VV156 Honors Calculus II
Fall 2021 — HW2 Solutions

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Exercise 2.1

i)

$$y = f(x) = \frac{1}{1+x^2} \Rightarrow$$
$$f'(x) = \frac{(1+x^2)(0) - 1(2x)}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2}$$

So the slope of the tangent line at the point $\left(-1, \frac{1}{2}\right)$ is

$$f'(-1) = \frac{2}{2^2} = \frac{1}{2}$$

and its equation is

$$y - \frac{1}{2} = \frac{1}{2}(x+1)$$

or

$$y = \frac{1}{2}x + 1$$

ii)

$$y = f(x) = \frac{x}{1+x^2} \Rightarrow$$
$$f'(x) = \frac{(1+x^2)1 - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

So the slope of the tangent line at the point $(3, 0.3)$ is

$$f'(3) = \frac{-8}{100}$$

and its equation is

$$y - 0.3 = -0.08(x-3)$$

or

$$y = -0.08x + 0.54$$

Exercise 2.2

i)

$$y = x^2 f(x) \Rightarrow y' = x^2 f'(x) + f(x)(2x)$$

ii)

$$y = \frac{f(x)}{x^2} \Rightarrow y' = \frac{x^2 f'(x) - f(x)(2x)}{(x^2)^2} = \frac{xf'(x) - 2f(x)}{x^3}$$

iii)

$$y = \frac{x^2}{f(x)} \Rightarrow y' = \frac{f(x)(2x) - x^2 f'(x)}{[f(x)]^2}$$

iv)

$$y = \frac{1 + xf(x)}{\sqrt{x}} \Rightarrow$$

$$\begin{aligned}
y' &= \frac{\sqrt{x}\,[xf'(x) + f(x)] - [1 + xf(x)]\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2} \\
&= \frac{x^{3/2}f'(x) + x^{1/2}f(x) - \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2}f(x)}{x} \cdot \frac{2x^{1/2}}{2x^{1/2}} = \frac{xf(x) + 2x^2f'(x) - 1}{2x^{3/2}}
\end{aligned}$$

Exercise 2.3

i)

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{g(x)}\right) &= \frac{g(x) \cdot \frac{d}{dx}(1) - 1 \cdot \frac{d}{dx}[g(x)]}{[g(x)]^2} \text{ [Quotient Rule]} \\
&= \frac{g(x) \cdot 0 - 1 \cdot g'(x)}{[g(x)]^2} = \frac{0 - g'(x)}{[g(x)]^2} = -\frac{g'(x)}{[g(x)]^2}
\end{aligned}$$

ii)

$$\begin{aligned}
\frac{d}{dx}\left(x^{-n}\right) &= \frac{d}{dx}\left(\frac{1}{x^n}\right) = -\frac{(x^n)'}{(x^n)^2} \text{ [Reciprocal Rule]} \\
&= -\frac{nx^{n-1}}{x^{2n}} = -nx^{n-1-2n} = -nx^{-n-1}
\end{aligned}$$

Exercise 2.4

i)

$$f(x) = \sqrt{x}\sin x \Rightarrow f'(x) = \sqrt{x}\cos x + \sin x\left(\frac{1}{2}x^{-1/2}\right) = \sqrt{x}\cos x + \frac{\sin x}{2\sqrt{x}}$$

$$\begin{aligned}
f''(x) &= (\sqrt{x}\cos x)' + \left(\frac{\sin x}{2\sqrt{x}}\right)' \\
&= \sqrt{x} \cdot (\cos x)' + \cos x \cdot (\sqrt{x})' + \frac{2\sqrt{x}(\sin x)' - \sin x(2\sqrt{x})'}{(2\sqrt{x})^2} \\
&= -\sqrt{x}\sin x + \frac{\cos x}{2\sqrt{x}} + \frac{2\sqrt{x}\cos x - \sin x \cdot \frac{1}{\sqrt{x}}}{4x} \\
&= -\sqrt{x}\sin x + \frac{\cos x}{\sqrt{x}} - \frac{\sin x}{4x\sqrt{x}}
\end{aligned}$$

ii)

$$f(x) = \sin x + \frac{1}{2}\cot x \Rightarrow f'(x) = \cos x - \frac{1}{2}\csc^2 x$$

$$\begin{aligned}
f''(x) &= (\cos x)' - \frac{1}{2}\left(\csc^2 x\right)' \\
&= -\sin x - \frac{1}{2} \cdot 2\csc x \cdot (-\cot x \cdot \csc x) \\
&= -\sin x + \csc^2 x \cdot \cot x
\end{aligned}$$

