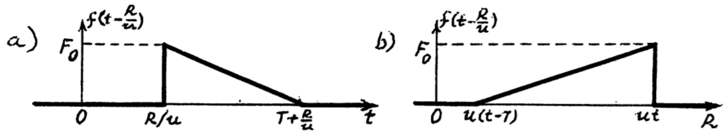


FIGURE 7–15
A triangular time function (Problem P.7–22).

Answer:



Exercise 7.23

The electric field of an electromagnetic wave

$$\mathbf{E} = \mathbf{a}_x E_0 \cos\left[10^{8^8}\pi\left(t - \frac{z}{c}\right) + \theta\right]$$

is the sum of

$$\mathbf{E}_1 = \mathbf{a}_x 0.03 \sin 10^8\pi\left(t - \frac{z}{c}\right)$$

and

$$\mathbf{E}_2 = \mathbf{a}_x 0.04 \cos\left[10^8\pi\left(t - \frac{z}{c}\right) - \frac{\pi}{3}\right]$$

Find E_0 and θ.

Answer:

$$\begin{aligned}
\bar{E}_1(z,t) &= \bar{a}_x 0.03 \sin 10^8\pi\left(t - \frac{z}{c}\right) = \bar{a}_x\left(\mathrm{Re}\left[0.03e^{-j\pi/2}e^{j10^8\pi(t-z/c)}\right]\right. \\
\bar{E}_2(z,t) &= \bar{a}_x 0.04 \cos\left[10^8\pi\left(t - \frac{z}{c}\right) - \frac{\pi}{3}\right] = \bar{a}_x\left(\mathrm{Re}\left[0.04e^{-j\pi/3}e^{j10^8\pi(t-z/c)}\right]\right. \\
\text{Phasors:} \quad \bar{E} &= \bar{E}_1 + \bar{E}_2 = \bar{a}_x\left[0.03e^{-j\pi/2} + 0.04e^{-j\pi/3}\right] \\
&= \bar{a}_x[-20.03 + (0.02 - j0.02\sqrt{3})] = \bar{a}_x\left(0.068e^{-j9.27}\right) = \bar{a}_x E_0 e^{jo}. \\
\therefore E_0 &= 0.068, \quad \theta = -1.27(\mathrm{rad}) \text{ or } -72.8^\circ
\end{aligned}$$

Exercise 7.24

Derive the general wave equations for $\mathbf{E}$ and $\mathbf{H}$ in a nonconducting simple medium where a charge distribution ρ and a current distribution $\mathbf{J}$ exist. Convert the wave equations to Helmholtz's equations for sinusoidal time dependence. Write the general solutions for $\mathbf{E}(R,t)$ and $\mathbf{H}(R,t)$ in terms of ρ and $\mathbf{J}$.

Answer:

$$\begin{array}{llll}
\bar{\nabla} \times \bar{E} = -\mu\frac{\partial \bar{H}}{\partial t} & (1) & \bar{\nabla} \times \bar{H} = \bar{J} + \epsilon\frac{\partial \bar{E}}{\partial t} & (2) \\
\bar{\nabla} \cdot \bar{E} = \frac{\rho}{\epsilon} & (3) & \bar{\nabla} \cdot \bar{H} = 0 & (4)
\end{array}$$

$$\bar{\nabla} \times (1) : \bar{\nabla} \times \bar{\nabla} \times \bar{E} = -\mu\frac{\partial}{\partial t}(\bar{\nabla} \times \bar{H}) = -\mu\frac{\partial}{\partial t}\left(\bar{J} + \epsilon\frac{\partial \bar{E}}{\partial t}\right) = \bar{\nabla}(\bar{\nabla} \cdot \bar{E}) - \bar{\nabla}^2\bar{E}$$

Wave equation for $\bar{E}$:

$$\bar{\nabla}^2\bar{E} - \mu\epsilon\frac{\partial^2 \bar{E}}{\partial t^2} = \mu\frac{\partial \bar{J}}{\partial t} + \frac{1}{\epsilon}\bar{\nabla}\rho$$

.

$$\bar{\nabla} \times (2) : \bar{\nabla} \times \bar{\nabla} \times \bar{H} = \bar{\nabla} \times \bar{J} + \epsilon\frac{\partial}{\partial t}(\bar{\nabla} \times \bar{E}) = \bar{\nabla} \times \bar{J} - \mu\epsilon\frac{\partial^2 \bar{H}}{\partial t^2} = \bar{\nabla}(\bar{\nabla} \cdot \bar{H}) - \bar{\nabla}^2\bar{H}$$

