# Chapter 6 Static Magnetic Fields

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Sung-Liang Chen

# 6-1 Introduction

- Electric charges at rest: $\nabla \cdot \mathbf{D} = \rho,$ $\nabla \times \mathbf{E} = 0.$
  - Constitutive relation: $\mathbf{D} = \epsilon \mathbf{E},$
  - Electric force $\mathbf{F}_e$: $\boxed{\mathbf{F}_e = q\mathbf{E} \qquad (\mathrm{N}).}$
- Electric charges in motion:
  - Magnetic force $\mathbf{F}_m$: $\boxed{\mathbf{F}_m = q\mathbf{u} \times \mathbf{B} \qquad (\mathrm{N}),}$

The force was found in experiments!
Defined as **B**: magnetic flux density (Wb/m$^2$ = Tesla)

- Lorentz's force equation: $\boxed{\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \qquad (\mathrm{N}),}$

# 6-2 Fundamental Postulates of Magnetostatics in Free Space

- Two postulates for **B** in free space:

$$\boxed{\nabla \cdot \mathbf{B} = 0,}$$

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.}$$

$\mu_0$ = $4\times10^{7}$ (H/m), permeability of free space

**J**: current density

Permeability is the measure of the ability of a material to support the formation of a magnetic field within itself. Hence, it is the degree of **magnetization** that a material obtains in response to an applied magnetic field.

Chap. 3

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}}$$

$$\boxed{\nabla \times \mathbf{E} = 0.}$$

$$\nabla \cdot (\nabla \times \mathbf{B}) = 0$$

$$\Downarrow$$

$$\nabla \cdot \mathbf{J} = 0, \qquad -\frac{\partial \rho}{\partial t}$$

# No Magnetic Charge

- Comparison of divergence **E** and **B**:

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \qquad\qquad \nabla \cdot \mathbf{B} = 0,$$

$\Rightarrow$ No magnetic charge

# The Law of Conservation of Magnetic Flux

- The integral form of **E**: $\boxed{\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0},}$

  ***Q* is the source** of the total outward electric flux through any closed surface.

$\nabla \cdot \vec{B} = 0$

- The integral form of **B**: $\boxed{\oint_S \mathbf{B} \cdot d\mathbf{s} = 0,}$

  The total outward magnetic flux through any closed surface is zero.

$\int_v \nabla \cdot \vec{B} \, dv = 0$

$\oint \vec{B} \cdot d\vec{S} = 0$

  - **No magnetic flow sources**
  - The magnetic flux lines always close upon themselves

# The Law of Conservation of Magnetic Flux

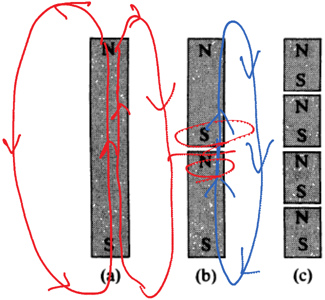
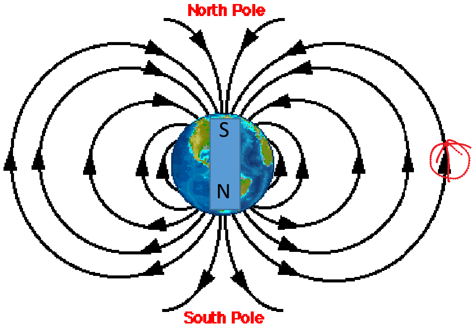



**FIGURE 6–1**
Successive division of a bar magnet.

The magnetic flux lines follow closed paths from one end of a magnet to the other end outside the magnet and then continue inside the magnet back to the first end.

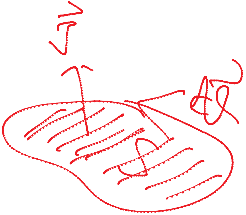
# The Earth's Magnetic Field



# Ampere's Circuital Law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{s}$$



$$\boxed{\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I,}$$

*C*: the contour bounding the surface S

*I*: the total current through S

Ampere's circuital law: the circulation of the magnetic flux density in free space around any closed path is equal to $\mu_0$ times the total current flowing through the surface bounded by the path.

Chap.3

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}}$$

Gauss's law

# A Summary

| Postulates of Magnetostatics in Free Space | |
|---|---|
| Differential Form | Integral Form |
| $\nabla \cdot \mathbf{B} = 0$ | $\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$ |
| $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ | $\oint_C \mathbf{B} \cdot d\ell = \mu_0 I$ |

$\nabla \cdot \vec{D} = \rho$

$\nabla \times \vec{E} = 0$

# 6-3 Vector Magnetic Potential

Chap.3

$$\nabla \cdot \mathbf{B} = 0$$

**B** is solenoidal
By null identity

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A} \quad \text{(T).}}$$

where **A**: **vector** magnetic potential (Wb/m)

$$\nabla \times \mathbf{E} = 0$$

$$\boxed{\mathbf{E} = -\nabla V}$$

In magnetostatics: **J** ➔ **A** ➔ **B**
In electrostatics: ρ ➔ V ➔ **E**

# $\nabla \cdot A = ?$

- To specify a vector, we should specify its curl and divergence.
  We have $\boxed{\mathbf{B} = \nabla \times \mathbf{A} \qquad \text{(T).}}$ How to choose $\nabla \cdot \mathbf{A}$?

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}}$$

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

$$\nabla \times \nabla \times \mathbf{A} = \mu_0 \mathbf{J}.$$

$\nabla^2 V =$

Vector identity:

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}.$$

Definition of Laplacian of **A**

$\nabla^2(A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z)$

$=$

- In Cartesian: $\nabla^2 \mathbf{A} = \mathbf{a}_x \nabla^2 A_x + \mathbf{a}_y \nabla^2 A_y + \mathbf{a}_z \nabla^2 A_z.$
  (Similar to Laplacian of $V$)

- In other coordinates: should use the definition of Laplacian of **A**

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}.$$

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} = \mu_0\mathbf{J}.$$

**To simplify the equation, we choose**

$$\boxed{\nabla \cdot \mathbf{A} = 0,}$$

Vector's Poisson's equation

$$\boxed{\nabla^2\mathbf{A} = -\mu_0\mathbf{J}.}$$

In Cartesian coordinate:

$$\nabla^2 A_x = -\mu_0 J_x,$$
$$\nabla^2 A_y = -\mu_0 J_y,$$
$$\nabla^2 A_z = -\mu_0 J_z.$$

By comparison

Solution:

$$A_x = \frac{\mu_0}{4\pi}\int_{V'} \frac{J_x}{R}\,dv'.$$

Combine 3 components

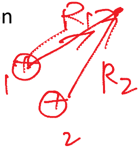
$A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$

$$\boxed{\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\mathbf{J}}{R}\,dv' \qquad \text{(Wb/m).}}$$

## Analogy in electrostatics: √

Scalar's Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$




Solution:

$$V = \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{\rho}{R}\,dv'.$$

V(space)

R(r')

# Review the Analogy

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J}}{R}\, dv' \qquad \text{(Wb/m).}$$

