Chapter 5 Steady Electric Currents

VE230 Summer 2021

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5-1 Introduction

- Charges at rest (Ch3 and Ch4); Charges in motion (Ch5)
- Different types of currents
  - Conduction currents:
    - ❖in conductors and semiconductors
    - ❖electrons and/or holes
  - Electrolytic currents:
    - ❖essentially in a liquid medium
    - ❖ions (e.g., Li-ion batteries)
  - Convection currents:
    - ❖in vacuum or rarefied gas
    - ❖electrons and/or ions

const. motion

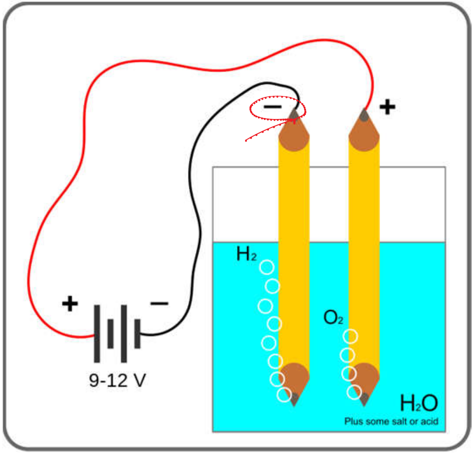
$\Rightarrow$ $v = $ const.

$I = $ const.

steady current.

$I(t)$

Electrolysis of Water



+

Oxidation at anode: $2\ H_2O(l) \rightarrow O_2(g) + 4\ H^+(aq) + 4e^-$

Reduction at cathode: $2\ H^+(aq) + 2e^- \rightarrow H_2(g)$

Topics for Conduction Currents

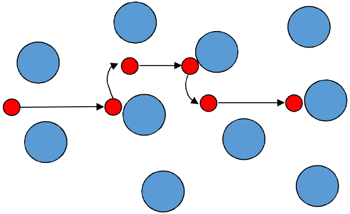
- Point form of Ohm's law
- Kirchhoff's voltage law
- Kirchhoff's current law
  - Conservation of charge
  - Equation of continuity
- Boundary conditions for current density

Conduction Currents

- For a conductor, atoms consist of positively charged nuclei surrounded by electrons
  - Inner shell: tightly bound charges
  - Outermost shell: loosely bound charges (valance or conduction electrons)
- Without external **E**, conduction electrons wander randomly ➔ **no net drift motion** of conduction electrons

Conduction Currents

- With external **E**, **organized motion** of conduction electrons
  - Very low drift velocity due to collision with atoms $\Rightarrow$ R
  - Conductor remains electrically neutral
    (electric forces prevent excess electrons from accumulating at any point of a conductor)



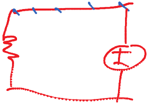
5-2 Current Density and Ohm's Law

$V = IR$

charge q across surface Δs with a velocity $\mathbf{u}$

N: #/volume

$\# \cdot q$ $\quad \Delta x \cdot \Delta S$



The amount of charge passing Δs $\quad \Delta Q = Nq\mathbf{u} \cdot \mathbf{a}_n \Delta s \, \Delta t \qquad$ (C).

differential volume with $\Delta \mathbf{s}$ along $\mathbf{a}_n$

ΔV

$\frac{\#}{\Delta V}$

$$\Delta I = \frac{\Delta Q}{\Delta t} = Nq\mathbf{u} \cdot \mathbf{a}_n \Delta s = Nq\mathbf{u} \cdot \Delta \mathbf{s} \qquad \text{(A).}$$

$\boxed{\mathbf{J} \equiv Nq\mathbf{u} \qquad (\mathrm{A/m^2}),}$ $\quad \boxed{\mathbf{J} = \rho\mathbf{u} \qquad (\mathrm{A/m^2}),}$

J defined as (volume) current density

$$\Delta I = \mathbf{J} \cdot \Delta \mathbf{s}.$$

$Nq = \frac{\#}{\Delta V} q = \frac{\Delta Q}{\Delta V} = \rho$

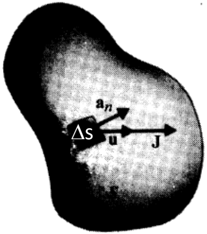


FIGURE 5–1
Conduction current due to drift motion of charge carriers across a surface.

$$\Delta I = \mathbf{J} \cdot \Delta \mathbf{s}.$$

Total current I flowing through a surface S

$$\boxed{I = \int_S \mathbf{J} \cdot d\mathbf{s} \qquad \text{(A)}.}$$

$J(s)$

Conduction Currents

- For more than one kind of charge carriers (electrons, holes, and ions) drifting, current density:

