# Chapter 5 Steady Electric Currents

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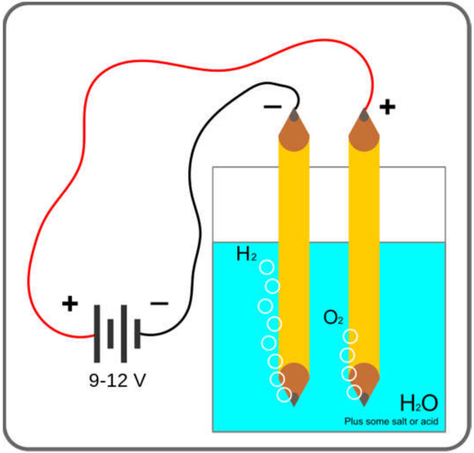
# 5-1 Introduction

- Charges at rest (Ch3 and Ch4); Charges in motion (Ch5)
- Different types of currents
  - Conduction currents:
    - in conductors and semiconductors
    - electrons and/or holes
  - Electrolytic currents:
    - essentially in a liquid medium
    - ions (e.g., Li-ion batteries)
  - Convection currents:
    - in vacuum or rarefied gas
    - electrons and/or ions

# Electrolysis of Water




Oxidation at anode: $2\ H_2O(l) \rightarrow \underline{O_2(g)} + 4\ H^+(aq) + 4e^-$

Reduction at cathode: $2\ H^+(aq) + 2e^- \rightarrow \underline{H_2(g)}$

# Topics for Conduction Currents

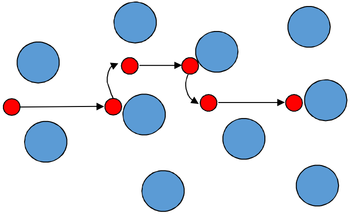
- Point form of Ohm’s law
- Kirchhoff’s voltage law
- Kirchhoff’s current law
  - Conservation of charge
  - Equation of continuity
- Boundary conditions for current density

# Conduction Currents

- For a conductor, atoms consist of positively charged nuclei surrounded by electrons
  - Inner shell: tightly bound charges
  - Outermost shell: loosely bound charges (valance or conduction electrons)
- Without external **E**, conduction electrons wander randomly ➔ **no net drift motion** of conduction electrons

# Conduction Currents

- With external **E**, **organized motion** of conduction electrons
  - Very low drift velocity due to collision with atoms
  - Conductor remains electrically neutral
    (electric forces prevent excess electrons from accumulating at any point of a conductor)



# 5-2 Current Density and Ohm's Law

charge $q$ across surface $\Delta$s with a velocity **u**
$N$: #/volume

The amount of charge passing $\Delta$s $\quad \Delta Q = Nq\underline{\mathbf{u} \cdot \mathbf{a}_n \Delta s \Delta t} \quad (\text{C}).$

differential volume with $\Delta$**s** along $\mathbf{a}_n$

$$\Delta I = \frac{\Delta Q}{\Delta t} = Nq\mathbf{u} \cdot \mathbf{a}_n \Delta s = \underline{Nq\mathbf{u}} \cdot \Delta\mathbf{s} \qquad (\text{A}).$$

$$\boxed{\mathbf{J} \equiv Nq\mathbf{u} \qquad (\text{A/m}^2),} \qquad \boxed{\mathbf{J} = \rho\mathbf{u} \qquad (\text{A/m}^2),}$$

**J** defined as (volume) current density

$$\Delta I = \mathbf{J} \cdot \Delta\mathbf{s}.$$

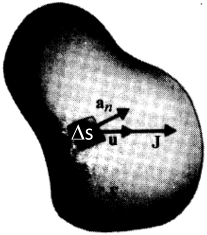



**FIGURE 5–1**
Conduction current due to drift motion of charge carriers across a surface.

$$\Delta I = \mathbf{J} \cdot \Delta \mathbf{s}.$$

$$\downarrow$$

Total current I flowing through a surface S

$$\boxed{I = \int_S \mathbf{J} \cdot d\mathbf{s} \qquad \text{(A).}}$$

# Conduction Currents

- For more than one kind of charge carriers (electrons, holes, and ions) drifting, current density:

$$\boxed{\mathbf{J} = Nq\mathbf{u} \qquad (\text{A/m}^2),} \quad \Rightarrow \quad \mathbf{J} = \sum_i N_i q_i \mathbf{u}_i \qquad (\text{A/m}^2).$$

