Chapter 2 Vector Analysis

Sung-Liang Chen

VE230 Summer 2021

2-1 Introduction

- Vector algebra: Addition, subtraction, and multiplication of vectors
- Orthogonal coordinate systems: Cartesian, cylindrical, and spherical coordinates
- Vector calculus: Differentiation and integration of vectors; line, surface, and volume integrals; “del” operator; gradient, divergence, and curl operations.

Definition - Merriam Webster

- Scalar: A quantity that has a magnitude describable by a real number and no direction
- Vector: A quantity that has magnitude and direction

Examples of Scalars and Vectors

- Today’s highest temperature 26°C, lowest temperature 18°C

- Wind level 3 to level 4 from south

2-2 Vector Addition and Subtraction

- Vector representation:

$$\mathbf{A} = \mathbf{a}_A A,$$

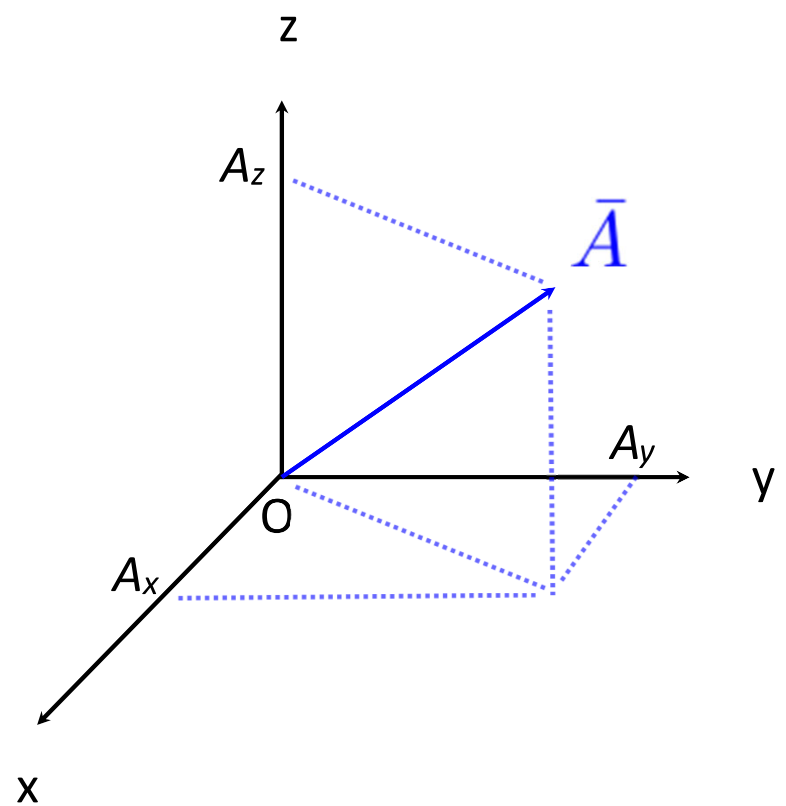
$$A = |\mathbf{A}|$$

Vector magnitude (same as scalar)

$$\mathbf{a}_A = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{A}}{A}$$

Unit vector: has magnitude of 1;
only denotes direction

Vector Representation - Cartesian Coordinate System

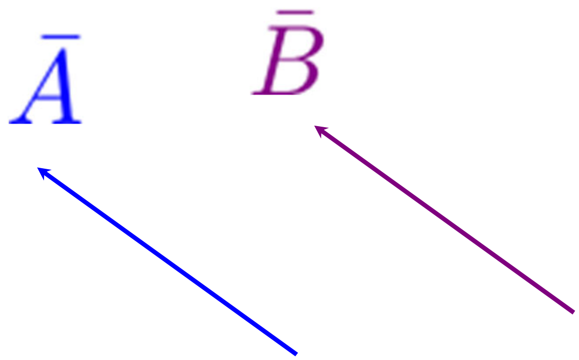


Hat to represent a unit vector

$$\vec{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$$

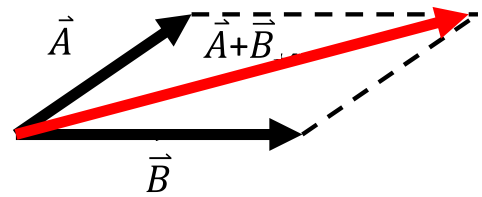
Equality of Two Vectors

- Two vectors ***A*** and ***B*** are said to be equal if they have equal magnitudes and unit vectors.

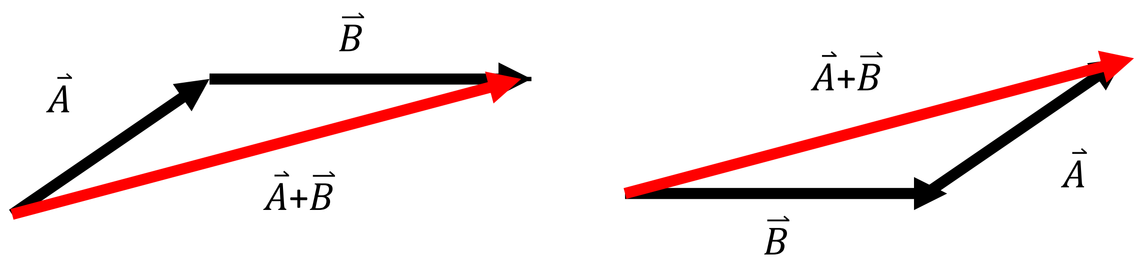


Vector Addition

parallelogram rule



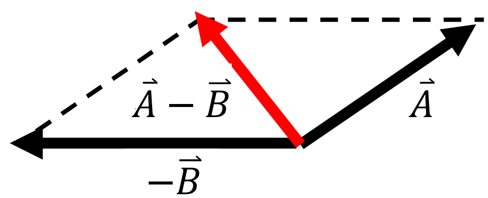
head-to-tail rule



Commutative law $\vec{A}+\vec{B} = \vec{B}+\vec{A}$

Vector Subtraction

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$



Vector subtraction is mathematically the same as vector addition, except the direction of the subtracted vector is reversed.

Vector Operations

Commutative Law of Vector Addition: $\vec{A}+\vec{B} = \vec{B}+\vec{A}$

The order of vector addition does not matter.

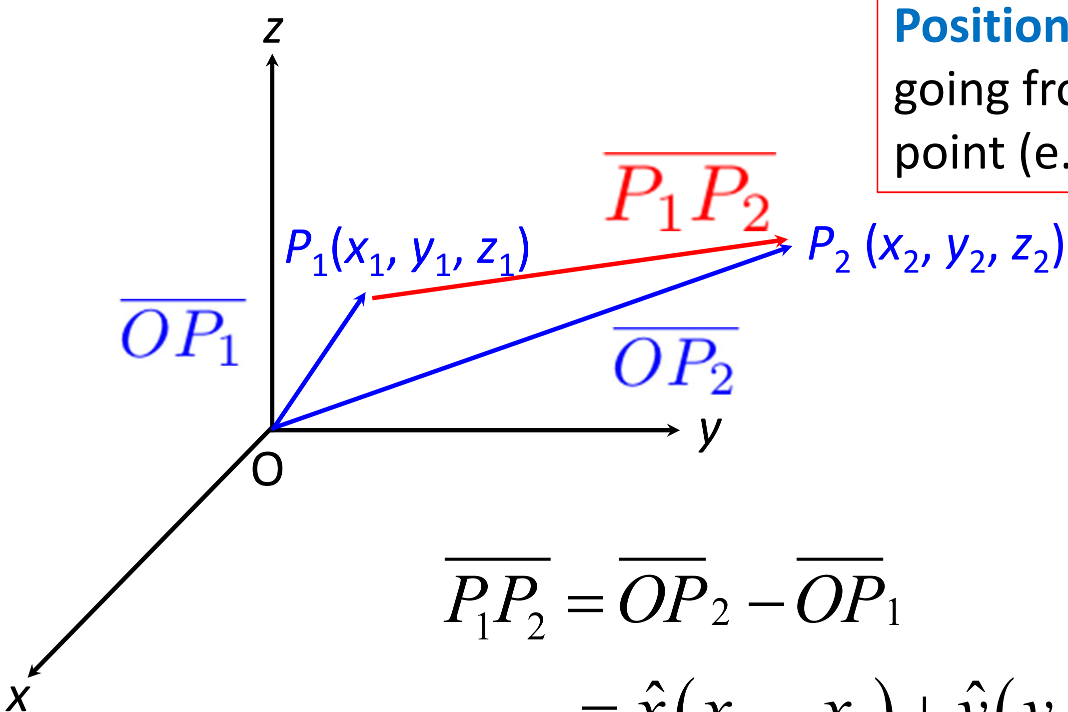
Associative Law of Vector Addition: $\vec{A}+\left(\vec{B} + \vec{C}\right) = \left(\vec{A} + \vec{B}\right) + \vec{C}$

Again, the order of vector addition does not matter.

Distributive Law of Vector Addition: $n(\vec{A}+\vec{B})= n\vec{A} + n\vec{B}$

The order of vector magnification does not matter.

Position and Distance



Position vector: The vector going from the origin to the point (e.g., $\overline{OP_1}$)

$$\overline{P_1P_2} = \overline{OP}_2 - \overline{OP}_1$$

$$= \hat{x}(x_2 - x_1) + \hat{y}(y_2 - y_1) + \hat{z}(z_2 - z_1)$$

$$\left|\overline{P_1P_2}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

2-3 Products of Vectors

- Vector Multiplication - Simple Product $\vec{B} = k\vec{A}$

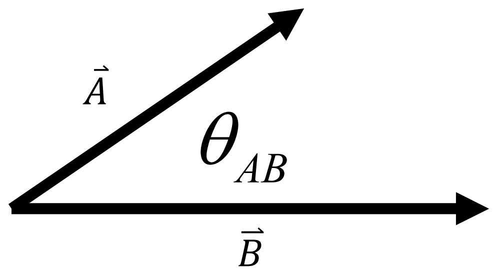
$$\vec{B} = B\hat{b} = k\vec{A} = kA\hat{a}$$

$$\therefore B = kA;\ \hat{b} = \hat{a}$$

does not change the direction but magnitude

Scalar Multiplication-Scalar or Dot Product

Dot Product: $\vec{A} \cdot \vec{B} = |A||B|cos\theta_{AB}$



The dot product (inner product) of two vectors produces a scalar quantity.

Commutative Law of Dot Product: $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

The order of dot product multiplication does not matter.

Distributive Law of Dot Product: $\vec{A} \cdot \left(\vec{B} + \vec{C}\right) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$

Scalar Product in XYZ Coordinate

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

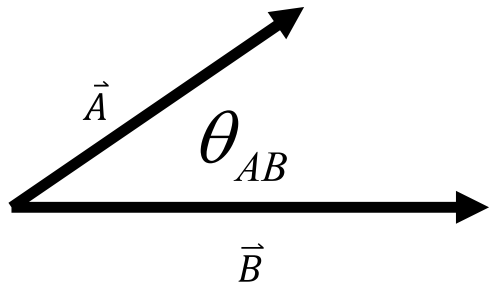
If the DOT product of two non-zero vectors is zero, then they are perpendicular to each other!

$$\hat{x} \cdot \hat{y} = \hat{y} \cdot \hat{z} = \hat{z} \cdot \hat{x} = 0$$

$$\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1$$

Vector Multiplication - Vector or Cross Product

Cross Product: $\vec{A} \times \vec{B} = |A||B| sin\theta_{AB}\hat{c}$ $\qquad$ $\hat{c} = \hat{a} \times \hat{b}$



Direction: The cross product (outer product) of two vectors produces a vector that is perpendicular to both of the original vectors and in a direction that obeys the permutation principle (right-hand rule).

Non-Commutative Law of Cross Product: $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$

$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ (anti-commutative)

Non-Associative Law of Cross Product:

$$\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$$

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) \neq -(\vec{A} \times \vec{B}) \times \vec{C}$$

The order of cross product multiplication **does matter**.

Associative Case of Triple Cross Product

$$\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$$

Associative when one of the vectors is zero, or when A//C

To prove this, note that $A \times (B \times C) = \underline{(A \cdot C)B} - (A \cdot B)C$ and likewise $(A \times B) \times C = -(C \cdot B)A + \underline{(C \cdot A)B}$

Assume the two are equal, and note that the dot product commutes, and we find

$$(A \cdot B)C = (C \cdot B)A$$

Then,

$$\frac{A}{A \cdot B} = \frac{C}{C \cdot B} \implies C = kA.$$

So C is some scalar multiple of A.

