# 11.1 Sequences

# Sequences

A **sequence** can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

The number $a_1$ is called the *first term*, $a_2$ is the *second term*, and in general $a_n$ is the *nth term*. We will deal exclusively with infinite sequences and so each term $a_n$ will have a successor $a_{n+1}$.

Notice that for every positive integer $n$ there is a corresponding number $a_n$ and so a sequence can be defined as a function whose domain is the set of positive integers.

# Sequences

But we usually write $a_n$ instead of the function notation $f(n)$ for the value of the function at the number $n$.

**Notation:** The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by

$$\{a_n\} \qquad \text{or} \qquad \{a_n\}_{n=1}^{\infty}$$

# Example 1

**Ways to define a** sequence:

1. Use the sequence brace notation
2. Define the *n*th term using a formula
3. Write the terms explicitly (with ellipses)

Note: the sequence start index doen't have to be 1.

(a) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$ $\quad a_n = \dfrac{n}{n+1}$ $\quad \left\{\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \ldots, \dfrac{n}{n+1}, \ldots\right\}$

(b) $\left\{\dfrac{(-1)^n(n+1)}{3^n}\right\}_{n=1}^{\infty}$ $\quad a_n = \dfrac{(-1)^n(n+1)}{3^n}$

$$\left\{-\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \ldots, \frac{(-1)^n(n+1)}{3^n}, \ldots\right\}$$

# Example 1

cont'd

(c) $\left\{\sqrt{n-3}\,\right\}_{n=3}^{\infty}$ $\qquad a_n = \sqrt{n-3},\ n \geqslant 3$

$$\left\{0, 1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n-3}, \ldots\right\}$$

(d) $\left\{\cos\dfrac{n\pi}{6}\right\}_{n=0}^{\infty}$ $\qquad a_n = \cos\dfrac{n\pi}{6},\ n \geqslant 0$

$$\left\{1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \ldots, \cos\frac{n\pi}{6}, \ldots\right\}$$

# Sequence visualization

A sequence such as the one in Example 1(a), $a_n = n/(n + 1)$, can be pictured either by plotting its terms on a number line, as in Figure 1, or by plotting its graph, as in Figure 2.

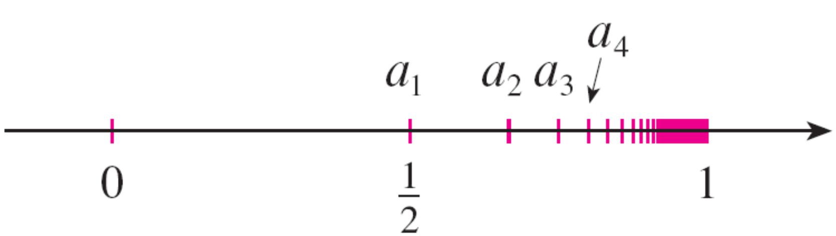



**Figure 1**

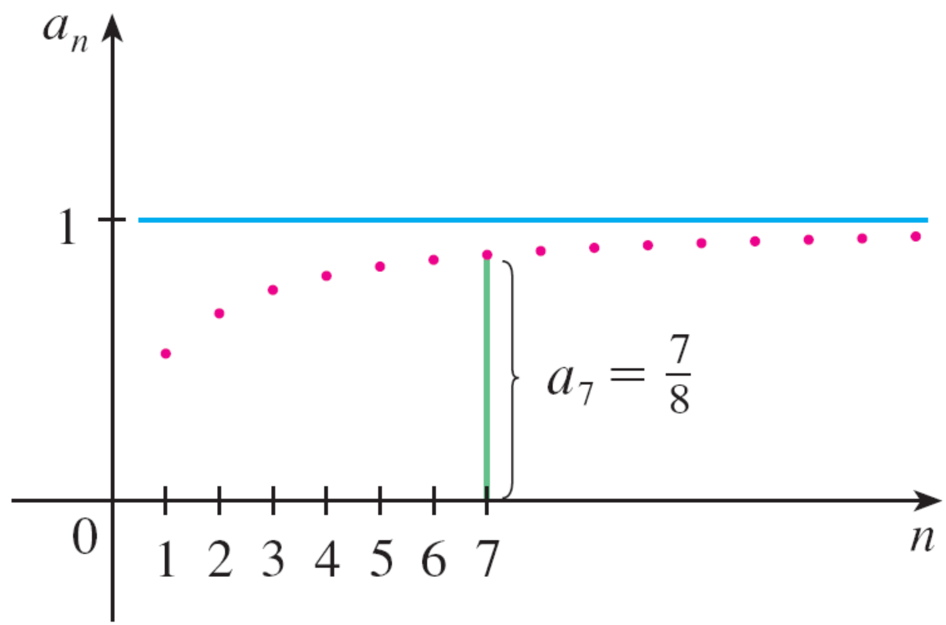



**Figure 2**

# Sequence limit, introduction

Note that, since a sequence is a function whose domain is the set of positive integers, its graph consists of isolated points with coordinates

$$(1, a_1) \quad (2, a_2) \quad (3, a_3) \quad . . . \quad (n, a_n) \quad . . .$$

From Figure 1 or Figure 2 it appears that the terms of the sequence $a_n = n/(n + 1)$ are approaching 1 as $n$ becomes large. In fact, the difference

$$1 - \frac{n}{n+1} = \frac{1}{n+1}$$

can be made as small as we like by taking $n$ sufficiently large.

# Sequence limit, introduction

We indicate this by writing

$$\lim_{n \to \infty} \frac{n}{n + 1} = 1$$

In general, the notation

$$\lim_{n \to \infty} a_n = L$$

means that the terms of the sequence $\{a_n\}$ approach $L$ as $n$ becomes large.

# Sequence limit, informal

Notice that the following definition of the limit of a sequence is very similar to the definition of a limit of a function at infinity.

$\boxed{1}$ **Definition** A sequence $\{a_n\}$ has the **limit** $L$ and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms $a_n$ as close to $L$ as we like by taking $n$ sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

# Sequence limit, informal

Figure 3 illustrates Definition 1 by showing the graphs of two sequences that have the limit *L*.

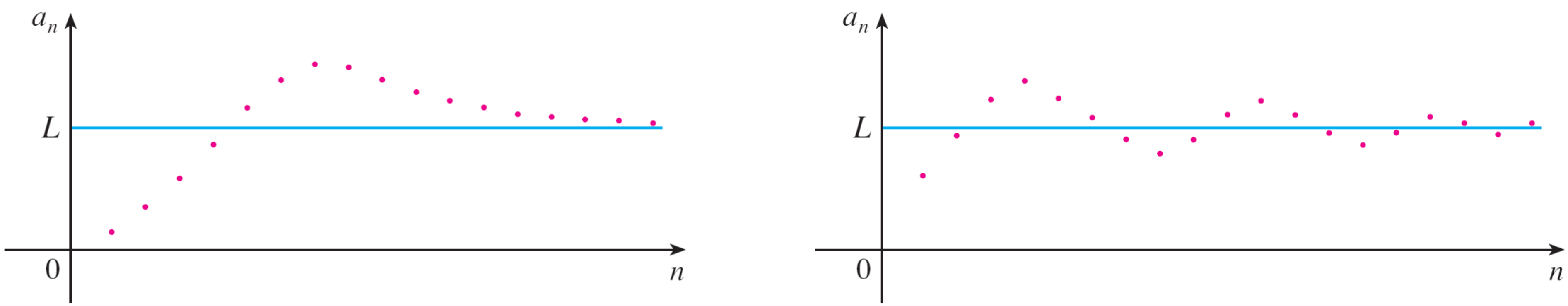



**Figure 3**
Graphs of two sequences with $\lim_{n \to \infty} a_n = L$

# Sequence limit, formal

A more precise version of Definition 1 is as follows.

