6-9 Behavior of Magnetic Materials

$$\boxed{\mathbf{M} = \chi_m \mathbf{H},}$$

$$\boxed{\mu_r = 1 + \chi_m = \frac{\mu}{\mu_0}}$$

χ_m: magnetic susceptibility
μ_r: relative permeability

Diamagnetic, if $\mu_r \lesssim 1$ (χ_m is a very small negative number). ⇒ $\chi_m < 0$ ⇒ $M < 0$ ⇒ decreased B
Paramagnetic, if $\mu_r \gtrsim 1$ (χ_m is a very small positive number).
Ferromagnetic, if $\mu_r \gg 1$ (χ_m is a large positive number).

$\mathbf{B}/\mu_0 = \mathbf{H} + \mathbf{M}$

$\varepsilon_0 \mathbf{E} = \mathbf{D} - \mathbf{P}$

⇒ $E_{total} < E_{ext}$

1. Diamagnetic

- Without external magnetic field ➔ no net magnetic dipole moments, **m** = 0

  $M=0$

- With external magnetic field ➔ a net magnetic dipole moment (induced magnetization **M**), **m** $\neq$ 0
- **M** opposes $\mathbf{B}_{ext}$, thus reducing the **B**. That is, $\chi_m<0$ ($\sim -10^{-5}$)

$$\mathbf{M} = \chi_m \mathbf{H},$$

$$\mathbf{B}/\mu_0 = \mathbf{H}+\mathbf{M}$$

$$\varepsilon_0\mathbf{E} = \mathbf{D}-\mathbf{P}$$

- Like **P** opposes **E**, reducing the **E** in dielectrics, **B** is reduced in diamagnetics. $\mathbf{M} = \chi_m \mathbf{H}$,

$$\mathbf{B}/\mu_0 = \mathbf{H}+\mathbf{M} \qquad\qquad \varepsilon_0\mathbf{E} = \mathbf{D}-\mathbf{P}$$

- No permanent magnetism: the induced magnetic moments disappears when applied field is withdrawn.
- Diamagnetic materials: bismuth, copper, lead, mercury, germanium, silver, gold, diamond
- Due to mainly orbiting electrons

2. Paramagnetic

- Without external magnetic field ➔ there is net magnetic dipole moments, **m** $\neq$ 0
- With external magnetic field ➔ a very weak induced magnetization **M** (similar to diamagnetic effect)
- **M** is in the direction of **B**$_{ext}$, thus increasing the **B**. $\chi_m > 0$ (~ 10^{-5})

$$\mathbf{M} = \chi_m \mathbf{H},$$

$\mathbf{B}/\mu_0 = \mathbf{H} + \mathbf{M}$

$\varepsilon_0 \mathbf{E} = \mathbf{D} - \mathbf{P}$

- Paramagnetic materials: aluminum, magnesium, titanium, and tungsten
- Due to mainly spinning electrons

3. Ferromagnetic

- Magnetization can be many orders of magnitude larger than that of paramagnetic substances.
- Magnetized domain:

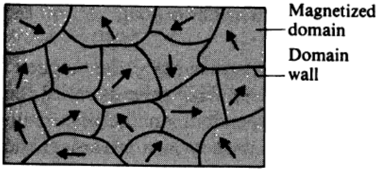


FIGURE 6–16
Domain structure of a polycrystalline ferromagnetic specimen.

- Without external magnetic field ➔ no net magnetization (due to random orientation in the various domains)
- With external magnetic field ➔ the domains aligned with applied magnetic field grow ➔ **B** is increased (χ_m>0) $\mathbf{M} = \chi_m \mathbf{H}$,

  $\mathbf{B}/\mu_0 = \mathbf{H} + \mathbf{M}$

- Due to mainly spinning electrons
- Ferromagnetic materials: cobalt, nickel, and iron

Q&A

- When a permanent magnet is placed against a steel (ferromagnetic material) refrigerator, it sticks. What is the best explanation of the physics involved?

Hysteresis for ferromagnetic materials

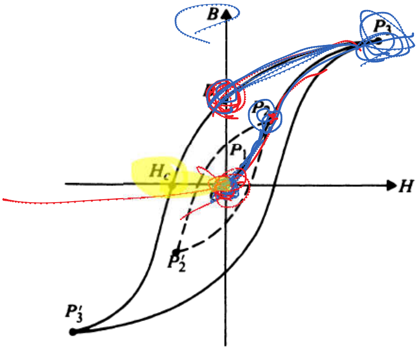


FIGURE 6–17
Hysteresis loops in the B–H plane for ferromagnetic material.

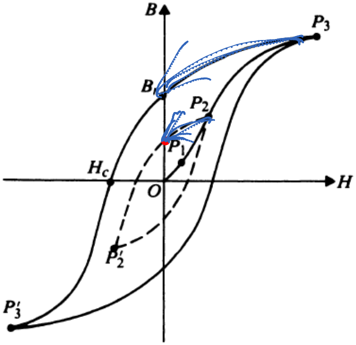


FIGURE 6–17
Hysteresis loops in the *B–H* plane for ferromagnetic material.

1. For weak applied fields (e.g., to P_1), magnetization are reversible.
2. For stronger applied fields (e.g., to P_2), magnetization are no longer reversible.
   - Forward: OP_1P_2
   - Backward: P_2P_2'

   Magnetization (*B*) lags the field (*H*), called **hysteresis** (“to lag” in Greek word)

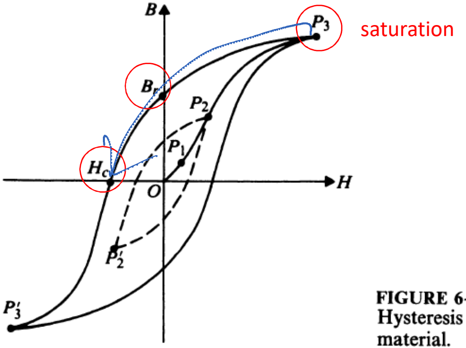


FIGURE 6–17
Hysteresis loops in the B–H plane for ferromagnetic material.

3. For much stronger fields (e.g., to P_3) ➔ total alignment of microscopic magnetic dipole moments with the applied field ➔ reached the saturation
   - **Permanent magnets**: If H is reduced to 0, B does not go to 0 but the value B_r (called residual or remnant B).
   - Coercive field intensity: to make B back to 0, a H_c in the opposite direction is necessary.

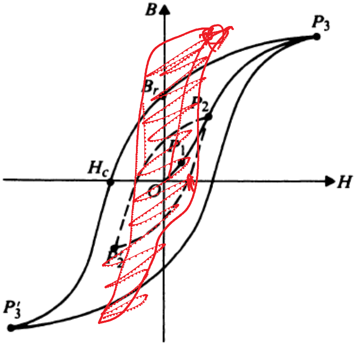


FIGURE 6–17
Hysteresis loops in the B–H plane for ferromagnetic material.

Soft materials

Q: How to have a large magnetization for a very small applied field?

A: **Tall narrow** hysteresis loops

The area of hysteresis loop = energy loss per unit volume per cycle (when the hysteresis loop is traced once per cycle)

$\frac{1}{2}\vec{B}\cdot\vec{H}$ W/m^3

hysteresis energy loss is the energy lost in the form of heat in overcoming the friction encountered during domain-wall motion and domain rotation.

