8-6 Normal Incidence at a Plane Conducting Boundary

- When an electromagnetic wave traveling in one medium impinges on another medium with a different intrinsic impedance, it experiences a reflection.
  - A plane conducting boundaries (8-6, 8-7)
  - An interface between two dielectric media (8-8, ~~8-9~~, 8-10)

Normal Incidence

- Assume
  - The incident wave ($\mathbf{E_i}$, $\mathbf{H_i}$) travels in a lossless medium ($\sigma_1 = 0$)
  - The boundary is an interface with a perfect conductor ($\sigma_2 = \infty$)

The Analogy between EM Waves and Transmission Lines

- EM waves:
  - Incoming wave with a certain frequency
  - Terminated by **a perfect conductor ($\eta = 0$)**
  - Waves are totally reflected.

- Transmission lines:
  - Voltage applied with a certain frequency
  - Terminated by **a short circuit ($Z = 0$)**
  - Voltage signals are totally reflected.

Medium 1 (lossless medium)

Incident waves

$$\mathbf{E}_i(z) = \mathbf{a}_x E_{i0} e^{\underline{-j\beta_1 z}},$$

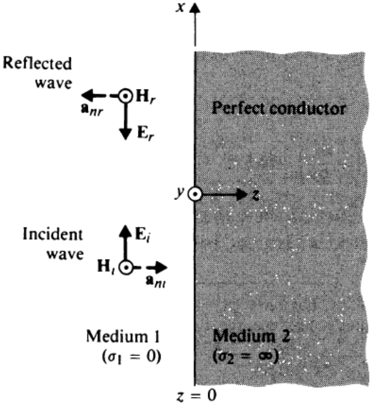
$$\mathbf{H}_i(z) = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z},$$

Travels in the +z direction

E_{i0}: magnitude of $\mathbf{E}_\mathrm{i}$

β_1: phase constant of medium 1

η_1: intrinsic impedance of medium 1



Poynting vector:

$$\mathscr{P}_i(z) = \mathbf{E}_i(z) \times \mathbf{H}_i(z),$$

In $\mathbf{a}_z$ direction (direction of energy propagation)

FIGURE 8–9
Plane wave incident normally on a plane conducting boundary.

Medium 2 (perfect conductor)

$\mathbf{E}_2 = 0$, $\mathbf{H}_2 = 0$

➔ No wave is transmitted

Incident wave is reflected ($\mathbf{E}_r$, $\mathbf{H}_r$)

$$\mathbf{E}_r(z) = \mathbf{a}_x E_{r0} e^{+j\beta_1 z},$$

Travels in the $-z$ direction

To be determined by B.C.

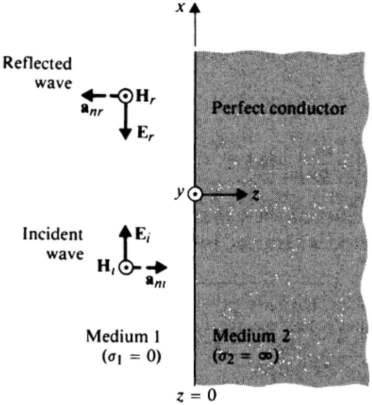


FIGURE 8–9
Plane wave incident normally on a plane conducting boundary.

Boundary

Total field in medium 1 = $\mathbf{E}_i + \mathbf{E}_r$

$$\mathbf{E}_1(z) = \mathbf{E}_i(z) + \mathbf{E}_r(z) = \mathbf{a}_x(E_{i0}e^{-j\beta_1 z} + E_{r0}e^{+j\beta_1 z}).$$

B.C.: $E_{1t} = E_{2t}$ at $z = 0$

$$\mathbf{E}_1(0) = \mathbf{a}_x(E_{i0} + E_{r0}) = \mathbf{E}_2(0) = 0,$$

Perfect conductor of medium 2

$$E_{r0} = -E_{i0}.$$

Thus, if $\mathbf{E}_i$ is along $\mathbf{a}_x$, $\mathbf{E}_r$ should be along $-\mathbf{a}_x$, as shown in the figure.

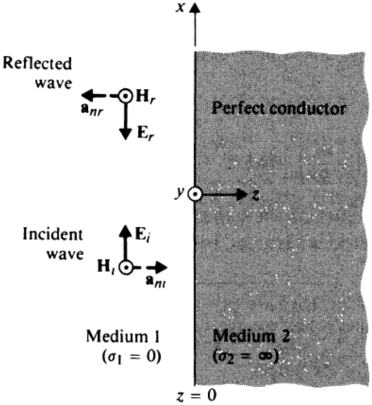


FIGURE 8–9
Plane wave incident normally on a plane conducting boundary.

$\mathbf{E}_1$

$$\boxed{\mathbf{E}_1(z) = \mathbf{E}_i(z) + \mathbf{E}_r(z) = \mathbf{a}_x(E_{i0}e^{-j\beta_1 z} + E_{r0}e^{+j\beta_1 z}).}$$

$$\boxed{E_{r0} = -E_{i0}.}$$

$$\begin{aligned}\mathbf{E}_1(z) &= \mathbf{a}_x E_{i0}(e^{-j\beta_1 z} - e^{+j\beta_1 z}) \\ &= -\mathbf{a}_x j2E_{i0} \sin \beta_1 z.\end{aligned}$$

$\boxed{\mathbf{H}_1}$

$$\boxed{\mathbf{E}_r(z) = \mathbf{a}_x E_{r0} e^{+j\beta_1 z},}$$

$$\boxed{\mathbf{H}(\mathbf{R}) = \frac{1}{\eta}\mathbf{a}_n \times \mathbf{E}(\mathbf{R})} \quad (8\text{-}29)$$

$$\mathbf{H}_r(z) = \frac{1}{\eta_1}\mathbf{a}_{nr} \times \mathbf{E}_r(z) = \frac{1}{\eta_1}(-\mathbf{a}_z) \times \mathbf{E}_r(z)$$

$$= -\mathbf{a}_y \frac{1}{\eta_1} E_{r0} e^{+j\beta_1 z} = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{+j\beta_1 z}.$$

$$\mathbf{H}_1(z) = \mathbf{H}_i(z) + \mathbf{H}_r(z)$$

$$\boxed{\mathbf{H}_i(z) = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z},}$$

$\mathbf{H}_r$ and $\mathbf{H}_i$ are both along $\mathbf{a}_y$, as shown in the figure

$$\mathbf{H}_1(z) = \mathbf{H}_i(z) + \mathbf{H}_r(z) = \mathbf{a}_y 2\frac{E_{i0}}{\eta_1}\cos\beta_1 z.$$

$$\boxed{\begin{aligned}\mathbf{E}_1(z) &= \mathbf{a}_x E_{i0}(e^{-j\beta_1 z} - e^{+j\beta_1 z}) \\ &= -\mathbf{a}_x j2E_{i0}\sin\beta_1 z.\end{aligned}}$$

$$\boxed{\mathbf{H}_1(z) = \mathbf{H}_i(z) + \mathbf{H}_r(z) = \mathbf{a}_y 2\frac{E_{i0}}{\eta_1}\cos\beta_1 z.}$$

$$\boxed{\mathscr{P}_{av} = \tfrac{1}{2}\mathscr{Re}(\mathbf{E}\times\mathbf{H}^*)} \qquad (8\text{-}96)$$

- $\mathbf{E_1}$ and $\mathbf{H_1}$ are in phase quadrature (i.e., in time quadrature)

- No average power is associated with the total electromagnetic wave in medium 1

Time-domain behavior

$$\boxed{\begin{aligned}\mathbf{E}_1(z) &= \mathbf{a}_x E_{i0}(e^{-j\beta_1 z} - e^{+j\beta_1 z})\\ &= -\mathbf{a}_x j2E_{i0}\sin\beta_1 z.\end{aligned}}$$

$$\boxed{\mathbf{H}_1(z) = \mathbf{H}_i(z) + \mathbf{H}_r(z) = \mathbf{a}_y 2\frac{E_{i0}}{\eta_1}\cos\beta_1 z.}$$

Total field in time domain

$$\mathbf{E}_1(z, t) = \mathscr{R}e[\mathbf{E}_1(z)e^{j\omega t}] = \mathbf{a}_x 2E_{i0}\sin\beta_1 z\,\underline{\sin\omega t},$$

$\boxed{=\cos(\omega t - \pi/2)}$

$$\mathbf{H}_1(z, t) = \mathscr{R}e[\mathbf{H}_1(z)e^{j\omega t}] = \mathbf{a}_y 2\frac{E_{i0}}{\eta_1}\cos\beta_1 z\,\underline{\cos\omega t}.$$

Zeros and maxima along *z* for all *t*

$$\left.\begin{array}{r}\text{Zeros of } \mathbf{E}_1(z, t)\\ \text{Maxima of } \mathbf{H}_1(z, t)\end{array}\right\}\text{occur at } \beta_1 z = -n\pi, \quad \text{or } z = -n\frac{\lambda}{2}, \qquad n = 0, 1, 2, \ldots$$

$$\left.\begin{array}{r}\text{Maxima of } \mathbf{E}_1(z, t)\\ \text{Zeros of } \mathbf{H}_1(z, t)\end{array}\right\}\text{occur at } \beta_1 z = -(2n+1)\frac{\pi}{2}, \quad \text{or } z = -(2n+1)\frac{\lambda}{4},$$

$$n = 0, 1, 2, \ldots$$

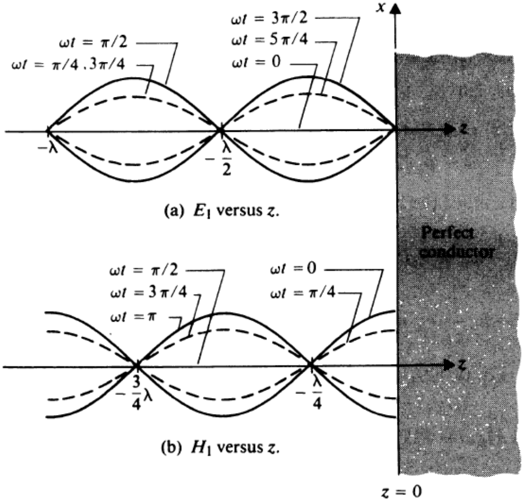


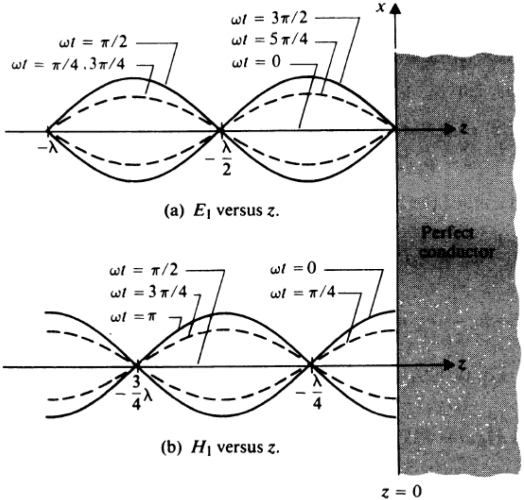
FIGURE 8–10
Standing waves of $\mathbf{E}_1 = \mathbf{a}_x E_1$ and $\mathbf{H}_1 = \mathbf{a}_y H_1$ for several values of ωt.

The total wave in medium 1 is a standing wave (not a traveling wave).

Space-time behavior:

1) $\mathbf{E_1}$ vanishes on the conducting boundary ($E_{r0} = -E_{i0}$) and at $-n\lambda/2$
2) $\mathbf{H_1}$ is a maximum on the conducting boundary ($H_{r0} = H_{i0} = E_{i0}/\eta_1$) and at $-n\lambda/2$
3) $\mathbf{E_1}$ and $\mathbf{H_1}$ are in time quadrature ($\pi/2$) and are shifted in space by $\lambda/4$

Recall:
$\mathbf{E}_i$: $\mathbf{a_x}$ and $\mathbf{E}_r$: $-\mathbf{a_x}$
$\mathbf{H}_i$: $\mathbf{a_y}$ and $\mathbf{H}_r$: $\mathbf{a_y}$



For a given t, both $\mathbf{E_1}$ and $\mathbf{H_1}$ vary sinusoidally with z

$$\mathbf{E}_1(z, t) = \mathbf{a}_x 2E_{i0} \sin \beta_1 z \sin \omega t,$$

$$\mathbf{H}_1(z, t) = \mathbf{a}_y 2 \frac{E_{i0}}{\eta_1} \cos \beta_1 z \cos \omega t.$$

FIGURE 8–10
Standing waves of $\mathbf{E}_1 = \mathbf{a}_x E_1$ and $\mathbf{H}_1 = \mathbf{a}_y H_1$ for several values of ωt.