- For most conducting materials,

$$\mathbf{u} = -\mu_e \mathbf{E} \qquad (\text{m/s}),$$

**u**: averaged drift velocity

$-\mu_e$: electron mobility (m²/V·s)

$$\boxed{\mathbf{J} = \rho\mathbf{u} \qquad (\mathrm{A/m^2}),}$$

$$\mathbf{u} = -\mu_e\mathbf{E} \qquad (\mathrm{m/s}),$$

$$\mathbf{J} = -\rho_e\mu_e\mathbf{E},$$

where $\rho_e = -Ne$

$$\boxed{\mathbf{J} = \sigma\mathbf{E} \qquad (\mathrm{A/m^2}),}$$

Such materials are called ohmic media

where $\sigma = -\rho_e\mu_e$

$\rho_e$<0 because of electrons

σ: conductivity (A/V·m or S/m)

For conductors, $\sigma = -\rho_e\mu_e$

For semiconductors, $\sigma = -\rho_e\mu_e + \rho_h\mu_h,$

- Circuit form of Ohm's law $V_{12} = RI.$

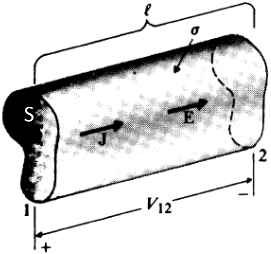
- Point form of Ohm's law $\boxed{\mathbf{J} = \sigma \mathbf{E} \qquad (\mathrm{A/m^2}),}$
  - Holds at all points
  - $\sigma$ can be a function of space

$$V_{12} = RI.$$

Integration?

$\mathbf{E} = (1/\sigma)\mathbf{J}$

# Ohm's Law: Point Form to Circuit Form




$$V_{12} = E\ell \quad \text{or} \quad E = \frac{V_{12}}{\ell}.$$

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} = JS \quad \text{or} \quad J = \frac{I}{S}.$$

**FIGURE 5–3**
Homogeneous conductor with a constant cross section.

$$\boxed{\mathbf{J} = \sigma \mathbf{E},}$$

$$J = \frac{I}{S}. \qquad E = \frac{V_{12}}{\ell}.$$

$$\frac{I}{S} = \sigma \frac{V_{12}}{\ell}$$

$$V_{12} = \left(\frac{\ell}{\sigma S}\right) I = RI,$$

The resistance

$$\boxed{R = \frac{\ell}{\sigma S} \qquad (\Omega).}$$

**EXAMPLE 5–1** In vacuum-tube diodes, electrons are emitted from a hot cathode at zero potential and collected by an anode maintained at a potential $V_0$, resulting in a convection current flow. Assuming that the cathode and the anode are parallel conducting plates and that the electrons leave the cathode with a zero initial velocity (space-charge limited condition), find the relation between the current density $J$ and $V_0$.

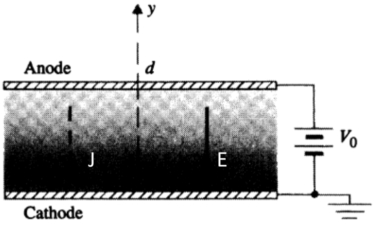



**FIGURE 5–2**
Space-charge-limited vacuum diode (Example 5–1).

Step1: $\mathbf{J} = \rho \mathbf{u}$
Step2: passion's equation (V- $\rho$ equation)

# 5-3 Electromotive Force and Kirchhoff's Voltage Law

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = 0.$$

For an ohmic material

$$\mathbf{J} = \sigma \mathbf{E},$$

$$\oint_C \frac{1}{\sigma} \mathbf{J} \cdot d\boldsymbol{\ell} = 0.$$

If **J** is constant, then **J** = 0.

