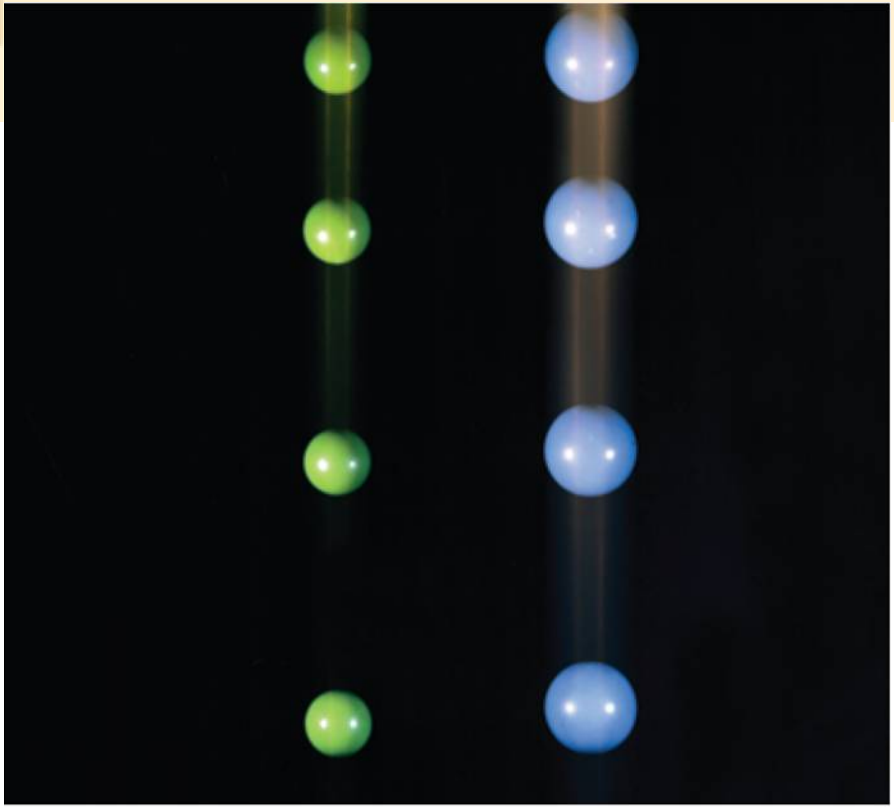
1

Functions and Limits



1.1 Four Ways to Represent a Function

Four Ways to Represent a Function

Functions arise whenever one quantity depends on another. Consider the following four situations.

A. The area *A* of a circle depends on the radius *r* of the circle. The rule that connects *r* and *A* is given by the equation $A = \pi r^2$. With each positive number *r* there is associated one value of *A*, and we say that *A* is a *function* of *r*.

Four Ways to Represent a Function

B. The human population of the world *P* depends on the time *t*. The table gives estimates of the world population *P*(*t*) at time *t*, for certain years. For instance,

$$P(1950) \approx 2{,}560{,}000{,}000$$

But for each value of the time *t* there is a corresponding value of *P*, and we say that *P* is a function of *t*.

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080

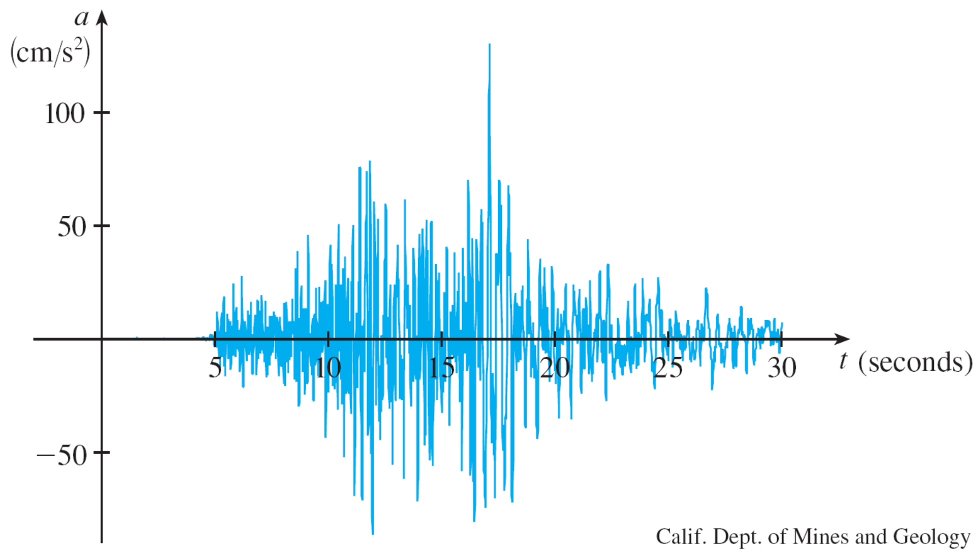
Four Ways to Represent a Function

C. The cost C of mailing a large envelope depends on the weight w of the envelope. Although there is no simple formula that connects w and C, the post office has a rule for determining C when w is known.

D. The vertical acceleration a of the ground as measured by a seismograph during an earthquake is a function of the elapsed time t.

Four Ways to Represent a Function

Figure 1 shows a graph generated by seismic activity during the Northridge earthquake that shook Los Angeles in 1994. For a given value of t, the graph provides a corresponding value of a.



Vertical ground acceleration during the Northridge earthquake

Figure 1

Four Ways to Represent a Function

A **function** f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

We usually consider functions for which the sets *D* and *E* are sets of real numbers. The set *D* is called the **domain** of the function.

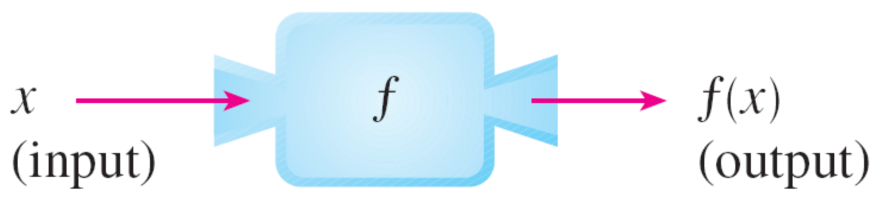
The number $f(x)$ is the **value of *f* at *x*** and is read "*f* of *x*." The **range** of *f* is the set of all possible values of $f(x)$ as *x* varies throughout the domain.

A symbol that represents an arbitrary number in the *domain* of a function *f* is called an **independent variable**.

Four Ways to Represent a Function

A symbol that represents a number in the *range* of f is called a **dependent variable**. In Example A, for instance, r is the independent variable and A is the dependent variable.

It's helpful to think of a function as a **machine** (see Figure 2).



Machine diagram for a function f

Figure 2

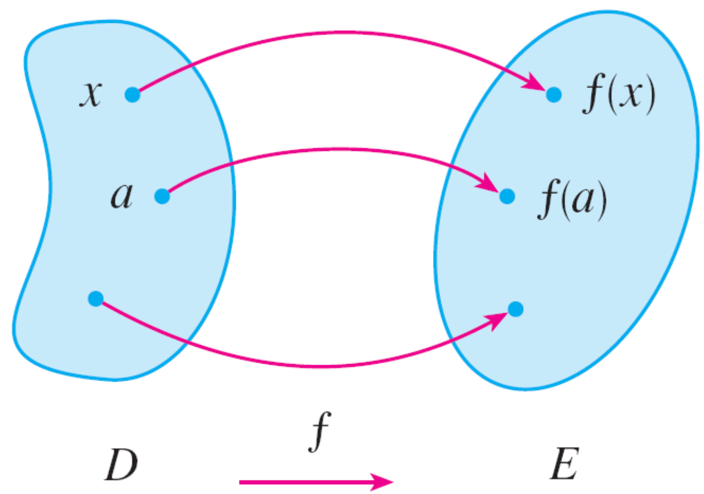
Four Ways to Represent a Function

If x is in the domain of the function f, then when x enters the machine, it's accepted as an input and the machine produces an output $f(x)$ according to the rule of the function.

Thus we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs.

Four Ways to Represent a Function

Another way to picture a function is by an **arrow diagram** as in Figure 3.



Arrow diagram for *f*

Figure 3

Each arrow connects an element of D to an element of E. The arrow indicates that $f(x)$ is associated with x, $f(a)$ is associated with a, and so on.

Four Ways to Represent a Function

The most common method for visualizing a function is its graph. If f is a function with domain D, then its **graph** is the set of ordered pairs

$$\{(x, f(x)) \mid x \in D\}$$

In other words, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f.

Four Ways to Represent a Function

The graph of a function *f* gives us a useful picture of the behavior or "life history" of a function. Since the y-coordinate of any point (x, y) on the graph is $y = f(x)$, we can read the value of $f(x)$ from the graph as being the height of the graph above the point x (see Figure 4).

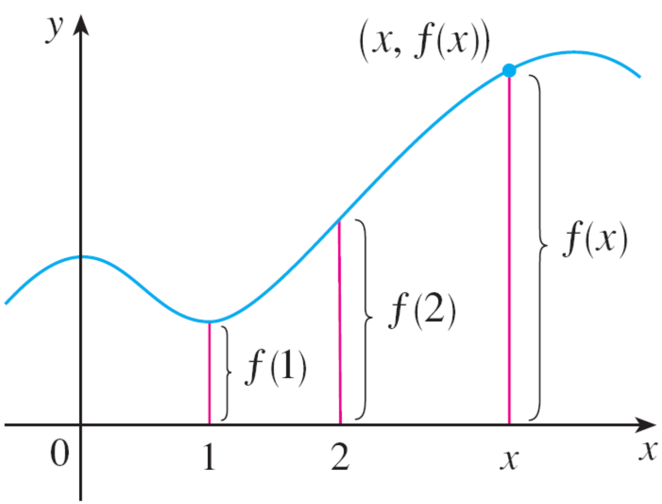


Figure 4

Four Ways to Represent a Function

The graph of *f* also allows us to picture the domain of *f* on the *x*-axis and its range on the *y*-axis as in Figure 5.

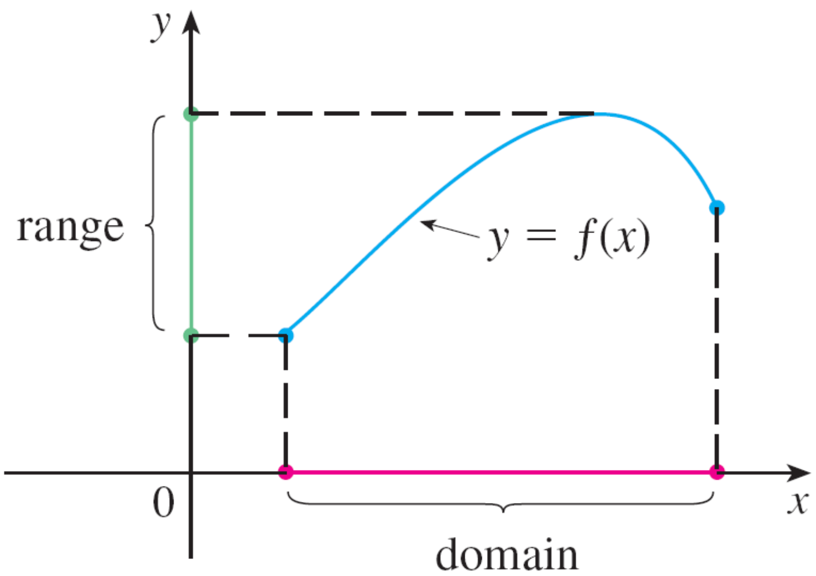
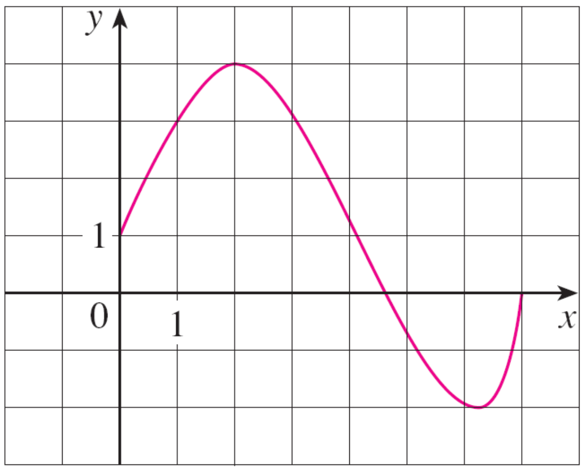


Figure 5

Example 1

The graph of a function f is shown in Figure 6.



The notation for intervals is given in Appendix A.

Figure 6

(a) Find the values of $f(1)$ and $f(5)$.

(b) What are the domain and range of f?

Example 1 – *Solution*

(a) We see from Figure 6 that the point (1, 3) lies on the graph of f, so the value of f at 1 is $f(1) = 3$. (In other words, the point on the graph that lies above $x = 1$ is 3 units above the x-axis.)

When $x = 5$, the graph lies about 0.7 unit below the x-axis, so we estimate that $f(5) \approx -0.7$.

(b) We see that $f(x)$ is defined when $0 \leq x \leq 7$, so the domain of f is the closed interval $[0, 7]$. Notice that f takes on all values from –2 to 4, so the range of f is

$$\{y \mid -2 \leq y \leq 4\} = [-2, 4]$$

Representations of Functions

Representations of Functions

There are four possible ways to represent a function:

- verbally (by a description in words)
- numerically (by a table of values)
- visually (by a graph)
- algebraically (by an explicit formula)

Example 4

When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running. Draw a rough graph of T as a function of the time t that has elapsed since the faucet was turned on.

Solution:

The initial temperature of the running water is close to room temperature because the water has been sitting in the pipes.

When the water from the hot-water tank starts flowing from the faucet, T increases quickly. In the next phase, T is constant at the temperature of the heated water in the tank.

Example 4 – *Solution*

cont'd

When the tank is drained, T decreases to the temperature of the water supply.

This enables us to make the rough sketch of T as a function of t in Figure 11.

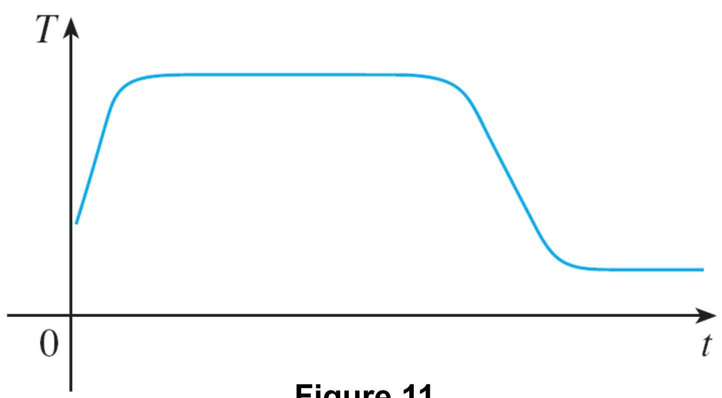


Figure 11

Representations of Functions

The graph of a function is a curve in the *xy*-plane. But the question arises: Which curves in the *xy*-plane are graphs of functions? This is answered by the following test.

The Vertical Line Test A curve in the *xy*-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

Representations of Functions

The reason for the truth of the Vertical Line Test can be seen in Figure 13.

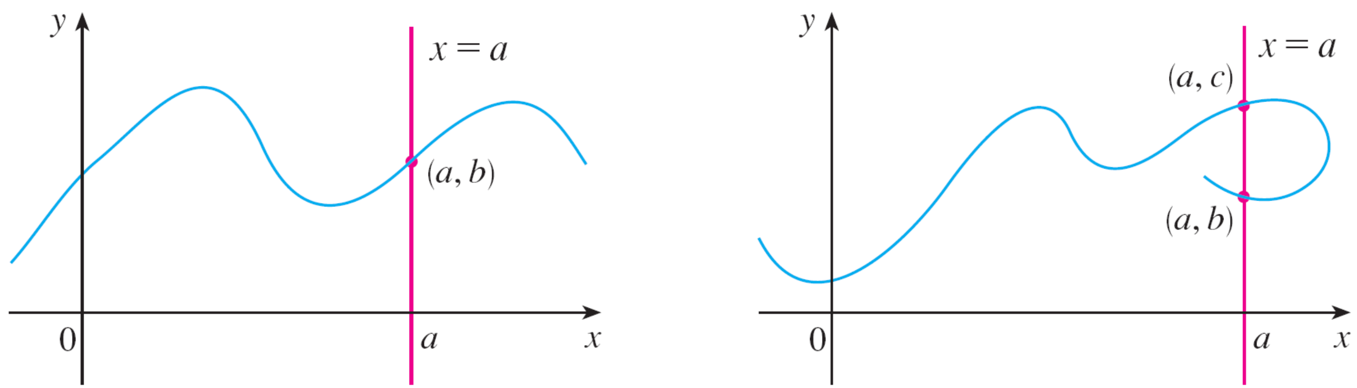
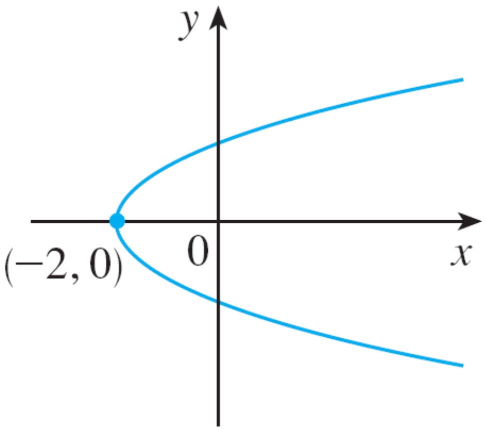


Figure 13

If each vertical line $x = a$ intersects a curve only once, at (a, b), then exactly one functional value is defined by $f(a) = b$. But if a line $x = a$ intersects the curve twice, at (a, b) and (a, c), then the curve can't represent a function because a function can't assign two different values to a.

Representations of Functions

For example, the parabola $x = y^2 - 2$ shown in Figure 14(a) is not the graph of a function of x because, as you can see, there are vertical lines that intersect the parabola twice. The parabola, however, does contain the graphs of *two* functions of x.

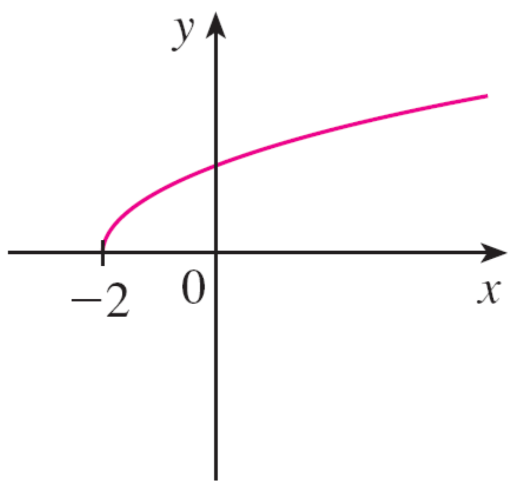


$x = y^2 - 2$

Figure 14(a)

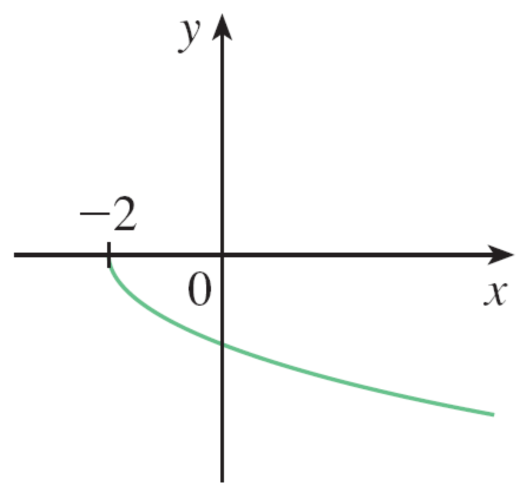
Representations of Functions

Notice that the equation $x = y^2 - 2$ implies $y^2 = x + 2$, so $y = \pm\sqrt{x + 2}$. Thus the upper and lower halves of the parabola are the graphs of the functions $f(x) = \sqrt{x + 2}$ and $g(x) = -\sqrt{x + 2}$. [See Figures 14(b) and (c).]



$y = \sqrt{x + 2}$

Figure 14(b)



$y = -\sqrt{x + 2}$

Figure 14(c)

Representations of Functions

We observe that if we reverse the roles of x and y, then the equation $x = h(y) = y^2 - 2$ *does* define x as a function of y (with y as the independent variable and x as the dependent variable) and the parabola now appears as the graph of the function h.

Piecewise Defined Functions

Example 7

A function *f* is defined by

$$f(x) = \begin{cases} 1 - x & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

Evaluate $f(-2)$, $f(-1)$, and $f(0)$ and sketch the graph.

Example 7 – *Solution*

Remember that a function is a rule. For this particular function the rule is the following:

First look at the value of the input x. If it happens that $x \leq -1$, then the value of $f(x)$ is $1 - x$.

On the other hand, if $x > -1$, then the value of $f(x)$ is x^2.

Since $-2 \leq -1$, we have $f(-2) = 1 - (-2) = 3$.

Since $-1 \leq -1$, we have $f(-1) = 1 - (-1) = 2$.

Since $2 > -1$, we have $f(0) = 0^2 = 0$.

Example 7 – *Solution*

cont'd

How do we draw the graph of f? We observe that if $x \leq 1$, then $f(x) = 1 - x$, so the part of the graph of f that lies to the left of the vertical line $x = -1$ must coincide with the line $y = 1 - x$, which has slope -1 and y-intercept 1.

If $x > 1$, then $f(x) = x^2$, so the part of the graph of f that lies to the right of the line $x = -1$ must coincide with the graph of $y = x^2$, which is a parabola.

Example 7 – *Solution*

cont'd

This enables us to sketch the graph in Figure 15.

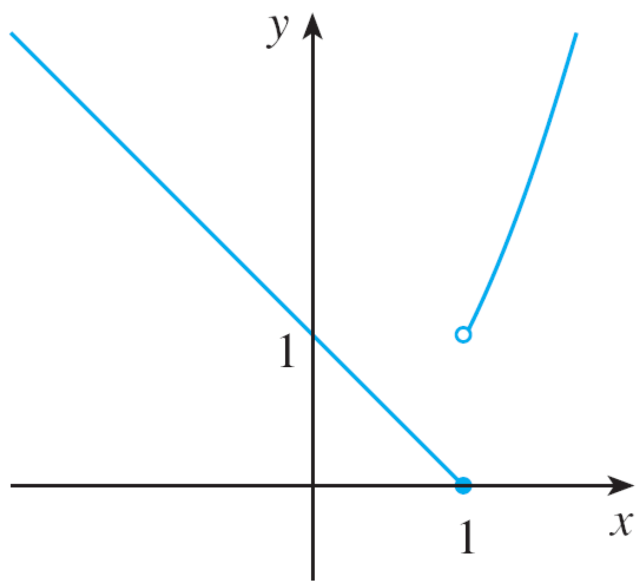


Figure 15

The solid dot indicates that the point (–1, 2) is included on the graph; the open dot indicates that the point (–1, 1) is excluded from the graph.

Piecewise Defined Functions

The next example of a piecewise defined function is the absolute value function. Recall that the **absolute value** of a number *a*, denoted by $|a|$, is the distance from *a* to 0 on the real number line. Distances are always positive or 0, so we have

$$|a| \geq 0 \qquad \text{for every number } a$$

For example,

$|3| = 3 \quad |-3| = 3 \quad |0| = 0 \quad |\sqrt{2} - 1| = \sqrt{2} - 1$

$|3 - \pi| = \pi - 3$

Piecewise Defined Functions

In general, we have

$$\boxed{\begin{aligned} |a| &= a \qquad &&\text{if } a \geq 0 \\ |a| &= -a &&\text{if } a < 0 \end{aligned}}$$

(Remember that if a is negative, then $-a$ is positive.)