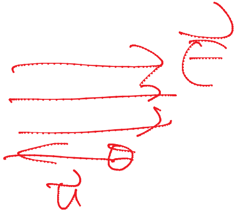
$$\boxed{\mathbf{J} = Nq\mathbf{u} \qquad (\mathrm{A/m^2}),} \Rightarrow \mathbf{J} = \sum_i N_i q_i \mathbf{u}_i \qquad (\mathrm{A/m^2}).$$

- For most conducting materials,

$$\mathbf{u} = -\mu_e \mathbf{E} \qquad (\mathrm{m/s}),$$

u: averaged drift velocity

$-\mu_e$: electron mobility ($\mathrm{m^2/V{\cdot}s}$)



$$\boxed{\mathbf{J} = \rho\mathbf{u} \qquad (\text{A/m}^2),}$$

$$\mathbf{u} = -\mu_e\mathbf{E} \qquad (\text{m/s}),$$

$$\mathbf{J} = -\rho_e\mu_e\mathbf{E},$$

where $\rho_e = -Ne$

$$\boxed{\mathbf{J} = \sigma\mathbf{E} \qquad (\text{A/m}^2),}$$

Such materials are called ohmic media

where $\sigma = -\rho_e\mu_e$

ρ_e<0 because of electrons

σ: conductivity (A/V·m or S/m)

For conductors, $\sigma = -\rho_e\mu_e$

For semiconductors, $\sigma = -\rho_e\mu_e + \rho_h\mu_h,$

- Circuit form of Ohm's law $V_{12} = RI.$
- Point form of Ohm's law $\boxed{\mathbf{J} = \sigma \mathbf{E} \qquad (\mathrm{A/m^2}),}$
  - Holds at all points
  - σ can be a function of space

$\sigma(R)$

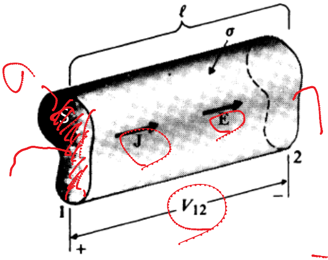
$$V_{12} = RI.$$

$$\uparrow \text{ Integration?}$$

$$\mathbf{E} = (1/\sigma)\mathbf{J}$$

$$V = -\int \vec{E} \cdot d\vec{l}$$

Ohm's Law: Point Form to Circuit Form



$$V_{12} = E\ell \quad \text{or} \quad E = \frac{V_{12}}{\ell}.$$

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} = JS \quad \text{or} \quad J = \frac{I}{S}.$$

FIGURE 5–3
Homogeneous conductor with a constant cross section.

$$\boxed{\mathbf{J} = \sigma \mathbf{E},}$$

$$J = \frac{I}{S}. \qquad E = \frac{V_{12}}{\ell}.$$

$$\frac{I}{S} = \sigma \frac{V_{12}}{\ell}$$

$$V_{12} = \left(\frac{\ell}{\sigma S}\right) I = RI,$$

The resistance

$$\boxed{R = \frac{\ell}{\sigma S} \qquad (\Omega).}$$

EXAMPLE 5–1 In vacuum-tube diodes, electrons are emitted from a hot cathode at zero potential and collected by an anode maintained at a potential V_0, resulting in a convection current flow. Assuming that the cathode and the anode are parallel conducting plates and that the electrons leave the cathode with a zero initial velocity (space-charge limited condition), find the relation between the current density J and V_0.

$J \sim V_0^{3/2}$

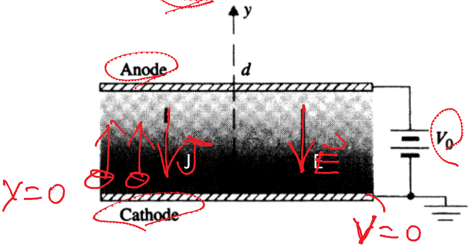


FIGURE 5–2
Space-charge-limited vacuum diode (Example 5–1).