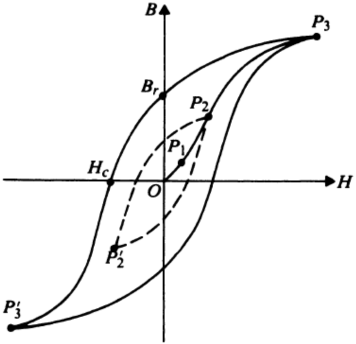


FIGURE 6–17
Hysteresis loops in the *B–H* plane for ferromagnetic material.

Hard materials

Q: How to have good permanent magnets?

A: **Fat** hysteresis loops (i.e., large H_c)

Temperature Effect to Ferromagnetic Material

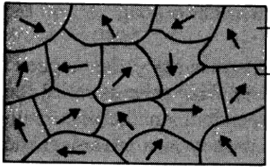
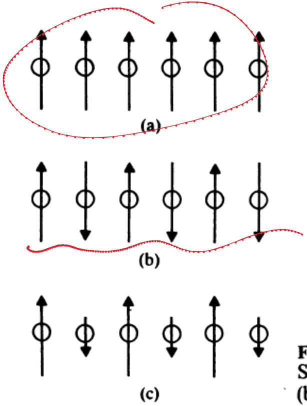


FIGURE 6–16
Domain structure of a polycrystalline ferromagnetic specimen.

- Under curie temperature T_c → well-defined magnetized domain (T_c = 770°C for iron)
- Above curie temperature → loses magnetization, reducing to paramagnetic substances

Anti-ferromagnetic and Ferri-magnetic



- (a) Ferromagnetic: parallel alignments of electron spins (in a magnetized domain)
- (b) Anti-ferromagnetic: antiparallel alignments of electron spins ➔ no net magnetic moment
- (c) Ferri-magnetic: alternating alignments of electron spins with unequal magnitudes ➔ nonzero net magnetic moment
  - Due to partial cancelation, $B_{ferri} \sim 1/10\ B_{ferro}$

FIGURE 6–18
Schematic atomic spin structures for (a) ferromagnetic, (b) antiferromagnetic, and (c) ferrimagnetic materials.

	Diamagnetic	Paramagnetic	Ferromagnetic
μ_r	≤ 1	≥ 1	$>>1$
χ_m	Small negative	Small positive	Larger positive
M ($\mathbf{M} = \chi_m \mathbf{H}$)	$//-\mathbf{H}$	$//\mathbf{H}$	$//\mathbf{H}$
B in the material ($\mathbf{B}/\mu_0 = \mathbf{H}+\mathbf{M}$)	Reduced	Increased	Increased
Mainly due to	Orbiting electrons	Spinning electrons	
When $\mathbf{B}_{ext}=0$	Net **m** = 0	Net m $\neq$ 0 (very weak)	Net m $\neq$ 0 (hysteresis)

6-10 Boundary Conditions for Magnetostatic Field

Magnetostatics

$$\nabla \cdot \mathbf{B} = 0$$

$$\boxed{B_{1n} = B_{2n} \qquad \text{(T)}.}$$

The normal component of **B** is continuous across an interface

For linear media,

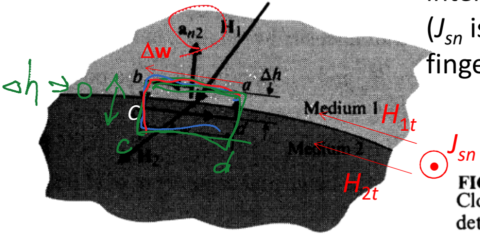
$$\mathbf{B}_1 = \mu_1 \mathbf{H}_1 \text{ and } \mathbf{B}_2 = \mu_2 \mathbf{H}_2,$$

$$\boxed{\mu_1 H_{1n} = \mu_2 H_{2n}.}$$

Electrostatics

$$\nabla \cdot \mathbf{D} = \rho$$

$$\boxed{D_{1n} - D_{2n} = \rho_s \qquad (\text{C/m}^2),}$$



J_{sn}: the surface current density on the interface **normal** to the contour C (J_{sn} is along the thumb when the fingers of right hand follow the path C)

FIGURE 6–19
Closed path about the interface of two media for determining the boundary condition of H_t.

Magnetostatics

$$\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I.$$

Let bc = da = $\Delta h \to 0$

$$\oint_{abcda} \mathbf{H} \cdot d\boldsymbol{\ell} = \mathbf{H}_1 \cdot \Delta\mathbf{w} + \mathbf{H}_2 \cdot (-\Delta\mathbf{w}) = J_{sn} \Delta w$$

$\overrightarrow{cd}$

$$H_{1t} - H_{2t} = J_{sn} \qquad \text{(A/m)},$$

along the finger – opposite to the finger

Electrostatics

$$\nabla \times \mathbf{E} = 0$$

$$\boxed{E_{1t} = E_{2t} \qquad \text{(V/m)},}$$

Or $$\boxed{\mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \qquad (\text{A/m}),}$$

The tangential component of the **H** field is discontinuous across an interface where a free surface current exists

- For finite σ of two media ➔ only J, no J_s ➔ H_t continuous
- If infinite σ for one medium ➔ J_s exists ➔ H_t discontinuous

Recall in Chap.5: $\boxed{J_{1n} = J_{2n} \qquad (\text{A/m}^2).}$ $\boxed{\frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}.}$ (see Prob. 6-30)

Analogous Boundary-value Problems

Magnetostatics

In current-free regions, $\nabla \times (\mathbf{B}/\mu)=0$

$$\mathbf{B} = -\mu \nabla V_m.$$

$$\nabla \cdot \mathbf{B} = 0$$

And assume a constant μ

$$\nabla^2 V_m = 0.$$

Thus, the techniques (method of images and method of separation of variables) discussed in Chap. 4 for solving boundary-value problems (BVPs) can be adapted to solving analogous magnetostatic BVPs.

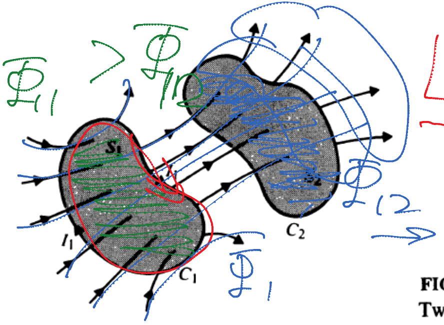
Electrostatics

In charge-free regions, $\nabla \times \mathbf{E}=0$

Laplace's equation

$$\nabla^2 V = 0,$$

6-11 Inductances and Inductors



$L = \frac{\Phi}{I}$

$C = \frac{Q}{V}$

FIGURE 6–22
Two magnetically coupled loops.

I_1 ➔ Φ_1 ➔ part of Φ_1 (Φ_{12}) passes through S_2

$S_2 + S_2 + S_2$

$$\Phi_{12} = \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s}_2 \quad \text{(Wb).}$$

$\Phi_{12} \sim B_1$ (fixed S_2)

and from Biot-Savart law: $B \sim I$

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\boldsymbol{\ell}' \times \mathbf{a}_R}{R^2} \quad \text{(T)}.$$

➔ $\Phi_{12} \sim I_1$

$\Phi_{12} \sim B_1$

$B_1 \sim I_1$

$\Rightarrow \Phi_{12} \sim I_1$

1 turn for C_2, mutual flux $\Phi_{12} = L_{12} I_1,$

The proportionality constant L_{12} (called mutual inductance)

$\times N$

If N turns for C_2, flux linkage $\Lambda_{12} = N_2 \Phi_{12}$ (Wb),

N $\quad$ 1

$\Phi \cong BS,$

➔ $\Phi \sim S$ (fixed B)