**2 Definition** A sequence $\{a_n\}$ has the **limit** $L$ and we write

$$\lim_{n \to \infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer $N$ such that

$$\text{if} \qquad n > N \qquad \text{then} \qquad |a_n - L| < \varepsilon$$

# Sequence limit, formal

Definition 2 is illustrated by Figure 4, in which the terms $a_1, a_2, a_3$ , . . . are plotted on a number line.

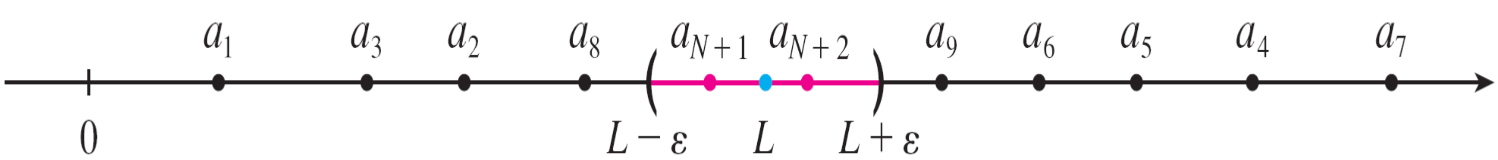



**Figure 4**

No matter how small an interval $(L - \varepsilon, L + \varepsilon)$ is chosen, there exists an $N$ such that all terms of the sequence from $a_{N+1}$ onward must lie in that interval.

# Sequence limit, formal

Another illustration of Definition 2 is given in Figure 5. The points on the graph of $\{a_n\}$ must lie between the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ if $n > N$. This picture must be valid no matter how small $\varepsilon$ is chosen, but usually a smaller $\varepsilon$ requires a larger $N$.

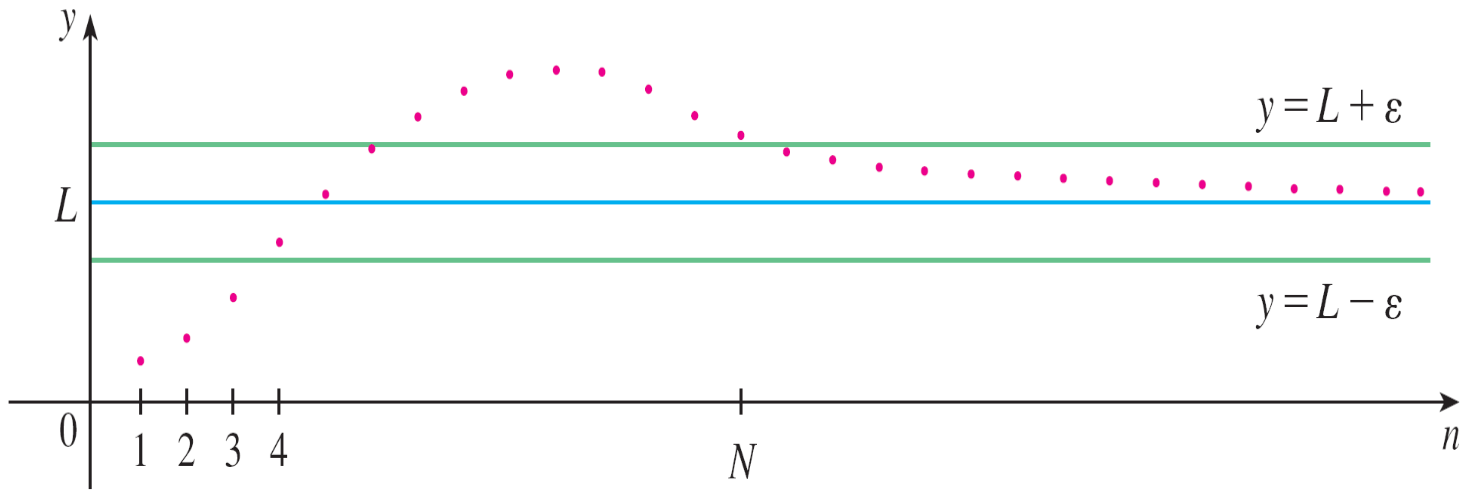



**Figure 5**

# Sequence limit from function limit

You will see that the only difference between $\lim_{n\to\infty} a_n = L$ and $\lim_{x\to\infty} f(x) = L$ is that $n$ is required to be an integer. Thus we have the following theorem, which is illustrated by Figure 6.

$\boxed{3}$ **Theorem** If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when $n$ is an integer, then $\lim_{n\to\infty} a_n = L$.

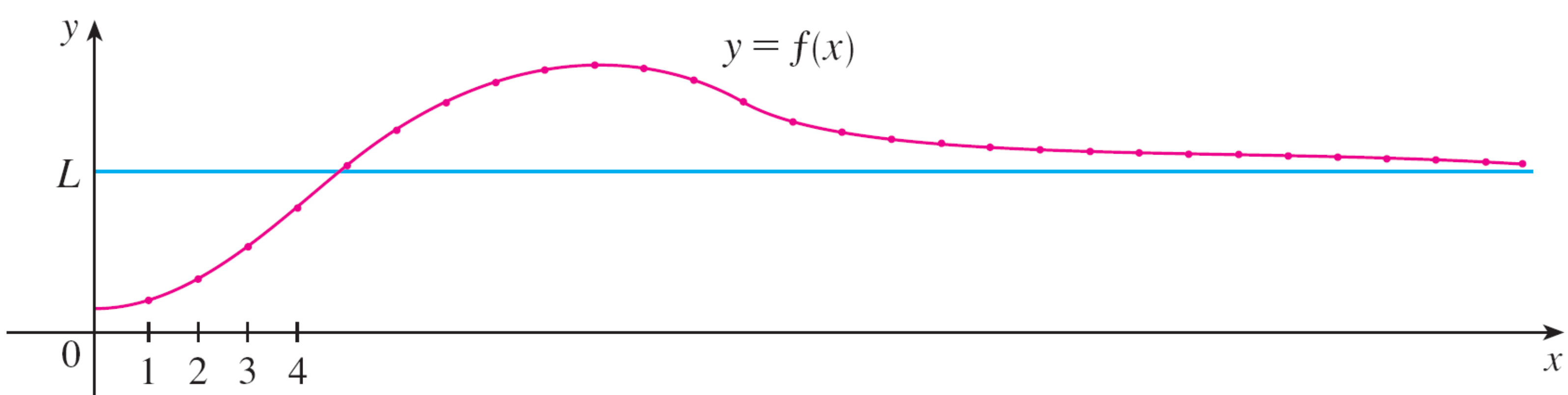



Figure 6

# Sequence limit from function limit

In particular, since we know that $\lim_{x \to \infty} (1/x^r) = 0$ when $r > 0$, we have

$$\boxed{4} \qquad \lim_{n\to\infty} \frac{1}{n^r} = 0 \qquad \text{if } r > 0$$

# Sequences diverging to infinity

If $a_n$ becomes large as $n$ becomes large, we use the notation $\lim_{n \to \infty} a_n = \infty$. Consider the definition

**5 Definition** $\lim_{n \to \infty} a_n = \infty$ means that for every positive number $M$ there is an integer $N$ such that

$$\text{if} \quad n > N \quad \text{then} \quad a_n > M$$

If $\lim_{n \to \infty} a_n = \infty$, then the sequence $\{a_n\}$ is divergent but in a special way. We say that $\{a_n\}$ diverges to $\infty$.