Step1: $\mathbf{J} = \rho \mathbf{u}$
Step2: passion's equation (V- ρ equation)

Assume ① $V(y=0) = 0$ —— ①

② $u(y=0) = 0$ —— ②

1° repulsion $\Rightarrow \ominus$ leaves from the cathode

$$\vec{E}(0) = \hat{a}_y E_y(0) = -\hat{a}_y \left.\frac{dV(y)}{dy}\right|_{y=0} = 0 \quad \text{—— } ③$$

$$V(y=0^+)=0$$

$$V(y=0^-)=0$$

$$\rho(y)<0$$

$\downarrow -\hat{a}_y$

$$\vec{J} = -\hat{a}_y J = -\hat{a}_y\underbrace{(-\rho(y))}_{}u(y)$$

$$= \hat{a}_y \rho(y) u(y)$$

2° Newton's law

$$m\frac{du(t)}{dt} = -eE = e\frac{dV(y)}{dy}$$

$m = 9.11 \times 10^{-31}$ kg

$e = 1.6 \times 10^{-19}$ C

$$\vdots$$

$$\Rightarrow \frac{d}{dy}\left(\frac{1}{2}mu^2\right) = e\frac{dV}{dy}$$

$$\int \sim \Rightarrow \frac{1}{2}mu^2 = eV \quad + \frac{1}{2}m\cancel{u^2(y=0)} \quad + e\cancel{V(y=0)}$$

2 1

$$\Rightarrow u = \sqrt{\frac{2eV}{m}}$$

3^0 Laplace's eq. ④ $\rho = -\frac{J}{u} = -J\sqrt{\frac{m}{2e}}\sqrt{\frac{1}{V}}$ — ④

$$\frac{d^2V}{dy^2} = -\frac{\rho}{\epsilon_0} \overset{④}{=} \frac{J}{\epsilon_0}\sqrt{\frac{m}{2e}}\sqrt{\frac{1}{V}}$$

$2\frac{dV}{dy} \times \sim \Rightarrow$

$\int \sim \Rightarrow \left(\frac{dV}{dy}\right)^2 = \frac{4J}{\epsilon_0}\sqrt{\frac{m}{2e}}\sqrt{V} + \underline{C}$

$\sqrt{\sim} \Rightarrow \frac{dV}{dy} = 2\sqrt{\frac{J}{\epsilon_0}}\left(\frac{m}{2e}\right)^{1/4}\underline{V}^{1/4} + C \Rightarrow$ using ① $V(y=0)=0$ ③ $\frac{dV}{dy}\Big|_{y=0} = 0$

$0 = 0 + C \Rightarrow C = 0$

$$V^{-1/4}\, dV = 2\sqrt{\frac{J}{\epsilon_0}}\left(\frac{m}{2e}\right)^{1/4} dy$$

$$\Rightarrow \int_{V=0}^{V_0} V^{-1/4}\, dV = \int_{y=0}^{d} 2\sqrt{\frac{J}{\epsilon_0}}\left(\frac{m}{2e}\right)^{1/4} dy$$

$$\Rightarrow \cdots$$

$$\Rightarrow J = \frac{4\epsilon_0}{9 d^2}\sqrt{\frac{2e}{m}}\, V_0^{3/2} \quad \#$$

5-3 Electromotive Force and Kirchhoff's Voltage Law

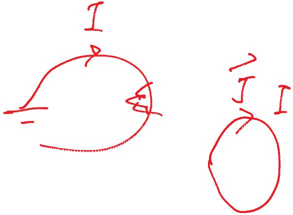
$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = 0.$$

For an ohmic material

$$\mathbf{J} = \sigma \mathbf{E},$$

$$\oint_C \frac{1}{\sigma} \mathbf{J} \cdot d\boldsymbol{\ell} = 0.$$

If **J** is constant, then **J** = 0.



A **steady** current **cannot** be maintained in the same direction in a closed circuit by **an electrostatic field** (**conservative field**)

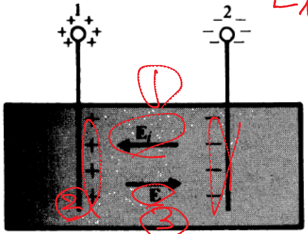
That is, to maintain a steady current in a closed circuit, there must be **non-conservative field** (e.g., electric batteries, etc.), which termed as impressed electric field intensity E_i

Electromotive Force

- Chemical action ($\mathbf{E}_i$)
  ➔ cumulation of + and – charges on electrodes due to $\mathbf{E}_i$
  ➔ $\mathbf{E}$
- Inside: $\mathbf{E}$ and $\mathbf{E}_i$
  - $\mathbf{E} = -\mathbf{E}_i$ due to I = 0 for open circuit
- Outside: $\mathbf{E}$ only

$\mathbf{E}$: electrostatic field
$\mathbf{E}_i$: nonconservative field

$E_{i,\text{outside}} = 0$



Electric battery

Open circuit

FIGURE 5–4
Electric fields inside an electric battery.