Wave equation for $\bar{H}$:

$$\bar{\nabla}^2\bar{H} - \mu\epsilon\frac{\partial^2 \bar{H}}{\partial t^2} = -\bar{\nabla} \times \bar{J}$$

For sinusoidal time dependence:

$$\frac{\partial}{\partial t} \longrightarrow j\omega, \frac{\partial^2}{\partial t^2} \longrightarrow -\omega^2$$

Helmholtz's equations: (for phasors)

$$\bar{\nabla}^2 \bar{E} + \omega^2 \mu \epsilon \bar{E} = j\omega\mu \bar{J} + \frac{1}{\epsilon} \bar{\nabla} \rho$$

$$\bar{\nabla}^2 \bar{H} + \omega^2 \mu \epsilon \bar{H} = -\bar{\nabla} \times \bar{J}$$

Exercise 7.25

Given that

$$\mathbf{E} = \mathbf{a}_y 0.1 \sin(10\pi x) \cos\left(6\pi 10^9 t - \beta z\right) \quad \text{(V/m)}$$

in air, find **H** and β.

Answer:

$$\bar{E} = \bar{a}_y 0.1 \sin(10\pi x) \cos\left(6\pi 10^9 t - \beta z\right) \quad (V/m).$$

Use phasors:

$$\bar{H} = -\frac{1}{j\omega\mu_0} \bar{V} \times \bar{E} = \frac{j}{\omega\mu_0} \left[\bar{a}_x j 0.1\beta \sin(10\pi x) + \bar{a}_z 0.1(10\pi) \cos(10\pi x)\right] \cdot e^{-j\beta z} (1)$$

$$\bar{E} = \frac{1}{j\omega\epsilon_0} \bar{\nabla} \times \bar{H} = \bar{a}_y \frac{0.1}{\omega^2 \mu_0 \epsilon_0} \left[(10\pi)^2 + \beta^2\right] \sin(10\pi x) e^{-j\beta z}. \ (2)$$

Phase form for given

$$\bar{E} : \bar{E} = \bar{a}_y 0.1 \sin(10\pi x) e^{-j\beta z} (3)$$

Equating (2) and (3):

$$(10\pi)^2 + \beta^2 = \omega^2 \mu_0 \epsilon_0 = 400\pi^2$$

$$\longrightarrow \beta = \sqrt{300}\pi = 54.4 \text{(rad/m)}$$

From (1):

$$\bar{H}(x, z; t) = \text{Re}\left(\bar{H} e^{j\omega t}\right)$$

$$\begin{aligned} &= -\bar{a}_x 2.30 \times 10^{-4} \sin(10\pi x) \cos\left(6\pi 10^9 t - 54.4z\right) \\ &\quad - \bar{a}_z 1.33 \times 10^{-4} \cos(10\pi x) \sin\left(6\pi 10^9 t - 54.4z\right) \text{(A/m)}. \end{aligned}$$

Exercise 7.26

Given that

$$\mathbf{H} = \mathbf{a}_y 2 \cos(15\pi x) \sin\left(6\pi 10^9 t - \beta z\right) \quad \text{(A/m)}$$

in air, find **E** and β.

Answer:

$$\bar{H}(x, z; t) = \bar{a}_y 2 \cos(15\pi x) \sin\left(6\pi 10^9 t - \beta z\right) \quad \text{(A/m)}.$$

$$\text{Phasor: } \bar{H} = \bar{a}_y 2\cos(15\pi x)e^{-j\beta 2}$$

$$(15\pi)^2 + \beta^2 = \omega^2\mu_0\epsilon_0 = \frac{\left(6\pi 10^9\right)^2}{\left(3\times 10^8\right)^2} = 400\pi^2.$$

$$\longrightarrow \beta = 13.2\pi = 41.6(\text{rad/m}).$$

$$\bar{E} = \frac{1}{j\omega t_0}\bar{\nabla}\times\bar{H} = \left[\bar{a}_x 158\pi\cos(15\pi x) + \bar{a}_z j180\pi\sin(15\pi x)\right]e^{-j\beta z}.$$

$$\bar{E}(x,z;t) = \text{Im}\left(\bar{E}e^{j\omega t}\right) = \bar{a}_x 496\cos(15\pi x)\sin\left(6\pi 10^9 t - 41.6z\right)$$

$$+\,\bar{a}_z 565\sin(15\pi x)\cos\left(6\pi 10^9 t - 41.6z\right)\,(V/m).$$

Exercise 7.27

It is known that the electric field intensity of a spherical wave in free space is

$$\mathbf{E} = \mathbf{a}_\theta \frac{E_0}{R}\sin\theta\cos(\omega t - kR).$$

Determine the magnetic field intensity $\mathbf{H}$ and the value of k.