# Sequence limit laws

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and $c$ is a constant, then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$

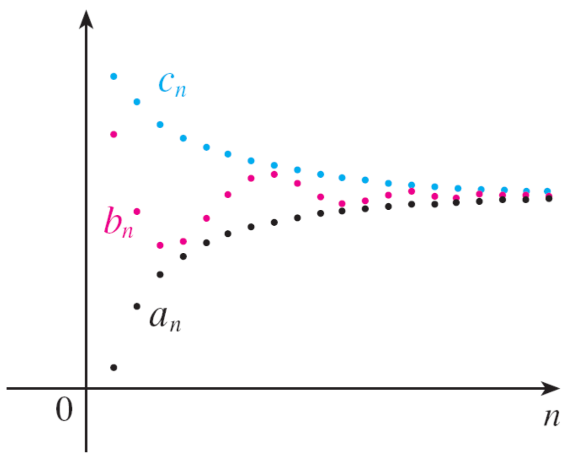
$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \text{ if } p > 0 \text{ and } a_n > 0$$

# Squeeze Theorem for sequences

The Squeeze Theorem can also be adapted for sequences as follows (see Figure 7).

## Squeeze Theorem for Sequences

If $a_n \leqslant b_n \leqslant c_n$ for $n \geqslant n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.




The sequence $\{b_n\}$ is squeezed between the sequences $\{a_n\}$ and $\{c_n\}$.

**Figure 7**

# Sequence limit theorems

If the magnitude of sequence terms goes to zero, then so do the terms themselves:

**6 Theorem** If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

If *f* is a continuous function evaluated on a sequence, the limit can be passed inside.

**7 Theorem** If $\lim_{n\to\infty} a_n = L$ and the function $f$ is continuous at $L$, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

# Example 11

For what values of $r$ is the sequence $\{r^n\}$ convergent ?

Solution:

We know that $\lim_{x \to \infty} a^x = \infty$ for $a > 1$, and $\lim_{x \to \infty} a^x = 0$ for $0 < a < 1$. Therefore, putting $a = r$ and using Theorem 3, we have

$$\lim_{n \to \infty} r^n = \begin{cases} \infty & \text{if } r > 1 \\ 0 & \text{if } 0 < r < 1 \end{cases}$$

It is obvious that

$$\lim_{n \to \infty} 1^n = 1 \quad \text{and} \quad \lim_{n \to \infty} 0^n = 0$$

If $-1 < r < 0$, then $0 < |r| < 1$, so

$$\lim_{n \to \infty} |r^n| = \lim_{n \to \infty} |r|^n = 0$$

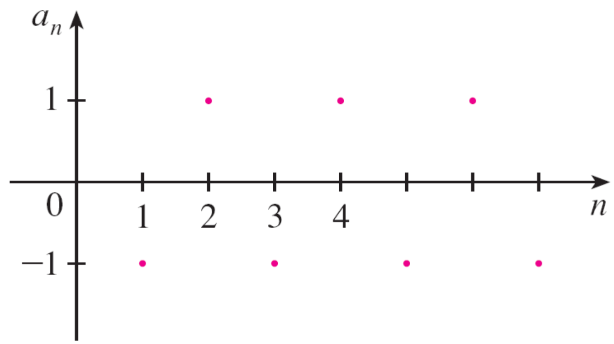
and therefore $\lim_{n \to \infty} r^n = 0$ by Theorem 6.

# Example 11 – *Solution*

cont'd

If $r \leq -1$, then $\{r^n\}$ diverges. Figure 11 shows the graphs for various values of $r$. (The case $r = -1$ is shown in Figure 8.)

**Figure 8**



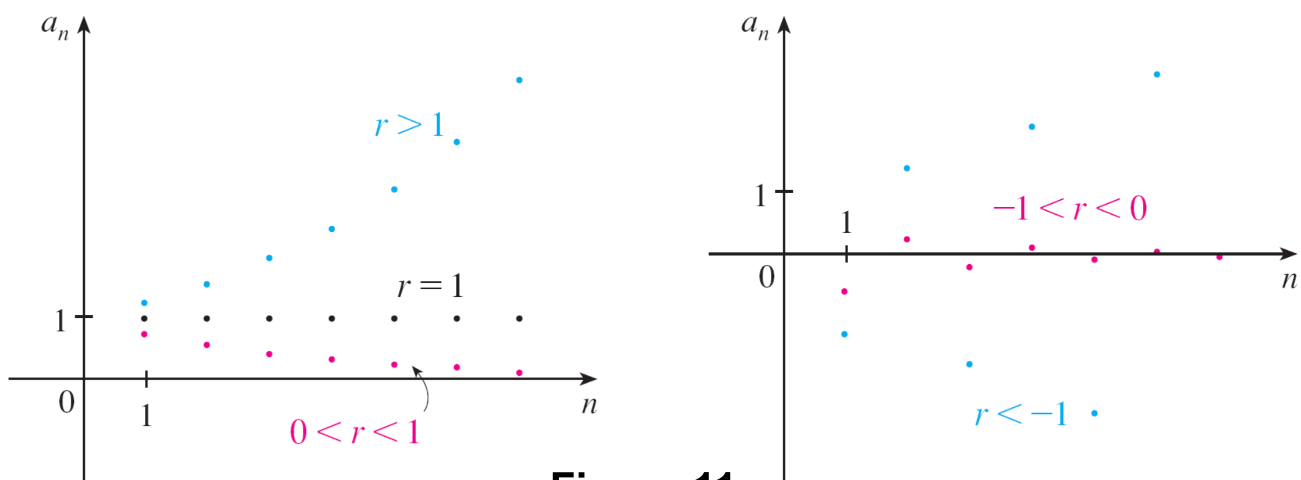




**Figure 11**
The sequence $a_n = r^n$

# Example 11: Summary

The results of Example 11 are summarized as follows.

**9** The sequence $\{r^n\}$ is convergent if $-1 < r \leqslant 1$ and divergent for all other values of $r$.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

# Monotonic sequences

A monotonic sequence has terms for which

•Every term is **greater than** the previous term,

OR

•Every term is **less than** the previous term

**Definition** A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geqslant 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geqslant 1$. A sequence is **monotonic** if it is either increasing or decreasing.

# Bounded sequences

**11 Definition** A sequence $\{a_n\}$ is **bounded above** if there is a number $M$ such that

$$a_n \leqslant M \qquad \text{for all } n \geqslant 1$$

It is **bounded below** if there is a number $m$ such that

$$m \leqslant a_n \qquad \text{for all } n \geqslant 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

For instance, the sequence $a_n = n$ is bounded below ($a_n > 0$) but not above. The sequence $a_n = n/(n + 1)$ is bounded because $0 < a_n < 1$ for all $n$.

# Bounded monotonic sequences

We know that not every bounded sequence is convergent [for instance, the sequence $a_n = (-1)^n$ satisfies $-1 \leq a_n \leq 1$ but is divergent,] and not every monotonic sequence is convergent ($a_n = n \to \infty$).

But if a sequence is both bounded *and* monotonic, then it must be convergent.

# Bounded monotonic sequences

This fact is stated without proof as Theorem 12, but intuitively you can understand why it is true by looking at Figure 12.

