Chapter 8 Plane Electromagnetic Waves

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8-1 Introduction

- In Chap. 7, homogeneous vector wave equations

$$\nabla^2 \mathbf{E} - \frac{1}{u^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0.$$

where $u = 1/\sqrt{\mu\epsilon}$,

- In free space the source-free wave equation

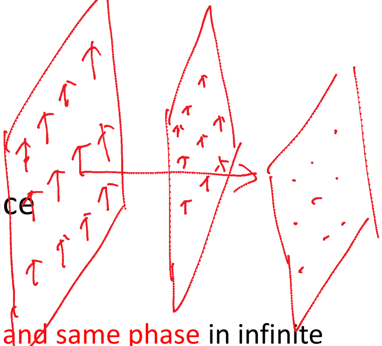
$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0,$$

where $c = \dfrac{1}{\sqrt{\mu_0 \epsilon_0}} \cong 3 \times 10^8 \text{ (m/s)} = 300 \quad \text{(Mm/s)}$

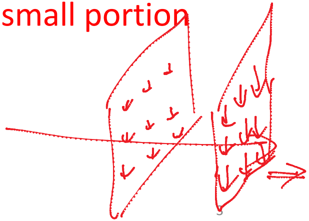
$4\pi\times10^{-7}$ $\quad \dfrac{10^{-9}}{36\pi}$

Plane Wave

$\vec{E}$ is const. over the plane



- Waves with one-dimensional spatial dependence
- A uniform plane wave:
  - a particular solution of Maxwell's equations
  - **E** (or **H**) with the same direction, same magnitude, and same phase in infinite planes perpendicular to the direction of propagation.
- If we are far enough away from a source, the wavefront (surface of constant phase) becomes almost spherical; and a very small portion of the surface of a giant sphere is very nearly a plane.



8-2 Plane Waves in Lossless Media

- Wave equation for source free, in free space:

Homogeneous vector Helmholtz's equation

$$\boxed{\nabla^2 \mathbf{E} + k_0^2 \mathbf{E} = 0,}$$

where k_0: free-space wavenumber

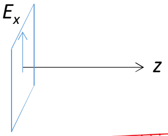
$$\boxed{k_0 = \omega\sqrt{\mu_0 \epsilon_0} = \frac{\omega}{c} \qquad \text{(rad/m).}}$$

In Cartesian coordinates

$$\left(\cancel{\frac{\partial^2}{\partial x^2}} + \cancel{\frac{\partial^2}{\partial y^2}} + \frac{\partial^2}{\partial z^2} + k_0^2\right)E_x = 0.$$

1D wave equation

Consider a uniform plane wave: uniform E_x (uniform magnitude and constant phase) over plane surfaces $\perp z$



E_x uniform in x and y; $E_x(z)$ ➔ $\partial^2 E_x/\partial x^2 = 0$ and $\partial^2 E_x/\partial y^2 = 0.$

$$\frac{d^2 E_x}{dz^2} + k_0^2 E_x = 0,$$

E_x: a phasor

2^{nd}-order ODE ➔ 2 integration constants

Solution

$$\begin{aligned} E_x(z) &= E_x^+(z) + E_x^-(z) \\ &= E_0^+ e^{-jk_0 z} + E_0^- e^{jk_0 z}, \end{aligned}$$

E_0^+, E_0^-: arbitrary complex constants, to be determined by boundary conditions

$$E_x(z) = E_x^+(z) + E_x^-(z)$$
$$= E_0^+ e^{-jk_0 z} + E_0^- e^{jk_0 z},$$

Check time-dependent E_x
(Phasor → time domain)

Time domain | phasor

$$E_x^+(z,t) = \mathcal{Re}[E_x^+(z)e^{j\omega t}]$$
$$= \mathcal{Re}[E_0^+ e^{j(\omega t - k_0 z)}]$$
$$= E_0^+ \cos(\omega t - k_0 z) \qquad \text{(V/m)}.$$

Assuming E_0^+ is a real constant

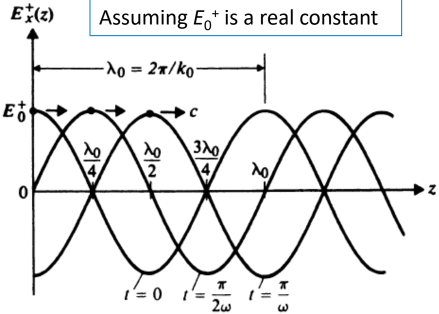


FIGURE 8–1
Wave traveling in positive z direction $E_x^+(z,t) = E_0^+ \cos(\omega t - k_0 z)$, for several values of t.

- Phase velocity: the velocity of propagation of a point of a particular phase on the wave

$$\cos(\omega t - k_0 z) = \text{a constant}$$

$$\omega t - k_0 z = \text{A constant phase,}$$

$$u_p = \frac{dz}{dt} = \frac{\omega}{k_0} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c.$$

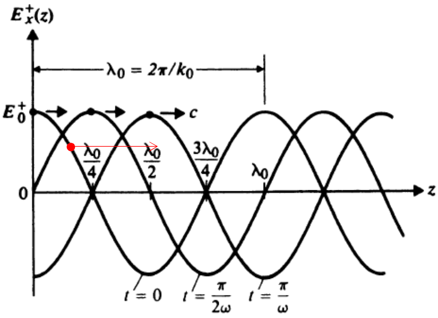


FIGURE 8–1
Wave traveling in positive z direction $E_x^+(z, t) = E_0^+ \cos(\omega t - k_0 z)$, for several values of t.

$$u_p = \frac{dz}{dt} = \frac{\omega}{k_0} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c.$$

$$k_0 = 2\pi f / c$$

Wavenumber: The number of wavelength in a complete cycle

$$\boxed{k_0 = \frac{2\pi}{\lambda_0} \qquad \text{(rad/m)},}$$

Inverse relation

$$\boxed{\lambda_0 = \frac{2\pi}{k_0} \qquad \text{(m)}.}$$

For lossless dielectrics:

$k=2\pi/\lambda$

$\lambda=2\pi/k$

$E_0^+ e^{-jk_0 z}$ A wave traveling in the +z direction $u_p = \frac{dz}{dt} = \frac{\omega}{k_0}$ >0

$E_0^- e^{jk_0 z}$ A wave traveling in the -z direction

The Associated Magnetic Fields H

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H},$$

Consider only the wave traveling in the +z direction

$$E_x(z) = E_x^+(z) = E_0^+ e^{-jk_0 z}$$

Only z dependence

$$\nabla \times \mathbf{E} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ E_x^+(z) & 0 & 0 \end{vmatrix} = -j\omega\mu_0(\mathbf{a}_x H_x^+ + \mathbf{a}_y H_y^+ + \mathbf{a}_z H_z^+),$$

Only E_x component

$$H_x^+ = 0,$$

$$H_y^+ = \frac{1}{-j\omega\mu_0}\frac{\partial E_x^+(z)}{\partial z},$$

$$H_z^+ = 0.$$

The only nonzero component H_y^+

$$H_y^+ = \frac{1}{-j\omega\mu_0}\frac{\partial E_x^+(z)}{\partial z},$$

$$E_x^+(z) = E_0^+ e^{-jk_0 z}$$

$$\frac{\partial E_x^+(z)}{\partial z} = \frac{\partial}{\partial z}(E_0^+ e^{-jk_0 z}) = -jk_0 E_x^+(z),$$

$$H_y^+(z) = \frac{k_0}{\omega\mu_0}E_x^+(z) = \frac{1}{\eta_0}E_x^+(z) \qquad \text{(A/m)}.$$

$$\frac{\omega\sqrt{\mu_0\epsilon_0}}{\omega\mu_0}$$

where

$$\boxed{\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \cong 120\pi \cong 377 \qquad (\Omega),}$$

intrinsic impedance of the free space

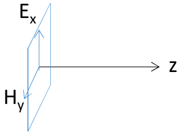
η_0 is real ➔ $H_y^+(z)$ is in phase with $E_x^+(z)$

Check time-domain **H**

$$\mathbf{H}(z,t) = \mathbf{a}_y H_y^+(z,t) = \mathbf{a}_y \mathscr{R}e[H_y^+(z)e^{j\omega t}]$$
$$= \mathbf{a}_y \frac{E_0^+}{\eta_0}\cos(\omega t - k_0 z) \qquad \text{(A/m)}.$$

For a Uniform Plane Wave

- $|\mathbf{E}|/|\mathbf{H}|=\eta_0$
- **H**, **E**, and the direction of propagation are perpendicular to each other.