EXAMPLE 8–8 The far field of a short vertical current element $I\,d\ell$ located at the origin of a spherical coordinate system in free space is

$$\mathbf{E}(R, \theta) = \mathbf{a}_\theta E_\theta(R, \theta) = \mathbf{a}_\theta\left(j\,\frac{60\pi I\,d\ell}{\lambda R}\sin\theta\right)e^{-j\beta R} \qquad \text{(V/m)}$$

and

$$\mathbf{H}(R, \theta) = \mathbf{a}_\phi\,\frac{E_\theta(R, \theta)}{\eta_0} = \mathbf{a}_\phi\left(j\,\frac{I\,d\ell}{2\lambda R}\sin\theta\right)e^{-j\beta R} \qquad \text{(A/m)},$$

where $\lambda = 2\pi/\beta$ is the wavelength.

a) Write the expression for instantaneous Poynting vector.

b) Find the total average power radiated by the current element.

(a) $\vec{P}(t) = \vec{E}(t) \times \vec{H}(t)$

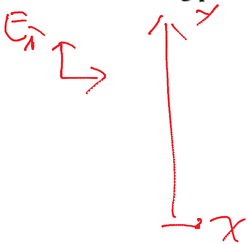
(b) $\vec{P}_{av} = \frac{1}{2} Re[\tilde{E} \times \tilde{H}^*]$

$P = \int \vec{P}_{av} \cdot d\vec{s}$

(a) $\vec{E}(t) = \hat{a}_\theta \frac{60\pi I dl}{\lambda R} \sin\theta \cos(\omega t - \beta R + \frac{\pi}{2})$

(b) $\tilde{H}^* = \hat{a}_\phi \; e^{j(\beta R - \frac{\pi}{2})}$

EXAMPLE 8–9 A y-polarized uniform plane wave ($\mathbf{E}_i$, $\mathbf{H}_i$) with a frequency 100 $= f$ (MHz) propagates in air in the $+x$ direction and impinges normally on a perfectly conducting plane at $x = 0$. Assuming the amplitude of $\mathbf{E}_i$ to be 6 (mV/m), write the E_{i0} phasor and instantaneous expressions for (a) $\mathbf{E}_i$ and $\mathbf{H}_i$ of the incident wave; (b) $\mathbf{E}_r$ and $\mathbf{H}_r$ of the reflected wave; and (c) $\mathbf{E}_1$ and $\mathbf{H}_1$ of the total wave in air. (d) Determine the location nearest to the conducting plane where E_1 is zero.



(a) E_i $\quad H_i$

$\vec{H}_i = \frac{1}{\eta}\hat{a}_n \times \vec{E}_i$

$\tilde{\vec{E}}_i = \hat{a}_y E_{i0} e^{-j\beta x}$

pol. $\quad$ mag. $\quad$ direction of propagation

$\Rightarrow \tilde{H}_i = \frac{1}{\eta_0}\hat{a}_x \times \tilde{\vec{E}}_i$

$\beta = \frac{\omega}{u_p} = \frac{2\pi \cdot f}{c} = \frac{2\pi \cdot 10^8}{3\times 10^8} = \frac{2\pi}{3}$

$\vec{E}_i(t) = \hat{a}_y E_{i0} \cos(\omega t - \beta x)$

$\vec{H}_i(t) = \ldots$

(b) $\tilde{\vec{E}}_r = \hat{a}_y E_{r0} e^{+j\beta x}$

$-x$ propagation

8-7 Oblique Incidence at a Plane Conducting Boundary

- The behavior of the reflected wave depends on the polarization of the incident wave in oblique incidence.
- Plane of incidence: the plane containing the direction of propagation (of the incident wave) and the normal of the boundary surface.
- Consider the two cases separately
  - $\mathbf{E}_i \perp$ plane of incidence
  - $\mathbf{E}_i$ // plane of incidence

8-7.1 Perpendicular Polarization

The incident wave

$$\mathbf{a}_{ni} = \mathbf{a}_x \sin\theta_i + \mathbf{a}_z \cos\theta_i,$$

$\mathbf{k} = \beta_1 \mathbf{a}_{ni}$
$\mathbf{R} = \mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z$

$$\mathbf{E}(\mathbf{R}) = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{R}} = \mathbf{E}_0 e^{-jk\mathbf{a}_n\cdot\mathbf{R}}$$

$$\mathbf{H}(\mathbf{R}) = \frac{1}{\eta}\mathbf{a}_n \times \mathbf{E}(\mathbf{R})$$

θ_i: angle of incidence

$$\mathbf{E}_i(x, z) = \mathbf{a}_y E_{i0} e^{-j\beta_1 \mathbf{a}_{ni}\cdot\mathbf{R}} = \mathbf{a}_y E_{i0} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)},$$

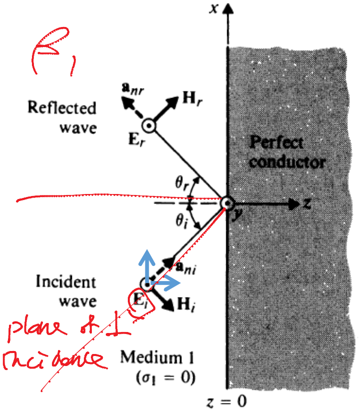
$$\mathbf{H}_i(x, z) = \frac{1}{\eta_1}[\mathbf{a}_{ni} \times \mathbf{E}_i(x, z)]$$

$$= \frac{E_{i0}}{\eta_1}(-\mathbf{a}_x \cos\theta_i + \mathbf{a}_z \sin\theta_i) e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}.$$

Different polarization

Same direction of propagation

$$(\hat{a}_x \sin\theta_i + \hat{a}_z \cos\theta_i) \times \hat{a}_y = \hat{a}_z \sin\theta_i + (-\hat{a}_x)\cos\theta_i$$



Perpendicular polarization:
also called horizontal polarization, E-polarization, s-polarization

FIGURE 8–11
Plane wave incident obliquely on a plane conducting boundary (perpendicular polarization).

The reflected wave

$$\mathbf{a}_{nr} = \mathbf{a}_x \sin\theta_r - \mathbf{a}_z \cos\theta_r,$$

$$\mathbf{E}_i(x, z) = \mathbf{a}_y E_{i0} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)},$$

$\mathbf{k} = \beta_1 \mathbf{a}_{nr}$
$\mathbf{R} = \mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z$

Assuming polarization in $\mathbf{a}_y$ ($\mathbf{a}_y$ or $-\mathbf{a}_y$ can be confirmed later)

$(\hat{a}_x \sin\theta_r - a_z \cos\theta_r)\cdot$
$(\hat{a}_x x + \hat{a}_y y + \hat{a}_z z)$
$= x\sin\theta_r$
$z\cos\theta_r$

$$\mathbf{E}_r(x, z) = \mathbf{a}_y E_{r0} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}.$$

θ_r: angle of reflection

B.C.: $E_{1t} = E_{2t} = 0$ at $z = 0$
For perpendicular polarization, only E tangential component
➔ Total **E** at boundary = 0

$$\mathbf{E}_1(x, 0) = \mathbf{E}_i(x, 0) + \mathbf{E}_r(x, 0)$$
$$= \mathbf{a}_y\left(E_{i0} e^{-j\beta_1 x\sin\theta_i} + E_{r0} e^{-j\beta_1 x\sin\theta_r}\right) = 0.$$

$E_{r0} = -E_{i0}$ $\quad \theta_r = \theta_i$ Snell's law of reflection

Thus, if $\mathbf{E}_i$ is in $\mathbf{a}_y$, $\mathbf{E}_r$ should be along $-\mathbf{a}_y$, which is different from the figure.

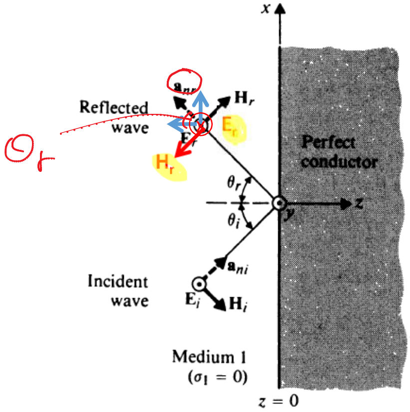


FIGURE 8–11
Plane wave incident obliquely on a plane conducting boundary (perpendicular polarization).

$$\boxed{\mathbf{E}_r(x, z) = \mathbf{a}_y E_{r0} e^{-j\beta_1(x \sin\theta_r - z\cos\theta_r)}}.$$

$$E_{r0} = -E_{i0} \qquad \theta_r = \theta_i$$

$\boxed{\mathbf{E}_r}$

$$\mathbf{E}_r(x, z) = -\mathbf{a}_y E_{i0} e^{-j\beta_1(x \sin\theta_i - z\cos\theta_i)}.$$

$$\mathbf{H}(\mathbf{R}) = \frac{1}{\eta}\mathbf{a}_n \times \mathbf{E}(\mathbf{R})$$

$\boxed{\mathbf{H}_r}$

$$\mathbf{H}_r(x, z) = \frac{1}{\eta_1}[\mathbf{a}_{nr} \times \mathbf{E}_r(x, z)]$$

$$= \frac{E_{i0}}{\eta_1}(-\mathbf{a}_x \cos\theta_i - \mathbf{a}_z \sin\theta_i)\underbrace{e^{-j\beta_1(x \sin\theta_i - z\cos\theta_i)}}_{\vec{E}_r}.$$

The total field

$$\mathbf{E}_i(x, z) = \mathbf{a}_y E_{i0} e^{-j\beta_1(x \sin\theta_i + z\cos\theta_i)},$$

$$\mathbf{E}_r(x, z) = -\mathbf{a}_y E_{i0} e^{-j\beta_1(x \sin\theta_i - z\cos\theta_i)}.$$

$\mathbf{E}_1 = \mathbf{E}_i + \mathbf{E}_r$

$$\begin{aligned}\mathbf{E}_1(x, z) &= \mathbf{E}_i(x, z) + \mathbf{E}_r(x, z)\\ &= \mathbf{a}_y E_{i0}(e^{-j\beta_1 z\cos\theta_i} - e^{j\beta_1 z\cos\theta_i})e^{-j\beta_1 x\sin\theta_i}\\ &= -\mathbf{a}_y j2E_{i0}\sin(\beta_1 z\cos\theta_i)e^{-j\beta_1 x\sin\theta_i}.\end{aligned}$$

$e^{i\alpha} = \cos\alpha + i\sin\alpha$

$$\mathbf{H}_i(x, z) = \frac{E_{i0}}{\eta_1}(-\mathbf{a}_x\cos\theta_i + \mathbf{a}_z\sin\theta_i)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}.$$

$$\mathbf{H}_r(x, z) = \frac{E_{i0}}{\eta_1}(-\mathbf{a}_x\cos\theta_i - \mathbf{a}_z\sin\theta_i)e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)}.$$

$\mathbf{H}_1 = \mathbf{H}_i + \mathbf{H}_r$

$$\begin{aligned}\mathbf{H}_1(x, z) = -2\frac{E_{i0}}{\eta_1}[&\mathbf{a}_x\cos\theta_i\cos(\beta_1 z\cos\theta_i)e^{-j\beta_1 x\sin\theta_i}\\ &+ \mathbf{a}_z j\sin\theta_i\sin(\beta_1 z\cos\theta_i)e^{-j\beta_1 x\sin\theta_i}].\end{aligned}$$

The total field

$$\mathbf{E}_1(x,z) = -\mathbf{a}_y j2E_{i0}\sin(\beta_1 z\cos\theta_i)e^{-j\beta_1 x\sin\theta_i}.$$