Vector Product in XYZ Coordinate

$$\hat{x} \times \hat{x} = \hat{y} \times \hat{y} = \hat{z} \times \hat{z} = 0$$

$$\hat{x} \times \hat{y} = \hat{z}$$

$$\hat{y} \times \hat{z} = \hat{x}$$

$$\hat{z} \times \hat{x} = \hat{y}$$

Vector Product in Matrix Format

$$\overline{A} \times \overline{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \hat{x}\left(A_y B_z - A_z B_y\right) + \hat{y}\left(A_z B_x - A_x B_z\right) + \hat{z}\left(A_x B_y - A_y B_x\right)$$

(see Example 2-4)

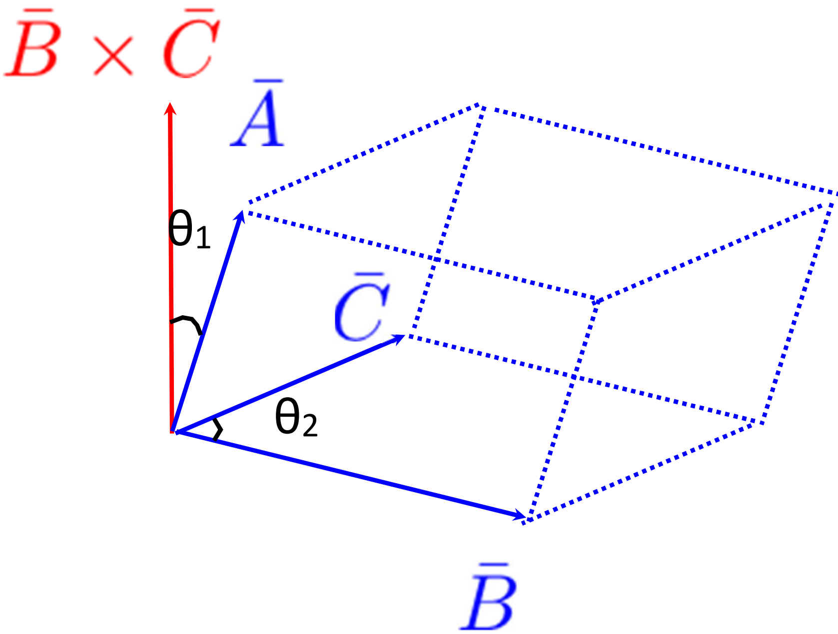
EXAMPLE 2–4 Given three vectors **A**, **B**, and **C**, obtain the expressions of (a) $\mathbf{A} \cdot \mathbf{B}$, (b) $\mathbf{A} \times \mathbf{B}$, and (c) $\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$ in the orthogonal curvilinear coordinate system (u_1, u_2, u_3).

Which one is NOT meaningful?

- (a) $\boldsymbol{A}(\boldsymbol{B} \cdot \boldsymbol{C})$
- (b) $\boldsymbol{A} \times (\boldsymbol{B} \cdot \boldsymbol{C})$
- (c) $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$
- (d) $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})$

Scalar Triple Product

- The product is a scalar, equal to the volume of the parallelepiped.



$$\bar{A} \cdot \left(\bar{B} \times \bar{C}\right) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Vector Triple Product

- The product is a vector.
- "Back-cab" rule

$$\bar{A} \times \left(\bar{B} \times \bar{C}\right) = \bar{B}\left(\bar{A} \cdot \bar{C}\right) - \bar{C}\left(\bar{A} \cdot \bar{B}\right)$$

(see Example 2-3)

EXAMPLE 2–3† Prove the back-cab rule of vector triple product.

$$\boxed{\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}).}$$

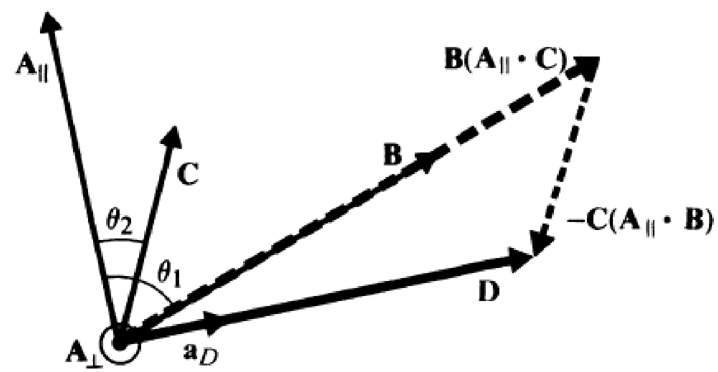


FIGURE 2–9
Illustrating the back-cab rule of vector triple product.

2-4 Orthogonal Coordinate Systems

- A tool to ease the solving process for problems with certain geometry
- The problem itself, EM quantities here, is not changed with different orthogonal coordinate systems.

Orthogonal Coordinate Systems

- u_1 = constant; u_2 = constant; u_3 = constant ➔ 3 surfaces
  (In Cartesian, $u_1 = x$, $u_2 = y$, $u_3 = z$)
- The intersection of 3 surfaces determine 1 point.
- Should u be length?
  Should the surface be planes?
- If 3 surfaces are perpendicular to each other, we have an orthogonal coordinate system.

General 3D, Right-handed, Curvilinear, and Orthogonal Coordinate System

Three curved surfaces: u_1, u_2, u_3

Three base vectors: $\hat{a}_{u_1}$ $\hat{a}_{u_2}$ $\hat{a}_{u_3}$

$$\bar{A} = \hat{a}_{u_1} A_{u_1} + \hat{a}_{u_2} A_{u_2} + \hat{a}_{u_3} A_{u_3}$$

***General expression*, convenient to write down (and derive) formulas in different coordinate systems.**

Base Vectors, Expression, and Length

$$\mathbf{a}_{u_1} \times \mathbf{a}_{u_2} = \mathbf{a}_{u_3},$$
$$\mathbf{a}_{u_2} \times \mathbf{a}_{u_3} = \mathbf{a}_{u_1},$$
$$\mathbf{a}_{u_3} \times \mathbf{a}_{u_1} = \mathbf{a}_{u_2}.$$

$$\mathbf{a}_{u_1} \cdot \mathbf{a}_{u_2} = \mathbf{a}_{u_2} \cdot \mathbf{a}_{u_3} = \mathbf{a}_{u_3} \cdot \mathbf{a}_{u_1} = 0$$

$$\mathbf{a}_{u_1} \cdot \mathbf{a}_{u_1} = \mathbf{a}_{u_2} \cdot \mathbf{a}_{u_2} = \mathbf{a}_{u_3} \cdot \mathbf{a}_{u_3} = 1.$$

$$\boxed{\mathbf{A} = \mathbf{a}_{u_1} A_{u_1} + \mathbf{a}_{u_2} A_{u_2} + \mathbf{a}_{u_3} A_{u_3}.}$$

$$A = |\mathbf{A}| = (A_{u_1}^2 + A_{u_2}^2 + A_{u_3}^2)^{1/2}.$$

Differential Elements

- Note: u_i may not be a length!

Differential length: $$dl_i = \underline{h_i} du_i$$

Metric coefficient

- Example:

In 2D polar coordinate: $(u_1, u_2) = (r, \phi)$
A differential change $d\phi\ (= du_2)$ in $\phi\ (= u_2)$ ➡ A differential length change: $d\ell_2 = r\, d\phi$

The metric coefficient for $\phi\ (= u_2)$: $h_2 = r$

h_2 in this case is a function of u_1, **not a constant**.

Differential Elements

Differential length: $d\overline{\boldsymbol{\ell}} = \mathbf{a}_{u_1}\, \underline{d\ell_1} + \mathbf{a}_{u_2}\, d\ell_2 + \mathbf{a}_{u_3}\, d\ell_3$

$$d\bar{l} = \hat{a}_{u_1}\left(\underline{h_1 du_1}\right) + \hat{a}_{u_2}\left(h_2 du_2\right) + \hat{a}_{u_3}\left(h_3 du_3\right)$$

Magnitude: $$\begin{aligned} d\ell &= [(d\ell_1)^2 + (d\ell_2)^2 + (d\ell_3)^2]^{1/2} \\ &= [(h_1\, du_1)^2 + (h_2\, du_2)^2 + (h_3\, du_3)^2]^{1/2}. \end{aligned}$$

Differential Elements

Differential surfaces: $d\bar{s} = \hat{a}_n ds$

$$ds_1 = dl_2 dl_3 = h_2 h_3 du_2 du_3$$

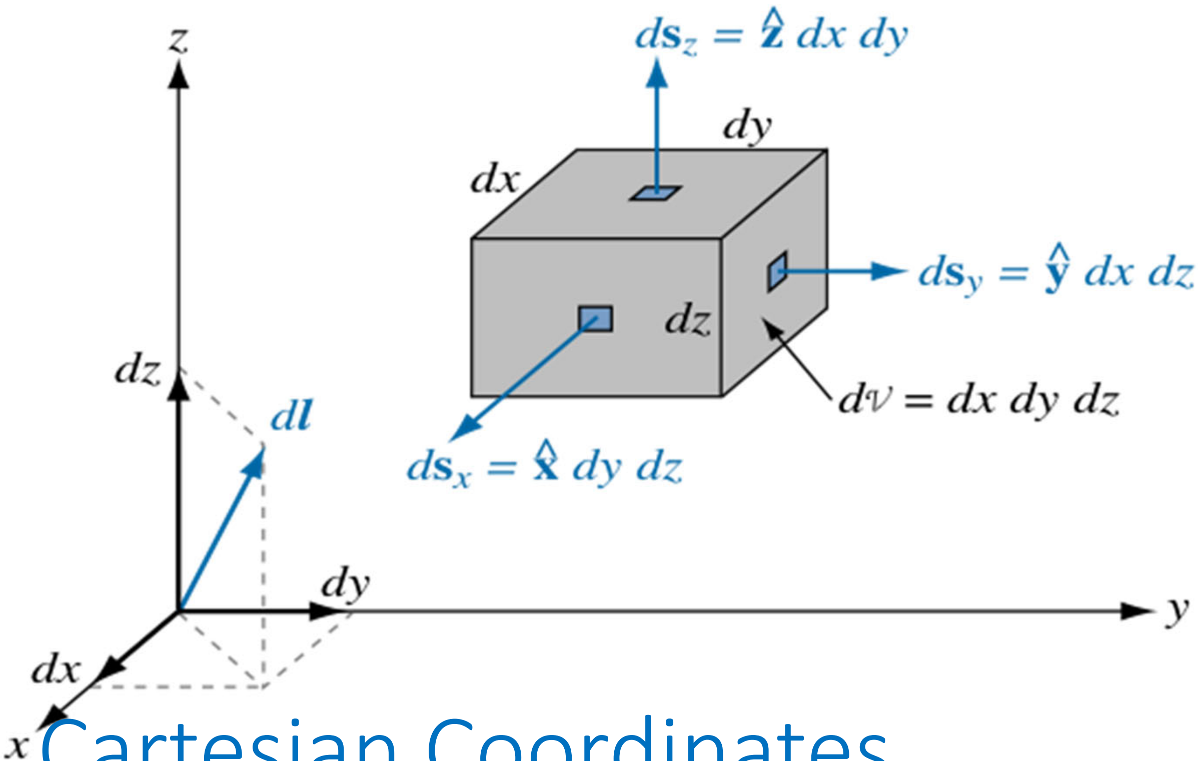
$$\boxed{ds_1 = h_2 h_3 \, du_2 \, du_3.}$$

$$\boxed{ds_2 = h_1 h_3 \, du_1 \, du_3}$$

$$\boxed{ds_3 = h_1 h_2 \, du_1 \, du_2.}$$

Differential volume:

$$\boxed{dv = h_1 h_2 h_3 \, du_1 \, du_2 \, du_3.}$$



Cartesian Coordinates

$$(x, y, z)$$

Cartesian Coordinate System

$$\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$$
$$\mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x$$
$$\mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y.$$

$$\overrightarrow{OP} = \mathbf{a}_x x_1 + \mathbf{a}_y y_1 + \mathbf{a}_z z_1.$$

$$\boxed{\mathbf{A} = \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z.}$$

$$\boxed{\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z,}$$

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= \mathbf{a}_x(A_y B_z - A_z B_y) + \mathbf{a}_y(A_z B_x - A_x B_z) + \mathbf{a}_z(A_x B_y - A_y B_x) \\ &= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}.\end{aligned}$$