In magnetostatics: **J** ➔ **A** ➔ **B**

In electrostatics: ρ ➔ V ➔ **E**

$$V = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho}{R}\, dv'.$$

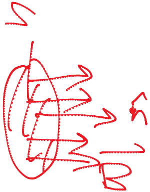
# Relation of Magnetic Flux $\Phi$ and Magnetic Vector Potential A

Wb/m²

$$\Phi = \int_S \mathbf{B}\cdot d\mathbf{s}$$

Magnetic flux

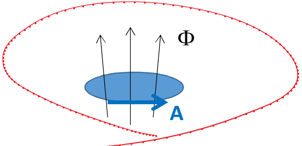
Magnetic flux density




$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

$$\Phi = \int_S (\nabla \times \mathbf{A})\cdot d\mathbf{s} = \oint_C \mathbf{A}\cdot d\boldsymbol{\ell} \qquad \text{(Wb)}.$$

Physical significance of **A**: line integral of **A** around any closed path = the total $\Phi$ passing through the area enclosed by the path

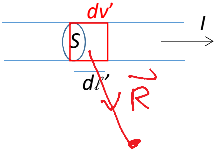


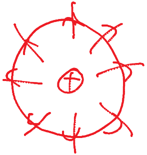
# 6-4 The Biot-Savart Law and Applications

- The magnetic field due to a current-carrying circuit.
- For a thin wire with cross-sectional area *S* (*dv*'=*Sdℓ*'=***S* • *dℓ*'**), we have

$$\mathbf{J}\,dv' = JS\,d\ell' = I\,d\ell',$$

J: (A/m$^2$)
*JS* = *I*



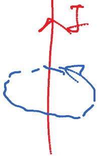



$$\boxed{\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'}\frac{\mathbf{J}}{R}\,dv' \quad \text{(Wb/m).}}$$

$$\boxed{\mathbf{A} = \frac{\mu_0 I}{4\pi}\oint_{C'}\frac{d\boldsymbol{\ell}'}{R(d\ell')} \quad \text{(Wb/m),}}$$

*C*' is closed because current must flow in a closed path.

$\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0 I$



$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

$$\boxed{\mathbf{A} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\boldsymbol{\ell}'}{R} \qquad \text{(Wb/m)},}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times \left[ \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\boldsymbol{\ell}'}{R} \right]$$

$$= \frac{\mu_0 I}{4\pi} \oint_{C'} \nabla \times \left( \frac{d\boldsymbol{\ell}'}{R} \right).$$

source: $\vec{J}$, $I$

fields: $\vec{A}$, $\vec{B}$

$$\frac{\partial}{\partial x} x' = 0$$

**unprimed** curl: to the field coordinate
**primed** integration: to the source coordinate

$$\nabla \times (f\mathbf{G}) = f\nabla \times \mathbf{G} + (\nabla f) \times \mathbf{G}.$$

with $f = 1/R$ and $\mathbf{G} = d\boldsymbol{\ell}'$,

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \left[ \underbrace{\frac{1}{R} \nabla \times d\boldsymbol{\ell}'}_{(1)} + \underbrace{\left( \nabla \frac{1}{R} \right) \times d\boldsymbol{\ell}'}_{(2)} \right].$$

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\oint_{C'}\left[\underbrace{\frac{1}{R}\nabla \times d\ell'}_{(1)} + \underbrace{\left(\nabla\frac{1}{R}\right)\times d\ell'}_{(2)}\right].$$

For term (1): primed and unprimed coordinates are independent ➔ 0

For term (2):

$$\frac{1}{R} = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{-1/2};$$

R is from source to field

$$\nabla\left(\frac{1}{R}\right) = \mathbf{a}_x\frac{\partial}{\partial x}\left(\frac{1}{R}\right) + \mathbf{a}_y\frac{\partial}{\partial y}\left(\frac{1}{R}\right) + \mathbf{a}_z\frac{\partial}{\partial z}\left(\frac{1}{R}\right)$$

$$= -\frac{\mathbf{a}_x(x - x') + \mathbf{a}_y(y - y') + \mathbf{a}_z(z - z')}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}}$$

$$= -\frac{\mathbf{R}}{R^3} = -\mathbf{a}_R\frac{1}{R^2},$$

Quotient rule and chain rule

$\mathbf{a}_R$: unit vector from source to field

$\nabla'\left(\frac{1}{R}\right)$

$= a_R \frac{1}{R^2}$

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \left[ \cancel{\frac{1}{R} \nabla \times d\boldsymbol{\ell}'} + \underline{\left( \nabla \frac{1}{R} \right)} \times d\boldsymbol{\ell}' \right].$$

$$\nabla \left( \frac{1}{R} \right) = -\mathbf{a}_R \frac{1}{R^2},$$

$$\boxed{\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\boldsymbol{\ell}' \times \mathbf{a}_R}{R^2} \qquad \text{(T).}}$$

Biot-Savart law: **B** due to a current element *I*d*ℓ'*

$P \to V \to \vec{E}$

B-S law

$\vec{J} \to \vec{A} \to \vec{B}$

$I$

Comparison with Ampere's circuital law: $\boxed{\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I,}$

Because **B** does not have to be constant along *C*, **Biot-Savart law is more general** to determine B than Ampere's circuital law (although the former is more difficult in calculation).

$V = \frac{Q}{4\pi\epsilon_0 R} \rightarrow V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho}{R} dv' \rightarrow \vec{J} dv' \rightarrow I d\vec{l}'$

$\vec{A} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}'}{R}$

**EXAMPLE 6–4** A direct current $I$ flows in a straight wire of length $2L$. Find the magnetic flux density $\mathbf{B}$ at a point located at a distance $r$ from the wire in the bisecting plane: (a) by determining the vector magnetic potential $\mathbf{A}$ first, and (b) by applying Biot-Savart law.

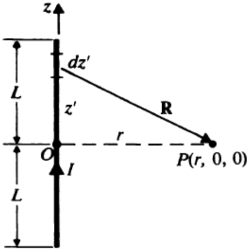
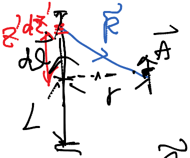



**FIGURE 6–5**
A current-carrying straight wire (Example 6–4).

Method I: $I d\vec{l}' \rightarrow \vec{A} \rightarrow \vec{B}$

Method 2: $I d\vec{l}' \xrightarrow{\text{B-S law}} \vec{B}$

(a) $\vec{A} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}'}{R}$




$d\vec{l}' = \hat{a}_z\, dz'$

$R = \sqrt{z'^2 + r^2}$

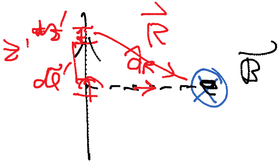
$\vec{A} = \frac{\mu_0 I}{4\pi}\int \frac{\hat{a}_z\, dz'}{\sqrt{z'^2+r^2}} = \cdots\cdots = \hat{a}_z \underbrace{\frac{\mu_0 I}{4\pi}\ln\left(\frac{\sqrt{L^2+r^2}+L}{\sqrt{L^2+r^2}-L}\right)}_{A_z}$