Φ: flux due to 1 turn

Λ: flux due to N turns

Thus, the general expression: $\Lambda_{12} = L_{12} I_1$ (Wb)

$$\boxed{L_{12} = \frac{\Lambda_{12}}{I_1} \quad \text{(H)}.}$$

The mutual inductance between two circuits is then the magnetic flux linkage with one circuit (Λ_{12}) per unit current in the other (I_1)

For linear media, μ is a constant

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\ell' \times \mathbf{a}_R}{R^2} \quad \text{(T)}.$$

$\mathbf{B} \sim I$

$$\Phi_{12} = L_{12} I_1, \quad \Rightarrow \quad \boxed{L_{12} = \frac{\Lambda_{12}}{I_1} \quad \text{(H)}.}$$

For linear media only

For nonlinear media, μ is a function of I

$\mathbf{B}(\mu, I)$

$$\Phi_{12}(\mu, I) \quad \Rightarrow \quad \boxed{L_{12} = \frac{d\Lambda_{12}}{dI_1} \quad \text{(H)}.}$$

In general

Self-inductance

$I_1 \rightarrow \Phi_1 \rightarrow$ part of Φ_1 (Φ_{11}) passes through S_1

$\Phi_1 \sim I_1$

If N_1 turns for C_1, flux linkage $\Lambda_{11} = N_1\Phi_{11} > N_1\Phi_{12}$.

N $\quad$ 1

.
.
.

Self-inductance $\quad \boxed{L_{11} = \frac{\Lambda_{11}}{I_1} \quad \text{(H)},}$ For linear media only

$\boxed{L_{11} = \frac{d\Lambda_{11}}{dI_1} \quad \text{(H)}.}$ In general

Inductor

- A conductor arranged in an appropriate shape (such as a conducting wire wound as a coil) to supply a certain amount of self-inductance is called an **inductor**.
- The procedure to determine self-inductance of an inductor: **From *I* to Λ**
  - 1. Choose an appropriate coordinate system
  - 2. Assume *I* in the conducting wire
  - 3. Find B from *I* by Ampere's circuital law (for the symmetric case) or Biot-Savart law (otherwise)

$$I \rightarrow B \rightarrow \Phi, \Lambda \Rightarrow \frac{\Lambda}{I} = L$$

- 4. Find the flux linkage with each turn, Φ, from **B** $\boxed{\Phi_{11} = \int_S \mathbf{B}_1 \cdot d\mathbf{s}_1}$
- 5. Find the total flux linkage Λ $\boxed{\Lambda = N\Phi}$
- 6. Find L by L = Λ/I

- The procedure to determine mutual-inductance L_{12}: slight modification

  I_1 ➔ B_1 ➔ **Φ_{12} by integrating B_1 over S_2** ➔ $\Lambda_{12} = N_2\Phi_{12}$ ➔ $L_{12} = \Lambda_{12}/I_1$

$$\Phi_{12} = \int_{S_2} \vec{B}_1 \cdot \vec{S}_2$$

$$L_{12} = L_{21}?$$

<Proof>

$$\Phi_{12} = \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s}_2 \qquad \text{(Wb)}.$$

$$\Lambda_{12} = N_2 \Phi_{12} \qquad \text{(Wb)},$$

$$L_{12} = \frac{\Lambda_{12}}{I_1} \qquad \text{(H)}.$$

Λ_{12}

$$L_{12} = \frac{N_2}{I_1} \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s}_2.$$

$$\mathbf{B}_1 = \nabla \times \mathbf{A}_1$$

$$L_{12} = \frac{N_2}{I_1} \int_{S_2} (\nabla \times \mathbf{A}_1) \cdot d\mathbf{s}_2$$

$$= \frac{N_2}{I_1} \oint_{C_2} \mathbf{A}_1 \cdot d\boldsymbol{\ell}_2.$$

$$\mathbf{A}_1 = \frac{\mu_0 N_1 I_1}{4\pi} \oint_{C_1} \frac{d\boldsymbol{\ell}_1}{R}.$$

$$L_{12} = \frac{\mu_0 N_1 N_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\boldsymbol{\ell}_1 \cdot d\boldsymbol{\ell}_2}{R},$$

1 turn for C_1

1 turn for C_2

Neumann formula for mutual inductance

$$\boxed{L_{12} = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\boldsymbol{\ell}_1 \cdot d\boldsymbol{\ell}_2}{R} \quad \text{(H)},}$$

N_1 turns for C_1

N_2 turns for C_2

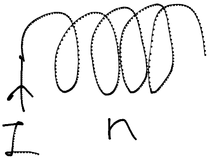
Mutual inductance:

- dependent on the geometrical shape and the physical arrangement of coupled circuits
- independent of currents (for linear media where μ is a constant)
- interchanging subscript 1 and 2 does not change the value ➔ $L_{12} = L_{21}$

Ex 6-3

: $\underline{B = \mu_0 n I}$ $\quad$ n: N/unit length

EXAMPLE 6–15 Find the inductance per ~~unit length~~ of a very long solenoid with air core having n turns per unit length.



$$\boxed{L' = \frac{\Lambda'}{I}}$$

$$\underline{L' = \mu_0 n^2 S}$$

(max.)

$$\underbrace{\Lambda'}_{N} = \underbrace{n}_{N/\text{unit length}} \Phi \ (\mathrm{Wb/m}) = nBS$$

$$= n(\mu_0 n I) S$$

$$= \mu_0 n^2 I S$$

max $L' = \mu_0 n^2 S$

for finite length solenoid: $L'_{actual} < L'_{max}$

EXAMPLE 6–16 An air coaxial transmission line has a solid inner conductor of radius a and a very thin outer conductor of inner radius b. Determine the inductance per unit length of the line.

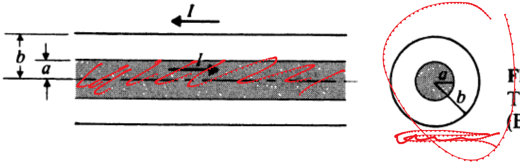


FIGURE 6–24
Two views of a coaxial transmission line (Example 6–16).

$$I \rightarrow B \rightarrow \phi \rightarrow L = \frac{\Lambda}{I}$$

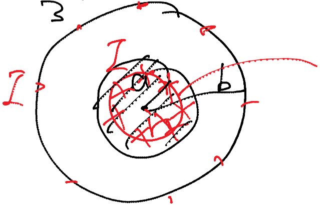
$$\Lambda$$

1° $0 \le r \le a$ $\vec{B}_1 = \hat{a}_\phi B_{\phi 1} = \hat{a}_\phi \frac{\mu_0 r I}{2\pi a^2}$ — ①

2° $a \le r \le b$ $\vec{B}_2 = \hat{a}_\phi B_{\phi 2} = \hat{a}_\phi \frac{\mu_0 I}{2\pi r}$ — ②

Ex 6-1

3°



$d\phi'$ due to an annular ring current I_{ar}

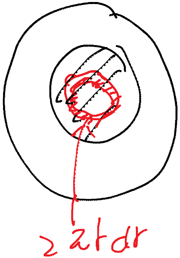
$$d\phi' = \int_0^r 0\, dr + \int_r^a B_{\phi 1}\, dr + \int_a^b B_{\phi 2}\, dr$$

①, ② $\rightarrow$ $= \cdots\cdots = \frac{\mu_0 I_{ar}}{4\pi a^2}(a^2 - r^2) + \frac{\mu_0 I_{ar}}{2\pi} \ln\left(\frac{b}{a}\right)$

$\phi = \int \vec{B} \cdot d\vec{s}$

$\phi' = \int \vec{B} \cdot d\vec{\ell}$

4° $d\Lambda'$ due to the total I



$$I_{ar} = I\frac{2\pi r\,dr}{2\pi a^2} = I\frac{2r\,dr}{a^2}$$

$I = JS$

$I \sim S$

$$\Lambda' = \int_{r=0}^{a} d\Phi'$$

$$= \int_{r=0}^{a}$$

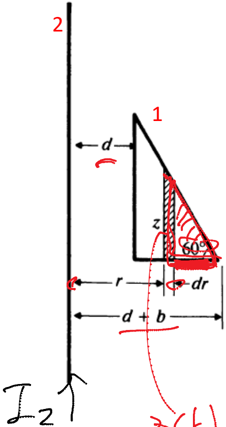
$= \cdots$

$$= \frac{\mu_0 I}{2\pi}\left[\frac{1}{4} + \ln\left(\frac{b}{a}\right)\right]$$

$$L' = \frac{\Lambda'}{I} = \cdots$$

EXAMPLE 6–19 Determine the mutual inductance between a conducting triangular loop and a very long straight wire as shown in Fig. 6–27.