EXAMPLE 8–1 A uniform plane wave with $\mathbf{E} = \mathbf{a}_x E_x$ propagates in a lossless simple medium ($\epsilon_r = 4$, $\mu_r = 1$, $\sigma = 0$) in the $+z$-direction. Assume that E_x is sinusoidal with a frequency 100 (MHz) and has a maximum value of $+10^{-4}$ (V/m) at $t = 0$ and $z = \frac{1}{8}$ (m).

a) Write the instantaneous expression for $\mathbf{E}$ for any t and z.

b) Write the instantaneous expression for $\mathbf{H}$.

c) Determine the locations where E_x is a positive maximum when $t = 10^{-8}$ (s).

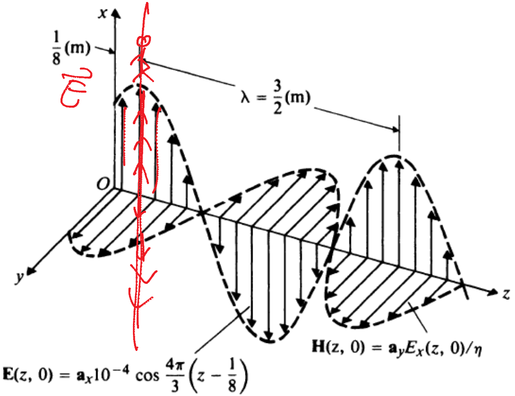


FIGURE 8–2
E and **H** fields of a uniform plane wave at $t = 0$ (Example 8–1).

(a) $\vec{E} = \hat{a}_x \underbrace{E_{amp}}_{10^{-4}} \cos(\underbrace{\omega}_{2\pi f = 100MHz} t - \beta z + \phi)$ → $+z$ traveling

$k = \omega\sqrt{\mu\epsilon} = \frac{\omega}{c}\sqrt{\mu_r\epsilon_r}$

$= \frac{\omega}{u_p} = \frac{2\pi\cdot 10^8}{3\times 10^8}\sqrt{4}$

$= \frac{4}{3}\pi \ (m^{-1})$

$= \hat{a}_x 10^{-4} \cos(2\pi\times 10^8 t - \frac{4}{3}\pi z + \phi)$ #

E_{max} at $t=0$

$z = \frac{1}{8}$

$0 + \frac{4}{3}\pi\cdot\frac{1}{8} + \phi = 0 \Rightarrow \phi = \frac{\pi}{6}$

(b) $\vec{H} = \hat{a}_y \frac{E_{amp}}{\eta} \cos(\ \ldots\ldots\ ) = \hat{a}_y \frac{10^{-4}}{60\pi}(\ \ldots\ldots\ )$ #

$\eta = \sqrt{\frac{\mu}{\epsilon}} = 120\pi\sqrt{\frac{\mu_r}{\epsilon_r}} = 60\pi$

(c) at $t = 10^{-8}$, z_{E+max}?

$\cos(\sim) = \pm 2n\pi \quad n = 0, 1, 2.$

$2\pi \times 10^{8} \cdot 10^{-8} - \frac{4}{3}\pi z_{E+max} + \frac{\pi}{6} = \pm 2n\pi$, $n = 0, 1, 2 \ldots$

$\Rightarrow z_{E+max} = -\frac{3}{4}(\pm 2n - \frac{13}{6}) = \pm \frac{3}{2} n + \frac{13}{8}$

$n = 0, 1, 2$

$\lambda = \frac{3}{2}$ (m)

$\lambda = \frac{2\pi}{k} = \frac{2\pi}{4/3\,\pi} = \frac{3}{2}$ (m)

8-2.1 Doppler Effect (excluded)

8-2.2 Transverse Electromagnetic Waves

- For a uniform plane wave, we have seen
  - $\mathbf{E} = \mathbf{a_x}E_x$; $\mathbf{H} = \mathbf{a_y}H_y$; direction of propagation in z
  - **E** and **H** are transvers to the direction of propagation, so it is called transverse electromagnetic (TEM) wave
  - Phasors **E** and **H** are functions of z only
- General case: Consider a uniform plane wave along an arbitrary direction (not necessarily coincide with a coordinate axis)

$\hat{a}_n$: direction of propagation

$\vec{E}$

$\vec{H}$

only $\vec{E} \perp \hat{a}_n$: TE

only $\vec{H} \perp \hat{a}_n$: TM

$\begin{cases} \vec{E} \perp \hat{a}_n \\ \vec{H} \perp \hat{a}_n \end{cases}$: TEM

For a uniform plane wave

$$\mathbf{E}(z) = \mathbf{E}_0 e^{-jkz},$$

Propagating in the +z direction

General case $\mathbf{E}(x, y, z) = \mathbf{E}_0 e^{-jk_x x - jk_y y - jk_z z}$

Propagating in the +x, +y, +z direction

Substitution in Helmholtz's equation

$$\nabla^2 \mathbf{E} + k_0^2 \mathbf{E} = 0,$$

$$k_x^2 + k_y^2 + k_z^2 = \omega^2 \mu \epsilon.$$

$$\boxed{\mathbf{E}(x, y, z) = \mathbf{E}_0 e^{-jk_x x - jk_y y - jk_z z}}$$

Define a wavenumber vector **k**:

$$\mathbf{k} = \mathbf{a}_x k_x + \mathbf{a}_y k_y + \mathbf{a}_z k_z = k\mathbf{a}_n$$

$$\mathbf{R} = \mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z,$$

$$\boxed{\mathbf{E}(\mathbf{R}) = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{R}} = \mathbf{E}_0 e^{-jk\mathbf{a}_n\cdot\mathbf{R}} \qquad (\mathrm{V/m}),}$$

$\mathbf{a}_n$: unit vector of **k**; direction of propagation (explained next)

$$\mathbf{k} = \mathbf{a}_x k_x + \mathbf{a}_y k_y + \mathbf{a}_z k_z = k\mathbf{a}_n \Rightarrow \begin{aligned} k_x &= \mathbf{k}\cdot\mathbf{a}_x = k\mathbf{a}_n\cdot\mathbf{a}_x, \\ k_y &= \mathbf{k}\cdot\mathbf{a}_y = k\mathbf{a}_n\cdot\mathbf{a}_y, \\ k_z &= \mathbf{k}\cdot\mathbf{a}_z = k\mathbf{a}_n\cdot\mathbf{a}_z, \end{aligned}$$

$$\vec{A}$$

$$\vec{A}\cdot\hat{x} = A_x$$

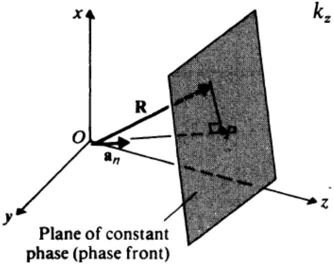


FIGURE 8–4
Radius vector and wave normal to a phase front of a uniform plane wave.

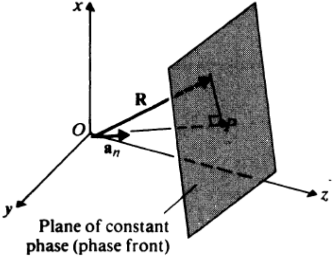


FIGURE 8–4
Radius vector and wave normal to a phase front of a uniform plane wave.