Cartesian Coordinate System

General expression

$$\boxed{d\ell = \mathbf{a}_{u_1}(h_1\, du_1) + \mathbf{a}_{u_2}(h_2\, du_2) + \mathbf{a}_{u_3}(h_3\, du_3).}$$

$$\boxed{ds_1 = h_2 h_3\, du_2\, du_3.}$$

$$\boxed{ds_2 = h_1 h_3\, du_1\, du_3}$$

$$\boxed{ds_3 = h_1 h_2\, du_1\, du_2.}$$

$$\boxed{dv = h_1 h_2 h_3\, du_1\, du_2\, du_3.}$$

$$\boxed{h_1 = h_2 = h_3 = 1}$$

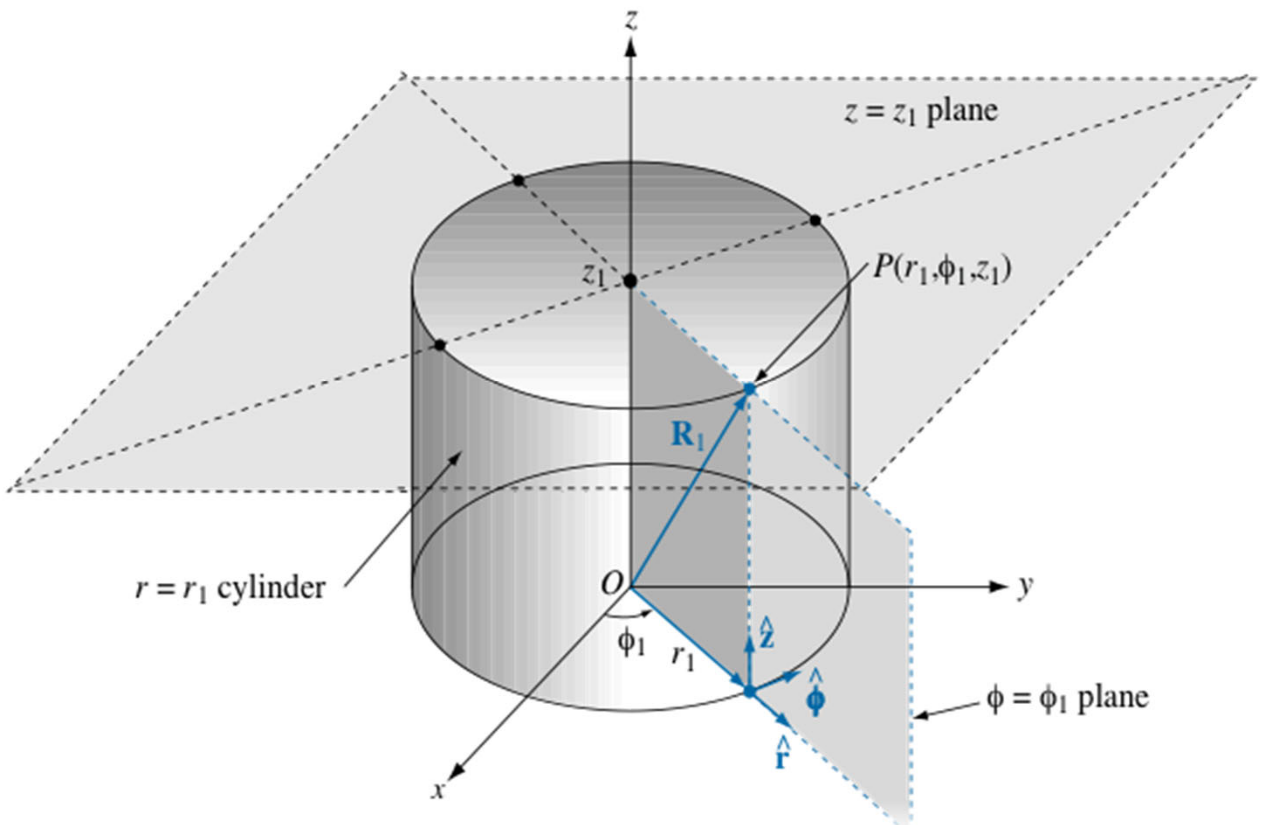
$$\Longrightarrow$$

Cartesian coordinate

$$\boxed{d\ell = \mathbf{a}_x\, dx + \mathbf{a}_y\, dy + \mathbf{a}_z\, dz;}$$

$$\boxed{\begin{aligned} ds_x &= dy\, dz, \\ ds_y &= dx\, dz, \\ ds_z &= dx\, dy; \end{aligned}}$$

$$\boxed{dv = dx\, dy\, dz.}$$



Cylindrical Coordinates

$$(r, \phi, z)$$

Cylindrical Coordinate System

$$\mathbf{a}_r \times \mathbf{a}_\phi = \mathbf{a}_z,$$
$$\mathbf{a}_\phi \times \mathbf{a}_z = \mathbf{a}_r,$$
$$\mathbf{a}_z \times \mathbf{a}_r = \mathbf{a}_\phi.$$

$$\boxed{\mathbf{A} = \mathbf{a}_r A_r + \mathbf{a}_\phi A_\phi + \mathbf{a}_z A_z.}$$

General expression

$$\boxed{d\ell = \mathbf{a}_{u_1}(h_1\, du_1) + \mathbf{a}_{u_2}(h_2\, du_2) + \mathbf{a}_{u_3}(h_3\, du_3).}$$

$$\boxed{ds_1 = h_2 h_3\, du_2\, du_3.}$$

$$\boxed{ds_2 = h_1 h_3\, du_1\, du_3}$$

$$\boxed{ds_3 = h_1 h_2\, du_1\, du_2.}$$

$$\boxed{dv = h_1 h_2 h_3\, du_1\, du_2\, du_3.}$$

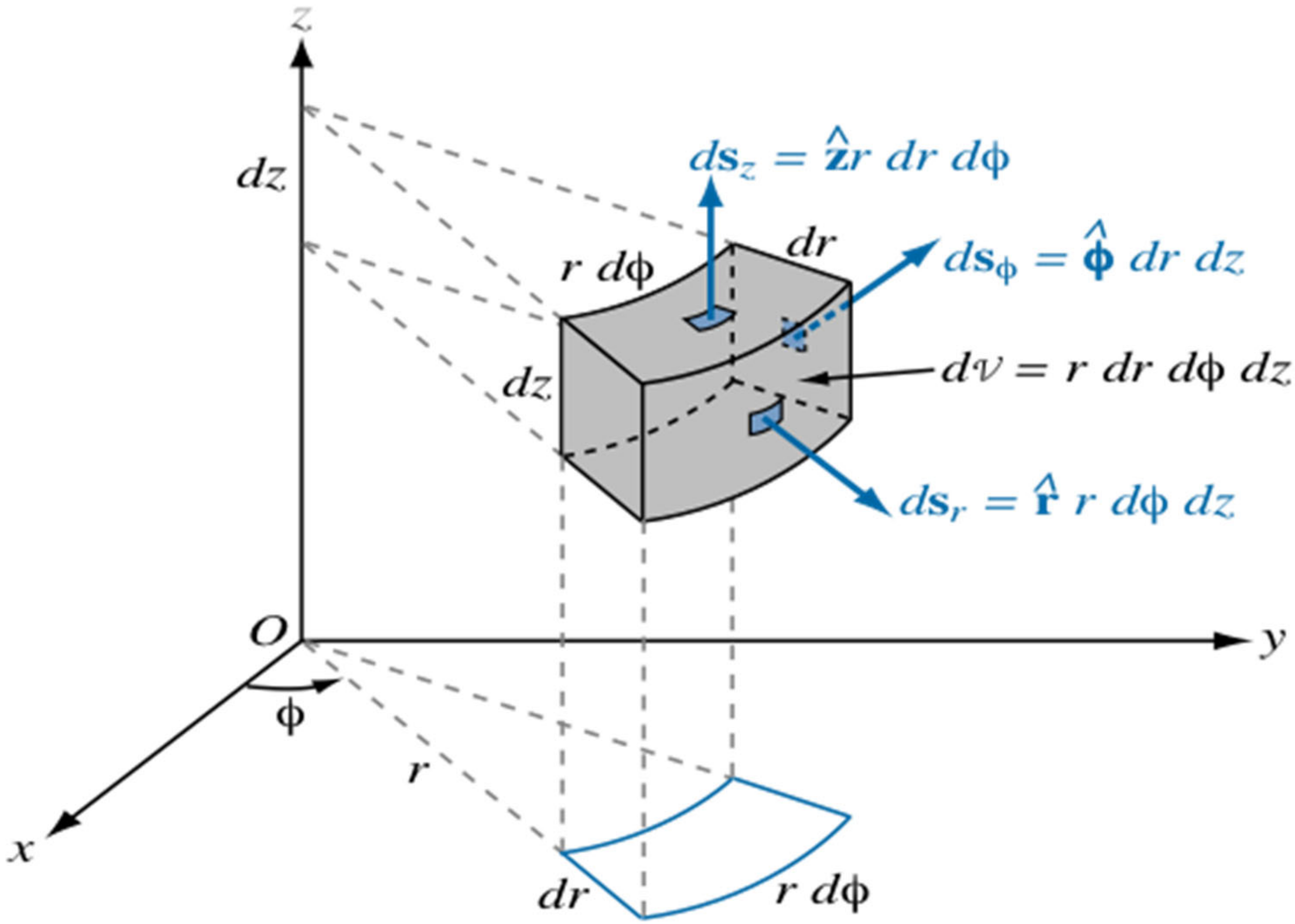
$$\boxed{\begin{aligned} h_1 &= h_3 = 1 \\ h_2 &= r \end{aligned}}$$

Cylindrical coordinate

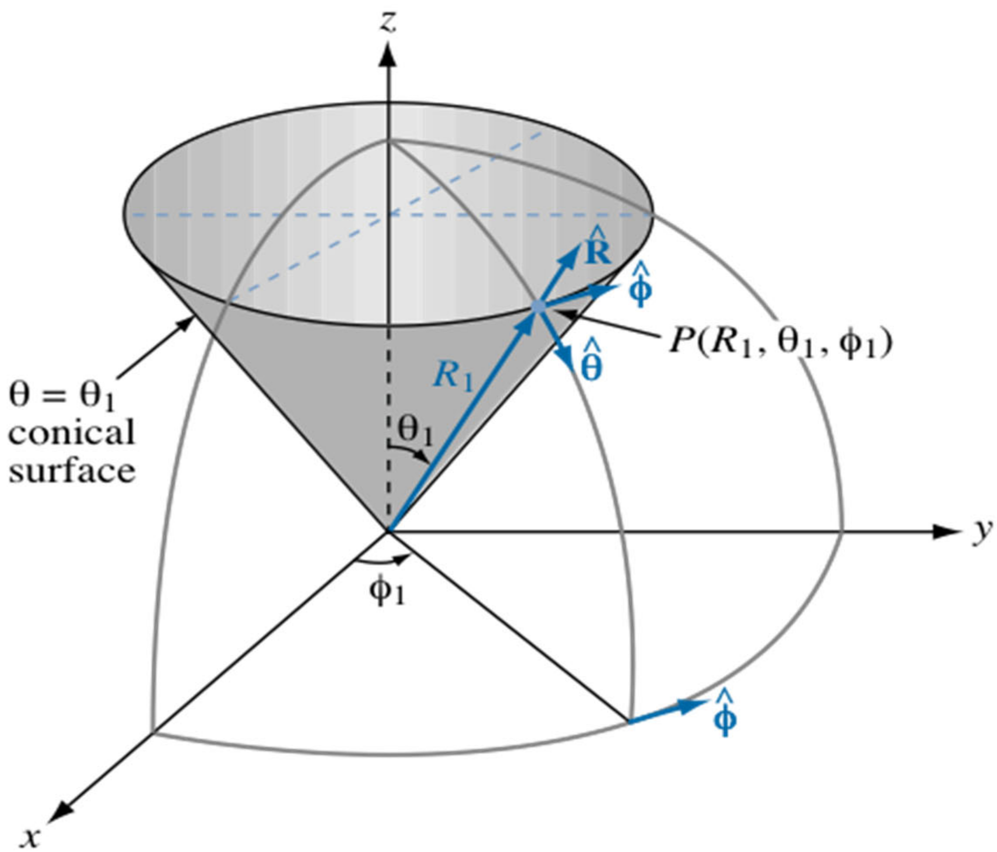
$$\boxed{d\ell = \mathbf{a}_r\, dr + \mathbf{a}_\phi \underline{r\, d\phi} + \mathbf{a}_z\, dz.}$$

$$\boxed{\begin{aligned} ds_r &= r\, d\phi\, dz, \\ ds_\phi &= dr\, dz, \\ ds_z &= r\, dr\, d\phi, \end{aligned}}$$

$$\boxed{dv = r\, dr\, d\phi\, dz.}$$



z
dz
$d\mathbf{s}_z = \hat{\mathbf{z}} r\, dr\, d\phi$
dr
r dϕ
$d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\, dr\, dz$
dz
$d\mathcal{V} = r\, dr\, d\phi\, dz$
$d\mathbf{s}_r = \hat{\mathbf{r}}\, r\, d\phi\, dz$
O
y
ϕ
r
x
dr
r dϕ



Spherical Coordinates

$$(R, \theta, \phi)$$

Spherical Coordinate System

$$\mathbf{a}_R \times \mathbf{a}_\theta = \mathbf{a}_\phi,$$
$$\mathbf{a}_\theta \times \mathbf{a}_\phi = \mathbf{a}_R,$$
$$\mathbf{a}_\phi \times \mathbf{a}_R = \mathbf{a}_\theta.$$

$$\boxed{\mathbf{A} = \mathbf{a}_R A_R + \mathbf{a}_\theta A_\theta + \mathbf{a}_\phi A_\phi.}$$