Example 8

Sketch the graph of the absolute value function $f(x) = |x|$.

Solution:

From the preceding discussion we know that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Example 8

Using the same method as in Example 7, we see that the graph of f coincides with the line $y = x$ to the right of the y-axis and coincides with the line $y = -x$ to the left of the y-axis (see Figure 16).

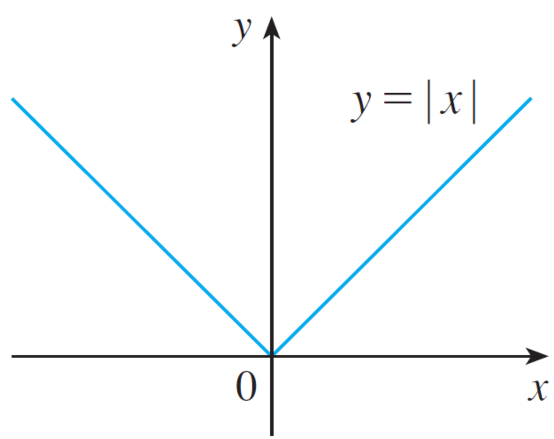


Figure 16

Example 10

We have considered the cost $C(w)$ of mailing a large envelope with weight w. This is a piecewise defined function and we have

$$C(w) = \begin{cases} 0.83 & \text{if } 0 < w \leq 1 \\ 1.00 & \text{if } 1 < w \leq 2 \\ 1.17 & \text{if } 2 < w \leq 3 \\ 1.34 & \text{if } 3 < w \leq 4 \\ \vdots & \end{cases}$$

Example 10

cont'd

The graph is shown in Figure 18.

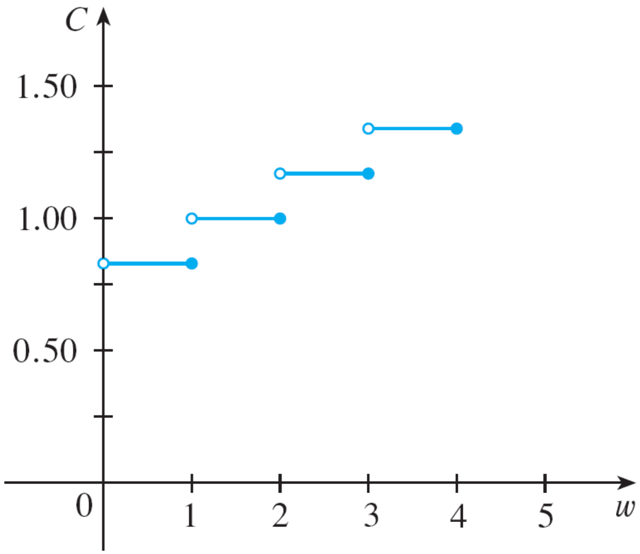


Figure 18

You can see why functions similar to this one are called **step functions**—they jump from one value to the next.

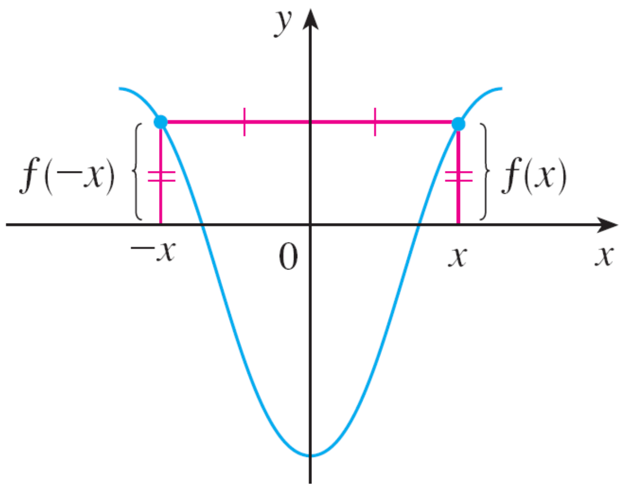
Symmetry

Symmetry

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an **even function**. For instance, the function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x)$$

The geometric significance of an even function is that its graph is symmetric with respect to the y-axis (see Figure 19).



An even function

Figure 19

Symmetry

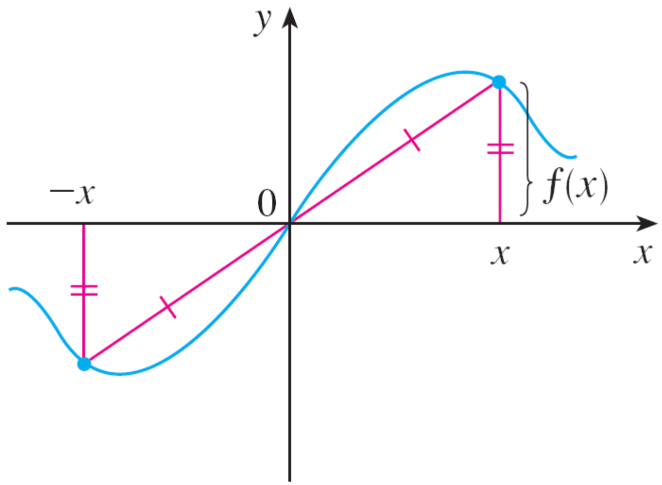
This means that if we have plotted the graph of f for $x \geq 0$, we obtain the entire graph simply by reflecting this portion about the y-axis.

If f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd function**. For example, the function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

Symmetry

The graph of an odd function is symmetric about the origin (see Figure 20).



An odd function

Figure 20

If we already have the graph of f for $x \geq 0$, we can obtain the entire graph by rotating this portion through 180° about the origin.

Example 11

Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ **(b)** $g(x) = 1 - x^4$ **(c)** $h(x) = 2x - x^2$

Solution:

(a) $f(-x) = (-x)^5 + (-x)$

$= (-1)^5 x^5 + (-x)$

$= -x^5 - x$

$= -(x^5 + x)$

$= -f(x)$

Therefore f is an odd function.

Example 11 – *Solution*

cont'd

(b) $g(-x) = 1 - (-x^4)$

$= 1 - x^4$

$= g(x)$

So g is even.

(c) $h(-x) = 2(-x) - (-x^2)$

$= -2x - x^2$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd.

Symmetry

The graphs of the functions in Example 11 are shown in Figure 21. Notice that the graph of *h* is symmetric neither about the *y*-axis nor about the origin.

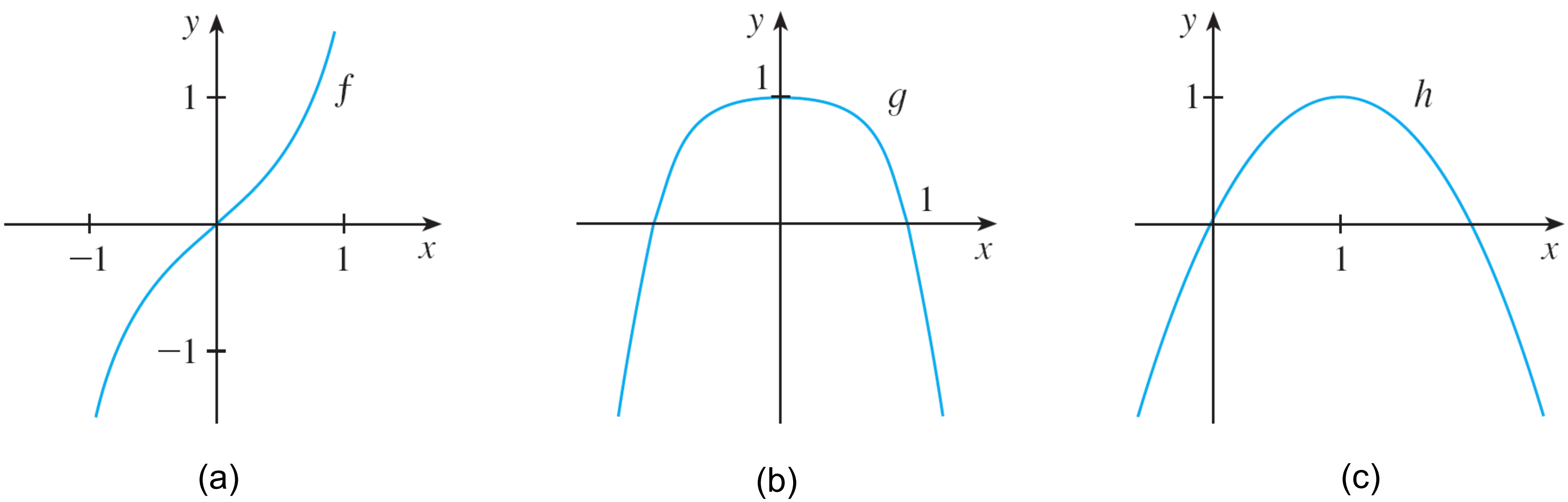


Figure 21

Increasing and Decreasing Functions

Increasing and Decreasing Functions

The graph shown in Figure 22 rises from *A* to *B*, falls from *B* to *C*, and rises again from *C* to *D*. The function *f* is said to be increasing on the interval $[a, b]$, decreasing on $[b, c]$, and increasing again on $[c, d]$.

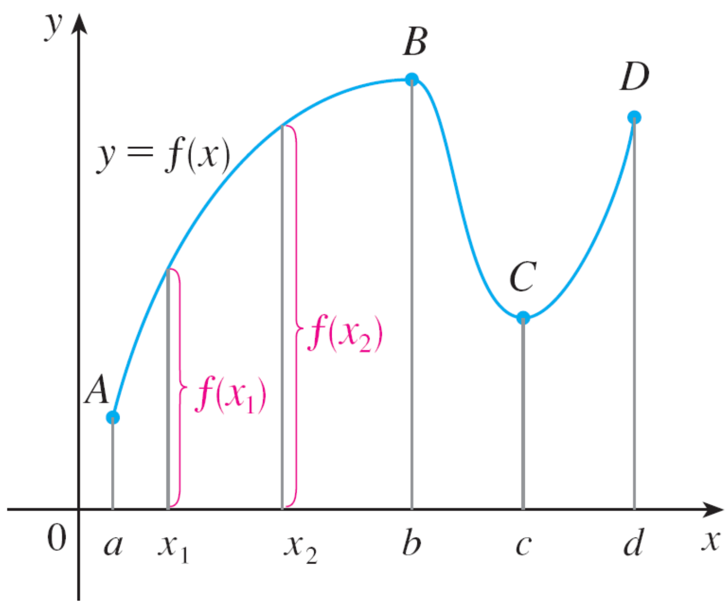


Figure 22

Increasing and Decreasing Functions

Notice that if x_1 and x_2 are any two numbers between a and b with $x_1 < x_2$, then $f(x_1) < f(x_2)$.

We use this as the defining property of an increasing function.

A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

It is called **decreasing** on I if

$$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

Increasing and Decreasing Functions

In the definition of an increasing function it is important to realize that the inequality $f(x_1) < f(x_2)$ must be satisfied for *every* pair of numbers x_1 and x_2 in I with $x_1 < x_2$.

You can see from Figure 23 that the function $f(x) = x^2$ is decreasing on the interval $(-\infty, 0]$ and increasing on the interval $[0, \infty)$.

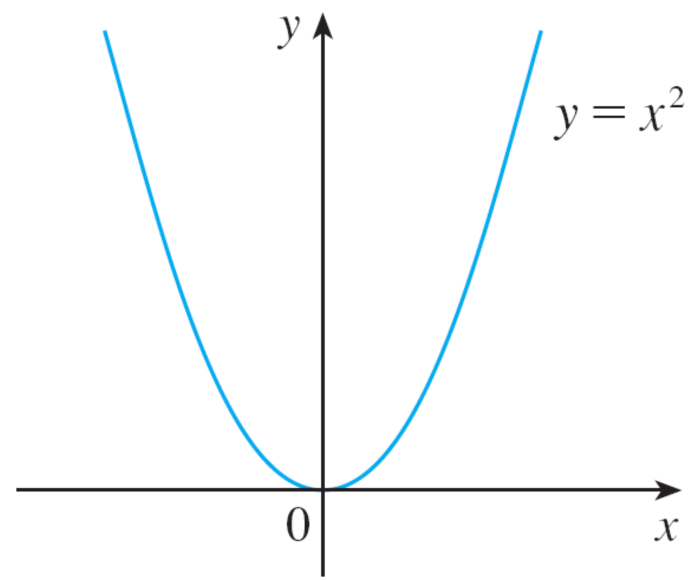


Figure 23

1.2 Mathematical Models: A Catalog of Essential Functions

Mathematical Models: A Catalog of Essential Functions

A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon such as the size of a population, the demand for a product, the speed of a falling object, the concentration of a product in a chemical reaction, the life expectancy of a person at birth, or the cost of emission reductions.

The purpose of the model is to understand the phenomenon and perhaps to make predictions about future behavior.

Figure 1 illustrates the process of mathematical modeling.

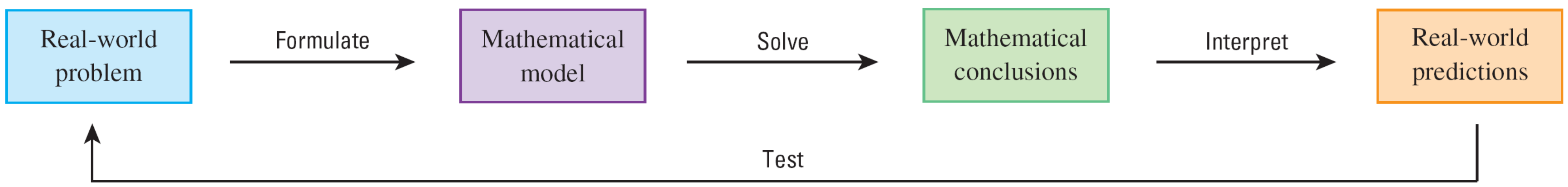


Figure 1

The modeling process

Mathematical Models: A Catalog of Essential Functions

A mathematical model is never a completely accurate representation of a physical situation—it is an *idealization*. A good model simplifies reality enough to permit mathematical calculations but is accurate enough to provide valuable conclusions.

It is important to realize the limitations of the model. In the end, Mother Nature has the final say.

There are many different types of functions that can be used to model relationships observed in the real world. In what follows, we discuss the behavior and graphs of these functions and give examples of situations appropriately modeled by such functions.

Linear Models

Linear Models

When we say that y is a **linear function** of x, we mean that the graph of the function is a line, so we can use the slope-intercept form of the equation of a line to write a formula for the function as

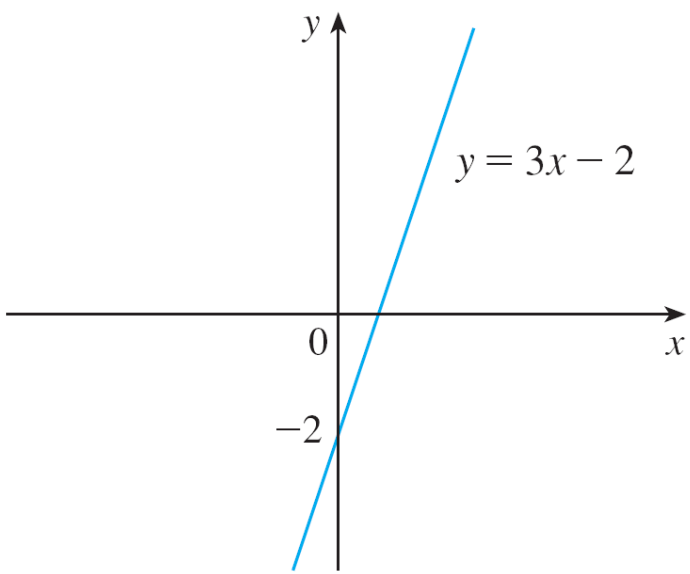
$$y = f(x) = mx + b$$

where m is the slope of the line and b is the y-intercept.

Linear Models

A characteristic feature of linear functions is that they grow at a constant rate.

For instance, Figure 2 shows a graph of the linear function $f(x) = 3x – 2$ and a table of sample values.



x	$f(x) = 3x - 2$
1.0	1.0
1.1	1.3
1.2	1.6
1.3	1.9
1.4	2.2
1.5	2.5

Figure 2

Linear Models

Notice that whenever x increases by 0.1, the value of $f(x)$ increases by 0.3.

So $f(x)$ increases three times as fast as x. Thus the slope of the graph $y = 3x - 2$, namely 3, can be interpreted as the rate of change of y with respect to x.

Example 1

(a) As dry air moves upward, it expands and cools. If the ground temperature is 20°C and the temperature at a height of 1 km is 10°C, express the temperature T (in °C) as a function of the height h (in kilometers), assuming that a linear model is appropriate.

(b) Draw the graph of the function in part (a). What does the slope represent?

(c) What is the temperature at a height of 2.5 km?

Example 1(a) – *Solution*

Because we are assuming that T is a linear function of h, we can write

$$T = mh + b$$

We are given that $T = 20$ when $h = 0$, so

$$20 = m \bullet 0 + b = b$$

In other words, the y-intercept is $b = 20$.

We are also given that $T = 10$ when $h = 1$, so

$$10 = m \bullet 1 + 20$$

The slope of the line is therefore $m = 10 - 20 = -10$ and the required linear function is

$$T = -10h + 20$$

Example 1(b) – *Solution*

cont'd

The graph is sketched in Figure 3.

The slope is m = –10°C/km, and this represents the rate of change of temperature with respect to height.

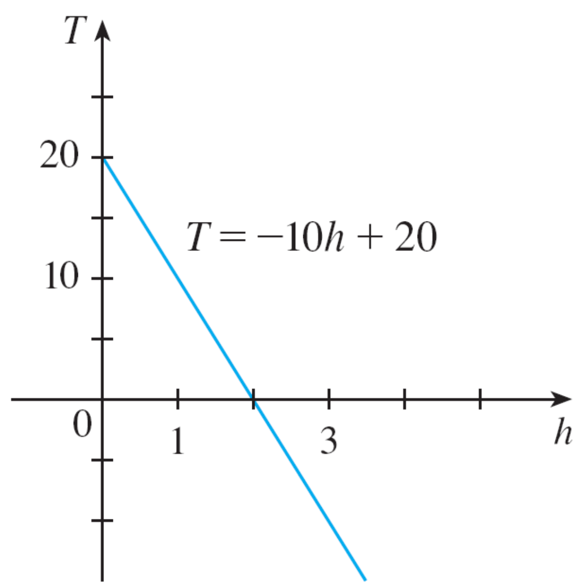


Figure 3

Example 1(c) – *Solution*

cont'd

At a height of $h = 2.5$ km, the temperature is

$$T = -10(2.5) + 20 = -5°\text{C}$$

Linear Models

If there is no physical law or principle to help us formulate a model, we construct an **empirical model**, which is based entirely on collected data.

We seek a curve that "fits" the data in the sense that it captures the basic trend of the data points.

Polynomials

Polynomials

A function P is called a **polynomial** if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are constants called the **coefficients** of the polynomial.

The domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$. If the leading coefficient $a_n \neq 0$, then the **degree** of the polynomial is n. For example, the function

$$P(x) = 2x^6 - x^4 + \tfrac{2}{5}x^3 + \sqrt{2}$$

is a polynomial of degree 6.

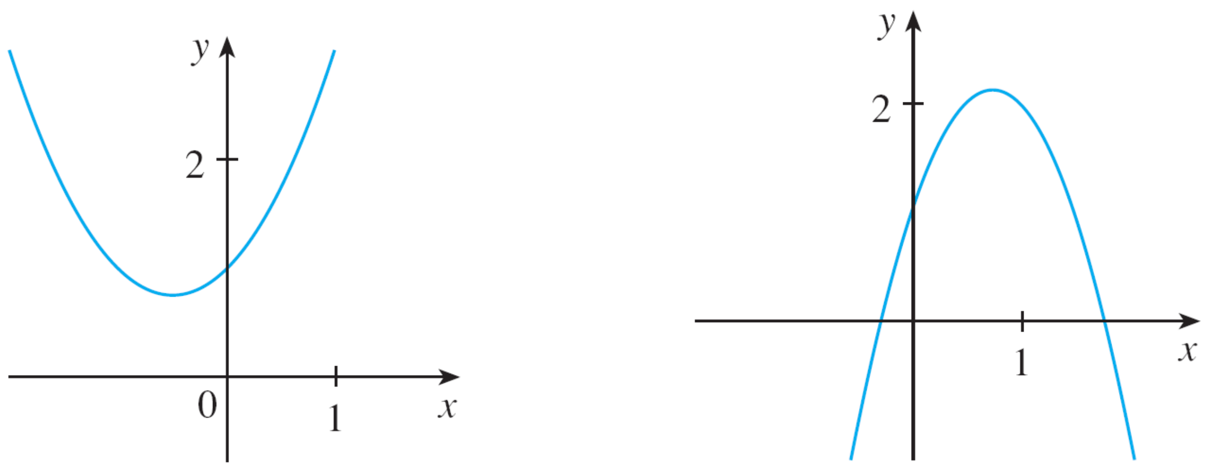
Polynomials

A polynomial of degree 1 is of the form $P(x) = mx + b$ and so it is a linear function.

A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$ and is called a **quadratic function**.

Polynomials

Its graph is always a parabola obtained by shifting the parabola $y = ax^2$. The parabola opens upward if $a > 0$ and downward if $a < 0$. (See Figure 7.)



(a) $y = x^2 + x + 1$

(b) $y = -2x^2 + 3x + 1$

The graphs of quadratic functions are parabolas.

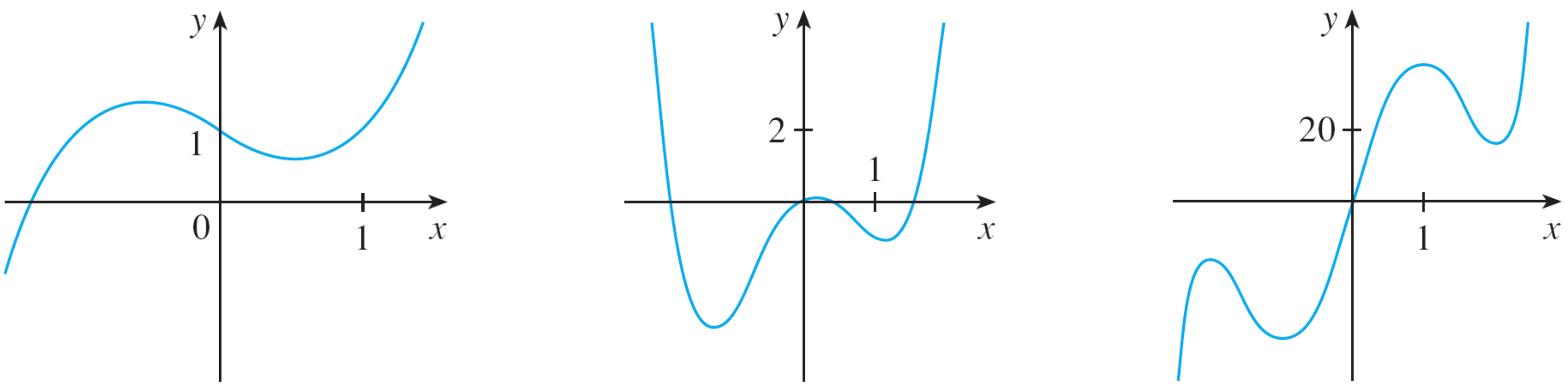
Figure 7

Polynomials

A polynomial of degree 3 is of the form

$$P(x) = ax^3 + bx^2 + cx + d \qquad a \neq 0$$

and is called a **cubic function**. Figure 8 shows the graph of a cubic function in part (a) and graphs of polynomials of degrees 4 and 5 in parts (b) and (c).