$\vec{B} = \nabla\times\vec{A} = \hat{a}_r \cancel{\frac{\partial A_z}{\partial\phi}}^{\,0} - r\hat{a}_\phi \frac{\partial A_z}{\partial r} = \cdots\cdots = \hat{a}_\phi \frac{\mu_0 I L}{2\pi r\sqrt{L^2+r^2}}$

$\nabla\times\vec{A} = \frac{1}{r}\begin{vmatrix} \hat{a}_r & r\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ 0 & 0 & A_z \end{vmatrix}$

(b)

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int_{C'} \frac{d\vec{l}' \times \hat{a}_R}{R^2} \Rightarrow \vec{B} \perp d\vec{l}', \quad \vec{B} \perp \hat{a}_R$$




$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho}{R^2}\, dv'\, \hat{a}_R$$

$$\downarrow$$

$$\vec{J}\, dv' \times \hat{a}_R$$

$$\downarrow$$

$$I\, d\vec{l}' \times \hat{a}_R$$

$$\hat{a}_R = \frac{\vec{R}}{R} \qquad \vec{R} = (\hat{a}_r r - \hat{a}_z z')$$

$$\frac{\hat{a}_R}{R^2} = \frac{\vec{R}}{R^3}$$

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int_{-L}^{L} \frac{\hat{a}_z\, dz' \times (\hat{a}_r r - \hat{a}_z z')}{(z'^2 + r^2)^{3/2}}$$

# 6-5 The Magnetic Dipole

- Example 6-7:

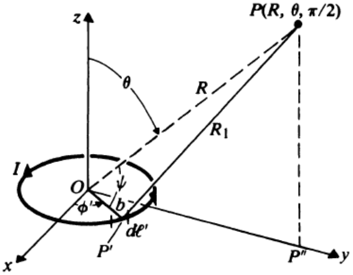



**FIGURE 6–8**
A small circular loop carrying current $I$ (Example 6–7).

$$\mathbf{A} = \mathbf{a}_\phi \frac{\mu_0 I b^2}{4R^2} \sin\theta.$$

$$\mathbf{B} = \frac{\mu_0 I b^2}{4R^3} (\mathbf{a}_R\, 2\cos\theta + \mathbf{a}_\theta \sin\theta),$$

$V = \ldots$

$\vec{E} = \ldots$

**EXAMPLE 6–7** Find the magnetic flux density at a distant point of a small circular loop of radius $b$ that carries current $I$ (a ***magnetic dipole***).

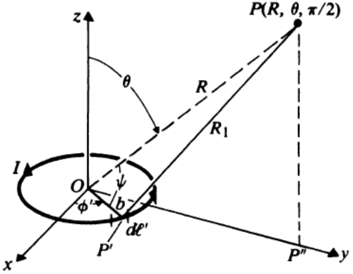
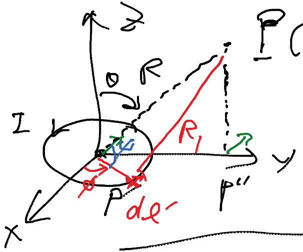



**FIGURE 6–8**
A small circular loop carrying current $I$ (Example 6–7).




$\rightarrow \vec{A} \rightarrow \vec{B}$

1° $\boxed{\vec{A} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}'}{R}}$

$d\vec{l}' = \hat{a}_\phi' dl' = (-\hat{a}_x' \sin\phi' + \hat{a}_y \cos\phi')\, b\, d\phi'$

$\vec{A} = \frac{\mu_0 I}{4\pi} \cdot 2 \int_{\pi/2}^{\pi/2} \frac{-\hat{a}_x' \sin\phi'}{R_1}\, b\, d\phi'$

$= \hat{a}_\phi \frac{\mu_0 I}{4\pi}\, 2 \int_{-\pi/2}^{\pi/2} \frac{b \sin\phi'}{R_1}\, d\phi'$ —— ①

$-\hat{a}_x' \Rightarrow \hat{a}_\phi$ general

Integrand:

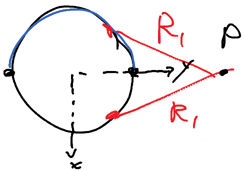
$\frac{\cos\phi'}{R_1}$ :

| | |
|---|---|
| − | − |
| + | + |

$\frac{\sin\phi'}{R_1}$ :

| | |
|---|---|
| − | + |
| − | + |

top view



$2^\circ$ $\frac{1}{R_1}$ : $R_1^2 = R^2 + b^2 - 2Rb\cos\psi$

$R\cos\psi$ = projection of $\overline{OP}$ on $\overline{OP'}$

= (projection of $\overline{OP}$ on $\overline{OP''}$) and

( " of $\overline{OP''}$ on $\overline{OP'}$ )

$= R\sin\theta\cos(\frac{\pi}{2} - \phi') = R\sin\theta\sin\phi'$

$\Rightarrow \cdots \Rightarrow \frac{1}{R_1} = \frac{1}{R}\left(1 + \frac{b^2}{R^2} - \frac{2b}{R}\sin\theta\sin\phi'\right)^{-1/2}$

when $R \gg b$ $\frac{1}{R_1} \approx \frac{1}{R}\left(1 - \frac{2b}{R}\sin\theta\sin\phi'\right)^{-1/2}$

when $R \gg b$ $\approx \frac{1}{R}\left(1 + \frac{b}{R}\sin\theta\sin\phi'\right)$ —— ②

② in ① $\Rightarrow \vec{A} = \hat{a}_\phi \frac{\mu_0 I b}{2\pi} \int_{-\pi/2}^{\pi/2} (1 + \frac{b}{R} \sin\theta \sin\phi') \sin\phi' d\phi' =$

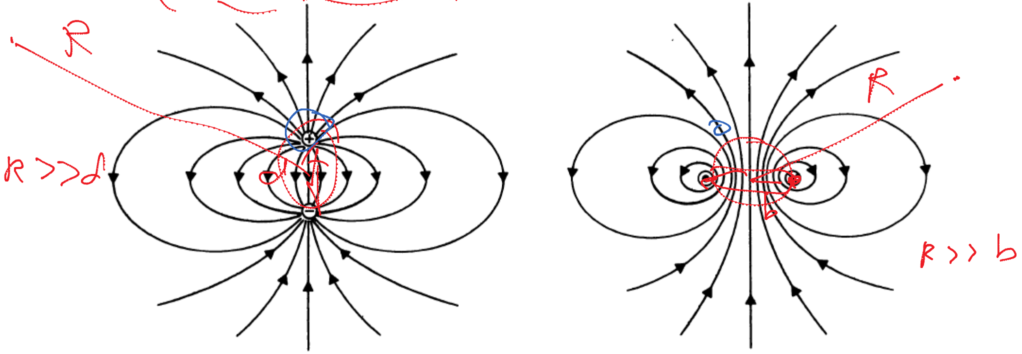
$\ldots$

$= \hat{a}_\phi \frac{\mu_0 I b^2}{4R^2} \sin\theta$

3° $\vec{B} = \nabla \times \vec{A} = \ldots \ldots = \frac{\mu_0 I b^2}{4R^3} (\hat{a}_R 2\cos\theta + \hat{a}_\theta \sin\theta)$

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 R^3}(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta \sin\theta).$$

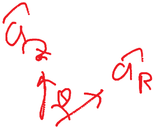
$$\mathbf{B} = \frac{\mu_0 I b^2}{4R^3}(\mathbf{a}_R\, 2\cos\theta + \mathbf{a}_\theta \sin\theta),$$




(a) Electric dipole. (b) Magnetic dipole.