Electromotive Force

- $\mathbf{E}_i$

Against $\mathbf{E}_i$ from 1 to 2

$$-\int_1^2 \mathbf{E}_i\, d\ell = \mathscr{V} = \int_2^1 \mathbf{E}_i \cdot d\ell = -\underbrace{\int_2^1 \mathbf{E} \cdot d\ell}_{\text{Inside the source}}.$$

Open circuit: $\mathbf{E}_i = -\mathbf{E}$

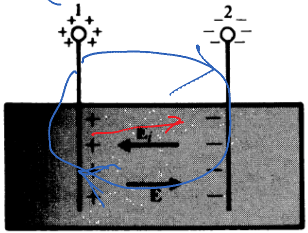
$\mathscr{V}$

- By definition: potential due to $\mathbf{E}_i$
- The electromotive force is a measure of the strength of the nonconservative source, **not a force**

- **E**

$$\oint_C \mathbf{E} \cdot d\ell = \underbrace{\int_1^2 \mathbf{E} \cdot d\ell}_{\substack{\text{Outside}\\ \text{the source}}} + \underbrace{\int_2^1 \mathbf{E} \cdot d\ell}_{\substack{\text{Inside}\\ \text{the source}}} = 0.$$

Because of conservative field



Electric battery

Open circuit

FIGURE 5–4
Electric fields inside an electric battery.

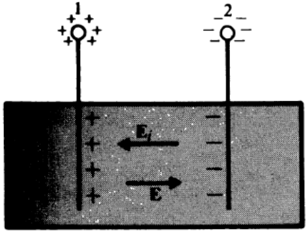
$$\boxed{\mathscr{V} = \int_2^1 \mathbf{E}_i \cdot d\boldsymbol{\ell} = -\underbrace{\int_2^1 \mathbf{E} \cdot d\boldsymbol{\ell}}_{\substack{\text{Inside}\\ \text{the source}}}.}$$

$$\boxed{\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = \underbrace{\int_1^2 \mathbf{E} \cdot d\boldsymbol{\ell}}_{\substack{\text{Outside}\\ \text{the source}}} + \underbrace{\int_2^1 \mathbf{E} \cdot d\boldsymbol{\ell}}_{\substack{\text{Inside}\\ \text{the source}}} = 0.}$$

$$\mathscr{V} = \underbrace{\int_1^2 \mathbf{E} \cdot d\boldsymbol{\ell}}_{\substack{\text{Outside}\\ \text{the source}}} = -\underbrace{\int_2^1 \mathbf{E} \cdot d\boldsymbol{\ell}}_{\substack{\text{Outside}\\ \text{the source}}} = V_{12} = V_1 - V_2.$$

emf = voltage rise between + and – terminals (outside)

$\mathbf{E}_i$ can produce a voltage difference



Electric battery

Open circuit

FIGURE 5–4
Electric fields inside an electric battery.

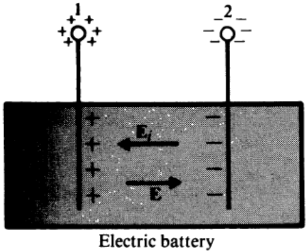
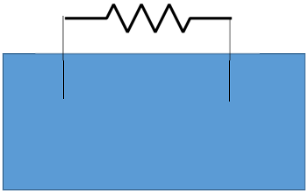


FIGURE 5–4
Electric fields inside an electric battery.

Open circuit ➔ No currents

If connected with a resistor ➔ Currents



Point form of Ohm's law

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{E}_i),$$

$$\mathbf{E} + \mathbf{E}_i = \frac{\mathbf{J}}{\sigma}.$$

$$\mathcal{V} = \oint_C (\mathbf{E} + \mathbf{E}_i)\cdot d\ell = \oint_C \frac{1}{\sigma}\mathbf{J}\cdot d\ell.$$

$$\boxed{\mathbf{E} + \mathbf{E}_i = \frac{\mathbf{J}}{\sigma}.}$$

Integration of 1st integrand = 0

$$\boxed{\oint_C \mathbf{E}\cdot d\ell = \underbrace{\int_1^2 \mathbf{E}\cdot d\ell}_{\substack{\text{Outside}\\ \text{the source}}} + \underbrace{\int_2^1 \mathbf{E}\cdot d\ell}_{\substack{\text{Inside}\\ \text{the source}}} = 0.}$$

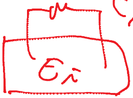
1

Integration of 2nd integrand

$$\oint_C \mathbf{E}_i\cdot d\ell = \underbrace{\cancel{\int_1^2 \mathbf{E}_i\cdot d\ell}}_{\substack{\text{Outside}\\ \text{the source}}}^{0} + \underbrace{\int_2^1 \mathbf{E}_i\cdot d\ell}_{\substack{\text{Inside}\\ \text{the source}}}$$