General expression

$$\boxed{d\ell = \mathbf{a}_{u_1}(h_1\,du_1) + \mathbf{a}_{u_2}(h_2\,du_2) + \mathbf{a}_{u_3}(h_3\,du_3).}$$

$$\boxed{ds_1 = h_2 h_3\,du_2\,du_3.}$$

$$\boxed{ds_2 = h_1 h_3\,du_1\,du_3}$$

$$\boxed{ds_3 = h_1 h_2\,du_1\,du_2.}$$

$$\boxed{dv = h_1 h_2 h_3\,du_1\,du_2\,du_3.}$$

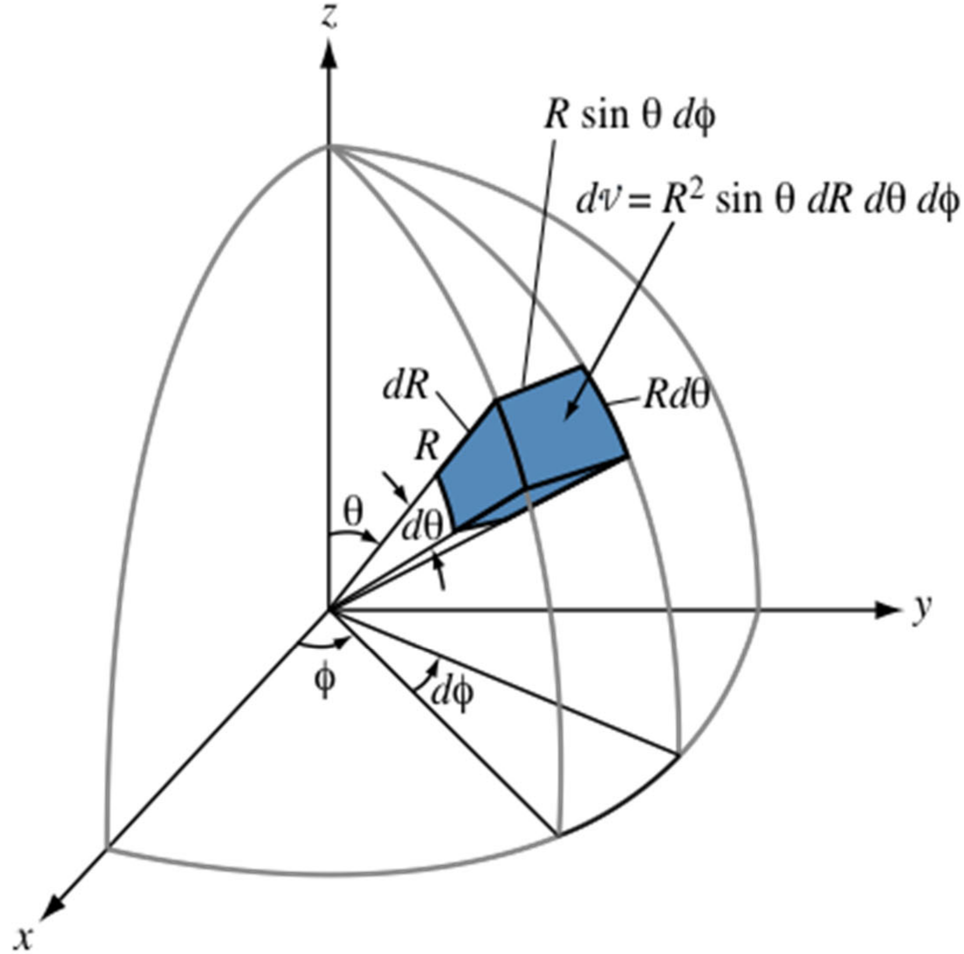
$$h_2 = R$$
$$h_3 = R\sin\theta$$

Spherical coordinate

$$\boxed{d\ell = \mathbf{a}_R\,dR + \mathbf{a}_\theta R\,d\theta + \mathbf{a}_\phi R\sin\theta\,d\phi.}$$

$$\boxed{\begin{aligned} ds_R &= R^2\sin\theta\,d\theta\,d\phi, \\ ds_\theta &= R\sin\theta\,dR\,d\phi, \\ ds_\phi &= R\,dR\,d\theta, \end{aligned}}$$

$$\boxed{dv = R^2\sin\theta\,dR\,d\theta\,d\phi.}$$


z
R sin θ dϕ
dV = R2 sin θ dR dθ dϕ
dR
Rdθ
R
θ
dθ
y
ϕ
dϕ
x

Metric Coefficients

TABLE 2–1
Three Basic Orthogonal Coordinate Systems

Coordinate System Relations		Cartesian Coordinates (x, y, z)	Cylindrical Coordinates (r, ϕ, z)	Spherical Coordinates (R, θ, ϕ)
Base vectors	$\mathbf{a}_{u_1}$	$\mathbf{a}_x$	$\mathbf{a}_r$	$\mathbf{a}_R$
	$\mathbf{a}_{u_2}$	$\mathbf{a}_y$	$\mathbf{a}_\phi$	$\mathbf{a}_\theta$
	$\mathbf{a}_{u_3}$	$\mathbf{a}_z$	$\mathbf{a}_z$	$\mathbf{a}_\phi$
Metric coefficients	h_1	1	1	1
	h_2	1	r	R
	h_3	1	1	$R \sin\theta$
Differential volume	dv	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2 \sin\theta\,dR\,d\theta\,d\phi$

Cartesian Coordinates: Vector Operation Example

$$\vec{A} = 4\hat{x} - 2\hat{y} + 3\hat{z} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} = (4, -2, 3) \qquad \vec{B} = 1\hat{x} - 1\hat{y} + 7\hat{z} = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix} = (1, -1, 7)$$

Vector Addition: $$\vec{A} + \vec{B} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 10 \end{bmatrix}$$

Vector Dot Product:

$$\vec{A} \cdot \vec{B} = \left(4\hat{x} - 2\hat{y} + 3\hat{z}\right) \cdot \left(1\hat{x} - 1\hat{y} + 7\hat{z}\right)$$

$$\vec{A} \cdot \vec{B} = 4\hat{x} \cdot \left(1\hat{x} - 1\hat{y} + 7\hat{z}\right) - 2\hat{y} \cdot \left(1\hat{x} - 1\hat{y} + 7\hat{z}\right) + 3\hat{z} \cdot \left(1\hat{x} - 1\hat{y} + 7\hat{z}\right)$$

$$\vec{A} \cdot \vec{B} = 4 + 2 + 21 = 27$$

Vector Operation Example (Continued)

$$\vec{A} = 4\hat{x} - 2\hat{y} + 3\hat{z} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} = (4,-2,3) \qquad \vec{B} = 1\hat{x} - 1\hat{y} + 7\hat{z} = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix} = (1,-1,7)$$

Cross Product:

$$\vec{A} \times \vec{B} = (4\hat{x} - 2\hat{y} + 3\hat{z}) \times (1\hat{x} - 1\hat{y} + 7\hat{z})$$

$$\vec{A} \times \vec{B} = 4\hat{x} \times (1\hat{x} - 1\hat{y} + 7\hat{z}) - 2\hat{y} \times (1\hat{x} - 1\hat{y} + 7\hat{z}) + 3\hat{z} \times (1\hat{x} - 1\hat{y} + 7\hat{z})$$

$$\vec{A} \times \vec{B} = (-28\hat{y} - 4\hat{z}) + (-14\hat{x} + 2\hat{z}) + (3\hat{x} + 3\hat{y})$$

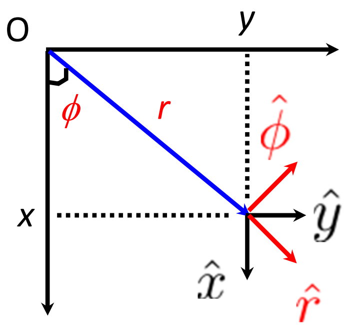
$$\vec{A} \times \vec{B} = (-11\hat{x} - 25\hat{y} - 2\hat{z})$$

The cross product is thus the determinant of the matrix:

$$\vec{A} \times \vec{B} = \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Coordinate Transformation - Cartesian to Cylindrical

$$\bar{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z = \hat{r}A_r + \hat{\phi}A_\phi + \hat{z}A_z$$
$$= \hat{x}x + \hat{y}y + \hat{z}z = \hat{r}r + \hat{\phi}\phi + \hat{z}z$$

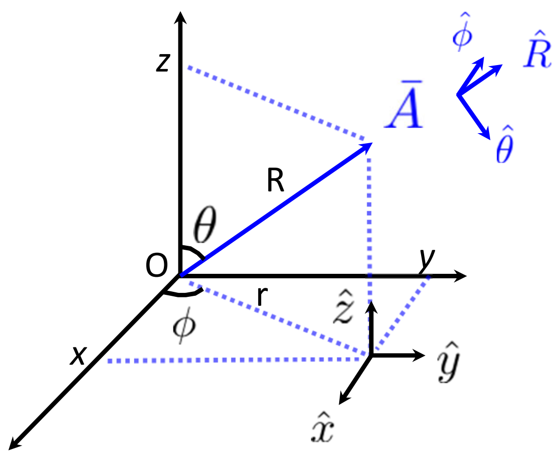


$$\boxed{\begin{aligned} x &= r\cos\phi, \\ y &= r\sin\phi, \\ z &= z. \end{aligned}}$$

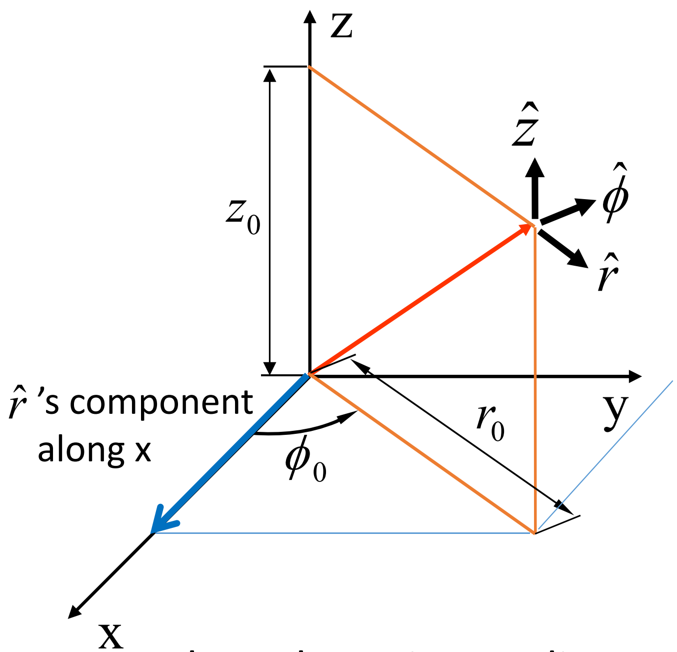
$$\boxed{\begin{aligned} r &= \sqrt{x^2 + y^2}, \\ \phi &= \tan^{-1}\frac{y}{x}, \\ z &= z. \end{aligned}}$$

Coordinate Transformation - Cartesian to Spherical

$$\bar{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z = \hat{r}A_R + \hat{\theta}A_\theta + \hat{\phi}A_\phi$$
$$= \hat{x}x + \hat{y}y + \hat{z}z = \hat{r}r + \hat{\theta}\theta + \hat{\phi}\phi$$



Coordinate Transformations: Cylindrical & Cartesian



The projection of the Cylindrical coordinates unit vectors along the Cartesian coordinates unit vector directions are:

$$\hat{r}\cdot\hat{x} = \cos\varphi_0 \qquad \hat{\phi}\cdot\hat{x} = -\sin\phi_0 \qquad \hat{z}\cdot\hat{x} = 0$$

$$\hat{r}\cdot\hat{y} = \sin\varphi_0 \qquad \hat{\phi}\cdot\hat{y} = \cos\phi_0 \qquad \hat{z}\cdot\hat{y} = 0$$

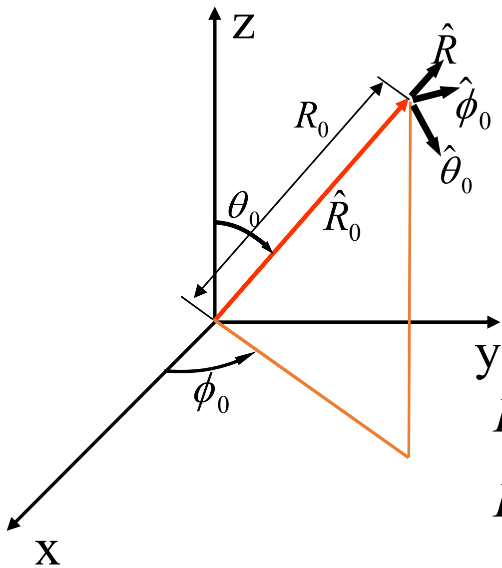
$$\hat{r}\cdot\hat{z} = 0 \qquad \hat{\phi}\cdot\hat{z} = 0 \qquad \hat{z}\cdot\hat{z} = 1$$

Thus, the unit coordinate transformation matrix is given by:

$$\begin{bmatrix}\hat{r}\\ \hat{\varphi}\\ \hat{z}\end{bmatrix} = \begin{bmatrix}\cos\varphi_0 & \sin\varphi_0 & 0\\ -\sin\varphi_0 & \cos\varphi_0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix} \qquad \begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix} = \begin{bmatrix}\cos\varphi_0 & -\sin\varphi_0 & 0\\ \sin\varphi_0 & \cos\varphi_0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{r}\\ \hat{\varphi}\\ \hat{z}\end{bmatrix}$$

$\hat{r}$ decompsed into $\hat{x}, \hat{y}, \hat{z}$components

Coordinate Transformations: Spherical & Cartesian



The projection of the Spherical coordinates unit vectors along the Cartesian coordinates unit vector directions are:

$$\hat{R}\cdot\hat{x} = \cos\varphi_0 \sin\theta_0 \qquad \hat{\theta}\cdot\hat{x} = \cos\phi_0 \cos\theta_0 \qquad \hat{\phi}\cdot\hat{x} = -\sin\phi_0$$

$$\hat{R}\cdot\hat{y} = \sin\varphi_0 \sin\theta_0 \qquad \hat{\theta}\cdot\hat{y} = \sin\phi_0 \cos\theta_0 \qquad \hat{\phi}\cdot\hat{y} = \cos\phi_0$$

$$\hat{R}\cdot\hat{z} = \cos\theta_0 \qquad \hat{\theta}\cdot\hat{z} = -\sin\theta_0 \qquad \hat{\phi}\cdot\hat{z} = 0$$

Thus, the unit coordinate transformation matrix is given by:

$$\begin{bmatrix} \hat{R} \\ \hat{\theta} \\ \hat{\varphi} \end{bmatrix} = \begin{bmatrix} \cos\varphi_0 \sin\theta_0 & \sin\varphi_0 \sin\theta_0 & \cos\theta_0 \\ \cos\varphi_0 \cos\theta_0 & \sin\varphi_0 \cos\theta_0 & -\sin\theta_0 \\ -\sin\varphi_0 & \cos\varphi_0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}$$

How about the conversion from spherical to Cartesian?