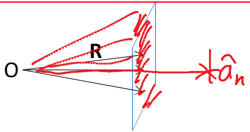
$$\mathbf{a}_n \cdot \mathbf{R} = \text{Length } \overline{OP} \quad \text{(a constant)}$$

$$\boxed{\mathbf{E}(\mathbf{R}) = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{R}} = \mathbf{E}_0 e^{-jk\mathbf{a}_n\cdot\mathbf{R}} \qquad \text{(V/m)},}$$

= phase

$$\vec{E}(t) = \vec{E}_0 \cos(\omega t - k\hat{a}_n\cdot\vec{R})$$

For constant phase, $\mathbf{a}_n \cdot \mathbf{R}$ = constant = $\overline{OP}$ ⇒ **R** forms a constant-phase plane
$\mathbf{a}_n$ // $\hat{n}$ of constant-phase plane // direction of propagation



In a charge-free region, $\nabla \cdot \mathbf{E} = 0$

$$\mathbf{E}(\mathbf{R}) = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{R}} = \mathbf{E}_0 e^{-jk\mathbf{a}_n\cdot\mathbf{R}}$$

$$\nabla \cdot (\mathbf{E}_0\, e^{-j\mathbf{k}\cdot\mathbf{R}}) = e^{-j\mathbf{k}\cdot\mathbf{R}}\, \cancel{\nabla\cdot\mathbf{E}_0} + \mathbf{E}_0 \cdot \nabla(e^{-j\mathbf{k}\cdot\mathbf{R}})$$

0

$$\mathbf{E}_0 \cdot \nabla(e^{-jk\mathbf{a}_n\cdot\mathbf{R}}) = 0.$$

$$\nabla(e^{-jk\mathbf{a}_n\cdot\mathbf{R}}) = \left(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}\right) e^{-j(k_x x + k_y y + k_z z)}$$

$$= -j(\mathbf{a}_x k_x + \mathbf{a}_y k_y + \mathbf{a}_z k_z) e^{-j(k_x x + k_y y + k_z z)}$$

$$= -jk\mathbf{a}_n e^{-jk\mathbf{a}_n\cdot\mathbf{R}},$$

$$-jk(\mathbf{E}_0 \cdot \mathbf{a}_n) e^{-jk\mathbf{a}_n\cdot\mathbf{R}} = 0,$$

which requires $\mathbf{a}_n \cdot \mathbf{E}_0 = 0.$

$\boxed{\text{Thus, for a plane-wave solution, } \mathbf{E}_0 \perp \mathbf{a}_n}$ $\Rightarrow \vec{E} \perp \hat{a}_n$ ①

The Associated Magnetic Fields H

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H},$$

$$\Downarrow$$

$$\mathbf{H}(\mathbf{R}) = -\frac{1}{j\omega\mu}\nabla \times \mathbf{E}(\mathbf{R})$$

$$\Downarrow$$

$\nabla \times \mathbf{E} = ?$

$$\mathbf{E}(\mathbf{R}) = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{R}} = \mathbf{E}_0 e^{-jk\mathbf{a}_n\cdot\mathbf{R}}$$

$$\nabla \times (\psi \mathbf{A}) = \psi \nabla \times \mathbf{A} + \nabla\psi \times \mathbf{A}$$

0 $\quad \nabla \times \vec{E}_0$ $\quad \vec{E}_0$ $\quad e^{-j\vec{k}\cdot\vec{R}}$

$$\boxed{\mathbf{H}(\mathbf{R}) = \frac{1}{\eta}\mathbf{a}_n \times \mathbf{E}(\mathbf{R}) \qquad (\mathrm{A/m}),}$$

where $\boxed{\eta = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\epsilon}} \qquad (\Omega)}$ the intrinsic impedance of the medium

$$\frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}$$

$$\vec{H}(\vec{R}) = -\frac{1}{j\omega\mu}(-jk\hat{a}_n)e^{-j\vec{k}\cdot\vec{R}} \times \vec{E}_0$$

$$= \frac{k}{\omega\mu}\hat{a}_n \times \vec{E}(\vec{R})$$

$\vec{E}_0 e^{-j\vec{k}\cdot\vec{R}}$

$$\boxed{\mathbf{H}(\mathbf{R}) = \frac{1}{\eta}\mathbf{a}_n \times \mathbf{E}(\mathbf{R}) \qquad \text{(A/m)},}$$

$\Longrightarrow$ $\vec{H} \perp \hat{a}_n$ ②

$\vec{H} \perp \vec{E}$ ③

$$\boxed{\mathbf{E}(\mathbf{R}) = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{R}} = \mathbf{E}_0 e^{-jk\mathbf{a}_n\cdot\mathbf{R}}}$$

$$\boxed{\mathbf{H}(\mathbf{R}) = \frac{1}{\eta}(\mathbf{a}_n \times \mathbf{E}_0)e^{-jk\mathbf{a}_n\cdot\mathbf{R}} \qquad \text{(A/m)}.}$$

A uniform plane wave propagating in an arbitrary direction, $\mathbf{a}_n$:

- a TEM wave
- $\mathbf{H} \perp \mathbf{E}$; $\mathbf{H} \perp \mathbf{a}_n$; $\mathbf{E} \perp \mathbf{a}_n$

③ ② ①

8-2.3 Polarization of Plane Waves

- Polarization of a uniform plane wave: time-varying behavior of **E** vector at a given point in space.
  - E.g., $\mathbf{E} = \mathbf{a_x} E_x$, the wave is linearly polarized in x direction

- In some cases, direction of **E** of a plane wave may change with time
  - Two linearly polarized waves in *x* and *y* direction

+z propagation

Phasor notation: $$\mathbf{E}(z) = \mathbf{a}_x E_1(z) + \mathbf{a}_y E_2(z)$$
$$= \mathbf{a}_x E_{10} e^{-jkz} - \mathbf{a}_y jE_{20} e^{-jkz},$$

$\boxed{\mathbf{a}_y \text{ lags } 90°}$

E_{10}, E_{20}: real numbers, denoting amplitudes

polarization

$$-j = e^{-\frac{\pi}{2}}$$

Time-domain expression: $$\mathbf{E}(z, t) = \mathcal{Re}\{[\mathbf{a}_x E_1(z) + \mathbf{a}_y E_2(z)]e^{j\omega t}\}$$
$$= \mathbf{a}_x E_{10} \cos(\omega t - kz) + \mathbf{a}_y E_{20} \cos\left(\omega t - kz - \frac{\pi}{2}\right).$$

  - Examine the direction change of **E** at a given point as *t* changes (*z* = 0 for convenience)

$$\mathbf{E}(0, t) = \mathbf{a}_x E_1(0, t) + \mathbf{a}_y E_2(0, t)$$
$$= \mathbf{a}_x E_{10} \cos \omega t + \mathbf{a}_y E_{20} \sin \omega t.$$

E(0, *t*): the sum of two linearly polarized waves in both space quadrature ($\mathbf{a}_x$ and $\mathbf{a}_y$) and time quadrature (cos ωt and sin ωt)

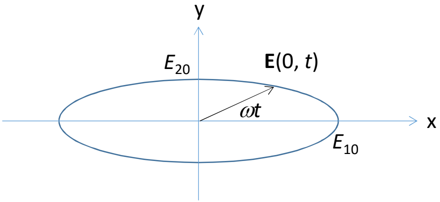
$$\mathbf{E}(0, t) = \mathbf{a}_x E_1(0, t) + \mathbf{a}_y E_2(0, t)$$
$$= \mathbf{a}_x E_{10} \cos \omega t + \mathbf{a}_y E_{20} \sin \omega t.$$

$$\cos \omega t = \frac{E_1(0, t)}{E_{10}}$$

$$\sin \omega t = \frac{E_2(0, t)}{E_{20}}$$
$$= \sqrt{1 - \cos^2 \omega t} = \sqrt{1 - \left[\frac{E_1(0, t)}{E_{10}}\right]^2},$$

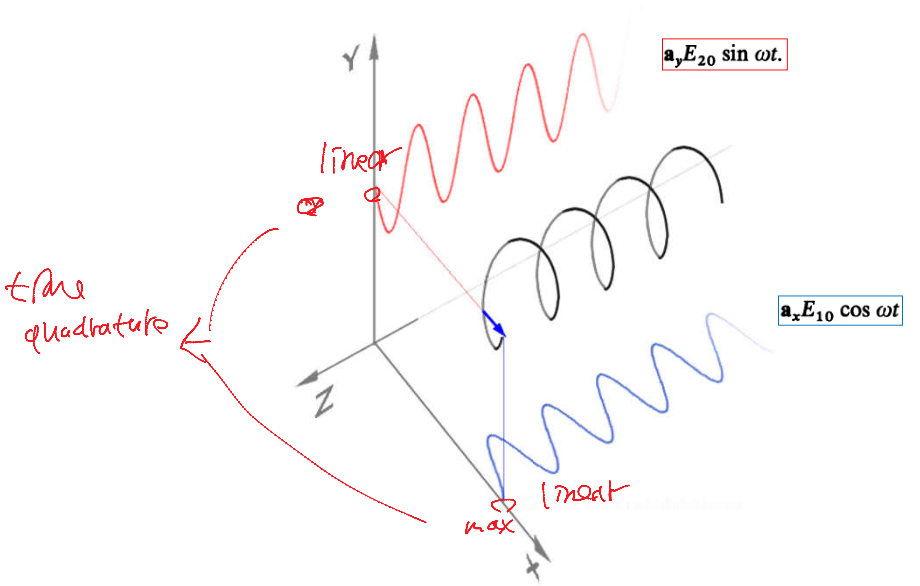
$$\left[\frac{E_2(0, t)}{E_{20}}\right]^2 + \left[\frac{E_1(0, t)}{E_{10}}\right]^2 = 1.$$