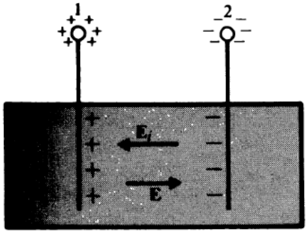
A **steady** current **cannot** be maintained in the same direction in a closed circuit by **an electrostatic field** (**conservative field**)

That is, to maintain a steady current in a closed circuit, there must be **non-conservative field** (e.g., electric batteries, etc.), which termed as impressed electric field intensity $E_i$

# Electromotive Force

- Chemical action ($\mathbf{E}_i$)
  ➔ cumulation of + and – charges on electrodes due to $\mathbf{E}_i$
  ➔ **E**
- Inside: **E** and $\mathbf{E}_i$
  - $\mathbf{E} = -\mathbf{E}_i$ due to I = 0 for open circuit
- Outside: **E** only

**E**: electrostatic field
$\mathbf{E}_i$: nonconservative field




Electric battery

Open circuit

**FIGURE 5–4**
Electric fields inside an electric battery.

# Electromotive Force

- $\mathbf{E}_i$

Against $\mathbf{E}_i$ from 1 to 2

$$\mathscr{V} = \int_2^1 \mathbf{E}_i \cdot d\ell = -\underbrace{\int_2^1 \mathbf{E} \cdot d\ell}_{\substack{\text{Inside} \\ \text{the source}}}.$$

Open circuit: $\mathbf{E}_i = -\mathbf{E}$

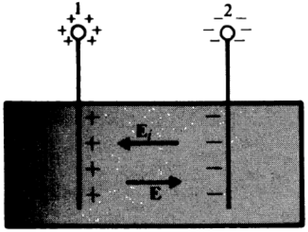
$\mathscr{V}$

- By definition: potential due to $\mathbf{E}_i$
- The electromotive force is a measure of the strength of the nonconservative source, **not a force**

- $\mathbf{E}$

$$\oint_C \mathbf{E} \cdot d\ell = \underbrace{\int_1^2 \mathbf{E} \cdot d\ell}_{\substack{\text{Outside} \\ \text{the source}}} + \underbrace{\int_2^1 \mathbf{E} \cdot d\ell}_{\substack{\text{Inside} \\ \text{the source}}} = 0.$$

Because of conservative field




Open circuit

**FIGURE 5–4**
Electric fields inside an electric battery.

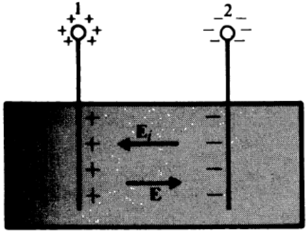
$$\boxed{\mathscr{V} = \int_2^1 \mathbf{E}_i \cdot d\ell = -\underbrace{\int_2^1 \mathbf{E} \cdot d\ell}_{\text{Inside the source}}.}$$

$$\boxed{\oint_C \mathbf{E} \cdot d\ell = \underbrace{\int_1^2 \mathbf{E} \cdot d\ell}_{\text{Outside the source}} + \underbrace{\int_2^1 \mathbf{E} \cdot d\ell}_{\text{Inside the source}} = 0.}$$

$$\mathscr{V} = \underbrace{\int_1^2 \mathbf{E} \cdot d\ell}_{\text{Outside the source}} = -\underbrace{\int_2^1 \mathbf{E} \cdot d\ell}_{\text{Outside the source}} = V_{12} = V_1 - V_2.$$

emf = voltage rise between + and – terminals (outside)

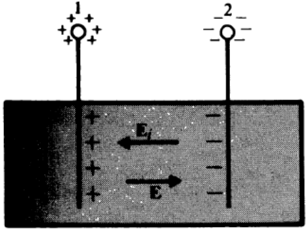
**$\mathbf{E}_i$ can produce a voltage difference**