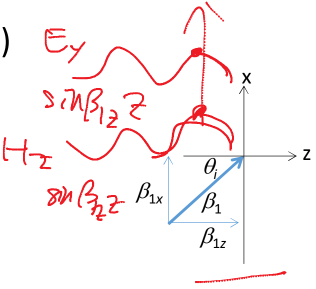
β_{1z}

$$\mathbf{H}_1(x,z) = -2\frac{E_{i0}}{\eta_1}[\mathbf{a}_x\cos\theta_i\cos(\beta_1 z\cos\theta_i)e^{-j\beta_1 x\sin\theta_i} + \mathbf{a}_z j\sin\theta_i\sin(\beta_1 z\cos\theta_i)e^{-j\beta_1 x\sin\theta_i}].$$

E_{1y}

H_{1x} H_{1z}

- 1. Power along the z direction ($\perp$ to boundary)
  - E_{1y}, H_{1x}
  - E_{1y}, H_{1x} maintain standing-wave patterns:
    - ❖ $E_{1y} \sim \sin(\beta_{1z}z)$, $H_{1x} \sim \cos(\beta_{1z}z)$, where $\beta_{1z} = \beta_1\cos\theta_i$
  - No average power in $+z$ direction
    - ❖ $\mathcal{P}_z = 1/2\mathrm{Re}[E_{1y}\times H_{1x}^*]$
    - ❖ $E_{1y} \sim -j\times\exp(-j\beta_{1x}x)$; $H_{1x}^* \sim \exp(+j\beta_{1x}x)$ ➔ $\mathcal{P}_{z,av} = 0$
- 2. Power along the x direction (// to boundary)
  - E_{1y}, H_{1z}
  - Propagation in x direction
    - ❖ $\mathcal{P}_{x,av} = 1/2\mathrm{Re}[E_{1y}\times H_{1z}^*] \neq 0$
    - ❖ E_{1y} and H_{1z} are in phase for both time and space (time: $-j$ and $-j$ ($\theta = -90$ degree); space: $\sin(\beta_{1z}z)$)



$$\mathbf{E}_1(x, z) = -\mathbf{a}_y j2E_{i0} \sin(\beta_1 z \cos\theta_i) e^{-j\beta_1 x \sin\theta_i}.$$

$$\mathbf{H}_1(x, z) = -2\frac{E_{i0}}{\eta_1}\left[\mathbf{a}_x \cos\theta_i \cos(\beta_1 z \cos\theta_i) e^{-j\beta_1 x \sin\theta_i} + \mathbf{a}_z j \sin\theta_i \sin(\beta_1 z \cos\theta_i) e^{-j\beta_1 x \sin\theta_i}\right].$$

E_{1y}

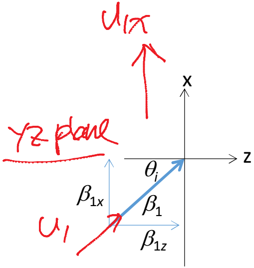
H_{1x} H_{1z}

- Phase velocity in x direction $u_p = \omega/\beta_{1x}$ (faster than u_1)

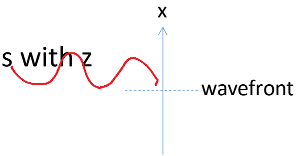
$$u_{1x} = \frac{\omega}{\beta_{1x}} = \frac{\omega}{\beta_1 \sin\theta_i} = \frac{u_1}{\sin\theta_i}.$$

- Wavelength in x direction $\lambda = 2\pi/\beta_{1x}$ (longer than λ_1)

$$\lambda_{1x} = \frac{2\pi}{\beta_{1x}} = \frac{\lambda_1}{\sin\theta_i}.$$



- 3. A nonuniform plane wave for the propagating wave in x direction
  - E_{1y} (or H_{1z}) ~ $\sin(\beta_{1z} z)$ ➔ Amplitude varies with z



$$\mathbf{E}_1(x, z) = -\mathbf{a}_y j2E_{i0} \underline{\sin(\beta_1 z \cos\theta_i)} e^{-j\beta_1 x \sin\theta_i}.$$

$$\mathbf{H}_1(x, z) = -2\frac{E_{i0}}{\eta_1}[\mathbf{a}_x \cos\theta_i \cos(\beta_1 z \cos\theta_i) e^{-j\beta_1 x \sin\theta_i} + \mathbf{a}_z j \sin\theta_i \sin(\beta_1 z \cos\theta_i) e^{-j\beta_1 x \sin\theta_i}].$$

E_{1y}

H_{1x} H_{1z}

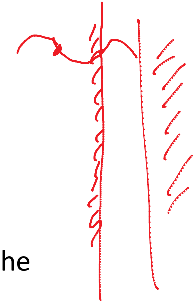
- 4. $\mathbf{E_1}$ = 0 for all x when $\sin(\beta_{1z} z)$=0

$$\beta_1 z \cos\theta_i = \frac{2\pi}{\lambda_1} z \cos\theta_i = -m\pi, \qquad m = 1, 2, 3, \ldots,$$

  - That is, a conducting plate could be inserted at

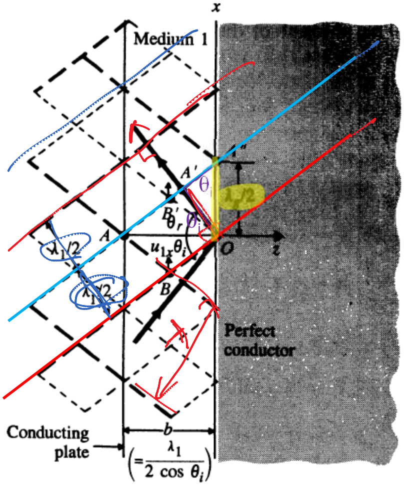
$$z = -\frac{m\lambda_1}{2\cos\theta_i}, \qquad m = 1, 2, 3, \ldots,$$

  without changing $\mathbf{E_1}$ between the conducting plate and the conducting boundary
  ➔ A transverse electric (TE) wave (**E** transverse to plan of propagation (xz plane) ➔ only $E_{1y} \neq 0$) would bounce back and forth (i.e., a waveguide).



Long (thick) dashed line: plane wave crests (peaks)
Short (thin) dashed line: plane wave troughs (valleys)

- Conducting surfaces: $\mathbf{E}_i$ and $\mathbf{E}_r$ are 180°out of phase ➔ $\mathbf{E}(z = 0) = 0$, such as O, A, and A''
- Intersections of two crests: $\mathbf{E}$ = max, along $\mathbf{a}_y$ (e.g., B)
  Intersections of two troughs: $\mathbf{E}$ = min, along $-\mathbf{a}_y$ (e.g., B')
- OA': reflected wave from a crest (red) to a trough (blue), $= \lambda_1/2$ ➔ $\overline{OA'} = \frac{\lambda_1}{2} = \frac{\pi}{\beta_1}$,



- OA: length from the inserted plate to the boundary, b ➔

$$\overline{OA} = b = \frac{\lambda_1}{2\cos\theta_i}.$$

Also recall $\boxed{z = -\frac{m\lambda_1}{2\cos\theta_i}, \qquad m = 1, 2, 3, \ldots,}$

- Wavelength in x direction (traveling wave in the parallel-plate waveguide), λ ·
  OA''= $\lambda_g/2$

$$\frac{\lambda_x}{2} = \frac{\lambda_g}{2}$$

$$\lambda_g = 2\overline{OA''} = 2\frac{\overline{OA'}}{\sin\theta_i} \qquad \frac{\lambda_1}{2}$$

$$= \frac{\lambda_1}{\sin\theta_i} > \lambda_1. \qquad \text{(longer)}$$

- At $\theta_i = 0$ (normal incidence) ➔ no propagating wave in x direction

FIGURE 8–12
Illustrating bouncing waves and interference patterns of oblique incidence at a plane conducting boundary (perpendicular polarization).

8-7.2 Parallel Polarization

The incident wave

$$\mathbf{a}_{ni} = \mathbf{a}_x \sin\theta_i + \mathbf{a}_z \cos\theta_i,$$

$\mathbf{k} = \beta_1 \mathbf{a}_{ni}$
$\mathbf{R} = \mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z$

$$\mathbf{E}(\mathbf{R}) = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{R}} = \mathbf{E}_0 e^{-jk\mathbf{a}_n\cdot\mathbf{R}}$$

$$\mathbf{H}(\mathbf{R}) = \frac{1}{\eta}\mathbf{a}_n \times \mathbf{E}(\mathbf{R})$$

or

$$\mathbf{H}(\mathbf{R}) = \mathbf{H}_0 e^{-jk\mathbf{a}_n\cdot\mathbf{R}},$$

$$\mathbf{E}(\mathbf{R}) = -\eta\mathbf{a}_n \times \mathbf{H}(\mathbf{R})$$

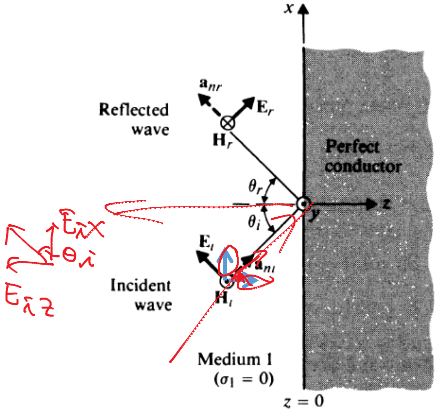
$\mathbf{E}_i$, $\mathbf{E}_r$: $\mathbf{a}_x$, $\mathbf{a}_z$ components
$\mathbf{H}_i$, $\mathbf{H}_r$: $\mathbf{a}_y$ component

$$\mathbf{E}_i(x, z) = E_{i0}(\mathbf{a}_x \cos\theta_i - \mathbf{a}_z \sin\theta_i)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)},$$

$$\mathbf{H}_i(x, z) = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}.$$

Same direction of propagation

Different polarization



Parallel polarization:
also called vertical polarization, H-polarization, p-polarization

FIGURE 8–13
Plane wave incident obliquely on a plane conducting boundary (parallel polarization).

The reflected wave

$$\mathbf{a}_{nr} = \mathbf{a}_x \sin\theta_r - \mathbf{a}_z \cos\theta_r,$$

$$\mathbf{E}_i(x,z) = E_{i0}(\mathbf{a}_x \cos\theta_i - \mathbf{a}_z \sin\theta_i) e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)},$$

$$\mathbf{H}_i(x,z) = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}.$$

1. Propagation direction in phase term
2. Denote polarization

$$\mathbf{E}_r(x,z) = E_{r0}(\mathbf{a}_x \cos\theta_r + \mathbf{a}_z \sin\theta_r) e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)},$$

$$\mathbf{H}_r(x,z) = -\mathbf{a}_y \frac{E_{r0}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}.$$

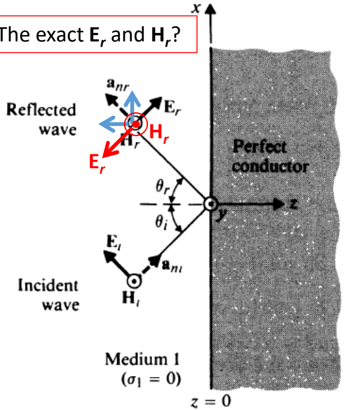
Polarization

Propagation (same direction)

$\vec{E} = -\eta\, \hat{a}_n \times \vec{H}$

θ_r: angle of reflection

The exact $\mathbf{E}_r$ and $\mathbf{H}_r$?



B.C.: $E_{1t} = E_{2t} = 0$

➔ Total E_{1x} at boundary = 0

$$E_{ix}(x,0) + E_{rx}(x,0) = 0.$$

$$(E_{i0}\cos\theta_i)e^{-j\beta_1 x\sin\theta_i} + (E_{r0}\cos\theta_r)e^{-j\beta_1 x\sin\theta_r} = 0,$$

Should be satisfied for all x

$$E_{r0} = -E_{i0} \qquad \theta_r = \theta_i$$

Snell's law of reflection

FIGURE 8–13
Plane wave incident obliquely on a plane conducting boundary (parallel polarization).

