VE230 Electromagnetics

Chapter 2

August 24, 2022

Exercise 2.1

Given three vectors $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ as follows,

$$\begin{aligned}\mathbf{A} &= \mathbf{a}_x + \mathbf{a}_y 2 - \mathbf{a}_z 3\\ \mathbf{B} &= -\mathbf{a}_y 4 + \mathbf{a}_z\\ \mathbf{C} &= \mathbf{a}_x 5 - \mathbf{a}_z 2\end{aligned}$$

find

a) a_A

b) $|\mathbf{A} - \mathbf{B}|$

c) $\mathbf{A} \cdot \mathbf{B}$

d) θ_{AB}

e) the component of $\mathbf{A}$ in the direction of $\mathbf{C}$

f) $\mathrm{A} \times \mathrm{C}$

g) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ and $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$

h) $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ and $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$

Answer:

a) $\bar{a}_A = \frac{\bar{A}}{A} = \frac{\bar{a}_x + \bar{a}_y 2 - \bar{a}_z 3}{\sqrt{1^2 + 2^2 + (-3)^2}} = \frac{1}{\sqrt{14}}(\bar{a}_x + \bar{a}_y 2 - \bar{a}_z 3)$.

b) $|\bar{A} - \bar{B}| = |\bar{a}_x + \bar{a}_y 6 - \bar{a}_2 4| = \sqrt{1^2 + 6^2 + (-4)^2} = \sqrt{53}$.

c) $\bar{A} \cdot \bar{B} = 0 + 2(-4) + (-3) = -11$.

d) $\theta_{AB} = \cos^{-1}(\bar{A} \cdot \bar{B}/AB) = \cos^{-1}(-11/\sqrt{14}\sqrt{17}) = 135.5°$.

e) $\bar{A} \cdot \bar{a}_c = \bar{A} \cdot \frac{\bar{c}}{c} = \bar{A} \cdot \frac{1}{\sqrt{29}}(\bar{a}_x 5 - \bar{a}_z 2) = \frac{11}{\sqrt{29}}$.

f) $\bar{A} \times \bar{c} = -\bar{a}_x 4 - \bar{a}_y 13 - \bar{a}_z 10$.

g) $\bar{A} \cdot (\bar{B} \times \bar{C}) = (\bar{A} \times \bar{B}) \cdot \bar{C} = -42$

h) $(\bar{A} \times \bar{B}) \times \bar{C} = \bar{B}(\bar{A} \cdot \bar{C}) - \bar{A}(\bar{C} \cdot \bar{B}) = \bar{a}_x 2 - \bar{a}_y 40 + \bar{a}_z 5$. $\bar{A} \times (\bar{B} \times \bar{C}) = \bar{B}(\bar{A} \cdot \bar{C}) - \bar{C}(\bar{A} \cdot \bar{B}) = \bar{a}_x 55 - \bar{a}_y 44 - \bar{a}_z 11$.

Exercise 2.2

Given

$$\mathbf{A} = \mathbf{a}_x - \mathbf{a}_y 2 + \mathbf{a}_z 3,$$
$$\mathbf{B} = \mathbf{a}_x + \mathbf{a}_y - \mathbf{a}_z 2,$$

find the expression for a unit vector **C** that is perpendicular to both **A** and **B**.

Answer:
Let $\bar{c} = \bar{a}_x c_x + \bar{a}_y c_y + \bar{a}_z C_z$, where $C_x^2 + c_y^2 + c_z^2 = 1$.
For $\bar{C} \perp \bar{A} : \bar{C} \cdot \bar{A} = 0 \rightarrow C_x - 2C_y + 3C_z = 0$.
For $\bar{C} \perp \bar{B} : \bar{C} \cdot \bar{B} = 0 \rightarrow C_x + C_y - 2C_z = 0$.
Solving (1), (2), and (3) simultaneously, we obtain

$$c_x = \frac{1}{\sqrt{35}}, \quad c_y = \frac{5}{\sqrt{35}}, \quad c_z = \frac{3}{\sqrt{35}},$$

and $\quad \bar{c} = \frac{1}{\sqrt{35}} (\bar{a}_x + \bar{a}_y 5 + \bar{a}_z 3)$.

Exercise 2.3

Two vector fields represented by $\mathbf{A} = \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z$ and $\mathbf{B} = \mathbf{a}_x B_x + \mathbf{a}_y B_y + \mathbf{a}_z B_z$, where all components may be functions of space coordinates. If these two fields are parallel to each other everywhere, what must be the relations between their components?

Answer:

For $\bar{A} \| \bar{B}$ everywhere, $\bar{A} \times \bar{B} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = 0$, which requires that

$$\frac{A_x}{B_x} = \frac{A_y}{B_y} = \frac{A_z}{B_z}$$

Exercise 2.4

Show that, if $\mathbf{A} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{C}$ and $\mathbf{A} \times \mathbf{B} = \mathbf{A} \times \mathbf{C}$, where **A** is not a null vector, then $\mathbf{B} = \mathbf{C}$.

Answer:
From $\bar{A} \cdot \bar{B} = \bar{A} \cdot \bar{C}$ we have $\bar{A} \cdot (\bar{B} - \bar{C}) = 0$. (1)
From $\bar{A} \times \bar{B} = \bar{A} \times \bar{C}$ we have $\bar{A} \times (\bar{B} - \bar{C}) = 0$. (2)
(1) implies $\bar{A} \perp (\bar{B} - \bar{C})$, and (2) implies $\bar{A} \| (\bar{B} - \bar{C})$.
Since A is not a null vector, (1) and (2) cannot hold at the same time unless $(\bar{B} - \bar{C})$ is a null vector. Thus, $\bar{B} - \bar{C} = 0$, or $\bar{B} = \bar{C}$.

Exercise 2.5

An unknown vector can be determined if both its scalar product and its vector product with a known vector are given. Assuming that **A** is a known vector, determine the unknown vector **X** if both p and **B** are given, where $p = \mathbf{A} \cdot \mathbf{X}$ and $\mathbf{B} = \mathbf{A} \times \mathbf{X}$.

