VV156 RC1
Functions

Yucheng Huang

University of Michigan
Shanghai Jiao Tong University
Joint Institute

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What will we learn in VV156/255/256

VV156

- Limits
- Derivatives and Integrals (VE203/VE401/VE215/VM211/VM235/VM250)
- Series
- Polar Coordinates

VV255

- Vectors
- Partial Derivatives
- Multiple Integrals
- Vector Calculus(VE230/VP260)

What will we learn in VV156/255/256

VV256

- Differential Equations (VE215/VE311/VE320/VV556/VV557)
- Linear Algebra (VV214/VV417)
- Fourier Transform and Laplace Transform (VE216)

Useful Softwares (3M)

(a) Matlab

(b) Mathematica

(c) Maple

Latex

https://liam.page/2014/09/08/latex-introduction/

Functions

Four ways to represent a function

- Verbally (by a description in words)
- Numerically (by a table of values)
- Visually (by a graph)
- Algebraically (by an explicit formula)

Domain and range

- Domain: D
- Range: E

Symmetry

- Even function: $f(-x) = f(x)$
- Odd function: $f(-x) = -f(x)$

Functions

A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

It is called **decreasing** on I if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

Exercise 1

Find the domain of these functions

(1) $y = \sin\sqrt{x}$
(2) $y = \tan(x+1)$
(3) $y = \arcsin(x-3)$

Exercise 1

Solution

(1) Since x is in the even root formula, it cannot take a negative value, $x \geqslant 0, D = \{x \mid x \geqslant 0\}$

(2) Since $x + 1 \neq k\pi + \pi/2(k = 0, \pm 1, \pm 2, \cdots)$,

$D = \{x \mid x \neq k\pi + \pi/2 - 1, k = 0, \pm 1, \pm 2, \cdots\}$

(3) $-1 \leqslant x - 3 \leqslant 1$, and $D = \{x \mid 2 \leqslant x \leqslant 4\}$

Exercise 1

How to solve these questions

- The denominator in the fractional function cannot be zero
- The quantity in the even root formula cannot take a negative value, that is, it should be greater than or equal to zero
- The antilogarithm of the logarithm cannot be negative and zero, that is, it must take a positive value
- The domain of the function $y = \arcsin x, y = \arccos x$ is $-1 \leqslant x \leqslant 1$
- $y = \tan x$, $x \neq k\pi + \pi/2, y = \cot x$, $x \neq k\pi$, k is integer

Exercise 2

Find the domain of these functions

(1) $h(x) = \frac{1}{\sqrt[4]{x^2-5x}}$

(2) $f(u) = \frac{u+1}{1+\frac{1}{u+1}}$

(3) $F(p) = \sqrt{2 - \sqrt{p}}$

Exercise 2

Solution

(1) $h(x) = 1/\sqrt[4]{x^2 - 5x}$ is defined when
$x^2 - 5x > 0 \Leftrightarrow x(x-5) > 0.$
Note that $x^2 - 5x \neq 0$ since that would result in division by zero.
The expression $x(x-5)$ is positive if $x < 0$ or $x > 5$. Thus, the domain is $(-\infty, 0) \cup (5, \infty)$.
(2) $f(u) = \frac{u+1}{1+\frac{1}{u+1}}$ is defined when $u + 1 \neq 0[u \neq -1]$ and
$1 + \frac{1}{u+1} \neq 0.$
Since $1 + \frac{1}{u+1} = 0 \Rightarrow \frac{1}{u+1} = -1 \Rightarrow 1 = -u - 1 \Rightarrow u = -2$, the domain is
$\{u \mid u \neq -2, u \neq -1\} = (-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$

Exercise 2

Solution

(3) $F(p) = \sqrt{2-\sqrt{p}}$ is defined when $p \geq 0$ and $2-\sqrt{p} \geq 0$. Since $2-\sqrt{p} \geq 0 \Rightarrow 2 \geq \sqrt{p} \Rightarrow \sqrt{p} \leq 2 \Rightarrow 0 \leq p \leq 4$, the domain is $[0, 4]$.