$\vec{\Phi}_{21}$



$$\vec{B}_2 \to \Phi_{21} \to L_{21} = \frac{\Phi_{21}}{I_2}$$

$$\vec{\Phi}_{21} = \int_{S_1} \vec{B}_2 \cdot d\vec{S}$$

$$\vec{B}_2 = \frac{\mu_0 I}{2\pi r}\hat{a}_\phi \Rightarrow \Phi_{21} = \int_{r=d}^{d+b} B_2\, \underbrace{z(r)\,dr}_{dS_1}$$

FIGURE 6–27
A conducting triangular loop and a long straight wire (Example 6–19).

$$z(r) = \sqrt{3}\cdot(d+b-r)$$

$$\Rightarrow \Phi_{21} = \ldots$$

$$\Rightarrow L_{21} = \Phi_{21}/I_2$$

6-12 Magnetic Energy

- For DC, the inductor behaves like short circuit.
- Exact AC case: retardation and radiation effects should be considered (Chaps. 7 and 8)
- Here, we consider **quasi-static conditions**: the current vary very slowly in time (low frequency, or long wavelength)

$\vec{J}(t)$ is with low frequency

- In section 3-11, work is required to assemble a group of charges (stored electric energy)
- Here, work is required to send currents into conducting loops (stored magnetic energy)

Stored Magnetic Energy

- A current generator increases the current i_1 from 0 to I_1:
- Work must be done to **overcome** this induced v_1

$\int P\,dt$

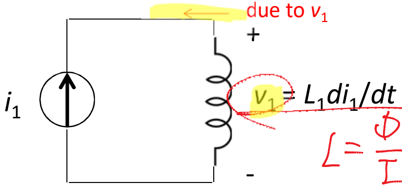
$$W_1 = \int v_1 i_1\,dt = L_1 \int_0^{I_1} i_1\,di_1 = \boxed{\tfrac{1}{2} L_1 I_1^2}.$$

For linear media

$L_1 = \Phi_1 / I_1$

$$W_1 = \tfrac{1}{2} I_1 \Phi_1,$$

W_m



$L = \frac{\Phi}{I}$ $C = \frac{Q}{V}$

Recall: in Chap.3

$$W_e = \frac{1}{2} \sum_{k=1}^{N} Q_k V_k \quad \text{(J)},$$

W_m

Two loops C_1 and C_2 to I_1 and I_2

- Initially, $i_1 = 0$, $i_2 = 0$
- Step 1: increase i_1 from 0 to I_1

$$W_1 = \tfrac{1}{2}L_1 I_1^2.$$

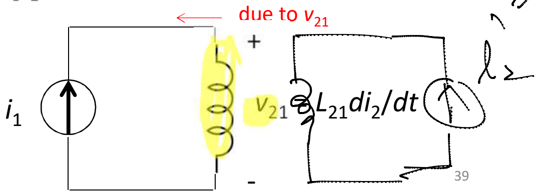
- Step 2: increase i_2 from 0 to I_2
  - Work must be done to **overcome** the induced $v_{21} = L_{21}di_2/dt$ (to keep i_1 constant at I_1)

$$W_{21} = \int v_{21} I_1 \, dt = L_{21} I_1 \int_0^{I_2} di_2 = L_{21} I_1 I_2.$$

  - Work in C_2

$$W_{22} = \tfrac{1}{2}L_2 I_2^2.$$

from 0 to I_2



$j=1,\ k=2$

$j=2,\ k=1$

Total work required: $W_2 = \frac{1}{2}L_1 I_1^2 + L_{21} I_1 I_2 + \frac{1}{2}L_2 I_2^2$

$$= \frac{1}{2}\sum_{j=1}^{2}\sum_{k=1}^{2} L_{jk} I_j I_k.$$

Generalization to a system of *N* loops carrying currents $I_1, I_2, \ldots, I_n$:

$$\boxed{W_m = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} L_{jk} I_j I_k \qquad \text{(J)},}$$

$N=1$

The stored magnetic energy for a current *I* through a single inductor with inductance *L*:

$$\boxed{W_m = \frac{1}{2} L I^2 \qquad \text{(J).}}$$

Alternative Derivation

Consider a kth loop of *N* magnetically coupled loops

The work done in the kth loop in time *dt*

$$dW_k = v_k i_k dt = i_k d\phi_k,$$

power

$$\boxed{v_k = d\phi_k/dt.}$$

$d\phi_k$: change in flux ϕ_k linking with the kth loop due to the change of currents in all the coupled loops
(di ➔ $d\phi$ ➔ v_k)

The differential work done to the system $dW_m = \sum_{k=1}^{N} dW_k = \sum_{k=1}^{N} i_k d\phi_k.$

I_k
Φ_k : const.

$$dW_m = \sum_{k=1}^{N} dW_k = \sum_{k=1}^{N} i_k \, d\phi_k.$$

$i_k = \alpha I_k,$ $\phi_k = \alpha \Phi_k$

I_k and Φ_k are final values (constants);
α increases from 0 to 1

α is from 0 to 1
$\Rightarrow i_k$ from 0 to I_k

The total magnetic energy

$$W_m = \int dW_m = \sum_{k=1}^{N} I_k \Phi_k \underbrace{\int_0^1 \alpha \, d\alpha}_{1/2}$$

$$\boxed{W_m = \frac{1}{2} \sum_{k=1}^{N} I_k \Phi_k \quad \text{(J)},}$$

For linear media

$$\Phi_k = \sum_{j=1}^{N} L_{jk} I_j,$$

$$\boxed{W_m = \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} L_{jk} I_j I_k \quad \text{(J)},}$$

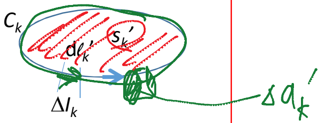
6-12.1 Magnetic Energy in Terms of Field Quantities

$$W_m = \frac{1}{2}\sum_{k=1}^{N} I_k \Phi_k \quad \text{(J)},$$

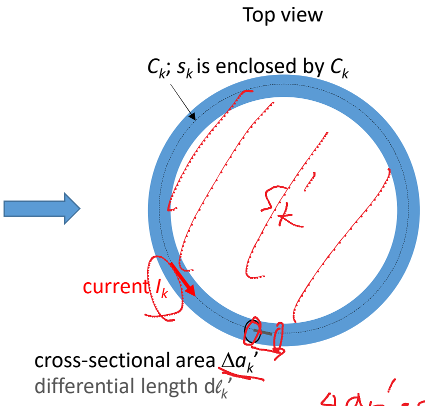
$$\Phi_k = \int_{S_k} \mathbf{B}\cdot\mathbf{a}_n\, ds'_k = \oint_{C_k} \mathbf{A}\cdot d\boldsymbol{\ell}'_k,$$