(a) $y = x^3 - x + 1$ (b) $y = x^4 - 3x^2 + x$ (c) $y = 3x^5 - 25x^3 + 60x$

Figure 8

Example 4

A ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground, and its height h above the ground is recorded at 1-second intervals in Table 2.

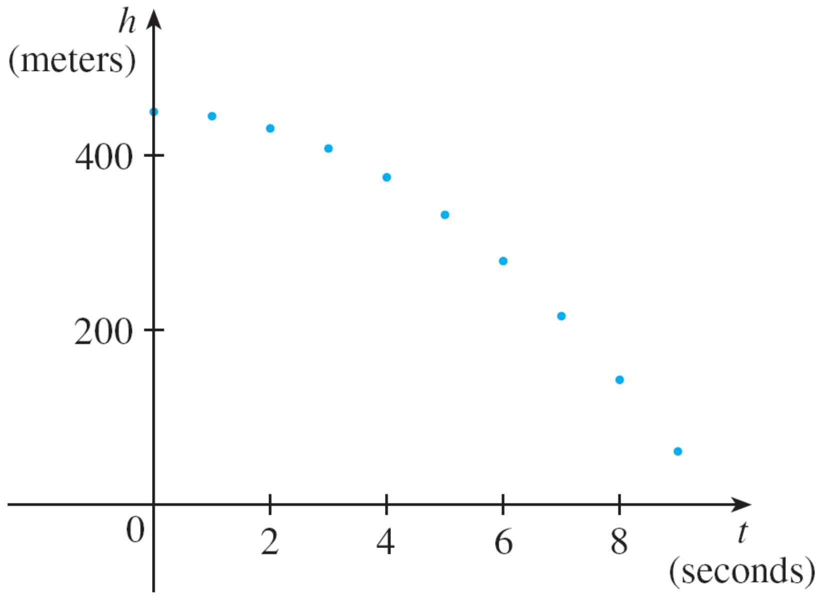
Find a model to fit the data and use the model to predict the time at which the ball hits the ground.

TABLE 2

Time (seconds)	Height (meters)
0	450
1	445
2	431
3	408
4	375
5	332
6	279
7	216
8	143
9	61

Example 4 – *Solution*

We draw a scatter plot of the data in Figure 9 and observe that a linear model is inappropriate.



Scatter plot for a falling ball

Figure 9

Example 4 – *Solution*

cont'd

But it looks as if the data points might lie on a parabola, so we try a quadratic model instead.

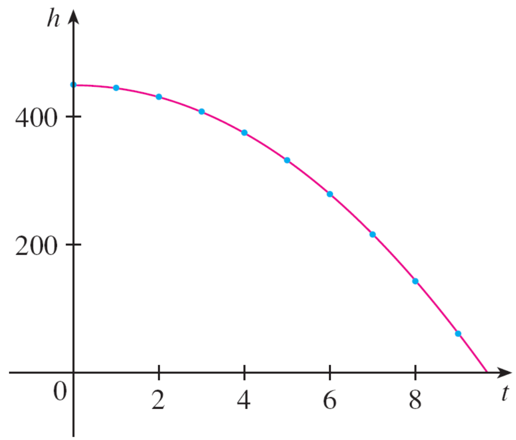
Using a graphing calculator or computer algebra system (which uses the least squares method), we obtain the following quadratic model:

$$\boxed{3} \qquad h = 449.36 + 0.96ts - 4.90t^2$$

Example 4 – *Solution*

cont'd

In Figure 10 we plot the graph of Equation 3 together with the data points and see that the quadratic model gives a very good fit.



Quadratic model for a falling ball

Figure 10

The ball hits the ground when $h = 0$, so we solve the quadratic equation

$$-4.90t^2 + 0.96t + 449.36 = 0$$

Example 4 – *Solution*

cont'd

The quadratic formula gives

$$t = \frac{-0.96 \pm \sqrt{(0.96)^2 - 4(-4.90)(449.36)}}{2(-4.90)}$$

The positive root is $t \approx 9.67$, so we predict that the ball will hit the ground after about 9.7 seconds.

Power Functions

Power Functions

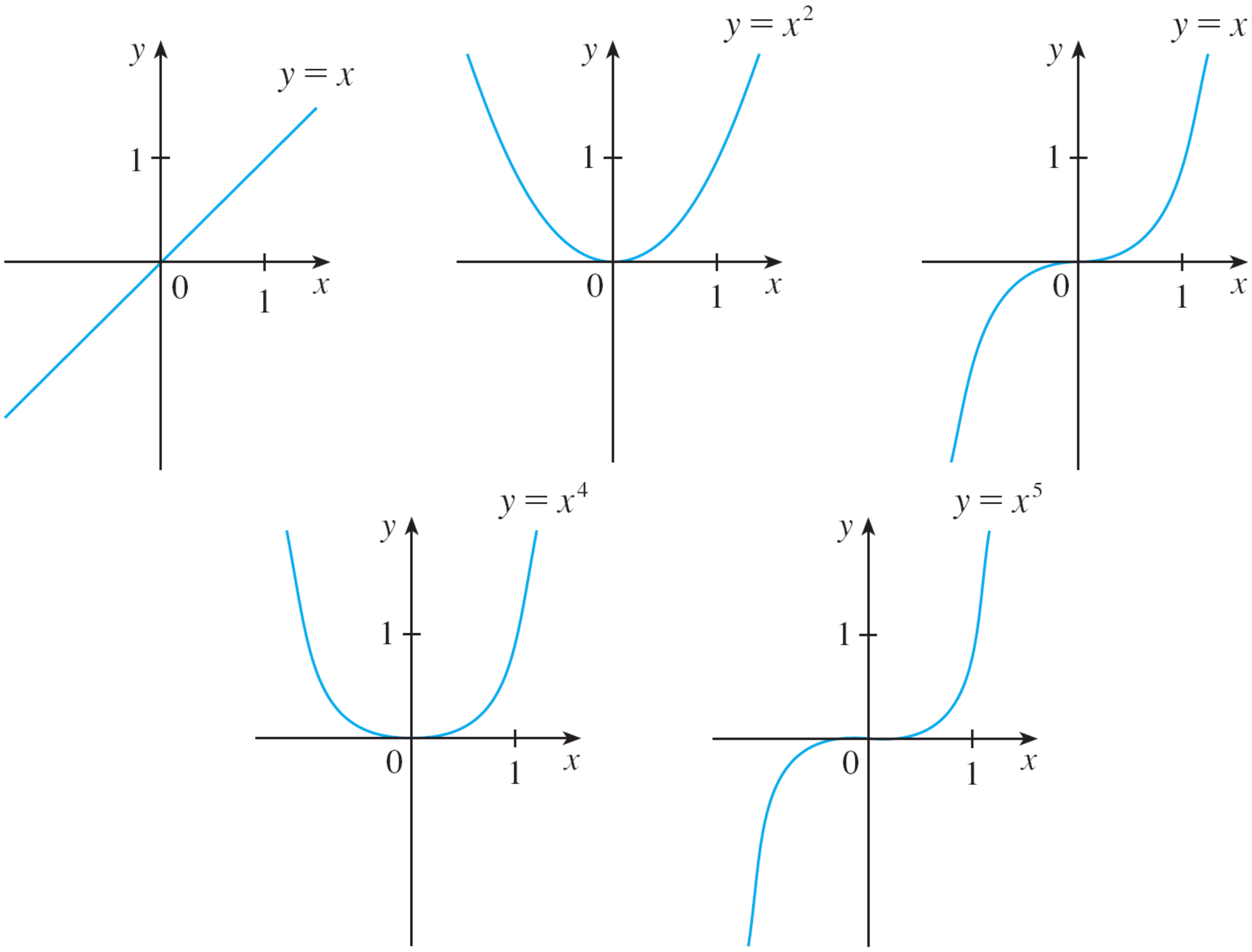
A function of the form $f(x) = x^a$, where a is a constant, is called a **power function**. We consider several cases.

(i) $a = n$, where n is a positive integer

The graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4$, and 5 are shown in Figure 11. (These are polynomials with only one term.)

We already know the shape of the graphs of $y = x$ (a line through the origin with slope 1) and $y = x^2$ (a parabola).

Power Functions



Graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4, 5$

Figure 11

Power Functions

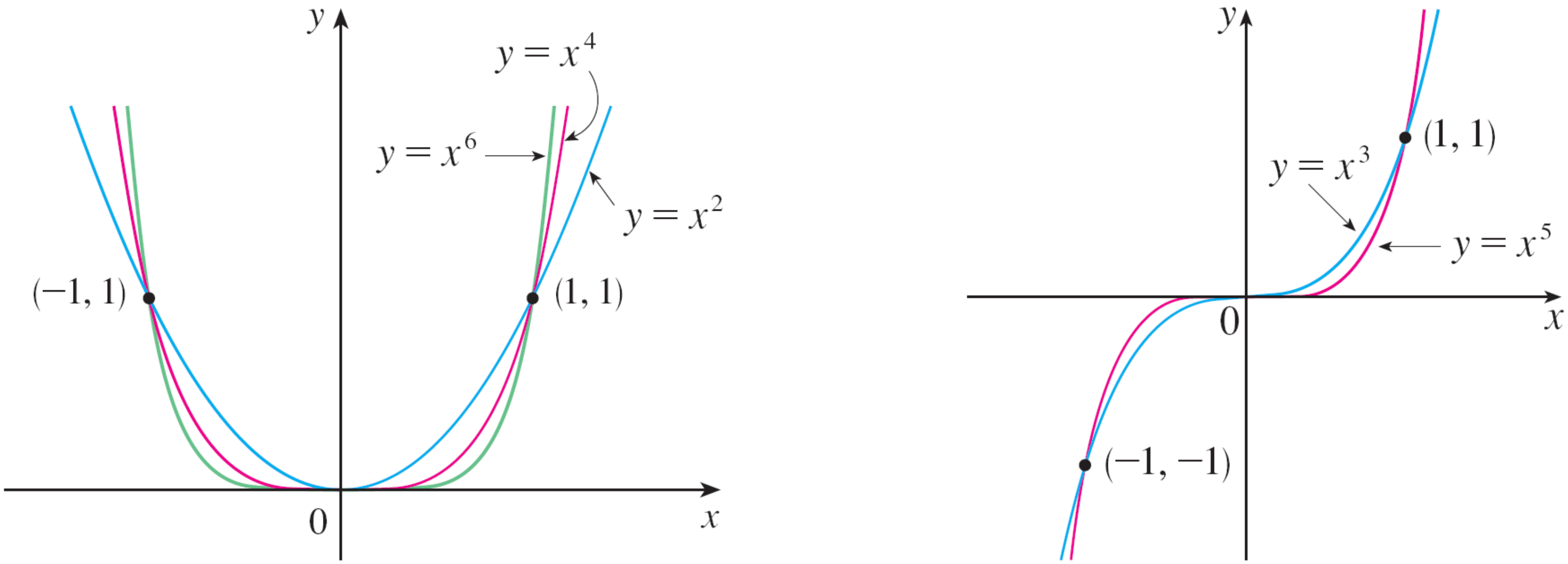
The general shape of the graph of $f(x) = x^n$ depends on whether n is even or odd.

If n is even, then $f(x) = x^n$ is an even function and its graph is similar to the parabola $y = x^2$.

If n is odd, then $f(x) = x^n$ is an odd function and its graph is similar to that of $y = x^3$.

Power Functions

Notice from Figure 12, however, that as *n* increases, the graph of $y = x^n$ becomes flatter near 0 and steeper when $|x| \geq 1$. (If *x* is small, then x^2 is smaller, x^3 is even smaller, x^4 is smaller still, and so on.)



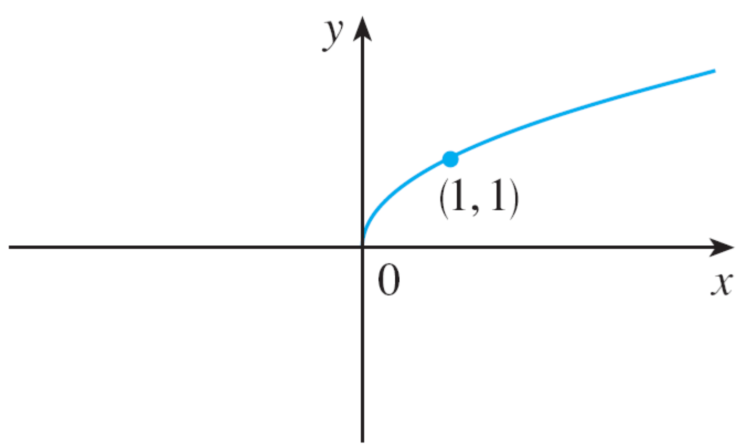
Families of power functions

Figure 12

Power Functions

(ii) *a* = 1/*n*, where *n* is a positive integer

The function $f(x) = x^{1/n} = \sqrt[n]{x}$ is a **root function**. For *n* = 2 it is the square root function $f(x) = \sqrt{x}$, whose domain is $[0, \infty)$ and whose graph is the upper half of the parabola $x = y^2$. [See Figure 13(a).]



$f(x) = \sqrt{x}$

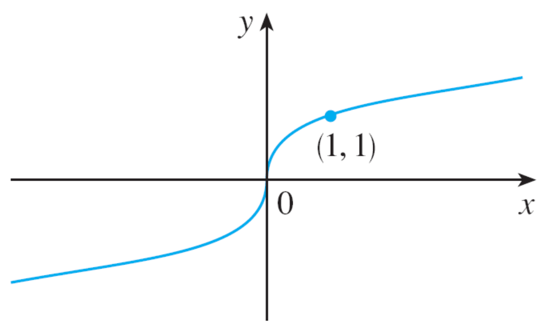
Graph of root function

Figure 13(a)

Power Functions

For other even values of *n*, the graph of $y = \sqrt[n]{x}$ is similar to that of $y = \sqrt{x}$.

For *n* = 3 we have the cube root function $f(x) = \sqrt[3]{x}$ whose domain is $\mathbb{R}$ (recall that every real number has a cube root) and whose graph is shown in Figure 13(b). The graph of $y = \sqrt[n]{x}$ for *n* odd (*n* > 3) is similar to that of $y = \sqrt[3]{x}$.



$f(x) = \sqrt[3]{x}$

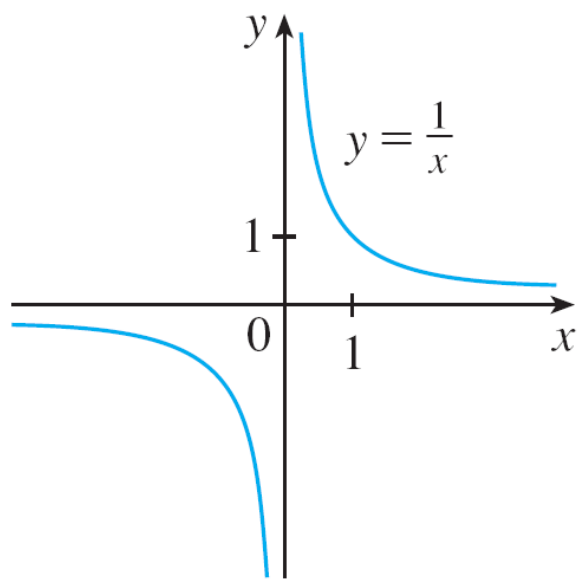
Graph of root function

Figure 13(b)

Power Functions

(iii) *a* = –1

The graph of the **reciprocal function** $f(x) = x^{-1} = 1/x$ is shown in Figure 14. Its graph has the equation $y = 1/x$, or $xy = 1$, and is a hyperbola with the coordinate axes as its asymptotes.



The reciprocal function

Figure 14

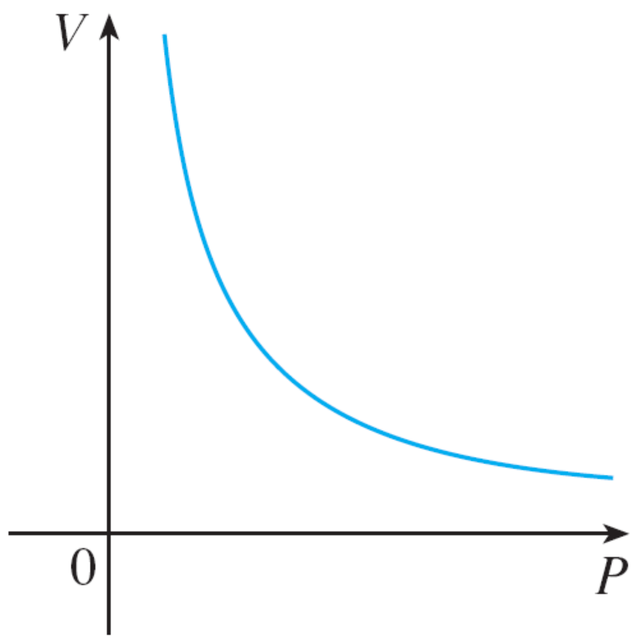
Power Functions

This function arises in physics and chemistry in connection with Boyle's Law, which says that, when the temperature is constant, the volume V of a gas is inversely proportional to the pressure P:

$$V = \frac{C}{P}$$

where C is a constant.

Thus the graph of V as a function of P (see Figure 15) has the same general shape as the right half of Figure 14.



Volume as a function of pressure at constant temperature

Figure 15

Rational Functions

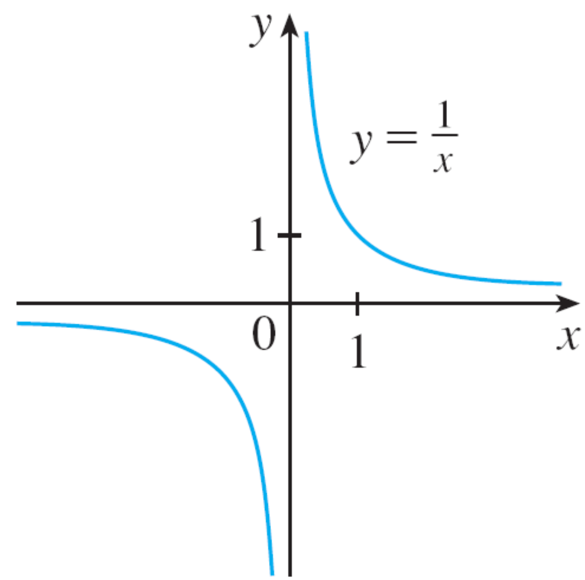
Rational Functions

A **rational function** *f* is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where *P* and *Q* are polynomials. The domain consists of all values of *x* such that $Q(x) \neq 0$.

A simple example of a rational function is the function $f(x) = 1/x$, whose domain is $\{x \mid x \neq 0\}$; this is the reciprocal function graphed in Figure 14.



The reciprocal function

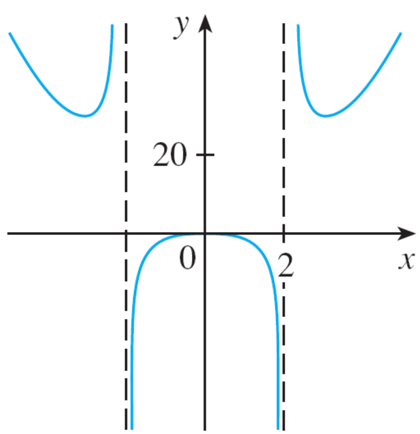
Figure 14

Rational Functions

The function

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

is a rational function with domain $\{x \mid x \neq \pm 2\}$. Its graph is shown in Figure 16.



$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

Figure 16

Algebraic Functions

Algebraic Functions

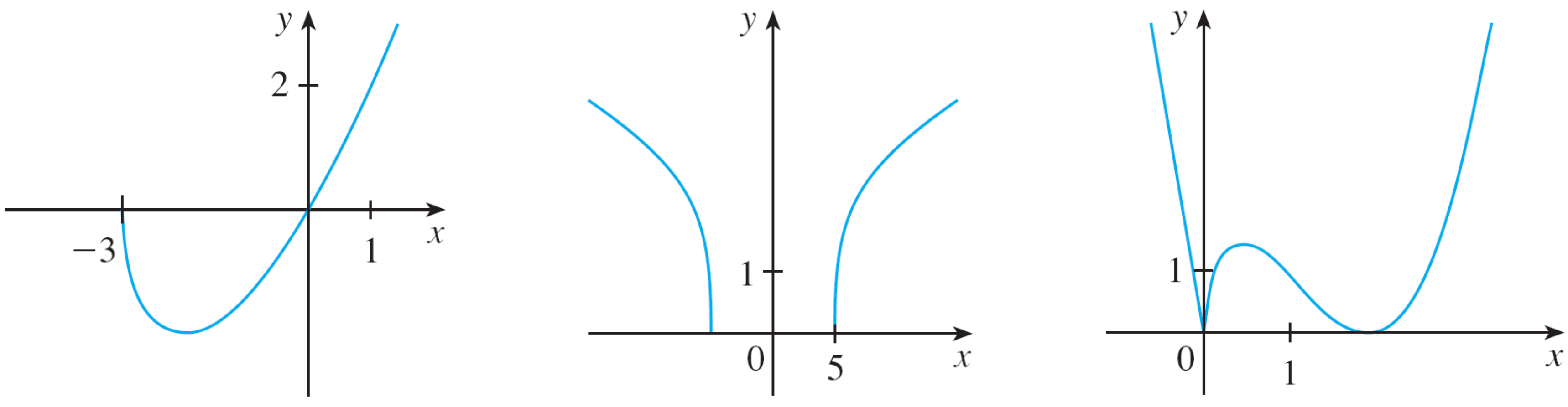
A function *f* is called an **algebraic function** if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with polynomials. Any rational function is automatically an algebraic function.

Here are two more examples:

$$f(x) = \sqrt{x^2 + 1} \qquad g(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 2)\sqrt[3]{x + 1}$$

Algebraic Functions

The graphs of algebraic functions can assume a variety of shapes. Figure 17 illustrates some of the possibilities.



(a) $f(x) = x\sqrt{x+3}$ (b) $g(x) = \sqrt[4]{x^2 - 25}$ (c) $h(x) = x^{2/3}(x-2)^2$

Figure 17

Algebraic Functions

An example of an algebraic function occurs in the theory of relativity. The mass of a particle with velocity v is

$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle and
$c = 3.0 \times 10^5$ km/s is the speed of light in a vacuum.

Trigonometric Functions

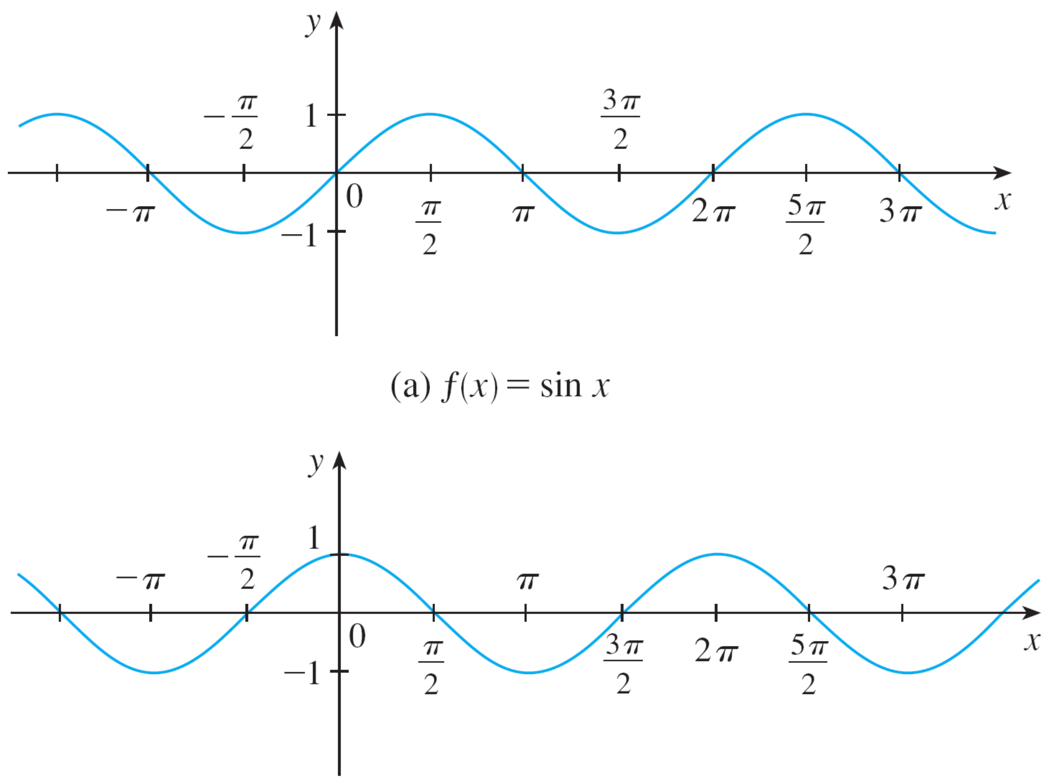
Trigonometric Functions

In calculus the convention is that radian measure is always used (except when otherwise indicated).