If $\{a_n\}$ is increasing and $a_n \leq M$ for all $n$, then the terms are forced to crowd together and approach some number $L$.

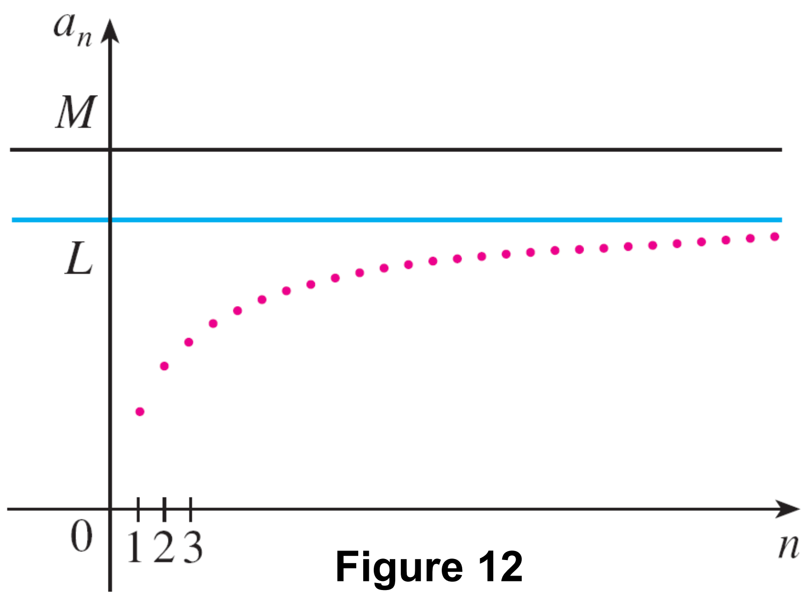



**Figure 12**

# Monotone sequence theorem

The proof of Theorem 12 is based on the **Completeness Axiom** for the set $\mathbb{R}$ of real numbers, which says that if $S$ is a nonempty set of real numbers that has an upper bound $M$ ($x \leq M$ for all $x$ in $S$), then $S$ has a **least upper bound** $b$.

(This means that $b$ is an upper bound for $S$, but if $M$ is any other upper bound, then $b \leq M$ .)

The Completeness Axiom is an expression of the fact that there is no gap or hole in the real number line.

$\boxed{12}$ **Monotonic Sequence Theorem** Every bounded, monotonic sequence is convergent.

# 11.2 Series

# Series

What do we mean when we express a number as an infinite decimal? For instance, what does it mean to write

$$\pi = 3.14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 50288\ldots$$

The convention behind our decimal notation is that any number can be written as an infinite sum. Here it means that

$$\pi = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} + \frac{6}{10^7} + \frac{5}{10^8} + \cdots$$

where the three dots $(\cdots)$ indicate that the sum continues forever, and the more terms we add, the closer we get to the actual value of $\pi$.

# Series

In general, if we try to add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$ we get an expression of the form

$$\boxed{1} \qquad a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

which is called an **infinite series** (or just a **series**) and is denoted, for short, by the symbol

$$\sum_{n=1}^{\infty} a_n \qquad \text{or} \qquad \sum a_n$$

# Series

It would be impossible to find a finite sum for the series

$$1 + 2 + 3 + 4 + 5 + \cdots + n + \cdots$$

because if we start adding the terms we get the cumulative sums 1, 3, 6, 10, 15, 21, . . . and, after the $n$th term, we get $n(n + 1)/2$, which becomes very large as $n$ increases.

However, if we start to add the terms of the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots + \frac{1}{2^n} + \cdots$$

we get $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \ldots, 1 - 1/2^n, \ldots$

# Series

The table shows that as we add more and more terms, these *partial sums* become closer and closer to 1.

| $n$ | Sum of first $n$ terms |
|---|---|
| 1 | 0.50000000 |
| 2 | 0.75000000 |
| 3 | 0.87500000 |
| 4 | 0.93750000 |
| 5 | 0.96875000 |
| 6 | 0.98437500 |
| 7 | 0.99218750 |
| 10 | 0.99902344 |
| 15 | 0.99996948 |
| 20 | 0.99999905 |
| 25 | 0.99999997 |

# Series

In fact, by adding sufficiently many terms of the series we can make the partial sums as close as we like to 1.

So it seems reasonable to say that the sum of this infinite series is 1 and to write

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 1$$

We use a similar idea to determine whether or not a general series (1) has a sum.

# Series

We consider the **partial sums**

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$s_4 = a_1 + a_2 + a_3 + a_4$$

and, in general,

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

These partial sums form a new sequence $\{s_n\}$, which may or may not have a limit.

# Series

If $\lim_{n \to \infty} s_n = s$ exists (as a finite number), then, as in the preceding example, we call it the sum of the infinite series $\Sigma\, a_n$.

$\boxed{2}$ **Definition** Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let $s_n$ denote its $n$th partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\Sigma\, a_n$ is called **convergent** and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n = s$$

The number $s$ is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

# Series

Thus the sum of a series is the limit of the sequence of partial sums.

So when we write $\sum_{n=1}^{\infty} a_n = s$ we mean that by adding sufficiently many terms of the series we can get as close as we like to the number *s*.

Notice that

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} \sum_{i=1}^{n} a_i$$

# Example 2

An important example of an infinite series is the **geometric series**

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \qquad a \neq 0$$

Each term is obtained from the preceding one by multiplying it by the **common ratio** *r*.

If $r = 1$, then $s_n = a + a + \cdots + a = na \to \pm\infty$.

Since $\lim_{n\to\infty} s_n$ doesn't exist, the geometric series diverges in this case.

# Example 2

cont'd

If $r \neq 1$, we have

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

and

$$rs_n = \qquad ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

Subtracting these equations, we get

$$s_n - rs_n = a - ar^n$$

$$\boxed{3} \qquad s_n = \frac{a(1 - r^n)}{1 - r}$$

# Example 2

cont'd

If $-1 < r < 1$, we know that as $r^n \to 0$ as $n \to \infty$,

so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{a}{1-r}\lim_{n\to\infty} r^n = \frac{a}{1-r}$$

Thus when $|r| < 1$ the geometric series is convergent and its sum is $a/(1-r)$.

If $r \le -1$ or $r > 1$, the sequence $\{r^n\}$ is divergent and so, by Equation 3, $\lim_{n\to\infty} s_n$ does not exist.

Therefore the geometric series diverges in those cases.

# Series

We summarize the results of Example 2 as follows.

$\boxed{4}$ The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| \geqslant 1$, the geometric series is divergent.

# Example 8

Show that the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent.

Solution:

For this particular series it's convenient to consider the partial sums $s_2$, $s_4$, $s_8$, $s_{16}$, $s_{32}$, . . . and show that they become large.

$$s_2 = 1 + \tfrac{1}{2}$$

$$s_4 = 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) > 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) = 1 + \tfrac{2}{2}$$

# Example 8 – *Solution*

cont'd

$$s_8 = 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\right)$$

$$> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}\right)$$

$$= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{3}{2}$$

$$s_{16} = 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{9} + \cdots + \tfrac{1}{16}\right)$$

$$> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{16} + \cdots + \tfrac{1}{16}\right)$$

$$= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{4}{2}$$

# Example 8 – *Solution*

cont'd

Similarly, $s_{32} > 1 + \frac{5}{2}$, $s_{64} > 1 + \frac{6}{2}$, and in general

$$s_{2^n} > 1 + \frac{n}{2}$$

This shows that $s_{2^n} \to \infty$ as $n \to \infty$ and so $\{s_n\}$ is divergent.