$\mathbf{B} = \nabla\times\mathbf{A}$

A single current-carrying loop



N contiguous filamentary current elements with a current ΔI_k (flowing in an infinitesimal cross-sectional area $\Delta a_k'$; $\Delta v_k' = \Delta a_k' d\ell_k'$)



$\Delta a_k' \cdot d\ell_k'$

$$W_m = \frac{1}{2} \sum_{k=1}^{N} \Delta I_k \oint_{C_k} \mathbf{A} \cdot d\ell'_k.$$

$$\Delta I_k \, d\ell'_k = J(\Delta a'_k)\, d\ell'_k = \mathbf{J} \Delta v'_k.$$

As $N \to \infty$, $\Delta v_k'$ becomes dv_k'

$$\boxed{W_m = \frac{1}{2} \int_{V'} \mathbf{A} \cdot \mathbf{J} \, dv' \qquad \text{(J)},}$$

Analogy to electric energy

$$\boxed{W_e = \frac{1}{2} \int_{V'} \rho V \, dv \qquad \text{(J)}.}$$

$J \mid \rho$

$A \mid V$

V': the volume of the loop or the linear medium in which **J** exists

By vector identity

$$\nabla \cdot (\mathbf{A} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{H}),$$

$$\mathbf{A} \cdot (\nabla \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{A}) - \nabla \cdot (\mathbf{A} \times \mathbf{H})$$

$$\mathbf{A} \cdot \mathbf{J} = \mathbf{H} \cdot \mathbf{B} - \nabla \cdot (\mathbf{A} \times \mathbf{H}).$$

$$W_m = \frac{1}{2} \int_{V'} \mathbf{H} \cdot \mathbf{B} \, dv' - \frac{1}{2} \oint_{S'} (\mathbf{A} \times \mathbf{H}) \cdot \mathbf{a}_n \, ds'.$$

$$W_m = \frac{1}{2}\int_{V'} \mathbf{H}\cdot\mathbf{B}\,dv' - \frac{1}{2}\oint_{S'} (\mathbf{A}\times\mathbf{H})\cdot\mathbf{a}_n\,ds',$$

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\mathbf{J}}{R}\,dv' \qquad \text{(Wb/m).}$$

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\oint_{C'} \frac{d\boldsymbol{\ell}'\times\mathbf{a}_R}{R^2} \qquad \text{(T).}$$

$|\mathbf{A}| \sim 1/R$

$|\mathbf{H}| \sim 1/R^2$

$S' \sim R^2$

As $R \to \infty$, 2nd term $\to 0$

$$\boxed{W_m = \frac{1}{2}\int_{V'} \mathbf{H}\cdot\mathbf{B}\,dv' \qquad \text{(J).}}$$

For linear media

$$\mathbf{H} = \mathbf{B}/\mu,$$

$$\boxed{W_m = \frac{1}{2}\int_{V'} \frac{B^2}{\mu}\,dv' \qquad \text{(J)}}$$

J/m^3

$$\boxed{W_m = \frac{1}{2}\int_{V'} \mu H^2\,dv' \qquad \text{(J).}}$$

$V \sim \frac{1}{R}$

$\vec{E} \sim \frac{1}{R^2}$

dl'

The magnetic energy density

$$W_m = \int_{V'} w_m \, dv',$$

$$w_m = \tfrac{1}{2}\mathbf{H}\cdot\mathbf{B} \qquad (\mathrm{J/m^3})$$

$$w_m = \frac{B^2}{2\mu} \qquad (\mathrm{J/m^3})$$

$$w_m = \tfrac{1}{2}\mu H^2 \qquad (\mathrm{J/m^3}).$$

L can be calculated more easily by W_m formula here than using flux linkage (Λ):

$$W_m = \tfrac{1}{2}LI^2 \quad (\mathrm{J}). \quad \Rightarrow \quad \boxed{L = \frac{2W_m}{I^2} \quad (\mathrm{H}).}$$

6-13 Magnetic Forces and Torques

- A magnetic force $\mathbf{F}_m$ on a moving charge q

$$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B} \qquad \text{(N)}.$$

6-13.1 Hall Effect

Applied: **B** and **J**

Results: V_h

$$\boxed{\mathbf{J} = \mathbf{a}_y J_0 = Nq\mathbf{u},}$$

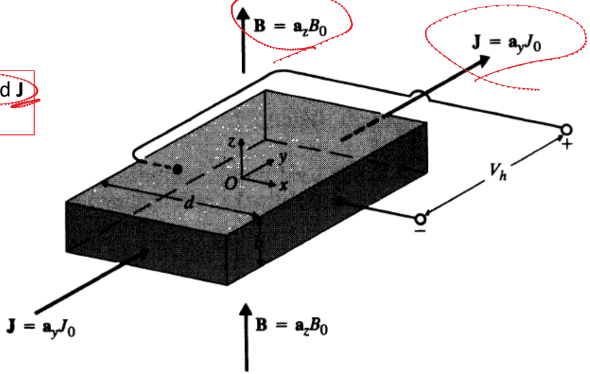
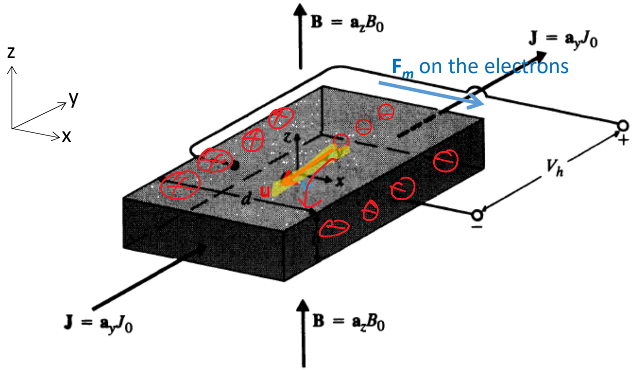


FIGURE 6–28
Illustrating the Hall effect.

From $\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$ (N). ➡ There is a force ⊥ **u**, ⊥ **B**

Considering a n-type semiconductor (carriers: electrons): q is negative



$$\boxed{\mathbf{F}_m = q\mathbf{u} \times \mathbf{B} \qquad \text{(N).}}$$

$\mathbf{F}_m \,//\, -(-\mathbf{a}_y)\times\mathbf{a}_z = \mathbf{a}_x$

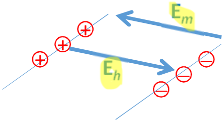
$\mathbf{E}_m = \mathbf{F}_m/q = \mathbf{u}\times\mathbf{B}$

$\mathbf{E}_m \,//\, (-\mathbf{a}_y)\times\mathbf{a}_z = -\mathbf{a}_x$

For electrons:

$\mathbf{u} = -\mathbf{a}_y u_0$

FIGURE 6–28
Illustrating the Hall effect.