Answer:
Expand $\bar{A} \times (\bar{A} \times \bar{X}) = \bar{A}(\bar{A} \cdot \bar{X}) - \bar{X}(\bar{A} \cdot \bar{A})$, or

$$\bar{A} \times \bar{B} = p\bar{A} - A^2\bar{X}.$$

$\therefore \quad \bar{X} = \frac{1}{A^2}(p\bar{A} + \bar{B} \times \bar{A})$.

Exercise 2.6

The three corners of a triangle are at $P_1(0,1,-2), P_2(4,1,-3)$, and $P_3(6,2,5)$.
a) Determine whether $\triangle P_1P_2P_3$ is a right triangle.
b) Find the area of the triangle.

Answer:
Position vectors of the three corners: $\overrightarrow{OP}_1 = \bar{a}_y - \bar{a}_z 2, \quad \overrightarrow{OP}_2 = \bar{a}_x 4 - \bar{a}_y - \bar{a}_z 3, \quad \overrightarrow{OP}_3 = \bar{a}_x 6 + \bar{a}_y 2 + \bar{a}_z 5$

Vectors representing the three sides of the triangle: $\vec{P_1}\vec{P_2} = \overrightarrow{OP}_2 - \overrightarrow{OP}_1 = \bar{a}_x 4 - \bar{a}_z, \vec{P_2}\vec{P_3} = \bar{a}_x 2 + \bar{a}_y + \bar{a}_z 8, \vec{P_3}\vec{P_1} = -\bar{a}_x 6 - \bar{a}_y - \bar{a}_z 7$.

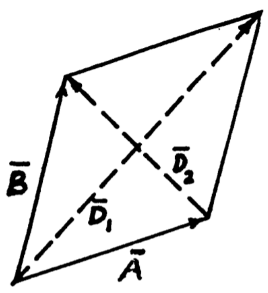
a) $\overrightarrow{P_1P_2} \cdot \overrightarrow{P_2P_3} = 0. \quad \therefore \Delta P_1P_2P_3$ is a right triangle.

b) Area of triangle $= \frac{1}{2}\left|\vec{P_1}\vec{P_2} \times \vec{P_2}\vec{P_3}\right| = 17.1$

Exercise 2.7

Show that the two diagonals of a rhombus are perpendicular to each other. (A rhombus is an equilateral parallelogram.)

Answer:



$$\bar{D}_1 = \bar{B} + \bar{A}, \quad \bar{D}_2 = \bar{B} - \bar{A}.$$

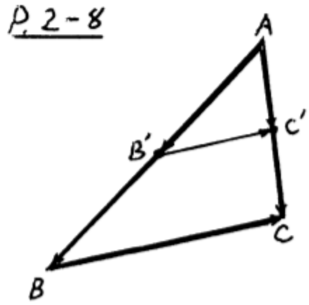
$$\begin{aligned} \bar{D}_1 \cdot \bar{D}_2 &= (\bar{B} + \bar{A}) \cdot (\bar{B} - \bar{A}) \\ &= \bar{B} \cdot \bar{B} - \bar{A} \cdot \bar{A} = 0 \end{aligned}$$

$$\therefore \quad \bar{D}_1 \perp \bar{D}_2.$$

Exercise 2.8

Prove that the line joining the midpoints of two sides of a triangle is parallel to and half as long as the third side.

Answer: Let A, B, and C denote the vertices of a triangle, and B' and C' be the



midpoints of sides AB and AC, respectively. The following vector relations hold:

$$\overrightarrow{AB'} = \frac{1}{2}\overrightarrow{AB}, \quad \overrightarrow{AC'} = \frac{1}{2}\overrightarrow{AC}$$
$$\overrightarrow{B'C'} = \overrightarrow{AC'} - \overrightarrow{AB'} = \frac{1}{2}(\overrightarrow{AC} - \overrightarrow{AB}) = \frac{1}{2}\overrightarrow{BC}$$

Exercise 2.9

Unit vectors $\mathbf{a}_A$ and $\mathbf{a}_B$ denote the directions of two-dimensional vectors $\mathbf{A}$ and $\mathbf{B}$ that make angles α and β, respectively, with a reference x-axis, as shown in Fig. 2-34.

a) Obtain a formula for the expansion of the cosine of the difference of two angles, $\cos(\alpha - \beta)$, by taking the scalar product $\mathbf{a}_A \cdot \mathbf{a}_B$.

b) Obtain a formula for $\sin(\alpha - \beta)$.

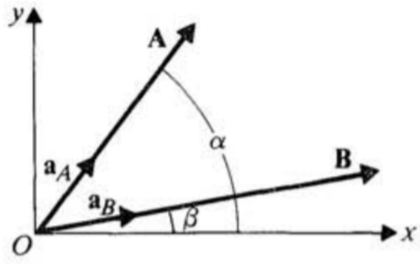


FIGURE 2–34
Graph for Problem P.2–9.

Answer:

$$\bar{a}_A = \bar{a}_x \cos\alpha + \bar{a}_y \sin\alpha,$$
$$\bar{a}_B = \bar{a}_x \cos\beta + \bar{a}_y \sin\beta.$$

a) $\bar{a}_A \cdot \bar{a}_B = \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$

b)

$$\bar{a}_B \times \bar{a}_A = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix} = \bar{a}_z(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$$
$$= \bar{a}_z \sin(\alpha - \beta).$$

$\therefore \sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$

Exercise 2.10

Prove the law of sines for a triangle.

Answer:

$$\bar{A}+\bar{B}+\bar{C}=0.$$
$$\bar{A}\times:\bar{A}\times\bar{B}=\bar{C}\times\bar{A}.$$
$$\bar{C}\times:\bar{C}\times\bar{A}=\bar{B}\times\bar{C}.$$
$$\bar{B}\times:\bar{B}\times\bar{C}=\bar{A}\times\bar{B}.$$

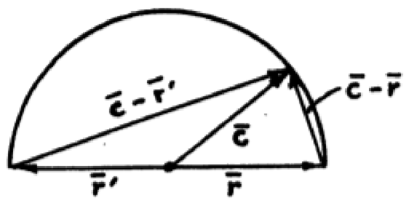
Magnitude relations:

$$AB\sin\theta_{AB}=CA\sin\theta_{CA}=BC\sin\theta_{BC}.$$
$$\frac{A}{\sin\theta_{BC}}=\frac{B}{\sin\theta_{CA}}-\frac{C}{\sin\theta_{AB}}\cdot\left(\begin{array}{c}\text{Law of}\\\text{sines.}\end{array}\right)$$

Exercise 2.11

Prove that an angle inscribed in a semicircle is a right angle.