Therefore the harmonic series diverges.

# Series

**6 Theorem** If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

The converse of Theorem 6 is not true in general.

If $\lim_{n\to\infty} a_n = 0$, we cannot conclude that $\Sigma\, a_n$ is convergent.

# Series

$\boxed{7}$ **The Test for Divergence** If $\lim_{n \to \infty} a_n$ does not exist or if $\lim_{n \to \infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

The Test for Divergence follows from Theorem 6 because, if the series is not divergent, then it is convergent, and so $\lim_{n \to \infty} a_n = 0$.

# Series

**$\boxed{8}$ Theorem** If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where $c$ is a constant), $\sum (a_n + b_n)$, and $\sum (a_n - b_n)$, and

(i) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

# 11.3 The Integral Test and Estimates of Sums

# The Integral Test and Estimates of Sums

In general, it is difficult to find the exact sum of a series. We were able to accomplish this for geometric series and the series $\Sigma\ 1/[n(n + 1)]$ because in each of those cases we could find a simple formula for the $n$th partial sum $s_n$.

But usually it isn't easy to discover such a formula.

# The Integral Test and Estimates of Sums

We begin by investigating the series whose terms are the reciprocals of the squares of the positive integers:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

There's no simple formula for the sum $s_n$ of the first $n$ terms, but the computer-generated table of values given to the right suggests that the partial sums are approaching a number near 1.64 as $n \to \infty$ and so it looks as if the series is convergent.

| $n$ | $s_n = \sum_{i=1}^{n} \frac{1}{i^2}$ |
|---|---|
| 5 | 1.4636 |
| 10 | 1.5498 |
| 50 | 1.6251 |
| 100 | 1.6350 |
| 500 | 1.6429 |
| 1000 | 1.6439 |
| 5000 | 1.6447 |

# The Integral Test and Estimates of Sums

We can confirm this impression with a geometric argument. Figure 1 shows the curve $y = 1/x^2$ and rectangles that lie below the curve.

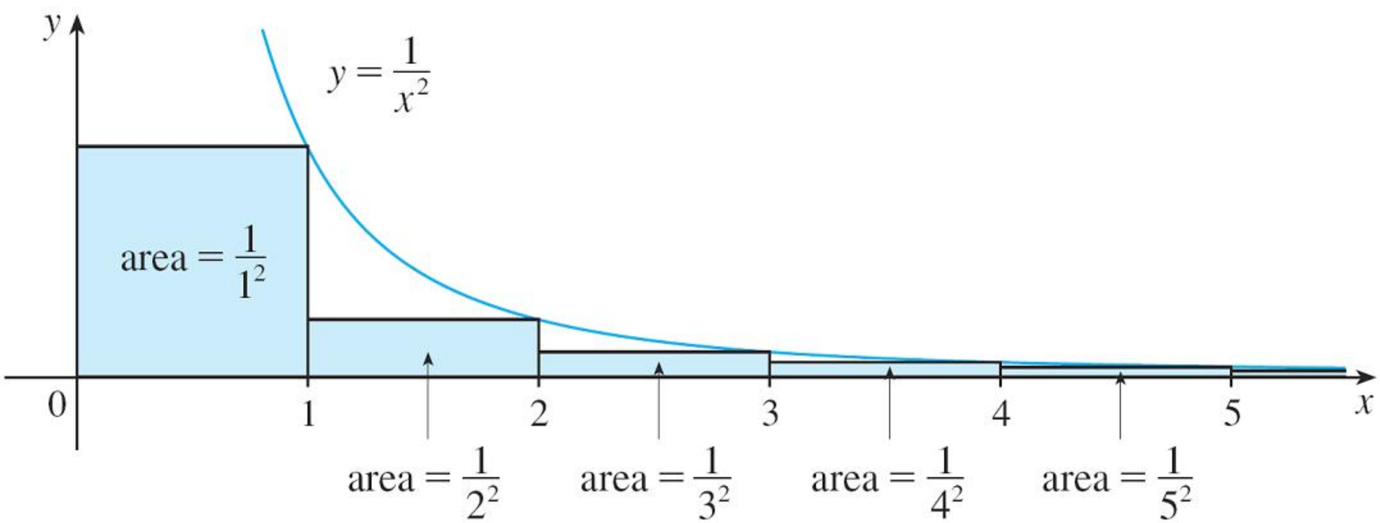



**Figure 1**

The base of each rectangle is an interval of length 1; the height is equal to the value of the function $y = 1/x^2$ at the right endpoint of the interval.

# The Integral Test and Estimates of Sums

So the sum of the areas of the rectangles is

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

If we exclude the first rectangle, the total area of the remaining rectangles is smaller than the area under the curve $y = 1/x^2$ for $x \geq 1$, which is the value of the integral $\int_1^\infty (1/x^2)\, dx$.

The improper integral is convergent and has value 1. So the picture shows that all the partial sums are less than

$$\frac{1}{1^2} + \int_1^\infty \frac{1}{x^2}\, dx = 2$$

# The Integral Test and Estimates of Sums

Thus the partial sums are bounded. We also know that the partial sums are increasing (because all the terms are positive). Therefore the partial sums converge (by the Mono tonic Sequence Theorem) and so the series is convergent. The sum of the series (the limit of the partial sums) is also less than 2:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots < 2$$

Now let's look at the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots$$

# The Integral Test and Estimates of Sums

The table of values of $s_n$ suggests that the partial sums aren't approaching a finite number, so we suspect that the given series may be divergent.

| $n$ | $s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$ |
|---|---|
| 5 | 3.2317 |
| 10 | 5.0210 |
| 50 | 12.7524 |
| 100 | 18.5896 |
| 500 | 43.2834 |
| 1000 | 61.8010 |
| 5000 | 139.9681 |

# The Integral Test and Estimates of Sums

Again we use a picture for confirmation. Figure 2 shows the curve $y = 1/\sqrt{x}$, but this time we use rectangles whose tops lie *above* the curve.