**FIGURE 6–9**
Electric field lines of an electric dipole and magnetic flux lines of a magnetic dipole.

- Difference: **E** terminated on the charges; **B** continuous
- Fields (**E** and **B**) are similar at far fields




$$\boxed{\mathbf{A} = \mathbf{a}_\phi \frac{\mu_0 I b^2}{4R^2} \sin\theta.}$$

$$\mathbf{A} = \mathbf{a}_\phi \frac{\mu_0 (I\pi b^2)}{4\pi R^2} \sin\theta$$

$$\boxed{\mathbf{a}_z \times \mathbf{a}_R = \mathbf{a}_\phi \sin\theta}$$

$$\boxed{\mathbf{A} = \frac{\mu_0 \mathbf{m} \times \mathbf{a}_R}{4\pi R^2} \quad \text{(Wb/m)},}$$

where $\mathbf{m} = \mathbf{a}_z I\pi b^2 = \mathbf{a}_z IS = \mathbf{a}_z m \quad (\text{A}\cdot\text{m}^2)$

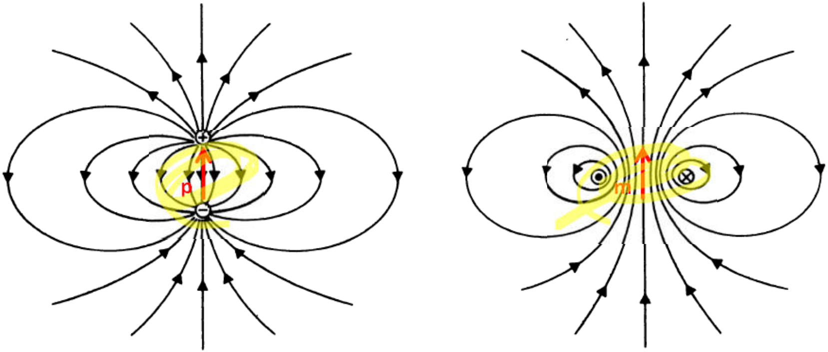
Defined as **magnetic dipole moment**

$$\boxed{\mathbf{B} = \frac{\mu_0 I b^2}{4R^3}(\mathbf{a}_R\, 2\cos\theta + \mathbf{a}_\theta \sin\theta),}$$

$$\boxed{\mathbf{B} = \frac{\mu_0 m}{4\pi R^3}(\mathbf{a}_R\, 2\cos\theta + \mathbf{a}_\theta \sin\theta) \quad \text{(T).}}$$

**p=qd**

Electric dipole moment




(a) Electric dipole. (b) Magnetic dipole.

Electric dipole moment **p**

Magnetic dipole moment **m**

**p** -> **m**
$1/\varepsilon_0$ -> $\mu_0$
• -> ×

Right-hand rule: **m** along thumb and **direction of current** along fingers

$$V = \frac{\mathbf{p} \cdot \mathbf{a}_R}{4\pi\epsilon_0 R^2} \quad \text{(V)},$$

$$\mathbf{A} = \frac{\mu_0 \mathbf{m} \times \mathbf{a}_R}{4\pi R^2} \quad \text{(Wb/m)},$$

We call a small current-carrying loop a **magnetic dipole**

# 6-5.1 Scalar Magnetic Potential

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

$\Downarrow$ **For a region J = 0**

$$\nabla \times \mathbf{B} = 0.$$

$\Downarrow$ By null identity

$$\mathbf{B} = -\mu_0 \nabla V_m,$$

$V_m$: scalar magnetic potential

$\nabla \times \vec{E} = 0$

$\Downarrow$

$$\boxed{\mathbf{E} = -\nabla V}$$

| Analogy | |
|---|---|
| $V_m$ | $V$ |
| $\mathbf{B}/\mu_0$ | $\mathbf{E}$ |

Analogous to electric potential

$$\boxed{\mathbf{E} = -\nabla V}$$

$$\mathbf{B} = -\mu_0 \nabla V_m,$$

$$V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\ell \quad \text{(V)}.$$

$$V_{m2} - V_{m1} = -\int_{P_1}^{P_2} \frac{1}{\mu_0}\mathbf{B} \cdot d\ell.$$

**E**

$\mathbf{B}/\mu_0$

$k = 1/(4\pi\varepsilon_0)$

$k = (\mu_0/(4\pi))\ /\mu_0$

$$V = \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{\rho}{R}\, dv' \quad \text{(V)}.$$

$$V_m = \frac{1}{4\pi}\int_{V'} \frac{\rho_m}{R}\, dv' \quad \text{(A)}.$$

No $\mu_0$ in $V_m$

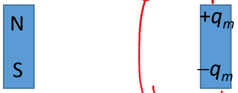
If there ***were*** magnetic charges

Fictitious magnetic charges: $\rho_m$

1. A mathematical (not physical) model
2. Helpful in discussion of magnetostatics from electrostatics

# Magnetic Source

- Macroscopic (traditional) view: magnetic pole (a magnetic dipole)




Fictitious magnetic charges

$$\mathbf{m} = q_m \mathbf{d} = \mathbf{a}_n I S.$$

- Microscopic view: circulating atomic current

# Potential due to a dipole

Electric dipole

$+q$

$-q$

See 3-5.1

$\vec{p} \to \vec{m}$

$\frac{1}{\epsilon_0} \to \mu_0$ $\quad \vec{A} \to \vec{B}$

$\to \times$

$$V = \frac{\mathbf{p} \cdot \mathbf{a}_R}{4\pi\epsilon_0 R^2} \qquad \text{(V)},$$

$\to V_m \to \vec{B} = -\mu_0 \nabla V_m$

Magnetic dipole

$+q_m$

$-q_m$

$$V_m = \frac{\mathbf{m} \cdot \mathbf{a}_R}{4\pi R^2} \qquad \text{(A)}.$$

$\boxed{\text{No } \mu_0 \text{ in } V_m}$

$$\boxed{\mathbf{B} = -\mu_0 \nabla V_m,}$$

$$\boxed{\mathbf{B} = \frac{\mu_0 m}{4\pi R^3}(\mathbf{a}_R\, 2\cos\theta + \mathbf{a}_\theta \sin\theta) \qquad \text{(T)}.}$$