Answer:



$$\overline{r'}=-\bar{r},\quad r'=r=c.$$
$$(\bar{c}-\bar{r})\cdot(\bar{c}-\bar{r})=(\bar{c}+\bar{r})\cdot(\bar{c}-\bar{r})$$
$$=0.$$
$$\therefore(\bar{c}-\overline{r'})\perp(\bar{c}-\bar{r}).$$

Exercise 2.12

Verify the back-cab rule of the vector triple product of three vectors, as expressed in Eq. $(2-20)$ in Cartesian coordinates.

Answer: 略

Exercise 2.13

Prove by vector relations that two lines in the xy-plane ($L_1: b_1x+b_2y=c; L_2: b_1'x+b_2'y=c'$) are perpendicular if their slopes are the negative reciprocals of each other.

Answer:

Consider line L_1 : $\quad b_1x + b_2y = c$, which has a slope equal to $-b_1/b_2$. Denote the shifted line passing fhrough the origin and parallel to L_1 as L_1^0 : $\quad b_1x + b_2y = 0$. The position vector of a point (x, y) on L_1^0 is

$$\bar{r}_i = \bar{a}_x x + \bar{a}_y y.$$

If we introduce the vector $\bar{n} = \bar{a}_x b_1 + \bar{a}_y b_2$, we can write the equation of L_i^0 as

$$\bar{n} \cdot \bar{r}_1 = 0.$$

Thus the vector $\bar{n}$ is $\perp$ to $\bar{r}_1$, and is normal to both L, and L_1^0. If follows that the two lines L_1 and L_2 are perpendicular to each other if and $0ny_$if their normal vectors $\bar{n}$ and $\bar{n}' = \bar{a}_x b_1' + \bar{a}_y b_2'$ are orthogonal: $\bar{n} \cdot \bar{n}' = 0$, which implies

$$b_1 b_1' + b_2 b_2' = 0, \text{ or } \frac{b_2'}{b_1'} = -\frac{b_1}{b_2};$$

that is, the slopes of lines L_2 and L_1 are the negative reciprocals of each other.

Exercise 2.14

1. Prove that the equation of any plane in space can be written in the form $b_1x + b_2y + b_3z = c$. (Hint: Prove that the dot product of the position vector to any point in the plane and a normal vector is a constant.)

2. Find the expression for the unit normal passing through the origin.

3. For the plane $3x - 2y + 6z = 5$, find the perpendicular distance from the origin to the plane.

Answer:

1. Letting the position vector of a point in the plane be $\bar{R} = \bar{a}_x x + \bar{a}_y y + \bar{a}_z z$ and introducing the vector $\bar{N} = \bar{a}_x b_1 + \bar{a}_y b_2 + \bar{a}_2 b_3$, we can write the given equation as $\bar{R} \cdot \bar{N} = c$ (a constant).

   This shows that the projection of the position vector to any point in the plane on $\bar{N}$ is a constant, and that $\bar{N}$ is a normal vector.

2. $$\bar{a}_N = \frac{\bar{N}}{|\bar{N}|} = \frac{\bar{a}_x b_1 + \bar{a}_y b_2 + \bar{a}_z b_3}{\sqrt{b_1^2 + b_2^2 + b_3^2}}.$$

3. The perpendicular distance from the origin to the plane is

   $$\bar{a}_N \cdot \bar{R} = \frac{c}{|\bar{N}|}.$$

   For aur case, $c = 5, |\bar{N}| = \sqrt{3^2 + (-2)^2 + 6^2} = 7$, and $\bar{a}_N \cdot \bar{R} = 5/7$.

Exercise 2.15

Find the component of the vector $\mathbf{A} = -\mathbf{a}_y z + \mathbf{a}_z y$ at the point $P_1(0, -2, 3)$, which is directed toward the point $P_2\left(\sqrt{3}, -60°, 1\right)$.

Answer:

$$\bar{A}_{p_1} = -\bar{a}_{y^3} - \bar{a}_2 2, \qquad \overrightarrow{OP}_1 = -\bar{a}_{y^2} + \bar{a}_z 3.$$

$$\overline{OP}_2 = \bar{a}_x(r\cos\phi) + \bar{a}_y(r\sin\phi) + \bar{a}_x = \bar{a}_x\frac{\sqrt{3}}{2} - \bar{a}_y\frac{3}{2} + \bar{a}_z.$$

$$\vec{P}_1 = \overrightarrow{OP_2} - \overrightarrow{OP}_1 = \bar{a}_x\frac{\sqrt{3}}{2} + \vec{a}_y\frac{1}{2} - \bar{a}_z 2, \quad \left|\overrightarrow{P_1P_2}\right| = \sqrt{5}$$

$$\bar{A}_{p_1} \cdot \bar{a}_{p_1p_2} = \vec{A}_{P_1} - \frac{\vec{P}_1 P_2}{\left|\vec{P}_1 P_2\right|} = \frac{\sqrt{5}}{2} = 1.12$$

Exercise 2.16

The position of a point in cylindrical coordinates is specified by $(4, 2\pi/3, 3)$. What is the location of the point
a) in Cartesian coordinates?
b) in spherical coordinates?

Answer:
a)

$$\begin{aligned} &x = r\cos\phi = 4\cos(2\pi/3) - -2 \\ &y = r\sin\phi = 4\sin(2\pi/3) = 2\sqrt{3} \\ &z = 3. \end{aligned}$$

b)

$$\begin{aligned} &R = \left(r^2 + z^2\right)^{1/2} = \left(4^2 + 3^2\right)^{1/2} = 5, \\ &\theta = \tan^{-1}(r/z) = \tan^{-1}(4/3) = 53.1° \\ &\phi = 2\pi/3 = 120°. \end{aligned}$$

Exercise 2.17

A field is expressed in spherical coordinates by $\mathbf{E} = \mathbf{a}_R\,(25/R^2)$.
a) Find $|\mathbf{E}|$ and E_x at the point $P(-3, 4, -5)$.
b) Find the angle that $\mathbf{E}$ makes with the vector $\mathbf{B} = \mathbf{a}_x 2 - \mathbf{a}_y 2 + \mathbf{a}_z$ at point P.