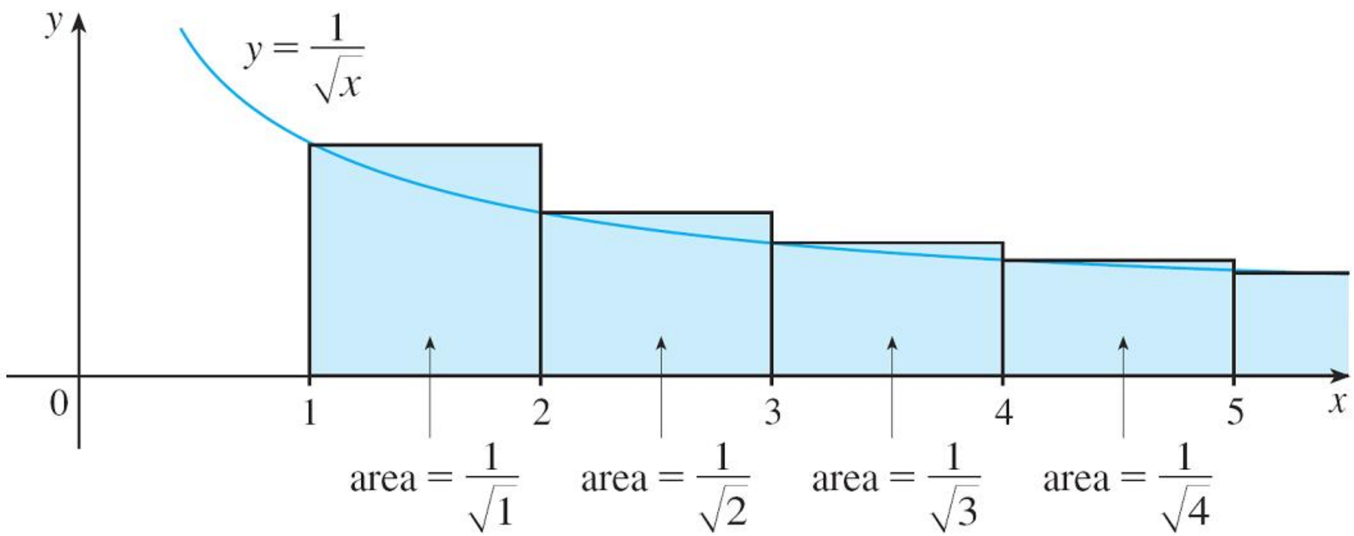



**Figure 2**

The base of each rectangle is an interval of length 1. The height is equal to the value of the function $y = 1/\sqrt{x}$ at the *left* endpoint of the interval.

# The Integral Test and Estimates of Sums

So the sum of the areas of all the rectangles is

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

This total area is greater than the area under the curve $y = 1/\sqrt{x}$, for $x \geq 1$, which is equal to the integral $\int_1^\infty \left(1/\sqrt{x}\,\right) dx$.

But we know that this improper integral is divergent. In other words, the area under the curve is infinite. So the sum of the series must be infinite, that is, the series is divergent.

# The Integral Test and Estimates of Sums

The same sort of geometric reasoning that we used for these two series can be used to prove the following test.

**The Integral Test** Suppose $f$ is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

(a) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(b) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

# The Integral Test and Estimates of Sums

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called the ***p*-series**.

$\boxed{1}$ The $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

# Example 4

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

Solution:

The function $f(x) = (\ln x)/x$ is positive and continuous for $x > 1$ because the logarithm function is continuous.

But it is not obvious whether or not $f$ is decreasing, so we compute its derivative:

$$f'(x) = \frac{x(1/x) - \ln x}{x^2}$$

$$= \frac{1 - \ln x}{x^2}$$

# Example 4 – *Solution*

cont'd

Thus $f'(x) < 0$ when $\ln x > 1$, that is, $x > e$. It follows that $f$ is decreasing when $x > e$ and so we can apply the Integral Test:

$$\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\frac{(\ln x)^2}{2}\bigg]_1^t$$

$$= \lim_{t\to\infty}\frac{(\ln t)^2}{2}$$

$$= \infty$$

Since this improper integral is divergent, the series $\Sigma\,(\ln n)/n$ is also divergent by the Integral Test.

# Estimating the Sum of a Series

# Estimating the Sum of a Series

Suppose we have been able to use the Integral Test to show that a series $\Sigma\, a_n$ is convergent and we now want to find an approximation to the sum $s$ of the series.

Of course, any partial sum $s_n$ is an approximation to $s$ because $\lim_{n \to \infty} s_n = s$. But how good is such an approximation? To find out, we need to estimate the size of the **remainder**

$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

The remainder $R_n$ is the error made when $s_n$, the sum of the first $n$ terms, is used as an approximation to the total sum.

# Estimating the Sum of a Series

We use the same notation and ideas as in the Integral Test, assuming that $f$ is decreasing on $[n, \infty)$. Comparing the areas of the rectangles with the area under $y = f(x)$ for $x > n$ in Figure 3, we see that

$$R_n = a_{n+1} + a_{n+2} + \cdots \leqslant \int_n^\infty f(x)\,dx$$

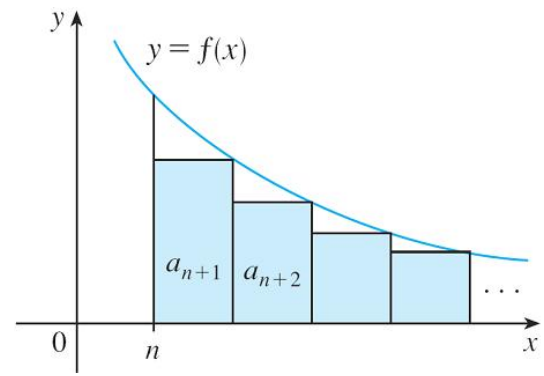


**Figure 3**

Similarly, we see from Figure 4 that

$$R_n = a_{n+1} + a_{n+2} + \cdots \geqslant \int_{n+1}^\infty f(x)\,dx$$

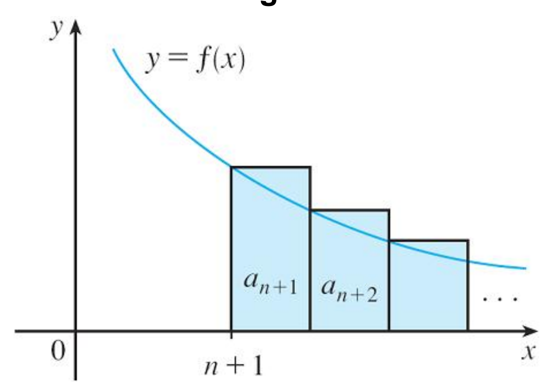


**Figure 4**

# Estimating the Sum of a Series

So we have proved the following error estimate.

**2** **Remainder Estimate for the Integral Test** Suppose $f(k) = a_k$, where $f$ is a continuous, positive, decreasing function for $x \geqslant n$ and $\Sigma\, a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \leqslant R_n \leqslant \int_{n}^{\infty} f(x)\,dx$$

# Example 5

**(a)** Approximate the sum of the series $\Sigma\ 1/n^3$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.

**(b)** How many terms are required to ensure that the sum is accurate to within 0.0005?

Solution:

In both parts (a) and (b) we need to know $\int_n^\infty f(x)\,dx$. With $f(x) = 1/x^3$, which satisfies the conditions of the Integral Test, we have

$$\int_n^\infty \frac{1}{x^3}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2x^2}\right]_n^t = \lim_{t\to\infty}\left(-\frac{1}{2t^2} + \frac{1}{2n^2}\right) = \frac{1}{2n^2}$$

# Example 5 – *Solution*

cont'd

**(a)** Approximating the sum of the series by the 10th partial sum, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx s_{10}$$

$$= \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{10^3}$$

$$\approx 1.1975$$

According to the remainder estimate in (2), we have

$$R_{10} \leqslant \int_{10}^{\infty} \frac{1}{x^3}\, dx = \frac{1}{2(10)^2} = \frac{1}{200}$$

So the size of the error is at most 0.005.

# Example 5 – *Solution*

cont'd

**(b)** Accuracy to within 0.0005 means that we have to find a value of $n$ such that $R_n \le 0.0005$. Since

$$R_n \leqslant \int_n^\infty \frac{1}{x^3}\,dx = \frac{1}{2n^2}$$

we want $$\frac{1}{2n^2} < 0.0005$$

Solving this inequality, we get

$$n^2 > \frac{1}{0.001} = 1000 \qquad \text{or} \qquad n > \sqrt{1000} \approx 31.6$$

We need 32 terms to ensure accuracy to within 0.0005.