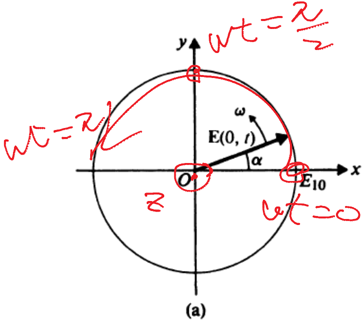
As ωt increases, **E**(0, *t*) will traverse an elliptical locus in the counterclockwise direction



If $E_{20} \neq E_{10}$, elliptical polarization

$$\mathbf{E}(0,t) = \mathbf{a}_x E_1(0,t) + \mathbf{a}_y E_2(0,t)$$
$$= \underline{\mathbf{a}_x E_{10}\cos\omega t} + \underline{\mathbf{a}_y E_{20}\sin\omega t}.$$





If $E_{20} = E_{10}$, circular polarization

$$\cos \omega t = \frac{E_1(0, t)}{E_{10}}$$

$$\sin \omega t = \frac{E_2(0, t)}{E_{20}}$$

And the angle $\alpha = \tan^{-1} \frac{E_2(0, t)}{E_1(0, t)} = \omega t,$

E(0, *t*) rotates counterclockwise

Right-hand (positive) circularly polarized wave:

- finger: direction of rotation of **E**
- thumb: direction of propagation (+z)

For $\mathbf{a}_y$ lags 90°

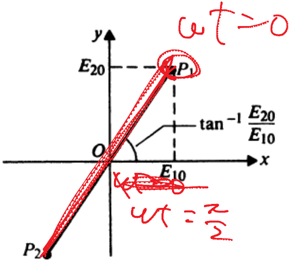
Left-hand (negative) circularly polarized wave:

For $\mathbf{a}_y$ leads 90°

FIGURE 8–5
Polarization diagrams for sum of two linearly polarized waves in space quadrature at $z = 0$: (a) circular polarization, $E(0, t) = E_{10}(\mathbf{a}_x \cos \omega t + \mathbf{a}_y \sin \omega t)$;

E(0,t): the sum of two linearly polarized waves in space quadrature (**a**$_x$ and **a**$_y$) but in time phase

$$\mathbf{E}(0, t) = (\mathbf{a}_x E_{10} + \mathbf{a}_y E_{20}) \cos \omega t.$$



(b)

tip at P_1 when $t = 0$;
tip at origin O when $\omega t = \pi/2$;
tip at P_2 when $\omega t = \pi$
➔ Linear polarization

FIGURE 8–5
Polarization diagrams for sum of two linearly polarized waves in space quadrature at $z = 0$:
(b) linear polarization,
$\mathbf{E}(0, t) = (\mathbf{a}_x E_{10} + \mathbf{a}_y E_{20}) \cos \omega t.$

Polarizations

- AM broadcast station: $\updownarrow$
- Television signals: $\longleftrightarrow$
- FM broadcast stations: $\bigcirc$
- Receiving antennas should have similar orientation to get the best signals

8-3 Plane Waves in Lossy Media

lossy
$\Downarrow$
$\sigma \neq 0$
$\Downarrow$
$\epsilon \rightarrow \epsilon_c$
$\Downarrow$
$k \rightarrow k_c$

- Wave equation for source free and in lossy media:

$k \rightarrow k_c$

$$\nabla^2 \mathbf{E} + k_c^2 \mathbf{E} = 0,$$

$$k_c = \omega\sqrt{\mu\epsilon_c}$$ A complex number

  - Conventional notation in transmission-line theory: propagation constant γ

1st convention

$\varepsilon = \varepsilon' - j\varepsilon''$

$$\boxed{\gamma = jk_c = j\omega\sqrt{\mu\epsilon_c} \qquad (\mathrm{m}^{-1}).}$$

γ is complex

$$\boxed{\begin{aligned} k_c &= \omega\sqrt{\mu\epsilon_c} \\ &= \omega\sqrt{\mu(\epsilon' - j\epsilon'')} \end{aligned}}$$

2nd convention

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon'}\left(1 - j\frac{\epsilon''}{\epsilon'}\right)^{1/2},$$

A special case:
$\varepsilon'' = \sigma/\omega$

$$\boxed{\epsilon_c = \epsilon - j\frac{\sigma}{\omega}}$$

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon}\left(1 + \frac{\sigma}{j\omega\epsilon}\right)^{1/2},$$

- For a lossless medium, σ=0
  - ❖ σ=0

$$\boxed{\gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon}\left(1 + \frac{\sigma}{j\omega\epsilon}\right)^{1/2},}$$

  - ❖ α=0,

$$\gamma = j\beta$$

$$\beta = k = \omega\sqrt{\mu\epsilon}$$

- Wave equation in lossy media expressed by γ:

$$\nabla^2\mathbf{E} + k_c^2\mathbf{E} = 0, \Rightarrow \nabla^2\mathbf{E} - \gamma^2\mathbf{E} = 0.$$

$$\boxed{\gamma = jk_c}$$

$$\frac{\partial^2\vec{E}}{\partial z^2} - \gamma^2\vec{E} = 0$$

- Solution of a uniform plane wave
  - ❖Propagating in +z
  - ❖Linearly polarized in *x*

$$\mathbf{E} = \mathbf{a}_x E_x = \mathbf{a}_x E_0 e^{-\gamma z},$$

$$\boxed{\gamma = \alpha + j\beta}$$

$$\hat{E}_x = E_0 e^{-\alpha z} e^{-j\beta z}.$$

Attenuation along +z

Propagation along +z

$$E(t) = Re[\hat{E}_x e^{j\omega t}]$$

$$= E_0 e^{-\alpha z} \cos(\omega t - \beta z)$$

- α>0, β>0
  - ❖$e^{-\alpha z}$ decreases as *z* increases, so it is called attenuation factor (α: attenuation constant)
  - ❖$e^{-j\beta z}$ determines the phase, so it is called phase factor (β: phase constant)

8-3.1 Low-Loss Dielectrics

- Low-loss dielectrics = good but imperfect insulator
  - Low $\sigma \rightarrow$ small current $\rightarrow$ low loss
  - $\varepsilon'' << \varepsilon'$ (or $\sigma/\omega\varepsilon << 1$)

$\varepsilon_c = \varepsilon' - j\varepsilon''$

$(1+x)^n \simeq 1+nx$

if $x \ll 1$

$$\boxed{\gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon'}\left(1 - j\frac{\epsilon''}{\epsilon'}\right)^{1/2},}$$

Binomial expansion

$\varepsilon''/\varepsilon' << 1 \rightarrow$ neglect H.O.T.

$$\gamma = \alpha + j\beta \cong j\omega\sqrt{\mu\epsilon'}\left[1 - j\frac{\epsilon''}{2\epsilon'} + \frac{1}{8}\left(\frac{\epsilon''}{\epsilon'}\right)^2\right],$$

$$\gamma = \alpha + j\beta \cong j\omega\sqrt{\mu\epsilon'}\left[1 - j\frac{\epsilon''}{2\epsilon'} + \frac{1}{8}\left(\frac{\epsilon''}{\epsilon'}\right)^2\right],$$

Attenuation constant $$\alpha \cong \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} \quad \text{(Np/m)}$$

Phase constant $$\beta \cong \omega\sqrt{\mu\epsilon'}\left[1 + \frac{1}{8}\left(\frac{\epsilon''}{\epsilon'}\right)^2\right] \quad \text{(rad/m).}$$

- α >0, α is proportional to ω
- When $\varepsilon''/\varepsilon'$ → 0, β reduces to the case of lossless dielectrics