Answer:
Use phasors: $\bar{E} = \bar{a}_\theta \frac{E_0}{R}\sin\theta\cdot e^{-jkR}$

$$\bar{\nabla}\times\bar{E} = \bar{a}_\phi \frac{1}{R}\frac{\partial}{\partial R}\left(RE_\theta\right) = \bar{a}_\phi(-jk)\frac{E_0}{R}\sin\theta\cdot e^{-jkR}$$

$$= -j\omega\mu_0\bar{H} \longrightarrow \bar{H} = \bar{a}_\phi\frac{kE_0}{\omega\mu_0 R}\sin\theta\cdot e^{-jkR}$$

In free space,

$$k = \omega\sqrt{\mu_0\epsilon_0} \rightarrow \bar{H} = \overline{a_\phi}\frac{E_0}{R}\sqrt{\frac{\epsilon_0}{\mu_0}}\sin\theta\cdot e^{-j\omega\sqrt{\mu_0\epsilon_0}R}$$

$$\bar{H}(R,\theta;t) = \bar{a}_\phi\frac{E_0}{R}\sqrt{\frac{\epsilon_0}{\mu_0}}\sin\theta\cos\omega\left(t - \sqrt{\mu_0\epsilon_0}R\right).$$

Exercise 7.28

In Section $7-4$ we indicated that $\mathbf{E}$ and $\mathbf{B}$ can be determined from the potentials V and $\mathbf{A}$, which are related by the Lorentz condition, Eq. (7-98), in the time-harmonic case. The vector potential $\mathbf{A}$ was introduced through the relation $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ because of the solenoidal nature of $\mathbf{B}$. In a source-free region, $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$, we can define another type of vector potential $\mathbf{A}_e$, such that $\mathbf{E} = \boldsymbol{\nabla}\times\mathbf{A}_e$. Assuming harmonic time dependence:
a) Express $\mathbf{H}$ in terms of $\mathbf{A}_e$.
b) Show that $\mathbf{A}_e$ is a solution of a homogeneous Helmholtz's equation.

Answer:

$$\text{Maxwell curl eqs: } \begin{array}{l}\bar{\nabla}\times\bar{E} = -j\omega\mu\bar{H},\\ \bar{\nabla}\times\bar{H} = j\omega\bar{E}\end{array}$$

From $\bar{\nabla}\cdot\bar{E} = 0$, define $\bar{A}_e$ such that

$$\bar{E} = \bar{\nabla} \times \bar{A}_e$$

From (1),

$$\bar{H} = \frac{j}{\omega\mu}\bar{\nabla} \times \bar{E} = \frac{j}{\omega\mu}\bar{\nabla} \times \bar{\nabla} \times \bar{A}_e$$

$$= \frac{j}{\omega\mu}\left[\bar{\nabla}\left(\bar{\nabla} \cdot \bar{A}_e\right) - \bar{\nabla}^2 \bar{A}_e\right]$$

From (2),

$$\bar{\nabla} \times \left(\bar{H} - j\omega\epsilon\bar{A}_e\right) = 0$$

Let

$$\bar{H} - j\omega\epsilon\bar{A}_e = -\bar{\nabla}V_m$$

Subtracting (5) from (4):

$$\omega\epsilon\bar{A}_e = \frac{1}{\omega\mu}\left[\bar{\nabla}\left(\bar{\nabla} \cdot \bar{A}_e\right) - \bar{V}^2 \bar{A}_e\right] - j\bar{\nabla}V_m$$

Choose

$$\bar{\nabla} \cdot \bar{A}_e = j\omega\mu V_m$$

Exercise 7.29

For a source-free polarized medium where $\rho = 0, \mathbf{J} = 0, \mu = \mu_0$, but where there is a volume density of polarization $\mathbf{P}$, a single vector potential π_e may be defined such that

$$\mathbf{H} = j\omega\epsilon_0 \nabla \times \pi_e.$$

a) Express electric field intensity $\mathbf{E}$ in terms of π_e and $\mathbf{P}$.

b) Show that π_e satisfies the nonhomogeneous Helmholtz's equation

$$\nabla^2 \pi_e + k_0^2 \pi_e = -\frac{\mathbf{P}}{\epsilon_0}.$$

The quantity π_e is known as the electric Hertz potential.