$\mathbf{E}_m$: due to moving charge ($\mathbf{u}\times\mathbf{B}$ magnetic force)
$\mathbf{E}_h$: due to charge accumulation

In the steady state, the net force on the charge carriers is zero.

$$\mathrm{F} = q(\mathbf{E}_m + \mathbf{E}_h) = 0$$

$$\mathbf{E}_m + \mathbf{E}_h = 0$$

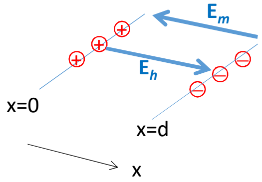
$$\mathbf{E}_h + \mathbf{u} \times \mathbf{B} = 0$$

or $\mathbf{E}_h = -\mathbf{u} \times \mathbf{B}.$

$\mathbf{E}_h$: Hall field

$$\boxed{\mathbf{u} = -\mathbf{a}_y u_0}$$

$$\mathbf{E}_h = -(-\mathbf{a}_y u_0) \times \mathbf{a}_z B_0$$
$$= \mathbf{a}_x u_0 B_0.$$



Correction: should be... $\int_d^0 \sim$

$$V_h = -\int_0^d E_h\,dx = u_0 B_0 d,$$

Hall effect can be used to measure the B field

V_h: Hall voltage

$$\mathbf{J} = \mathbf{a}_y J_0 = Nq\mathbf{u},$$

$$\mathbf{E}_h = -(-\mathbf{a}_y u_0) \times \mathbf{a}_z B_0 = \mathbf{a}_x u_0 B_0.$$

Scalars only:

E_h/JB

$= E_x/J_yB_z$

$= u_0B_0/(\rho u_0)B_0$

$= 1/\rho$

V_h

$$E_x/J_yB_z = 1/Nq$$

Hall coefficient, a characteristic of the material (ρ)

① main carriers

② charge density

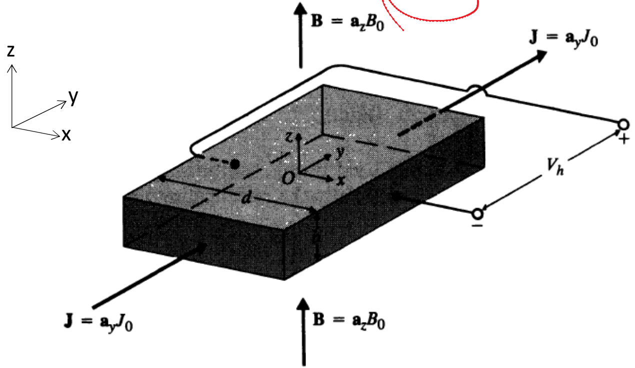


FIGURE 6–28
Illustrating the Hall effect.

- Considering a p-type semiconductor (carriers: + charges): $\mathbf{E}_h$ will be reversed, V_h will be in opposite polarity (see Fig. 6-28)
- Hall effect can be used to determine the sign of predominant charge carriers.

6-13.2 Forces and Torques on Current-Carrying Conductors

Magnetic force

$\vec{F}_m = q\vec{u} \times \vec{B}$

dQ

$$d\mathbf{F}_m = -NeS|d\ell|\mathbf{u} \times \mathbf{B}$$
$$= -NeS|\mathbf{u}|\,d\boldsymbol{\ell} \times \mathbf{B},$$

$= JS = I \qquad (J = -Ne|\mathbf{u}|)$

I

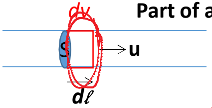
$$\boxed{d\mathbf{F}_m = I\,d\boldsymbol{\ell} \times \mathbf{B} \quad (\text{N}).}$$

For a complete circuit

$$\boxed{\mathbf{F}_m = I \oint_C d\boldsymbol{\ell} \times \mathbf{B} \quad (\text{N}).}$$

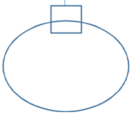
N: number density

Part of a conductor



$d\vec{\ell} \parallel \vec{u}$

u and ***dℓ*** are in the same direction



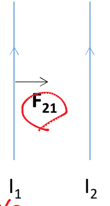
A complete circuit

Two Circuits Carrying Currents

$$\mathbf{F}_{21} = I_1 \oint_{C_1} d\boldsymbol{\ell}_1 \times \mathbf{B}_{21},$$

Force $\mathbf{F}_{21}$ on circuit C_1

$\mathbf{B}$ from I_2 on circuit 1



$\vec{F}_{21} = -\vec{F}_{12}$

Biot-Savart law (source: 2)

$$\mathbf{B}_{21} = \frac{\mu_0 I_2}{4\pi} \oint_{C_2} \frac{d\boldsymbol{\ell}_2 \times \mathbf{a}_{R_{21}}}{R_{21}^2}.$$

2: Source

1: field

$\mathbf{a}_{R21}$: from 2 (source) to 1 (field)

The Ampere's law of force

$$\mathbf{F}_{21} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1} \oint_{C_2} \frac{d\boldsymbol{\ell}_1 \times (d\boldsymbol{\ell}_2 \times \mathbf{a}_{R_{21}})}{R_{21}^2} \quad \text{(N)},$$

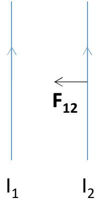
Comparison with Coulomb's law:

$$\mathbf{F}_{12} = \mathbf{a}_R \frac{q_1 q_2}{4\pi\epsilon_0 R^2}$$

$$\boxed{\mathbf{F}_{21} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1} \oint_{C_2} \frac{d\boldsymbol{\ell}_1 \times (d\boldsymbol{\ell}_2 \times \mathbf{a}_{R_{21}})}{R_{21}^2} \quad (\mathrm{N}),}$$

Interchanging subscript 1 and 2

$$\mathbf{F}_{12} = \frac{\mu_0}{4\pi} I_2 I_1 \oint_{C_2} \oint_{C_1} \frac{d\boldsymbol{\ell}_2 \times (d\boldsymbol{\ell}_1 \times \mathbf{a}_{R_{12}})}{R_{12}^2}.$$



<Proof> Newton's third law in this case

First, $d\boldsymbol{\ell}_2 \times (d\boldsymbol{\ell}_1 \times \mathbf{a}_{R_{12}}) \neq -d\boldsymbol{\ell}_1 \times (d\boldsymbol{\ell}_2 \times \mathbf{a}_{R_{21}})$,

So we need to check if $F_{12} = -F_{21}$!?

Expand the left side by back-cab rule:

$$\frac{d\boldsymbol{\ell}_1 \times (d\boldsymbol{\ell}_2 \times \mathbf{a}_{R_{21}})}{R_{21}^2} = \frac{d\boldsymbol{\ell}_2 (d\boldsymbol{\ell}_1 \cdot \mathbf{a}_{R_{21}})}{R_{21}^2} - \frac{\mathbf{a}_{R_{21}} (d\boldsymbol{\ell}_1 \cdot d\boldsymbol{\ell}_2)}{R_{21}^2}.$$

$$\frac{d\ell_1 \times (d\ell_2 \times \mathbf{a}_{R_{21}})}{R_{21}^2} = \frac{d\ell_2(d\ell_1 \cdot \mathbf{a}_{R_{21}})}{R_{21}^2} - \frac{\mathbf{a}_{R_{21}}(d\ell_1 \cdot d\ell_2)}{R_{21}^2}.$$

Double closed line integration

$$\oint_{C_1}\oint_{C_2} \frac{d\ell_2(d\ell_1 \cdot \mathbf{a}_{R_{21}})}{R_{21}^2} = \oint_{C_2} d\ell_2 \oint_{C_1} \frac{d\ell_1 \cdot \mathbf{a}_{R_{21}}}{R_{21}^2}$$

$$\boxed{\nabla_1(1/R_{21}) = -\mathbf{a}_{R_{21}}/R_{21}^2.}$$

$$= \oint_{C_2} d\ell_2 \oint_{C_1} d\ell_1 \cdot \left(-\nabla_1 \frac{1}{R_{21}}\right)$$

$$\vec{R}_{21} = \vec{R}_1 - \vec{R}_2$$

$$= -\oint_{C_2} d\ell_2 \underbrace{\oint_{C_1} d\left(\frac{1}{R_{21}}\right)}_{=0} = 0.$$