2

$= \mathcal{V}$



$E_{i,\text{outside}} = 0$

For a resistor with uniform cross section:

$$J = I/S \qquad R = \frac{\ell}{\sigma S}$$

$$\mathcal{V} = RI.$$

$$\mathscr{V} = RI.$$

For more-than-one emf and more-than-one resistor connected in series

$$\boxed{\sum_j \mathscr{V}_j = \sum_k R_k I_k \qquad \text{(V).}}$$

Kirchhoff's voltage law: around a closed path in an electric circuit, the algebraic sum of the emf's (voltage rises) is equal to the algebraic sum of the voltage drops across the resistances

5-4 Equation of Continuity and Kirchhoff's Current Law

- Principle of conservation of charge: electric charges may not be created or destroyed

$\hat{n}$

$$I = \oint_S \mathbf{J} \cdot d\mathbf{s} = -\frac{dQ}{dt} = -\frac{d}{dt} \int_V \rho \, dv.$$

Current leaving a volume

Rate of charge decrease in the volume

$$\int_V \nabla \cdot \mathbf{J} \, dv = -\int_V \frac{\partial \rho}{\partial t} \, dv.$$

$$\int_V \nabla \cdot \mathbf{J}\, dv = -\int_V \frac{\partial \rho}{\partial t}\, dv.$$

Holds for arbitrary volume V

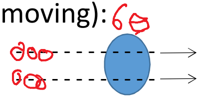
Equation of continuity

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \qquad (\mathrm{A/m^3}).}$$

= conservation of charge

- For **steady current (I = constant)**, charge density does not vary with time (or charge in a volume is a constant over time although charge is moving): $\boxed{\partial\rho/\partial t = 0.}$

$\nabla\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$



$$\nabla\cdot\mathbf{J} = 0.$$ (divergenceless: streamlines of steady currents close upon themselves)

$$\oint_S \mathbf{J}\cdot d\mathbf{s} = 0,$$ (integral form)

For $S \rightarrow 0$ (i.e., the volume shrinks to a point)

$$\boxed{\sum_j I_j = 0 \qquad \text{(A)}.}$$

Kirchhoff's current law: the algebraic sum of all the currents flowing **out of a junction** (a small volume) in an electric circuit is zero.

Time to Reach Equilibrium in a Conductor

- Inside a conductor, $\rho = 0$, $\mathbf{E} = 0$ under equilibrium conditions (Chap. 3)
- Time to reach equilibrium?

Equation of continuity

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \qquad (\mathrm{A/m^3}).}$$

Assume a homogeneous σ

$$\sigma \nabla \cdot \mathbf{E} = -\frac{\partial \rho}{\partial t}.$$

$$\sigma \nabla \cdot \mathbf{E} = -\frac{\partial \rho}{\partial t}.$$

In a simple medium
$\nabla \cdot \mathbf{E} = \rho/\epsilon$

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon}\rho = 0.$$

Solution: $\boxed{\rho = \rho_0 e^{-(\sigma/\epsilon)t} \quad (\mathrm{C/m^3}),}$

$\rho = \rho_0 e^{-\frac{t}{\tau}}$

Charge density inside a conductor will decrease with time exponentially.

Relaxation time: time for ρ_0 to decay to $1/e \times \rho_0$

$$\downarrow \tau = \frac{\epsilon \downarrow}{\sigma \uparrow} \quad (\mathrm{s}).$$

$\sigma \to \infty$ perfect conductor

practical conductor : $\varepsilon \neq 0$

? insulator : $\sigma \neq 0$

5-5 Power Dissipation and Joule's Law

- Power dissipation:

  External **E**
  ➔ drift motion of electrons, which collide with atoms on lattice sites
  ➔ thermal energy

- Power by **E** to move a charge q

$$p = \lim_{\Delta t \to 0} \frac{\Delta w}{\Delta t} = q\mathbf{E} \cdot \mathbf{u},$$

u: drift velocity

$\frac{\#}{dv} q_i \, dv = \# q_i$

Differential power in a volume dv $$dP = \sum_i p_i = \mathbf{E} \cdot \left(\sum_i N_i q_i \mathbf{u}_i \right) dv,$$

Total *Q* in a volume *dv*

$$dP = \sum_i p_i = \mathbf{E} \cdot \left(\boxed{\sum_i N_i q_i \mathbf{u}_i}\right) dv,$$

$$\vec{J} = \rho \vec{u}$$

$$\boxed{\mathbf{J} = \sum_i \underbrace{N_i q_i}_{\rho} \mathbf{u}_i \qquad (\text{A/m}^2).}$$

$$dP = \mathbf{E} \cdot \mathbf{J}\, dv$$

$$\frac{dP}{dv} = \underline{\mathbf{E} \cdot \mathbf{J}} \qquad (\text{W/m}^3).$$