Same as in Example 6-7

Electric potential

$$V = \frac{\mathbf{p} \cdot \mathbf{a}_R}{4\pi\epsilon_0 R^2} \qquad (\mathrm{V}),$$

$$\nabla \times \mathbf{E} = 0.$$

J = 0

**Fictitious magnetic charge** and magnetic scalar potential

$$V_m = \frac{\mathbf{m} \cdot \mathbf{a}_R}{4\pi R^2} \qquad (\mathrm{A}).$$

$$\nabla \times \mathbf{B} = 0.$$

$V_m$ holds at any points **with no currents**

**B** is conservative

J ≠ 0

**Circulating current** and magnetic vector potential

$$\mathbf{A} = \frac{\mu_0 \mathbf{m} \times \mathbf{a}_R}{4\pi R^2} \qquad (\mathrm{Wb/m}),$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

**B** is nonconservative

# 6-6 Magnetization and Equivalent Current Densities

- Microscopic viewpoint: orbiting electrons ➔ atomic currents ➔ magnetic dipoles **m**
- Without external **B**: random orientation of magnetic dipoles ➔ no net magnetic dipole moment, ∑**m** = 0
- With external **B**: alignment of magnetic dipoles ➔ induced magnetic dipole moment **m** ≠ 0

$\frac{\sum \vec{p}}{\Delta V}$

$\vec{E} \to \vec{P}$

$\frac{\sum m}{\Delta V}$

$\vec{B} \to \vec{M}$

Let $\mathbf{m}_k$: magnetic dipole moment of an atom

Define magnetization vector **M**

$N$ - #

| |
|---|
| $n$: number density |
| $n\Delta v = N$: total # |

$$\mathbf{M} \overset{D}{=} \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\,\Delta v} \mathbf{m}_k}{\Delta v} \qquad \text{(A/m)},$$

**M**: density of total magnetic dipoles

$$d\mathbf{m} = \mathbf{M}\, dv'$$

$$\boxed{\mathbf{A} = \frac{\mu_0 \mathbf{m} \times \mathbf{a}_R}{4\pi R^2} \qquad \text{(Wb/m)},}$$

$$\boxed{d\mathbf{m} = \mathbf{M}\, dv'}$$

$$d\mathbf{A} = \frac{\mu_0 \mathbf{M} \times \mathbf{a}_R}{4\pi R^2}\, dv'.$$

Recall polarization vector **P**

$\vec{P} =$

$d\vec{p} = \vec{P}\, dv'$

$$V = \frac{\mathbf{p} \cdot \mathbf{a}_R}{4\pi\epsilon_0 R^2} \qquad \text{(V)},$$

$$\nabla'\left(\frac{1}{R}\right) = \frac{\mathbf{a}_R}{R^2}.$$

**R**: source to field

$$dA = \frac{\mu_0}{4\pi}\mathbf{M} \times \nabla'\left(\frac{1}{R}\right) dv'.$$

$$\mathbf{A} = \int_{V'} d\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \mathbf{M} \times \nabla'\left(\frac{1}{R}\right) dv',$$

By using the vector identity

$$\mathbf{M} \times \nabla'\left(\frac{1}{R}\right) = \frac{1}{R}\nabla' \times \mathbf{M} - \nabla' \times \left(\frac{\mathbf{M}}{R}\right)$$

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\nabla' \times \mathbf{M}}{R} dv' - \frac{\mu_0}{4\pi}\int_{V'} \nabla' \times \left(\frac{\mathbf{M}}{R}\right) dv'.$$

By using the vector identity (Prob. 6-20)

$$\int_{V'} \nabla' \times \mathbf{F}\, dv' = -\oint_{S'} \mathbf{F} \times d\mathbf{s}',$$

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\nabla' \times \mathbf{M}}{R} dv' + \frac{\mu_0}{4\pi}\oint_{S'} \frac{\mathbf{M} \times \mathbf{a}'_n}{R} ds',$$

See section 3-7.1

$$V = \frac{1}{4\pi\epsilon_0}\int_{V'} \mathbf{P} \cdot \nabla'\left(\frac{1}{R}\right) dv'.$$

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\nabla' \times \mathbf{M}}{R}\, dv' + \frac{\mu_0}{4\pi} \oint_{S'} \frac{\mathbf{M} \times \mathbf{a}'_n}{R}\, ds',$$

$A/m^2 \cdot m^3$

$\downarrow$

$A \cdot m$

$A/m \cdot m^2$

$\downarrow$

$A \cdot m$

Comparisons with

$$\boxed{\mathbf{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J}}{R}\, dv'}$$

$$\boxed{\mathbf{J}_m = \nabla \times \mathbf{M} \qquad (\text{A/m}^2)}$$

$$\boxed{\mathbf{J}_{ms} = \mathbf{M} \times \mathbf{a}_n \qquad (\text{A/m}).}$$

Prime is omitted for simplicity

For a given **M**

➔ Find equivalent magnetization current densities $\mathbf{J}_m$ and $\mathbf{J}_{ms}$

➔ **A**

➔ **B** = ∇×**A**

$\vec{M} \rightarrow \vec{J}_m, \vec{J}_{ms} \longrightarrow \vec{A} \rightarrow \vec{B}$

See section 3-7.1

$$\boxed{\rho_p = -\nabla \cdot \mathbf{P}.}$$

$$\boxed{\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n}$$

## Magnetization **M**

homogeneous

$J_m = \nabla \times \vec{M} = 0$

$J_{ms} = \vec{M} \times \hat{a}_n$

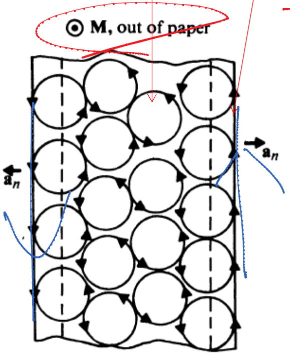



**FIGURE 6–10**
A cross section of a magnetized material.

## Polarization **P**

$\rho_p$

$\rho_{ps}$

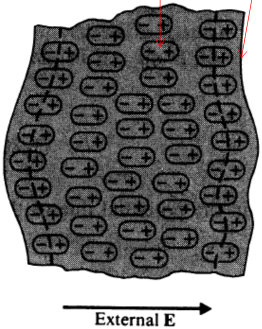
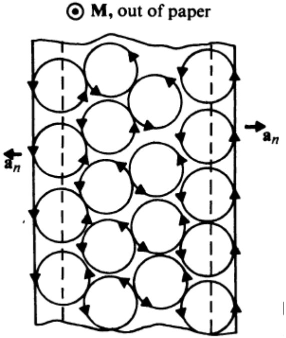



**FIGURE 3–20**
A cross section of a polarized dielectric medium.




$$\boxed{\mathbf{J}_m = \nabla \times \mathbf{M} \qquad (\mathrm{A/m^2})}$$

$$\boxed{\mathbf{J}_{ms} = \mathbf{M} \times \mathbf{a}_n \qquad (\mathrm{A/m}).}$$

