VE230 Electromagnetics

Chapter 4

September 12, 2022

Exercise 4.1

The upper and lower conducting plates of a large parallel-plate capacitor are separated by a distance d and maintained at potentials V_0 and 0 , respectively. A dielectric slab of dielectric constant 6.0 and uniform thickness $0.8d$ is placed over the lower plate. Assuming negligible fringing effect, determine

a) the potential and electric field distribution in the dielectric slab,

b) the potential and electric field distribution in the air space between the dielectric slab and the upper plate,

c) the surface charge densities on the upper and lower plates.

d) Compare the results in part (b) with those without the dielectric slab.

Answer:

Use subscripts d and $\underline{\bar{\nabla}^2}$ to denote dielectric and air regions respectively, $\bar{\nabla}^2 V = 0$ in both regions.

$$V_d = c_1 y + c_2, \quad \bar{E}_d = -\bar{a}_y c_1, \bar{D}_d = -\bar{a}_y \epsilon_0 \epsilon_r c_1$$

$$V_a = c_3 y + c_4, \quad \bar{E}_a = -\bar{a}_y c_3, \bar{D}_a = -\bar{a}_y \epsilon_0 c_3$$

B.C: At $y = 0, V_d = 0$; at $y = d, \quad V_a = V_0$;
at $y = 0.8d: \quad V_d = V_a, \bar{D}_d = \bar{D}_a$.
Solving:

$$c_1 = \frac{V_0}{(0.8 + 0.2\epsilon_r) d}, \quad c_2 = 0, \quad c_3 = \frac{\epsilon_r V_0}{(0.8 + 0.2\epsilon_r) d}, \quad c_4 = \frac{(1 - \epsilon_r) V_0}{1 + 0.25\epsilon_r}$$

a)

$$V_d = \frac{5yV_0}{(4 + \epsilon_r) d}, \quad \bar{E}_d = -\bar{a}_y \frac{5V_0}{(4 + \epsilon_r) d}$$

b)

$$V_a = \frac{5\epsilon_r y - 4(\epsilon_r - 1) d}{(4 + \epsilon_r) d} V_b, \quad \bar{E}_a = -\bar{a}_y \frac{5\epsilon_r V_0}{(4 + \epsilon_r) d}$$

c)

$$(\rho_s)_{y=d} = -(D_a)_{y=d} = \frac{5\epsilon_0 \epsilon_r V_0}{(4 + \epsilon_r) d}$$

$$(\rho_s)_{y=0} = (D_d)_{y=0} = -\frac{5\epsilon_0 \epsilon_r V_0}{(4 + \epsilon_r) d}$$

Exercise 4.2

Prove that the scalar potential V in Eq. (3-61)

$$V = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho}{R} dv'$$

satisfies Poisson's equation, Eq. (4-6)

$$\nabla^2 V = -\frac{\rho}{\epsilon}$$

Answer:
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Exercise 4.3

Prove that a potential function satisfying Laplace's equation in a given region possesses no maximum or minimum within the region.

Answer:
At a point where V is a maximum (minimum) the second derivatives of V with respect to x, y, and z would. all be negative (positive); their sum could not vanish, as required by Laplace's equation.

Exercise 4.4

Verify that

$$V_1 = C_1/R \quad \text{and} \quad V_2 = C_2 z/\left(x^2+y^2+z^2\right)^{3/2},$$

where C_1 and C_2 are arbitrary constants, are solutions of Laplace's equation.

Answer:
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Exercise 4.5

Assume a point charge Q above an infinite conducting plane at $y=0$.

a) Prove that $V(x, y, z)$ in Eq. (4-37) satisfies Laplace's equation if the conducting plane is maintained at zero potential.

b) What should the expression for $V(x, y, z)$ be if the conducting plane has a nonzero potential V_0 ?

c) What is the electrostatic force of attraction between the charge Q and the conducting plane?

Answer:
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Exercise 4.6

Assume that the space between the inner and outer conductors of a long coaxial cylindrical structure is filled with an electron cloud having a volume density of charge $\rho = A/r$ for $a < r < b$, where a and b are, the radii of the inner and outer conductors, respectively. The inner conductor is maintained at a potential V_0, and the outer conductor is grounded. Determine the potential distribution in the region $a < r < b$ by solving Poisson's equation.

Answer:

Poisson's eq. $\bar{\nabla}^2 V = -\frac{A}{\epsilon r} \longrightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = -\frac{A}{\epsilon r}$.
Solution: $V = -\frac{A}{\epsilon} r + c_1 \ln r + c_2$.

B.C.: $\begin{cases} \text{At } r = a, \quad V_0 = -\frac{A}{\epsilon} a + c_1 \ln a + c_2. & c_1 = \frac{\frac{A}{\epsilon}(b-a) - V_0}{\ln(b/a)}, \\ \text{At } r = b, \quad 0 = -\frac{A}{\epsilon} b + c_1 \ln b + c_2. & c_2 = \frac{V_0 \ln b + \frac{A}{\epsilon}(a \ln b - b \ln a)}{\ln(b/a)}. \end{cases}$

Exercise 4.7

A point charge Q exists at a distance d above a large grounded conducting plane. Determine

a) the surface charge density ρ_s,

b) the total charge induced on the conducting plane.

Answer:

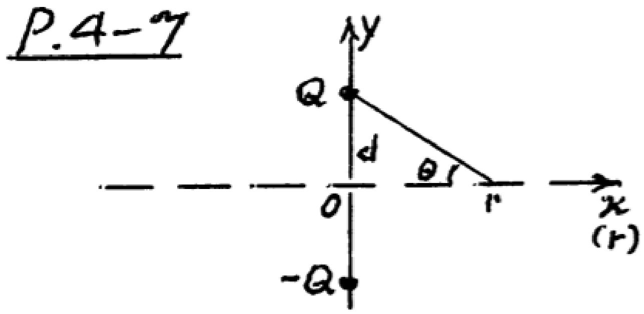
$$\bar{E}\Big|_{y=0} = -\bar{a}_y \frac{Q}{4\pi\epsilon R^2}(2\sin\theta) = -\bar{a}_y \frac{Qd}{2\pi\epsilon\left(d^2 + r^2\right)^{3/2}}$$

a)

$$\rho_s = \bar{a}_y \cdot \epsilon \bar{E}\Big|_{y=0} = -\frac{Qd}{2\pi\left(d^2 + r^2\right)^{3/2}}$$

b)

$$\int_0^\infty \rho_s 2\pi r dr = -Q$$



Exercise 4.8

For a positive point charge Q located at distances d_1 and d_2, respectively, from two grounded perpendicular conducting half-planes shown in Fig. 4-4(a), find the expressions for

a) the potential and the electric field intensity at an arbitrary point $P(x, y)$ in the first quadrant,

b) the surface charge densities induced on the two half-planes. Sketch the variations of the surface charge densities in the xy-plane.