Answer:
a)

$$\bar{E}_p = \bar{a}_R\frac{25}{(-3)^2 + 4^2 + (-5)^2} = \bar{a}_R\frac{1}{2}.$$

$$(E_p)_x = \frac{1}{2}\left(\frac{-3}{\sqrt{(-3)^2 + 4^2 + (-5)^2}}\right) = -0.212.$$

b)

$$\bar{a}_R = \frac{1}{\sqrt{50}}\left(-\bar{a}_x 3 + \bar{a}_y 4 - \bar{a}_z 5\right), \bar{a}_B = \frac{\bar{B}}{B} = \frac{1}{3}\left(\bar{a}_x 2 - \bar{a}_y 2 + \bar{a}_z\right).$$

$$\theta = \cos^{-1}\left(\bar{a}_R \cdot \bar{a}_B\right) = \cos^{-1}\left(-\frac{19}{3\sqrt{50}}\right) = 154°.$$

Exercise 2.18

Express the base vectors $\mathbf{a}_R, \mathbf{a}_\theta$, and $\mathbf{a}_\phi$ of a spherical coordinate system in Cartesian coordinates.

Answer:

$$\bar{a}_R = \bar{a}_x \sin\theta\cos\phi + \bar{a}_y \sin\theta\sin\phi + \bar{a}_z\cos\theta = \frac{\overline{a_x}x + \bar{a}_y y + \bar{a}_z z}{\sqrt{x^2+y^2+z^2}},$$

$$\bar{a}_\theta = \bar{a}_x \cos\theta\cos\phi + \bar{a}_y \cos\theta\sin\phi - \bar{a}_x\sin\theta = \frac{\bar{a}_x xz + \bar{a}_y yz - \bar{a}_z\left(x^2+y^2\right)}{\sqrt{\left(x^2+y^2\right)\left(x^2+y^2+z^2\right)}},$$

$$\bar{a}_\phi = -\bar{a}_x\sin\phi + \bar{a}_y\cos\phi = \frac{-\bar{a}_x y + \bar{a}_y x}{\sqrt{x^2+y^2}}.$$

Exercise 2.19

Determine the values of the following products of base vectors:

a) $a_x \cdot a_\phi$

b) $a_\theta \cdot a_y$

c) $a_r \times a_x$

d) $a_R \cdot a_r$

e) $a_y \cdot a_R$

f) $a_R \cdot a_z$

g) $a_R \times a_z$

h) $a_\theta \cdot a_z$

i) $\mathrm{a}_z \times \mathrm{a}_\theta$

Answer:

a) $\bar{a}_x \cdot \bar{a}_\phi = -\sin\phi$

b) $\bar{a}_\theta \cdot \bar{a}_y = \cos\theta\sin\phi$

c) $\bar{a}_r \times \bar{a}_{x'} = -\bar{a}_z\sin\phi$

d) $\bar{a}_k \cdot \bar{a}_r = \sin\theta$

e) $\bar{a}_y \cdot \bar{a}_R = \sin\theta\sin\phi$

f) $\bar{a}_R \cdot \bar{a}_z = \cos\theta$

g) $\bar{a}_R \times \bar{a}_z = -\bar{a}_\phi \sin\theta$

h) $\bar{a}_\theta \cdot \bar{a}_z = -\sin\theta$

i) $\bar{a}_z \times \bar{a}_\theta = \bar{a}_\phi \cos\theta$

Exercise 2.20

Given a vector function $\mathbf{F} = \mathbf{a}_x xy + \mathbf{a}_y (3x - y^2)$, evaluate the integral $\int \mathbf{F} \cdot d\ell$ from $P_1(5,6)$ to $P_2(3,3)$ in Fig. $2-35$
a) along the direct path P_1P_2,
b) along path P_1AP_2.

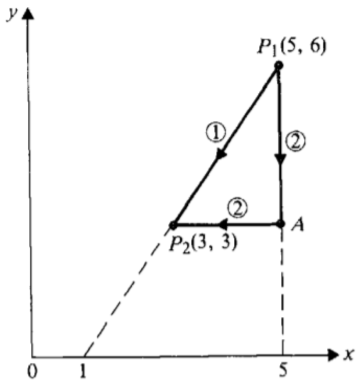


FIGURE 2–35
Paths of integration for Problem P.2–20.

Answer:

$$\begin{aligned}\bar{F} \cdot d\bar{l} &= \left[\bar{a}_x xy + \bar{a}_y \left(3x - y^2\right)\right] \cdot \left(\bar{a}_x dx + \bar{a}_y dy\right) \\ &= xydx + \left(3x - y^2\right) dy.\end{aligned}$$

a) Along direct path (1). The equation of P_1P_2 is

$$y = \frac{3}{2}(x-1).$$

$$\begin{aligned}\int_{p_1}^{p_2} \bar{F} \cdot d\bar{l} &= \int_{p_1}^{p_2} \left[xydx + \left(3x - y^2\right) dy\right] \\ &= \int_5^3 \frac{3}{2}x(x-1)dx + \int_6^3 \left(2y + 3 - y^2\right) dy \\ &= -37 + 27 = -10.\end{aligned}$$

b) Along path (2). This path has two straight-line Segments. From P_1 to $A : x = 5, dx = 0, \bar{F} \cdot d\bar{l} = (15 - y^2)\, dy$. From A to $P_z : y = 3, dy = 0, \bar{F} \cdot d\bar{l} = 3xdx$. Hence,

$$\begin{aligned}\int_{P_1}^{\rho_2} \bar{F} \cdot d\bar{l} &= \int_6^3 \left(15 - y^2\right) dy + \int_5^3 3xdx = 18 - 24 = -6. \\ &\neq \int_{P_1}^{P_2} \bar{F} \cdot d\bar{l} \longrightarrow \text{ Vector field } \bar{F} \text{ is not conservative.}\end{aligned}$$

Exercise 2.21

Given a vector function $\mathbf{E} = \mathbf{a}_x y + \mathbf{a}_y x$, evaluate the scalar line integral $\int \mathbf{E} \cdot d\ell$ from $P_1(2, 1, -1)$ to $P_2(8, 2, -1)$

a) along the parabola $x = 2y^2$,

b) along the straight line joining the two points.