- Check directions of $\mathbf{J}_{ms}$
- If **M** is uniform inside ➔ currents inside cancel each other ➔ $\mathbf{J}_m = 0$ (i.e., $\nabla\times$(constant) = 0 )

# 6-6.1 Equivalent Magnetization Charge Densities

- In a **current-free** region, we may define $V_m$
- **B** can be found by $\mathbf{B} = -\mu_0 \nabla V_m,$

$$\boxed{V_m = \frac{\mathbf{m} \cdot \mathbf{a}_R}{4\pi R^2} \qquad \text{(A).}}$$

$\vec{A} =$ $\vec{m}$

$\downarrow$

$\boxed{d\mathbf{m} = \mathbf{M}\, dv'}$

$d\vec{m} =$

$$dV_m = \frac{\mathbf{M} \cdot \mathbf{a}_R}{4\pi R^2}\, dv'$$

$\vec{J}_m = \nabla \times \vec{M}$

$\vec{J}_{ms} = \vec{M} \times \hat{a}_n$

$\downarrow$

$\rho_m$

$\rho_{ms}$

integration

$$V_m = \frac{1}{4\pi}\int_{V'} \frac{\mathbf{M}\cdot\mathbf{a}_R}{R^2}\,dv'.$$

$$V = \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{\mathbf{P}\cdot\mathbf{a}_R}{R^2}\,dv',$$

Following similar steps
as in section 3-7.1

$$V_m = \frac{1}{4\pi}\oint_{S'} \frac{\overset{A/m}{\mathbf{M}\cdot\mathbf{a}_n'}}{R}\,ds' + \frac{1}{4\pi}\int_{V'} \frac{\overset{A/m^2}{-(\nabla'\cdot\mathbf{M})}}{R}\,dv',$$

$\vec{M}$ $\longleftrightarrow$ $\vec{P}$

$\rho_m \cdot A/m^2$

$$\boxed{\rho_{ms} = \mathbf{M}\cdot\mathbf{a}_n \qquad (\mathrm{A/m})}$$

$$\boxed{\rho_m = -\nabla\cdot\mathbf{M} \qquad (\mathrm{A/m^2}).}$$

$$V = \frac{1}{4\pi\epsilon_0}\oint_{S'} \frac{\mathbf{P}\cdot\mathbf{a}_n'}{R}\,ds' + \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{(-\nabla'\cdot\mathbf{P})}{R}\,dv',$$

$$\boxed{\rho_{ps} = \mathbf{P}\cdot\mathbf{a}_n}$$

$$\boxed{\rho_p = -\nabla\cdot\mathbf{P}.}$$

- A **polarized dielectric** may be replaced by an equivalent $\rho_p$ and $\rho_{ps}$

Polarization charge density

$$\boxed{\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n}$$

$$\boxed{\rho_p = -\nabla \cdot \mathbf{P}.}$$

- A **magnetized body** may be replaced by an equivalent $\rho_m$ and $\rho_{ms}$

Magnetization charge density

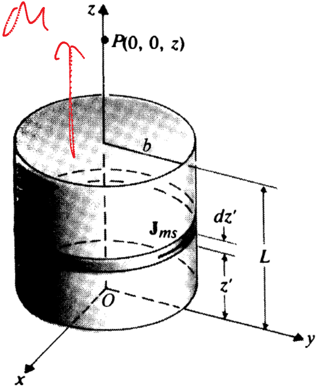
$$\boxed{\rho_{ms} = \mathbf{M} \cdot \mathbf{a}_n \qquad (\mathrm{A/m})}$$

$$\boxed{\rho_m = -\nabla \cdot \mathbf{M} \qquad (\mathrm{A/m^2}).}$$

$\vec{B}$

**EXAMPLE 6–8** Determine the magnetic flux density on the axis of a uniformly magnetized circular cylinder of a magnetic material. The cylinder has a radius $b$, length $L$, and axial magnetization $\mathbf{M} = \mathbf{a}_z M_0$.

$\vec{M}$




Question: **B** on z axis?

Methods:

(1) Calculate $J_m$ and $J_{ms}$

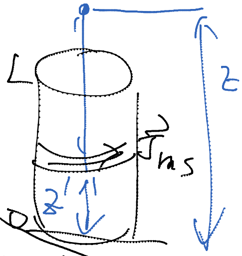
(2) **B** due to J

$$\vec{M} \to \begin{matrix}\vec{J}_m \\ \vec{J}_{ms}\end{matrix} \to \vec{A} \to \vec{B}$$

**FIGURE 6–11**
A uniformly magnetized circular cylinder (Example 6–8).

1° $\vec{J}_m = \nabla' \times \vec{M} = 0$

$\vec{J}_{ms} = \vec{M} \times \underbrace{\hat{a}_n}_{\text{side surface}} = (\hat{a}_z M_0) \times (\hat{a}_r) = \hat{a}_\phi M_0$




2°

$J_{ms}\, dz' = M_0\, dz'$

$$d\vec{B} = \hat{a}_z \frac{\mu_0\, dI'\, b^2}{2(z^2+b^2)^{3/2}}$$

z: distance between source and field

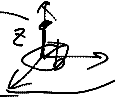
$$\vec{B} = \int d\vec{B} = \hat{a}_z \int_0^L \frac{\mu_0 M_0 b^2}{2[(z-z')^2+b^2]^{3/2}}\, dz'$$

$= \dots = \dots$

Ex6-6:

$\vec{B}$ on axis due to a circular loop current:

$$\vec{B} = \hat{a}_z \frac{\mu_0 I b^2}{2(z^2+b^2)^{3/2}}$$



# 6-7 Magnetic Field Intensity and Relative Permeability

- Review: External $\mathbf{E}_{ext}$ applied to a dielectric material ➔ induced dipole moments inside the dielectric material (and thus, $\mathbf{E}_{induced}$)

➡ **E** inside the dielectric $\mathbf{E}_{total} = \mathbf{E}_{ext} + \mathbf{E}_{induced}$

➡ $\boxed{\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}(\rho + \rho_p).}$

$\rho$

$\rho_p$ For dielectrics, we consider $\rho_p$

$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \Longrightarrow$ modify to $\nabla \cdot \vec{D} = \rho$

# Magnetic Side

- External $\mathbf{B}_{ext}$ applied to a **magnetic** material ➔ induced dipole moments inside the **magnetic** material (and thus, $\mathbf{B}_{induced}$)

➔ **B** inside the magnetic $\mathbf{B}_{total} = \mathbf{B}_{ext} + \mathbf{B}_{induced}$

➔ $$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{J} + \mathbf{J}_m = \mathbf{J} + \nabla \times \mathbf{M}$$

$$\nabla \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = \mathbf{J}.$$

Defined as **H**: magnetic field intensity

$$\boxed{\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \qquad (\mathrm{A/m}).}$$

| | |
|---|---|
| $\vec{H}$ | $\vec{D}$ |
| $\vec{B}$ | $\vec{E}$ |
| $\frac{1}{\mu_0}$ | $\varepsilon_0$ |
| $-\vec{M}$ | $\vec{P}$ |