Answer:
Consider the conditions in the xy-plane ($z = 0$).

a) $V_p = \frac{Q}{4\pi\epsilon}\left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{1}{R_3} - \frac{1}{R_4}\right)$, where

$$\begin{aligned} R_1 &= \left[(x-d_1)^2 + (y-d_2)^2\right]^{1/2}, \quad R_2 = \left[(x-d_1)^2 + (y+d_2)^2\right]^{1/2}, \\ R_3 &= \left[(x+d_1)^2 + (y+d_2)^2\right]^{1/2}, \quad R_4 = \left[(x+d_1)^2 + (y-d_2)^2\right]^{1/2}. \\ \bar{E}_p = & -\bar{\nabla} V_p = -\bar{a}_x \frac{\partial V_P}{\partial x} - \bar{a}_y \frac{\partial V_P}{\partial y} \\ = & \ \bar{a}_x \frac{Q}{4\pi\epsilon}\left[-\frac{x-d_1}{R_1^3} + \frac{x-d_1}{R_2^3} - \frac{x+d_1}{R_3^3} + \frac{x+d_1}{R_4^3}\right] \\ & +\bar{a}_y \frac{Q}{4\pi\epsilon}\left[-\frac{y-d_2}{R_1^3} + \frac{y+d_2}{R_2^3} - \frac{y+d_2}{R_3^3} + \frac{y-d_2}{R_4^3}\right]. \end{aligned}$$

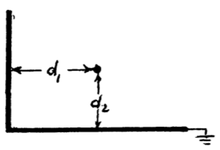
E_p will have a z-component if the point P does not lie in the xy-plane.

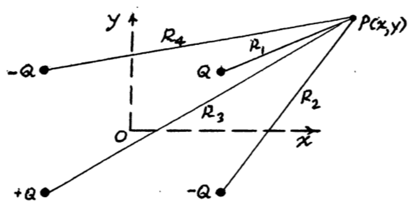
b) On the conducting half-planes, $\rho_s = D_n = \epsilon E_n$.

Along the x-axis, $y = 0 : R_1 = \left[(x-d_1)^2 + d_2^2\right]^{1/2} = R_2$, and $R_3 = \left[(x+d_1)^2 + d_2^2\right]^{1/2} = R_4$. $E_x = 0, E_y = \frac{Q}{2\pi E}\left[\frac{d_2}{R_1^3} - \frac{d_2}{R_3^3}\right]$.

$$\begin{aligned} \therefore \rho_s(y=0) &= \frac{Qd_2}{2\pi}\left\{\frac{1}{\left[(x-d_1)^2 + d_2^2\right]^{3/2}} - \frac{1}{\left[(x+d_1)^2 + d_2^2\right]^{3/2}}\right\} \\ &= \begin{cases} 0, \text{ at } x = 0. \\ \text{max, at } x = d_1. \end{cases} \end{aligned}$$

Similarly for $\rho_s(x=0)$ on the vertical conducting
Conducting plane by changing x to y and $d_1 \leftrightarrow d_2$.





Exercise 4.9

Determine the systems of image charges that will replace the conducting boundaries that are maintained at zero potential for

a) a point charge Q located between two large, grounded, parallel conducting planes as shown in Fig. 4-22(a),

b) an infinite line charge ρ_ℓ located midway between two large, intersecting conducting planes forming a 60-degree angle, as shown in Fig. 4-22(b).

Answer:
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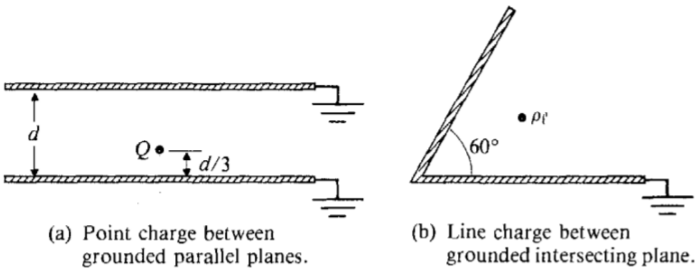


FIGURE 4–22
Diagrams for Problem P.4–9.

Exercise 4.10

A straight conducting wire of radius a is parallel to and at height h from the surface of the earth. Assuming that the earth is perfectly conducting, determine the capacitance and the force per unit length between the wire and the earth.

Answer:
Refer to Example 4-4.

$$C' = \frac{2\pi\epsilon_0}{\ln\left[(h/a) + \sqrt{(h/a)^2 - 1}\right]} = \frac{2\pi\epsilon_0}{\cosh^{-1}(h/a)} \quad \text{(F/m)}$$

Exercise 4.11

A very long two-wire transmission line, each wire of radius a and separated by a distance d, is supported at a height h above a flat conducting ground. Assuming both d and h to be much larger than a, find the capacitance per unit length of the line.

Answer:

$$\text{Same as } C_{12} \text{ in problem } P3 - 38$$

Exercise 4.12

For the pair of equal and opposite line charges shown in Fig. 4-7,
a) write the expression for electric field intensity **E** at point $P(x, y)$ in Cartesian coordinates,
b) find the equation of the electric field lines sketched in Fig. 4-8.

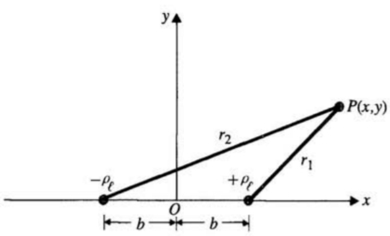


FIGURE 4–7
Cross section of a pair of line charges.