The total field

$$\mathbf{E}_i(x, z) = E_{i0}(\mathbf{a}_x \cos\theta_i - \mathbf{a}_z \sin\theta_i)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)},$$

$$\mathbf{E}_r(x, z) = E_{r0}(\mathbf{a}_x \cos\theta_r + \mathbf{a}_z \sin\theta_r)e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)},$$

$$E_{r0} = -E_{i0} \qquad \theta_r = \theta_i.$$

$\mathbf{E}_1 = \mathbf{E}_i + \mathbf{E}_r$

$$\begin{aligned}\mathbf{E}_1(x, z) &= \mathbf{E}_i(x, z) + \mathbf{E}_r(x, z)\\ &= \mathbf{a}_x E_{i0}\cos\theta_i(e^{-j\beta_1 z\cos\theta_i} - e^{j\beta_1 z\cos\theta_i})e^{-j\beta_1 x\sin\theta_i}\\ &\quad - \mathbf{a}_z E_{i0}\sin\theta_i(e^{-j\beta_1 z\cos\theta_i} + e^{j\beta_1 z\cos\theta_i})e^{-j\beta_1 x\sin\theta_i}\end{aligned}$$

or
$$\begin{aligned}\mathbf{E}_1(x, z) = -2E_{i0}[&\mathbf{a}_x j\cos\theta_i \sin(\beta_1 z\cos\theta_i)\\ &+ \mathbf{a}_z \sin\theta_i \cos(\beta_1 z\cos\theta_i)]e^{-j\beta_1 x\sin\theta_i}.\end{aligned}$$

$$\mathbf{H}_i(x, z) = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}.$$

$$\mathbf{H}_r(x, z) = -\mathbf{a}_y \frac{E_{r0}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}.$$

$\mathbf{H}_1 = \mathbf{H}_i + \mathbf{H}_r$

$$\begin{aligned}\mathbf{H}_1(x, z) &= \mathbf{H}_i(x, z) + \mathbf{H}_r(x, z)\\ &= \mathbf{a}_y 2\frac{E_{i0}}{\eta_1}\cos(\beta_1 z\cos\theta_i)e^{-j\beta_1 x\sin\theta_i}.\end{aligned}$$

Similar notes as in perpendicular polarization

$$\mathbf{E}_1(x,z) = -2E_{i0}[\mathbf{a}_x j\cos\theta_i \sin(\beta_1 z\cos\theta_i) + \mathbf{a}_z \sin\theta_i \cos(\beta_1 z\cos\theta_i)]e^{-j\beta_1 x\sin\theta_i}.$$

$$\mathbf{H}_1(x,z) = \mathbf{a}_y 2\frac{E_{i0}}{\eta_1}\cos(\beta_1 z\cos\theta_i)e^{-j\beta_1 x\sin\theta_i}.$$

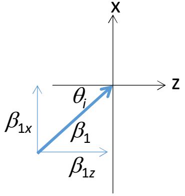
E_{1x} E_{1z}

H_{1y}

- 1. Power along the z direction ($\perp$ to boundary)
  - E_{1x}, H_{1y}
  - E_{1x}, H_{1y} maintain standing-wave patterns:
    - ❖ $E_{1x} \sim \sin(\beta_{1z}z)$, $H_{1y} \sim \cos(\beta_{1z}z)$, where $\beta_{1z} = \beta_1\cos\theta_i$
  - No average power in +z direction
    - ❖ $\mathcal{P}_z = 1/2\mathrm{Re}[E_{1x}\times H_{1y}^*]$
    - ❖ $E_{1x} \sim -j\times\exp(-j\beta_{1x}x)$; $H_{1y}(t)^* \sim \exp(+j\beta_{1x}x)$ ➔ $\mathcal{P}_z = 0$

av

- 2. Power along the x direction (// to boundary)
  - E_{1z}, H_{1y}
  - Propagation in x direction (E_{1z}, H_{1y})
    - ❖ $\mathcal{P}_x = 1/2\mathrm{Re}[E_{1z}\times H_{1y}^*] \neq 0$
    - ❖ E_{1z} and H_{1y} are in phase in both time and space (time: $\theta = 0$; space: $\cos(\beta_{1z}z)$)



$$\mathbf{E}_1(x, z) = -2E_{i0}[\mathbf{a}_x j \cos\theta_i \sin(\beta_1 z \cos\theta_i) + \mathbf{a}_z \sin\theta_i \cos(\beta_1 z \cos\theta_i)]e^{-j\beta_1 x \sin\theta_i}.$$

$$\mathbf{H}_1(x, z) = \mathbf{a}_y 2 \frac{E_{i0}}{\eta_1} \cos(\beta_1 z \cos\theta_i) e^{-j\beta_1 x \sin\theta_i}.$$

E_{1x} E_{1z}

H_{1y}

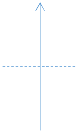
- Phase velocity in x direction $u_p = \omega/\beta_{1x}$ (faster than u_1) $u_{1x} = u_1/\sin\theta_i$
- Wavelength in x direction $\lambda = 2\pi/\beta_{1x}$ (longer than λ_1)

$$\lambda_{1x} = \frac{2\pi}{\beta_{1x}} = \frac{\lambda_1}{\sin\theta_i}.$$

Same as perpendicular polarization

- 3. A nonuniform plane wave for the propagating wave in x direction
  - H_{1y} (or E_{1z}) ~ $\cos(\beta_{1z} z)$ ➔ Amplitude varies with z

Same as perpendicular polarization



$$\mathbf{E}_1(x, z) = -2E_{i0}[\mathbf{a}_x j \cos\theta_i \sin(\beta_1 z \cos\theta_i) + \mathbf{a}_z \sin\theta_i \cos(\beta_1 z \cos\theta_i)]e^{-j\beta_1 x \sin\theta_i}.$$

$$\mathbf{H}_1(x, z) = \mathbf{a}_y 2 \frac{E_{i0}}{\eta_1} \cos(\beta_1 z \cos\theta_i) e^{-j\beta_1 x \sin\theta_i}.$$

E_{1x} E_{1z}

H_{1y}

- 4. $\mathbf{E}_{1x}$ = 0 for all x when $\sin(\beta_{1x} z) = 0$

$$\beta_1 z \cos\theta_i = \frac{2\pi}{\lambda_1} z \cos\theta_i = -m\pi, \qquad m = 1, 2, 3, \ldots,$$

  - That is, a conducting plate could be inserted at

$$z = -\frac{m\lambda_1}{2\cos\theta_i}, \qquad m = 1, 2, 3, \ldots,$$

without changing $\mathbf{E}_{1x}$ (tangential component) between the conducting plate and the conducting boundary
➔ A transverse magnetic (TM) wave (H transverse to plan of propagation (xz plane) ➔ only $H_{1y} \neq 0$) would bounce back and forth (i.e., a waveguide).

A Summary of Oblique Incidence

- Perpendicular polarization

$\bar{E} \times \bar{H}$

$$\mathbf{E}_1(x, z) = -\mathbf{a}_y j2E_{i0} \sin(\beta_1 z \cos\theta_i) e^{-j\beta_1 x \sin\theta_i}.$$

$$\mathbf{H}_1(x, z) = -2\frac{E_{i0}}{\eta_1}\left[\mathbf{a}_x \cos\theta_i \cos(\beta_1 z \cos\theta_i) e^{-j\beta_1 x \sin\theta_i} + \mathbf{a}_z j \sin\theta_i \sin(\beta_1 z \cos\theta_i) e^{-j\beta_1 x \sin\theta_i}\right].$$

E_{1y} $\hat{z}$

H_{1x} H_{1z} $\hat{x}$

- Parallel polarization

$$\mathbf{E}_1(x, z) = -2E_{i0}[\mathbf{a}_x j \cos\theta_i \sin(\beta_1 z \cos\theta_i) + \mathbf{a}_z \sin\theta_i \cos(\beta_1 z \cos\theta_i)] e^{-j\beta_1 x \sin\theta_i}.$$

$$\mathbf{H}_1(x, z) = \mathbf{a}_y 2\frac{E_{i0}}{\eta_1} \cos(\beta_1 z \cos\theta_i) e^{-j\beta_1 x \sin\theta_i}.$$

E_{1x} E_{1z} $\hat{z}$

H_{1y} $\hat{x}$

8-8 Normal Incidence at a Plane Dielectric Boundary

- When an electromagnetic wave is incident on the surface of a dielectric medium that has an intrinsic impedance different from that of the medium in which the wave is originated, part of incident power is reflected and part is transmitted.
- For a normal plane wave incident on a plane dielectric medium:
  - Dissipationless media ($\sigma_1 = \sigma_2 = 0$)
  - Normal incidence (8-8); Oblique incidence (8-10)

The incident wave

$$\mathbf{E}_i(z) = \mathbf{a}_x E_{i0} e^{-j\beta_1 z},$$

$$\mathbf{H}_i(z) = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z}.$$

Propagating in +z

$$\vec{H} = \frac{1}{\eta}\hat{a}_n \times \vec{E}$$

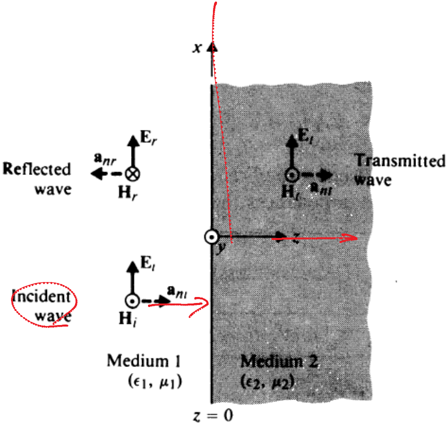


FIGURE 8–14
Plane wave incident normally on a plane dielectric boundary.

The reflected wave

$$\mathbf{E}_r(z) = \mathbf{a}_x E_{r0} e^{j\beta_1 z},$$

$$\mathbf{H}_r(z) = (-\mathbf{a}_z) \times \frac{1}{\eta_1} \mathbf{E}_r(z) = -\mathbf{a}_y \frac{E_{r0}}{\eta_1} e^{j\beta_1 z}.$$

Propagating in –z

$$\vec{H} = \frac{1}{\eta} \hat{a}_n \times \vec{E}$$

$$(-\hat{a}_z)$$

The transmitted wave

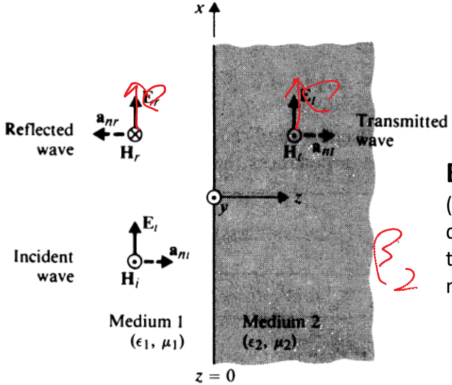
$$\mathbf{E}_t(z) = \mathbf{a}_x E_{t0} e^{-j\beta_2 z},$$

$$\mathbf{H}_t(z) = \mathbf{a}_z \times \frac{1}{\eta_2} \mathbf{E}_t(z) = \mathbf{a}_y \frac{E_{t0}}{\eta_2} e^{-j\beta_2 z},$$

Propagating in +z

E_{t0}: magnitude of $\mathbf{E}_t$
β_2: phase constant of medium 2
η_2: intrinsic impedance of medium 2

β_1



β_2

$\mathbf{E}_r$ and $\mathbf{E}_t$ are drawn arbitrarily

(E_{r0} and E_{t0} may be positive or negative, depending on the relative magnitudes of the constitutive parameters of the two media.)