The intrinsic impedance $\eta_c = (\mu/\varepsilon)^{1/2} = (\mu/(\varepsilon' - j\varepsilon''))^{1/2}$

$$\eta_c = \sqrt{\frac{\mu}{\epsilon'}}\left(1 - j\frac{\epsilon''}{\epsilon'}\right)^{-1/2}$$

$$\cong \sqrt{\frac{\mu}{\epsilon'}}\left(1 + j\frac{\epsilon''}{2\epsilon'}\right) \quad (\Omega).$$

For $\varepsilon'' << \varepsilon'$

For a uniform plane wave $\eta_c = E_x/H_y$

- In lossless case, η is real, E_x and H_y are in time phase
- In low-loss case, η_c is complex, E_x/H_y are out of phase

$$\frac{\tilde{E}}{\tilde{H}} = \eta$$

The phase velocity $u_p = \omega/\beta$

$$u_p = \frac{\omega}{\beta} \cong \frac{1}{\sqrt{\mu\epsilon'}}\left[1 - \frac{1}{8}\left(\frac{\epsilon''}{\epsilon'}\right)^2\right] \quad \text{(m/s)}.$$

$$\boxed{\beta \cong \omega\sqrt{\mu\epsilon'}\left[1 + \frac{1}{8}\left(\frac{\epsilon''}{\epsilon'}\right)^2\right]}$$

8-3.2 Good Conductors

- A good conductor
  - $\sigma / \omega\varepsilon >> 1$

$$\frac{\sigma}{\omega\varepsilon} << 1$$

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon}\left(1 + \frac{\sigma}{j\omega\epsilon}\right)^{1/2},$$

$\sigma / \omega\varepsilon >> 1$

$$\sqrt{j} = (e^{j\pi/2})^{1/2} = e^{j\pi/4} = (1 + j)/\sqrt{2}$$

$$\gamma \cong j\omega\sqrt{\mu\epsilon}\sqrt{\frac{\sigma}{j\omega\epsilon}} = \sqrt{j}\sqrt{\omega\mu\sigma} = \frac{1+j}{\sqrt{2}}\sqrt{\omega\mu\sigma}$$

$\omega = 2\pi f.$

$$\gamma = \alpha + j\beta \cong (1 + j)\sqrt{\pi f\mu\sigma},$$

low-loss : $\alpha \sim f$

good conductor: $\alpha \sim \sqrt{f}$

$$\boxed{\gamma = \alpha + j\beta \cong (1+j)\sqrt{\pi f \mu \sigma},}$$

$$\boxed{\alpha = \beta = \sqrt{\pi f \mu \sigma}.}$$

α and β are approximately equal

$\alpha, \beta \sim f^{1/2}, \mu^{1/2}, \sigma^{1/2}$

The intrinsic impedance $\eta_c = (\mu/\varepsilon)^{1/2} = (\mu/\varepsilon_c)^{1/2}$

$$\eta_c = \sqrt{\frac{\mu}{\epsilon_c}} \cong \sqrt{\frac{j\omega\mu}{\sigma}} = (1+j)\sqrt{\frac{\pi f \mu}{\sigma}} = \underline{(1+j)\frac{\alpha}{\sigma}} \quad (\Omega),$$

$\angle \eta_c = 45°$

$\boxed{\varepsilon_c \cong -j\sigma/\omega \text{ for a good conductor } (\sigma/\omega >> \varepsilon')}$

For a uniform plane wave $\eta_c = E_x/H_y$

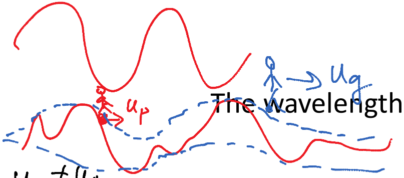
- For a good conductor ($\angle\eta_c = 45°$), H_y lags E_x by 45°

The phase velocity $u_p = \omega/\beta$

$$u_p = \frac{\omega}{\beta} \cong \sqrt{\frac{2\omega}{\mu\sigma}} \quad \text{(m/s)},$$

lossless: $k = \beta$

lossy: $\gamma = jk_c = \alpha + j\beta$ → u_p, λ



$u_p \neq u_g$

The wavelength of a plane wave

$$\boxed{\lambda = \frac{2\pi}{\beta}} = \frac{u_p}{f} = 2\sqrt{\frac{\pi}{f\mu\sigma}} \quad \text{(m)}.$$

Example: copper, f = 3MHz

1. u_p

$$\sigma = 5.80 \times 10^{7} \quad \text{(S/m)},$$
$$\mu = 4\pi \times 10^{-7} \quad \text{(H/m)},$$

$$\boxed{u_p = \frac{\omega}{\beta} \cong \sqrt{\frac{2\omega}{\mu\sigma}} \quad \text{(m/s)},}$$

$$u_p = 720 \text{ (m/s)} \quad \text{at} \quad 3 \text{ (MHz)},$$

Much slower than the velocity of light in air

2. λ

$$\boxed{\lambda = u_p/f}$$

$$\lambda = 0.24 \text{ (mm)}.$$

Much shorter than electromagnetic wave in air (λ = 100m)

3. α

$$\boxed{\alpha = \beta = \sqrt{\pi f \mu \sigma}.}$$

$$\alpha = \sqrt{\pi(3 \times 10^{6})(4\pi \times 10^{-7})(5.80 \times 10^{7})} = \underline{2.62 \times 10^{4}} \quad \text{(Np/m)}.$$

$e^{-\alpha z}$

Very large attenuation

Skin Depth (Depth of Penetration)

- Attenuation: $e^{-\alpha z}$, $= e^{-1}$
- When $z = 1/\alpha$, the intensity reduces to e^{-1}
  ➔ The amplitude of a wave will be attenuated by a factor of e^{-1} (= 0.368) when it travels a distance $\delta = 1/\alpha$ (skin depth)

$$\boxed{\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f \mu \sigma}} \quad \text{(m)}.}$$

  - For copper, f = 3MHz: $\delta = 1/\alpha = 1/(2.62\times10^4)$ (m) = 0.038 (mm)
  - For copper, f = 10GHz: δ = 0.66 (μm) (very small distance)

- Thus, a high-frequency electromagnetic wave is attenuated very rapidly as it propagates in a good conductor. $\delta \sim \frac{1}{\sqrt{f}}$
- At microwave frequencies, the fields and currents can be considered confined in a very thin layer (i.e., in the skin) of the conductor surface.
- For a good conductor, $\alpha = \beta$, $\delta = 1/\beta$

$$\boxed{\delta = \frac{1}{\beta} = \frac{\lambda}{2\pi} \qquad (\mathrm{m}).}$$

TABLE 8–1
Skin Depths, δ in (mm), of Various Materials

Material	σ (S/m)	$f = 60$ (Hz)	1 (MHz)	1 (GHz)
Silver	6.17×10^7	8.27 (mm)	0.064 (mm)	0.0020 (mm)
Copper	5.80×10^7	8.53	0.066	0.0021
Gold	4.10×10^7	10.14	0.079	0.0025
Aluminum	3.54×10^7	10.92	0.084	0.0027
Iron ($\mu_r \cong 10^3$)	1.00×10^7	0.65	0.005	0.00016
Seawater	4	32 (m)	0.25 (m)	†

$\frac{\sigma}{\omega\varepsilon} = \ldots = 200 \gg 1 \Rightarrow$ approximated as a good conductor

$\sigma = 4$

$\omega = 10^7\pi$

$\varepsilon = \frac{10^{-9}}{36\pi} \cdot 72$

EXAMPLE 8–4 The electric field intensity of a linearly polarized uniform plane wave propagating in the $+z$-direction in seawater is $\mathbf{E} = \mathbf{a}_x 100 \cos(10^7 \pi t)$ (V/m) at $z = 0$. The constitutive parameters of seawater are $\epsilon_r = 72$, $\mu_r = 1$, and $\sigma = 4$ (S/m). (a) Determine the attenuation constant, phase constant, intrinsic impedance, phase velocity, wavelength, and skin depth. (b) Find the distance at which the amplitude of $\mathbf{E}$ is 1% of its value at $z = 0$. (c) Write the expressions for $\mathbf{E}(z, t)$ and $\mathbf{H}(z, t)$ at $z = 0.8$ (m) as functions of t.

(a) α? β? η_c? u_p? λ? δ?