Exercise 3

Prove or Disprove

- If f and g are both even functions, is $f + g$ even? If f and g are both odd functions, is $f + g$ odd? What if f is even and g is odd? Justify your answers.
- If f and g are both even functions, is the product fg even? If f and g are both odd functions, is fg odd? What if f is even and g is odd? Justify your answers.

Exercise 3

Solution for 1

(i) If f and g are both even functions, then $f(-x) = f(x)$ and $g(-x) = g(x)$. Now $(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x)$, so $f+g$ is an even function.

(ii) If f and g are both odd functions, then $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Now $(f+g)(-x) = f(-x) + g(-x) = -f(x) + [-g(x)] = -[f(x) + g(x)] = -(f+g)(x)$, so $f+g$ is an odd function.

(iii) If f is an even function and g is an odd function, then $(f+g)(-x) = f(-x) + g(-x) = f(x) + [-g(x)] = f(x) - g(x)$, which is not $(f+g)(x)$ nor $-(f+g)(x)$, so $f+g$ is neither even nor odd. (Exception: if f is the zero function, then $f+g$ will be odd. If g is the zero function, then $f+g$ will be even.)

Exercise 3

Solution for 2

(i) If f and g are both even functions, then $f(-x) = f(x)$ and $g(-x) = g(x)$. Now $(fg)(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x)$, so fg is an even function.

(ii) If f and g are both odd functions, then $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Now $(fg)(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = f(x)g(x) = (fg)(x)$, so fg is an even function.

(iii) If f is an even function and g is an odd function, then $(fg)(-x) = f(-x)g(-x) = f(x)[-g(x)] = -[f(x)g(x)] = -(fg)(x)$, so fg is an odd function.

Function Transformations

Vertical and Horizontal Shifts, suppose $c > 0$

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward
$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward
$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right
$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

Function Transformations

Vertical and Horizontal Stretching and Reflecting, suppose $c > 1$

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c
$y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c
$y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c
$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c
$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis
$y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis

Combinations of Functions

Definition:
Given two functions f and g, the composite function $f \circ g$ (also called the composition of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h, then g, and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

Exercise 4

Graph the functions step by step

$(1) y = 1 - 2\sqrt{x+3}$

$(2) y = |\cos \pi x|$

Exercise 4

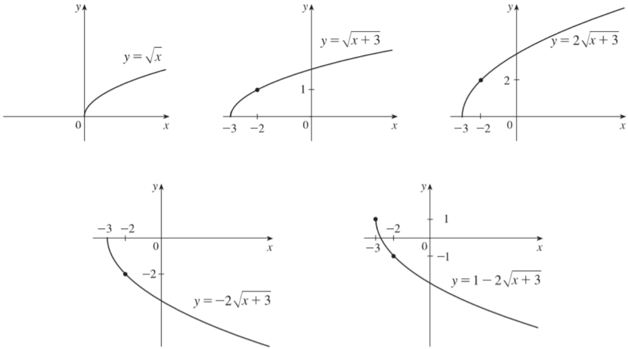

Figure: Solution for the first function

Exercise 4

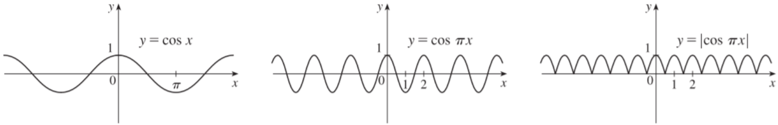

Figure: Solution for the second function

Exercise 5

Find the function (a) $f \circ g$, (b) $g \circ f$, (c) $f \circ f$, and (d) $g \circ g$ and their domains.