Electric battery

Open circuit

**FIGURE 5–4**
Electric fields inside an electric battery.

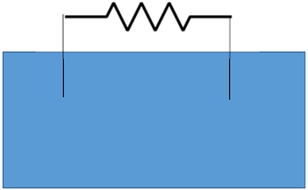



Electric battery

Open circuit → No currents

**FIGURE 5–4**
Electric fields inside an electric battery.

If connected with a resistor → Currents



Point form of Ohm's law

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{E}_i),$$

$$\mathbf{E} + \mathbf{E}_i = \frac{\mathbf{J}}{\sigma}.$$

$$\mathcal{V} = \oint_C (\mathbf{E} + \mathbf{E}_i) \cdot d\boldsymbol{\ell} = \oint_C \frac{1}{\sigma} \mathbf{J} \cdot d\boldsymbol{\ell}.$$

$$\boxed{\mathbf{E} + \mathbf{E}_i = \frac{\mathbf{J}}{\sigma}.}$$

Integration of 1st integrand = 0

$$\boxed{\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = \underbrace{\int_1^2 \mathbf{E} \cdot d\boldsymbol{\ell}}_{\substack{\text{Outside} \\ \text{the source}}} + \underbrace{\int_2^1 \mathbf{E} \cdot d\boldsymbol{\ell}}_{\substack{\text{Inside} \\ \text{the source}}} = 0.}$$

Integration of 2nd integrand

$$\oint_C \mathbf{E}_i \cdot d\boldsymbol{\ell} = \underbrace{\int_1^2 \mathbf{E}_i \cdot d\boldsymbol{\ell}}_{\substack{\text{Outside} \\ \text{the source}}}^{0} + \underbrace{\int_2^1 \mathbf{E}_i \cdot d\boldsymbol{\ell}}_{\substack{\text{Inside} \\ \text{the source}}}$$

$= \mathcal{V}$

For a resistor with uniform cross section:

$$J = I/S \qquad R = \frac{\ell}{\sigma S}$$

$$\mathcal{V} = RI.$$

$$\mathscr{V} = RI.$$

For more-than-one emf and more-than-one resistor connected in series

$$\boxed{\sum_j \mathscr{V}_j = \sum_k R_k I_k \qquad \text{(V).}}$$

Kirchhoff's voltage law: around a closed path in an electric circuit, the algebraic sum of the emf's (voltage rises) is equal to the algebraic sum of the voltage drops across the resistances

# 5-4 Equation of Continuity and Kirchhoff's Current Law

- Principle of conservation of charge: electric charges may not be created or destroyed

$$\underbrace{I = \oint_S \mathbf{J} \cdot d\mathbf{s}}_{\text{Current leaving a volume}} = \underbrace{-\frac{dQ}{dt}}_{\text{Rate of charge decrease in the volume}} = -\frac{d}{dt}\int_V \rho \, dv.$$

$$\int_V \nabla \cdot \mathbf{J} \, dv = -\int_V \frac{\partial \rho}{\partial t} \, dv.$$

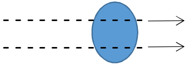
$$\int_V \nabla \cdot \mathbf{J}\, dv = -\int_V \frac{\partial \rho}{\partial t}\, dv.$$

Holds for arbitrary volume *V*

Equation of continuity

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \qquad (\mathrm{A/m^3}).}$$

- For **steady current (I = constant)**, charge density does not vary with time (or charge in a volume is a constant over time although charge is moving): $\boxed{\partial\rho/\partial t = 0.}$



$$\nabla \cdot \mathbf{J} = 0.$$ (divergenceless: streamlines of steady currents close upon themselves)

$$\oint_S \mathbf{J} \cdot d\mathbf{s} = 0,$$ (integral form)

For $S$ →0 (i.e., the volume shrinks to a point)

$$\boxed{\sum_j I_j = 0 \qquad \text{(A).}}$$

Kirchhoff's current law: the algebraic sum of all the currents flowing **out of a junction** (a small volume) in an electric circuit is zero.

# Time to Reach Equilibrium in a Conductor

- Inside a conductor, $\rho = 0$, $\mathbf{E} = 0$ under equilibrium conditions (Chap. 3)
- Time to reach equilibrium?