A• ~ on both sides

e.g., $A \cdot \hat{x} = A_x$

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \cos\varphi_0 & -\sin\varphi_0 & 0 \\ \sin\varphi_0 & \cos\varphi_0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\varphi} \\ \hat{z} \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}.}$$

Coordinate Transformations: Cylindrical & Cartesian

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}.$$

For cylindrical to Cartesian conversion, A_r, A_ϕ, and A_z may themselves be functions of r, ϕ, z. In that case, they should be converted into functions of x, y, and z (using the below relations) in the final answer.

$$r = \sqrt{x^2 + y^2},$$

$$\phi = \tan^{-1}\frac{y}{x},$$

$$z = z.$$

(see Example 2-9)

EXAMPLE 2–9 Express the vector

$$\mathbf{A} = \mathbf{a}_r(3 \cos \phi) - \mathbf{a}_\phi 2r + \mathbf{a}_z 5$$

in Cartesian coordinates.

2-5 Integrals Containing Vector Functions

$$\int_V \mathbf{F}\, dv,$$

$$\int_C V\, d\boldsymbol{\ell},$$

$$\oint_C V\, d\boldsymbol{\ell}.$$

$$\int_C \mathbf{F} \cdot d\boldsymbol{\ell},$$

$$\int_S \mathbf{A} \cdot d\mathbf{s}.$$

$$\int_C V\, d\boldsymbol{\ell} = \int_C V(x, y, z)[\mathbf{a}_x\, dx + \mathbf{a}_y\, dy + \mathbf{a}_z\, dz],$$

$$\int_C V\, d\boldsymbol{\ell} = \mathbf{a}_x \int_C V(x, y, z)\, dx + \mathbf{a}_y \int_C V(x, y, z)\, dy + \mathbf{a}_z \int_C V(x, y, z)\, dz.$$

$$\int_S \mathbf{A} \cdot d\mathbf{s},$$

Flux of vector field **A** flowing through the area S

$$d\mathbf{s} = \mathbf{a}_n \, ds$$

1. If S is a closed surface → $\mathbf{a}_n$ is in the outward direction
2. If S is an open surface → $\mathbf{a}_n$ is decided by right-hand rule (thumb)

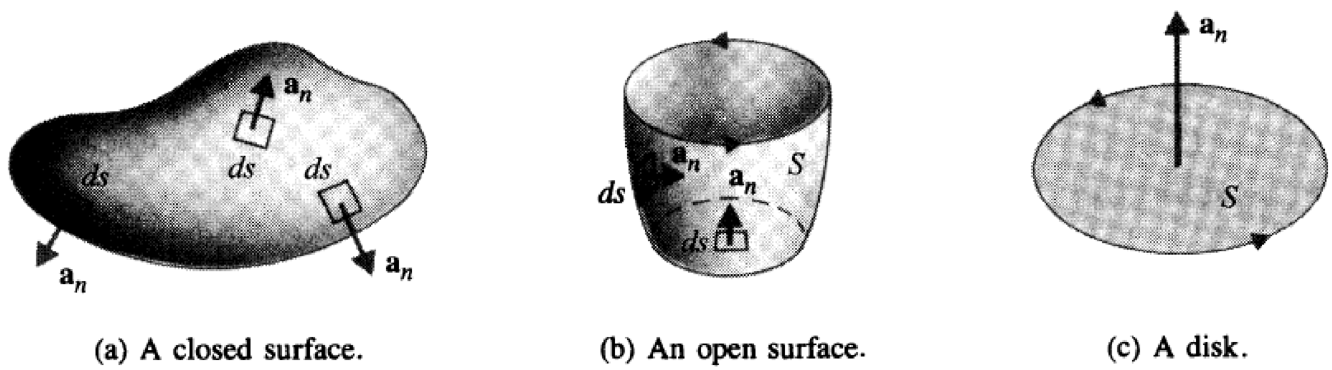
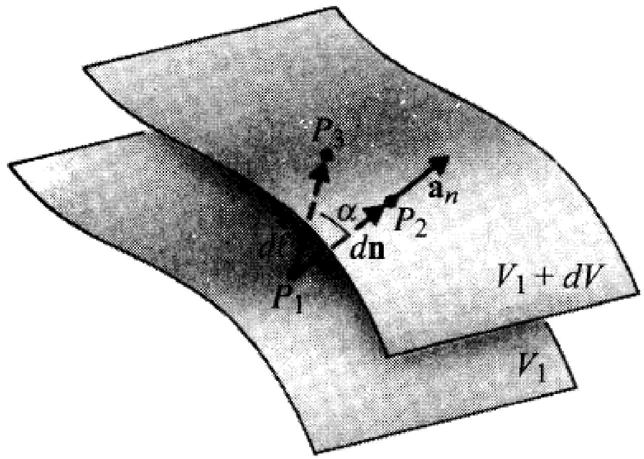


FIGURE 2–22
Illustrating the positive direction of $\mathbf{a}_n$ in scalar surface integral.

2-6 Gradient of a Scalar Field

- In general, fields(t, u_1, u_2, u_3)
- At a given time, fields(u_1, u_2, u_3)
  - Considering a scalar field, we have $V(u_1, u_2, u_3)$,



$V(u_1, u_2, u_3)$,

FIGURE 2–24
Concerning gradient of a scalar.

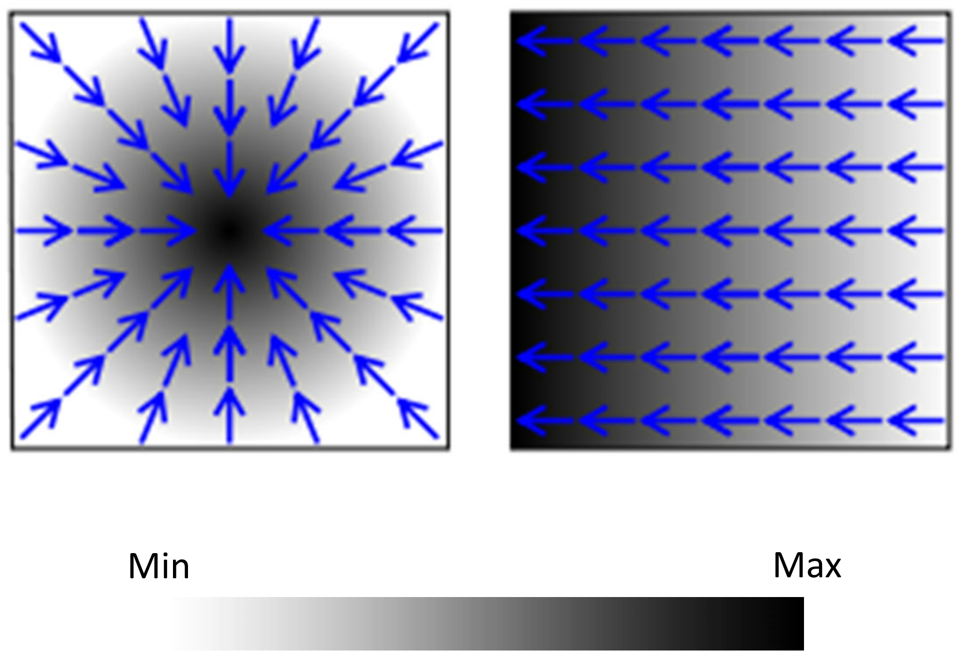
- 2 constant-*V* surfaces
- P_1 on surface V_1; P_2 and P_3 on surface V_1+dV
- Space rate of change: d$V/d\ell$
- *d*n is the shortest distance between the two surfaces
- Maximum space rate of change? ➔ dV/dn
- What's the ratio between dV/dn (P_1 to P_2) and $dV/d\ell$ (P_1 to P_3)? ➔ cosα

Gradient of a Scalar Field

- Describes the maximum space rate of change of a scalar field at a given time, which is a vector along that direction.

$$\boxed{\mathbf{grad}\ V \triangleq \mathbf{a}_n \frac{dV}{dn}.}$$

$$\boxed{\nabla V \triangleq \mathbf{a}_n \frac{dV}{dn}.}$$


Min
Max

$$\frac{dV}{d\ell} = \frac{dV}{dn}\frac{dn}{d\ell} = \frac{dV}{dn}\cos\alpha$$

$$= \frac{dV}{dn}\mathbf{a}_n \cdot \mathbf{a}_\ell = (\nabla V) \cdot \mathbf{a}_\ell.$$

dV/dℓ: the component of ∇*V* along **a**$_\ell$ direction

$$d\bar{\ell} = \mathbf{a}_\ell \, d\ell.$$

$$\boxed{dV = (\nabla V) \cdot d\bar{\ell},}$$

$$dV = \frac{\partial V}{\partial \ell_1} d\ell_1 + \frac{\partial V}{\partial \ell_2} d\ell_2 + \frac{\partial V}{\partial \ell_3} d\ell_3,$$

Total differential change

partial change at one coordinate direction

$$\boxed{dV = (\nabla V) \cdot d\bar{\ell},}$$

$$dV = \frac{\partial V}{\partial \ell_1} d\ell_1 + \frac{\partial V}{\partial \ell_2} d\ell_2 + \frac{\partial V}{\partial \ell_3} d\ell_3,$$

$$\begin{aligned} d\bar{\ell} &= \mathbf{a}_{u_1} d\ell_1 + \mathbf{a}_{u_2} d\ell_2 + \mathbf{a}_{u_3} d\ell_3 \\ &= \mathbf{a}_{u_1}(h_1\, du_1) + \mathbf{a}_{u_2}(h_2\, du_2) + \mathbf{a}_{u_3}(h_3\, du_3). \end{aligned}$$

$$\nabla V = \underline{\mathbf{a}_{u_1} \frac{\partial V}{\partial \ell_1} + \mathbf{a}_{u_2} \frac{\partial V}{\partial \ell_2} + \mathbf{a}_{u_3} \frac{\partial V}{d\ell_3}}$$

$$\boxed{\nabla V = \mathbf{a}_{u_1} \frac{\partial V}{h_1\, \partial u_1} + \mathbf{a}_{u_2} \frac{\partial V}{h_2\, \partial u_2} + \mathbf{a}_{u_3} \frac{\partial V}{h_3\, \partial u_3}.}$$

take V out

Gradient operator
(general expression)

$$\nabla \equiv \left(\mathbf{a}_{u_1} \frac{\partial}{h_1\, \partial u_1} + \mathbf{a}_{u_2} \frac{\partial}{h_2\, \partial u_2} + \mathbf{a}_{u_3} \frac{\partial}{h_3\, \partial u_3} \right)$$

E.g., in Cartesian coordinate

$$\boxed{\nabla \equiv \mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}.}$$

Similarly, in cylindrical coordinate

$$h_1 = h_3 = 1$$
$$h_2 = r$$

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}$$

$$\nabla \equiv \left(\mathbf{a}_{u_1}\frac{\partial}{h_1\,\partial u_1} + \mathbf{a}_{u_2}\frac{\partial}{h_2\,\partial u_2} + \mathbf{a}_{u_3}\frac{\partial}{h_3\,\partial u_3}\right)$$

In spherical coordinate

$$h_2 = R$$
$$h_3 = R\sin\theta$$

$$\nabla = \hat{R}\frac{\partial}{\partial R} + \hat{\theta}\frac{1}{R}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{R\sin\theta}\frac{\partial}{\partial \phi}$$

Integrating Gradients Along an Open Path

For the special case of the integral of a gradient function, the integral is simply the difference of the value of the function at the endpoints:

$$W = \int_L \underline{\nabla V \cdot d\vec{l}}$$

Recap:

$$\boxed{dV = (\nabla V) \cdot d\bar{\ell},}$$

$$W = \int_L \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz = \int_L \underline{dV} = V(\vec{r}_f) - V(\vec{r}_i)$$

This result is known as the fundamental theorem of calculus. This also implies that the integral of a gradient function over any closed integral vanishes.