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

$$\nabla \times \left( \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \right) = \mathbf{J}.$$

$$\boxed{\mathbf{H} \equiv \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \qquad (\text{A/m}).}$$

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} \qquad (\text{A/m}^2),}$$

**J**: volume density of **free** current

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho.$$

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \qquad (\text{C/m}^2).}$$

$$\boxed{\nabla \cdot \mathbf{D} = \rho \qquad (\text{C/m}^3),}$$

ρ: volume density of **free** charges

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} \qquad (\mathrm{A/m^2}),}$$

integration

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{s} = \int_S \mathbf{J} \cdot d\mathbf{s}$$

Stoke's theorem

$$\boxed{\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I \qquad (\mathrm{A}),}$$

*C*: the contour bounding the surface S

Another form of Ampere's circuital law: the circulation of the magnetic field intensity around any closed path is equal to the free current flowing through the surface bounded by the path.

$$\boxed{\nabla \cdot \mathbf{D} = \rho \qquad (\mathrm{C/m^3}),}$$

$$\int_V \nabla \cdot \mathbf{D}\, dv = \int_V \rho\, dv$$

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \qquad (\mathrm{C}).}$$

Another form of Gauss's law: …

If the magnetic properties of the medium is *linear* and *isotropic*, then
**M ~ H**

$$\mathbf{M} = \chi_m \mathbf{H},$$

$\chi_m$: magnetic susceptibility

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E},$$

$$\boxed{\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \qquad \text{(A/m)}.}$$

$$\mathbf{M} = \chi_m \mathbf{H},$$

$$\mathbf{B} = \mu_0(1 + \chi_m)\mathbf{H} = \mu_0 \mu_r \mathbf{H} = \mu \mathbf{H} \qquad (\text{Wb/m}^2)$$

$$\vec{H} = \frac{\vec{B}}{\mu}$$

| | |
|---|---|
| $\vec{H}$ | $\vec{D}$ |
| $\vec{B}$ | $\vec{E}$ |
| $\frac{1}{\mu}$ | $\varepsilon$ |

$$\mathbf{D} = \epsilon_0(1 + \chi_e)\mathbf{E} = \epsilon_0 \epsilon_r \mathbf{E} = \epsilon \mathbf{E} \qquad (\text{C/m}^2),$$

$$\vec{D} = \varepsilon \vec{E}$$

$$\boxed{\begin{aligned}\mathbf{B} &= \mu_0(1+\chi_m)\mathbf{H} \\ &= \mu_0\mu_r\mathbf{H} = \mu\mathbf{H} \qquad (\mathrm{Wb/m^2})\end{aligned}}$$

Or $$\boxed{\mathbf{H} = \frac{1}{\mu}\mathbf{B} \qquad (\mathrm{A/m}),}$$

where $$\boxed{\mu_r = 1 + \chi_m = \frac{\mu}{\mu_0}}$$

$\mu$: permeability
$\mu_r$: relative permeability

For a simple medium (linear, isotropic, homogeneous), $\chi_m$ and $\mu_r$ are constants.

$$\boxed{\begin{aligned}\mathbf{D} &= \epsilon_0(1+\chi_e)\mathbf{E} \\ &= \epsilon_0\epsilon_r\mathbf{E} = \epsilon\mathbf{E} \qquad (\mathrm{C/m^2}),\end{aligned}}$$

$$\epsilon_r = 1 + \chi_e = \frac{\epsilon}{\epsilon_0}$$

# Relative permeability $\mu_r$

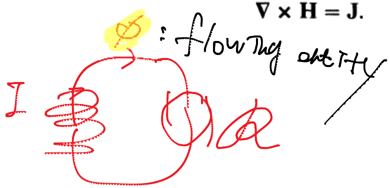
- Ferromagnetic materials: iron, nickel, and cobalt. $\mu_r$ is very large (i.e., easy to be magnetized)

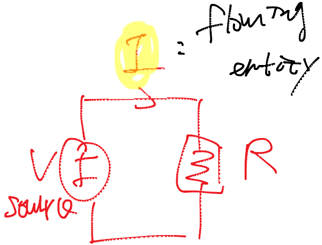
# Analogy between electrostatics and magnetostatics

| Electrostatics | Magnetostatics |
|---|---|
| $\mathbf{E}$ | $\mathbf{B}$ |
| $\mathbf{D}$ | $\mathbf{H}$ |
| $\epsilon$ | $\dfrac{1}{\mu}$ |
| $\mathbf{P}$ | $-\mathbf{M}$ |
| $\rho$ | $\mathbf{J}$ |
| $V$ | $\mathbf{A}$ |
| $\cdot$ | $\times$ |
| $\times$ | $\cdot$ |

# 6-8 Magnetic Circuits

- Magnetic circuits: transformers, generators, motors, relays, magnetic recording devices, and so on.
- Electric circuits: to find $V$ (and **E**) and $I$
  Magnetic circuits: to find $I$ (and **H**) and $\Phi$
- Analysis of magnetic circuits: $$\nabla \cdot \mathbf{B} = 0,$$ $$\nabla \times \mathbf{H} = \mathbf{J}.$$



# Magnetotive Force (mmf)

- Analogous to electromotive force (emf)
- Not a force in Newtons, but in ampere (A)

$$\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I \quad \text{(A)},$$

For $N$ turns (see Example 6-10)

$$\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = NI = \mathcal{V}_m.$$

mmf

# A toroidal core with $N$ turns of wires wound

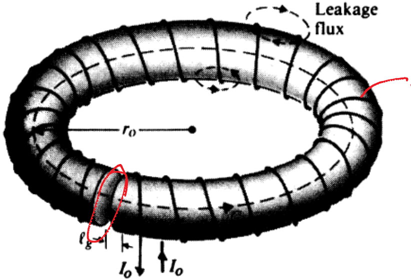



**FIGURE 6–13**
Coil on ferromagnetic toroid with air gap (Example 6–10).

In the ferromagnetic core

$$\mathbf{B}_f = \mathbf{a}_\phi \frac{\mu_0 \mu N I_o}{\mu_0(2\pi r_o - \ell_g) + \mu \ell_g}.$$

$$\mathbf{H}_f = \mathbf{a}_\phi \frac{\mu_0 N I_o}{\mu_0(2\pi r_o - \ell_g) + \mu \ell_g}.$$

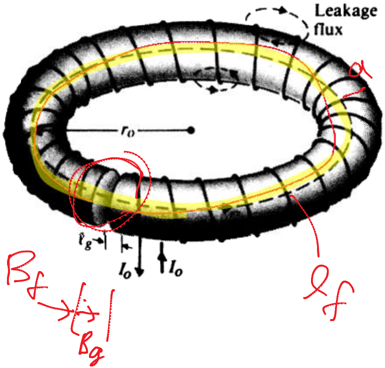
In the air gap

$$\mathbf{H}_g = \mathbf{a}_\phi \frac{\mu N I_o}{\mu_0(2\pi r_o - \ell_g) + \mu \ell_g}.$$

**EXAMPLE 6–10** Assume that $N$ turns of wire are wound around a toroidal core of a ferromagnetic material with permeability $\mu$. The core has a mean radius $r_o$, a circular cross section of radius $a$ ($a \ll r_o$), and a narrow air gap of length $\ell_g$, as shown in Fig. 6–13. A steady current $I_o$ flows in the wire. Determine (a) the magnetic flux density, $\mathbf{B}_f$, in the ferromagnetic core; (b) the magnetic field intensity, $\mathbf{H}_f$, in the core; and (c) the magnetic field intensity, $\mathbf{H}_g$, in the air gap.