(b) z where E reduces to 1%?

$e^{-\alpha z} = 1\% \Rightarrow z = \ldots$

(c) $E(t)$ and $H(t)$? when $z = 0.8$

$$E(t) = \hat{a}_x \, 100 \, e^{-\alpha z} \cos(\omega t - \beta z)$$

$z = 0$ &

$$H(t) = \hat{a}_y \frac{100}{|\eta_c|} e^{-\alpha z} \cos(\omega t - \beta z - \angle \eta_c)$$

8-3.3 Ionized Gases (excluded)

8-4 Group Velocity

- Phase velocity: velocity of propagation of an equiphase wavefront

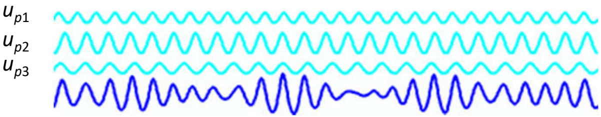
$k = \beta$

  - For plane waves in lossless media:
    β is a linear function of ω $\beta = \omega\sqrt{\mu\epsilon}$
    ➔ $u_p = 1/\sqrt{\mu\epsilon}$ a constant, independent of ω

$$\boxed{u_p = \frac{\omega}{\beta} \quad \text{(m/s).}} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu\varepsilon}}$$

  - In some cases (e.g., in lossy dielectrics):
    β is not a linear function of ω
    ➔ $u_p(\omega)$

- Dispersion: signal distortion due to $u_p(\omega)$
  - Waves of the component frequencies travel with different phase velocities
    ➔ a distortion in the signal wave shape
  - A lossy dielectric is a dispersive medium



a distortion in waveform

Through a non-dispersive medium

Through a dispersive medium

$u_p(\omega)$

Waveform is changing with time when they propagate (i.e., distortion)

Group Velocity

- An information bearing signal has a small spread of frequencies (Δf) around a high carrier frequency (f_c).
- Group velocity u_g: the velocity of propagation of the wave-packet envelope of a group of frequencies

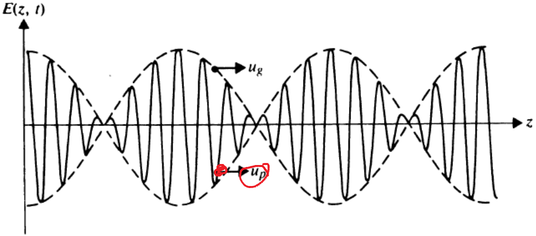


FIGURE 8–6
Sum of two time-harmonic traveling waves of equal amplitude and slightly different frequencies at a given t.

- Consider the simplest case of a wave packet
  - Two travelling waves with equal amplitude and slightly different angular frequencies $\omega_0 + \Delta\omega \qquad \omega_0 - \Delta\omega \qquad (\Delta\omega \ll \omega_0)$

    ➔ the corresponding phase constants

$$\beta_0 + \Delta\beta \qquad \beta_0 - \Delta\beta.$$

$$\Downarrow$$

$$\begin{aligned} E(z,t) &= E_0 \cos\left[(\omega_0 + \Delta\omega)t - (\beta_0 + \Delta\beta)z\right] \\ &\quad + E_0 \cos\left[(\omega_0 - \Delta\omega)t - (\beta_0 - \Delta\beta)z\right] \\ &= 2E_0 \cos(t\,\Delta\omega - z\,\Delta\beta)\cos(\omega_0 t - \beta_0 z). \end{aligned}$$

$$E(z,t) = E_0 \cos[(\omega_0 + \Delta\omega)t - (\beta_0 + \Delta\beta)z] + E_0 \cos[(\omega_0 - \Delta\omega)t - (\beta_0 - \Delta\beta)z] = 2E_0 \cos(t\,\Delta\omega - z\,\Delta\beta)\cos(\omega_0 t - \beta_0 z).$$

$\Delta\omega \ll \omega_0$

A slowly-varying envelope ($\Delta\omega$)

group velocity $u_g = \frac{dz}{dt} = \frac{\Delta\omega}{\Delta\beta} = \frac{1}{\Delta\beta/\Delta\omega}.$

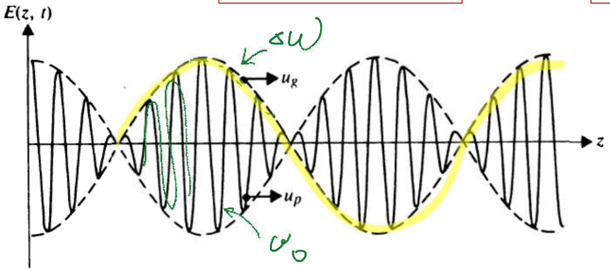
Velocity of the envelope

A rapidly oscillating wave (ω_0)

$\omega_0 t - \beta_0 z = \text{Constant}$

$u_p = \frac{dz}{dt} = \frac{\omega_0}{\beta_0}.$

Velocity of the carrier



Group velocity: the velocity of a point on the envelope of the wave packet

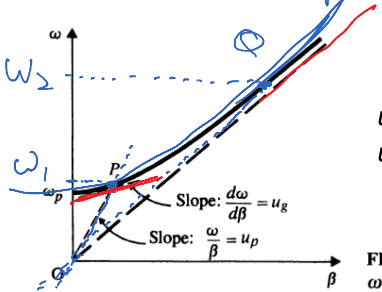
FIGURE 8–6
Sum of two time-harmonic traveling waves of equal amplitude and slightly different frequencies at a given t.

$$u_g = \frac{dz}{dt} = \frac{\Delta\omega}{\Delta\beta} = \frac{1}{\Delta\beta/\Delta\omega}.$$

In the limit $\Delta\omega \to 0$ (narrow-band signal)

$$\boxed{u_g = \frac{1}{d\beta/d\omega} \qquad \text{(m/s).}}$$

dispersive



$u_p(\omega_1) > u_p(\omega_2)$

u_p: the slope drawn from the origin to a point

u_g: local slope

at ω_1, $u_p > u_g$

FIGURE 8–7
ω–β graph for ionized gas.

- In an ionized medium:

8-3.3

$$\boxed{\gamma = j\omega\sqrt{\mu\epsilon_0}\sqrt{1-\left(\frac{f_p}{f}\right)^2},}$$

$$\beta = \omega\sqrt{\mu\epsilon_0}\sqrt{1-\left(\frac{f_p}{f}\right)^2}$$

$$= \frac{\omega}{c}\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}.$$

plasma frequency

<1 $\Rightarrow \beta$ real

- At $\omega = \omega_p$, $\beta = 0$.
- At $\omega > \omega_p$, wave propagation is possible

$$\boxed{u_g = \frac{1}{d\beta/d\omega}}$$

$$u_p = \frac{\omega}{\beta} = \frac{c}{\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}}. \qquad u_g = c\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}.$$

$\boxed{u_p \geq c \text{ and } u_g \leq c; u_p u_g = c^2}$

Similar situation exists in waveguides!

- A general relation between u_g and u_p:

$$\boxed{u_p = \frac{\omega}{\beta}} \qquad \boxed{u_g = \frac{1}{d\beta/d\omega}}$$

$\omega \uparrow \Rightarrow \lambda \downarrow \Rightarrow$ violet $\Rightarrow u_p \downarrow \Rightarrow n \uparrow$

$$\frac{d\beta}{d\omega} = \frac{d}{d\omega}\left(\frac{\omega}{u_p}\right) = \frac{1}{u_p} - \frac{\omega}{u_p^2}\frac{du_p}{d\omega}.$$

$\omega \downarrow \Rightarrow \lambda \uparrow \Rightarrow$ red $\Rightarrow u_p \uparrow \Rightarrow n \downarrow$

$$\boxed{u_g = \frac{1}{d\beta/d\omega}}$$

$n = \frac{c}{u_p}$

$$u_g = \frac{u_p}{1 - \dfrac{\omega}{u_p}\dfrac{du_p}{d\omega}}.$$

< 0

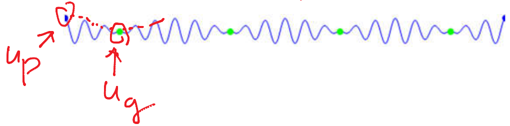
a) No dispersion: $\frac{du_p}{d\omega} = 0$ $\quad u_g = u_p.$ $\quad u_p$ independent of ω

b) Normal dispersion: $\frac{du_p}{d\omega} < 0$ $\quad u_g < u_p.$ $\quad u_p$ decreasing with ω

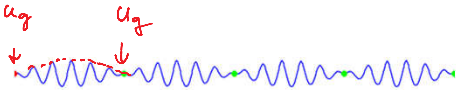
c) Anomalous dispersion: $\frac{du_p}{d\omega} > 0$ $\quad u_g > u_p.$ $\quad u_p$ increasing with ω

glass — Example of a material with normal dispersion?