$f(x) = \frac{x}{1+x}, \quad g(x) = \sin 2x$

Exercise 5

Solutions

$f(x) = \frac{x}{1+x}, D = \{x \mid x \neq -1\}; \quad g(x) = \sin 2x, D = \mathbb{R}$

(a) $(f \circ g)(x) = f(g(x)) = f(\sin 2x) = \frac{\sin 2x}{1+\sin 2x}$

Domain: $1 + \sin 2x \neq 0 \quad \Rightarrow \quad \sin 2x \neq -1 \quad \Rightarrow \quad 2x \neq \frac{3\pi}{2} + 2\pi n \Rightarrow x \neq \frac{3\pi}{4} + \pi n$ [n an integer]

(b) $(g \circ f)(x) = g(f(x)) = g\left(\frac{x}{1+x}\right) = \sin\left(\frac{2x}{1+x}\right)$

Domain: $\{x \mid x \neq -1\}$

(c) $(f \circ f)(x) = f(f(x)) = f\left(\frac{x}{1+x}\right) = \frac{\frac{x}{1+x}}{1+\frac{x}{1+x}} = \frac{\left(\frac{x}{1+x}\right)\cdot(1+x)}{\left(1+\frac{x}{1+x}\right)\cdot(1+x)} = \frac{x}{1+x+x} = \frac{x}{2x+1}$ Since $f(x)$ is not defined for $x = -1$, and $f(f(x))$ is not defined for $x = -\frac{1}{2}$, the domain of $(f \circ f)(x)$ is $D = \left\{x \mid x \neq -1, -\frac{1}{2}\right\}$

(d) $(g \circ g)(x) = g(g(x)) = g(\sin 2x) = \sin(2 \sin 2x)$ Domain: $\mathbb{R}$

Exercise 6

Find $f \circ g \circ h$

$$f(x) = \tan x, \quad g(x) = \frac{x}{x-1}, \quad h(x) = \sqrt[3]{x}$$

Exercise 6

Solutions

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(\sqrt[3]{x})) = f\left(\frac{\sqrt[3]{x}}{\sqrt[3]{x}-1}\right) =$$
$$\tan\left(\frac{\sqrt[3]{x}}{\sqrt[3]{x}-1}\right)$$

Exercise 7

Composite Function

(a) If $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$, find a function f such that $f \circ g = h$. (Think about what operations you would have to perform on the formula for g to end up with the formula for h.)
(b) If $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$, find a function g such that $f \circ g = h$.

Exercise 7

Solutions

(a) By examining the variable terms in g and h, we deduce that we must square g to get the terms $4x^2$ and $4x$ in h. If we let $f(x) = x^2 + c$, then
$(f \circ g)(x) = f(g(x)) = f(2x+1) = (2x+1)^2 + c = 4x^2 + 4x + (1+c)$.
Since $h(x) = 4x^2 + 4x + 7$, we must have $1 + c = 7$. So $c = 6$ and $f(x) = x^2 + 6$
(b) We need a function g so that $f(g(x)) = 3(g(x)) + 5 = h(x)$.
But $h(x) = 3x^2 + 3x + 2 = 3\left(x^2 + x\right) + 2 = 3\left(x^2 + x - 1\right) + 5$, so we see that $g(x) = x^2 + x - 1$

Special Functions

Dirichlet Function

$$1_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Special Functions

Impulse Function

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

Step Function

$$H[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

Ramp Function

$$R(x) := \begin{cases} x, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Special Functions

Hyperbolic Function

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \cosh(x) = \frac{e^x + e^{-x}}{2}, \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

For more detail, see in textbook section 3.9 and exercise.

Inverse trigonometric function

$$\arcsin(x), \arccos(x), \arctan(x)$$

$$\operatorname{arsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\operatorname{arcosh}(x) = \ln\left(x + \sqrt{x^2-1}\right)$$

$$\operatorname{artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Geogebra and Desmos

https://www.geogebra.org/calculator
https://www.desmos.com/

Q&A

Q&A