Thus,

$$\boxed{\mathbf{F}_{21} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1}\oint_{C_2} \frac{d\ell_1 \times (d\ell_2 \times \mathbf{a}_{R_{21}})}{R_{21}^2}}$$

$$\mathbf{F}_{21} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1}\oint_{C_2} \frac{\mathbf{a}_{R_{21}}(d\ell_1 \cdot d\ell_2)}{R_{21}^2},$$

Interchanging subscript 1 and 2

$$\mathbf{a}_{R_{12}} = -\mathbf{a}_{R_{21}}$$

$$\mathbf{F}_{21} = -\mathbf{F}_{12}$$

A Circular Circuit Carrying Currents

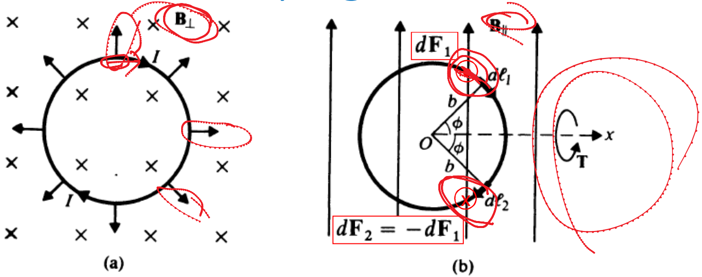
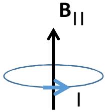


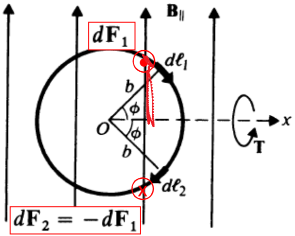
FIGURE 6–30
A circular loop in a uniform magnetic field $\mathbf{B} = \mathbf{B}_\perp + \mathbf{B}_{\|}$.

(a) $\mathbf{B}_\perp$ ➔ $\mathbf{F}_m$ tends to expand the loop

(b) $\mathbf{B}_{//}$ ➔ $\mathbf{F}_m$ tends to rotates the loop about *x* axis

(or, tends to align the $\mathbf{B}_I$ (due to I) with $\mathbf{B}_{||}$)





The torque by $d\mathbf{F}_1$ and $d\mathbf{F}_2$

$$dF = |d\mathbf{F}_1| = |d\mathbf{F}_2|$$

$$d\ell = |d\ell_1| = |d\ell_2| = b\,d\phi.$$

Torque due to $d\ell_1$ and $d\ell_2$

arm

$$d\mathbf{T} = \mathbf{a}_x(dF)2b\sin\phi$$

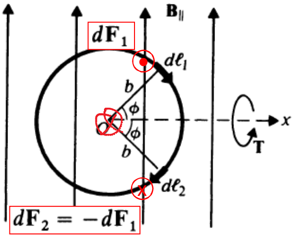
$$= \mathbf{a}_x(I\,d\ell\,B_{||}\sin\phi)2b\sin\phi$$

$$= \mathbf{a}_x 2Ib^2 B_{||}\sin^2\phi\,d\phi,$$

$dF = I d\ell \times B_{||}$

$$\mathbf{T} = \int d\mathbf{T} = \mathbf{a}_x 2Ib^2 B_{||}\int_0^{\pi}\sin^2\phi\,d\phi$$

$$= \mathbf{a}_x I(\pi b^2)B_{||}.$$



$$\mathbf{T} = \int d\mathbf{T} = \mathbf{a}_x 2Ib^2 B_{||} \int_0^{\pi} \sin^2\phi \, d\phi$$
$$= \mathbf{a}_x I(\pi b^2) B_{||}.$$

$\hat{a}_x m B_{||}$

By definition of magnetic dipole moment **m**

$$\boxed{\mathbf{m} = \mathbf{a}_n I(\pi b^2) = \mathbf{a}_n IS,}$$

$$\mathbf{m} \times (\mathbf{B}_\perp + \mathbf{B}_{||}) = \mathbf{m} \times \mathbf{B}_{||}.$$

$m \times \vec{B}_\perp = 0$

$$\boxed{\mathbf{T} = \mathbf{m} \times \mathbf{B} \qquad (\text{N}\cdot\text{m}).}$$

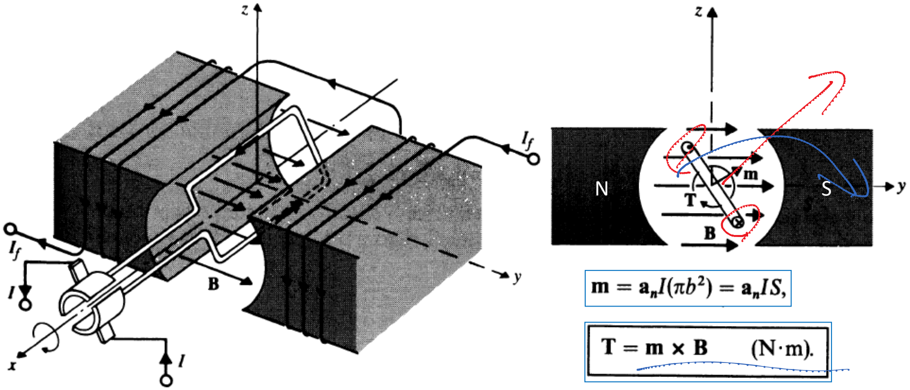
$\hat{a}_x |mB|$ holds also for a planar loop of any shape under a uniform **B**

Microscopically, an applied $\mathbf{B}_{ext}$

➔ **T** to align magnetic dipoles **m** along $\mathbf{B}_{ext}$ in magnetic materials

➔ magnetization in the material

DC Motor



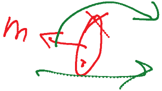
$$\mathbf{m} = \mathbf{a}_n I(\pi b^2) = \mathbf{a}_n IS,$$

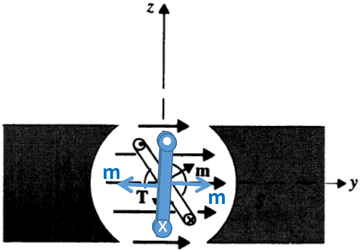
$$\boxed{\mathbf{T} = \mathbf{m} \times \mathbf{B} \qquad (\mathrm{N \cdot m}).}$$

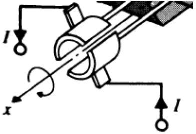
(a) Perspective view.

(b) Schematic view from $+x$ direction.

FIGURE 6–32
Illustrating the principle of operation of d-c motor.