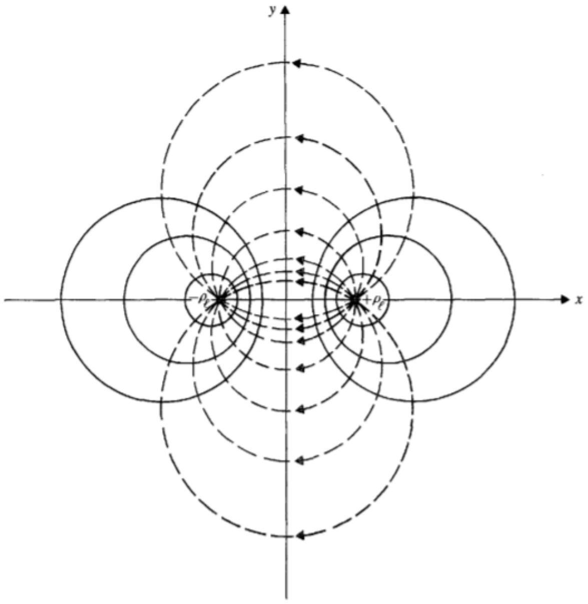


FIGURE 4–8
Equipotential (solid) and electric field (dashed) lines around a pair of line charges.

Answer:
a)

$$\text{From } V_P = \frac{\rho_\ell}{2\pi\epsilon_0}\ln\frac{r_2}{r_1} \text{ and } \frac{r_2}{r_1} = \frac{\sqrt{(x+b)^2+y^2}}{\sqrt{(x-b)^2+y^2}} = k:$$

$$V_p = \frac{\rho_l}{2\pi\epsilon_0}\ln\frac{\sqrt{(x+b)^2+y^2}}{\sqrt{(x-b)^2+y^2}}.$$

$$\bar{E}_p = -\bar{a}_x\frac{\partial V_p}{\partial x} - \bar{a}_y\frac{\partial V_p}{\partial y} = \frac{\rho_l}{2\pi\epsilon_0}\left\{\frac{\bar{a}_x 2b\left(y^2+b^2-x^2\right) - \bar{a}_y 4bxy}{\left[(x+b)^2+y^2\right]\left[y^2+(x-b)^2\right]}\right\}$$

b) Equation for lines everywhere tangent to the electric field lines is obtained by requiring

$$\frac{dy}{dx} = \frac{E_y}{E_x} = -\frac{2xy}{y^2+b^2-x^2},$$

Which reduces to $\frac{d\left(x^2+y^2\right)}{\left(x^2+y^2\right)-b^2} = \frac{dy}{y}$. Integrating, we obtain $x^2+y^2-2ky = b^2$, or $x^2+(y-k)^2 = b^2+k^2$, where K is a constant. Circles of radii $\sqrt{b^2+k^2}$ having centers at $(0, K)$.

Exercise 4.13

Determine the capacitance per unit length of a two-wire transmission line with parallel conducting cylinders of different radii a_1 and a_2, their axes being separated by a distance D (where $D > a_1 + a_2$).

Answer:

$$V_1 = -\frac{\rho_l}{2\pi r\epsilon_0}\ln\frac{a_1}{d_1}, \quad V_2 = \frac{\rho_l}{2\pi\epsilon_0}\ln\frac{a_2}{d_2}$$

Capacitance per unit length

$$C' = \frac{\rho_l}{V_1 - V_2} = \frac{2\pi\epsilon_0}{\ln\frac{d_1 d_2}{a_1 a_2}}.$$

Four equations:

$$a_1^2 = d_{i1}d_1, \qquad a_2^2 = d_{i_2}d_2,$$
$$d_1 + d_{i_2} = D, \quad d_2 + d_{i_1} = D.$$

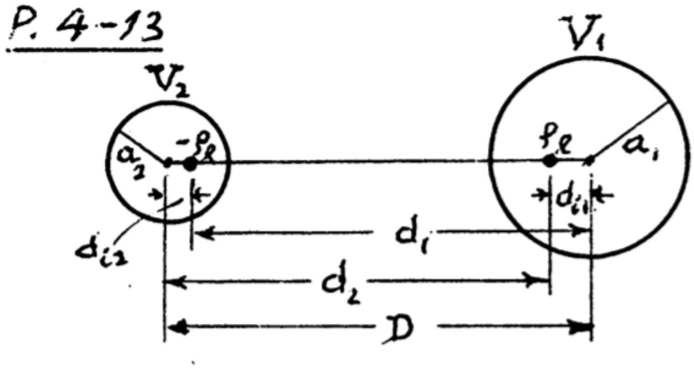
We obtain

$$\frac{d_1 d_2}{a_1 a_2} = \frac{a_1 a_2}{d_{i_1} d_{i2}} \text{ and } a_1^2 + a_2^2 + d_1 d_2 + d_{i1} d_{i2} = D^2$$

$$\frac{d_1 d_2}{a_1 a_2} = \frac{D^2}{2a_1 a_2} - \frac{a_1}{2a_2} - \frac{a_2}{2a_1} + \sqrt{\left(\frac{D^2}{2a_1 a_2} - \frac{a_1}{2a_2} - \frac{a_2}{2a_1}\right)^2 - 1}$$

$$\therefore \quad C' = \frac{2\pi\epsilon_0}{\ln\left[\frac{1}{2}\left(\frac{D^2}{a_1 a_2} - \frac{a_1}{a_2} - \frac{a_2}{a_1}\right) + \sqrt{\frac{1}{4}\left(\frac{D^2}{a_1 a_2} - \frac{a_1}{a_2} - \frac{a_2}{a_1}\right)^2 - 1}\right]}$$

$$= \frac{2\pi\epsilon_0}{\cosh^{-1}\left[\frac{1}{2}\left(\frac{D^2}{a_1 a_2} - \frac{a_1}{a_2} - \frac{a_2}{a_1}\right)\right]} \quad (F/m).$$



Exercise 4.14

A long wire of radius a_1 lies inside a conducting circular tunnel of radius a_2, as shown in Fig. 4-10(a). The distance between their axes is D.

a) Find the capacitance per unit length.

b) Determine the force per unit length on the wire if the wire and the tunnel carry equal and opposite line charges of magnitude ρ_ℓ.