Power density

or

$$\boxed{P = \int_V \mathbf{E} \cdot \mathbf{J}\, dv \qquad (\text{W}).}$$

Joule's law

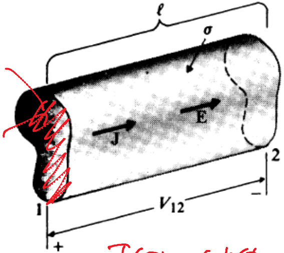
$$\boxed{P = \int_V \mathbf{E} \cdot \mathbf{J}\, dv \qquad \text{(W)}.}$$

In a conductor with constant cross section

$dv = ds\, d\ell$

$$P = \int_L E\, d\ell \int_S J\, ds = VI,$$

We get the familiar expression.



$I = JS$

$J(S) = \text{const.}$

$E(S) = \text{const.}$

5-6 Boundary Conditions for Current Density

- **Steady** current density **J** on boundaries without nonconservative energy source

Governing Equations for Steady Current Density	
Differential Form	Integral Form
$\nabla \cdot \mathbf{J} = 0$	$\oint_S \mathbf{J} \cdot d\mathbf{s} = 0$
$\nabla \times \left(\frac{\mathbf{J}}{\sigma}\right) = 0$	$\oint_C \frac{1}{\sigma} \mathbf{J} \cdot d\boldsymbol{\ell} = 0$

$\nabla \times \vec{E} = 0$

conservative

Chapter 3

Here

The **normal** component of a **divergenceless** vector field is continuous

$$\nabla \cdot \mathbf{D} = \rho$$

$$\boxed{D_{1n} - D_{2n} = \rho_s \qquad (\text{C/m}^2),}$$

$$\nabla \cdot \mathbf{J} = 0$$

$$\boxed{J_{1n} = J_{2n} \qquad (\text{A/m}^2).}$$

The **tangential** component of a **curl-free** vector field is continuous across an interface

$$\nabla \times \mathbf{E} = 0$$

$$\boxed{E_{1t} = E_{2t} \qquad (\text{V/m}),}$$

$$\nabla \times \left(\frac{\mathbf{J}}{\sigma}\right) = 0$$

$$\boxed{\frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}.}$$

The ratio of J_t at two sides of an interface is equal to **the ratio of the conductivities**

$$\nabla \times \mathbf{E} = 0 \qquad\qquad \nabla \times \left(\frac{\mathbf{J}}{\sigma}\right) = 0$$

$$\nabla\times(\mathbf{D}/\varepsilon)=0$$

Analogy

Interface of dielectric media	Interface of conducting media
D	**J**
ε	σ

A homogeneous conducting medium

Electrostatics analogy

$$\nabla \times \left(\frac{\mathbf{J}}{\sigma}\right) = 0$$

If σ is a constant (homogeneous)

$$\nabla \times \mathbf{J} = 0.$$

$\nabla \times \mathbf{E} = 0$

By null identity

$$\mathbf{J} = -\nabla \psi.$$

$\mathbf{E} = -\nabla V$

$\nabla \cdot \vec{E} = 0$ if $\rho = 0$

Laplace's eq.: $\nabla^2 \psi = 0.$

$\nabla^2 V = 0$

$\mathbf{J} = \sigma \mathbf{E}$

$$\psi = \sigma V.$$

Boundary Condition between Two Lossy Dielectrics (for a Steady Current)

- Two lossy dielectrics:

ϵ_1 and ϵ_2 $\boxed{\sigma_1 \text{ and } \sigma_2}$

ϵ_1 and ϵ_2	σ_1 and σ_2
$E_{2t} = E_{1t}$	$J_{1t}/\sigma_1 = J_{2t}/\sigma_2$
$D_{1n} - D_{2n} = \rho_s \rightarrow \epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s$	$J_{1n} = J_{2n} \rightarrow \sigma_1 E_{1n} = \sigma_2 E_{2n}$

$\rho \neq 0$

$\Downarrow$

$p = \vec{E} \cdot \vec{J}$

$\Downarrow$

$\sigma \neq 0$

$$\rho_s = \left(\epsilon_1 \frac{\sigma_2}{\sigma_1} - \epsilon_2\right) E_{2n} = \left(\epsilon_1 - \epsilon_2 \frac{\sigma_1}{\sigma_2}\right) E_{1n}.$$

$E_n \neq 0$, $\rho_s = 0$

when $\frac{\sigma_1}{\sigma_2} = \frac{\epsilon_1}{\epsilon_2}$

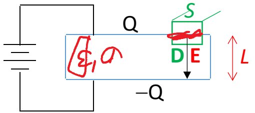
If $E_{1n} \neq 0$ or $E_{2n} \neq 0$, in most cases, a surface charge exists at the interface unless $\sigma_2/\sigma_1 = \varepsilon_2/\varepsilon_1$

What if medium 2 is a perfect conductor?