iii)

$$y = 2\sec x - \csc x \quad \Rightarrow \quad y' = 2(\sec x \tan x) - (-\csc x \cot x) = 2\sec x \tan x + \csc x \cot x$$

$$\begin{aligned} f''(x) &= 2(\sec x \cdot \tan x)' + (\csc x \cdot \cos x)' \\ &= 2\left[\sec x \cdot (\tan x)' + \tan x \cdot (\sec x)'\right] + \csc x \cdot (\cot x)' + \cot x \cdot (\sec x)' \\ &= 2\sec x \cdot \sec^2 x + 2\tan x \cdot \sec x \cdot \tan x + \csc x \cdot \left(-\csc^2 x\right) + \cot x(-\csc x \cdot \cot x) \\ &= 2\sec^3 x + 2\sec x \cdot \tan^2 x - \csc^3 x - \csc x \cdot \cot^2 x \end{aligned}$$

iv)

$$y = \frac{x}{2 - \tan x} \Rightarrow y' = \frac{(2 - \tan x)(1) - x\,(-\sec^2 x)}{(2 - \tan x)^2} = \frac{2 - \tan x + x\sec^2 x}{(2 - \tan x)^2}$$

$$f''(x) = \frac{4\cos x - 2\sin x + 4x\sin x + 2x\cos x}{(2\cos x - \sin x)^3}$$

v)

$$\begin{aligned} f(x) &= \frac{\sec x}{1 + \sec x} \Rightarrow \\ f'(x) &= \frac{(1 + \sec x)(\sec x \tan x) - (\sec x)(\sec x \tan x)}{(1 + \sec x)^2} \\ &= \frac{(\sec x \tan x)[(1 + \sec x) - \sec x]}{(1 + \sec x)^2} = \frac{\sec x \tan x}{(1 + \sec x)^2} \end{aligned}$$

$$\begin{aligned} f''(x) &= \frac{(\cos x + 1)^2 \cdot (\sin x)' - \sin x \cdot ((\cos x + 1)^2)'}{(\cos x + 1)^4} \\ &= \frac{\cos x \cdot (\cos x + 1)^2 - 2\sin x \cdot (\cos x + 1) \cdot (-\sin x)}{(\cos x + 1)^4} \\ &= \frac{\cos^3 x + 2\sin^2 x \cos x + \cos x + 2}{(\cos x + 1)^4} \end{aligned}$$

vi)

$$\begin{aligned} y &= \frac{x \sin x}{1 + x} \Rightarrow \\ y' &= \frac{(1 + x)(x\cos x + \sin x) - x\sin x(1)}{(1 + x)^2} \\ &= \frac{x\cos x + \sin x + x^2\cos x + x\sin x - x\sin x}{(1 + x)^2} = \frac{(x^2 + x)\cos x + \sin x}{(1 + x)^2} \end{aligned}$$

$$f''(x) = \frac{(2x + 2)\cos x - (x^3 + 2x^2 + x + 2)\sin x}{(x + 1)^3}$$

vii)

$$\begin{aligned} y &= \frac{1 - \sec x}{\tan x} \Rightarrow \\ y' &= \frac{\tan x(-\sec x \tan x) - (1 - \sec x)\,(\sec^2 x)}{(\tan x)^2} \\ &= \frac{\sec x\,(-\tan^2 x - \sec x + \sec^2 x)}{\tan^2 x} = \frac{\sec x(1 - \sec x)}{\tan^2 x} \end{aligned}$$

$$\begin{aligned} f''(x) &= -\left(\csc^2 x\right)' + (\csc x \cdot \cot x)' \\ &= -2\csc x(-\csc x \cdot \cot x) + \csc x\left(-\csc^2 x\right) + \cot x(-\csc x \cdot \cot x) \\ &= 2\csc^2 x \cot x - \csc^3 x - \csc x \cot^2 x \end{aligned}$$

viii)

$$f(x) = x^2 \sin x \tan x \Rightarrow$$

$$\begin{aligned}
f'(x) &= \left(x^2\right)' \sin x \tan x + x^2 (\sin x)' \tan x + x^2 \sin x (\tan x)' \\
&= 2x \sin x \tan x + x^2 \cos x \tan x + x^2 \sin x \sec^2 x \\
&= 2x \sin x \tan x + x^2 \sin x + x^2 \sin x \sec^2 x \\
&= x \sin x \left(2 \tan x + x + x \sec^2 x\right)
\end{aligned}$$

$$\begin{aligned}
f''(x) &= x^2 \cos(x) + x^2 \sec^2(x) \cos(x) \\
&\quad + 2x^2 \sec^2(x) \sin(x) \tan(x) + 4x \sin(x) + 4x \sec^2(x) \sin(x) + 2 \sin(x) \tan(x)
\end{aligned}$$