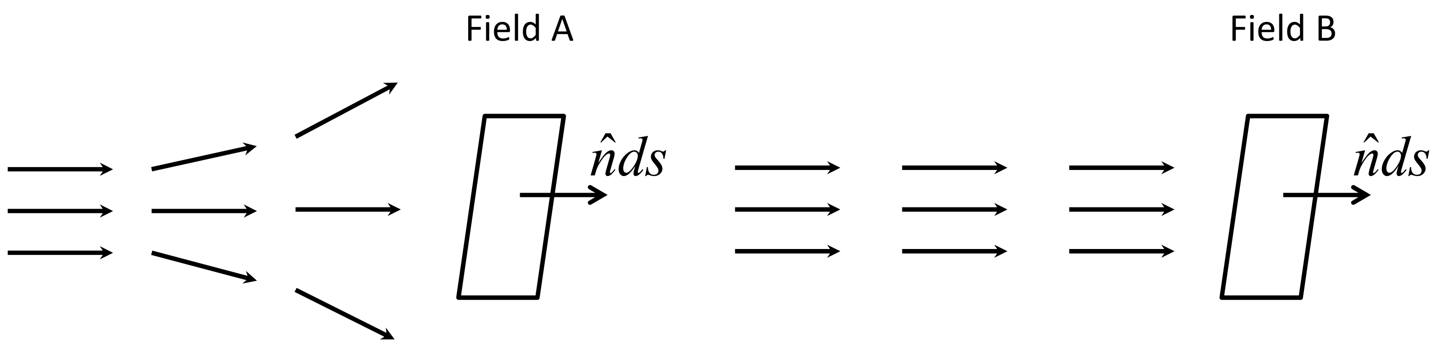
$$W = \oint \nabla V \cdot d\vec{l} = \oint dV = V(\vec{r}_i) - V(\vec{r}_i) = 0$$

2-7 Divergence of a Vector Field

- Spatial derivative of a **vector** field
  → Divergence, Curl

- Divergence: A concept on the (flow) source or sink

Flux Lines for A Vector Field



The vector field strength is measured by the number of flux lines passing through a unit surface normal to the vector (i.e., density).

Flux and Flux Density

- **Flux**:
  - A flow (e.g., an incompressible fluid such as water)
  - In general, a continuous moving on or passing by as of a stream

$$\overline{A} \cdot d\overline{S}$$

E.g.: $\boldsymbol{v} \bullet \boldsymbol{dS}$, water amount/time (where $\boldsymbol{v}$ is the flow speed), m^3/s

- **Flux density**: $$\frac{\overline{A} \cdot d\overline{S}}{\left|d\overline{S}\right|} = \frac{\overline{A} \cdot \hat{n} dS}{dS}$$

- The **total flux** crossing an enclosed surface S which bounds a volume *v* is $\oint_s \overline{A} \cdot d\overline{S}$

E.g.: Water amount/time through (outward or inward) *v*, m^3/s

Divergence

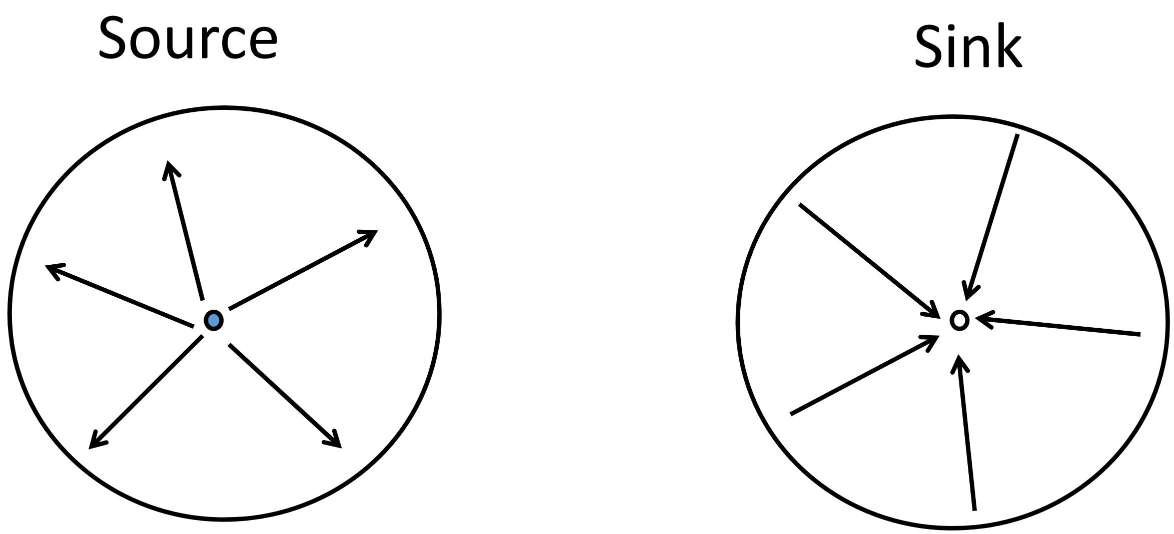
- The net outward flux of ***A*** per unit volume as the volume about the point tends to zero.

$$div\bar{A} = \lim \frac{\oint \bar{A} \cdot d\bar{S}}{\Delta V}$$

flux "volume" density

- diverge – to move or extend in different directions from a common point; move away from.

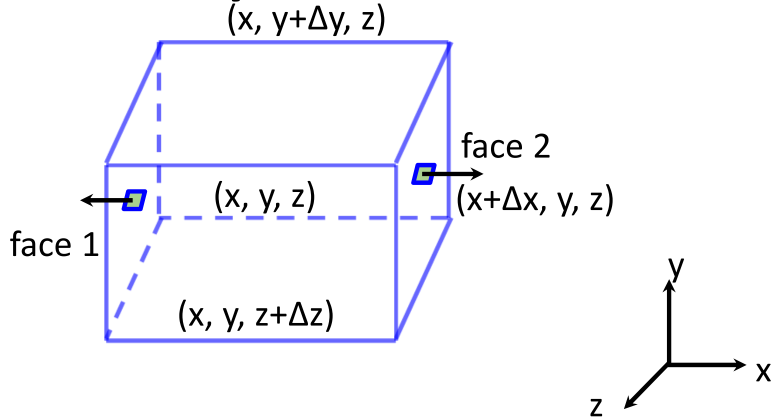
Source or Sink of a Vector Field



1. Divergence >0, source exists
2. Divergence <0, sink exists
3. Divergence = 0, divergenceless or solenoidal (pipe)

net outward flux > inward flux ➔ source ➔ divergence > 0
net outward flux < inward flux ➔ sink ➔ divergence < 0

Derivation of Divergence in the XYZ Coordinate System



Face 1: $$F_1 = \int_s \bar{A}\cdot\hat{n}_1\, dS = \int_x \left(\hat{x}A_x + \hat{y}A_y + \hat{z}A_z\right)\cdot\left(-\hat{x}dydz\right) = -A_x(1)\Delta y\Delta z$$

where $A_x(1)$ is the average value of A_x on face 1

Face 2: $$F_2 = \int_s \bar{A}\cdot\hat{n}_s\, dS = \int_x \left(\hat{x}A_x + \hat{y}A_y + \hat{z}A_z\right)\cdot\left(\hat{x}dydz\right) = A_x(2)\Delta y\Delta z$$

$$A_x(2) = A_x(1) + \frac{\partial A_x}{\partial x}\Delta x$$ (H.O.T.: See textbook, p. 48)

$$F_1 + F_2 = -A_x(1)\Delta y\Delta z + \left(A_x(1) + \frac{\partial A_x}{\partial x}\Delta x\right)\Delta y\Delta z = \frac{\partial A_x}{\partial x}\Delta V$$

Derivation of Divergence in the XYZ Coordinate System

Similarly, $F_3 + F_4 = \frac{\partial A_y}{\partial y}\Delta V$ $F_5 + F_6 = \frac{\partial A_z}{\partial z}\Delta V$

So the total flux is $$\sum_{i=1}^{6} F_i = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\Delta V$$

And the divergence is defined as

$$div\overline{A} = \lim \frac{\oint \overline{A} \cdot d\overline{S}}{\Delta V}$$

$$div\overline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \nabla \cdot \overline{A}$$

General Expression

$$\boxed{\nabla \cdot \mathbf{A} \equiv \operatorname{div} \mathbf{A}.}$$

$$\boxed{\nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_1 h_3 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right].}$$

In cylindrical coordinate system:

$$\boxed{\begin{aligned} h_1 &= h_3 = 1 \\ h_2 &= r \end{aligned}}$$

$$\nabla \cdot \bar{A} = \frac{1}{r}\frac{\partial}{\partial r}\left(rA_r\right) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

In spherical coordinate system:

$$\boxed{\begin{aligned} h_2 &= R \\ h_3 &= R \sin\theta \end{aligned}}$$

$$\nabla \cdot \bar{A} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 A_R\right) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}\left(A_\theta \sin\theta\right) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

2-8 Divergence Theorem

- The volume integral of the divergence of a vector field equals the total outward flux of the vector through the surface that bounds the volume.

$$\int_{v} \nabla \cdot \overline{A} dV = \oint_{s} \overline{A} \cdot d\overline{S}$$

flux volume density × volume = flux
E.g.: outward water density/time × volume = outward water amount/time (m^3/s)

- Conversion between volume integral and surface integral.

Proof

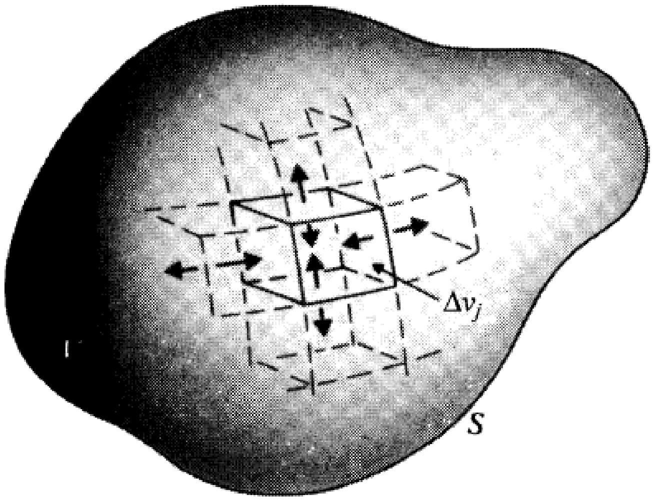


FIGURE 2–27
Subdivided volume for proof of divergence theorem.

$$(\nabla \cdot \mathbf{A})_j \Delta v_j = \oint_{s_j} \mathbf{A} \cdot d\mathbf{s}.$$

$$\lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} (\nabla \cdot \mathbf{A})_j \Delta v_j \right] = \lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} \oint_{s_j} \mathbf{A} \cdot d\mathbf{s} \right].$$

Internal surfaces cancelled

$$\int_V (\nabla \cdot \mathbf{A})\, dv. \qquad \oint_S \mathbf{A} \cdot d\mathbf{s}.$$

EXAMPLE 2–20 Given $\mathbf{F} = \mathbf{a}_R kR$, determine whether the divergence theorem holds for the shell region enclosed by spherical surfaces at $R = R_1$ and $R = R_2 (R_2 > R_1)$ centered at the origin, as shown in Fig. 2–29.

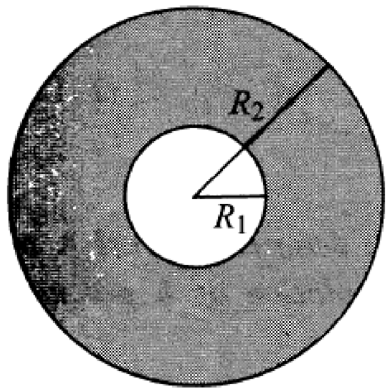


FIGURE 2–29
A spherical shell region (Example 2–20).

2-9 Curl of a Vector Field

- Divergence indicates whether there is a flow source or not.
- Curl indicates whether there is a vortex source or not.

Water whirling down a sink drain is an example of a vortex sink causing a circulation of fluid velocity

Circulation

- The net circulation of a vector field around a closed path is defined as:

$$\oint_c \overline{A} \cdot d\overline{l}$$

Example 1

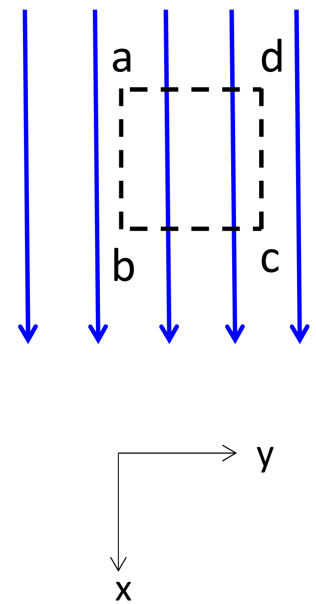
Given a uniform magnetic field $\overline{B} = \hat{x}B_0$

$$Circulation = \oint_s \overline{B} \cdot d\overline{l}$$

$$= \int_a^b \hat{x}B_0 \cdot \hat{x}\,dx + \int_b^c \hat{x}B_0 \cdot \hat{y}\,dy +$$

$$\int_c^d \hat{x}B_0 \cdot \hat{x}\,dx + \int_d^a \hat{x}B_0 \cdot \hat{y}\,dy$$

$$= B_0\Delta x - \underline{B_0\Delta x} = 0$$



No net circulation exists!