For example, when we use the function $f(x) = \sin x$, it is understood that $\sin x$ means the sine of the angle whose radian measure is x.

Trigonometric Functions

Thus the graphs of the sine and cosine functions are as shown in Figure 18.



(a) $f(x) = \sin x$

(b) $g(x) = \cos x$

Figure 18

Trigonometric Functions

Notice that for both the sine and cosine functions the domain is $(-\infty, \infty)$ and the range is the closed interval [–1, 1].

Thus, for all values of *x*, we have

$$\boxed{-1 \leqslant \sin x \leqslant 1 \qquad -1 \leqslant \cos x \leqslant 1}$$

or, in terms of absolute values,

$$|\sin x| \leq 1 \qquad |\cos x| \leq 1$$

Trigonometric Functions

Also, the zeros of the sine function occur at the integer multiples of π; that is,

$$\sin x = 0 \qquad \text{when} \qquad x = n\pi \qquad n \text{ an integer}$$

An important property of the sine and cosine functions is that they are periodic functions and have period 2π.

This means that, for all values of x,

$$\boxed{\sin(x + 2\pi) = \sin x \qquad \cos(x + 2\pi) = \cos x}$$

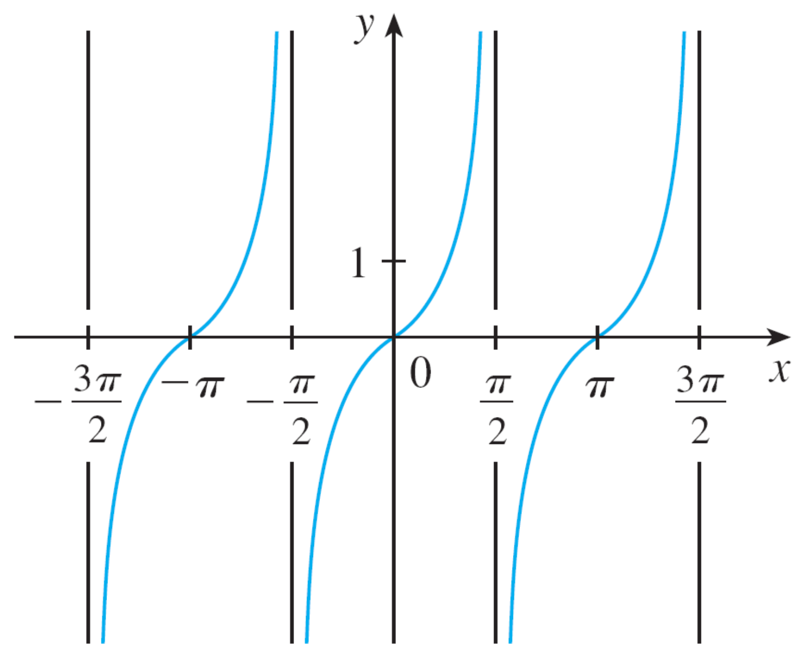
Trigonometric Functions

The tangent function is related to the sine and cosine functions by the equation

$$\tan x = \frac{\sin x}{\cos x}$$

and its graph is shown in Figure 19. It is undefined whenever $\cos x = 0$, that is, when $x = \pm\pi/2, \pm 3\pi/2, \ldots$.

Its range is $(-\infty, \infty)$.



$y = \tan x$

Figure 19

Trigonometric Functions

Notice that the tangent function has period π:

$$\tan(x + \pi) = \tan x \qquad \text{for all } x$$

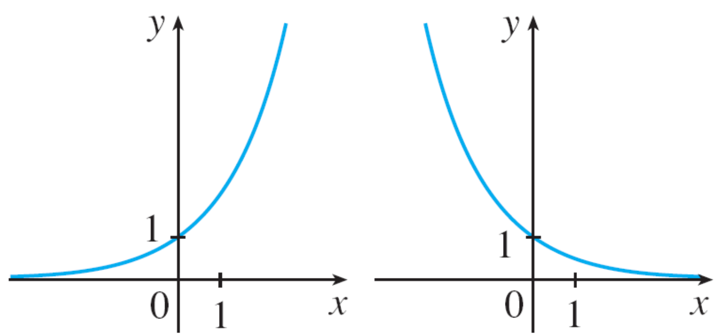
The remaining three trigonometric functions (cosecant, secant, and cotangent) are the reciprocals of the sine, cosine, and tangent functions.

Exponential Functions

Exponential Functions

The **exponential functions** are the functions of the form $f(x) = a^x$, where the base a is a positive constant.

The graphs of $y = 2^x$ and $y = (0.5)^x$ are shown in Figure 20. In both cases the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.



(a) $y = 2^x$ (b) $y = (0.5)^x$

Figure 20

Exponential Functions

Exponential functions are useful for modeling many natural phenomena, such as population growth (if $a > 1$) and radioactive decay (if $a < 1$).

Logarithmic Functions

Logarithmic Functions

The **logarithmic functions** $f(x) = \log_a x$, where the base a is a positive constant, are the inverse functions of the exponential functions. Figure 21 shows the graphs of four logarithmic functions with various bases.

In each case the domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the function increases slowly when $x > 1$.

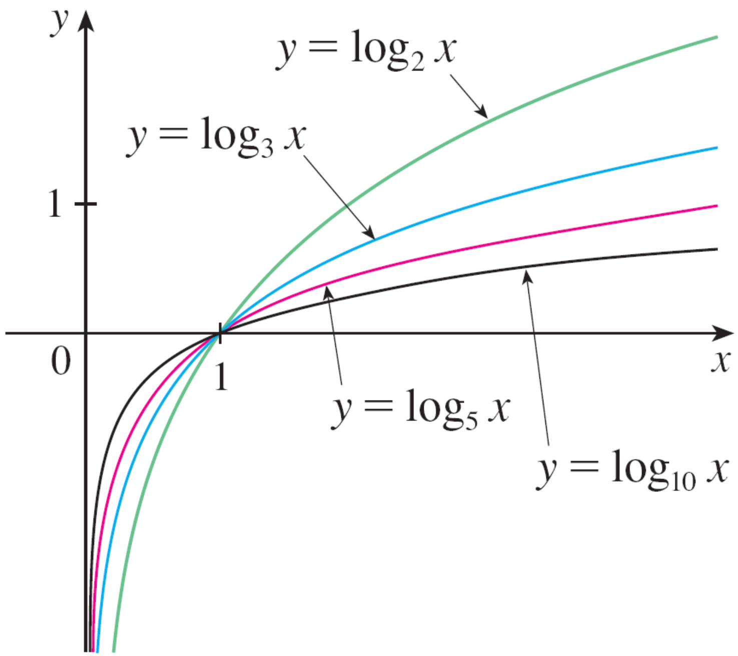


Figure 21

Example 5

Classify the following functions as one of the types of functions that we have discussed.

(a) $f(x) = 5^x$

(b) $g(x) = x^5$

(c) $h(x) = \dfrac{1 + x}{1 - \sqrt{x}}$

(d) $u(t) = 1 - t + 5t^4$

Example 5 – *Solution*

(a) $f(x) = 5^x$ is an exponential function.
(The x is the exponent.)

(b) $g(x) = x^5$ is a power function. (The x is the base.)
We could also consider it to be a polynomial of degree 5.

(c) $h(x) = \dfrac{1 + x}{1 - \sqrt{x}}$ is an algebraic function.

(d) $u(t) = 1 - t + 5t^4$ is a polynomial of degree 4.

1.3

New Functions from Old Functions

Transformations of Functions

Transformations of Functions

By applying certain transformations to the graph of a given function we can obtain the graphs of certain related functions.

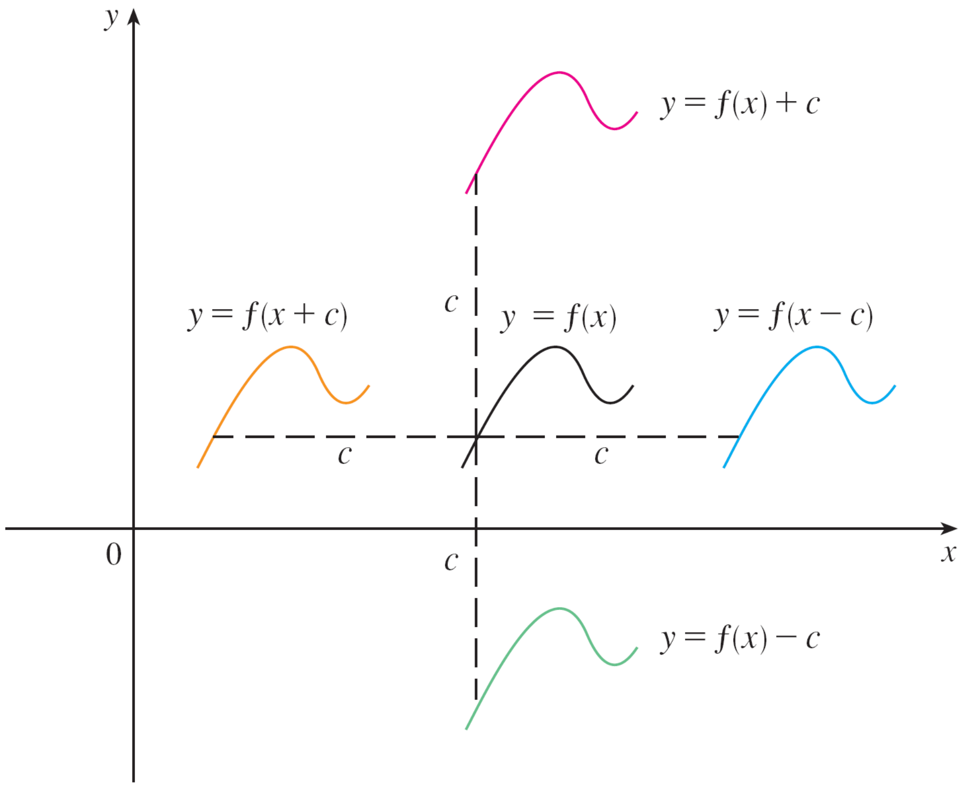
This will give us the ability to sketch the graphs of many functions quickly by hand. It will also enable us to write equations for given graphs.

Let's first consider **translations**. If c is a positive number, then the graph of $y = f(x) + c$ is just the graph of $y = f(x)$ shifted upward a distance of c units (because each y-coordinate is increased by the same number c).

Transformations of Functions

Likewise, if $g(x) = f(x - c)$, where $c > 0$, then the value of g at x is the same as the value of f at $x - c$ (c units to the left of x).

Therefore the graph of $y = f(x - c)$, is just the graph of $y = f(x)$ shifted c units to the right (see Figure 1).



Translating the graph of f

Figure 1

Transformations of Functions

Vertical and Horizontal Shifts Suppose $c > 0$. To obtain the graph of

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward

$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward

$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right

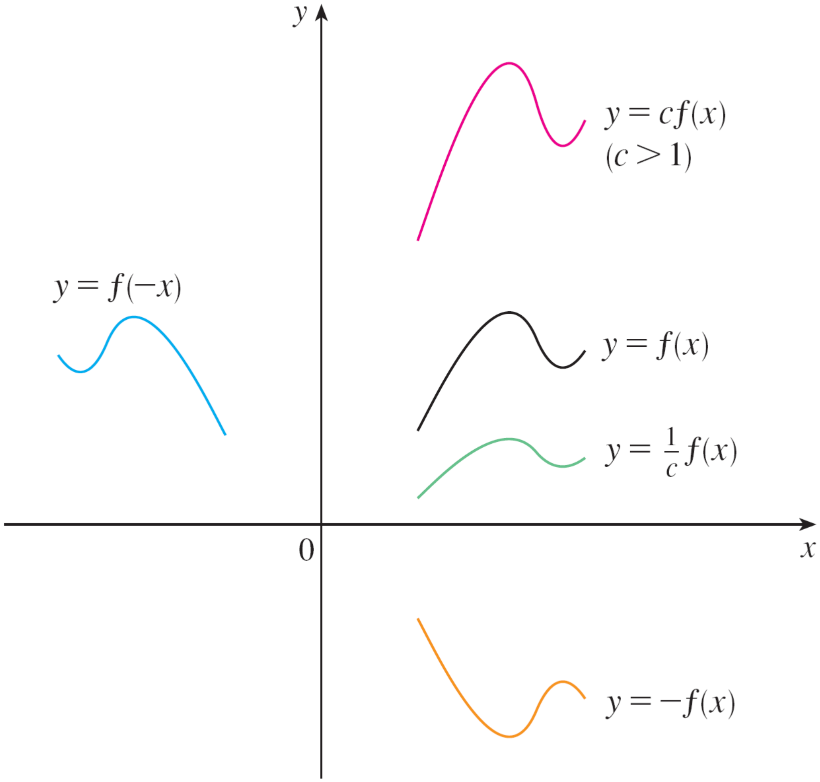
$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

Now let's consider the **stretching** and **reflecting** transformations. If $c > 1$, then the graph of $y = cf(x)$ is the graph of $y = f(x)$ stretched by a factor of c in the vertical direction (because each y-coordinate is multiplied by the same number c).

Transformations of Functions

The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x-axis because the point (x, y) is replaced by the point $(x, -y)$.

(See Figure 2 and the following chart, where the results of other stretching, shrinking, and reflecting transformations are also given.)



Stretching and reflecting the graph of f

Figure 2

Transformations of Functions

Vertical and Horizontal Stretching and Reflecting Suppose $c > 1$. To obtain the graph of

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c

$y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c

$y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c

$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c

$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis

$y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis

Transformations of Functions

Figure 3 illustrates these stretching transformations when applied to the cosine function with c = 2.

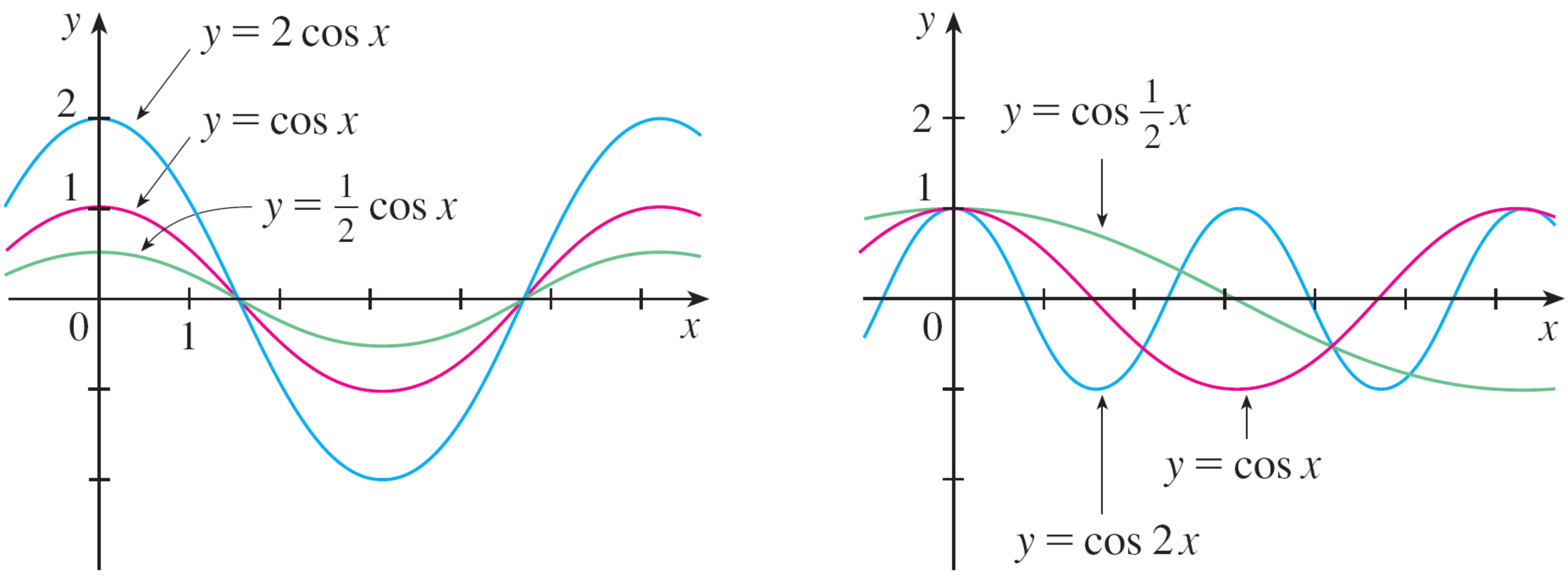


Figure 3

Transformations of Functions

For instance, in order to get the graph of $y = 2 \cos x$ we multiply the y-coordinate of each point on the graph of $y = \cos x$ by 2.

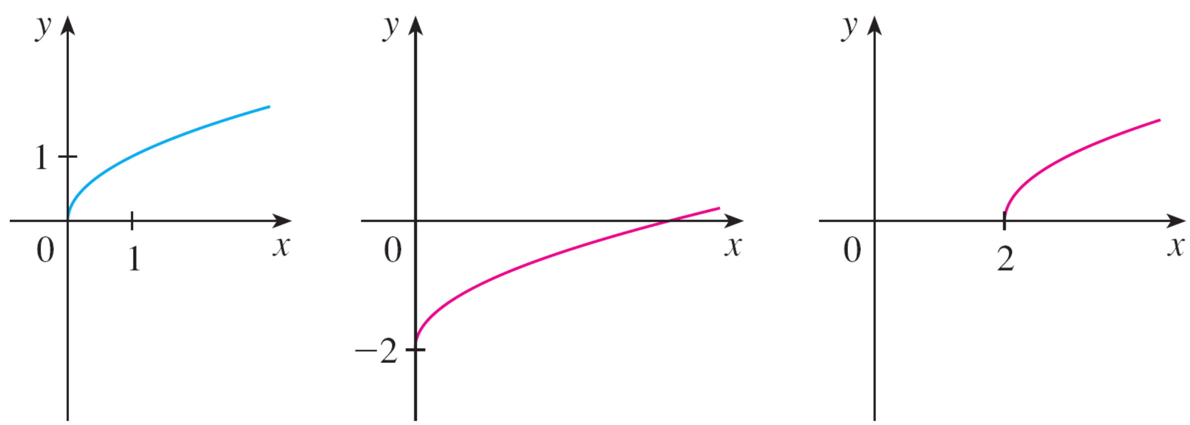
This means that the graph of $y = \cos x$ gets stretched vertically by a factor of 2.

Example 1

Given the graph of $y = \sqrt{x}$, use transformations to graph $y = \sqrt{x} - 2$, $y = \sqrt{x - 2}$, $y = -\sqrt{x}$, $y = 2\sqrt{x}$, and $y = \sqrt{-x}$.

Solution:

The graph of the square root function $y = \sqrt{x}$, is shown in Figure 4(a).



(a) $y = \sqrt{x}$ (b) $y = \sqrt{x} - 2$ (c) $y = \sqrt{x - 2}$

Figure 4

Example 1 – *Solution*

cont'd

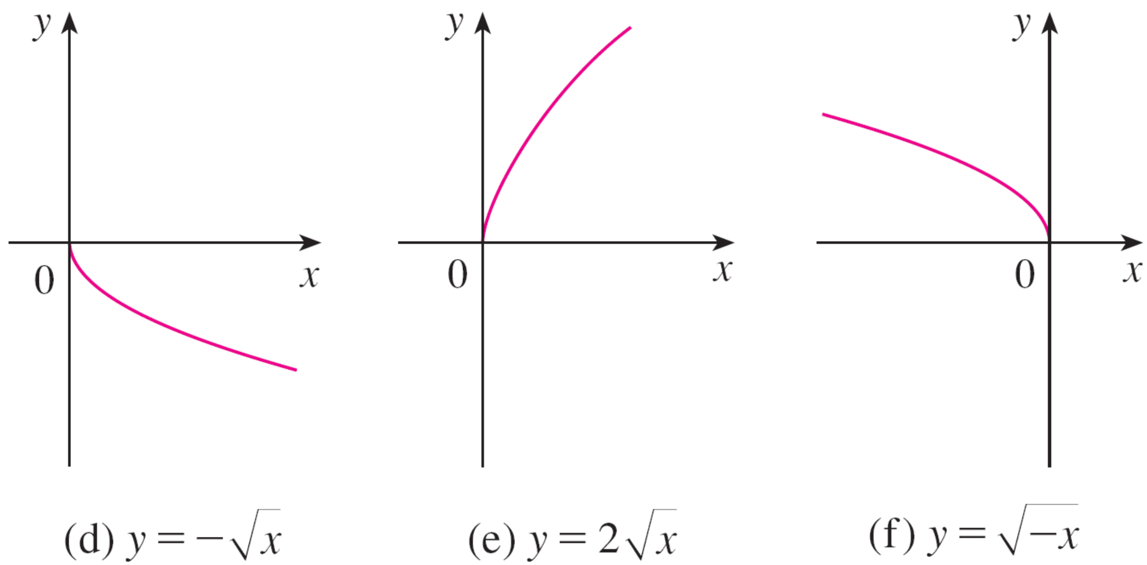


Figure 4

In the other parts of the figure we sketch $y = \sqrt{x} - 2$ by shifting 2 units downward, $y = \sqrt{x - 2}$ by shifting 2 units to the right, $y = -\sqrt{x}$ by reflecting about the *x*-axis, $y = 2\sqrt{x}$ by stretching vertically by a factor of 2, and $y = \sqrt{-x}$ by reflecting about the *y*-axis.

Transformations of Functions

Another transformation of some interest is taking the *absolute value* of a function. If $y = |f(x)|$, then according to the definition of absolute value, $y = f(x)$ when $f(x) \geq 0$ and $y = -f(x)$ when $f(x) < 0$.

This tells us how to get the graph of $y = |f(x)|$ from the graph of $y = f(x)$: The part of the graph that lies above the x-axis remains the same; the part that lies below the x-axis is reflected about the x-axis.

Combinations of Functions

Combinations of Functions

Two functions f and g can be combined to form new functions $f + g$, $f - g$, fg, and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers. The sum and difference functions are defined by

$$(f + g)(x) = f(x) + g(x) \qquad\qquad (f - g)(x) = f(x) - g(x)$$

If the domain of f is A and the domain of g is B, then the domain of $f + g$ is the intersection $A \cap B$ because both $f(x)$ and $g(x)$ have to be defined.