Exercise 2.5

i)

$$f(x) = \frac{\tan x - 1}{\sec x} \Rightarrow$$

$$f'(x) = \frac{\sec x\,(\sec^2 x) - (\tan x - 1)(\sec x \tan x)}{(\sec x)^2} = \frac{\sec x\,(\sec^2 x - \tan^2 x + \tan x)}{\sec^2 x} = \frac{1 + \tan x}{\sec x}$$

ii)

$$f(x) = \frac{\tan x - 1}{\sec x} = \frac{\frac{\sin x}{\cos x} - 1}{\frac{1}{\cos x}} = \frac{\frac{\sin x - \cos x}{\cos x}}{\frac{1}{\cos x}} = \sin x - \cos x$$

$$\Rightarrow \quad f'(x) = \cos x - (-\sin x) = \cos x + \sin x$$

iii) From part (a),

$$f'(x) = \frac{1 + \tan x}{\sec x} = \frac{1}{\sec x} + \frac{\tan x}{\sec x} = \cos x + \sin x$$

, which is the expression for $f'(x)$ in part (b).

Exercise 2.6

i)

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin 3x}{x} &= \lim_{x \to 0} \frac{3 \sin 3x}{3x} && \text{[multiply numerator and denominator by 3]} \\
&= 3 \lim_{3x \to 0} \frac{\sin 3x}{3x} && [\text{ as } x \to 0, 3x \to 0] \\
&= 3 \lim_{\theta \to 0} \frac{\sin \theta}{\theta} && [\text{ let } \theta = 3x] \\
&= 3(1) && [\text{ Equation 2}] \\
&= 3
\end{aligned}$$

ii)

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin 4x}{\sin 6x} &= \lim_{x \to 0} \left(\frac{\sin 4x}{x} \cdot \frac{x}{\sin 6x}\right) \\
&= \lim_{x \to 0} \frac{4 \sin 4x}{4x} \cdot \lim_{x \to 0} \frac{6x}{6 \sin 6x} \\
&= 4 \lim_{x \to 0} \frac{\sin 4x}{4x} \cdot \frac{1}{6} \lim_{x \to 0} \frac{6x}{\sin 6x} \\
&= 4(1) \cdot \frac{1}{6}(1) \\
&= \frac{2}{3}
\end{aligned}$$

iii)

$$\lim_{t\to 0}\frac{\tan 6t}{\sin 2t}=\lim_{t\to 0}\left(\frac{\sin 6t}{t}\cdot\frac{1}{\cos 6t}\cdot\frac{t}{\sin 2t}\right)=\lim_{t\to 0}\frac{6\sin 6t}{6t}\cdot\lim_{t\to 0}\frac{1}{\cos 6t}\cdot\lim_{t\to 0}\frac{2t}{2\sin 2t}$$
$$=6\lim_{t\to 0}\frac{\sin 6t}{6t}\cdot\lim_{t\to 0}\frac{1}{\cos 6t}\cdot\frac{1}{2}\lim_{t\to 0}\frac{2t}{\sin 2t}=6(1)\cdot\frac{1}{1}\cdot\frac{1}{2}(1)=3$$

iv) Consider the left sided limit.

$$\lim_{x\to 0^-}\cot(x)$$

As the x values approach 0 from the left, the function values decrease without bound.

$-\infty$

Consider the right sided limit.

$$\lim_{x\to 0^+}\cot(x)$$

As the x values approach 0 from the right, the function values increase without bound.