No dispersion: $\dfrac{du_p}{d\omega} = 0$ $\quad u_g = u_p.$ $\quad u_p$ independent of ω



Normal dispersion: $\dfrac{du_p}{d\omega} < 0$ $\quad u_g < u_p.$ $\quad u_p$ decreasing with ω



Red dot: u_p; Green dot: u_g

loss tangent $\triangleq \frac{\varepsilon''}{\varepsilon'} = 0.2 < 1$

$\left(\frac{\varepsilon''}{\varepsilon'}\right)^2 \ll 1 \Rightarrow$ low-loss dielectrics

neglect H.O.T. $\left(\frac{\varepsilon''}{\varepsilon'}\right)^3, \left(\frac{\varepsilon''}{\varepsilon'}\right)^4, \ldots$

(a) $\boxed{\alpha \cong \frac{\omega\varepsilon''}{2}\sqrt{\frac{\mu}{\varepsilon'}}} = \frac{\cdots}{\cdots} = 1.82\times10^{-3}$ (Np/m)

EXAMPLE 8–6 A narrow-band signal propagates in a lossy dielectric medium which has a loss tangent 0.2 at 550 (kHz), the carrier frequency of the signal. The dielectric constant of the medium is 2.5. (a) Determine α and β. (b) Determine u_p and u_g. Is the medium dispersive?

ε_r $\qquad$ f

$\boxed{\beta \cong \omega\sqrt{\mu\varepsilon'}\left[1+\frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right]}$ 0.04

$\omega = 2\pi \cdot 550\times10^3$

$= \cdots = 0.0183$ (rad/m)

$\varepsilon' = \varepsilon_0\varepsilon_r$

$= 2.5\varepsilon_0$

$\varepsilon'' = 0.2\varepsilon' = 0.5\varepsilon_0$

$\mu = \mu_0$

(b) $u_p = \dfrac{\omega}{\beta} = \dfrac{\omega}{\underset{\text{(a)}}{\uparrow}} = \dfrac{1}{\underline{\sqrt{\mu\varepsilon'}\left[1+\frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right]}} = 1.\cancel{88}8 \times 10^8 \ (m/s)$

$u_g = \dfrac{d\omega}{d\beta}$

$\underline{\beta \simeq \omega\sqrt{\mu\varepsilon'}}$

$\dfrac{d\beta}{d\omega} \simeq \sqrt{\mu\varepsilon'}\underbrace{\left[1+\frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right]}_{} = \dfrac{1}{u_p}$ $\Rightarrow \underline{u_g \simeq u_p}$

+ $\mathcal{O}(\delta^3)$

not dispersive

for high-loss dielectrics $\Rightarrow u_g \neq u_p$, dispersive

8-5 Flow of Electromagnetic Power and the Poynting Vector

- Electromagnetic waves carry with them electromagnetic power.
- Energy is transported through space to distant receiving points by electromagnetic waves.

Vector identity

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}).$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{E} \cdot \mathbf{J}.$$

$(fg)' = f'g + g'f$

In a simple medium, whose ε, μ, and σ do not change with time

Product rule

μ is const. of time

$$\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = \mathbf{H} \cdot \frac{\partial (\mu \mathbf{H})}{\partial t} = \frac{1}{2} \frac{\partial (\mu \mathbf{H} \cdot \mathbf{H})}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2} \mu H^2\right),$$

$$\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = \mathbf{E} \cdot \frac{\partial (\epsilon \mathbf{E})}{\partial t} = \frac{1}{2} \frac{\partial (\epsilon \mathbf{E} \cdot \mathbf{E})}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2} \epsilon E^2\right),$$

$$\mathbf{E} \cdot \mathbf{J} = \mathbf{E} \cdot (\sigma \mathbf{E}) = \sigma E^2.$$

$\vec{A} \cdot \vec{A} = A^2$

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\frac{\partial}{\partial t}\left(\frac{1}{2} \epsilon E^2 + \frac{1}{2} \mu H^2\right) - \sigma E^2,$$

A point-function relationship

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right) - \sigma E^2,$$

Integration over the volume of concern & Divergence theorem

Watt/m^2

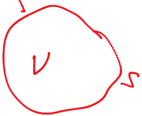
energy density : J/m^3

P_{EM}

An integral form

$$\oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s} = -\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right) dv - \int_V \sigma E^2 \, dv,$$

Watt



The time-rate of change of the energy stored in the electric and magnetic fields, respectively

Ohmic power dissipation (due to conduction current)

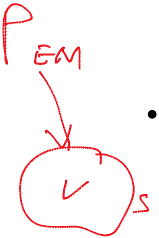
Law of conservation of energy

- Right side: the rate of decrease of the electric and magnetic energies stored, subtracted by the ohmic power dissipated as heat in the volume V
- Left side: power (rate of energy) leaving the volume through its surface

Power flow per unit area $\boxed{\mathscr{P} = \mathbf{E} \times \mathbf{H} \qquad (\mathrm{W/m^2}).}$

Known as Poynting vector, a power density vector associated with electromagnetic field

- Poynting theorem: the surface integral of ***P*** (=**E**×**H**) over a closed surface = power leaving the enclosed volume



$$\oint_S (\mathbf{E}\times\mathbf{H})\cdot ds = -\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right) dv - \int_V \sigma E^2\, dv,$$

- Another form

$$-\oint_S \mathcal{P}\cdot ds = \frac{\partial}{\partial t}\int_V (w_e + w_m)\,dv + \int_V p_\sigma\, dv,$$

Watt

where
$$w_e = \tfrac{1}{2}\epsilon E^2 = \tfrac{1}{2}\epsilon\mathbf{E}\cdot\mathbf{E}^* = \text{Electric energy density},$$
$$w_m = \tfrac{1}{2}\mu H^2 = \tfrac{1}{2}\mu\mathbf{H}\cdot\mathbf{H}^* = \text{Magnetic energy density},$$
$$p_\sigma = \sigma E^2 = J^2/\sigma = \sigma\mathbf{E}\cdot\mathbf{E}^* = \mathbf{J}\cdot\mathbf{J}^*/\sigma = \text{Ohmic power density}.$$

  - The total power flowing into a closed surface at any instant = the sum of rates of increase of the stored electric and magnetic energies and the ohmic power dissipated within the enclosed volume

$$-\oint_S \mathscr{P} \cdot d\mathbf{s} = \frac{\partial}{\partial t}\int_V (w_e + w_m)\, dv + \int_V p_\sigma\, dv,$$

0

for static

- ***P*** **(=E×H)**
  - ***P*⊥E, *P*⊥H**
- Lossless case: σ=0
  - Right side: only the rate of increase of the stored electric and magnetic energies
- Static case: $\partial/\partial t$ = 0
  - Right side: only the ohmic power dissipated in the enclosed volume

8-5.1 Instantaneous and Average Power Densities

- Phasor: $\mathbf{E}(z) = \mathbf{a}_x E_x(z) = \mathbf{a}_x E_0 e^{-(\alpha + j\beta)z}$,

  Instantaneous expression: $\mathbf{E}(z, t) = \mathcal{R}e[\mathbf{E}(z)e^{j\omega t}] = \mathbf{a}_x E_0 e^{-\alpha z}\, \mathcal{R}e[e^{j(\omega t - \beta z)}]$
  $= \mathbf{a}_x E_0 e^{-\alpha z} \cos(\omega t - \beta z).$

- For a uniform plane wave propagating in a lossy medium in the +z direction, the **H** field:

  $\vec{H} = \frac{1}{\eta}\hat{a}_n \times \vec{E}$

  Phasor $\mathbf{H}(z) = \mathbf{a}_y H_y(z) = \mathbf{a}_y \frac{E_0}{|\eta|} e^{-\alpha z} e^{-j(\beta z + \theta_\eta)}$,

  Due to lossy media, $\exp(-j\vartheta_\eta)$

  The intrinsic impedance of the medium $\eta = |\eta| e^{j\theta_\eta}$

  +z propagation

  Instantaneous expression $\mathbf{H}(z, t) = \mathcal{R}e[\mathbf{H}(z)e^{j\omega t}] = \mathbf{a}_y \frac{E_0}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta).$

$$\mathcal{R}e[\mathbf{E}(z)e^{j\omega t}] \times \mathcal{R}e[\mathbf{H}(z)e^{j\omega t}] \neq \mathcal{R}e[\cancel{\mathbf{E}(z) \times \mathbf{H}(z)}e^{j\omega t}].$$

Or $a\cdot\cos\omega t \times b\cdot\cos\omega t \neq ab\cos\omega t$

To get time-domain Poynting vector $\mathcal{P}(z, t)$, one cannot do the simple cross product of **E** and **H** in phasor domain and then change it to time-domain expression!