For example, the domain of $f(x) = \sqrt{x}$ is $A = [0, \infty)$ and the domain of $g(x) = \sqrt{2 - x}$ is $B = (-\infty, 2]$, so the domain of $(f + g)(x) = \sqrt{x} + \sqrt{2 - x}$ is $A \cap B = [0, 2]$.

Combinations of Functions

Similarly, the product and quotient functions are defined by

$$(fg)(x) = f(x)g(x) \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

The domain of fg is $A \cap B$, but we can't divide by 0 and so the domain of f/g is $\{x \in A \cap B \mid g(x) \neq 0\}$.

For instance, if $f(x) = x^2$ and $g(x) = x - 1$, then the domain of the rational function $(f/g)(x) = x^2/(x - 1)$ is $\{x \mid x \neq 1\}$, or $(-\infty, 1) \cup (1, \infty)$.

Combinations of Functions

There is another way of combining two functions to obtain a new function. For example, suppose that $y = f(u) = \sqrt{u}$ and $u = g(x) = x^2 + 1$.

Since y is a function of u and u is, in turn, a function of x, it follows that y is ultimately a function of x. We compute this by substitution:

$$y = f(u) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

The procedure is called *composition* because the new function is *composed* of the two given functions f and g.

Combinations of Functions

In general, given any two functions f and g, we start with a number x in the domain of g and find its image $g(x)$. If this number $g(x)$ is in the domain of f, then we can calculate the value of $f(g(x))$.

The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f. It is called the *composition* (or *composite*) of f and g *and* is denoted by $f \circ g$ ("f circle g").

Definition Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

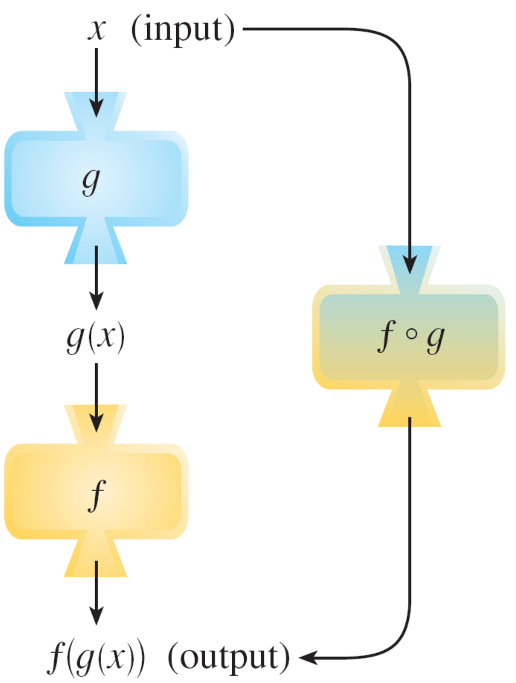
$$(f \circ g)(x) = f(g(x))$$

Combinations of Functions

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined.

Figure 11 shows how to picture $f \circ g$ in terms of machines.



The $f \circ g$ machine is composed of the g machine (first) and then the f machine.

Figure 11

Example 6

If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$.

Solution:

We have

$$(f \circ g)(x) = f(g(x)) = f(x - 3) = (x - 3)^2$$

$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 3$$

Combinations of Functions

Remember, the notation $f \circ g$ means that the function g is applied first and then f is applied second. In Example 6, $f \circ g$ is the function that *first* subtracts 3 and *then* squares; $g \circ f$ is the function that *first* squares and *then* subtracts 3.

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h, then g, and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

1.4

The Tangent and Velocity Problems

The Tangent Problem

The Tangent Problems

The word *tangent* is derived from the Latin word *tangens*, which means “touching.”

Thus a tangent to a curve is a line that touches the curve.

In other words, a tangent line should have the same direction as the curve at the point of contact.

The Tangent Problems

For a circle we could simply follow Euclid and say that a tangent is a line that intersects the circle once and only once, as in Figure 1(a).

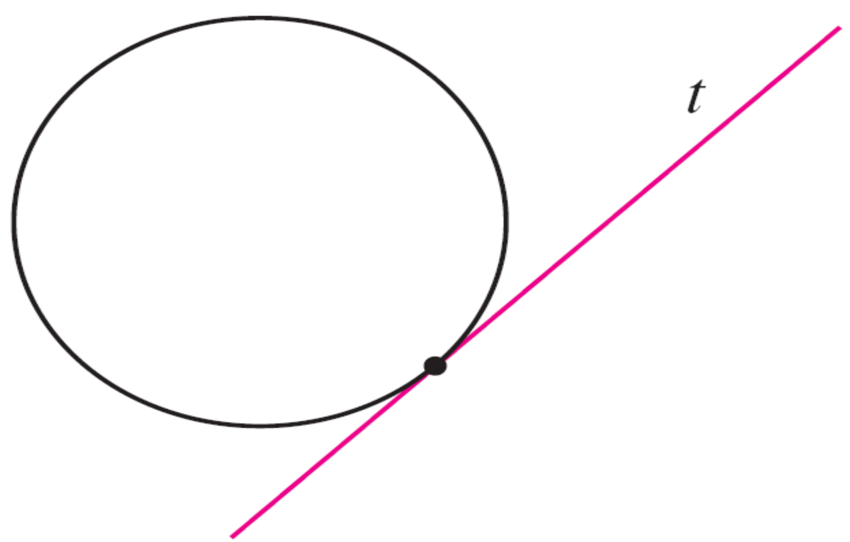


Figure 1(a)

The Tangent Problems

For more complicated curves this definition is inadequate. Figure 1(b) shows two lines *l* and *t* passing through a point *P* on a curve *C*.

The line *l* intersects *C* only once, but it certainly does not look like what we think of as a tangent.

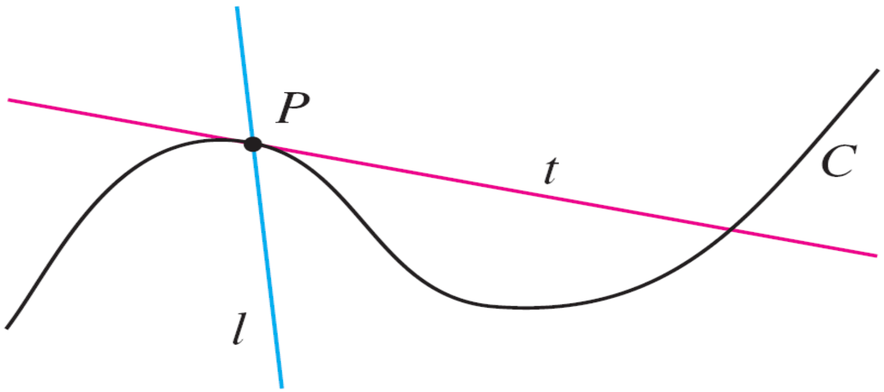


Figure 1(b)

The line *t*, on the other hand, looks like a tangent but it intersects *C* twice.

Example 1

Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Solution:

We will be able to find an equation of the tangent line t as soon as we know its slope m.

The difficulty is that we know only one point, P, on t, whereas we need two points to compute the slope.

Example 1 – *Solution*

cont'd

But observe that we can compute an approximation to m by choosing a near by point $Q(x, x^2)$ on the parabola (as in Figure 2) and computing the slope m_{PQ} of the secant line PQ. [A **secant line**, from the Latin word *secans*, meaning cutting, is a line that cuts (intersects) a curve more than once.]

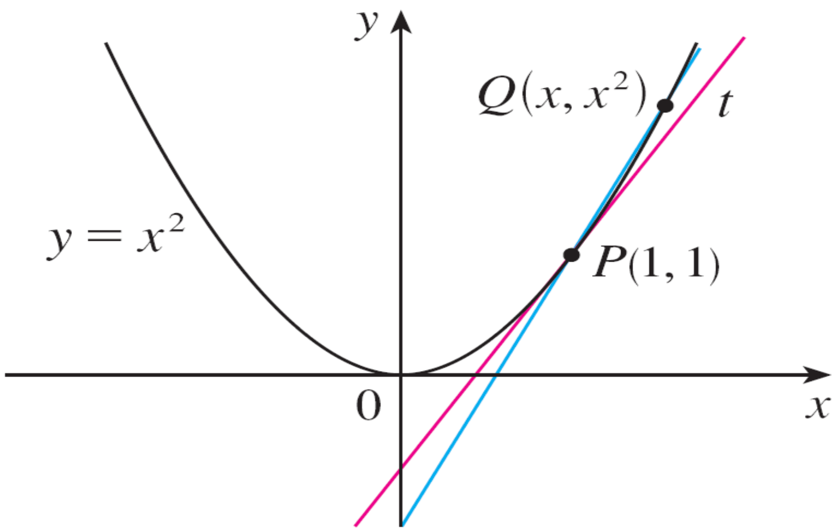


Figure 2

Example 1 – *Solution*

cont'd

We choose $x \neq 1$ so that $Q \neq P$. Then

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

For instance, for the point $Q(1.5, 2.25)$ we have

$$m_{PQ} = \frac{2.25 - 1}{1.5 - 1}$$

$$= \frac{1.25}{0.5}$$

$$= 2.5$$

Example 1 – *Solution*

cont'd

The tables in the margin show the values of m_{PQ} for several values of x close to 1.

The closer Q is to P, the closer x is to 1 and, it appears from the tables, the closer m_{PQ} is to 2.

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

Example 1 – *Solution*

cont'd

This suggests that the slope of the tangent line *t* should be $m = 2$.

We say that the slope of the tangent line is the *limit* of the slopes of the secant lines, and we express this symbolically by writing

$$\lim_{Q \to P} m_{PQ} = m \qquad \text{and} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

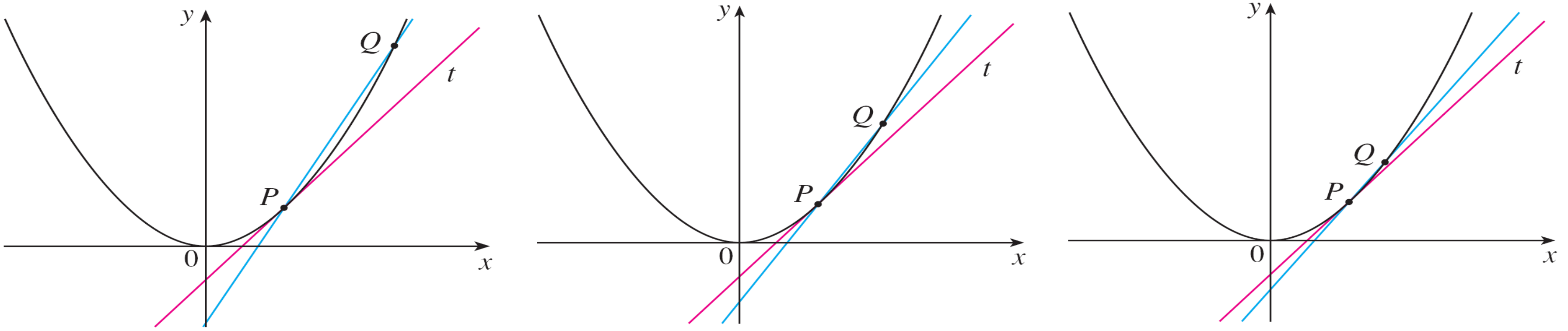
Assuming that the slope of the tangent line is indeed 2, we use the point-slope form of the equation of a line to write the equation of the tangent line through (1, 1) as

$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1$$

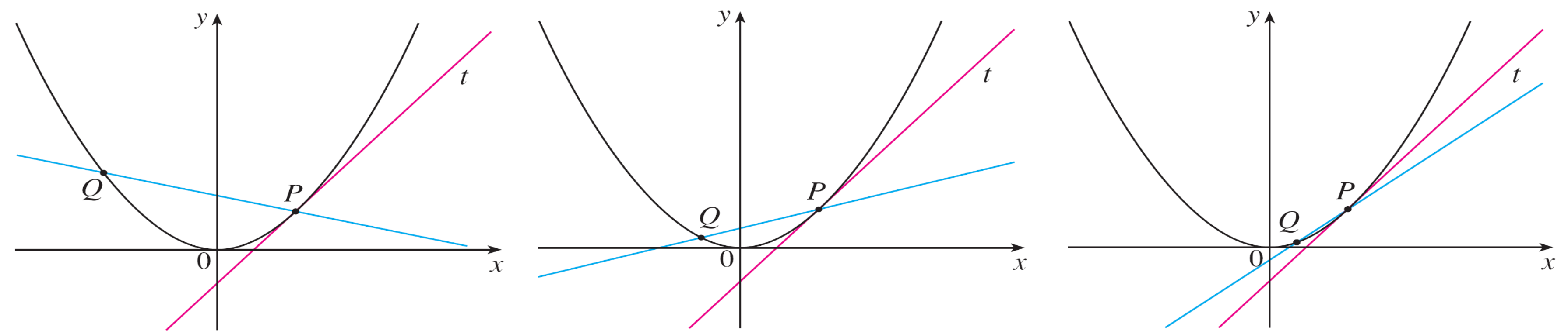
Example 1 – *Solution*

cont'd

Figure 3 illustrates the limiting process that occurs in this example.



Q approaches *P* from the right



Q approaches *P* from the left

Figure 3

Example 1 – *Solution*

cont'd

As Q approaches P along the parabola, the corresponding secant lines rotate about P and approach the tangent line t.

The Velocity Problem

Example 3

Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

Solution:

Through experiments carried out four centuries ago, Galileo discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling. (This model for free fall neglects air resistance.)

If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo's law is expressed by the equation $s(t) = 4.9t^2$

Example 3 – *Solution*

cont'd

The difficulty in finding the velocity after 5 s is that we are dealing with a single instant of time ($t = 5$), so no time interval is involved.

However, we can approximate the desired quantity by computing the average velocity over the brief time interval of a tenth of a second from $t = 5$ to $t = 5.1$:

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}}$$

$$= \frac{s(5.1) - s(5)}{0.1}$$

$$= \frac{4.9(5.1)^2 - 4.9(5)^2}{0.1} = 49.49 \text{ m/s}$$

Example 3 – *Solution*

cont'd

The following table shows the results of similar calculations of the average velocity over successively smaller time periods.

Time interval	Average velocity (m/s)
$5 \leqslant t \leqslant 6$	53.9
$5 \leqslant t \leqslant 5.1$	49.49
$5 \leqslant t \leqslant 5.05$	49.245
$5 \leqslant t \leqslant 5.01$	49.049
$5 \leqslant t \leqslant 5.001$	49.0049

Example 3 – *Solution*

cont'd

It appears that as we shorten the time period, the average velocity is becoming closer to 49 m/s.

The **instantaneous velocity** when $t = 5$ is defined to be the limiting value of these average velocities over shorter and shorter time periods that start at $t = 5$.

Thus the (instantaneous) velocity after 5 s is

$$v = 49 \text{ m/s}$$

1.5 The Limit of a Function

The Limit of a Function

To find the tangent to a curve or the velocity of an object, we now turn our attention to limits in general and numerical and graphical methods for computing them.

Let's investigate the behavior of the function f defined by $f(x) = x^2 - x + 2$ for values of x near 2.

The Limit of a Function

The following table gives values of $f(x)$ for values of x close to 2 but not equal to 2.

x	$f(x)$	x	$f(x)$
1.0	2.000000	3.0	8.000000
1.5	2.750000	2.5	5.750000
1.8	3.440000	2.2	4.640000
1.9	3.710000	2.1	4.310000
1.95	3.852500	2.05	4.152500
1.99	3.970100	2.01	4.030100
1.995	3.985025	2.005	4.015025
1.999	3.997001	2.001	4.003001

The Limit of a Function

From the table and the graph of *f* (a parabola) shown in Figure 1 we see that when *x* is close to 2 (on either side of 2), *f*(*x*) is close to 4.

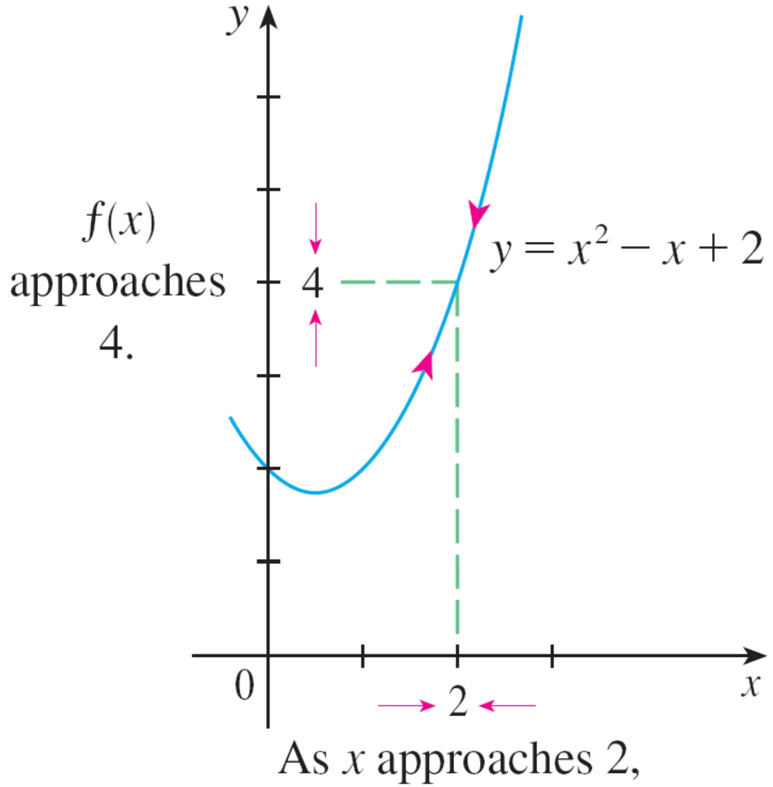


Figure 1

The Limit of a Function

In fact, it appears that we can make the values of $f(x)$ as close as we like to 4 by taking x sufficiently close to 2.

We express this by saying "the limit of the function $f(x) = x^2 - x + 2$ as x approaches 2 is equal to 4."

The notation for this is

$$\lim_{x \to 2} (x^2 - x + 2) = 4$$

The Limit of a Function

In general, we use the following notation.

1 Definition We write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

This says that the values of $f(x)$ tend to get closer and closer to the number L as x gets closer and closer to the number a (from either side of a) but $x \neq a$.

The Limit of a Function

An alternative notation for

$$\lim_{x \to a} f(x) = L$$

is $\qquad f(x) \to L \qquad$ as $\qquad x \to a$

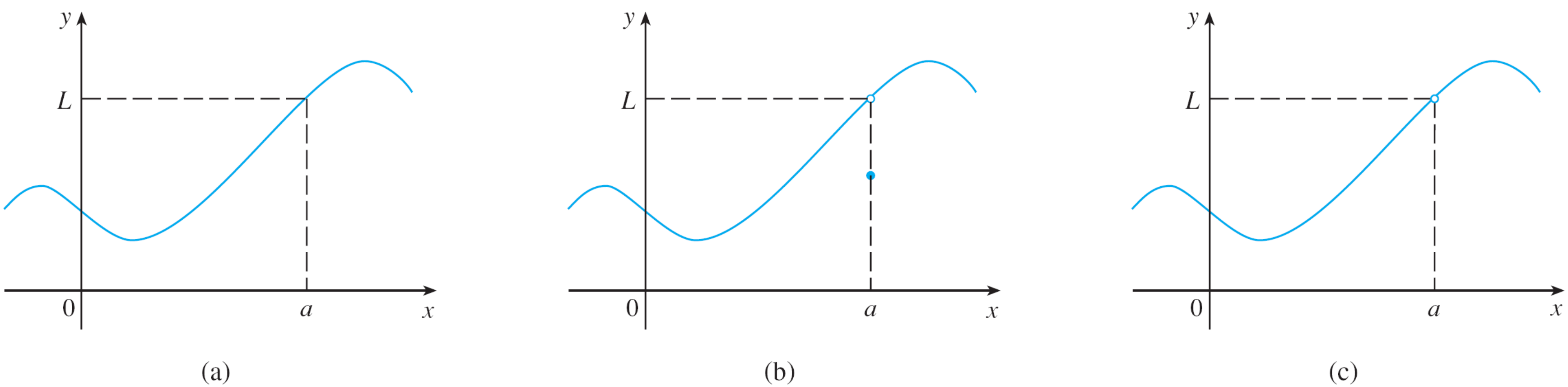
which is usually read "$f(x)$ approaches L as x approaches a."

Notice the phrase "but $x \neq a$" in the definition of limit. This means that in finding the limit of $f(x)$ as x approaches a, we never consider $x = a$. In fact, $f(x)$ need not even be defined when $x = a$. The only thing that matters is how f is defined *near a*.

The Limit of a Function

Figure 2 shows the graphs of three functions. Note that in part (c), $f(a)$ is not defined and in part (b), $f(a) \neq L$.

But in each case, regardless of what happens at a, it is true that $\lim_{x \to a} f(x) = L$.



$\lim_{x \to a} f(x) = L$ in all three cases

Figure 2

Example 1

Guess the value of $\lim_{x \to 1} \dfrac{x - 1}{x^2 - 1}$.

Solution:

Notice that the function $f(x) = (x - 1)/(x^2 - 1)$ is not defined when $x = 1$, but that doesn't matter because the definition of $\lim_{x \to a} f(x)$ says that we consider values of x that are close to a but not equal to a.

Example 1 – *Solution*

cont'd

The tables below give values of $f(x)$ (correct to six decimal places) for values of x that approach 1 (but are not equal to 1).

$x < 1$	$f(x)$
0.5	0.666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025

$x > 1$	$f(x)$
1.5	0.400000
1.1	0.476190
1.01	0.497512
1.001	0.499750
1.0001	0.499975

On the basis of the values in the tables, we make the guess that

$$\lim_{x \to 1} \frac{x - 1}{x^2 - 1} = 0.5$$

The Limit of a Function

Example 1 is illustrated by the graph of *f* in Figure 3. Now let's change *f* slightly by giving it the value 2 when $x = 1$ and calling the resulting function *g*:

$$g(x) = \begin{cases} \dfrac{x-1}{x^2-1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$

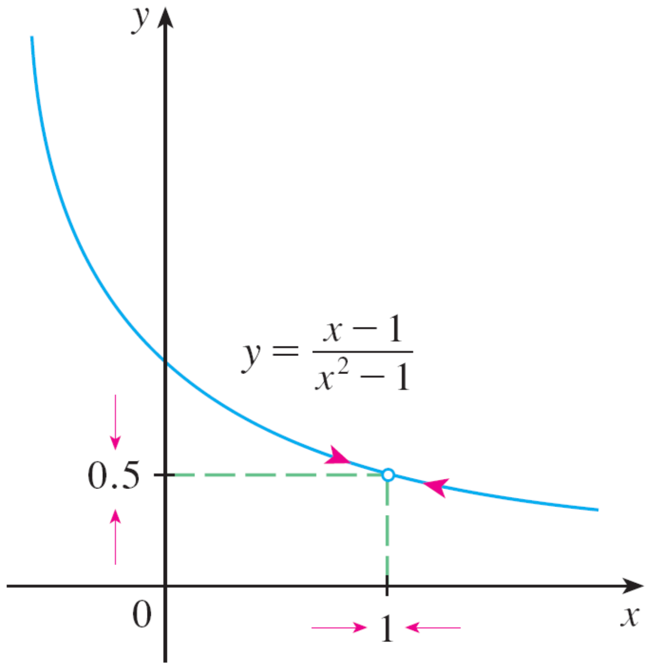


Figure 3

The Limit of a Function

This new function g still has the same limit as x approaches 1. (See Figure 4.)