Answer:

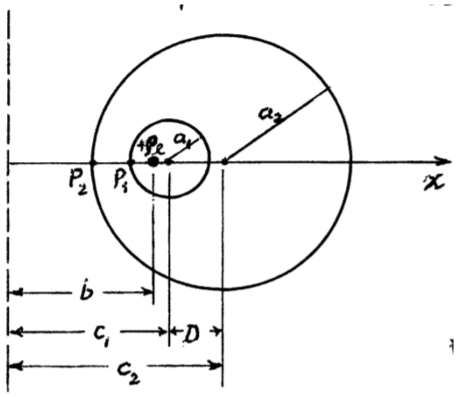
$$\text{Eq. } (4-61): \quad c_1 = \frac{1}{2D}\left(a_2^2 - a_1^2 - D^2\right); \quad E_q(4-62): c_2 = \frac{1}{2D}\left(a_2^2 - a_1^2 + D^2\right).$$

$$\text{Eq. } (4-55): b^2 = c_1^2 - a_1^2;$$
$$\text{Eq. } (4-56): b^2 = c_2^2 - a_2^2.$$
$$\text{a) } V = \frac{\varphi_l}{2\pi\epsilon_0}\ln\frac{r_2}{r_1}$$
At $P_1: r_2 = b + (c_1 - a_1), r_1 = b - (c_1 - a_1)$.
At $P_2: r_2 = b + (c_2 - a_2), r_1 = b - (c_2 - a_2)$.
$$V_1 - V_2 = \frac{\rho_l}{2\pi\epsilon_0}\ln\left[\frac{b+(c_1-a_1)}{b-(c_1-a_1)}\frac{b-(c_2-a_2)}{b+(c_2-a_2)}\right].$$
Expressing $b, c_1\&c_2$ in terms of $D, a_1\&a_2$

and simplifying: $V_1 - V_2 = \frac{\rho_l}{2\pi\epsilon_0}\ln\left\{\left(\frac{a_1^2+a_2^2-D^2}{2a_1a_2}\right) + \left[\left(\frac{a_1^2+a_2^2-D^2}{2a_1a_2}\right)^2 - 1\right]^{1/2}\right\}$

$$C' = \frac{\rho_l}{V_1 - V_2} = \frac{2\pi\epsilon_0}{\ln\left\{\left(\frac{a_1^2+a_2^2-D^2}{2a_1a_2}\right) + \left[\left(\frac{a_1^2+a_2^2-D^2}{2a_1a_2}\right)^2 - 1\right]^{1/2}\right\}} = \frac{2\pi\epsilon_0}{\cosh^{-1}\left(\frac{a_1^2+a_2^2-D^2}{2a_1a_2}\right)} \quad (F/m).$$

b) Force per unit length $F' = \frac{\rho_l^2}{2\pi\epsilon_0(4b^2)} = \frac{D^2\rho_l^2}{2\pi\epsilon_0\left[\left(a_1^2+a_2^2-D^2\right)^2 - 4a_1^2D^2\right]}$(N/m).



Exercise 4.15

A point charge Q is located inside and at distance d from the center of a grounded spherical conducting shell of radius b (where $b > d$). Use the method of images to determine

a) the potential distribution inside the shell,

b) the charge density ρ_s induced on the inner surface of the shell.

Answer:

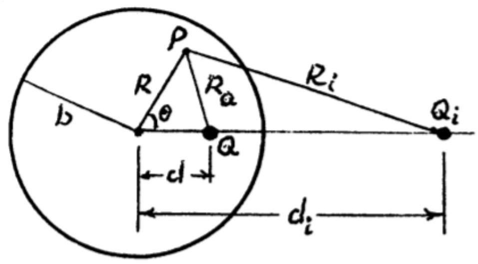
$$Q_i = -\frac{b}{d}Q, \quad d_i = \frac{b^2}{d}.$$

a) $V_p = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_Q} - \frac{b}{dR_i}\right)$;

$$R_Q = \left(R^2 + d^2 - 2Rd\cos\theta\right)^{1/2}$$

$$R_i = \left(R^2 + d_i^2 - 2Rd_i \cdot \cos\theta\right)^{1/2}$$

b) $\rho_s = -\left.\epsilon_0 \frac{\partial V}{\partial R}\right|_{R=b} = -\frac{Q\left(b^2-d^2\right)}{4\pi b(b^2+d^2-2bd\cos\theta)^{3/2}}.$



Exercise 4.16

Two conducting spheres of equal radius a are maintained at potentials V_0 and 0 , respectively. Their centers are separated by a distance D.

a) Find the image charges and their locations that can electrically replace the two spheres.

b) Find the capacitance between the two spheres.

Answer:

a) Q_0 and system of image changes:

In left sphere Q_0 at $d_0 = 0$.

$$Q_2 = \frac{a^2}{D\left(D-d_1\right)} Q_0 \text{ at } d_2$$

$$Q_4 = \frac{a^4}{D\left(D-d_1\right)\left(D-d_2\right)\left(D-d_3\right)} Q_0 \text{ at } d_4$$

In right sphere

$-Q_1 = -\frac{a}{D} Q_0$ at d_1

$-Q_3 = -\frac{a}{D-d_2} Q_2 = -\frac{a^3}{D(D-d_1)(D-d_2)} Q_0$ at d_3

$$Q_{Ln} = Q_0 \prod_{\substack{m=1 \\ (n=2,4,6,\cdots)}}^{n} \frac{a}{D-d_{m-1}} \text{ at } d_n$$

$$-Q_{Rn} = -Q_0 \prod_{m=1}^{n} \frac{a}{D-d_{m-1}} \text{ at } d_n$$

$$d_m = \frac{a^2}{D-d_{m-1}}, m = 1, 2, 3, \cdots; d_0 = 0$$

(b)

$$C = \frac{Q_0 + \sum_n Q_{Ln}}{V_0} = 4\pi\epsilon_0 a \left[1 + \sum_{n=2,4,\ldots} \left(\prod_{m=1}^{n} \frac{a}{D-d_{m-1}}\right)\right]$$

Exercise 4.17

Two dielectric media with dielectric constants ϵ_1 and ϵ_2 are separated by a plane boundary at $x = 0$, as shown in Fig. 4-23. A point charge Q exists in medium 1 at distance d from the boundary.

a) Verify that the field in medium 1 can be obtained from Q and an image charge $-Q_1$, both acting in medium 1.

b) Verify that the field in medium 2 can be obtained from Q and an image charge $+Q_2$ coinciding with Q, both acting in medium 2 .

c) Determine Q_1 and Q_2. (Hint: Consider neighboring points P_1 and P_2 in media 1 and 2 , respectively, and require the continuity of the tangential component of the E-field and of the normal component of the D-field.)