# Estimating the Sum of a Series

If we add $s_n$ to each side of the inequalities in (2), we get

**3**
$$s_n + \int_{n+1}^{\infty} f(x)\,dx \leqslant s \leqslant s_n + \int_{n}^{\infty} f(x)\,dx$$

because $s_n + R_n = s$. The inequalities in (3) give a lower bound and an upper bound for $s$.

They provide a more accurate approximation to the sum of the series than the partial sum $s_n$ does.

# 11.4 The Comparison Tests

# The Comparison Tests

In the comparison tests the idea is to compare a given series with a series that is known to be convergent or divergent. For instance, the series

$$\boxed{1} \qquad \sum_{n=1}^{\infty} \frac{1}{2^n + 1}$$

reminds us of the series $\sum_{n=1}^{\infty} 1/2^n$, which is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$ and is therefore convergent. Because the series $\boxed{1}$ is so similar to a convergent series, we have the feeling that it too must be convergent. Indeed, it is.

# The Comparison Tests

The inequality

$$\frac{1}{2^n + 1} < \frac{1}{2^n}$$

shows that our given series $\boxed{1}$ has smaller terms than those of the geometric series and therefore all its partial sums are also smaller than 1 (the sum of the geometric series).

This means that its partial sums form a bounded increasing sequence, which is convergent. It also follows that the sum of the series is less than the sum of the geometric series:

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1} < 1$$

# The Comparison Tests

Similar reasoning can be used to prove the following test, which applies only to series whose terms are positive. The first part says that if we have a series whose terms are *smaller* than those of a known *convergent* series, then our series is also convergent.

The second part says that if we start with a series whose terms are *larger* than those of a known *divergent* series, then it too is divergent.

**The Comparison Test** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leqslant b_n$ for all $n$, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geqslant b_n$ for all $n$, then $\sum a_n$ is also divergent.

# The Comparison Tests

In using the Comparison Test we must, of course, have some known series $\Sigma\, b_n$ for the purpose of comparison. Most of the time we use one of these series:

- A *p*-series [$\Sigma\, 1/n^p$ converges if $p > 1$ and diverges if $p \leq 1$]

- A geometric series [$\Sigma\, ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$]

# Example 1

Determine whether the series $\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$ converges or diverges.

Solution:

For large *n* the dominant term in the denominator is $2n^2$, so we compare the given series with the series $\Sigma\ 5/(2n^2)$. Observe that

$$\frac{5}{2n^2 + 4n + 3} < \frac{5}{2n^2}$$

because the left side has a bigger denominator. (In the notation of the Comparison Test, $a_n$ is the left side and $b_n$ is the right side.)

# Example 1 – *Solution*

cont'd

We know that

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent because it's a constant times a $p$-series with $p = 2 > 1$.

Therefore

$$\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$$

is convergent by part (i) of the Comparison Test.

# The Comparison Tests

**Note 1:**

Although the condition $a_n \leq b_n$ or $a_n \geq b_n$ in the Comparison Test is given for all $n$, we need verify only that it holds for $n \geq N$, where $N$ is some fixed integer, because the convergence of a series is not affected by a finite number of terms.

# The Comparison Tests

**Note 2:**

The terms of the series being tested must be smaller than those of a convergent series or larger than those of a divergent series.

If the terms are larger than the terms of a convergent series or smaller than those of a divergent series, then the Comparison Test doesn’t apply.

# The Comparison Tests

Consider, for instance, the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

The inequality

$$\frac{1}{2^n - 1} > \frac{1}{2^n}$$

is useless as far as the Comparison Test is concerned because $\sum b_n = \sum \left(\frac{1}{2}\right)^n$ is convergent and $a_n > b_n$.

# The Comparison Tests

Nonetheless, we have the feeling that $\Sigma\ 1/(2^n - 1)$ ought to be convergent because it is very similar to the convergent geometric series $\sum \left(\frac{1}{2}\right)^n$.

In such cases the following test can be used.

**The Limit Comparison Test** Suppose that $\Sigma\ a_n$ and $\Sigma\ b_n$ are series with positive terms. If

$$\lim_{n \to \infty} \frac{a_n}{b_n} = c$$

where $c$ is a finite number and $c > 0$, then either both series converge or both diverge.

# Example 3

Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence.

Solution:

We use the Limit Comparison Test with

$$a_n = \frac{1}{2^n - 1} \qquad b_n = \frac{1}{2^n}$$

and obtain

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(2^n - 1)}{1/2^n}$$

$$= \lim_{n\to\infty} \frac{2^n}{2^n - 1}$$

# Example 3 – *Solution*

cont'd

$$= \lim_{n \to \infty} \frac{1}{1 - 1/2^n}$$

$$= 1 > 0$$

Since this limit exists and $\Sigma\ 1/2^n$ is a convergent geometric series, the given series converges by the Limit Comparison Test.

# Estimating Sums

# Estimating Sums

If we have used the Comparison Test to show that a series $\Sigma\, a_n$ converges by comparison with a series $\Sigma\, b_n$, then we may be able to estimate the sum $\Sigma\, a_n$ by comparing remainders.

We consider the remainder

$$R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots$$

# Estimating Sums

For the comparison series $\Sigma\, b_n$ we consider the corresponding remainder

$$T_n = t - t_n = b_{n+1} + b_{n+2} + \cdots$$

Since $a_n \leq b_n$ for all $n$, we have $R_n \leq T_n$. If $\Sigma\, b_n$ is a $p$-series, we can estimate its remainder $T_n$.

If $\Sigma\, b_n$ is a geometric series, then $T_n$ is the sum of a geometric series and we can sum it exactly.

In either case we know that $R_n$ is smaller than $T_n$.

# Example 5

Use the sum of the first 100 terms to approximate the sum of the series $\Sigma\ 1/(n^3 + 1)$. Estimate the error involved in this approximation.

Solution:

Since

$$\frac{1}{n^3 + 1} < \frac{1}{n^3}$$

the given series is convergent by the Comparison Test.

# Example 5 – *Solution*

cont'd

The remainder $T_n$ for the comparison series $\Sigma\ 1/n^3$ was estimated earlier using the Remainder Estimate for the Integral Test.

There we found that

$$T_n \leqslant \int_n^\infty \frac{1}{x^3}\,dx = \frac{1}{2n^2}$$

Therefore the remainder $R_n$ for the given series satisfies

$$R_n \leqslant T_n \leqslant \frac{1}{2n^2}$$

# Example 5 – *Solution*

cont'd

With $n$ = 100 we have

$$R_{100} \leqslant \frac{1}{2(100)^2} = 0.00005$$

Using a programmable calculator or a computer, we find that

$$\sum_{n=1}^{\infty} \frac{1}{n^3 + 1} \approx \sum_{n=1}^{100} \frac{1}{n^3 + 1} \approx 0.6864538$$

with error less than 0.00005.

# 11.5 Alternating Series

# Alternating Series

In this section we learn how to deal with series whose terms are not necessarily positive. Of particular importance are *alternating series,* whose terms alternate in sign.

An **alternating series** is a series whose terms are alternately positive and negative. Here are two examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \cdots = \sum_{n=1}^{\infty} (-1)^{n} \frac{n}{n+1}$$

# Alternating Series

We see from these examples that the *n*th term of an alternating series is of the form

$$a_n = (-1)^{n-1} b_n \qquad \text{or} \qquad a_n = (-1)^n b_n$$

where $b_n$ is a positive number. (In fact, $b_n = |a_n|$.)

The following test says that if the terms of an alternating series decrease toward 0 in absolute value, then the series converges.