$B_f$

$H_f$, $H_g$




$\ell_g + \ell_f = 2\pi r_o$

**FIGURE 6–13**
Coil on ferromagnetic toroid with air gap (Example 6–10).

Ampere's circuital law: $\oint_C \vec{H}\cdot d\vec{l} = NI_0$


$$H_g l_g + H_f l_f = NI_0 \quad \text{①}$$

normal B should be continuous $\Rightarrow$ $B_f = B_g = \hat{a}_\phi B$ ②

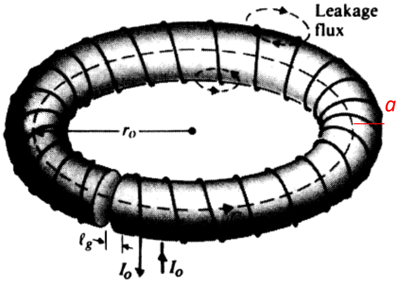
① and ② $\Rightarrow$

$$\frac{B}{\mu_0} l_g + \frac{B}{\mu_f} l_f = NI_0$$

$$\Rightarrow \vec{B} = \hat{a}_\phi \frac{\mu\mu_0 N I_0}{\mu_0 l_f + \mu l_g} = \vec{B}_f = \vec{B}_g \quad ※$$

$$\vec{H}_f = \frac{\vec{B}}{\mu_f}$$

$$\vec{H}_g = \frac{\vec{B}}{\mu_0}$$




$a$: radius of the coil
$S$: cross section of the toroid core

**FIGURE 6–13**
Coil on ferromagnetic toroid with air gap (Example 6–10).

If $a \ll r_0$, **B** ~ constant,

$$\Phi \cong BS,$$

$$\boxed{\mathbf{B}_f = \mathbf{a}_\phi \frac{\mu_0 \mu N I_o}{\mu_0(2\pi r_o - \ell_g) + \mu \ell_g}.}$$

$$\Phi = \frac{N I_o}{(2\pi r_o - \ell_g)/\mu S + \ell_g/\mu_0 S}.$$

$$\Phi = \frac{NI_o}{(2\pi r_o - \ell_g)/\mu S + \ell_g/\mu_0 S}.$$

$$\Phi = \frac{\mathcal{V}_m}{\mathcal{R}_f + \mathcal{R}_g},$$

$$I = \frac{V}{R}$$

| electric circuit | magnetic circuit |
|---|---|
| $I$ | $\phi$ |
| $V$ | $I$ |
| $R$ | $\mathcal{R}$ |
| $\ell$ | $\ell$ |
| $S$ | $S$ |
| $\sigma$ | $\mu$ |

where $\mathcal{R}_f = \dfrac{2\pi r_o - \ell_g}{\mu S} = \dfrac{\ell_f}{\mu S},$ $\qquad \mathcal{R}_g = \dfrac{\ell_g}{\mu_0 S}.$

$$\ell_f = 2\pi r_o - \ell_g$$

$\ell_f$: length of the ferromagnetic core

$R_f$: reluctance of the ferromagnetic core ($H^{-1}$)
$R_g$: reluctance of the air gap

Ohm's law: $I = V/R$ Resistance $R = \ell/(\sigma S)$

# Analogous to Electric Circuit

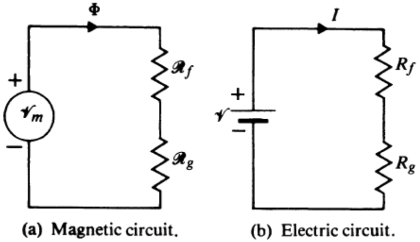



**FIGURE 6–14**
Equivalent magnetic circuit and analogous electric circuit for toroidal coil with air gap in Fig. 6–13.

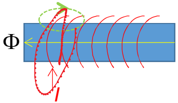
$$\Phi = \frac{\mathscr{V}_m}{\mathscr{R}_f + \mathscr{R}_g}, \qquad I = \frac{\mathscr{V}}{R_f + R_g}.$$

| Magnetic Circuits | Electric Circuits |
|---|---|
| mmf, $\mathscr{V}_m$ $(=NI)$ | emf, $\mathscr{V}$ |
| magnetic flux, $\Phi$ | electric current, $I$ |
| reluctance, $\mathscr{R}$ | resistance, $R$ |
| permeability, $\mu$ | conductivity, $\sigma$ |

# Difficulty in Analysis of Magnetic Circuits

- 1. Very difficult to account for leakage fluxes

Magnetic circuits



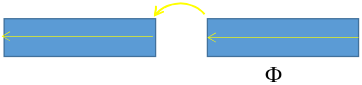

Leakage fluxes $\Phi$ outside $\neq 0$
due to $\mu_0 \neq 0$ (in air)

Electric circuits


Current $I$ outside = 0
because of $\sigma = 0$ (in air)

- 2. Difficult to account for fringing effect (at the air gap)

Magnetic circuits



- 3. *B* and *H* have a nonlinear relationship
  - *B*(*H*) or $\mu$ is not a constant

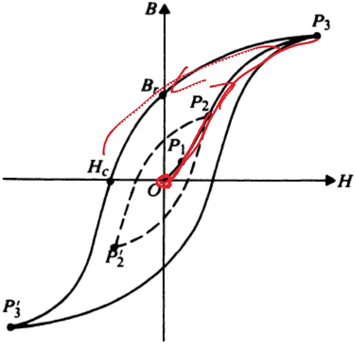



**FIGURE 6–17**
Hysteresis loops in the *B–H* plane for ferromagnetic material.

# KVL and KCL in Magnetic Circuits

$\mathcal{V}_m = (\mathcal{R}_f + \mathcal{R}_g)\Phi$

$$\Phi = \frac{\mathcal{V}_m}{\mathcal{R}_f + \mathcal{R}_g},$$

$$\boxed{\sum_j N_j I_j = \sum_k \mathcal{R}_k \Phi_k.}$$

KVL: around a closed path in a magnetic circuit the algebraic sum of ampere-terns is equal to the algebraic sum of the products of the reluctances and fluxes

$$\nabla \cdot \mathbf{B} = 0$$

$$\boxed{\sum_j \Phi_j = 0,}$$

KCL: the algebraic sum of all the magnetic fluxes flowing out of a junction in a magnetic circuit is zero

See Chap. 5

$$I = \frac{\mathcal{V}}{R_f + R_g}.$$

$$\nabla \cdot \mathbf{J} = 0.$$