Is this $\mathbf{E}$ a conservative field?

Answer:

$$\int_{P_1}^{P_2} \bar{E} \cdot d\bar{l} = \int_{P_1}^{P_2} (ydx + xdy).$$

a) $x = 2y^2, dx = 4ydy; \int_{P_1}^{P_2} \bar{E} \cdot dl = \int_1^2 (4y^2dy + 2y^2dy) = 14$

b) $x = 6y - 4, dx = 6dy; \int_{P_1}^{P_2} \bar{E} \cdot dt = \int_1^2 [6ydy + (6y - 4)]dy = 14.$

Equal line integrals along two specific paths do not necessarily imply a conservative field. $\bar{E}$ is a conservative field in this case because $\bar{E} = \bar{\nabla}(xy + c)$.

Exercise 2.22

For the E of Problem P.2-21, evaluate $\int \mathbf{E} \cdot d\ell$ from $P_3(3, 4, -1)$ to $P_4(4, -3, -1)$ by converting both $\mathbf{E}$ and the positions of P_3 and P_4 into cylindrical coordinates.

Answer:

$$\begin{bmatrix} E_r \\ E_\phi \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} r\sin\phi \\ r\cos\phi \end{bmatrix}.$$

$\bar{E} = \bar{a}_\mu, r\sin 2\phi + \bar{a}_\phi r\cos 2\phi,$

$\bar{E} \cdot d\bar{l} = r\sin 2\phi dr + r^2\cos 2\phi d\phi.$

$P_3(3, 4, -1) = P_3\left(5, 53.1^\circ, 1\right); \quad P_4(4, -3, -1) = P_4\left(5, -36.9^\circ, -1\right).$

There is 170 change in $r(= 5)$ from P_3 to P_4.

$$\therefore \int_{P_3}^{P_4} \bar{E} \cdot d\bar{C} = 5^2 \int_{53.1^\circ}^{36.4^\circ} \cos 2\phi d\phi = -24.$$

Exercise 2.23

Given a scalar function

$$V = \left(\sin\frac{\pi}{2}x\right)\left(\sin\frac{\pi}{3}y\right)e^{-z}$$

determine

a) the magnitude and the direction of the maximum rate of increase of V at the point $P(1, 2, 3)$

b) the rate of increase of V at P in the direction of the origin.

Answer:

$$\text{a) } \bar{\nabla}V = \left[\bar{a}_x\left(\frac{\pi}{2}\cos\frac{\pi}{2}x\right)\left(\sin\frac{\pi}{3}y\right) + \bar{a}_y\left(\sin\frac{\pi}{2}x\right)\left(\frac{\pi}{3}\cos\frac{\pi}{3}y\right)\right.$$
$$\left. -\bar{a}_z\left(\sin\frac{\pi}{2}x\right)\left(\sin\frac{\pi}{3}y\right)\right]e^{-z}.$$

$$(\bar{\nabla}V)_p = -\left(\bar{a}_y\frac{\pi}{6} + \bar{a}_z\frac{\sqrt{3}}{2}\right)e^{-3} = -\left(\bar{a}_y 0.026 + \bar{a}_z 0.043\right).$$

$$\text{b) } \overrightarrow{PO} = -\bar{a}_x - \bar{a}_y 2 - \bar{a}_z 3; \quad \bar{a}_{p0} = -\frac{1}{\sqrt{14}}\left(\bar{a}_x + \bar{a}_y 2 + \bar{a}_z 3\right).$$

$$\therefore (\bar{\nabla}V)_p \cdot \bar{a}_{p0} = \frac{1}{\sqrt{14}}\left(\frac{\pi}{3} + \frac{3\sqrt{3}}{2}\right)e^{-3} = 0.0485.$$

Exercise 2.24

Evaluate

$$\oint_S (\mathbf{a}_R 3\sin\theta)\cdot d\mathbf{s}$$

over the surface of a sphere of a radius 5 centered at the origin.

Answer: On the surface of the sphere, $R = 5$.

$$\oint_S (\bar{a}_R 3\sin\theta)\cdot d\bar{s} = \int_0^{2\pi}\int_0^{\pi}(\bar{a}_R 3\sin\theta)\cdot\left(a_R 5^2\sin\theta\right)d\theta d\phi$$
$$= \int_0^{2\pi}\int_0^{\pi} 75\sin^2\theta d\theta d\phi = 75\pi^2.$$

Exercise 2.25

The equation in space of a plane containing the point (x_1, y_1, z_1) can be written as

$$\ell(x - x_1) + m(y - y_1) + p(z - z_1) = 0,$$

where ℓ, m, and p are direction cosines of a unit normal to the plane:

$$\mathbf{a}_n = \mathbf{a}_x\ell + \mathbf{a}_y m + \mathbf{a}_z p.$$

Given a vector field $\mathbf{F} = \mathbf{a}_x + \mathbf{a}_y 2 + \mathbf{a}_z 3$, evaluate the integral $\int_S \mathbf{F}\cdot d\mathbf{s}$ over the square plane surface whose corners are at $(0,0,2), (2,0,2), (2,2,0)$, and $(0,2,0)$.

Answer:

The first step is to find the expression for the unit normal $\bar{a}_n = \bar{a}_x l + \bar{a}_y m + \bar{a}_z p$ to the given surface. The given four corner points of the surface lead to the following four equations:

Corner$(0,0,2)$:

$$lx + my + p(z-2) = 0$$

Corner $(2,0,2)$:

$$l(x-2) + my + p(z-2) = 0.$$

Corner $(2,2,0)$:

$$l(x-2) + m(y-z) + pz = 0.$$

Corner $(0,2,0)$:

$$lx + m(y-2) + pz = 0.$$

The direction cosines satisfy the condition:

From (1)-(5) we obtain $l=0$, and $m = p = 1/\sqrt{2}$.