FIGURE 8–14
Plane wave incident normally on a plane dielectric boundary.

$$\mathbf{E}_i(z) = \mathbf{a}_x E_{i0} e^{-j\beta_1 z},$$

$$\mathbf{H}_i(z) = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z}.$$

$$\mathbf{E}_r(z) = \mathbf{a}_x E_{r0} e^{j\beta_1 z},$$

$$\mathbf{H}_r(z) = (-\mathbf{a}_z) \times \frac{1}{\eta_1} \mathbf{E}_r(z) = -\mathbf{a}_y \frac{E_{r0}}{\eta_1} e^{j\beta_1 z}.$$

$$\mathbf{E}_t(z) = \mathbf{a}_x E_{t0} e^{-j\beta_2 z},$$

$$\mathbf{H}_t(z) = \mathbf{a}_z \times \frac{1}{\eta_2} \mathbf{E}_t(z) = \mathbf{a}_y \frac{E_{t0}}{\eta_2} e^{-j\beta_2 z},$$

Two unknowns: E_{r0} and E_{t0}

Two B.C. equations: $E_{1t} = E_{2t}$; $H_{1t} = H_{2t}$ (J_s =0)

$z=0$

$$\mathbf{E}_i(0) + \mathbf{E}_r(0) = \mathbf{E}_t(0) \quad \text{or} \quad E_{i0} + E_{r0} = E_{t0}$$

$$\mathbf{H}_i(0) + \mathbf{H}_r(0) = \mathbf{H}_t(0) \quad \text{or} \quad \frac{1}{\eta_1}(E_{i0} - E_{r0}) = \frac{E_{t0}}{\eta_2}.$$

$$E_{r0} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E_{i0},$$

$$E_{t0} = \frac{2\eta_2}{\eta_2 + \eta_1} E_{i0}.$$

$$E_{r0} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E_{i0},$$

$$E_{t0} = \frac{2\eta_2}{\eta_2 + \eta_1} E_{i0}.$$

Reflection coefficient
$= E_{r0}/E_{i0}$

$$\boxed{\Gamma = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \quad \text{(Dimensionless)}}$$

Transmission coefficient
$= E_{t0}/E_{i0}$

$$\boxed{\tau = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2}{\eta_2 + \eta_1} \quad \text{(Dimensionless).}}$$

Γ can be positive or negative
τ is always positive

$$\boxed{1 + \Gamma = \tau \quad \text{(Dimensionless).}}$$

For dissipative media (η_1 and η_2 are complex), Γ and τ equations still apply*.
➔ Γ and τ are complex in the general case
➔ a phase shift is introduced at the interface upon reflection (or transmission)

*: the previous equations can be derived by considering complex η_c for **lossy media**, and complex $\mathbf{k}_c$ for **wave with attenuation** (they are all connected).

$$\Gamma = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \qquad \text{(Dimensionless)}$$

$$\tau = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2}{\eta_2 + \eta_1} \qquad \text{(Dimensionless).}$$

→ | PEC

If medium 2 is a perfect conductor ($\eta_2 = 0$)

➔ $\Gamma = -1$, $\tau = 0$

➔ $E_{r0} = -E_{i0}$, $E_{t0} = 0$

➔ The incident wave is totally reflected (as discussed in Section 8-6)

If medium 2 is NOT a perfect conductor

➔ Partial reflection, partial transmission

➔ Total field in medium 1

E_{r0} replaced

Propagation –z

$$\begin{aligned}\mathbf{E}_1(z) &= \mathbf{E}_i(z) + \mathbf{E}_r(z) = \mathbf{a}_x E_{i0}(e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z}) \\ &= \mathbf{a}_x E_{i0}[(1+\Gamma)e^{-j\beta_1 z} + \Gamma(e^{j\beta_1 z} - e^{-j\beta_1 z})] \\ &= \mathbf{a}_x E_{i0}[(1+\Gamma)e^{-j\beta_1 z} + \Gamma(j2\sin\beta_1 z)]\end{aligned}$$

$$1 + \Gamma = \tau$$

$$\mathbf{E}_1(z) = \mathbf{a}_x E_{i0}[\tau e^{-j\beta_1 z} + \Gamma(j2\sin\beta_1 z)].$$

$$\mathbf{E}_1(z) = \mathbf{a}_x E_{i0}[\tau e^{-j\beta_1 z} + \Gamma(j2\sin\beta_1 z)].$$

1st term: a traveling wave with an amplitude τE_{i0}

2nd term: a standing wave with an amplitude $2\Gamma E_{i0}$

How to know?

Check in time domain:

traveling wave: $\cos(\omega t - \beta_1 z)$

standing wave: $\sin(\beta_1 z)\,(-\sin(\omega t))$ ⇒ $|\mathbf{E}_1|$ has locations of max. and min. values?

$$\mathbf{E}_1(z) = \mathbf{a}_x E_{i0}(e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z})$$

$$\boxed{\Gamma = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}}$$

$$\mathbf{E}_1(z) = \mathbf{a}_x E_{i0} e^{-j\beta_1 z}(1 + \Gamma e^{j2\beta_1 z}).$$

$$\boxed{\tau = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2}{\eta_2 + \eta_1}}$$

For dissipationless media, η_1 and η_2 are real >0

➔ Γ and τ are real; Γ can be positive or negative

Check amplitude $|\mathbf{E}_1(z)|$

(1) $\Gamma > 0$ ($\eta_2 > \eta_1$)

max. of $|\mathbf{E}_1(z)|$: $E_{i0}(1+1\Gamma)$, when $2\beta_1 z_{\max} = -2n\pi \; (n = 0, 1, 2, \ldots)$,

or $\quad z_{\max} = -\frac{n\pi}{\beta_1} = -\frac{n\lambda_1}{2}, \qquad n = 0, 1, 2, \ldots$

min. of $|\mathbf{E}_1(z)|$: $E_{i0}(1-1\Gamma)$, when $2\beta_1 z_{\min} = -(2n+1)\pi$ $\quad -\pi, -3\pi, -5\pi \ldots$

or $\quad z_{\min} = -\frac{(2n+1)\pi}{2\beta_1} = -\frac{(2n+1)\lambda_1}{4}, \qquad n = 0, 1, 2, \ldots$

(2) $\Gamma < 0$ ($\eta_2 < \eta_1$)

max. of $|\mathbf{E}_1(z)|$: $E_{i0}(1-1\Gamma)$,

min. of $|\mathbf{E}_1(z)|$: $E_{i0}(1+1\Gamma)$,

In other words, the location for $|E_1(z)|_{\max}$ and $|E_1(z)|_{\min}$ when $\Gamma > 0$ are interchanged when $\Gamma > 0$.

Standing-wave ratio (SWR): ratio of maximum value to the minimum value of |**E**| of a standing wave

$$\boxed{S = \frac{|E|_{\max}}{|E|_{\min}} = \frac{1+|\Gamma|}{1-|\Gamma|} \qquad \text{(Dimensionless).}}$$

Inverse relation of Γ and S

$$\boxed{|\Gamma| = \frac{S-1}{S+1} \qquad \text{(Dimensionless).}}$$

Range of Γ: -1 to 1

Range of S: 1 to ∞

The magnetic field in medium 1:

$$\mathbf{H}_i(z) = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z}.$$

$$\mathbf{H}_r(z) = -\mathbf{a}_y \frac{E_{r0}}{\eta_1} e^{j\beta_1 z}.$$

$$\mathbf{H}_1(z) = \mathbf{a}_y \frac{E_{i0}}{\eta_1} (e^{-j\beta_1 z} - \Gamma e^{j\beta_1 z})$$
$$= \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z}(1 - \Gamma e^{j2\beta_1 z}).$$

Compared with

$$\mathbf{E}_1(z) = \mathbf{a}_x E_{i0} e^{-j\beta_1 z}(1 + \Gamma e^{j2\beta_1 z}).$$

Compared with $\mathbf{E}_1(z)$: In a dissipationless medium, Γ is real.
$|\mathbf{H}_1(z)|$ is max. at locations where $|\mathbf{E}_1(z)|$ is min.
$|\mathbf{H}_1(z)|$ is min. at locations where $|\mathbf{E}_1(z)|$ is max.

The magnetic field in medium 2 (expressed in terms of E_{i0} and τ):

$$\mathbf{H}_t(z) = \mathbf{a}_y \frac{E_{t0}}{\eta_2} e^{-j\beta_2 z},$$

$$\mathbf{H}_t(z) = \mathbf{a}_y \frac{\tau}{\eta_2} E_{i0} e^{-j\beta_2 z}.$$

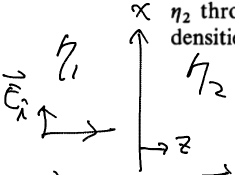
$$\tau = \frac{E_{t0}}{E_{i0}}$$

$$\mathbf{E}_t(z) = \mathbf{a}_x E_{t0} e^{-j\beta_2 z},$$

$$\mathbf{E}_t(z) = \mathbf{a}_x \tau E_{i0} e^{-j\beta_2 z}.$$

$$\vec{P}_{av,2} = \vec{P}_{av,1} \Rightarrow \frac{1-\Gamma^2}{\eta_1} = \frac{\tau^2}{\eta_2}$$

EXAMPLE 8–11 A uniform plane wave in a lossless medium with intrinsic impedance η_1 is incident normally onto another lossless medium with intrinsic impedance η_2 through a plane boundary. Obtain the expressions for the time-average power densities in both media.



$$\tilde{\vec{E}}_i(z) = \hat{a}_x E_{i0} e^{-j\beta_1 z}$$

$$\boxed{\begin{aligned} \vec{E}_1(z) &= \hat{a}_x E_{i0} e^{-j\beta_1 z}(1+\Gamma e^{j2\beta_1 z}) \\ \vec{H}_1(z) &= \hat{a}_y \;--\; --- \end{aligned}}$$

$\vec{P}_{av,1}$ $\qquad$ $\vec{P}_{av,2}$

$$\vec{P}_{av,1} = \frac{1}{2}\mathrm{Re}[\tilde{\vec{E}}_1 \times \tilde{\vec{H}}_1^*]$$

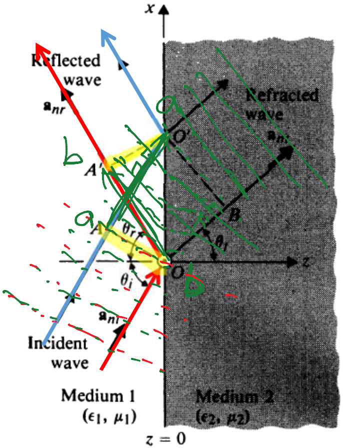
$$= \dots$$

8-9 Normal Incidence at Multiple Dielectric Interfaces (excluded)

8-10 Oblique Incidence at a Plane Dielectric Boundary

- Oblique incidence on a plane interface between two dielectric media.
  - Lossless media assumed

Intersection of wavefronts (surfaces of constant) with the plane of incidence
AO: incident waves
O’A’: reflected waves
O’B: refracted waves



In medium 1, incident and reflected waves propagate with the same u_{p1}

$b \to b$ $OA' = AO'$ $a \to a$

$$\overline{OO'} \sin \theta_r = \overline{OO'} \sin \theta_i$$

or $\boxed{\theta_r = \theta_i.}$ Snell’s law of reflection

FIGURE 8–16
Uniform plane wave incident obliquely on a plane dielectric boundary.

In medium 1, incident waves propagate with u_{p1}
In medium 2, refracted waves propagate with u_{p2}
The same time is taken for OB and AO’

t_{OB} $\qquad$ $OB/u_{p2} = AO'/u_{p1}$ $\qquad$ $t_{AO'}$

$$\frac{\overline{OB}}{\overline{AO'}} = \frac{\overline{OO'}\sin\theta_t}{\overline{OO'}\sin\theta_i} = \frac{u_{p2}}{u_{p1}},$$

$$\boxed{\frac{\sin\theta_t}{\sin\theta_i} = \frac{u_{p2}}{u_{p1}} = \frac{\beta_1}{\beta_2} = \frac{n_1}{n_2},}$$

Snell’s law of refraction

$u_p = \dfrac{\omega}{\beta}$

n: the index of refraction
By definition, n is the ratio of the speed of light in free space to that in the medium ➔ $n = c/u_p$

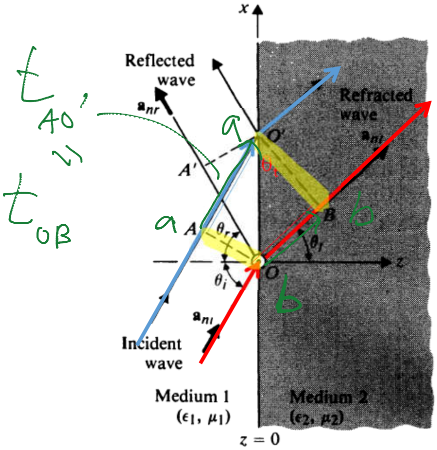


FIGURE 8–16
Uniform plane wave incident obliquely on a plane dielectric boundary.