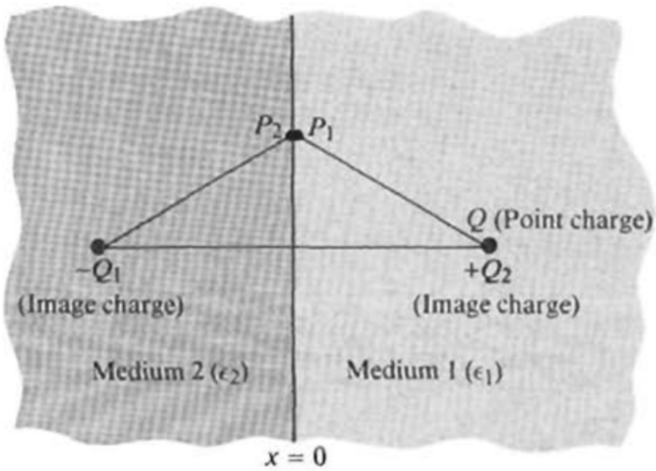


FIGURE 4–23
Image charges in dielectric media (Problem P.4–17).

Answer:
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Exercise 4.18

Describe the geometry of the region in which the potential function can be represented by a single term as follows:

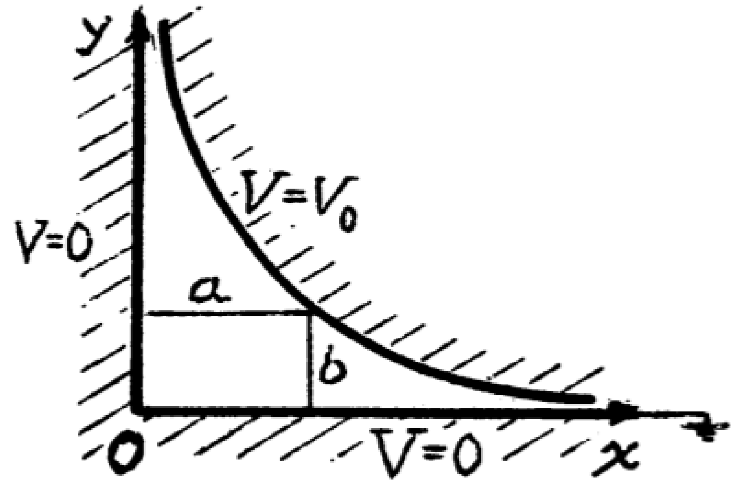
a) $V(x, y) = c_1 xy$,

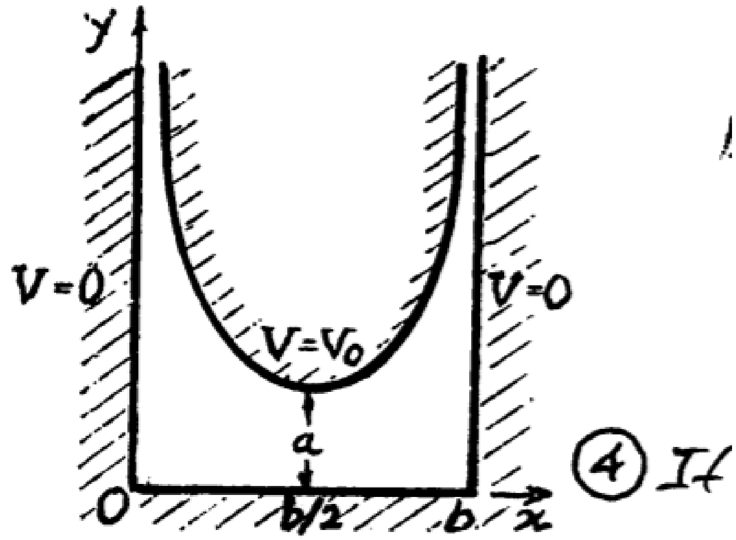
b) $V(x, y) = c_2 \sin kx \sinh ky$.

Find c_1, c_2, and k in terms of the dimensions and a fixed potential V_0

Answer:
a) $V(x, y) = c, xy$ Required boundary conditions:
(1) $V(0, y) = 0 \rightarrow$ Grounded conducting plane at $x = 0$.
(2) $V(x, 0) = 0 \rightarrow$ Grounded conducting plane at $y = 0$.
If $V(a, b) = c, ab = V_0 \longrightarrow c_1 = \frac{V_0}{ab} \rightarrow V(x, y) = \frac{V_0}{ab} xy$
(3) Shape of curved boundary defined by:
$V(x, y) = V_0 = \left(\frac{V_0}{ab}\right) xy$, or $xy = ab$ (a hyperbola).
b) $V(x, y) = c_2 \sin kx \sinh y$. Required boundary conditions:

(1) $V(0,y)=0 \longrightarrow$ Grounded plane at $x=0$.
(2) $V(\pi/k,y)=0 \rightarrow$ Grounded plane at $x=\pi/k=b$.
(3) $V(x,0)=0 \longrightarrow$ Grounded plane at $y=0$.
(4) If $V(b/2,a)=c_2 \sin\left(\frac{\pi}{b}\right)\left(\frac{b}{2}\right)\sinh\left(\frac{\pi}{b}\right)a=c_2\sinh(a\pi/b)=V_0$,
$V(x,y)=\frac{V_0}{\sinh(a\pi/b)}\sin(\pi x/b)\sinh(\pi y/b)$. curved bounded $=\sinh\left(\frac{\pi}{b}y\right)=\frac{\sinh(a\pi/b)}{\sin(\pi x/b)}$.





Exercise 4.19

In what way should we modify the solution in Eq. (4-114)

$$\begin{aligned} V(x,y) &= \sum_{n=1}^{\infty} C_n' \sinh\frac{n\pi}{b}(x-a)\sin\frac{n\pi}{b}y \\ &= \frac{4V_0}{\pi}\sum_{n=\text{ odd}}^{\infty}\frac{\sinh[n\pi(a-x)/b]}{n\sinh(n\pi a/b)}\sin\frac{n\pi}{b}y, \\ & n=1,3,5,\ldots, \\ & 0<x<a \quad \text{and} \quad 0<y<b. \end{aligned}$$

for Example $4-7$ if the boundary conditions on the top, bottom, and right planes in Fig. 4-17 are $\partial V/\partial n = 0$?