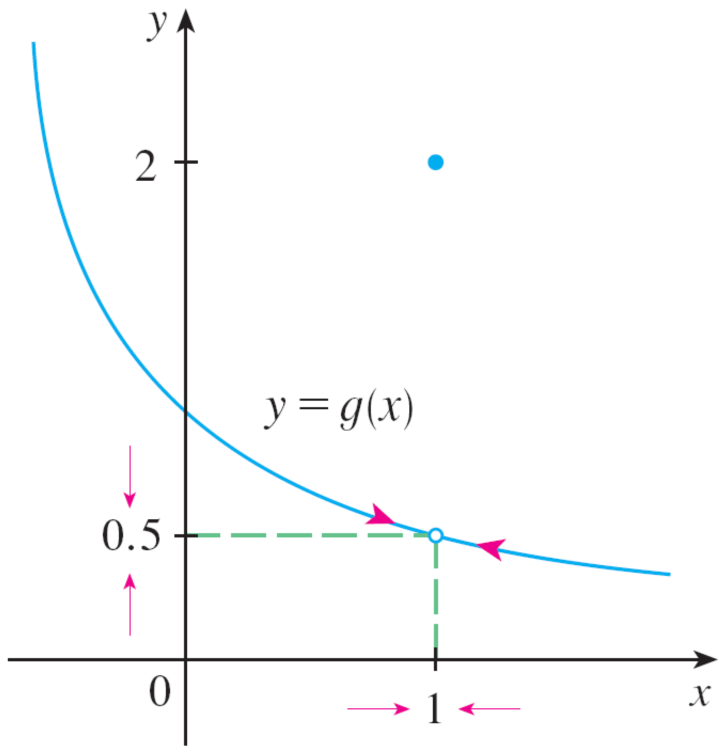


Figure 4

One-Sided Limits

One-Sided Limits

The function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geqslant 0 \end{cases}$$

$H(t)$ approaches 0 as t approaches 0 from the left and $H(t)$ approaches 1 as t approaches 0 from the right.

We indicate this situation symbolically by writing

$$\lim_{t \to 0^-} H(t) = 0 \qquad \text{and} \qquad \lim_{t \to 0^+} H(t) = 1$$

One-Sided Limits

The symbol "$t \to 0^-$" indicates that we consider only values of t that are less than 0.

Likewise, "$t \to 0^+$" indicates that we consider only values of t that are greater than 0.

One-Sided Limits

2 Definition We write

$$\lim_{x \to a^-} f(x) = L$$

and say the **left-hand limit of $f(x)$ as x approaches a** [or the **limit of $f(x)$ as x approaches a from the left**] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than a.

Notice that Definition 2 differs from Definition 1 only in that we require *x* to be less than *a*.

One-Sided Limits

Similarly, if we require that x be greater than a, we get "the **right-hand limit of $f(x)$ as x approaches a** is equal to L" and we write

$$\lim_{x \to a^+} f(x) = L$$

Thus the symbol "$x \to a^+$" means that we consider only $x > a$. These definitions are illustrated in Figure 9.

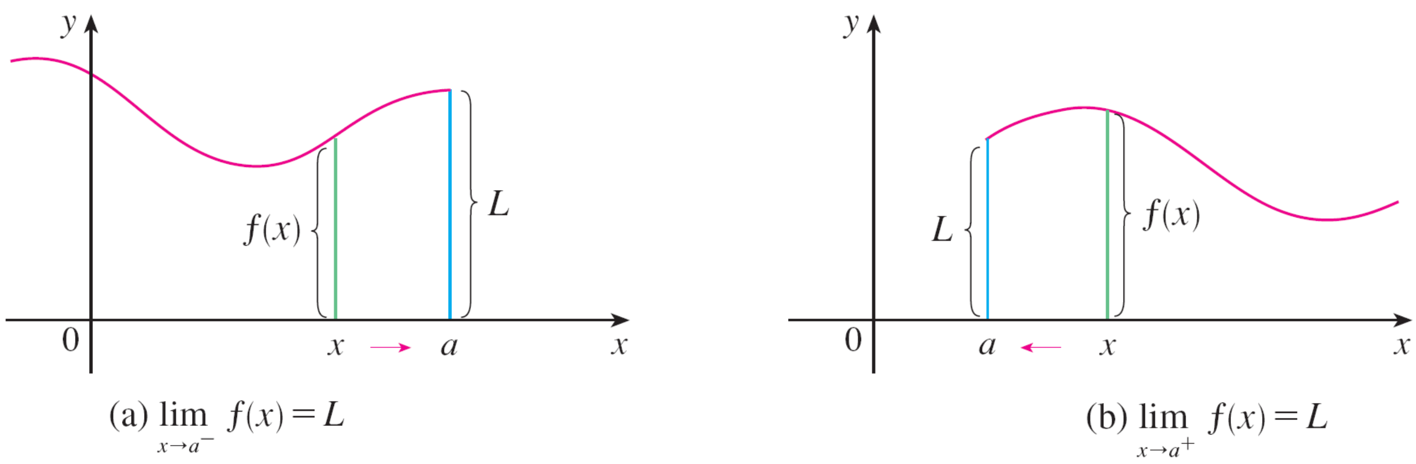


Figure 9

One-Sided Limits

By comparing Definition 1 with the definitions of one-sided limits, we see that the following is true.

3 $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

Example 7

The graph of a function *g* is shown in Figure 10. Use it to state the values (if they exist) of the following:

(a) $\lim_{x \to 2^-} g(x)$ (b) $\lim_{x \to 2^+} g(x)$ (c) $\lim_{x \to 2} g(x)$

(d) $\lim_{x \to 5^-} g(x)$ (e) $\lim_{x \to 5^+} g(x)$ (f) $\lim_{x \to 5} g(x)$

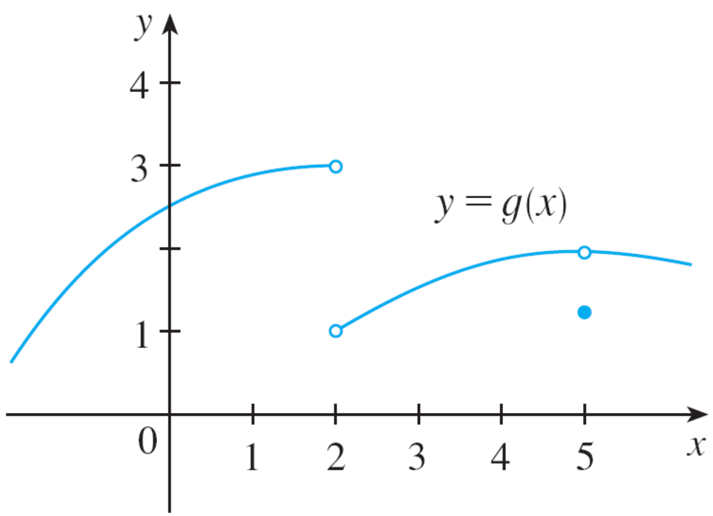


Figure 10

Example 7 – *Solution*

From the graph we see that the values of $g(x)$ approach 3 as x approaches 2 from the left, but they approach 1 as x approaches 2 from the right.

Therefore

(a) $\lim_{x \to 2^-} g(x) = 3$ and (b) $\lim_{x \to 2^+} g(x) = 1$

(c) Since the left and right limits are different, we conclude from $\boxed{3}$ that $\lim_{x \to 2} g(x)$ does not exist.

Example 7 – *Solution*

cont'd

The graph also shows that

$$\text{(d)} \quad \lim_{x \to 5^-} g(x) = 2 \qquad \text{and} \qquad \text{(e)} \quad \lim_{x \to 5^+} g(x) = 2$$

(f) This time the left and right limits are the same and so, by $\boxed{3}$, we have

$$\lim_{x \to 5} g(x) = 2$$

Despite this fact, notice that $g(5) \neq 2$.

Infinite Limits

Infinite Limits

4 Definition Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

Another notation for $\lim_{x \to a} f(x) = \infty$ is

$$f(x) \to \infty \text{ as } x \to a$$

Infinite Limits

Again, the symbol ∞ is not a number, but the expression $\lim_{x \to a} f(x) = \infty$ is often read as

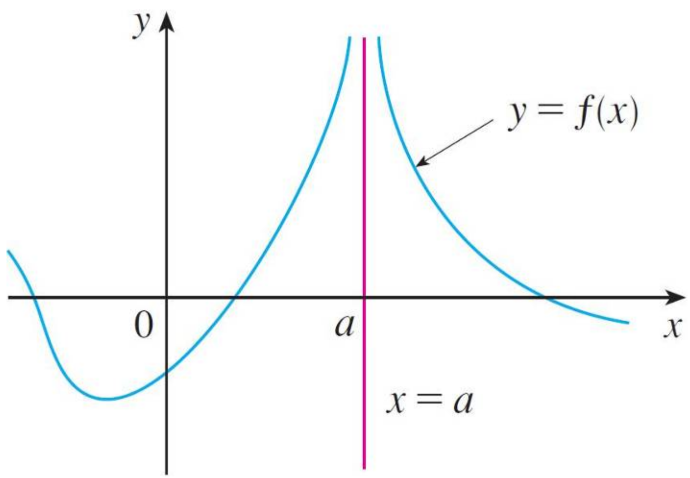
"the limit of $f(x)$, as approaches a, is infinity"

or "$f(x)$ becomes infinite as approaches a"

or "$f(x)$ increases without bound as approaches a"

Infinite Limits

This definition is illustrated graphically in Figure 12.

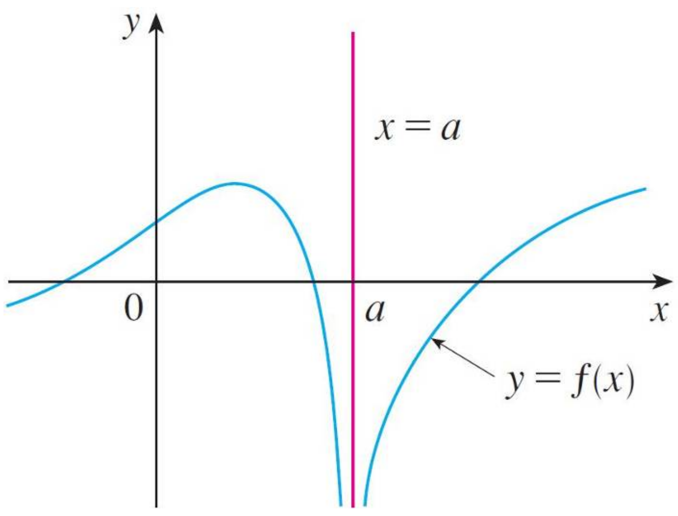


$$\lim_{x \to a} f(x) = \infty$$

Figure 12

Infinite Limits

A similar sort of limit, for functions that become large negative as *x* gets close to *a*, is defined in Definition 5 and is illustrated in Figure 13.



$$\lim_{x \to a} f(x) = -\infty$$

Figure 13

Infinite Limits

5 Definition Let f be defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

The symbol $\lim_{x \to a} f(x) = -\infty$ can be read as "the limit of $f(x)$, as x approaches a, is negative infinity" or "$f(x)$ decreases without bound as x approaches a." As an example we have

$$\lim_{x \to 0} \left(-\frac{1}{x^2} \right) = -\infty$$

Infinite Limits

Similar definitions can be given for the one-sided infinite limits

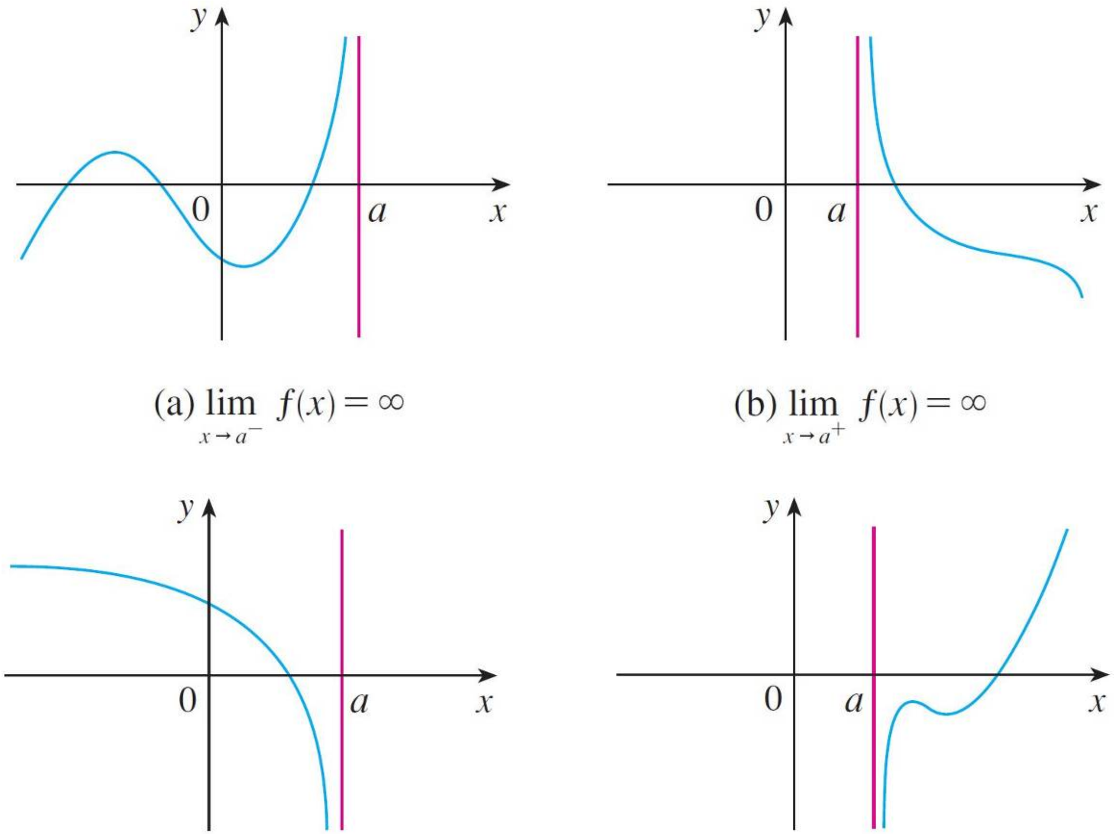
$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

remembering that "$x \to a^-$" means that we consider only values of that are less than a, and similarly "$x \to a^+$" means that we consider only $x > a$.

Infinite Limits

Illustrations of these four cases are given in Figure 14.



(a) $\lim_{x \to a^-} f(x) = \infty$

(b) $\lim_{x \to a^+} f(x) = \infty$

(c) $\lim_{x \to a^-} f(x) = -\infty$

(d) $\lim_{x \to a^+} f(x) = -\infty$

Figure 14

Infinite Limits

6 Definition The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

Example 10

Find the vertical asymptotes of $f(x) = \tan x$.

Solution:

Because

$$\tan x = \frac{\sin x}{\cos x}$$

there are potential vertical asymptotes where $\cos x = 0$.

In fact, since $\cos x \to 0^+$ as $x \to (\pi/2)^+$, whereas $\sin x$ is positive when x is near $\pi/2$, we have

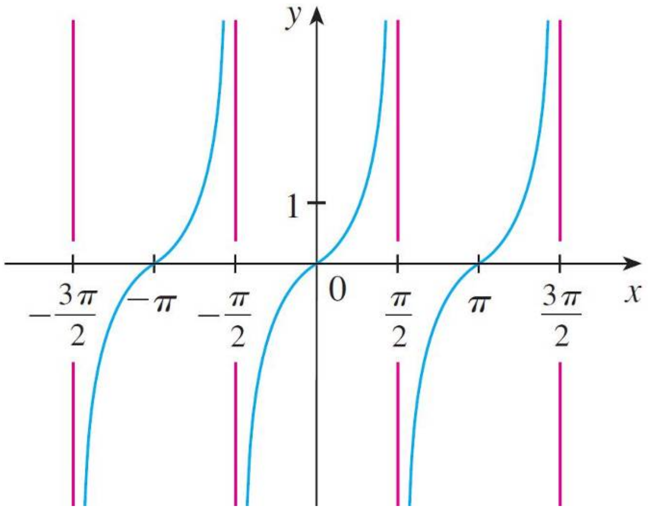
$$\lim_{x \to (\pi/2)^-} \tan x = \infty \qquad \text{and} \qquad \lim_{x \to (\pi/2)^+} \tan x = -\infty$$

Example 10 – *Solution*

cont'd

This shows that the line $x = \pi/2$ is a vertical asymptote. Similar reasoning shows that the lines $x = (2n + 1)\pi/2$, where n is an integer, are all vertical asymptotes of $f(x) = \tan x$.

The graph in Figure 16 confirms this.



$y = \tan x$

Figure 16

1.6

Calculating Limits Using the Limit Laws

Calculating Limits Using the Limit Laws

In this section we use the following properties of limits, called the *Limit Laws,* to calculate limits.

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} g(x)$$

exist. Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$

4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{if } \lim_{x \to a} g(x) \neq 0$

Calculating Limits Using the Limit Laws

These five laws can be stated verbally as follows:

Sum Law

1. The limit of a sum is the sum of the limits.

Difference Law

2. The limit of a difference is the difference of the limits.

Constant Multiple Law

3. The limit of a constant times a function is the constant times the limit of the function.

Calculating Limits Using the Limit Laws

Product Law **4.** The limit of a product is the product of the limits.

Quotient Law **5.** The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

For instance, if $f(x)$ is close to L and $g(x)$ is close to M, it is reasonable to conclude that $f(x) + g(x)$ is close to $L + M$.

Example 1

Use the Limit Laws and the graphs of *f* and *g* in Figure 1 to evaluate the following limits, if they exist.

(a) $\lim_{x \to -2} [f(x) + 5g(x)]$ (b) $\lim_{x \to 1} [f(x)g(x)]$ (c) $\lim_{x \to 2} \frac{f(x)}{g(x)}$

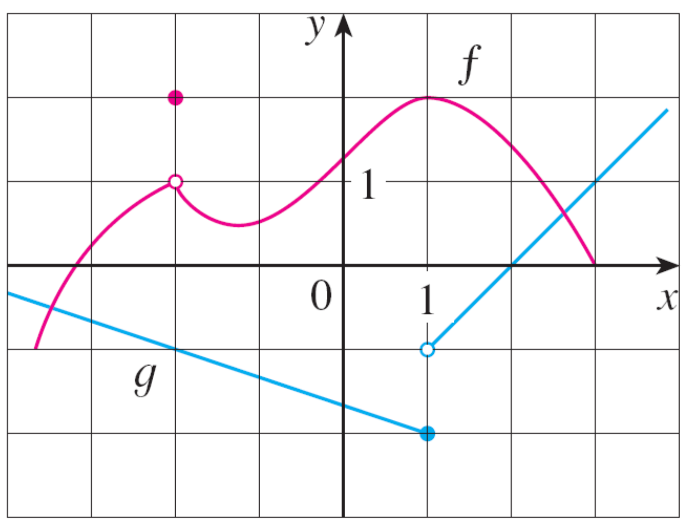


Figure 1

Example 1(a) – *Solution*

From the graphs of *f* and *g* we see that

$$\lim_{x \to -2} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to -2} g(x) = -1$$

Therefore we have

$$\lim_{x \to -2} [f(x) + 5g(x)] = \lim_{x \to -2} f(x) + \lim_{x \to -2} [5g(x)] \quad \text{(by Law 1)}$$

$$= \lim_{x \to -2} f(x) + 5 \lim_{x \to -2} g(x) \quad \text{(by Law 3)}$$

$$= 1 + 5(-1)$$

$$= -4$$

Example 1(b) – *Solution*

cont'd

We see that $\lim_{x \to 1} f(x) = 2$. But $\lim_{x \to 1} g(x)$ does not exist because the left and right limits are different:

$$\lim_{x \to 1^-} g(x) = -2 \qquad \lim_{x \to 1^+} g(x) = -1$$

So we can't use Law 4 for the desired limit. But we *can* use Law 4 for the one-sided limits:

$$\lim_{x \to 1^-} [f(x)g(x)] = 2 \cdot (-2) = -4 \qquad \lim_{x \to 1^+} [f(x)g(x)] = 2 \cdot (-1) = -2$$

The left and right limits aren't equal, so $\lim_{x \to 1} [f(x)g(x)]$ does not exist.

Example 1(c) – *Solution*

cont'd

The graphs show that

$$\lim_{x \to 2} f(x) \approx 1.4 \quad \text{and} \quad \lim_{x \to 2} g(x) = 0$$

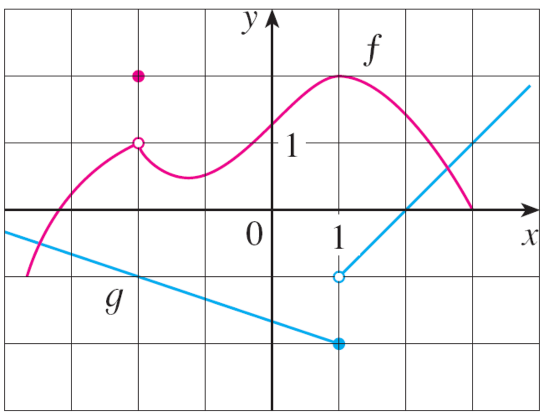


Figure 1

Because the limit of the denominator is 0, we can't use Law 5.

The given limit does not exist because the denominator approaches 0 while the numerator approaches a nonzero number.

Calculating Limits Using the Limit Laws

If we use the Product Law repeatedly with $g(x) = f(x)$, we obtain the following law.

Power Law

6. $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ where n is a positive integer

In applying these six limit laws, we need to use two special limits:

7. $\lim_{x \to a} c = c$

8. $\lim_{x \to a} x = a$

These limits are obvious from an intuitive point of view (state them in words or draw graphs of $y = c$ and $y = x$).

Calculating Limits Using the Limit Laws

If we now put $f(x) = x$ in Law 6 and use Law 8, we get another useful special limit.

9. $\lim_{x \to a} x^n = a^n$ where n is a positive integer

A similar limit holds for roots as follows.

10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

(If n is even, we assume that $a > 0$.)

More generally, we have the following law.

Root Law

11. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer

[If n is even, we assume that $\lim_{x \to a} f(x) > 0$.]

Calculating Limits Using the Limit Laws

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

Functions with the Direct Substitution Property are called *continuous at a.*

In general, we have the following useful fact.

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, provided the limits exist.

Calculating Limits Using the Limit Laws

Some limits are best calculated by first finding the left- and right-hand limits. The following theorem says that a two-sided limit exists if and only if both of the one-sided limits exist and are equal.

1 Theorem $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

When computing one-sided limits, we use the fact that the Limit Laws also hold for one-sided limits.

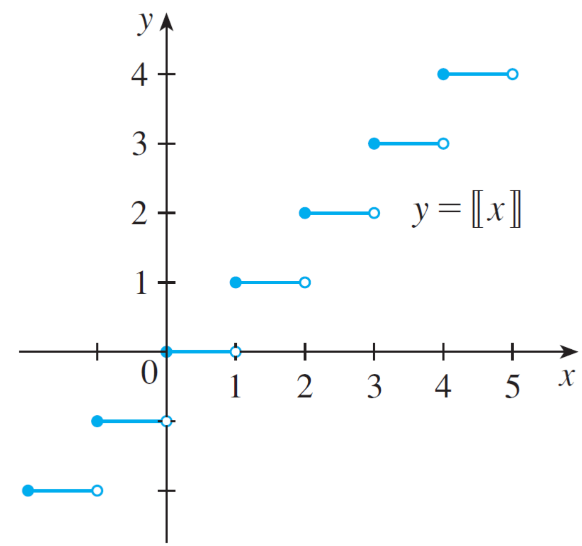
Example 10

The **greatest integer function** is defined $[\![x]\!]$ = the largest integer that is less than or equal to *x*. (For instance, $[\![4]\!] = 4$, $[\![4.8]\!] = 4$, $[\![\pi]\!] = 3$, $[\![\sqrt{2}\,]\!] = 1$, $[\![-\frac{1}{2}]\!] = -1$.) Show that $\lim_{x \to 3} [\![x]\!]$ does not exist.