Snell's law of refraction: at an interface between two dielectric media, the ratio of the sine of the angle of refraction (transmission) in medium 2 to the sine of the angle of incidence in medium 1 is equal to the inverse ratio of indices of refraction n_1/n_2

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{n_1}{n_2}$$

$u_p \sim \frac{1}{\sqrt{\epsilon}}$

For nonmagnetic material, $\mu_1 = \mu_2 = \mu_0$

$$\boxed{\frac{\sin\theta_t}{\sin\theta_i} = \sqrt{\frac{\epsilon_1}{\epsilon_2}} = \sqrt{\frac{\epsilon_{r1}}{\epsilon_{r2}}} = \frac{n_1}{n_2} = \frac{\eta_2}{\eta_1},}$$

$$\boxed{\frac{\sin\theta_t}{\sin\theta_i} = \frac{u_{p2}}{u_{p1}} \qquad u_p = \frac{1}{\sqrt{\epsilon\mu}}}$$

$$\boxed{\eta = \sqrt{\frac{\mu}{\epsilon}}}$$

- $n, \sqrt{\epsilon}, \beta$
- $\sin\theta, u_p, \eta$

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{u_{p1}}{u_{p2}} = \frac{\eta_1}{\eta_2} = \frac{n_2}{n_1} = \sqrt{\frac{\epsilon_2}{\epsilon_1}} = \frac{\beta_2}{\beta_1}$$

For nonmagnetic material, $\mu_1 = \mu_2 = \mu_0$

And if medium 1 is free space: $\varepsilon_{r1} = 1$, $n_1 = 1$, $\eta_1 = 120\pi$

$$\boxed{\frac{\sin\theta_t}{\sin\theta_i} = \frac{1}{\sqrt{\epsilon_{r2}}} = \frac{1}{n_2} = \frac{\eta_2}{120\pi}.}$$

$n_2 > 1 \rightarrow \theta_t < \theta_i$

$\rightarrow$ Wave will be bent toward normal (for oblique incidence to a denser medium)

- In these derivation, no indications of the wave polarizations have been made.

- Snell's law of reflection and Snell's law of refraction are independent of wave polarization.

8-10.1 Total Reflection

$\omega\sqrt{\mu\varepsilon}$

$n_2\downarrow \Rightarrow \theta_t\uparrow$

$u_p = \frac{1}{\sqrt{\mu\varepsilon}}$

$\frac{c}{u_p}$

- For $\varepsilon_1 > \varepsilon_2$:
  - wave in medium 1 is incident on a less dense medium 2
  - $\theta_t > \theta_i$
  - θ_t increases with θ_i; When $\theta_t = \pi/2$, the refracted wave will glaze along the interface.
  - A further increase in θ_i ➔ no refracted wave, and the incident wave is totally reflected.
  - Critical angle θ_c: the angle of θ_i corresponds to $\theta_t = \pi/2$ (threshold of total reflection)

- n, $\sqrt{\epsilon}$, β
- $\sin\theta$, u_p, η

$\sqrt{\frac{\mu}{\varepsilon}}$

Unit vectors for propagation direction
$\mathbf{a}_{ni}$: direction of incident waves
$\mathbf{a}_{nr}$: direction of reflected waves
$\mathbf{a}_{nt}$: direction of transmitted waves

$$\frac{\sin\theta_t}{\sin\theta_i} = \sqrt{\frac{\epsilon_1}{\epsilon_2}} = \frac{n_1}{n_2}$$

$\theta_i = \theta_c,\ \theta_t = \pi/2$

$$\sin\theta_c = \sqrt{\frac{\epsilon_2}{\epsilon_1}}$$

or

$$\boxed{\theta_c = \sin^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}} = \sin^{-1}\left(\frac{n_2}{n_1}\right).}$$

<1

- $n,\ \sqrt{\epsilon},\ \beta$
- $\sin\theta,\ u_p,\ \eta$

$\theta_i = \theta_c$

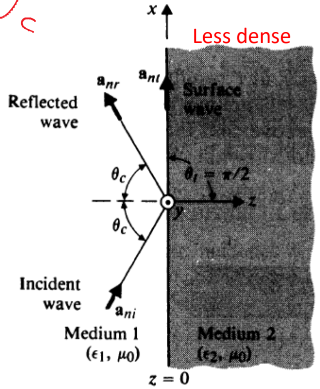


FIGURE 8–17
Plane wave incident at critical angle, $\epsilon_1 > \epsilon_2$.

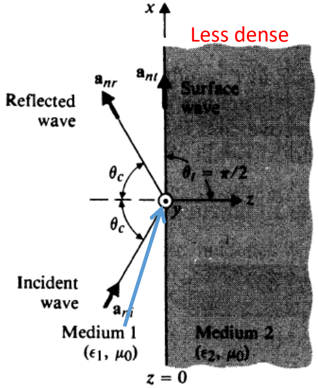
What happens mathematically if $\theta_i > \theta_c$?

$$\boxed{\frac{\sin\theta_t}{\sin\theta_i} = \sqrt{\frac{\epsilon_1}{\epsilon_2}} = \frac{n_1}{n_2}}$$

$$\sin\theta_i > \sin\theta_c = \sqrt{\epsilon_2/\epsilon_1}$$

$$\sin\theta_t = \sqrt{\frac{\epsilon_1}{\epsilon_2}}\sin\theta_i > 1,$$

$$> \sqrt{\frac{\epsilon_2}{\epsilon_1}}$$

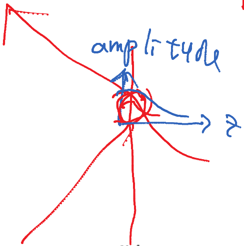


θ_t: not real

$\sin\theta_t$: still real

$$\cos\theta_t = \sqrt{1 - \sin^2\theta_t} = \pm j\sqrt{\frac{\epsilon_1}{\epsilon_2}\sin^2\theta_i - 1}.$$

FIGURE 8–17
Plane wave incident at critical angle, $\epsilon_1 > \epsilon_2$.



$\mathbf{a}_{nt} = \mathbf{a}_x \sin\theta_t + \mathbf{a}_z \cos\theta_t.$

$\mathbf{E}_t, \mathbf{H}_t \sim \exp(-j\,\beta_2\, \mathbf{a}_{nt}\cdot\mathbf{R})$

$$e^{-j\beta_2 \mathbf{a}_{nt}\cdot\mathbf{R}} = e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$-j\beta_2 z\cos\theta_t$$

$$= -j\beta_2 z(-j)\sqrt{\frac{\epsilon_1}{\epsilon_2}\sin^2\theta_i - 1}$$

$$= -z\left(\beta_2\sqrt{\frac{\epsilon_1}{\epsilon_2}\sin^2\theta_i - 1}\right)$$

α_2

$\sin\theta_t = \sqrt{\dfrac{\epsilon_1}{\epsilon_2}}\sin\theta_i$	real
$\cos\theta_t = \pm j\sqrt{\dfrac{\epsilon_1}{\epsilon_2}\sin^2\theta_i - 1}.$	imaginary

$(-j)\times(-j) = -1$

$\beta_2\sqrt{\frac{\epsilon_1}{\epsilon_2}}\sin\theta_i$

$= \beta_2 \sin\theta_t = \beta_{2x}$

$$e^{-\alpha_2 z}e^{-j\beta_{2x}x},$$

where $\alpha_2 = \beta_2\sqrt{(\epsilon_1/\epsilon_2)\sin^2\theta_i - 1}$

$$\beta_{2x} = \beta_2\sqrt{\epsilon_1/\epsilon_2}\,\sin\theta_i.$$

$+j$

$\Downarrow$

$+\alpha_2 z$

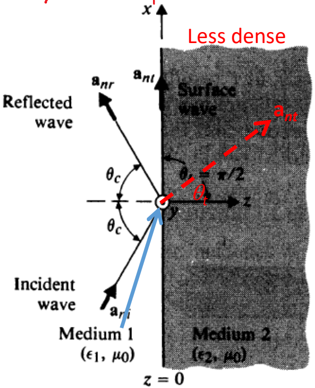


FIGURE 8–17
Plane wave incident at critical angle, $\epsilon_1 > \epsilon_2$.

Evanescent wave exists along the interface (in the x direction) $e^{-\alpha_2 z} e^{-j\beta_{2x} x}$,

$e^{-\alpha_2 z}$ The evanescent wave attenuated exponentially (rapidly) in medium 2 in the normal direction (z direction);
No power is transmitted into medium 2

$e^{-j\beta_{2x} x}$ The wave is tightly bound to the interface and is called a surface wave (Not a uniform plane wave due to $\exp(-\alpha_2 z)$)

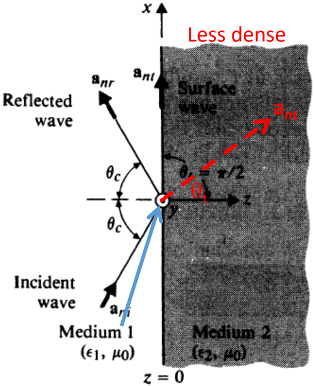


FIGURE 8–17
Plane wave incident at critical angle, $\epsilon_1 > \epsilon_2$.

Evanescent wave:
Attenuation in z direction
Propagation along x direction

8-10.2 Perpendicular Polarization

- **E** $\perp$ the plane of incidence
- Also called s-polarization (German origin: s = senkrecht = perpendicular)
- TE

$\perp$ electric field

The incident fields

$$\mathbf{E}_i(x, z) = \mathbf{a}_y E_{i0} e^{-j\beta_1(x \sin\theta_i + z\cos\theta_i)}$$

$$\mathbf{H}_i(x, z) = \frac{E_{i0}}{\eta_1}(-\mathbf{a}_x \cos\theta_i + \mathbf{a}_z \sin\theta_i) e^{-j\beta_1(x \sin\theta_i + z\cos\theta_i)}.$$

$$\mathbf{H}(\mathbf{R}) = \frac{1}{\eta}\mathbf{a}_n \times \mathbf{E}(\mathbf{R})$$

polarization

propagation

$$a_{ni} = a_z \cos\theta_i + a_x \sin\theta_i$$

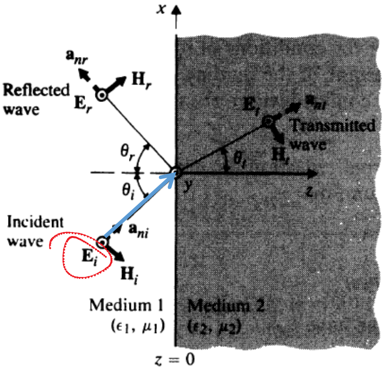


FIGURE 8–20
Plane wave incident on a plane dielectric boundary (perpendicular polarization).

The reflected fields

$$\mathbf{E}_r(x, z) = \mathbf{a}_y E_{r0} e^{-j\beta_1(x \sin\theta_r - z \cos\theta_r)}$$

$$\mathbf{H}_r(x, z) = \frac{E_{r0}}{\eta_1}(\mathbf{a}_x \cos\theta_r + \mathbf{a}_z \sin\theta_r) e^{-j\beta_1(x \sin\theta_r - z \cos\theta_r)}.$$

polarization

propagation

$$\mathbf{H}(\mathbf{R}) = \frac{1}{\eta}\mathbf{a}_n \times \mathbf{E}(\mathbf{R})$$

method 1 to get $\vec{H}$

method 2 to get $\vec{H}$: refer to the figure

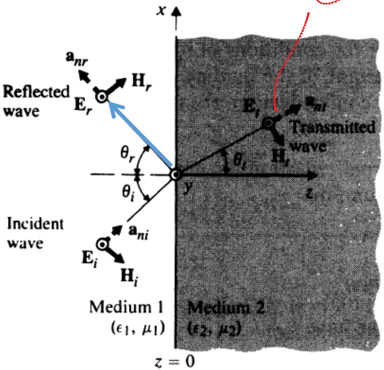


FIGURE 8–20
Plane wave incident on a plane dielectric boundary (perpendicular polarization).