Equation of continuity

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \qquad (\mathrm{A/m^3}).}$$

Assume a homogeneous $\sigma$

$$\sigma \nabla \cdot \mathbf{E} = -\frac{\partial \rho}{\partial t}.$$

$$\sigma \nabla \cdot \mathbf{E} = -\frac{\partial \rho}{\partial t}.$$

In a simple medium

$$\nabla \cdot \mathbf{E} = \rho/\epsilon$$

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon}\rho = 0.$$

Solution: $$\boxed{\rho = \rho_0 e^{-(\sigma/\epsilon)t} \qquad (\mathrm{C/m^3}),}$$

Charge density inside a conductor will decrease with time exponentially.

**Relaxation time**: time for $\rho_0$ to decay to $1/e \times \rho_0$

$$\tau = \frac{\epsilon}{\sigma} \qquad (\mathrm{s}).$$

# 5-5 Power Dissipation and Joule's Law

- Power dissipation:

  External **E**
  ➔ drift motion of electrons, which collide with atoms on lattice sites
  ➔ thermal energy

- Power by **E** to move a charge q

$$p = \lim_{\Delta t \to 0} \frac{\Delta w}{\Delta t} = q\mathbf{E} \cdot \mathbf{u},$$

**u**: drift velocity

Differential power in a volume dv

$$dP = \sum_i p_i = \mathbf{E} \cdot \left( \sum_i N_i q_i \mathbf{u}_i \right) dv,$$

$\boxed{\text{Total } Q \text{ in a volume } dv}$

$$dP = \sum_i p_i = \mathbf{E} \cdot \left( \sum_i N_i q_i \mathbf{u}_i \right) dv,$$

$$\boxed{\mathbf{J} = \sum_i N_i q_i \mathbf{u}_i \qquad (\mathrm{A/m^2}).}$$

$$dP = \mathbf{E} \cdot \mathbf{J}\, dv$$

$$\frac{dP}{dv} = \underline{\mathbf{E} \cdot \mathbf{J}} \qquad (\mathrm{W/m^3}).$$

Power density

or

$$\boxed{P = \int_V \mathbf{E} \cdot \mathbf{J}\, dv \qquad (\mathrm{W}).}$$

Joule's law

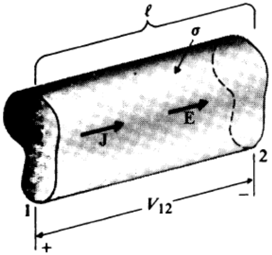
$$\boxed{P = \int_V \mathbf{E} \cdot \mathbf{J}\, dv \qquad \text{(W)}.}$$

In a conductor with constant cross section

$$dv = ds\, d\ell$$

$$P = \int_L E\, d\ell \int_S J\, ds = VI,$$

We get the familiar expression.



# 5-6 Boundary Conditions for Current Density

- **Steady** current density **J** on boundaries without nonconservative energy source

| Governing Equations for Steady Current Density | |
|---|---|
| Differential Form | Integral Form |
| $\nabla \cdot \mathbf{J} = 0$ | $\oint_S \mathbf{J} \cdot d\mathbf{s} = 0$ |
| $\nabla \times \left(\dfrac{\mathbf{J}}{\sigma}\right) = 0$ | $\oint_C \dfrac{1}{\sigma} \mathbf{J} \cdot d\boldsymbol{\ell} = 0$ |

Chapter 3

Here

The **normal** component of a **divergenceless** vector field is continuous

$$\nabla \cdot \mathbf{D} = \rho$$

$$\boxed{D_{1n} - D_{2n} = \rho_s \qquad (\mathrm{C/m^2}),}$$

$$\nabla \cdot \mathbf{J} = 0$$

$$\boxed{J_{1n} = J_{2n} \qquad (\mathrm{A/m^2}).}$$

The **tangential** component of a **curl-free** vector field is continuous across an interface

$$\nabla \times \mathbf{E} = 0$$

$$\boxed{E_{1t} = E_{2t} \qquad (\mathrm{V/m}),}$$

$$\nabla \times \left(\frac{\mathbf{J}}{\sigma}\right) = 0$$

$$\boxed{\frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}.}$$

**The ratio of $J_t$** at two sides of an interface is equal to **the ratio of the conductivities**

$$\nabla \times \mathbf{E} = 0$$

$$\nabla \times \left(\frac{\mathbf{J}}{\sigma}\right) = 0$$

$\nabla \times (\mathbf{D}/\varepsilon) = 0$

Analogy

| Interface of dielectric media | Interface of conducting media |
|---|---|
| **D** | **J** |
| $\varepsilon$ | $\sigma$ |