Thus, $\bar{a}_n = \frac{1}{\sqrt{2}}\left(\bar{a}_y + \bar{a}_x\right)$, $\bar{F}\cdot\bar{a}_n = \frac{5}{\sqrt{2}}$ (a constant), and $\int_S F\cdot d\bar{s} = \frac{5}{\sqrt{2}}S = \frac{5}{\sqrt{2}}(2\times 2\sqrt{2}) = 20$.

Exercise 2.26

Find the divergence of the following radial vector fields:
a) $f_1(\mathbf{R}) = \mathbf{a}_R R^n$,
b) $f_2(\mathbf{R}) = \mathbf{a}_R \frac{k}{R^2}$.

Answer:
In spherical coordinates, $\bar{\nabla}\cdot\bar{A} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 A_R\right)$, if $\bar{A} = \bar{a}_R A_R$.
a) $\bar{A} = f_1(\bar{R}) = \bar{a}_R R^n$, $A_R = R^n$.

$$\bar{\nabla}\cdot\bar{A} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^{n+2}\right) = (n+2)R^{n-1}.$$

b) $\bar{A} = f_2(\bar{R}) = \bar{a}_R\frac{k}{R^2}$, $A_R = kR^{-2}$.

$$\bar{\nabla}\cdot\bar{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(k) = 0.$$

Exercise 2.27

Show that $\frac{1}{3}\oint_S \mathbf{R}\cdot d\mathbf{s} = V$, where $\mathbf{R}$ is the radial vector and V is the volume of the region enclosed by surface S.

Answer:
For radial vector $\bar{R} = \bar{a}_R R, \bar{\nabla}\cdot\bar{R} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\cdot R\right) = 3$. Using divergence theorem, we have

$$\frac{1}{3}\oint_s \bar{R}\cdot d\bar{s} = \frac{1}{3}\int_V \bar{\nabla}\cdot\bar{R}dV = \frac{1}{3}(3V) = V.$$

Exercise 2.28

For a scalar function f and a vector function $\mathbf{A}$, prove that

$$\boldsymbol{\nabla}\cdot(f\mathbf{A}) = f\boldsymbol{\nabla}\cdot\mathbf{A} + \mathbf{A}\cdot\boldsymbol{\nabla}f$$

in Cartesian coordinates.

Answer: 略

Exercise 2.29

For vector function $\mathbf{A} = \mathbf{a}_r r^2 + \mathbf{a}_z 2z$, verify the divergence theorem for the circular cylindrical region enclosed by $r=5, z=0$, and $z=4$.

Answer:

$$\oint_S \bar{A}\cdot d\bar{s} = \left(\int_{\substack{\text{top}\\\text{face}}} + \int_{\substack{\text{bottom}\\\text{face}}} + \int_{\text{walls}}\right)\bar{A}\cdot d\bar{s}$$

Top face

$$(z=4): \bar{A} = \bar{a}_r r^2 + \bar{a}_z 8, \quad d\bar{s} = \bar{a}_z ds$$

Bottom face

$$(z=0): \bar{A} = \bar{a}_r r^2, \quad d\bar{s} = -\bar{a}_z ds$$

$$\int_{\substack{\text{bottom}\\\text{face}}} \bar{A}\cdot d\bar{s} = 0$$

Walls

$$(r=5): \bar{A} = \bar{a}_r 25 + \bar{a}_z 2z, \quad d\bar{s} = \bar{a}_r ds$$

$$\int_{\text{walls}} \bar{A}\cdot d\bar{s} = 25\int_{\text{walls}} ds = 25(2\pi 5\times 4) = 1000\pi$$

$$\therefore \oint_S \bar{A}\cdot d\bar{s} = 200\pi + 0 + 1000\pi = 1,200\pi$$

$$\bar{\nabla}\cdot\bar{A} = 3r+2, \int_V \bar{\nabla}\cdot\bar{A}dv = \int_0^4\int_0^{2\pi}\int_0^5 \bar{\nabla}\cdot\bar{A} r dr d\phi dz = 1,200\pi$$
$$= \oint_s \bar{A}\cdot d\bar{s}.$$

Exercise 2.30

For the vector function $\mathbf{F} = \mathbf{a}_r k_1/r + \mathbf{a}_z k_2 z$ given in Example 2-15 (page 41) evaluate $\int \nabla\cdot\mathbf{F}dv$ over the volume specified in that example. Explain why the divergence theorem fails here.

Answer:
$\bar{\nabla}\cdot\bar{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{\partial}{\partial z}F_z = k_2.$

$$\int_V \bar{\nabla}\cdot\bar{F}dv = k_2 V = k_2\left(\pi 2^2\times 6\right) = 24\pi k_2 \neq \oint_S \bar{A}\cdot d\bar{s}.$$

Divergence theorem fails here because $\bar{F}$ has a singularity inside the volume at $r=0$.

Exercise 2.31

Use the definition in Eq. (2-98) to derive the expression of $\boldsymbol{\nabla}\cdot\mathbf{A}$ for a vector field $\mathbf{A} = \mathbf{a}_r A_r + \mathbf{a}_\phi A_\phi + \mathbf{a}_z A_z$ in cylindrical coordinates.

Answer:
$Eq\cdot(2-98): \quad \bar{\nabla}\cdot\bar{A} = \lim_{\Delta v\to 0}\frac{\oint_s \bar{A}\cdot d\bar{s}}{\Delta v}.$
Referring to Fig. 2-15, we note that the areas on the opposite sides of a differential volume in cylindrical coordinates are the same in ϕ - and z-directions, but are different in the r-direction.