Solution:

The graph of the greatest integer function is shown in Figure 6.
Since $[\![x]\!] = 3$ for $3 \le x < 4$,
we have

$$\lim_{x \to 3^+} [\![x]\!] = \lim_{x \to 3^+} 3 = 3$$



Greatest integer function

Figure 6

Example 10 – *Solution*

cont'd

Since $[\![x]\!] = 2$ for 2 ≤ *x* < 3, we have

$$\lim_{x \to 3^-} [\![x]\!] = \lim_{x \to 3^-} 2 = 2$$

Because these one-sided limits are not equal, $\lim_{x \to 3} [\![x]\!]$ does not exist by Theorem 1.

Calculating Limits Using the Limit Laws

The next two theorems give two additional properties of limits.

2 Theorem If $f(x) \leqslant g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leqslant \lim_{x \to a} g(x)$$

3 The Squeeze Theorem If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$

Calculating Limits Using the Limit Laws

The Squeeze Theorem, which is sometimes called the Sandwich Theorem or the Pinching Theorem, is illustrated by Figure 7.

It says that if $g(x)$ is squeezed between $f(x)$ and $h(x)$ near a, and if f and h have the same limit L at a, then g is forced to have the same limit L at a.

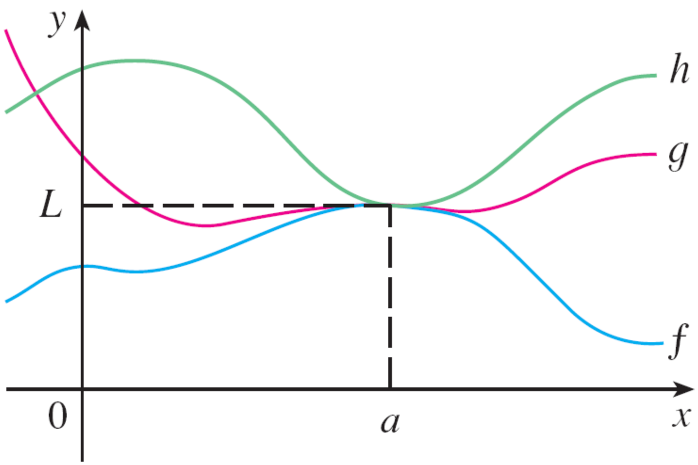


Figure 7

1.7

The Precise Definition of a Limit

The Precise Definition of a Limit

The intuitive definition of a limit is inadequate for some purposes because such phrases as "*x* is close to 2" and "*f*(*x*) gets closer and closer to *L*" are vague.

In order to be able to prove conclusively that

$$\lim_{x \to 0} \left(x^3 + \frac{\cos 5x}{10{,}000} \right) = 0.0001 \qquad \text{or} \qquad \lim_{x \to 0} \frac{\sin x}{x} = 1$$

we must make the definition of a limit precise.

The Precise Definition of a Limit

To motivate the precise definition of a limit, let's consider the function

$$f(x) = \begin{cases} 2x - 1 & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$$

Intuitively, it is clear that when x is close to 3 but $x \neq 3$, then $f(x)$ is close to 5, and so $\lim_{x \to 3} f(x) = 5$.

To obtain more detailed information about how $f(x)$ varies when x is close to 3, we ask the following question:
How close to 3 does x have to be so that $f(x)$ differs from 5 by less than 0.1?

The Precise Definition of a Limit

The distance from x to 3 is $|x - 3|$ and the distance from $f(x)$ to 5 is $|f(x) - 5|$, so our problem is to find a number δ such that

$$|f(x) - 5| < 0.1 \quad \text{if} \quad |x - 3| < \delta \qquad \text{but } x \neq 3$$

If $|x - 3| > 0$, then $x \neq 3$, so an equivalent formulation of our problem is to find a number δ such that

$$|f(x) - 5| < 0.1 \quad \text{if} \quad 0 < |x - 3| < \delta$$

The Precise Definition of a Limit

Notice that if $0 < |x - 3| < (0.1)/2 = 0.05$ then

$$|f(x) - 5| = |(2x - 1) - 5| = |2x - 6|$$

$$= 2|x - 3| < 2(0.05) = 0.1$$

that is,

$$|f(x) - 5| < 0.1 \qquad \text{if} \qquad 0 < |x - 3| < 0.05$$

Thus an answer to the problem is given by $\delta = 0.05$; that is, if x is within a distance of 0.05 from 3, then $f(x)$ will be within a distance of 0.1 from 5.

The Precise Definition of a Limit

If we change the number 0.1 in our problem to the smaller number 0.01, then by using the same method we find that $f(x)$ will differ from 5 by less than 0.01 provided that x differs from 3 by less than $(0.01)/2 = 0.005$:

$$|f(x) - 5| < 0.01 \qquad \text{if} \qquad 0 < |x - 3| < 0.005$$

Similarly,

$$|f(x) - 5| < 0.001 \qquad \text{if} \qquad 0 < |x - 3| < 0.0005$$

The numbers 0.1, 0.01 and 0.001 that we have considered are *error tolerances* that we might allow.

The Precise Definition of a Limit

For 5 to be the precise limit of $f(x)$ as x approaches 3, we must not only be able to bring the difference between $f(x)$ and 5 below each of these three numbers; we must be able to bring it below *any* positive number.

And, by the same reasoning, we can! If we write ε (the Greek letter epsilon) for an arbitrary positive number, then we find as before that

$$\boxed{1} \qquad |f(x) - 5| < \varepsilon \quad \text{if} \quad 0 < |x - 3| < \delta = \frac{\varepsilon}{2}$$

The Precise Definition of a Limit

This is a precise way of saying that $f(x)$ is close to 5 when x is close to 3 because $\boxed{1}$ says that we can make the values of $f(x)$ within an arbitrary distance ε from 5 by taking the values of x within a distance $\varepsilon/2$ from 3 (but $x \neq 3$).

Note that $\boxed{1}$ can be rewritten as follows: if

$$3 - \delta < x < 3 + \delta \qquad (x \neq 3)$$

then

$$5 - \varepsilon < f(x) < 5 + \varepsilon$$

and this is illustrated in Figure 1.

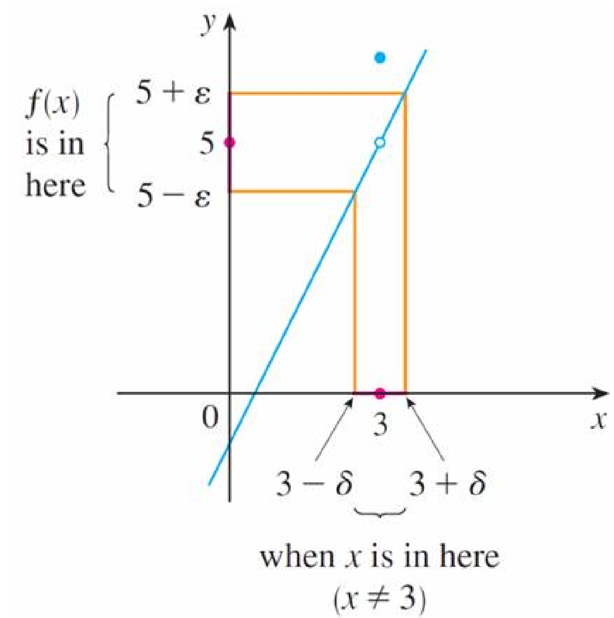


Figure 1

The Precise Definition of a Limit

By taking the values of x ($\neq 3$) to lie in the interval $(3 - \delta, 3 + \delta)$ we can make the values of $f(x)$ lie in the interval $(5 - \varepsilon, 5 + \varepsilon)$.

Using $\boxed{1}$ as a model, we give a precise definition of a limit.

$\boxed{2}$ **Definition** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

The Precise Definition of a Limit

Since $|x - a|$ is the distance from x to a and $|f(x) - L|$ is the distance from $f(x)$ to L, and since ε can be arbitrarily small, the definition of a limit can be expressed in words as follows:

$$\lim_{x \to a} f(x) = L$$

means that the distance between $f(x)$ and L can be made arbitrarily small by taking the distance from x to a sufficiently small (but not 0).

The Precise Definition of a Limit

Alternatively,

$$\lim_{x \to a} f(x) = L$$

the values of $f(x)$ can be made as close as we please to L by taking x close enough to a (but not equal to a).

The Precise Definition of a Limit

We can also reformulate Definition 2 in terms of intervals by observing that the inequality $|x - a| < \delta$ is equivalent to $-\delta < x - a < \delta$, which in turn can be written as
$a - \delta < x < a + \delta$.

Also $0 < |x - a|$ is true if and only if $x - a \neq 0$, that is, $x \neq a$.

The Precise Definition of a Limit

Similarly, the inequality $|f(x) - L| < \varepsilon$ is equivalent to the pair of inequalities $L - \varepsilon < f(x) < L + \varepsilon$. Therefore, in terms of intervals, Definition 2 can be stated as follows:

$$\lim_{x \to a} f(x) = L$$

means that for every $\varepsilon > 0$ (no matter how small ε is) we can find $\delta > 0$ such that if x lies in the open interval $(a - \delta, a + \delta)$ and $x \neq a$, then $f(x)$ lies in the open interval $(L - \varepsilon, L + \varepsilon)$.

The Precise Definition of a Limit

We interpret this statement geometrically by representing a function by an arrow diagram as in Figure 2, where *f* maps a subset of $\mathbb{R}$ onto another subset of $\mathbb{R}$.

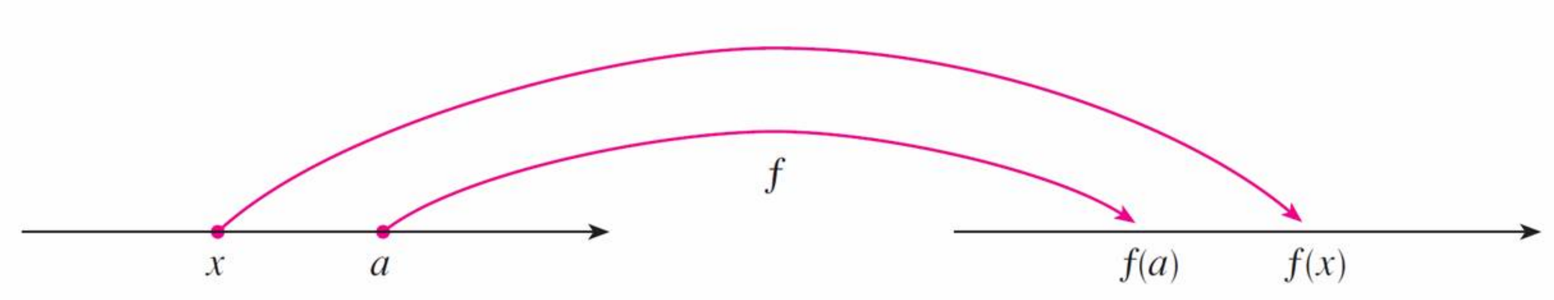


Figure 2

The Precise Definition of a Limit

The definition of limit says that if any small interval $(L - \varepsilon, L + \varepsilon)$ is given around L, then we can find an interval $(a - \delta, a + \delta)$ around a such that f maps all the points in $(a - \delta, a + \delta)$ (except possibly a) into the interval $(L - \varepsilon, L + \varepsilon)$. (See Figure 3.)

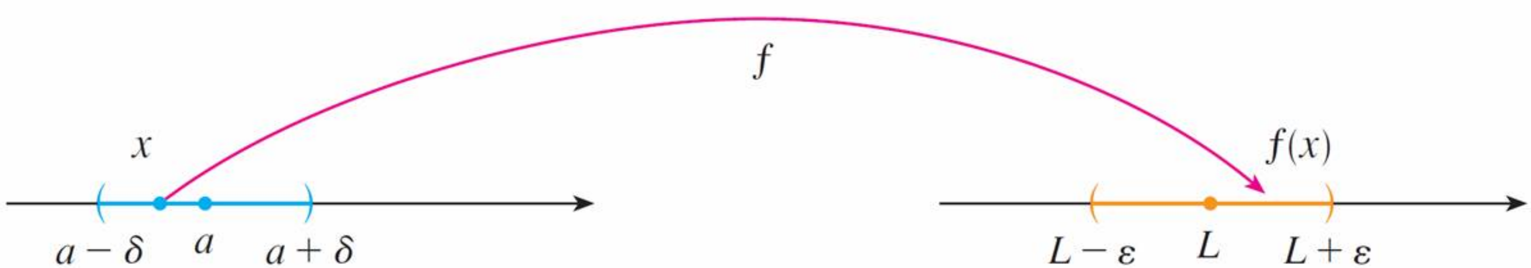


Figure 3

The Precise Definition of a Limit

Another geometric interpretation of limits can be given in terms of the graph of a function. If $\varepsilon > 0$ is given, then we draw the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ and the graph of f. (See Figure 4.)

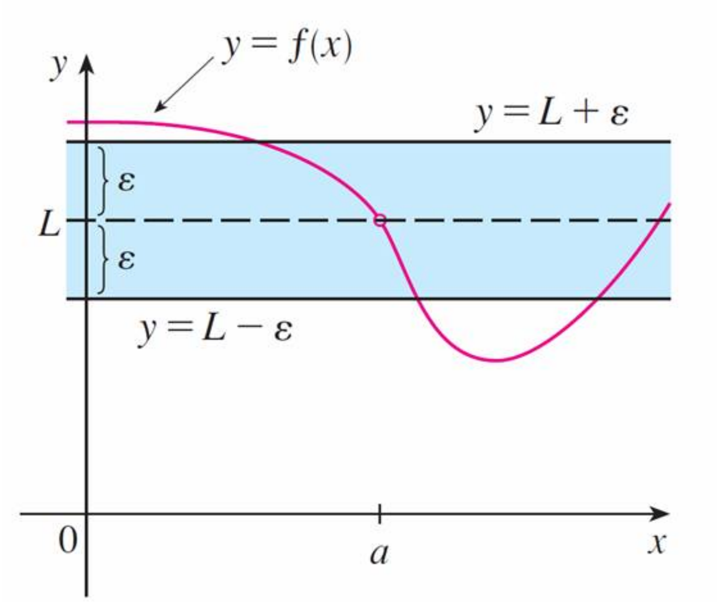


Figure 4

The Precise Definition of a Limit

If $\lim_{x \to a} f(x) = L$, then we can find a number $\delta > 0$ such that if we restrict x to lie in the interval $(a - \delta, a + \delta)$ and take $x \neq a$, then the curve $y = f(x)$ lies between the lines $y = L - \varepsilon$ and $y = L + \varepsilon$ (See Figure 5.) You can see that if such a δ has been found, then any smaller δ will also work.

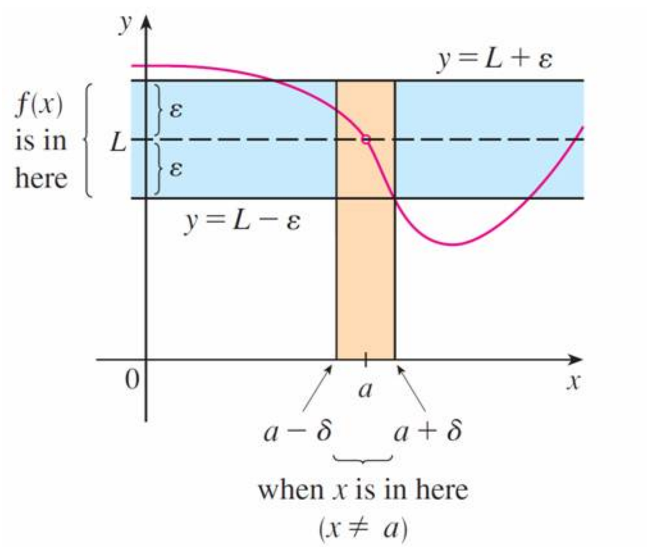


Figure 5

The Precise Definition of a Limit

It is important to realize that the process illustrated in Figures 4 and 5 must work for *every* positive number ε, no matter how small it is chosen. Figure 6 shows that if a smaller ε is chosen, then a smaller δ may be required.

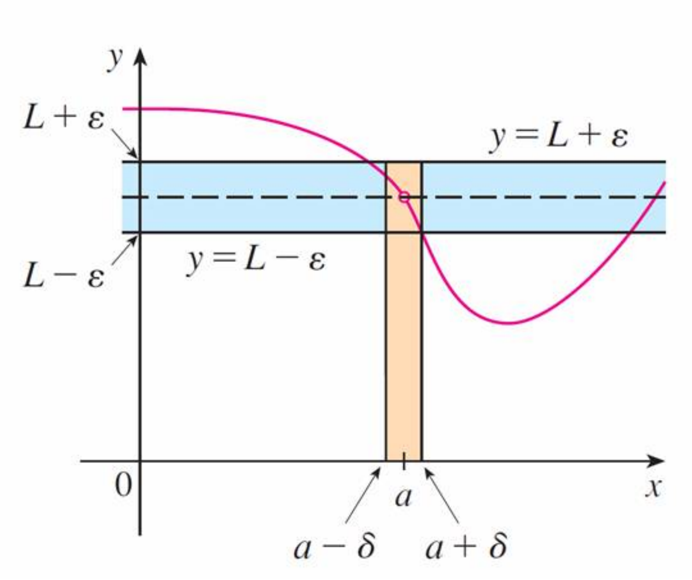


Figure 6

Example 1

Use a graph to find a number δ such that

$$\text{if } |x - 1| < \delta \quad \text{then} \quad |(x^3 - 5x + 6) - 2| < 0.2$$

In other words, find a number δ that corresponds to $\varepsilon = 0.2$ in the definition of a limit for the function $f(x) = x^3 - 5x + 6$ with $a = 1$ and $L = 2$.

Example 1 – *Solution*

cont'd

A graph of f is shown in Figure 7; we are interested in the region near the point (1, 2).

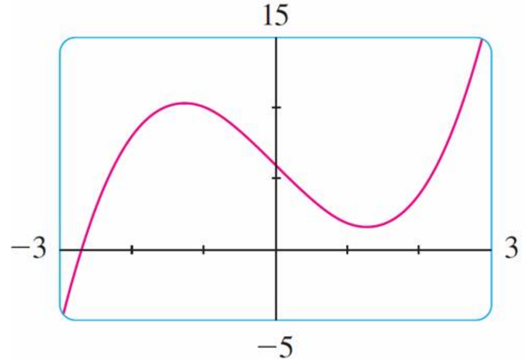


Figure 7

Notice that we can rewrite the inequality

$$|(x^3 - 5x + 6) - 2| < 0.2$$

as $$1.8 < x^3 - 5x + 6 < 2.2$$

Example 1 – *Solution*

cont'd

So we need to determine the values of for x which the curve $y = x^3 - 5x + 6$ lies between the horizontal lines $y = 1.8$ and $y = 2.2$.

Therefore we graph the curves $y = x^3 - 5x + 6$, $y = 1.8$, and $y = 2.2$ near the point (1,2) in Figure 8.

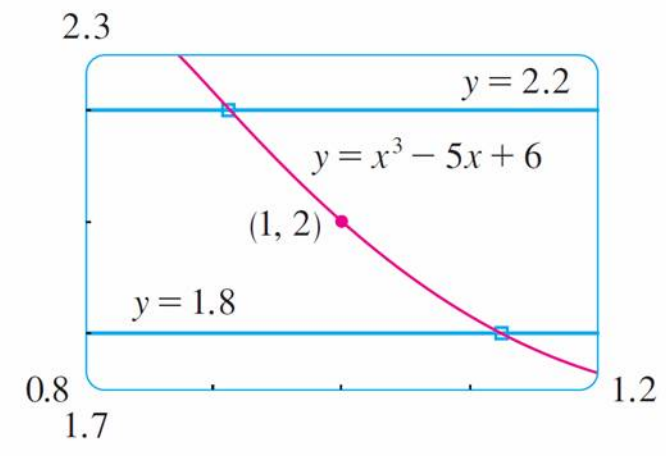


Figure 8

Example 1 – *Solution*

cont'd

Then we use the cursor to estimate that the x-coordinate of the point of intersection of the line $y = 2.2$ and the curve $y = x^3 - 5x + 6$ is about 0.911. Similarly, $y = x^3 - 5x + 6$ intersects the line $y = 1.8$ when $x \approx 1.124$. So, rounding to be safe, we can say that

$$\text{if} \quad 0.92 < x < 1.12 \quad \text{then} \quad 1.8 < x^3 - 5x + 6 < 2.2$$

This interval (0.92, 1.12) is not symmetric about $x = 1$. The distance from $x = 1$ to the left endpoint is $1 - 0.92 = 0.08$ and the distance to the right endpoint is 0.12. We can choose δ to be the smaller of these numbers, that is, $\delta = 0.08$.

Example 1 – *Solution*

cont'd

Then we can rewrite our inequalities in terms of distances as follows:

$$\text{if} \quad |x - 1| < 0.08 \qquad \text{then} \quad |(x^3 - 5x + 6) - 2| < 0.2$$

This just says that by keeping x within 0.08 of 1, we are able to keep $f(x)$ within 0.2 of 2.

Although we chose $\delta = 0.08$, any smaller positive value of δ would also have worked.

Example 2

Prove that $\lim_{x \to 3} (4x - 5) = 7$.

Solution:

1. *Preliminary analysis of the problem* (*guessing a value for* δ). Let ε be a given positive number. We want to find a number δ such that

$$\text{if} \quad 0 < |x - 3| < \delta \qquad \text{then} \qquad |(4x - 5) - 7| < \varepsilon$$

But $|(4x - 5) - 7| = |4x - 12| = |4(x - 3)| = 4|x - 3|$.

Example 2 – *Solution*

cont'd

Therefore we want δ such that

$$\text{if} \quad 0 < |x - 3| < \delta \quad \text{then} \quad 4|x - 3| < \varepsilon$$

that is, $\quad$ if $\quad 0 < |x - 3| < \delta \quad$ then $\quad |x - 3| < \dfrac{\varepsilon}{4}$

This suggests that we should choose $\delta = \varepsilon/4$.

Example 2 – *Solution*

cont'd

2. *Proof* (*showing that this works*). Given $\varepsilon > 0$, choose $\delta = \varepsilon/4$. If $0 < |x - 3| < \delta$, then

$$|(4x - 5) - 7| = |4x - 12| = 4|x - 3| < 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon$$

Thus

$$\text{if} \quad 0 < |x - 3| < \delta \qquad \text{then} \qquad |(4x - 5) - 7| < \varepsilon$$

Example 2 – *Solution*

cont'd

Therefore, by the definition of a limit,

$$\lim_{x \to 3} (4x - 5) = 7$$

This example is illustrated by Figure 9.

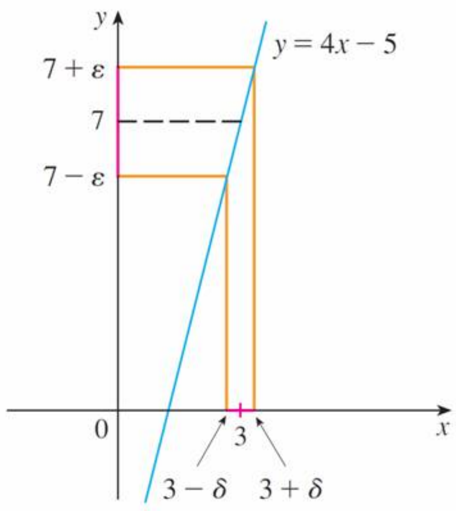


Figure 9

The Precise Definition of a Limit

The intuitive definitions of one-sided limits can be precisely reformulated as follows.