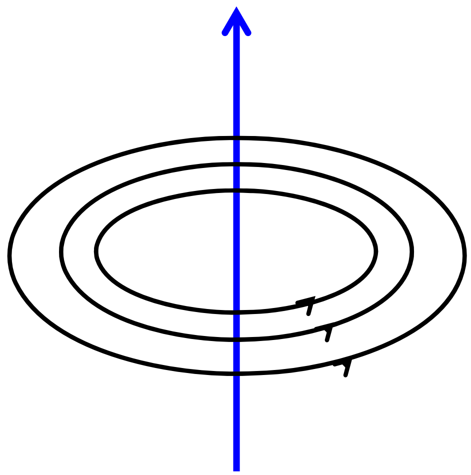
Example 2

Magnetic field induced by an infinite wire carrying a DC current *I* in the positive z-axis direction, then

$$\overline{B} = \hat{\phi}\frac{\mu_0 I}{2\pi r}$$

$$Circulation = \oint_c \overline{B} \cdot d\overline{l}$$

$$= \int_0^{2\pi} \hat{\phi}\frac{\mu_0 I}{2\pi r} \cdot \hat{\phi} r \, d\phi = \mu_0 I$$



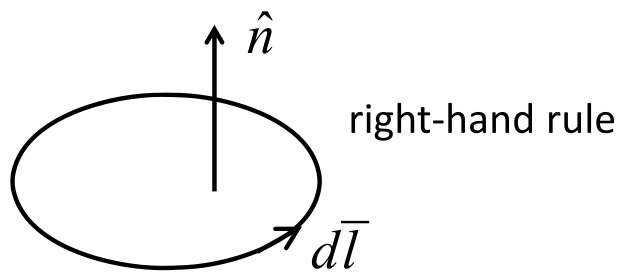
A net circulation exists!

Curl

- Curl is the circulation per unit area in short.

$$\nabla \times \overline{A} = \lim \frac{1}{\Delta s}\left(\hat{n} \oint_{c} \overline{A} \cdot d\overline{l} \right)$$

- Its magnitude is the **maximum** net circulation of A per unit area as the area tends to zero.
- Its direction is the normal direction of the area.



Curl: circulation "surface" density (over a tiny surface)

Component of Curl

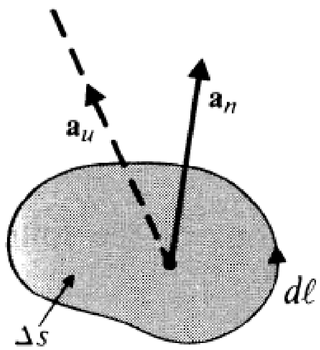


FIGURE 2–30
Relation between $\mathbf{a}_n$ and $d\ell$ in defining curl.

$$(\nabla \times \mathbf{A})_u = \mathbf{a}_u \cdot (\nabla \times \mathbf{A}) = \lim_{\Delta s_u \to 0} \frac{1}{\Delta s_u} \left(\oint_{C_u} \mathbf{A} \cdot d\ell \right),$$

Normal of Δs_u is $\mathbf{a}_u$

The component of $\nabla\times\mathbf{A}$ along $\mathbf{a}_u$ direction

Proof of Curl in Cartesian Coordinate

$$\nabla \times \bar{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

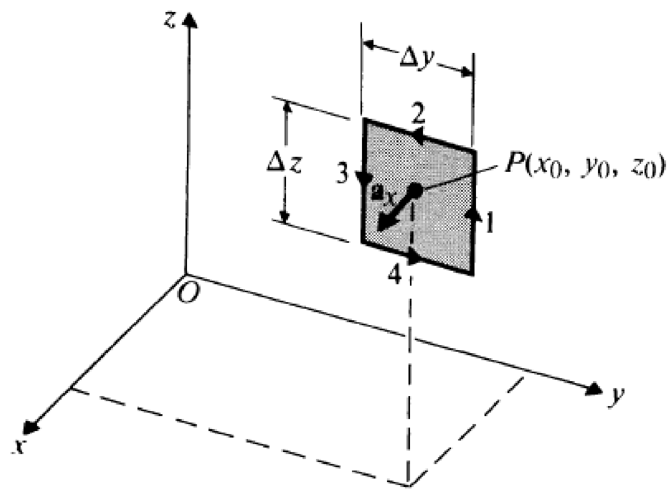


FIGURE 2–31
Determining $(\nabla \times \mathbf{A})_x$.

$$(\nabla \times \mathbf{A})_x = \lim_{\Delta y \Delta z \to 0} \frac{1}{\Delta y \, \Delta z} \left(\oint_{\substack{\text{sides} \\ 1,\,2,\,3,\,4}} \mathbf{A} \cdot d\ell \right).$$

$$\mathbf{A} = \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z.$$

Side 1: $\quad d\boldsymbol{\ell} = \mathbf{a}_z \Delta z,\ \mathbf{A}\cdot d\boldsymbol{\ell} = A_z\left(x_0, y_0 + \dfrac{\Delta y}{2}, z_0\right)\Delta z,$

where $A_z\left(x_0, y_0 + \dfrac{\Delta y}{2}, z_0\right)$ can be expanded as a Taylor series:

$$A_z\left(x_0, y_0 + \frac{\Delta y}{2}, z_0\right) = A_z(x_0, y_0, z_0) + \frac{\Delta y}{2}\frac{\partial A_z}{\partial y}\bigg|_{(x_0, y_0, z_0)} + \underline{\text{H.O.T.}},$$

$(\Delta y)^2$, $(\Delta y)^3$, etc.

$$\int_{\text{side 1}} \mathbf{A}\cdot d\boldsymbol{\ell} = \left\{A_z(x_0, y_0, z_0) + \frac{\Delta y}{2}\frac{\partial A_z}{\partial y}\bigg|_{(x_0, y_0, z_0)} + \text{H.O.T.}\right\}\Delta z.$$

Side 3: $\quad d\boldsymbol{\ell} = -\mathbf{a}_z \Delta z,\ \mathbf{A}\cdot d\boldsymbol{\ell} = A_z\left(x_0, y_0 - \dfrac{\Delta y}{2}, z_0\right)(-\Delta z)$

where

$$A_z\left(x_0, y_0 - \frac{\Delta y}{2}, z_0\right) = A_z(x_0, y_0, z_0) - \frac{\Delta y}{2}\frac{\partial A_z}{\partial y}\bigg|_{(x_0, y_0, z_0)} + \text{H.O.T.};$$

$$\int_{\text{side 3}} \mathbf{A}\cdot d\boldsymbol{\ell} = \left\{A_z(x_0, y_0, z_0) - \frac{\Delta y}{2}\frac{\partial A_z}{\partial y}\bigg|_{(x_0, y_0, z_0)} + \text{H.O.T.}\right\}(-\Delta z).$$

$$\int_{\text{side 1}} \mathbf{A} \cdot d\boldsymbol{\ell} = \left\{ \cancel{A_z(x_0, y_0, z_0)} + \frac{\Delta y}{2} \frac{\partial A_z}{\partial y}\bigg|_{(x_0, y_0, z_0)} + \text{H.O.T.} \right\} \Delta z.$$

$$\int_{\text{side 3}} \mathbf{A} \cdot d\boldsymbol{\ell} = \left\{ \cancel{A_z(x_0, y_0, z_0)} - \frac{\Delta y}{2} \frac{\partial A_z}{\partial y}\bigg|_{(x_0, y_0, z_0)} + \text{H.O.T.} \right\} (-\Delta z).$$

$$\int_{\substack{\text{sides} \\ 1\ \&\ 3}} \mathbf{A} \cdot d\boldsymbol{\ell} = \left(\frac{\partial A_z}{\partial y} + \underline{\text{H.O.T.}} \right)\bigg|_{(x_0, y_0, z_0)} \Delta y \, \Delta z.$$

$\propto \Delta y, \Delta y^2$, etc.

Similarly, $$\int_{\substack{\text{sides} \\ 2\ \&\ 4}} \mathbf{A} \cdot d\boldsymbol{\ell} = \left(-\frac{\partial A_y}{\partial z} + \underline{\text{H.O.T.}} \right)\bigg|_{(x_0, y_0, z_0)} \Delta y \, \Delta z.$$

$\propto \Delta z, \Delta z^2$, etc.

As $\Delta y \rightarrow 0$ and $\Delta z \rightarrow 0$, H.O.T. $\rightarrow 0$

$$(\nabla \times \mathbf{A})_x = \lim_{\Delta y \, \Delta z \to 0} \frac{1}{\Delta y \, \Delta z} \left(\oint_{\substack{\text{sides} \\ 1,\, 2,\, 3,\, 4}} \mathbf{A} \cdot d\boldsymbol{\ell} \right).$$

$$(\nabla \times \mathbf{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}.$$

General Expression

$$\boxed{\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \mathbf{a}_{u_1} h_1 & \mathbf{a}_{u_2} h_2 & \mathbf{a}_{u_3} h_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}.}$$

2-10 Stoke's Theorem

- The surface integral of the curl of a vector field over an open surface is equal to the closed line integral of the vector along the contour bounding the surface.

$$\int_s \left(\nabla \times \bar{A}\right) \cdot d\bar{S} = \oint_c \bar{A} \cdot d\bar{l}$$

Circulation "surface" density $\times$ area = circulation

- Conversion between surface integral and line integral.

Comparisons

Divergence theorem

$$\int_v \nabla \cdot \bar{A}\, dV = \oint_s \bar{A} \cdot d\bar{S}$$

Flux “volume” density × volume = flux

Stoke’s theorem

$$\int_s \left(\nabla \times \bar{A}\right) \cdot d\bar{S} = \oint_c \bar{A} \cdot d\bar{l}$$

circulation “surface” density × area = circulation

Proof

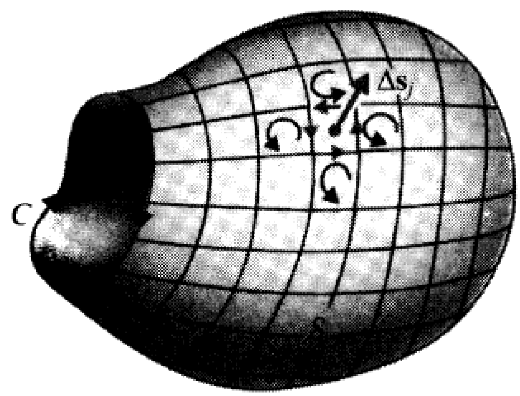


FIGURE 2–32
Subdivided area for proof of Stokes's theorem.

$$(\nabla \times \mathbf{A})_j \cdot (\Delta \mathbf{s}_j) = \oint_{C_j} \mathbf{A} \cdot d\boldsymbol{\ell}.$$

$$\lim_{\Delta s_j \to 0} \sum_{j=1}^{N} (\nabla \times \mathbf{A})_j \cdot (\Delta \mathbf{s}_j) = \underline{\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s}}.$$

$$\lim_{\Delta s_j \to 0} \sum_{j=1}^{N} \left(\oint_{c_j} \mathbf{A} \cdot d\boldsymbol{\ell} \right) = \underline{\oint_C \mathbf{A} \cdot d\boldsymbol{\ell}}.$$

Interior line integrals cancel each other

2-11 Two Null Identities

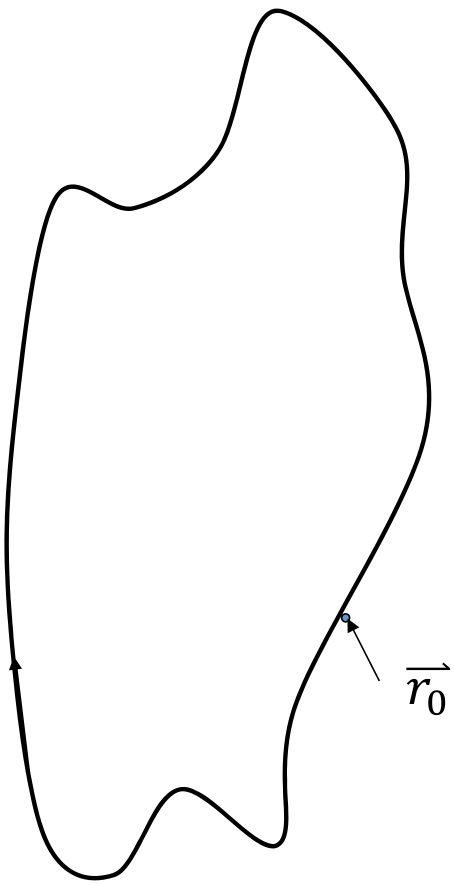
$$\nabla \times \left(\nabla V\right) = 0$$

Therefore, if a vector field is curl-free, then it can be expressed as the gradient of a scalar field.

$$\nabla \cdot \left(\nabla \times \boldsymbol{A}\right) = 0$$

Therefore, if a vector field is divergenceless, then it can be expressed as the curl of another vector field.