Let us first evaluate the contributions to $\oint_s \bar{A} \cdot d\bar{s}$ of the inside and outside faces: On the inside face:

$$\int_{\substack{\text{inside}\\\text{face}}} \bar{A} \cdot d\bar{s} = \bar{A}_{\substack{\text{inside}\\\text{face}}} \Delta \bar{s}_{\substack{\text{inside}\\\text{face}}} = -A_r\left(r_0 - \frac{\Delta r}{2}, \phi_0, z_0\right)\left(r_0 - \frac{\Delta r}{2}\right)\Delta\phi\Delta z$$

$$= -\left[A_r\left(r_0, \phi_0, z_0\right) - \frac{\Delta r}{2}\frac{\partial A_r}{\partial r}\bigg|_{(r_0,\phi_0,z_0)} + H.O.T\right]\left(r_0 - \frac{\Delta r}{2}\right)\Delta\phi\Delta z.$$

On the outside face:

$$\int_{\text{outside}} \bar{A} \cdot d\bar{s} = A_r\left(r_0 + \frac{\Delta r}{Z}, \phi_0, z_0\right)\left(r_0 + \frac{\Delta r}{2}\right)\Delta\phi\Delta z$$

$= \Big[A_r\left(r_0, \phi_0, z_0\right) + \frac{\Delta r}{2}\frac{\partial A_r}{\partial r}\big|_{(r_0,\phi_0,z_0)} +$ H. O.T. $]\left(r_0 - \frac{\Delta r}{2}\right)_{\Delta\phi\Delta z}$ (3)

Adding (2) and (3), we have

$$\left[\int_{\substack{\text{inside}\\\text{face}}} + \int_{\substack{\text{outside}\\\text{face}}}\right]\bar{A} \cdot d\bar{s} = \left(A_r + r_0\frac{\partial A_r)}{\partial r}\right)\bigg|_{(r_0,\phi_0:z_0)} \Delta r\Delta\phi\Delta z + H.O.T.$$

$$= \frac{\partial}{\partial r}\left(rA_r\right)\bigg|_{(r_0,\phi_0,Z_0)} \Delta r\Delta\phi\Delta Z + H.O.T(4)$$

where H.O.T. contain second and higher powers of Δ. The sum of the contributions of the front and back faces (differential area $= \Delta r\Delta z$) is

$$\left[\int_{\substack{\text{front}\\\text{face}}} + \int_{\substack{\text{back}\\\text{fave}}}\right]\bar{A} \cdot d\bar{s} = \frac{\partial A_\psi}{\partial\phi}\bigg|_{r_0,\varphi_0,z_s)} \Delta r\Delta\phi\Delta z + H.O.T.\ (5)$$

where H,O.T. contain second and higher powers of $\Delta\phi$. Similarly, the sum of the contributions of the top and bottom faces (differential area $= r_0\Delta r\Delta\phi$) is

$$\left[\int_{\substack{\text{top}\\\text{face}}} + \int_{\text{bottom face}}\right]\bar{A} \cdot d\bar{s} = \left(r\frac{\partial A_z}{\partial z}\right)\bigg|_{r_0,\phi_0,z_d)} \Delta r\Delta\phi\Delta z + H \cdot O \cdot T,\ (6)$$

where H. O.T. contain second and higher powers of ΔZ. Combining (4), (5), and (6) in (1), dividing by $\Delta v = r_0\Delta r\Delta\phi\Delta z$, and letting $\Delta r\Delta\phi\Delta z \to 0$, we get

$$\bar{\nabla} \cdot \bar{A} = \frac{1}{r}\frac{\partial}{\partial r}\left(rA_r\right) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z},$$

where the subscript o has been dropped for simplicity.

Exercise 2.32

A vector field $\mathbf{D} = \mathrm{a}_R\left(\cos^2\phi\right)/R^3$ exists in the region between two spherical shells defined by $R = 1$ and $R = 2$. Evaluate

a) $\oint \mathbf{D} \cdot d\mathbf{s}$

b) $\int \nabla \cdot \mathbf{D}dv$

Answer:

a)

$$\bar{D} = \bar{a}_R \frac{\cos^2\phi}{R^3}, \quad ds = R^2 \sin\theta d\theta d\phi$$

$$\oint \bar{D} \cdot d\bar{s} = \int_0^{2\pi} \int_0^{\pi} \left(\frac{1}{2} - 1\right) \sin\theta d\theta \cos^2\phi d\phi = -\pi$$

b)

$$\bar{\nabla} \cdot \bar{D} = -\frac{\cos^2\phi}{R^4}, \quad dv = R^2 \sin\theta dR d\theta d\phi$$

$$\int_V \bar{\nabla} \cdot \bar{D} dv = \int_0^{2\pi} \int_0^{2} \int_1^{2} \cdot \left(-\frac{\cos^2\phi}{R^2}\right) \sin\theta dR d\theta d\phi = -\pi$$

Exercise 2.33

For two differentiable vector functions **E** and **H**, prove that

$$\boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\boldsymbol{\nabla} \times \mathbf{E}) - \mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathrm{H}).$$

Answer: 略

Exercise 2.34

Assume the vector function $\mathbf{A} = \mathbf{a}_x 3x^2y^3 - \mathbf{a}_y x^3y^2$
a) Find $\oint \mathbf{A} \cdot d\ell$ around the triangular contour shown in Fig. 2-36.
b) Evaluate $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{s}$ over the triangular area.
c) Can A be expressed as the gradient of a scalar? Explain.

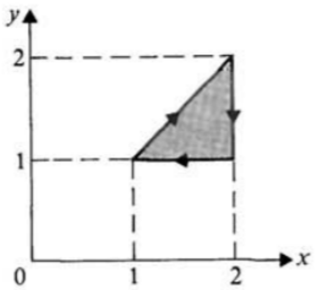


FIGURE 2–36
Graph for Problem P.2–34.

Answer:

$$\bar{A} = \bar{a}_x 3x^2y^3 - \bar{a}_y x^3y^2; d\bar{l} = \bar{a}_x dx + \bar{a}_y dy$$

a)

$$\bar{A} \cdot d\bar{l} = 3x^2y^3 dx - x^3y^2 dy$$

Path (1):

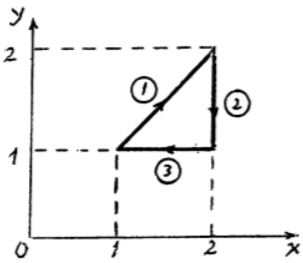
$$x = y, \int_{(1)} \bar{A} \cdot d\bar{t} = \int_1^2 2x^5 dx = 21$$

Path (2) :

$$x=2, dx=0, \int_{(2)} \bar{A} \cdot d\bar{l} = \int_{2}^{1} \left(-2^{3} y^{2}\right) dy = \frac{56}{3}$$