3 Definition of Left-Hand Limit

$$\lim_{x \to a^-} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad a - \delta < x < a \quad \text{then} \quad |f(x) - L| < \varepsilon$$

The Precise Definition of a Limit

4 Definition of Right-Hand Limit

$$\lim_{x \to a^+} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \qquad a < x < a + \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

Example 3

Use Definition 4 to prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.

Example 3 – *Solution*

1. *Guessing a value for δ.* Let ε be a given positive number. Here $a = 0$ and $L = 0$, so we want to find a number δ such that

$$\text{if} \quad 0 < x < \delta \quad \text{then} \quad |\sqrt{x} - 0| < \varepsilon$$

that is,

$$\text{if} \quad 0 < x < \delta \quad \text{then} \quad \sqrt{x} < \varepsilon$$

or, squaring both sides of the inequality $\sqrt{x} < \varepsilon$, we get

$$\text{if} \quad 0 < x < \delta \quad \text{then} \quad x < \varepsilon^2$$

This suggests that we should choose $\delta = \varepsilon^2$.

Example 3 – *Solution*

cont'd

2. *Showing that this δ works.* Given $\varepsilon > 0$, let $\delta = \varepsilon^2$. If $0 < x < \delta$, then

$$\sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$$

so $$|\sqrt{x} - 0| < \varepsilon$$

According to Definition 4, this shows that

$$\lim_{x \to 0^+} \sqrt{x} = 0.$$

The Precise Definition of a Limit

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ both exist, then

$$\lim_{x \to a} [f(x) + g(x)] = L + M$$

Infinite Limits

Infinite Limits

Infinite limits can also be defined in a precise way.

$\boxed{6}$ **Definition** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad f(x) > M$$

Infinite Limits

This says that the values of $f(x)$ can be made arbitrarily large (larger than any given number M) by taking x close enough to a (within a distance δ, where δ depends on M, but with $x \neq a$). A geometric illustration is shown in Figure 10.

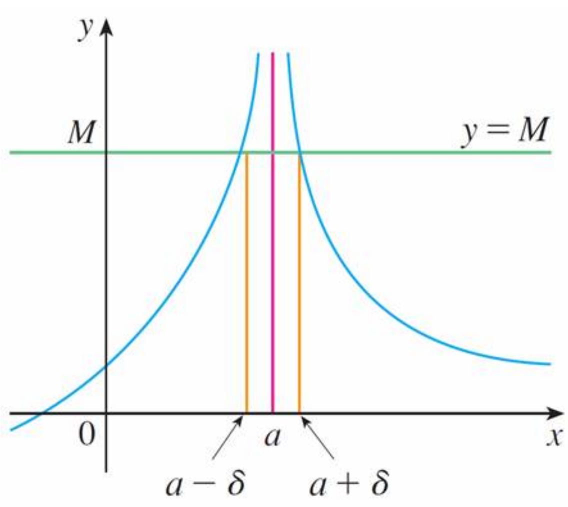


Figure 10

Infinite Limits

Given any horizontal line $y = M$, we can find a number $\delta > 0$ such that if we restrict to x lie in the interval $(a - \delta, a + \delta)$ but $x \neq a$, then the curve $y = f(x)$ lies above the line $y = M$.

You can see that if a larger M is chosen, then a smaller δ may be required.

Example 5

Use Definition 6 to prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Solution:

Let M be a given positive number. We want to find a number δ such that

if $0 < |x| < \delta$ then $1/x^2 > M$

But $\frac{1}{x^2} > M \iff x^2 < \frac{1}{M} \iff |x| < \frac{1}{\sqrt{M}}$

So if we choose $\delta = 1/\sqrt{M}$ and $0 < |x| < \delta = 1/\sqrt{M}$, then $1/x^2 > M$. This shows that as $1/x^2 \to \infty$ as $x \to 0$.

Infinite Limits

7 Definition Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that for every negative number N there is a positive number δ such that

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad f(x) < N$$

1.8 Continuity

Continuity

The limit of a function as *x* approaches *a* can often be found simply by calculating the value of the function at *a*. Functions with this property are called *continuous at a*.

We will see that the mathematical definition of continuity corresponds closely with the meaning of the word *continuity* in everyday language. (A continuous process is one that takes place gradually, without interruption or abrupt change.)

1 Definition A function f is **continuous at a number *a*** if

$$\lim_{x \to a} f(x) = f(a)$$

Continuity

Notice that Definition 1 implicitly requires three things if *f* is continuous at *a*:

1. $f(a)$ is defined (that is, *a* is in the domain of *f*)

2. $\lim_{x \to a} f(x)$ exists

3. $\lim_{x \to a} f(x) = f(a)$

The definition says that *f* is continuous at *a* if $f(x)$ approaches $f(a)$ as *x* approaches *a*. Thus a continuous function *f* has the property that a small change in *x* produces only a small change in $f(x)$.

Continuity

In fact, the change in $f(x)$ can be kept as small as we please by keeping the change in x sufficiently small.

If f is defined near a (in other words, f is defined on an open interval containing a, except perhaps at a), we say that f is **discontinuous at *a*** (or f has a **discontinuity** at a) if f is not continuous at a.

Physical phenomena are usually continuous. For instance, the displacement or velocity of a vehicle varies continuously with time, as does a person’s height. But discontinuities do occur in such situations as electric currents.

Continuity

Geometrically, you can think of a function that is continuous at every number in an interval as a function whose graph has no break in it. The graph can be drawn without removing your pen from the paper.

Example 1

Figure 2 shows the graph of a function *f*. At which numbers is *f* discontinuous? Why?

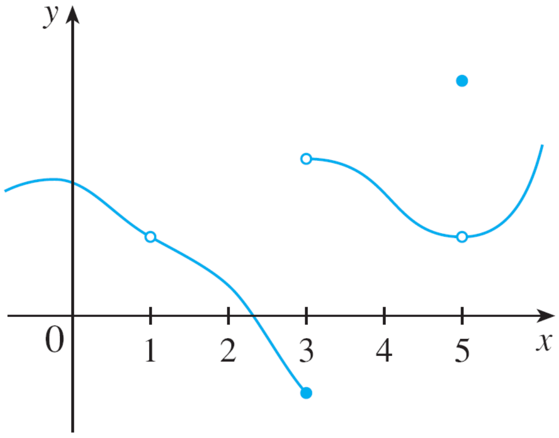


Figure 2

Solution:

It looks as if there is a discontinuity when $a = 1$ because the graph has a break there. The official reason that *f* is discontinuous at 1 is that $f(1)$ is not defined.

Example 1 – *Solution*

cont'd

The graph also has a break when $a = 3$, but the reason for the discontinuity is different. Here, $f(3)$ is defined, but $\lim_{x \to 3} f(x)$ does not exist (because the left and right limits are different). So f is discontinuous at 3.

What about $a = 5$? Here, $f(5)$ is defined and $\lim_{x \to 5} f(x)$ exists (because the left and right limits are the same).

But

$$\lim_{x \to 5} f(x) \neq f(5)$$

So f is discontinuous at 5.

Example 2

Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

(d) $f(x) = [\![x]\!]$

Solution:

(a) Notice that $f(2)$ is not defined, so f is discontinuous at 2. Later we'll see why f is continuous at all other numbers.

Example 2 – *Solution*

cont'd

(b) Here $f(0) = 1$ is defined but

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2}$$

does not exist. So f is discontinuous at 0.

(c) Here $f(2) = 1$ is defined and

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2}$$

$$= \lim_{x \to 2} \frac{(x - 2)(x + 1)}{x - 2}$$

$$= \lim_{x \to 2} (x + 1)$$

$= 3$ exists.

Example 2 – *Solution*

cont'd

But

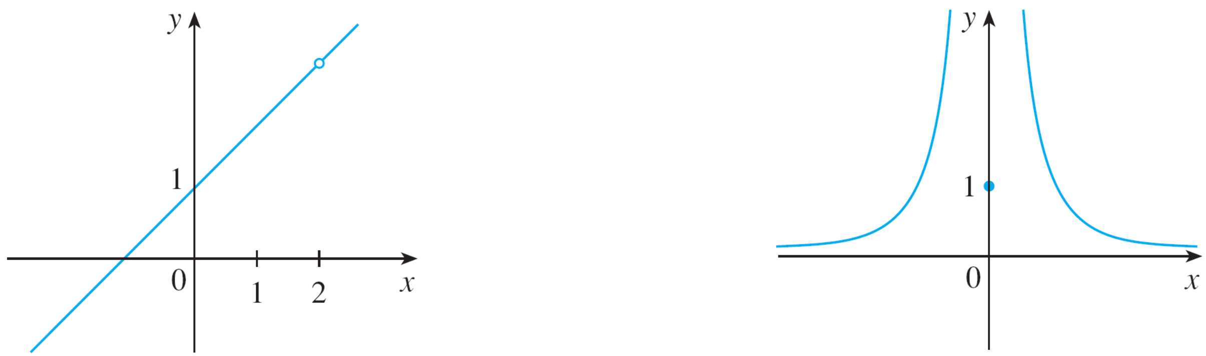
$$\lim_{x \to 2} f(x) \neq f(2)$$

so *f* is not continuous at 2.

(d) The greatest integer function $f(x) = [\![x]\!]$ has discontinuities at all of the integers because $\lim_{x \to n} [\![x]\!]$ does not exist if *n* is an integer.

Continuity

Figure 3 shows the graphs of the functions in Example 2.



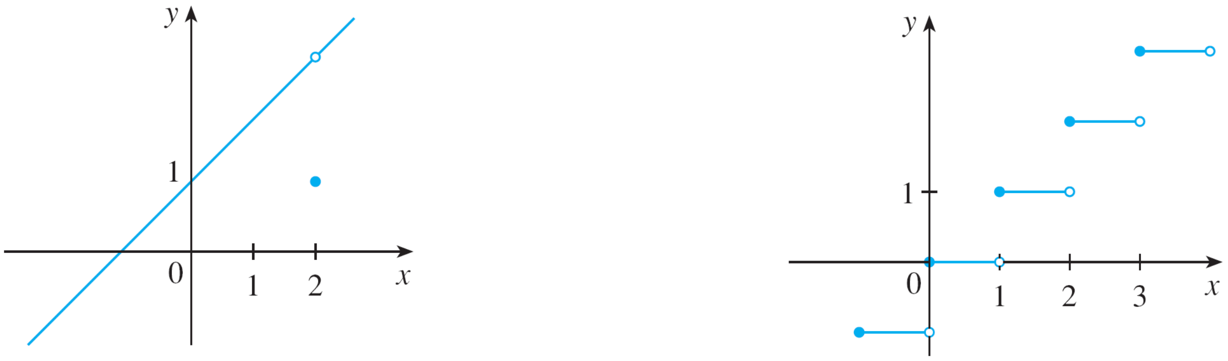
(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Graphs of the functions in Example 2

Figure 3

Continuity



(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

(d) $f(x) = [\![x]\!]$

Graphs of the functions in Example 2

Figure 3

Continuity

In each case the graph can't be drawn without lifting the pen from the paper because a hole or break or jump occurs in the graph.

The kind of discontinuity illustrated in parts (a) and (c) is called **removable** because we could remove the discontinuity by redefining *f* at just the single number 2. [The function $g(x) = x + 1$ is continuous.]

The discontinuity in part (b) is called an **infinite discontinuity**. The discontinuities in part (d) are called **jump discontinuities** because the function "jumps" from one value to another.

Continuity

2 **Definition** A function f is **continuous from the right at a number *a*** if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at *a*** if

$$\lim_{x \to a^-} f(x) = f(a)$$

3 **Definition** A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left*.)

Continuity

Instead of always using Definitions 1, 2, and 3 to verify the continuity of a function, it is often convenient to use the next theorem, which shows how to build up complicated continuous functions from simple ones.

4 **Theorem** If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$ **2.** $f - g$ **3.** cf

4. fg **5.** $\dfrac{f}{g}$ if $g(a) \neq 0$

Continuity

It follows from Theorem 4 and Definition 3 that if *f* and *g* are continuous on an interval, then so are the functions $f + g$, $f - g$, cf, fg, and (if g is never 0) f/g.

The following theorem was stated as the Direct Substitution Property.

5 Theorem

(a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

Continuity

As an illustration of Theorem 5, observe that the volume of a sphere varies continuously with its radius because the formula $V(r) = \frac{4}{3}\pi r^3$ shows that V is a polynomial function of r.

Likewise, if a ball is thrown vertically into the air with a velocity of 50 ft/s, then the height of the ball in feet t seconds later is given by the formula $h = 50t - 16t^2$.

Again this is a polynomial function, so the height is a continuous function of the elapsed time.

Continuity

It turns out that most of the familiar functions are continuous at every number in their domains.

From the appearance of the graphs of the sine and cosine functions, we would certainly guess that they are continuous.

We know from the definitions of sin θ and cos θ that the coordinates of the point P in Figure 5 are (cos θ, sin θ). As $\theta \to 0$, we see that P approaches the point (1, 0) and so cos $\theta \to 1$ and sin $\theta \to 0$.

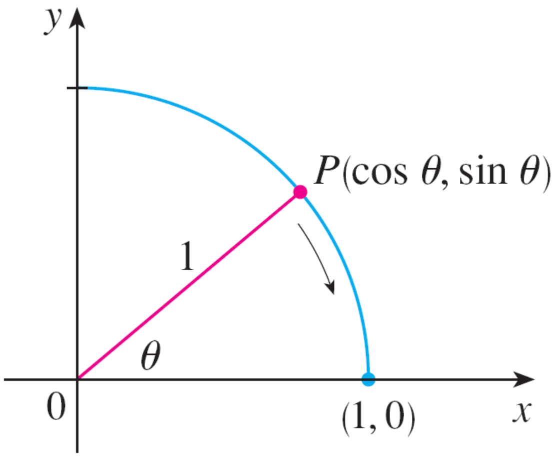


Figure 5

Continuity

Thus

$$\boxed{6} \qquad \lim_{\theta \to 0} \cos \theta = 1 \qquad \lim_{\theta \to 0} \sin \theta = 0$$

Since cos 0 = 1 and sin 0 = 0, the equations in $\boxed{6}$ assert that the cosine and sine functions are continuous at 0.

The addition formulas for cosine and sine can then be used to deduce that these functions are continuous everywhere.

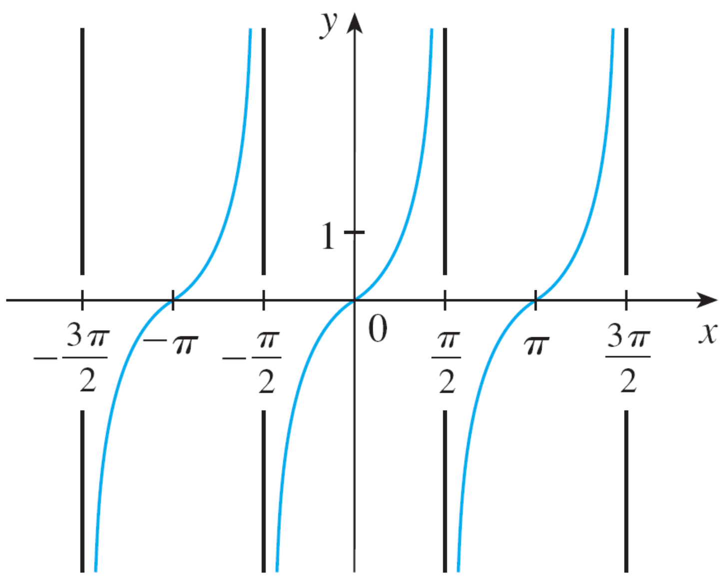
It follows from part 5 of Theorem 4 that

$$\tan x = \frac{\sin x}{\cos x}$$

is continuous except where cos x = 0.

Continuity

This happens when x is an odd integer multiple of $\pi/2$, so $y = \tan x$ has infinite discontinuities when $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2$, and so on (see Figure 6).



$y = \tan x$

Figure 6

Continuity

> $\boxed{7}$ **Theorem** The following types of functions are continuous at every number in their domains:
>
> | | |
> |---|---|
> | polynomials | rational functions |
> | root functions | trigonometric functions |

Another way of combining continuous functions *f* and *g* to get a new continuous function is to form the composite function $f \circ g$. This fact is a consequence of the following theorem.

> $\boxed{8}$ **Theorem** If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. In other words,
>
> $$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

Continuity

Intuitively, Theorem 8 is reasonable because if x is close to a, then $g(x)$ is close to b, and since f is continuous at b, if $g(x)$ is close to b, then $f(g(x))$ is close to $f(b)$.

9 Theorem If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

An important property of continuous functions is expressed by the following theorem, whose proof is found in more advanced books on calculus.

10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

Continuity

The Intermediate Value Theorem states that a continuous function takes on every intermediate value between the function values *f*(*a*) and *f*(*b*). It is illustrated by Figure 7.

Note that the value *N* can be taken on once [as in part (a)] or more than once [as in part (b)].

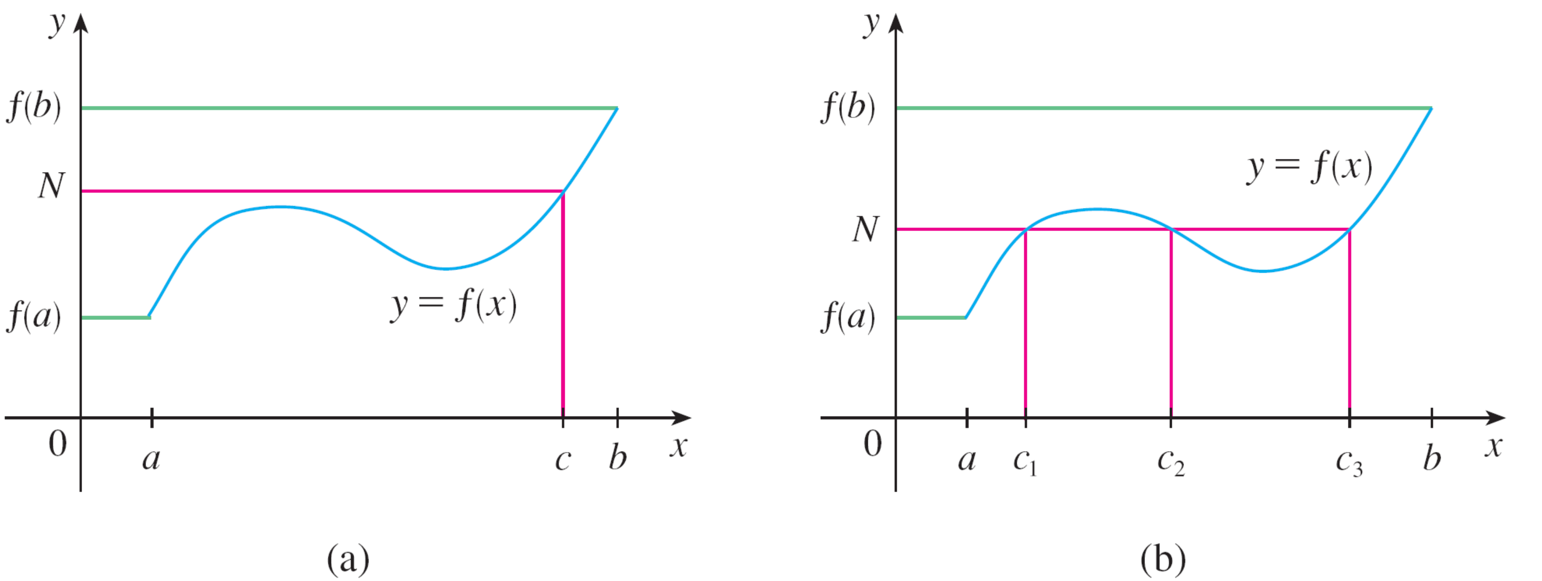


Figure 7

Continuity

If we think of a continuous function as a function whose graph has no hole or break, then it is easy to believe that the Intermediate Value Theorem is true.

In geometric terms it says that if any horizontal line $y = N$ is given between $y = f(a)$ and $y = f(b)$ as in Figure 8, then the graph of f can't jump over the line. It must intersect $y = N$ somewhere.

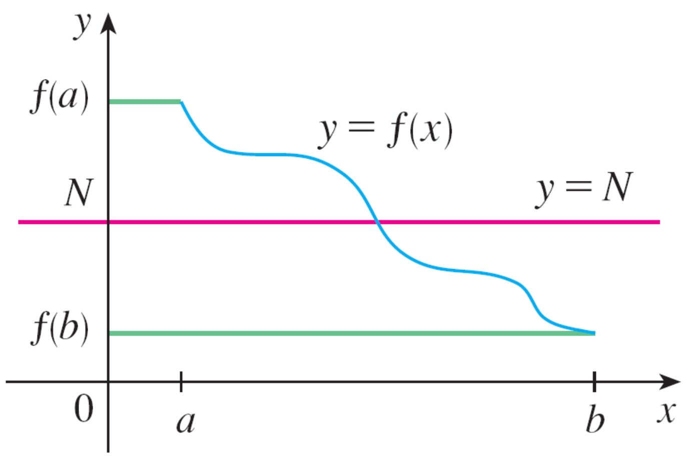


Figure 8

Continuity

It is important that the function *f* in Theorem 10 be continuous. The Intermediate Value Theorem is not true in general for discontinuous functions.

We can use a graphing calculator or computer to illustrate the use of the Intermediate Value Theorem.

Figure 9 shows the graph of *f* in the viewing rectangle [–1, 3] by [–3, 3] and you can see that the graph crosses the *x*-axis between 1 and 2.

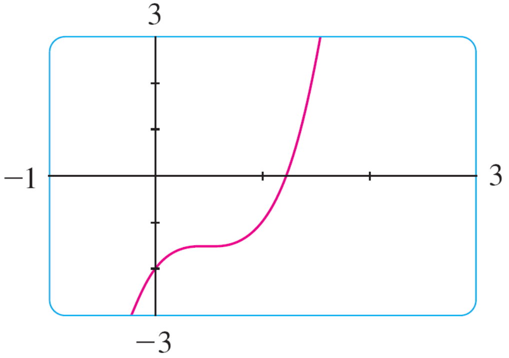


Figure 9

Figure 10 shows the result of zooming in to the viewing rectangle [1.2, 1.3] by [–0.2, 0.2].

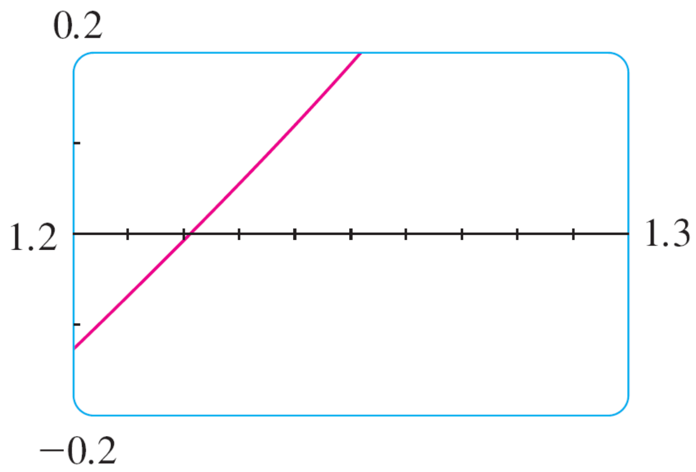


Figure 10

In fact, the Intermediate Value Theorem plays a role in the very way these graphing devices work.

Continuity

A computer calculates a finite number of points on the graph and turns on the pixels that contain these calculated points.

It assumes that the function is continuous and takes on all the intermediate values between two consecutive points.

The computer therefore connects the pixels by turning on the intermediate pixels.