∞

Since the left sided and right sided limits are not equal, the limit does not exist.

v)

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 3x}{5x^3-4x}&=\lim_{x\to 0}\left(\frac{\sin 3x}{3x}\cdot\frac{3}{5x^2-4}\right)\\&=\lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\lim_{x\to 0}\frac{3}{5x^2-4}\\&=1\cdot\left(\frac{3}{-4}\right)\\&=-\frac{3}{4}\end{aligned}$$

vi)

$$\lim_{x\to 0}\frac{\sin 3x\sin 5x}{x^2}=\lim_{x\to 0}\left(\frac{3\sin 3x}{3x}\cdot\frac{5\sin 5x}{5x}\right)=\lim_{x\to 0}\frac{3\sin 3x}{3x}\cdot\lim_{x\to 0}\frac{5\sin 5x}{5x}$$
$$=3\lim_{x\to 0}\frac{\sin 3x}{3x}\cdot 5\lim_{x\to 0}\frac{\sin 5x}{5x}=3(1)\cdot 5(1)=15$$

vii)

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta+\tan\theta}=\lim_{\theta\to 0}\frac{\frac{\sin\theta}{\theta}}{1+\frac{\sin\theta}{\theta}\cdot\frac{1}{\cos\theta}}=\frac{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}}{1+\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\lim_{\theta\to 0}\frac{1}{\cos\theta}}=\frac{1}{1+1\cdot 1}=\frac{1}{2}$$

viii)

$$\lim_{x\to 0}\frac{2x}{x+\sin x}=\lim_{x\to 0}\frac{2\frac{x}{\sin(x)}}{\frac{x}{\sin(x)}+1}=1$$

ix)

$$\lim_{x\to\pi/4}\frac{1-\tan x}{\sin x-\cos x} = \lim_{x\to\pi/4}\frac{\left(1-\frac{\sin x}{\cos x}\right)\cdot\cos x}{(\sin x-\cos x)\cdot\cos x}$$
$$= \lim_{x\to\pi/4}\frac{\cos x-\sin x}{(\sin x-\cos x)\cos x}$$
$$= \lim_{x\to\pi/4}\frac{-1}{\cos x}$$
$$= \frac{-1}{1/\sqrt{2}}$$
$$= -\sqrt{2}$$

x)

$$\lim_{x\to 1}\frac{\sin(x-1)}{x^2+x-2} = \lim_{x\to 1}\frac{\sin(x-1)}{(x+2)(x-1)} = \lim_{x\to 1}\frac{1}{x+2}\lim_{x\to 1}\frac{\sin(x-1)}{x-1} = \frac{1}{3}\cdot 1 = \frac{1}{3}$$

Exercise 2.7

i) Let

$$\theta = \frac{1}{x}$$

Then as

$$x\to\infty, \theta\to 0^+$$

and

$$\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{\theta\to 0^+}\frac{1}{\theta}\sin\theta = \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

.

ii) Since

$$-1\le\sin(1/x)\le 1$$
$$-|x|\le x\sin(1/x)\le |x|$$

We know that

$$\lim_{x\to 0}(|x|) = 0$$

and

$$\lim_{x\to 0}(-|x|) = 0$$

so by the Squeeze Theorem,

$$\lim_{x\to 0} x\sin(1/x) = 0$$

iii) Plot

Exercise 2.8

$$y = A\sin x + B\cos x \Rightarrow y' = A\cos x - B\sin x \Rightarrow y'' = -A\sin x - B\cos x$$

Substituting these expressions for y, y', and y'' into the given differential equation

$$y'' + y' - 2y = \sin x$$

gives us

$$(-A\sin x - B\cos x) + (A\cos x - B\sin x) - 2(A\sin x + B\cos x) = \sin x \Leftrightarrow$$

$$-3A\sin x - B\sin x + A\cos x - 3B\cos x = \sin x \Leftrightarrow \quad (-3A - B)\sin x + (A - 3B)\cos x = 1\sin x$$

, so we must have $-3A - B = 1$ and $A - 3B = 0$ (since 0 is the coefficient of $\cos x$ on the right side). Solving for A and B, we add the first equation to three times the second to get

$$B = -\frac{1}{10}$$

and

$$A = -\frac{3}{10}$$

.

Exercise 2.9 Since $|f(x)| \leq x^2$, we must have $|f(0)| \leq 0^2 = 0$, but since it is definitely non-negative, it must be 0 .

$$f(0) = 0$$

Now

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h}$$

Thus

$$\begin{aligned} |f'(0)| &= \lim_{h\to 0} \left| \frac{f(h)}{h} \right| = \lim_{h\to 0} \frac{|f(h)|}{h} \\ &\leq \lim_{h\to 0} \frac{h^2}{h} = \lim_{h\to 0} h = 0 \end{aligned}$$

Hence $|f'(0)| \leq 0 \Rightarrow |f'(0)| = 0$ Hence the derivative at $x = 0$ exists and is 0 .