Path (3):

$$y=1, dy=0: \int \bar{A} \cdot d\bar{l} = \int_{1}^{1} 3x^{2} dx = -7$$



$$\therefore \oint \bar{A} \cdot d\bar{l} = 21 + \frac{56}{3} - 7 = \frac{98}{3} = 32\frac{2}{3}$$

b)

$$\bar{\nabla} \times \bar{A} = -\bar{a}_{z} 12 x^{2} y^{2}, \quad d\bar{s} = -\bar{a}_{z} dx dy.$$

$$\int_{S} (\bar{\nabla} \times \bar{A}) \cdot d\bar{s} = 12 \int_{1}^{2} x^{2} dx \int_{1}^{x} y^{2} dy = 32\frac{2}{3}$$

Exercise 2.35

Use the definition in Eq. $(2-126)$ to derive the expression of the $\mathbf{a}_R$-component of $\nabla \times \mathrm{A}$ in spherical coordinates for a vector field $\mathrm{A} = \mathrm{a}_R A_R + \mathrm{a}_\theta A_\theta + \mathrm{a}_\phi A_\phi$

Answer: 略

Exercise 2.36

Given the vector function $\mathbf{A} = \mathrm{a}_\phi \sin(\phi/2)$, verify Stokes's theorem over the hemispherical surface and its circular contour that are shown in Fig. 2-37.

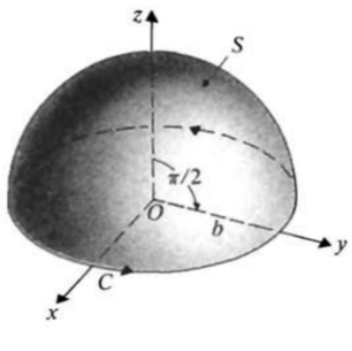


FIGURE 2–37
Graph for Problem P.2–36.

Answer:

$$
\bar{\nabla} \times \bar{A} = \frac{1}{R\sin\theta}\left(\bar{a}_R \cos\theta \sin\frac{\phi}{2} - \bar{a}_\theta \sin\theta \sin\frac{\phi}{2}\right),
$$

$$
\int_S (\bar{\nabla} \times \bar{A}) \cdot d\bar{s} = \int_0^{2\pi} \int_0^{\pi/2} (\bar{\nabla} \times \bar{A}), b_{R=b} \cdot \left(\bar{a}_R b^2 \sin\theta d\theta d\phi\right) = 4b.
$$

$$
\oint_c \bar{A} \cdot d\bar{l} = \int_0^{2\pi} (\bar{A})_{\substack{R=b/2 \\ \theta=\pi/2}} (\overline{a_\phi} b d\phi) = \int_0^{2\pi} b \sin\frac{\phi}{2} d\phi = 4b.
$$

Exercise 2.37

For a scalar function f and a vector function $\mathbf{G}$, prove that

$$
\nabla \times (f\mathbf{G}) = f\nabla \times \mathbf{G} + (\nabla f) \times \mathbf{G}
$$

in Cartesian coordinates.

Answer: 略

Exercise 2.38

Verify the null identities:
a) $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} V) \equiv 0$
b) $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) \equiv 0$
by expansion in general orthogonal curvilinear coordinates.

Answer: 略

Exercise 2.39

Given a vector function $\mathbf{F} = \mathbf{a}_x (x + c_1 z) + \mathbf{a}_y (c_2 x - 3z) + \mathbf{a}_z (x + c_3 y + c_4 z)$.
a) Determine the constants c_1, c_2, and c_3 if $\mathbf{F}$ is irrotational.
b) Determine the constant c_4 if $\mathbf{F}$ is also solenoidal.
c) Determine the scalar potential function V whose negative gradient equals $\mathbf{F}$.

Answer:

$$
\bar{F} = \bar{a}_x (x + c_i z) + \bar{a}_y (c_2 x - 3z) + \bar{a}_z (x + c_4 y + c_5 z).
$$

a) $\bar{F}$ irrotational $\longrightarrow \bar{\nabla} \times \bar{F} = 0$, or

$$
\bar{a}_x \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \bar{a}_y \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \bar{a}_z \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial F_y}\right) = 0,
$$

which gives three equations:

$$
\frac{\partial}{\partial y}(x + c_3 y + c_4 z) - \frac{\partial}{\partial z}(c_2 x - 3z) = 0 \to c_3 + 3 = 0 \to c_3 = -3.
$$

$$
\frac{\partial}{\partial z}(x + c_1 x) - \frac{\partial}{\partial x}(x + c_3 y + c_4 z) = 0 \to c_1 - 1 = 0 \to c_1 = 1.
$$

$$
\frac{\partial}{\partial x}(c_2 x - 3z) - \frac{\partial}{\partial y}(x + c_1 z) = 0 \to c_2 = 0.
$$

b) $\bar{F}$ also soleneictal $\longrightarrow \bar{\nabla} \cdot \bar{F} = 0$, or

$$\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 0$$

$$\frac{\partial}{\partial x}(x + c, z) + \frac{\partial}{\partial y}(c_2 x - 3z) + \frac{\partial}{\partial x}(x + c_3 y + c_4 z) = 0,$$

$$1 + c_4 = 0 \quad \longrightarrow c_4 = -1.$$

c)

$$\bar{F} = -\bar{\nabla}\bar{V} \longrightarrow \bar{a}_x(x + z) - \bar{a}_y 3z + \bar{a}_z(x - 3y - z)$$

$$= -a_x \frac{\partial V}{\partial x} - a_y \frac{\partial V}{\partial y} - a_z \frac{\partial V}{\partial z}.$$

$$\frac{\partial V}{\partial x} = -(x + z) \longrightarrow V = -\frac{x^2}{2} - xz + f_1(y, z).$$

$$\frac{\partial V}{\partial y} = 3z \quad \longrightarrow V = 3yz + f_2(x, z).$$

$$\frac{\partial V}{\partial z} = -x + 3y + z \longrightarrow V = -xz + 3yz + \frac{z^2}{2} + f_3(x, y).$$

$$\therefore V = -\frac{x^2}{2} - xz + 3yz + \frac{z^2}{2}.$$

Reference

1. Cheng, David Keun. Field and wave electromagnetics. Pearson Education India, 1989.