5-7 Resistance Calculations

- Capacitance between two conductors:

$$C = \frac{Q}{V} = \frac{\oint_S \mathbf{D} \cdot d\mathbf{s}}{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}} = \frac{\oint_S \epsilon\mathbf{E} \cdot d\mathbf{s}}{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}},$$

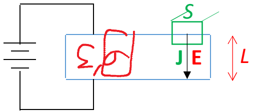


Numerator: surface integral over a surface enclosing the positive conductor

- Resistance between two conductors (medium between is lossy):

$$R = \frac{V}{I} = \frac{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}}{\oint_S \mathbf{J} \cdot d\mathbf{s}} = \frac{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}}{\oint_S \sigma\mathbf{E} \cdot d\mathbf{s}},$$

$$\boxed{\mathbf{J} = \sigma\mathbf{E},}$$



Denominator: the same surface as in the numerator of the above equation

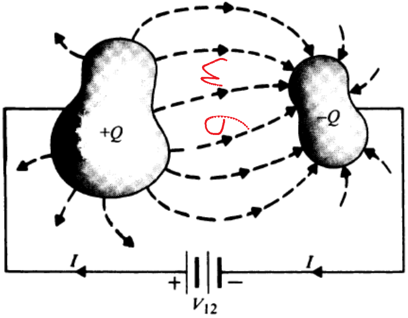


FIGURE 5–7
Two conductors in a lossy dielectric medium.

$$C = \frac{Q}{V} = \frac{\oint_S \mathbf{D} \cdot d\mathbf{s}}{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}} = \frac{\oint_S \epsilon \mathbf{E} \cdot d\mathbf{s}}{\cancel{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}}},$$

$\times$

$$R = \frac{V}{I} = \frac{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}}{\oint_S \mathbf{J} \cdot d\mathbf{s}} = \frac{\cancel{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}}}{\oint_S \sigma \mathbf{E} \cdot d\mathbf{s}},$$

If ε and σ of the medium have the same space dependence or if the medium is homogeneous

$$\boxed{RC = \frac{C}{G} = \frac{\epsilon}{\sigma}.}$$

$$\rho = \rho_0 e^{-\frac{t}{\tau}} \qquad \tau = \frac{\epsilon}{\sigma}$$

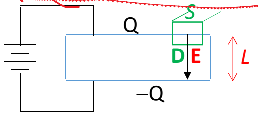
$$\rho = \rho_0 e^{-\frac{t}{RC}}$$

C_ℓ and R_ℓ

- C_ℓ: capacitance per unit length
  (ℓ longer → area S larger → C larger)
  $C = C_\ell \ell$ ➔ $C_\ell = C/\ell$ (F/m)

$C = \varepsilon S/d$

ℓ: in/out plane direction

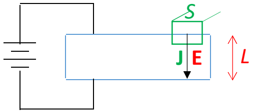


$\ell \neq L$

- R_ℓ: Resistance per unit length
  (ℓ longer → area S larger → R smaller)
  $R = R_\ell/\ell$ ➔ $R_\ell = R\ell$ (Ω·m)

$R = \varepsilon L/S$

ℓ: in/out plane direction



Note that ℓ and L are different!

$$\boxed{RC = \frac{C}{G} = \frac{\epsilon}{\sigma}.}$$

$$\boxed{R_\ell C_\ell = RC = \varepsilon/\sigma}$$

Difference between J and D

- Current flow can be confined strictly within a conductor
- Electric flux usually **cannot** be contained within a dielectric slab (of finite dimensions)

$\sigma = 0 \Rightarrow \vec{J} = 0$

$\varepsilon \neq 0 \Rightarrow$

A conductor

A dielectric

Procedure to Compute R between Specified Equi-potential Surfaces

- Procedure 1: ***V_0* to I**
  - 1. Choose a coordinate
  - 2. Assume potential difference V_0 between conductors
  - 3. Find **E** between conductors
    - ❖ $\nabla^2 V = 0 \rightarrow \mathbf{E} = -\nabla V$
  - 4. Find current $I = \int_S \mathbf{J} \cdot d\mathbf{s} = \int_S \sigma \mathbf{E} \cdot d\mathbf{s},$
  - 5. Find $R = V_0/I$