The transmitted fields

$$\mathbf{E}_t(x, z) = \mathbf{a}_y E_{t0} e^{-j\beta_2(x \sin\theta_t + z \cos\theta_t)}$$

$$\mathbf{H}_t(x, z) = \frac{E_{t0}}{\eta_2}\underbrace{(-\mathbf{a}_x \cos\theta_t + \mathbf{a}_z \sin\theta_t)}_{\text{polarization}} \underbrace{e^{-j\beta_2(x \sin\theta_t + z \cos\theta_t)}}_{\text{propagation}}.$$

$$\mathbf{H}(\mathbf{R}) = \frac{1}{\eta}\mathbf{a}_n \times \mathbf{E}(\mathbf{R})$$

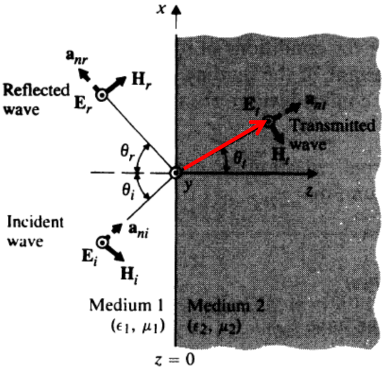


FIGURE 8–20
Plane wave incident on a plane dielectric boundary (perpendicular polarization).

4 unknowns: E_{r0}, E_{t0}, θ_r, θ_t

B.C.: tangential **E** and **H** should be continuous

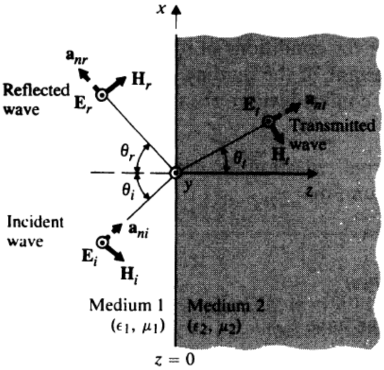


FIGURE 8–20
Plane wave incident on a plane dielectric boundary (perpendicular polarization).

$$\mathbf{E}_i(x, z) = \mathbf{a}_y E_{i0} e^{-j\beta_1(x \sin\theta_i + z\cos\theta_i)}$$

$$\mathbf{H}_i(x, z) = \frac{E_{i0}}{\eta_1}(-\mathbf{a}_x \cos\theta_i + \mathbf{a}_z \sin\theta_i) e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}.$$

$$\mathbf{E}_r(x, z) = \mathbf{a}_y E_{r0} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\mathbf{H}_r(x, z) = \frac{E_{r0}}{\eta_1}(\mathbf{a}_x \cos\theta_r + \mathbf{a}_z \sin\theta_r) e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}.$$

$$\mathbf{E}_t(x, z) = \mathbf{a}_y E_{t0} e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$\mathbf{H}_t(x, z) = \frac{E_{t0}}{\eta_2}(-\mathbf{a}_x \cos\theta_t + \mathbf{a}_z \sin\theta_t) e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}.$$

Tangential **E** and **H** should be continuous

$\boxed{E_{1y}}$

$$E_{iy}(x, 0) + E_{ry}(x, 0) = E_{ty}(x, 0)$$

$$\Downarrow$$

$$E_{i0} e^{-j\beta_1 x \sin\theta_i} + E_{r0} e^{-j\beta_1 x\sin\theta_r} = E_{t0} e^{-j\beta_2 x \sin\theta_t}.$$

$\beta_1 \sin\theta_i = \beta_1 \sin\theta_r = \beta_2 \sin\theta_t$

$\boxed{H_{1x}}$

$$H_{ix}(x, 0) + H_{rx}(x, 0) = H_{tx}(x, 0)$$

$$\Downarrow$$

$$\frac{1}{\eta_1}(-E_{i0}\cos\theta_i e^{-j\beta_1 x\sin\theta_i} + E_{r0}\cos\theta_r e^{-j\beta_1 x \sin\theta_r}) = -\frac{E_{t0}}{\eta_2}\cos\theta_t e^{-j\beta_2 x\sin\theta_t}.$$

$$E_{i0}e^{-j\beta_1 x\sin\theta_i} + E_{r0}e^{-j\beta_1 x\sin\theta_r} = E_{t0}e^{-j\beta_2 x\sin\theta_t}. \qquad (8\text{-}202)$$

$$\frac{1}{\eta_1}\left(-E_{i0}\cos\theta_i e^{-j\beta_1 x\sin\theta_i} + E_{r0}\cos\theta_r e^{-j\beta_1 x\sin\theta_r}\right) = -\frac{E_{t0}}{\eta_2}\cos\theta_t e^{-j\beta_2 x\sin\theta_t}. \qquad (8\text{-}203)$$

The 2 equations are to be satisfied for all *x* (boundary)
➔ Exponential terms that are functions of *x* (phase terms) must be equal (phase matching)

$$\beta_1 x\sin\theta_i = \beta_1 x\sin\theta_r = \beta_2 x\sin\theta_t,$$

$$\theta_r = \theta_i \qquad \sin\theta_t/\sin\theta_i = \beta_1/\beta_2 = n_1/n_2$$

Snell's law of reflection
Snell's law of refraction

Substitute in Eqs. (8-202) and (8-203)

$$E_{i0} + E_{r0} = E_{t0}$$

$$\frac{1}{\eta_1}(E_{i0} - E_{r0})\cos\theta_i = \frac{E_{t0}}{\eta_2}\cos\theta_t,$$

$$E_{i0} + E_{r0} = E_{t0}$$

$$\frac{1}{\eta_1}(E_{i0} - E_{r0}) \cos\theta_i = \frac{E_{t0}}{\eta_2} \cos\theta_t,$$

Derivation

Express E_{r0} and E_{t0}

$$\Gamma_\perp = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

$$= \frac{(\eta_2/\cos\theta_t) - (\eta_1/\cos\theta_i)}{(\eta_2/\cos\theta_t) + (\eta_1/\cos\theta_i)}$$

Fresnel's equations

$$\tau_\perp = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

$$= \frac{2(\eta_2/\cos\theta_t)}{(\eta_2/\cos\theta_t) + (\eta_1/\cos\theta_i)}.$$

Comparison with normal incidence

Normal incidence

$$\Gamma = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

$$\tau = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2}{\eta_2 + \eta_1}$$

$$1 + \Gamma = \tau$$

$$\eta_1 \rightarrow (\eta_1/\cos\theta_i)$$

$$\eta_2 \rightarrow (\eta_2/\cos\theta_t)$$

Oblique incidence

$$\Gamma_\perp = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} = \frac{(\eta_2/\cos\theta_t) - (\eta_1/\cos\theta_i)}{(\eta_2/\cos\theta_t) + (\eta_1/\cos\theta_i)}$$

$$\tau_\perp = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} = \frac{2(\eta_2/\cos\theta_t)}{(\eta_2/\cos\theta_t) + (\eta_1/\cos\theta_i)}.$$

$$\boxed{1 + \Gamma_\perp = \tau_\perp,}$$

When $\theta_i = 0$, $\theta_r = \theta_t = 0$
➔ reduce to normal incidence

If medium 2 is a perfect conductor, η_2=0

$$\Gamma_\perp = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} = \frac{(\eta_2/\cos\theta_i) - (\eta_1/\cos\theta_t)}{(\eta_2/\cos\theta_i) + (\eta_1/\cos\theta_t)}$$

$$\tau_\perp = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} = \frac{2(\eta_2/\cos\theta_t)}{(\eta_2/\cos\theta_t) + (\eta_1/\cos\theta_i)}.$$

$\xrightarrow{\eta_2=0}$

$$\Gamma_\perp = -1 \; (E_{r0} = -E_{i0})$$

$$\tau_\perp = 0 \; (E_{t0} = 0)$$

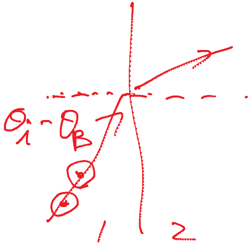
E tangential on the surface of conductor = 0. No energy is transmitted across a perfectly conducting boundary (as was noted).

When reflection = 0 ?

$$\Gamma_\perp = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

$$= \frac{(\eta_2/\cos\theta_t) - (\eta_1/\cos\theta_i)}{(\eta_2/\cos\theta_t) + (\eta_1/\cos\theta_i)} = 0$$

θ_B



$\Gamma_\perp = 0$

Denote the $\theta_i = \theta_{B\perp}$ for no reflection

$$\eta_2 \cos\theta_{B\perp} = \eta_1 \cos\theta_t.$$

Derivation

$$\cos\theta_t = \sqrt{1 - \sin^2\theta_t} = \sqrt{1 - \frac{n_1^2}{n_2^2}\sin^2\theta_i}$$

By Snell's law of refraction

$$\sin^2\theta_{B\perp} = \frac{1 - \mu_1\epsilon_2/\mu_2\epsilon_1}{1 - (\mu_1/\mu_2)^2}.$$

$\theta_{B\perp}$: Brewster angle of no reflection of s-polarization

For nonmagnetic material, $\mu_1 = \mu_2 = \mu_0$, $\theta_{B\perp}$ does not exist.

For materials $\varepsilon_1 = \varepsilon_2$ and $\mu_1 \neq \mu_2$ (**very rare situation**), $\theta_{B\perp}$ exists: $\sin\theta_{B\perp} = \frac{1}{\sqrt{1 + (\mu_1/\mu_2)}}$,

8-10.3 Parallel Polarization

- **E** // the plane of incidence
- p-polarization
- TM

$\perp$ magnetic

$\boxed{\text{The incident fields}}$

$$\mathbf{E}_i(x, z) = E_{i0}(\mathbf{a}_x \cos\theta_i - \mathbf{a}_z \sin\theta_i)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\mathbf{H}_i(x, z) = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$\boxed{\text{polarization}}$ $\boxed{\text{propagation}}$

$$\vec{E} = -\eta\,\hat{a}_n \times \vec{H}$$

method I

$$\mathbf{H}(\mathbf{R}) = \frac{1}{\eta}\mathbf{a}_n \times \mathbf{E}(\mathbf{R})$$

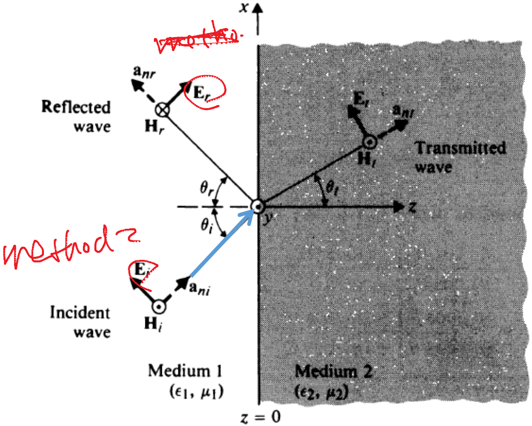


FIGURE 8–21
Plane wave incident obliquely on a plane dielectric boundary (parallel polarization).

The reflected fields

$$\mathbf{E}_r(x, z) = E_{r0}(\mathbf{a}_x \cos\theta_r + \mathbf{a}_z \sin\theta_r)e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)},$$

$$\mathbf{H}_r(x, z) = -\mathbf{a}_y \frac{E_{r0}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}.$$

polarization

propagation

$$\mathbf{H}(\mathbf{R}) = \frac{1}{\eta}\mathbf{a}_n \times \mathbf{E}(\mathbf{R})$$

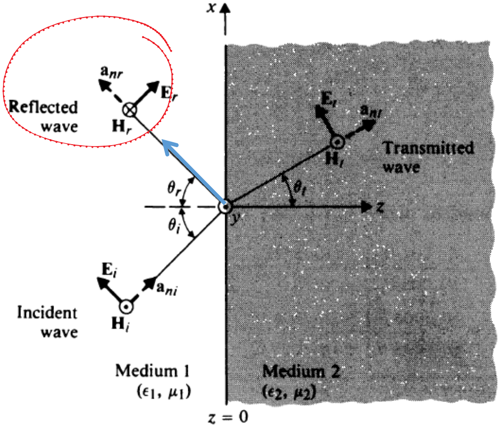


FIGURE 8–21
Plane wave incident obliquely on a plane dielectric boundary (parallel polarization).