Answer:

$$\bar{H} = j\omega\epsilon_0 \bar{\nabla} \times \bar{\pi}_e$$

$$\bar{\nabla} \times \bar{E} = -j\omega\mu_0\bar{H} = \omega^2\mu_0\epsilon_0\bar{\nabla} \times \bar{\pi}_e$$

$$\longrightarrow \bar{\nabla} \times \left(\bar{E} - k_0^2\bar{\pi}_e\right) = 0$$

Let

$$\bar{E} - k_0^2\bar{\pi}_e = \bar{\nabla}V_e$$

$$\bar{\nabla} \times \bar{H} = j\omega\bar{D} = j\omega\left(\epsilon_0\bar{E} + \bar{P}\right) = j\omega\epsilon_0\left(\bar{E} + \frac{\bar{\rho}}{\epsilon_0}\right)$$

Substituting (1) and (2) in (3):

$$j\omega\epsilon_0\bar{\nabla} \times \bar{\nabla} \times \bar{\pi}_e = j\omega\epsilon_0\left(k_0^2\bar{\pi}_e + \bar{\nabla}V_e + \frac{\bar{p}}{\epsilon_0}\right)$$

$$= j\omega\epsilon_0 \left(\bar{\nabla}\bar{\nabla}\cdot\bar{\pi}_e - \bar{\nabla}^2\bar{\pi}_e\right)$$

Choose $\bar{\nabla}\cdot\bar{\pi}_e = V_e$. Eq. (4) becomes

b)

$$\bar{\nabla}^2\bar{\pi}_e + k_0^2\bar{\pi}_e = -\frac{\bar{p}}{\epsilon_0}$$

a) Eq. (2) becomes

$$\begin{aligned}\bar{E} &= k_0^2\bar{\pi}_e + \bar{\nabla}\bar{\nabla}\cdot\bar{\pi}_e\\ &= k_0^2\bar{\pi}_e + \left(\bar{\nabla}^2\bar{\pi}_e + \bar{\nabla}\times\bar{\nabla}\times\bar{\pi}_e\right).\end{aligned}$$

Combination of Egs. $(7-119)$ and (5) gives

$$\bar{E} = \bar{\nabla}\times\bar{\nabla}\times\bar{\pi}_e - \frac{\bar{P}}{\epsilon_0}.$$

Exercise 7.30

Calculations concerning the electromagnetic effect of currents in a good conductor usually neglect the displacement current even at microwave frequencies.

a) Assuming $\epsilon_r = 1$ and $\sigma = 5.70\times 10^7(\text{ S/m})$ for copper, compare the magnitude of the displacement current density with that of the conduction current density at 100 (GHz).

b) Write the governing differential equation for magnetic field intensity **H** in a source-free good conductor.

Answer:

a)

$$\begin{aligned}\left|\frac{\text{Displacement current}}{\text{Conduction current}}\right| = \frac{\omega\epsilon}{\sigma} &= \frac{(2\pi\times 100\times 10^9)\times\frac{1}{36\pi}\times 10^{-9}}{5.90\times 10^7}\\ &= 9.75\times 10^{-8}.\end{aligned}$$

b) In a source-free conductor:

$$\begin{aligned}\bar{\nabla}\times\bar{H} &= \sigma\bar{E},\\ \bar{\nabla}\times\bar{E} &= -j\omega\mu\bar{H}.\\ \bar{\nabla}\times(1):\bar{\nabla}\times\bar{\nabla}\times\bar{H} &= \bar{\nabla}(\bar{\nabla}\cdot\bar{H}) - \bar{\nabla}^2\bar{H} = \sigma\bar{\nabla}\times\bar{E}.\end{aligned}$$

But $\bar{\nabla}\cdot\bar{H} = 0$, Eq (3) becomes

$$\bar{\nabla}^2\bar{H} + \sigma\bar{\nabla}\times\bar{E} = 0.$$

Combining (2) and (4):

$$\bar{\nabla}^2\bar{H} - j\cos\mu\sigma\bar{H} = 0.$$

Reference

1. Cheng, David Keun. Field and wave electromagnetics. Pearson Education India, 1989.