$\vec{E}(t) \times \vec{H}(t)$

$//$

$\vec{P}(t,z)$

$\vec{P}_{av}(z)$

Method 1: check in time domain directly

Thus, for the instantaneous expression for Poynting vector:

$$\mathbf{E}(z, t) = \mathbf{a}_x E_0 e^{-\alpha z} \cos(\omega t - \beta z).$$

$$\mathbf{H}(z, t) = \mathbf{a}_y \frac{E_0}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta).$$

$$\mathcal{P}(z, t) = \mathbf{E}(z, t) \times \mathbf{H}(z, t) = \mathcal{R}e[\mathbf{E}(z)e^{j\omega t}] \times \mathcal{R}e[\mathbf{H}(z)e^{j\omega t}]$$

E, **H** in phasors

$$= \mathbf{a}_z \frac{E_0^2}{|\eta|} e^{-2\alpha z} \cos(\omega t - \beta z)\cos(\omega t - \beta z - \theta_\eta)$$

$$= \mathbf{a}_z \frac{E_0^2}{2|\eta|} e^{-2\alpha z}[\cos\theta_\eta + \cos(2\omega t - 2\beta z - \theta_\eta)].$$

Correct expression for instantaneous $\mathcal{P}$

Obviously, not equal to

$$\cancel{\mathcal{R}e[\mathbf{E}(z) \times \mathbf{H}(z)e^{j\omega t}] = \mathbf{a}_z \frac{E_0^2}{|\eta|} e^{-2\alpha z} \cos(\omega t - 2\beta z - \theta_\eta)},$$

Time-average Poynting vector, $\boldsymbol{P}_{\text{av}}(z)$:

$$\mathscr{P}(z, t) = \mathbf{a}_z \frac{E_0^2}{2|\eta|} e^{-2\alpha z}[\cos\theta_\eta + \cos(2\omega t - 2\beta z - \theta_\eta)].$$

Independent of t

Average = 0 over $T/2$

Integration over a period T

$$\mathscr{P}_{\text{av}}(z) = \frac{1}{T}\int_0^T \mathscr{P}(z, t)\,dt = \mathbf{a}_z \frac{E_0^2}{2|\eta|} e^{-2\alpha z}\cos\theta_\eta \quad (\text{W/m}^2),$$

where $T = 2\pi/\omega$

As far as the power transmitted by an electromagnetic wave is concerned, its average value is a more significant quantity than its instantaneous value.

Method 2: check in time domain with phasor expression

$\vec{\mathcal{P}}_{av}$

Consider two general complex vectors **A** and **B**:

*: complex conjugate of

$$\mathcal{R}e(\mathbf{A}) = \tfrac{1}{2}(\mathbf{A} + \mathbf{A}^*) \quad \text{and} \quad \mathcal{R}e(\mathbf{B}) = \tfrac{1}{2}(\mathbf{B} + \mathbf{B}^*),$$

$$\begin{aligned}\mathcal{R}e(\mathbf{A}) \times \mathcal{R}e(\mathbf{B}) &= \tfrac{1}{2}(\mathbf{A} + \mathbf{A}^*) \times \tfrac{1}{2}(\mathbf{B} + \mathbf{B}^*) \\ &= \tfrac{1}{4}[(\mathbf{A} \times \mathbf{B}^* + \mathbf{A}^* \times \mathbf{B}) + (\mathbf{A} \times \mathbf{B} + \mathbf{A}^* \times \mathbf{B}^*)] \\ &= \tfrac{1}{2}\mathcal{R}e(\mathbf{A} \times \mathbf{B}^* + \mathbf{A} \times \mathbf{B}).\end{aligned}$$

(C+C*)=2Re[C]

Express the instantaneous Poynting vector (in phasors):

$$\begin{aligned}\mathcal{P}(z,t) &= \mathcal{R}e[\mathbf{E}(z)e^{j\omega t}] \times \mathcal{R}e[\mathbf{H}(z)e^{j\omega t}] \\ &= \tfrac{1}{2}\mathcal{R}e[\mathbf{E}(z) \times \mathbf{H}^*(z) + \mathbf{E}(z) \times \mathbf{H}(z)e^{j2\omega t}].\end{aligned}$$

$\vec{A}$ $\vec{B}$

$\vec{A} \times \vec{B}^*$

Let **A** = **E**$e^{j\omega t}$; **B** = **H**$e^{j\omega t}$

Independent of t

Integrating $\mathcal{P}(z, t)$ over a fundamental period T

Average of the last term ($e^{j2\omega t}$) vanishes

Time-average Poynting vector, $\mathcal{P}_{av}(z)$:

$$\mathcal{P}_{av}(z) = \tfrac{1}{2}\mathcal{R}e[\tilde{\mathbf{E}}(z) \times \tilde{\mathbf{H}}^*(z)].$$

The general formula for computing the average power density in a propagating wave $\mathcal{P}_{av}$:

$$\boxed{\mathcal{P}_{av}(z) = \tfrac{1}{2}\mathcal{R}e[\mathbf{E}(z) \times \mathbf{H}^*(z)].}$$

Not necessarily propagating in *z* direction

General expression

$$\boxed{\mathcal{P}_{av} = \tfrac{1}{2}\mathcal{R}e(\mathbf{E} \times \mathbf{H}^*) \qquad (\mathrm{W/m^2}),}$$

Point form:

$\mathcal{P}$: power density vector

Watt/m²

for a certain point

Recall in circuits: $P_{av} = ½ \, Re(VI^*)$

P: Power

Watt

for a certain element

Analogy between electromagnetics and circuits:

	Electromagnetics	Circuits
	E	*V*
	H	*I*
Impedance	η = E/H	*Z* = *V*/*I*

- Power density vector, $\mathcal{P}$: EM
  - W/m^2
  - vector (energy propagation direction)
  - a point value
- Power, P: circuits
  - Watt
  - scalar
  - a value for a certain volume

EXAMPLE 8–7 Find the Poynting vector on the surface of a long, straight conducting wire (of radius b and conductivity σ) that carries a direct current I. Verify Poynting's theorem.

$$\vec{J} = \hat{a}_z \frac{I}{\pi b^2}$$

$$\vec{E} = \frac{\vec{J}}{\sigma} = \hat{a}_z \frac{I}{\sigma \pi b^2}$$

$$\vec{H} = \hat{a}_\phi \frac{I}{2\pi b}$$

$$\vec{J} = \sigma \vec{E}$$

$$\vec{P} = \vec{E} \times \vec{H} = -\hat{a}_r \frac{I^2}{2\pi^2 \sigma b^3} \ (W/m^2)$$

$$-\oint \vec{P} \cdot d\vec{s} = \int_0^{2\pi} \int_0^{\ell} \frac{I^2}{2\pi^2 \sigma b^3} \, dz \, b \, d\phi = \dots = I^2 \frac{\ell}{\pi \sigma b^2} = I^2 R$$

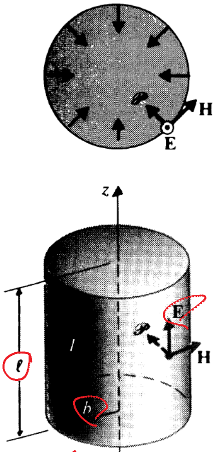


FIGURE 8–8
Illustrating Poynting's theorem (Example 8–7).

where $R = \frac{\ell}{\sigma A}$ $A = \pi b^2$

verified. #