The transmitted fields

$$\mathbf{E}_t(x, z) = E_{t0}(\mathbf{a}_x \cos\theta_t - \mathbf{a}_z \sin\theta_t)e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)},$$

$$\mathbf{H}_t(x, z) = \mathbf{a}_y \frac{E_{t0}}{\eta_2} e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}.$$

$$\mathbf{H}(\mathbf{R}) = \frac{1}{\eta}\mathbf{a}_n \times \mathbf{E}(\mathbf{R})$$

polarization

propagation

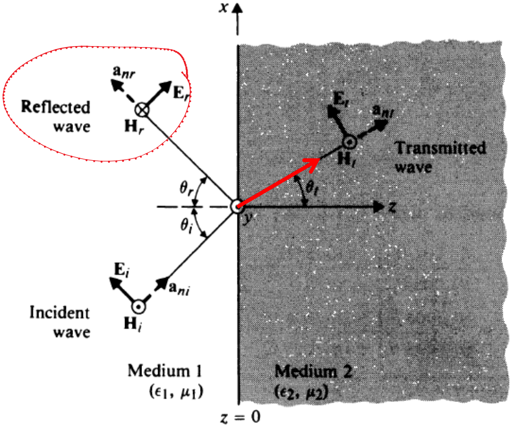


FIGURE 8–21
Plane wave incident obliquely on a plane dielectric boundary (parallel polarization).

$$\mathbf{E}_i(x, z) = E_{i0}(\mathbf{a}_x \cos\theta_i - \mathbf{a}_z \sin\theta_i)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\mathbf{H}_i(x, z) = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\mathbf{E}_r(x, z) = E_{r0}(\mathbf{a}_x \cos\theta_r + \mathbf{a}_z \sin\theta_r)e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)},$$

$$\mathbf{H}_r(x, z) = -\mathbf{a}_y \frac{E_{r0}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}.$$

$$\mathbf{E}_t(x, z) = E_{t0}(\mathbf{a}_x \cos\theta_t - \mathbf{a}_z \sin\theta_t)e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)},$$

$$\mathbf{H}_t(x, z) = \mathbf{a}_y \frac{E_{t0}}{\eta_2} e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}.$$

Tangential **E** and **H** should be continuous at z = 0

$\Downarrow$

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$\Downarrow$

$$(E_{i0} + E_{r0})\cos\theta_i = E_{t0}\cos\theta_t,$$

$$\frac{1}{\eta_1}(E_{i0} - E_{r0}) = \frac{1}{\eta_2}E_{t0}.$$

$$(E_{i0} + E_{r0}) \cos \theta_i = E_{t0} \cos \theta_t,$$

$$\frac{1}{\eta_1}(E_{i0} - E_{r0}) = \frac{1}{\eta_2} E_{t0}.$$

Express E_{r0} and E_{t0}

$$\Gamma_{||} = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 \cos \theta_t - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i}$$

$$\tau_{||} = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i}.$$

Fresnel's equations

$$\boxed{1 + \Gamma_{||} = \tau_{||}\left(\frac{\cos \theta_t}{\cos \theta_i}\right).}$$

Different from the case in s-polarization

$$\boxed{1 + \Gamma_{\perp} = \tau_{\perp},}$$

When $\theta_i = 0$, $\theta_r = \theta_t = 0$

➔ reduce to normal incidence

If medium 2 is a perfect conductor, η_2=0

$$\boxed{\Gamma_{||} = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}}$$

$$\boxed{\tau_{||} = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}.}$$

$\xrightarrow{\eta_2=0}$

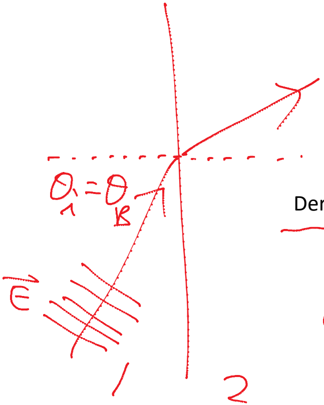
$$\Gamma_{||} = -1$$

$$\tau_{||} = 0$$

E tangential on the surface of conductor = 0.
No energy is transmitted across a perfectly conducting boundary (as was noted).

When is reflection = 0 ?

$$\Gamma_{||} = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$



$\Gamma_{||} = 0$

Denote the $\theta_i = \theta_{B||}$ for no reflection

$$\eta_2 \cos\theta_t = \eta_1 \cos\theta_{B||},$$

Derivation

$$\cos\theta_t = \sqrt{1 - \sin^2\theta_t} = \sqrt{1 - \frac{n_1^2}{n_2^2}\sin^2\theta_i}$$

By Snell's law of refraction

$$\boxed{\sin^2\theta_{B||} = \frac{1 - \mu_2\epsilon_1/\mu_1\epsilon_2}{1 - (\epsilon_1/\epsilon_2)^2}.}$$

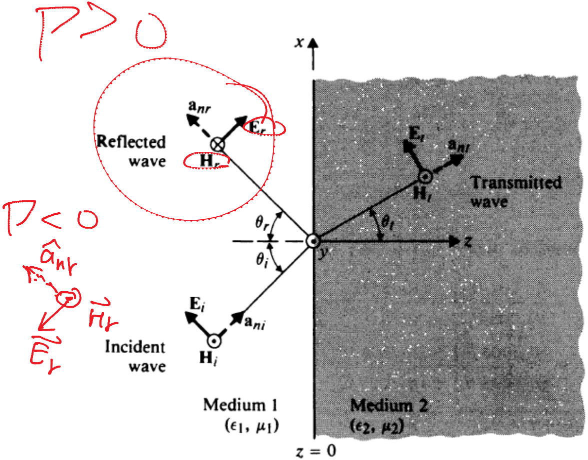
$\theta_{B||}$: Brewster angle of no reflection of p-polarization

$\theta_{B\perp}$

$\theta_{B//}$

Directions of **E**$_r$, **H**$_r$ in figures 8-11, 8-13, 8-20, and 8-21 are chosen arbitrarily.
The actual directions depend on the sign of the expression.

- In Figs. 8-11 and 8-13, actual directions of **E**$_r$ **H**$_r$ are opposite to those chosen because $E_{r0} = -E_{i0}$
- In Figs. 8-20 and 8-21, actual directions of **E**$_r$ **H**$_r$ depends on the sign of $\Gamma_\perp$ and $\Gamma_{||}$, respectively



$$\mathbf{E}_r(x, z) = E_{r0}(\mathbf{a}_x \cos\theta_r + \mathbf{a}_z \sin\theta_r)e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)},$$

$$\mathbf{H}_r(x, z) = -\mathbf{a}_y \frac{E_{r0}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}.$$

$$\Gamma_{||} = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

- If $\Gamma_{||} > 0$, **H**$_r$ is in $-\mathbf{a}_y$ direction (same as shown in figure)
- If $\Gamma_{||} < 0$, **H**$_r$ is in $+\mathbf{a}_y$ direction (opposite to that shown in figure)

FIGURE 8–21
Plane wave incident obliquely on a plane dielectric boundary (parallel polarization).

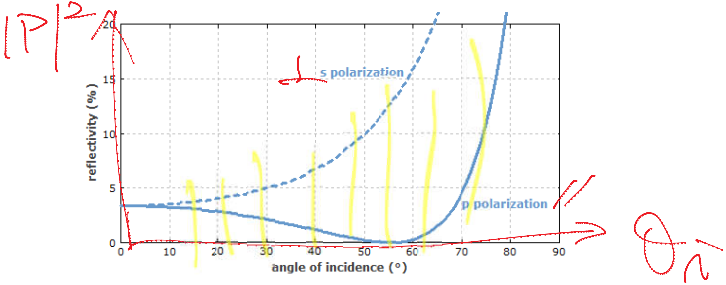
Comparison of $|\Gamma_\perp|^2$ and $|\Gamma_{||}|^2$

$$\Gamma_\perp = \frac{E_{r0}}{E_{i0}} = \frac{(\eta_2/\cos\theta_t) - (\eta_1/\cos\theta_i)}{(\eta_2/\cos\theta_t) + (\eta_1/\cos\theta_i)}$$

$$\Gamma_{||} = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

$|\Gamma_\perp|^2$ is always larger than $|\Gamma_{||}|^2$

➔ When an unpolarized light strikes a plane dielectric interface, the reflected wave will contain more power in s-polarization than p-polarization.

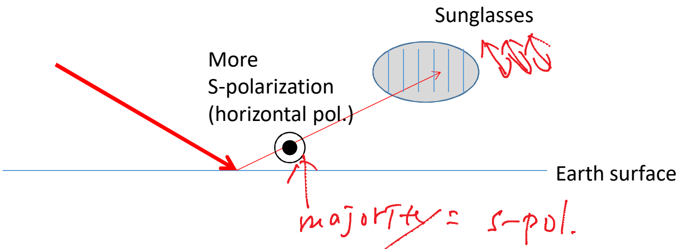


$\mu_1 = \mu_2 = \mu_0$
$n_1 = 1$
$n_2 = 1.45$

Power reflectivity of the interface for s and p polarization, if a beam is incident from air onto a medium with refractive index 1.45 (e.g., silica at 1064 nm).

http://www.rp-photonics.com/fresnel_equations.html

Polaroid Sunglasses



The light reaching the eye is predominately s-polarization
(i,e, **E** $\perp$ plane of reflection, or **E** field is parallel to the earth surface)

Polaroid sunglasses (a polarizer) are designed to filter out this component $\mathbf{E}_{\perp}$
As a result, a dim light of $\mathbf{E}_{//}$ will penetrate into the sunglasses.

For materials $\mu_1 = \mu_2$, $\theta_{B||}$:

$$\boxed{\sin\theta_{B||} = \frac{1}{\sqrt{1+(\epsilon_1/\epsilon_2)}}.} \qquad (\mu_1 = \mu_2)$$

or

$$\boxed{\theta_{B||} = \tan^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}} = \tan^{-1}\left(\frac{n_2}{n_1}\right).} \qquad (\mu_1 = \mu_2)$$

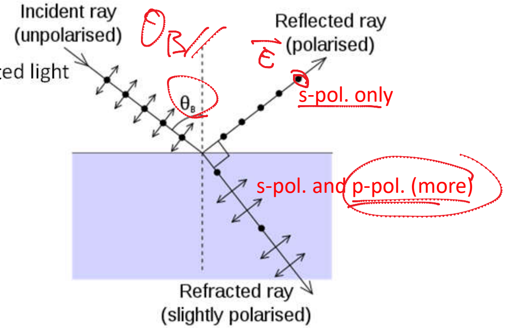
Because of different formulas for Brewster angles for s- and p-polarization, it is possible to separate these 2 types of polarizations from an unpolarized light.

E.g.,

Light is incident at angle $\theta_{B||}$

➔ no p-polarization component in the reflected light

or only s-polarization in the reflected light



(a) nonmagnetic. $\theta_{B,/\!/} = \sin^{-1}\frac{1}{\sqrt{1+\frac{\epsilon_1}{\epsilon_2}}} = \cdots = 83.6^\circ$ *

$\epsilon_1 = \epsilon_0 \cdot 1$

$\epsilon_2 = \epsilon_0 \cdot 80$

$\epsilon_r =$

EXAMPLE 8–15 The dielectric constant of pure water is 80. (a) Determine the Brewster angle for parallel polarization, $\theta_{B\parallel}$, and the corresponding angle of transmission. (b) A plane wave with perpendicular polarization is incident from air on water surface at $\theta_i = \theta_{B\parallel}$. Find the reflection and transmission coefficients.

θ_t

(b) $\Gamma_\perp = \dfrac{\eta_2/\cos\theta_t - \eta_1/\cos\theta_{B/\!/}}{\eta_2/\cos\theta_t - \eta_1/\cos\theta_{B/\!/}}$ ← (a)

$\eta_1 = 120\pi$

$\eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}} = \sqrt{\frac{\mu_0}{\epsilon_0 \cdot 80}} = \frac{120\pi}{\sqrt{80}}$

$\tau_\perp =$