Answer:

$$V_n(x,y) = C'_n \cosh \frac{n\pi}{b}(x-a) \cos \frac{n\pi}{b} y$$

Exercise 4.20

In what way should we modify the solution in Eq. (4-114)

$$\begin{aligned} V(x,y) &= \sum_{n=1}^{\infty} C'_n \sinh \frac{n\pi}{b}(x-a) \sin \frac{n\pi}{b} y \\ &= \frac{4V_0}{\pi} \sum_{n=\text{ odd}}^{\infty} \frac{\sinh[n\pi(a-x)/b]}{n \sinh(n\pi a/b)} \sin \frac{n\pi}{b} y, \\ & n = 1, 3, 5, \ldots, \\ & 0 < x < a \quad \text{ and } \quad 0 < y < b. \end{aligned}$$

for Example 4-7 if the top, bottom, and left planes in Fig. 4-17 are grounded ($V = 0$) and an end plate on the right is maintained at a constant potential V_0 ?

Answer:

$$V_n(x,y) = C_n \sinh \frac{n\pi}{b} x \sin \frac{n\pi}{b} y$$

B.C.

$$\text{at } x = a : V_0 = \sum_{n=1}^{\infty} V_n(a,y) = \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi}{b} a \sin \frac{n\pi}{b} y$$

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=\text{ odd}}^{\infty} \frac{\sinh(n\pi x/b)}{n \sinh(n\pi a/b)} \sin \frac{n\pi}{b} y$$

Exercise 4.21

Consider the rectangular region shown in Fig. 4-17 as the cross section of an enclosure formed by four conducting plates. The left and right plates are grounded, and the top and bottom plates are maintained at constant potentials V_1 and V_2, respectively. Determine the potential distribution inside the enclosure.

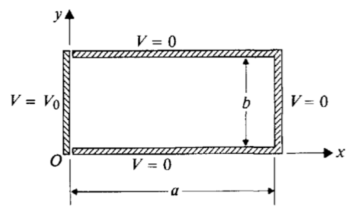


FIGURE 4–17
Cross-sectional figure for Example 4–7.

Answer:

$$V(x,y)=\sum_n \sin\frac{n\pi}{a}x\left[A_n\sinh\frac{n\pi}{a}y+B_n\cosh\frac{n\pi}{a}y\right]$$

$$\text{At } y=0,\quad V(x,0)=V_2=\sum_n B_n\sin\frac{n\pi}{a}x\rightarrow B_n=\begin{cases}\frac{4V_2}{n\pi}, n=\text{ odd.}\\ 0,\quad n=\text{ even.}\end{cases}$$

$$\text{At } y=b, V(x,b)=V_1=\sum_n \sin\frac{n\pi}{a}x\left[A_n\sinh\frac{n\pi}{a}b+B_n\cosh\frac{n\pi}{a}b\right]$$

$$\therefore A_n\sinh\frac{n\pi}{a}b+B_n\cosh\frac{n\pi}{a}b=\begin{cases}\frac{4V_1}{n\pi}, n=\text{ odd.}\\ 0, n=\text{ even}\end{cases}$$

$$\therefore A_n=\begin{cases}\frac{4}{n\pi\sinh(n\pi b/a)}\left(V_1-V_2\cosh\frac{n\pi}{a}b\right), n=\text{ odd.}\\ 0,\quad n=\text{ even.}\end{cases}$$

Exercise 4.22

Consider a metallic rectangular box with sides a and b and height c. The side walls and the bottom surface are grounded. The top surface is isolated and kept at a constant potential V_0. Determine the potential distribution inside the box.

Answer:
$V(x,y,z)=\sum_m\sum_n C_{mn}\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\sinh k_{mn}z$, Where $k_{mn}=\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}$.
At $z=c$

$$V(x,y,c)=V_0=\sum_m\sum_n C_{mn}\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\sinh k_{mn}c.$$

$$\longrightarrow C_{mn}=\begin{cases}\frac{16V_0}{mn\pi^2\sinh k_{mn}c}; m,n=odd.\\ 0,\quad m,n=even.\end{cases}$$

Exercise 4.23

Two infinite insulated conducting planes maintained at potentials 0 and V_0 form a wedge-shaped configuration, as shown in Fig. 4-24. Determine the potential distributions for the regions: (a) $0<\phi<\alpha$, and (b) $\alpha<\phi<2\pi$.

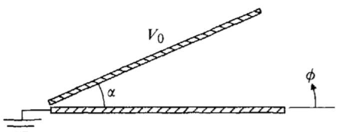


FIGURE 4-24
Two infinite insulated conducting planes maintained at constant potentials (Problem P.4-23).

Answer:

$$\text{Solution: } V(\phi)=A_0\phi+B_0$$

$$\left.\begin{array}{ll}\text{a) B.C .(1):} & V(0)=0\longrightarrow B_0=0.\\ \text{B.C. (2):} & V(\alpha)=V_0=A_0\alpha\longrightarrow A_0=\frac{V_0}{\alpha}.\end{array}\right\}$$

$$\therefore V(\phi) = \frac{V_0}{\alpha}\phi$$
$$0 \leqslant \phi \leqslant \alpha$$

b)

$$\left.\begin{array}{ll} \text{B.C. (1):} & V(\alpha) = V_0 = A_1\alpha + B_1 \\ \text{B.C. (2):} & V(2\pi) = 0 = 2\pi A_1 + B \end{array}\right\}$$

$$A_1 = -\frac{V_0}{2\pi - \alpha}, B_1 = \frac{2\pi V_0}{2\pi - \alpha}$$

$$\therefore V(\phi) = \frac{V_0}{2\pi - \alpha}(2\pi - \phi), \quad \alpha \leqslant \phi \leqslant 2\pi$$

Exercise 4.24

An infinitely long, thin conducting circular cylinder of radius b is split in four quarter-cylinders, as shown in Fig. 4-25. The quarter-cylinders in the second and fourth quadrants are grounded, and those in the first and third quadrants are kept at potentials V_0 and $-V_0$, respectively. Determine the potential distribution both inside and outside the cylinder.