# A homogeneous conducting medium

$$\nabla \times \left(\frac{\mathbf{J}}{\sigma}\right) = 0$$

If $\sigma$ is a constant (homogeneous)

$$\nabla \times \mathbf{J} = 0.$$

By null identity

$$\mathbf{J} = -\nabla\psi.$$

Laplace's eq.: $$\nabla^2\psi = 0.$$

Electrostatics analogy

$$\nabla\times\mathbf{E} = 0$$

$$\mathbf{E} = -\nabla V$$

$$\nabla^2 V=0$$

$$\mathbf{J} = \sigma\mathbf{E}$$

$$\psi = \sigma V.$$

# Boundary Condition between Two Lossy Dielectrics (for a Steady Current)

- Two lossy dielectrics: $\epsilon_1$ and $\epsilon_2$ $\boxed{\sigma_1 \text{ and } \sigma_2}$

| $\epsilon_1$ and $\epsilon_2$ | $\sigma_1$ and $\sigma_2$ |
|---|---|
| $E_{2t} = E_{1t}$ | $J_{1t}/\sigma_1 = J_{2t}/\sigma_2$ |
| $D_{1n} - D_{2n} = \rho_s \rightarrow \epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s,$ | $J_{1n} = J_{2n} \rightarrow \sigma_1 E_{1n} = \sigma_2 E_{2n}$ |

$$\rho_s = \left(\epsilon_1 \frac{\sigma_2}{\sigma_1} - \epsilon_2\right) E_{2n} = \left(\epsilon_1 - \epsilon_2 \frac{\sigma_1}{\sigma_2}\right) E_{1n}.$$

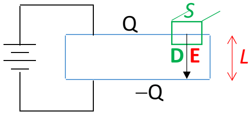
If $E_{1n} \neq 0$ or $E_{2n} \neq 0$, in most cases, a surface charge exists at the interface unless $\sigma_2/\sigma_1 = \varepsilon_2/\varepsilon_1$

What if medium 2 is a perfect conductor?

# 5-7 Resistance Calculations

- Capacitance between two conductors:

$$C = \frac{Q}{V} = \frac{\oint_S \mathbf{D} \cdot d\mathbf{s}}{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}} = \frac{\oint_S \epsilon \mathbf{E} \cdot d\mathbf{s}}{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}},$$

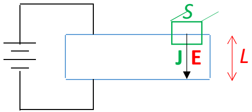



Numerator: surface integral over a surface enclosing the positive conductor

- Resistance between two conductors (medium between is lossy):

$$R = \frac{V}{I} = \frac{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}}{\oint_S \mathbf{J} \cdot d\mathbf{s}} = \frac{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}}{\oint_S \sigma \mathbf{E} \cdot d\mathbf{s}},$$

$$\boxed{\mathbf{J} = \sigma \mathbf{E},}$$




Denominator: the same surface as in the numerator of the above equation

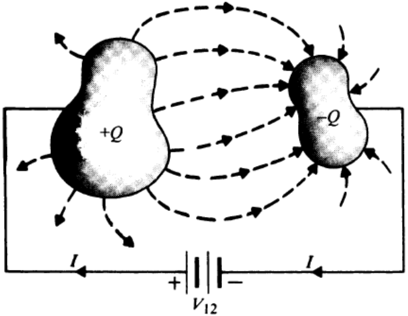



**FIGURE 5–7**
Two conductors in a lossy dielectric medium.

$$C = \frac{Q}{V} = \frac{\oint_S \mathbf{D} \cdot d\mathbf{s}}{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}} = \frac{\oint_S \epsilon \mathbf{E} \cdot d\mathbf{s}}{-\int_L \cancel{\mathbf{E} \cdot d\boldsymbol{\ell}}},$$

$$\times$$

$$R = \frac{V}{I} = \frac{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}}{\oint_S \mathbf{J} \cdot d\mathbf{s}} = \frac{-\cancel{\int_L \mathbf{E} \cdot d\boldsymbol{\ell}}}{\oint_S \sigma \mathbf{E} \cdot d\mathbf{s}},$$