$V_0 \to V(R) \to \vec{E}(R) \to \vec{J}(R) \to I$

- Procedure 2: **I to V_0**
  - Assume $I \rightarrow \mathbf{J} \rightarrow \mathbf{E} \rightarrow V_0$
    - $\vec{J}(R)$
  - $R = V_0/I$

if **J** can be determined easily from I

EXAMPLE 5–4 An emf $\mathcal{V}$ is applied across a parallel-plate capacitor of area S. The space between the conducting plates is filled with two different lossy dielectrics of thicknesses d_1 and d_2, permittivities ϵ_1 and ϵ_2, and conductivities σ_1 and σ_2, respectively. Determine (a) the current density between the plates, (b) the electric field intensities in both dielectrics, and (c) the surface charge densities on the plates and at the interface.

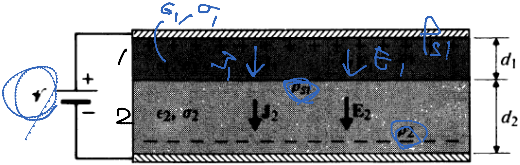


FIGURE 5–6 Parallel-plate capacitor with two lossy dielectrics (Example 5–4).

(a) $\vec{J}$?

(b) $\vec{E}_1$? $\vec{E}_2$?

(c) $\rho_{s1}, \rho_{si}, \rho_{s2}$?

(a) ohm's law :

$$\mathcal{V} = RI = \underbrace{(R_1 + R_2)}_{\text{series connection}} I = \left(\frac{d_1}{\sigma_1 S} + \frac{d_2}{\sigma_2 S}\right) I$$

$$\Rightarrow J = \frac{I}{S} = \cdots$$

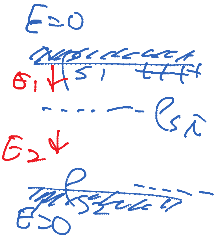
(b) $V = E_1 d_1 + \underset{\sim}{E_2} d_2$

J_n continuous $\Rightarrow J_1 = \sigma_1 E_1 = \sigma_2 E_2 = J_2$

$\Rightarrow V = E_1 d_1 + \left(\frac{\sigma_1}{\sigma_2} E_1\right) d_2 \Rightarrow E_1 = \cdots$ #

$E_2 = \cdots$ #

(c) top surface



$D_1 - \underline{0} = \rho_{s1} \Rightarrow \epsilon_1 E_1 = \rho_{s1}$

$D_2 - D_1 = \rho_{si} \Rightarrow \rho_{si} = \epsilon_2 E_2 - \epsilon_1 E_1$

$0 - D_2 = \rho_{s2} \Rightarrow \rho_{s2} = -\epsilon_2 E_2$

$\rho_{s1} + \rho_{si} + \rho_{s2} = 0$

EXAMPLE 5–6 A conducting material of uniform thickness h and conductivity σ has the shape of a quarter of a flat circular washer, with inner radius a and outer radius b, as shown in Fig. 5–8. Determine the resistance between the end faces.

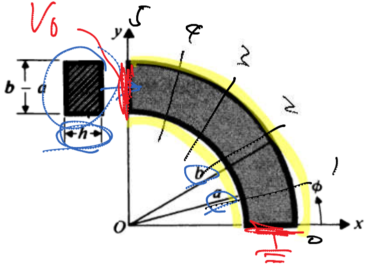


FIGURE 5–8
A quarter of a flat conducting circular washer (Example 5–6).

1° $V=0$ at $\phi=0$

$V=V_0$ at $\phi=\pi/2$

2° $V(\phi)$ only

$$\nabla^2 V=0 \Rightarrow \frac{\partial^2 V}{\partial\phi^2}=0 \Rightarrow V(\phi)=C_1\phi+C_2 \Rightarrow V(\phi)=\frac{2V_0}{\pi}\phi$$

3°
$$\vec{E}=-\nabla V=-\hat{a}_\phi\frac{\partial}{r\partial\phi}\left(\frac{2V_0}{\pi}\phi\right)=-\hat{a}_\phi\frac{2V_0}{r\pi}$$

4°
$$\vec{J}=\sigma\vec{E}=-\hat{a}_\phi\frac{2\sigma V_0}{r\pi}$$

$$I=\int\vec{J}\cdot d\vec{S}, \quad d\vec{S}=-\hat{a}_\phi\,dr\,dz$$

$$=\int_0^h\int_a^b J_\phi\,dr\,dz$$

$$=h\int_a^b+\frac{2\sigma V_0}{r\pi}\,dr$$

$$=\frac{2\sigma V_0 h}{\pi}\ln\left(\frac{b}{a}\right)$$

5° $R=\frac{V}{I}=\ldots$