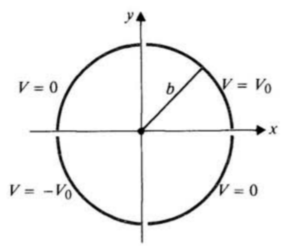


FIGURE 4–25
Cross section of long circular cylinder split in four quarters (Problem P.4–24).

Answer:
The solution is the superposition of that for Example 4-9 and that for Fig.4-19 rotated 90° in the clockwise direction. (In both cases V_0 should be replaced by $V_0/2$.)
Inside:

$$V(r,\phi) = \frac{2V_0}{\pi}\sum_{n=odd}\frac{1}{n}\left(\frac{r}{b}\right)^n\left[\sin n\phi + \sin n\left(\phi + \frac{\pi}{2}\right)\right], \quad r < b$$

Outside:

$$V(r,\phi) = \frac{2V_0}{\pi}\sum_{n=odd}\frac{1}{n}\left(\frac{b}{r}\right)^n\left[\sin n\phi + \sin n\left(\phi + \frac{\pi}{2}\right)\right], \quad r > b$$

Exercise 4.25

A long, grounded conducting cylinder of radius b is placed along the z-axis in an initially uniform electric field $\mathbf{E}_0 = \mathbf{a}_x E_0$. Determine potential distribution $V(r,\phi)$ and electric field intensity $\mathbf{E}(r,\phi)$ outside the cylinder. Show that the electric field intensity at the surface of the cylinder may be twice as high as that in the distance, which may cause a local breakdown or corona. (This phenomenon of corona discharge along the rigging and spars of ships and on airplanes near storms is known as ***St***. Elmo's fire. †)

Answer:

$$V(r,\phi) = -E_0 r\cos\phi + \sum_{n=1}^{\infty} B_n r^{-n}\cos n\phi. \left(\begin{array}{c}\text{At } r >> b\\ E = \bar{a}_k E_0, V = -E_0 r\cos\phi\end{array}\right)$$

$$\text{At } r = b, V(b,\phi) = -E_0 b\cos\phi + \sum_{n=1}^{\infty} B_n b^{-n}\cos n\phi$$

$$\longrightarrow B_1 = E_0 b^2; \quad B_n = 0$$

for

$$n \neq 1.$$

Outside the cylinder,

$$r \geqslant b : V(r,\phi) = -E_0 r\left(1 - \frac{b^2}{r^2}\right)\cos\phi$$

$$\bar{E}(r,\phi) = -\bar{\nabla}V = \bar{a}_r E_0\left(\frac{b^2}{r^2} + 1\right)\cos\phi + \bar{a}_\phi E_0\left(\frac{b^2}{r^2} - 1\right)\sin\phi.$$

$$(\text{At } r = b, \phi = 0, \pi : |E| = 2E_0.)$$

Exercise 4.26

A long dielectric cylinder of radius b and dielectric constant ϵ_r is placed along the z-axis in an initially uniform electric field $\mathbf{E}_0 = \mathbf{a}_x E_0$. Determine $V(r,\phi)$ and $\mathbf{E}(r,\phi)$ both inside and outside the dielectric cylinder.

Answer:

$$r \geq b, \quad V_0(r,\phi) = -E_0 r\cos\phi + \sum_{n=1}^{\infty} B_n r^{-n}\cos n\phi$$

$$r \leq b, \quad V_i(r,\phi) = \sum_{n=1}^{\infty} A_n r^n \cos n\phi.$$

$$\text{At } r = b : V_0(b,\phi) = V_i(b,\phi) \longrightarrow -E_0 b + B, b^{-1} = A_1 b; B_n b^{-n} = A_n b_n^n n \neq 1$$

$$-\left.\frac{\partial V_0}{\partial r}\right|_{r=b} = -\left.\epsilon_r\frac{\partial V_i}{\partial r}\right|_{r=b} \longrightarrow E_0 + B_1 b^{-2} = -\epsilon_r A_1; nB_n b^{-(n+1)} = -\epsilon_r n A_n b^{n-1}, n \neq 1.$$

Solving:

$$A_1 = -\frac{2E_0}{\epsilon_r + 1}, B_1 = \frac{\varepsilon_r - 1}{\epsilon_r + 1} b^2 E_0$$

$$A_n = B_n = 0 \text{for } n \neq 1$$

$$V_0(r,\phi) = -\left(1 - \frac{\epsilon_r - 1}{\epsilon_r + 1}\frac{b^2}{r^2}\right) E_0 r \cos\phi$$

$$V_i(r,\phi) = -\frac{2}{\epsilon_r + 1} E_0 r \cos\phi$$

$$\bar{E} = -\bar{\nabla} V = -\bar{a}_r \frac{\partial V}{\partial r} - \bar{a}_\phi \frac{\partial V}{r \partial \phi}$$

$$\bar{E}_0 = \bar{a}_r E_0 \left(1 + \frac{\epsilon_r - 1}{\epsilon_r + 1}\frac{b^2}{r^2}\right) \cos\phi - \bar{a}_\phi E_0 \left(1 - \frac{\epsilon_r - 1}{\epsilon_r + 1}\frac{b^2}{r^2}\right) \sin\phi$$

$$\bar{E}_i = \frac{2}{\epsilon_r + 1} E_0 \left(\bar{a}_r \cos\phi - \bar{a}_\phi \sin\phi\right)$$

Exercise 4.27

An infinite conducting cone of half-angle α is maintained at potential V_0 and insulated from a grounded conducting plane, as illustrated in Fig. 4-26. Determine
a) the potential distribution $V(\theta)$ in the region $\alpha < \theta < \pi/2$,
b) the electric field intensity in the region $\alpha < \theta < \pi/2$,
c) the charge densities on the cone surface and on the grounded plane.

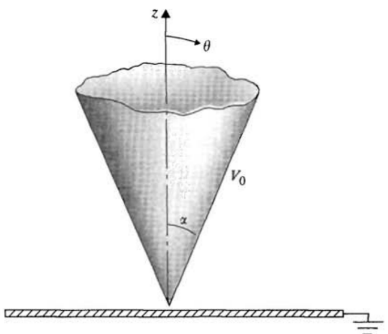


FIGURE 4–26
An infinite conducting cone and a grounded conducting plane (Problem P.4–27).