If $\varepsilon$ and $\sigma$ of the medium have the same space dependence or if the medium is homogeneous

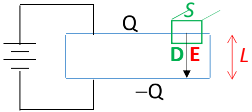
$$\boxed{RC = \frac{C}{G} = \frac{\epsilon}{\sigma}.}$$

# $C_\ell$ and $R_\ell$

- $C_\ell$: capacitance per unit length
  ($\ell$ longer → area $S$ larger → $C$ larger)
  $C = C_\ell \ell$ ➔ $C_\ell = C/\ell$ (F/m)

$C = \varepsilon S/d$

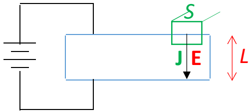
$\ell$: in/out plane direction




- $R_\ell$: Resistance per unit length
  ($\ell$ longer → area $S$ larger → $R$ smaller)
  $R = R_\ell/\ell$ ➔ $R_\ell = R\ell$ ($\Omega$·m)

$R = \varepsilon L/S$

$\ell$: in/out plane direction




Note that $\ell$ and L are different!

$$\boxed{RC = \frac{C}{G} = \frac{\epsilon}{\sigma}.}$$

$$\boxed{R_\ell C_\ell = RC = \varepsilon/\sigma}$$

# Difference between J and D

- Current flow can be confined strictly within a conductor
- Electric flux usually **cannot** be contained within a dielectric slab (of finite dimensions)

# Procedure to Compute R between Specified Equi-potential Surfaces

- Procedure 1: **$V_0$ to I**
  - 1. Choose a coordinate
  - 2. Assume potential difference $V_0$ between conductors
  - 3. Find **E** between conductors
    - $\nabla^2 V = 0 \rightarrow \mathbf{E} = -\nabla V$
  - 4. Find current $I = \int_S \mathbf{J} \cdot d\mathbf{s} = \int_S \sigma \mathbf{E} \cdot d\mathbf{s},$
  - 5. Find $R = V_0/I$
- Procedure 2: **I to $V_0$**
  - Assume $I \rightarrow \mathbf{J} \rightarrow \mathbf{E} \rightarrow V_0$
  - $R = V_0/I$

if **J** can be determined easily from I

**EXAMPLE 5–4** An emf $\mathscr{V}$ is applied across a parallel-plate capacitor of area $S$. The space between the conducting plates is filled with two different lossy dielectrics of thicknesses $d_1$ and $d_2$, permittivities $\epsilon_1$ and $\epsilon_2$, and conductivities $\sigma_1$ and $\sigma_2$, respectively. Determine (a) the current density between the plates, (b) the electric field intensities in both dielectrics, and (c) the surface charge densities on the plates and at the interface.

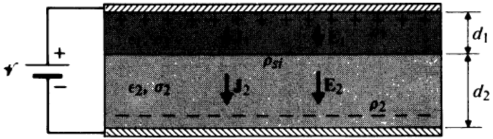



**FIGURE 5–6**
Parallel-plate capacitor with two lossy dielectrics (Example 5–4).

**EXAMPLE 5–6** A conducting material of uniform thickness $h$ and conductivity $\sigma$ has the shape of a quarter of a flat circular washer, with inner radius $a$ and outer radius $b$, as shown in Fig. 5–8. Determine the resistance between the end faces.

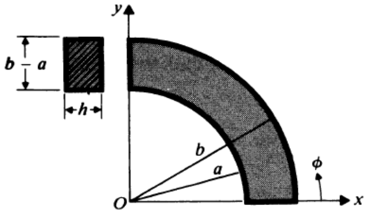



**FIGURE 5–8**
A quarter of a flat conducting circular washer (Example 5–6).