<u>Proof: The curl of a gradient of a scalar is always zero</u>



$$\oint_L \nabla V(\vec{r}) \cdot d\vec{l} = V(\vec{r}_0) - V(\vec{r}_0) = 0$$

Recap $\boxed{dV = \nabla V \cdot d\vec{l}}$

Now consider $\nabla \times \nabla V(\vec{r})$

$$\hat{n} \cdot \nabla \times \nabla V(\vec{r}) = \lim_{S_n \to 0} \frac{\oint_{L_n} \nabla V(\vec{r}) \cdot d\vec{l}}{S_n} = \lim_{S_n \to 0} \frac{0}{S_n} = 0$$

Thus $\boxed{\nabla \times \nabla V(\vec{r}) = 0}$

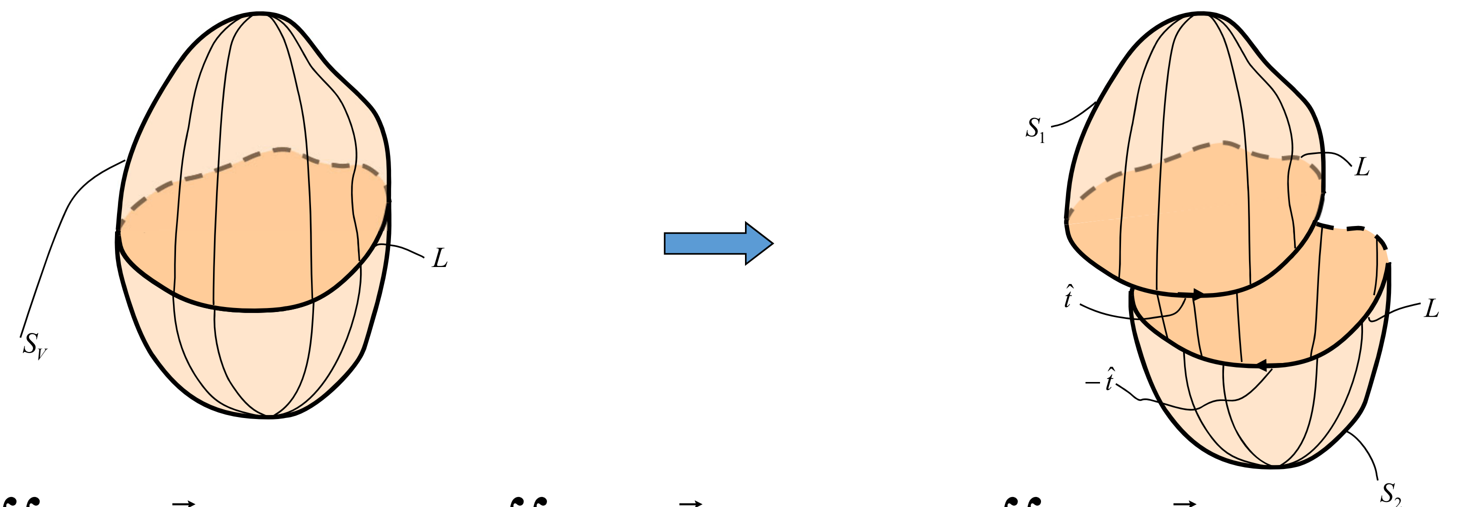
$$\nabla \times \underbrace{\left(\nabla V\right)}_{\boldsymbol{A}} = 0$$

A: Curl free = irrotational field = Conservative field

A conservative field can always be expressed:

$$\boldsymbol{A} = \nabla V$$

Proof: The divergence of a curl of a vector is always zero



$$\oiint_{S_V} \nabla \times \vec{A}(\vec{r}) \cdot \hat{n} dS = \iint_{S_1} \nabla \times \vec{A}(\vec{r}) \cdot \hat{n} dS + \iint_{S_2} \nabla \times \vec{A}(\vec{r}) \cdot \hat{n} dS =$$

$$\oint_{L_1} \vec{A}(\vec{r}) \cdot \hat{t} dl + \oint_{L_2} \vec{A}(\vec{r}) \cdot \left(-\hat{t}\right) dl = 0$$

$\hat{n}$ of S_1 is upward

$\hat{n}$ of S_2 is downward

$$\nabla \cdot \left(\nabla \times \vec{A}(\vec{r})\right) = \lim_{V \to 0} \frac{\oiint_{S_V} \nabla \times \vec{A}(\vec{r}) \cdot \hat{n} dS}{V} = \lim_{V \to 0} \frac{0}{V} = 0$$

$$\nabla \cdot \big(\underbrace{\nabla \times V}_{\boldsymbol{B}}\big) = 0$$

B: divergenceless

⇓

A divergenceless field can always be expressed:

$$\boldsymbol{B} = \nabla \times V$$

Combination of Vector Operators: Laplacian

One of the important vector operators that we will commonly use is the divergence of the gradient of a scalar function:

$$\nabla \cdot \nabla V = \nabla^2 V$$

Let's first analyze its form in Cartesian coordinates. Knowing that the divergence of a vector is:

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

And assuming that ***A*** can be written as the gradient of a scalar:

$$\vec{A} = \nabla V = \hat{x}\frac{\partial V}{\partial x} + \hat{y}\frac{\partial V}{\partial y} + \hat{z}\frac{\partial V}{\partial z} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$$

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}}$$

Laplacian in Cylindrical Coordinates

$$\nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial\left(rA_r\right)}{\partial r} + \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$

$$\vec{A} = \nabla V = \hat{r}\frac{\partial V}{\partial r} + \hat{\varphi}\frac{1}{r}\frac{\partial V}{\partial \varphi} + \hat{z}\frac{\partial V}{\partial z}$$

$$\boxed{\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial^2 \varphi} + \frac{\partial^2 V}{\partial z^2}}$$

Laplacian in Spherical Coordinates

$$\nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial\left(r^2 A_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta A_\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$$

$$\vec{A} = \nabla V = \hat{r}\frac{\partial V}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial V}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial V}{\partial\varphi}$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2}$$

2-12 Helmholtz Theorem

- Divergenceless ⇔ solenoidal ⇔ no (flow) source
- Curl free ⇔ irrotational ⇔ no circulation

- 4 cases:

$$\nabla \cdot \mathbf{F} = 0 \quad \text{and} \quad \nabla \times \mathbf{F} = 0.$$

$$\nabla \cdot \mathbf{F} = 0 \quad \text{and} \quad \nabla \times \mathbf{F} \neq 0.$$

$$\nabla \times \mathbf{F} = 0 \quad \text{and} \quad \nabla \cdot \mathbf{F} \neq 0.$$

$$\nabla \cdot \mathbf{F} \neq 0 \quad \text{and} \quad \nabla \times \mathbf{F} \neq 0.$$

Helmholtz Theorem

- A vector field (vector point function) is determined to within an additive constant if both its divergence and its curl are specified everywhere.
- Intuitive illustration:
  - Divergence ⇔ flow source
  - Curl ⇔ vortex source
  - If both the flow source and vortex source of a vector are specified, the vector is determined

Helmholtz Theorem-Equations

$$\mathbf{F} = \underline{\mathbf{F}_i} + \underline{\mathbf{F}_s},$$

Irrotational = Flow source only

$$\begin{cases} \nabla \times \mathbf{F}_i = 0 \\ \nabla \cdot \mathbf{F}_i = g \end{cases}$$

Solenoidal = Vortex source only

$$\begin{cases} \nabla \cdot \mathbf{F}_s = 0 \\ \nabla \times \mathbf{F}_s = \mathbf{G}, \end{cases}$$

$$\Downarrow$$

$$\mathbf{F}_i = -\nabla V.$$

$$\nabla \cdot \mathbf{F} = \nabla \cdot \mathbf{F}_i = g$$

$$\mathbf{F}_s = \nabla \times \mathbf{A}.$$

$$\nabla \times \mathbf{F} = \nabla \times \mathbf{F}_s = \mathbf{G}.$$

$$\Downarrow$$

$$\mathbf{F} = -\nabla V + \nabla \times \mathbf{A}.$$

F is unique given known *g* and **G**

Proof of Uniqueness

Uniqueness: How do we know if there is more than one solution to the vector field equations? Let's suppose that there are two possible solutions, $\mathbf{F_1}$ and $\mathbf{F_2}$ which satisfy the same source and circulation distributions.

$$\nabla \cdot \overrightarrow{F_1}(\vec{r}) = g(\vec{r}) \qquad \nabla \cdot \overrightarrow{F_2}(\vec{r}) = g(\vec{r})$$

$$\nabla \times \overrightarrow{F_1}(\vec{r}) = \vec{G}(\vec{r}) \qquad \nabla \times \overrightarrow{F_2}(\vec{r}) = \vec{G}(\vec{r})$$

Let's define a new function ***W***, which is the difference between the two vector fields $\mathbf{F_1}$ and $\mathbf{F_2}$:

$$\overrightarrow{W}(\vec{r}) = \overrightarrow{F_1}(\vec{r}) - \overrightarrow{F_2}(\vec{r})$$

$$\nabla \cdot \overrightarrow{W}(\vec{r}) = \nabla \cdot \overrightarrow{F_1}(\vec{r}) - \nabla \cdot \overrightarrow{F_2}(\vec{r}) = g(\vec{r}) - g(\vec{r}) = 0 \quad \text{(1)}$$

$$\nabla \times \overrightarrow{W}(\vec{r}) = \nabla \times \overrightarrow{F_1}(\vec{r}) - \nabla \times \overrightarrow{F_2}(\vec{r}) = \vec{G}(\vec{r}) - \vec{G}(\vec{r}) = 0 \quad \text{(2)}$$

The goal is to prove that vector field ***W*** is identically zero everywhere in space.

First, let's define the vector field ***W*** produced solely by sources as the gradient of a scalar potential *V*.

(2) and Null identity $\Rightarrow$ $\overrightarrow{W}(\vec{r}) = -\nabla V(\vec{r}) \longrightarrow \nabla \cdot \overrightarrow{W}(\vec{r}) = -\nabla^2 V(\vec{r}) \underset{\text{By (1)}}{=} 0 \quad \text{(3)}$

Let’s now apply Gauss’s divergence theorem to the vector:

$$\oiint V(\vec{r})\nabla V(\vec{r})\cdot\hat{n}\,dS = \iiint \nabla\cdot\left(V(\vec{r})\nabla V(\vec{r})\right)dV = \iiint \left(\overset{\text{By (3)}}{\cancel{V(\vec{r})\nabla^2 V(\vec{r})}} + \nabla V(\vec{r})\cdot\nabla V(\vec{r})\right)dV$$

$$\boxed{\nabla\cdot(\psi\mathbf{A}) \;=\; \psi\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla\psi}$$

$$= \iiint (\nabla V(\vec{r}))^2 dV \overset{\text{By (2)}}{=} \iiint (\vec{W})^2 dV$$

One very important assumption for vector fields is that all sources of potential are finite and no sources exist at infinity. This also implies that the potential at infinity is also zero. Therefore, the surface integral on the left hand side will vanish if we integrate over all space.

$$\oiint_{R\to\infty} \underbrace{V(\vec{r})\nabla V(\vec{r})}_{=0}\cdot\hat{n}\,dS = 0 = \iiint (\vec{W})^2 dV \qquad \therefore \vec{W} = 0 \Rightarrow \vec{F_1}(\vec{r}) = \vec{F_2}(\vec{r})$$

This condition is satisfied everywhere

Other Useful Vector Identities

$$\nabla(\psi\,\phi) = \phi\,\nabla\psi + \psi\,\nabla\phi$$

$$\nabla\cdot(\psi\mathbf{A}) \;=\; \psi\,\nabla\cdot\mathbf{A} \,+\, \mathbf{A}\cdot\nabla\psi$$

$$\nabla\times(\psi\mathbf{A}) \;=\; \psi\,(\nabla\times\mathbf{A}) \,+\, \nabla\psi\times\mathbf{A}$$

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0$$

$$\nabla\times(\nabla\psi) = \mathbf{0}$$

$\nabla\cdot(\nabla\psi) = \nabla^2\psi$ (scalar Laplacian)

$\nabla\,(\nabla\cdot\mathbf{A}) - \nabla\times(\nabla\times\mathbf{A}) = \nabla^2\mathbf{A}$ (vector Laplacian)

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) \;=\; \mathbf{B}\cdot(\nabla\times\mathbf{A}) \,-\, \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) \;=\; \mathbf{A}\,(\nabla\cdot\mathbf{B}) \,-\, \mathbf{B}\,(\nabla\cdot\mathbf{A}) \,+\, (\mathbf{B}\cdot\nabla)\,\mathbf{A} \,-\, (\mathbf{A}\cdot\nabla)\,\mathbf{B}$$

$$\nabla\,(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\,\mathbf{B} + (\mathbf{B}\cdot\nabla)\,\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A})$$