Answer:
V and $\bar{E}$ depend only on θ. $\rightarrow E_q(4-9) : \frac{d}{d\theta}\left(\sin\theta \frac{dV}{d\theta}\right) = 0$

$$\text{a) Solution: } \frac{dV}{d\theta} = \frac{C_1}{\sin\theta} \longrightarrow V(\theta) = C_1 \ln\left(\tan\frac{\theta}{2}\right) + C_2$$

$$(1) V(\alpha) = V_0 = C_1 \ln\left(\tan\frac{\theta}{2}\right) + C_2$$

$$(2) V\left(\frac{\pi}{2}\right) = 0 = C_1 \ln\left(\tan\frac{\pi}{4}\right) + C_2 \longrightarrow C_2 = 0$$

$$C_1 = \frac{V_0}{\ln[\tan(\alpha/2)]} \longrightarrow V(\theta) = \frac{V_0 \ln[\tan(\theta/2)]}{\ln[\tan(\alpha/2)]}$$

$$\text{b) } \bar{E} = -\overline{a_\theta}\frac{dV}{R d\theta} = -\bar{a}_\theta \frac{V_0}{R \ln[\tan(\alpha/2)] \sin\theta}$$

c) On the cone: $\theta=\alpha, \rho_s=\epsilon_0 E(\alpha)=-\dfrac{t_0 V_0}{R\,\mathrm{sn}[\tan(\alpha/2)]\sin\theta}$

On the grounded plane:

$$\theta=\pi/2, \rho_s=-\epsilon_0 E\left(\frac{\pi}{2}\right)=\frac{\epsilon_0 V_0}{R\ln[\tan(\alpha/2)]}$$

Exercise 4.28

Rework Example $4-10$, assuming that $V(b,\theta)=V_0$ in Eq. (4-155a) $V(b,\theta)=0^{\dagger}$

EXAMPLE 4-10 An uncharged conducting sphere of radius b is placed in an initially uniform electric field $\mathbf{E}_0=\mathbf{a}_z E_0$. Determine (a) the potential distribution $V(R,\theta)$, and (b) the electric field intensity $\mathbf{E}(R,\theta)$ after the introduction of the sphere.

Answer:

Starting from Eq. $(4-157)$ and applying the b.c. $V(b,\theta)=V_0$:

$$V_0=\frac{B_0}{b}+\left(\frac{B_1}{b^2}-E_0 b\right)\cos\theta-\sum_{n=2}^{\infty}B_n b^{-(n+1)}P_n(\cos\theta), \quad R\geq b.$$

$\longrightarrow B_0=bV_0, \quad B_1=E_0 b^3, \quad B_n=0$ for $n\geqslant 2$. $\therefore V(R,\theta)=\frac{b}{R}V_0-E_0\left[1-\left(\frac{b}{R}\right)^3\right]R\cos\theta, \quad R\geqslant b.$

$$\bar{E}(R,\theta)=\bar{a}_R\left\{\frac{bV_0}{R^2}+E_0\left[1+2\left(\frac{b}{R}\right)^3\right]\cos\theta\right\}-\bar{a}_\theta E_0\left[1-\left(\frac{b}{R}\right)^3\right]R\sin\theta, R\geq b.$$

$$\rho_s=\epsilon_0 E_R|_{R=b}=\epsilon_0\frac{V_0}{b}+3\epsilon_0 E_0\cos\theta.$$

Exercise 4.29

A dielectric sphere of radius b and dielectric constant ϵ_r is placed in an initially uniform electric field, $\mathbf{E}_0=\mathbf{a}_z E_0$, in air. Determine $V(R,\theta)$ and $\mathbf{E}(R,\theta)$ both inside and outside the dielectric sphere.

Answer:

$R\leqslant b: V_i(R,\theta)=\sum_{n=0}^{\infty}A_n R^n P_n(\cos\theta); R\geqslant b, V_0(R,\theta)=\sum_{n=0}^{\infty}\left(B_n R^n+C_n R^{-(n+1)}\right)P_n(\cos\theta).$

For $R\gg b, V_0(R,\theta)=-E_0 Z=-E_0 R\cos\theta\longrightarrow B_1=-E_0; B_n=c_n=0$ for $n\neq 1$.

$\therefore V_0(R,\theta)=-E_0 R\cos\theta+C_1 R^{-2}\cos\theta.$

B.C. (1) $V_i(b,\theta)=V_0(b,\theta)\longrightarrow A_1 b=-E_0 b+C,b^{-2}\}\ A_1=-\dfrac{3E_0}{\epsilon_r+2},$

(2) $\epsilon_r\left.\dfrac{\partial V_i}{\partial R}\right|_{R=b}=\left.\dfrac{\partial V_0}{\partial R}\right|_{k=b}\longrightarrow\epsilon_r A_1=-E_0-2c_1 b^{-3}\}\ c_1=\dfrac{\epsilon_r-1}{\epsilon_r+2}E_0 b^3.$

$V_i(R,\theta)=-\dfrac{3E_0}{\epsilon_r+2}R\cos\theta, \quad V_0(R,\theta)=-E_0 R\cos\theta+\dfrac{(\epsilon_r-1)\,b^3}{(\epsilon_r+2)\,R^2}E_0\cos\theta.$

$\bar{E}_i(R,\theta)=-\bar{\nabla}V_i=\dfrac{3E_0}{\epsilon_r+2}\left(\bar{a}_R\cos\theta-\bar{a}_\theta\sin\theta\right)=\bar{a}_z\dfrac{3\epsilon_0}{\epsilon_r+2}.$

$\bar{E}_0(R,\theta)=-\bar{\nabla}V_0=\bar{a}_R\left[1+\dfrac{2\,(\epsilon_r-1)\,b^3}{(\epsilon_r+2)\,R^3}\right]E_0\cos\theta-\bar{a}_\theta\left[1-\dfrac{(\epsilon_r-1)\,b^3}{(\epsilon_r+2)\,R^3}\right]E_0\sin\theta.$

Reference

1. Cheng, David Keun. Field and wave electromagnetics. Pearson Education India, 1989.