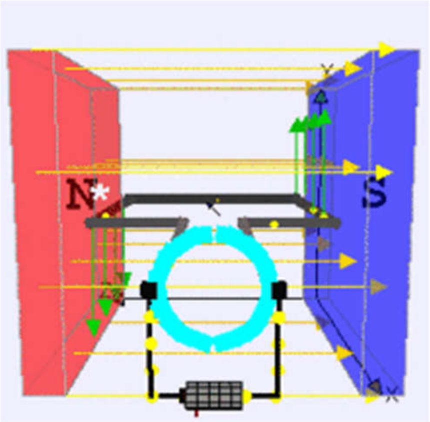






When **m** // **B**, a split ring with brushes is used to reverse the direction of currents

➔ **T** always in the same direction (clockwise here)



Yellow: **B**
Green: force, **F**

When **m** ⊥ **B**, |T| max
When **m** // **B**, the direction of currents reverses

6-13.4 Forces and Torques in Terms of Mutual Inductance

- All current-carrying conductors and circuits experience $\mathbf{F}_m$ when situated in a magnetic field.
- In general, determination of $\mathbf{F}_m$ is tedious by Ampere's law of force (except for special symmetrical cases).

$$\boxed{\mathbf{F}_{21} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1} \oint_{C_2} \frac{d\boldsymbol{\ell}_1 \times (d\boldsymbol{\ell}_2 \times \mathbf{a}_{R_{21}})}{R_{21}^2} \quad \text{(N)},}$$

Ampere's law of force

- Alternative method: principle of virtual displacement

Case 1: System of Circuits with **Constant Flux Linkages**

Assume a virtual differential displacement $d\boldsymbol{\ell}$

Assume **Constant Flux Linkages**

➔ $\Delta\Phi = 0$

➔ emf = $d\Phi/dt = 0$

The source supplies no energy to the system

The mechanical work done by the system

$\mathbf{F}_\Phi \cdot d\boldsymbol{\ell}$,

0 from emf → system → Mechanical work, dW

→ Stored magnetic energy, dW_m

$dW+dW_m = 0$

The mechanical work (increase) is provided by the stored magnetic energy (decrease, $dW_m<0$)

$$\mathbf{F}_\Phi \cdot d\boldsymbol{\ell} = -dW_m = -(\nabla W_m) \cdot d\boldsymbol{\ell},$$

$$\mathbf{F}_\Phi \cdot d\boldsymbol{\ell} = -dW_m = -(\nabla W_m) \cdot d\boldsymbol{\ell},$$

$$\boxed{\mathbf{F}_\Phi = -\nabla W_m \qquad \text{(N).}}$$

In Cartesian coordinates,

$$(F_\Phi)_x = -\frac{\partial W_m}{\partial x},$$

$$(F_\Phi)_y = -\frac{\partial W_m}{\partial y},$$

$$(F_\Phi)_z = -\frac{\partial W_m}{\partial z}.$$

For the case with a virtual rotation $d\phi$ about an axis (say z axis)

$$\boxed{(T_\Phi)_z = -\frac{\partial W_m}{\partial \phi} \qquad (\text{N}\cdot\text{m}),}$$

Case 2: System of Circuits with **Constant Currents**

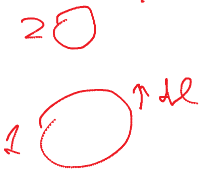
Assume a virtual differential displacement $d\ell$

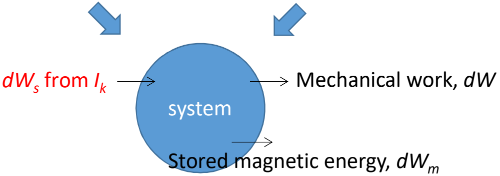
$d\ell \rightarrow d\Phi$

$$dW_s = \sum_k I_k \, d\Phi_k .$$

The source supplies energy to the system

$d\Phi_{12} \neq 0$





$dW_s = dW + dW_m$

The mechanical work (increase) and the stored magnetic energy (increase, dW_m>0) are provided by dW_s

$$dW_s = dW + dW_m.$$

$$\boxed{W_m = \frac{1}{2}\sum_{k=1}^{N} I_k\Phi_k \qquad \text{(J)},}$$

$$\boxed{dW_s = \sum_k I_k\, d\Phi_k.}$$

$$dW_m = \frac{1}{2}\sum_k I_k\, d\Phi_k = \frac{1}{2}\, dW_s.$$

$$dW = \mathbf{F}_I \cdot d\boldsymbol{\ell} = dW_m$$
$$= (\nabla W_m) \cdot d\boldsymbol{\ell}$$

$$\boxed{\mathbf{F}_I = \nabla W_m \qquad \text{(N)},}$$

Similar to case 1 except for a sign change

For the case with a virtual rotation $d\phi$ about an axis (say z axis)

$$\boxed{(T_I)_z = \frac{\partial W_m}{\partial \phi} \qquad (\text{N}\cdot\text{m}).}$$

- Method of virtual displacement ($d\ell$) for constant currents is powerful to determine the **F** and **T** between rigid-carrying circuits.
- The magnetic energy of two circuits with currents I_1 and I_2:

L_1 I_1

virtual displacement
constant I

$$W_m = \frac{1}{2}L_1 I_1^2 + L_{12} I_1 I_2 + \frac{1}{2}L_2 I_2^2.$$

L_2 I_2 $d\ell$

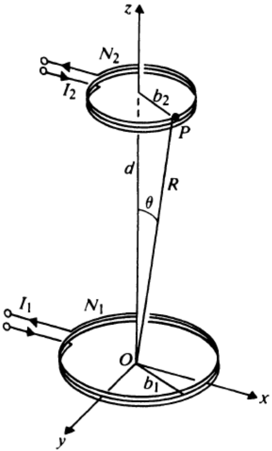
$$\boxed{\mathbf{F}_I = \nabla W_m \qquad \text{(N)},} \qquad \boxed{(T_I)_z = \frac{\partial W_m}{\partial \phi} \qquad \text{(N·m)}.}$$

$$\boxed{\mathbf{F}_I = I_1 I_2 (\nabla L_{12}) \qquad \text{(N)}.} \qquad \boxed{(T_I)_z = I_1 I_2 \frac{\partial L_{12}}{\partial \phi} \qquad \text{(N·m)}.}$$

Given a virtual displacement $d\ell$:
L_1 and L_2 (self inductance) remain constants
L_{12} changes

$$F_I = \nabla W_m$$
$$= \underbrace{\nabla\left(\tfrac{1}{2}L_1 I_1^2\right)}_{0} + \nabla(L_{12} I_1 I_2) + \underbrace{\nabla\left(\tfrac{1}{2}L_2 I_2^2\right)}_{0}$$

EXAMPLE 6–24 Determine the force between two coaxial circular coils of radii b_1 and b_2 separated by a distance d that is much larger than the radii ($d \gg b_1, b_2$). The coils consist of N_1 and N_2 closely wound turns and carry currents I_1 and I_2, respectively.



$$\vec{F} = I_1 I_2 \nabla L_{12}$$

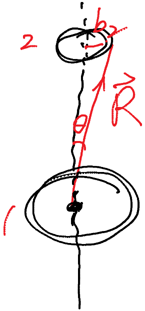
FIGURE 6–34
Coaxial current-carrying circular loops (Example 6–24).

$\vec{A}_{12} \rightarrow \Phi_{12} \rightarrow \Lambda_{12} = N_2 \Phi_{12} \Rightarrow L_{12} = \frac{\Lambda_{12}}{I_1}$

1° 2° 3° 4°

current area

I_1 S_2



1° Ex6-7: $\vec{A}_{12} = \hat{a}_\phi \frac{\mu_0 N_1 I_1 b_1^2}{4R^2} \sin\theta$

$R^2 = z^2 + b_2^2$ $\sin\theta = \frac{b_2}{R}$

2° $\Phi_{12} = \int \vec{A}_{12} \cdot d\vec{l}_2$

$= \int_0^{2\pi}$ ____ $b_2 d\phi$

$=$ ____

3° Λ_{12}

4° L_{12}

$$L_{12} = \frac{\Lambda_{12}}{I_1} = \frac{N_2 \Phi_{12}}{I_1} = \frac{\mu_0 N_1 N_2 \pi b_1^2 b_2^2}{2(z^2 + b_2^2)^{3/2}}$$

$$F_I = I_1 I_2 \nabla L_{12} = I_1 I_2 \hat{a}_z \frac{\partial}{\partial z} \left(\quad \right)$$