# Alternating Series

**Alternating Series Test** If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

satisfies

$$\text{(i)} \quad b_{n+1} \leqslant b_n \qquad \text{for all } n$$

$$\text{(ii)} \quad \lim_{n \to \infty} b_n = 0$$

then the series is convergent.

# Example 1

The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

satisfies

(i) $b_{n+1} < b_n$ because $\dfrac{1}{n+1} < \dfrac{1}{n}$

$$\text{(ii)} \quad \lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n} = 0$$

so the series is convergent by the Alternating Series Test.

# Estimating Sums

# Estimating Sums

A partial sum $s_n$ of any convergent series can be used as an approximation to the total sum $s$, but this is not of much use unless we can estimate the accuracy of the approximation. The error involved in using $s \approx s_n$ is the remainder $R_n = s - s_n$.

The next theorem says that for series that satisfy the conditions of the Alternating Series Test, the size of the error is smaller than $b_{n+1}$, which is the absolute value of the first neglected term.

# Estimating Sums

**Alternating Series Estimation Theorem** If $s = \Sigma (-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

$$\text{(i)}\ \ b_{n+1} \leqslant b_n \qquad \text{and} \qquad \text{(ii)}\ \ \lim_{n \to \infty} b_n = 0$$

then $$|R_n| = |s - s_n| \leqslant b_{n+1}$$

# Example 4

Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

Solution:

We first observe that the series is convergent by the Alternating Series Test because

$$\text{(i)} \quad \frac{1}{(n+1)!} = \frac{1}{n!\,(n+1)} < \frac{1}{n!}$$

$$\text{(ii)} \quad 0 < \frac{1}{n!} < \frac{1}{n} \to 0 \quad \text{so} \quad \frac{1}{n!} \to 0 \ \text{as} \ n \to \infty$$

# Example 4 – *Solution*

cont'd

To get a feel for how many terms we need to use in our approximation, let's write out the first few terms of the series:

$$s = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \cdots$$

$$= 1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24} - \tfrac{1}{120} + \tfrac{1}{720} - \tfrac{1}{5040} + \cdots$$

# Example 4 – *Solution*

cont'd

Notice that

$$b_7 = \frac{1}{5040} < \frac{1}{5000} = 0.0002$$

and

$$s_6 = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}$$

$$\approx 0.368056$$

# Example 4 – *Solution*

cont'd

By the Alternating Series Estimation Theorem we know that

$$|s - s_6| \le b_7 < 0.0002$$

This error of less than 0.0002 does not affect the third decimal place, so we have $s \approx 0.368$ correct to three decimal places.

# Estimating Sums

**Note:**

The rule that the error (in using $s_n$ to approximate $s$) is smaller than the first neglected term is, in general, valid only for alternating series that satisfy the conditions of the Alternating Series Estimation Theorem. The rule does not apply to other types of series.

# 11.6 Absolute Convergence and the Ratio and Root Tests

# Absolute Convergence and the Ratio and Root Tests

Given any series $\Sigma\, a_n$, we can consider the corresponding series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

whose terms are the absolute values of the terms of the original series.

**1 Definition** A series $\Sigma\, a_n$ is called **absolutely convergent** if the series of absolute values $\Sigma\, |a_n|$ is convergent.

# Absolute Convergence and the Ratio and Root Tests

Notice that if $\Sigma\ a_n$ is a series with positive terms, then $|a_n| = a_n$ and so absolute convergence is the same as convergence in this case.

# Example 1

The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

is absolutely convergent because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

is a convergent *p*-series ($p = 2$).

# Example 2

We know that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is convergent, but it is not absolutely convergent because the corresponding series of absolute values is

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

which is the harmonic series ($p$-series with $p = 1$) and is therefore divergent.

**2 Definition** A series $\Sigma\, a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

Example 2 shows that the alternating harmonic series is conditionally convergent. Thus it is possible for a series to be convergent but not absolutely convergent. However, the next theorem shows that absolute convergence implies convergence.

**3 Theorem** If a series $\Sigma\, a_n$ is absolutely convergent, then it is convergent.

# Example 3

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$$

is convergent or divergent.

Solution:

This series has both positive and negative terms, but it is not alternating. (The first term is positive, the next three are negative, and the following three are positive: The signs change irregularly.)

# Example 3 – *Solution*

cont'd

We can apply the Comparison Test to the series of absolute values

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$$

Since $|\cos n| \leq 1$ for all $n$, we have

$$\frac{|\cos n|}{n^2} \leqslant \frac{1}{n^2}$$

We know that $\Sigma\ 1/n^2$ is convergent ($p$-series with $p = 2$) and therefore $\Sigma\ |\cos n|/n^2$ is convergent by the Comparison Test.

# Example 3 – *Solution*

cont'd

Thus the given series $\Sigma$ (cos $n$)/$n^2$ is absolutely convergent and therefore convergent by Theorem 3.

# Absolute Convergence and the Ratio and Root Tests

The following test is very useful in determining whether a given series is absolutely convergent.

**The Ratio Test**

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

**Note:**

Part (iii) of the Ratio Test says that if $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$, the test gives no information. For instance, for the convergent series $\Sigma\ 1/n^2$ we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\dfrac{1}{(n+1)^2}}{\dfrac{1}{n^2}} = \frac{n^2}{(n+1)^2} = \frac{1}{\left(1+\dfrac{1}{n}\right)^2} \to 1 \quad \text{as } n \to \infty$$

whereas for the divergent series $\Sigma\ 1/n$ we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\dfrac{1}{n+1}}{\dfrac{1}{n}} = \frac{n}{n+1} = \frac{1}{1+\dfrac{1}{n}} \to 1$$

Therefore, if $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$, the series $\Sigma\ a_n$ might converge or it might diverge. In this case the Ratio Test fails and we must use some other test.

# Example 5

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

Solution:

Since the terms $a_n = n^n/n!$ are positive, we don't need the absolute value signs.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}$$

$$= \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n}$$

# Example 5 – *Solution*

cont'd

$$= \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n \to e \qquad \text{as } n \to \infty$$

Since $e > 1$, the given series is divergent by the Ratio Test.

**Note:**
Although the Ratio Test works in Example 5, an easier method is to use the Test for Divergence. Since

$$a_n = \frac{n^n}{n!} = \frac{n \cdot n \cdot n \cdot \cdots \cdot n}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n} \geqslant n$$

it follows that $a_n$ does not approach 0 as $n \to \infty$. Therefore the given series is divergent by the Test for Divergence.

# Absolute Convergence and the Ratio and Root Tests

The following test is convenient to apply when *n*th powers occur.

**The Root Test**

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

# Absolute Convergence and the Ratio and Root Tests

If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then part (iii) of the Root Test says that the test gives no information. The series $\Sigma\, a_n$ could converge or diverge.

(If $L = 1$ in the Ratio Test, don't try the Root Test because $L$ will again be 1. And if $L = 1$ in the Root Test, don't try the Ratio Test because it will fail too.)

# Example 6

Test the convergence of the series $\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^n$.

Solution:

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n$$

$$\sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} = \frac{2+\dfrac{3}{n}}{3+\dfrac{2}{n}} \to \frac{2}{3} < 1$$

Thus the given series converges by the Root Test.

# Rearrangements

# Rearrangements

The question of whether a given convergent series is absolutely convergent or conditionally convergent has a bearing on the question of whether infinite sums behave like finite sums.

If we rearrange the order of the terms in a finite sum, then of course the value of the sum remains unchanged. But this is not always the case for an infinite series.

# Rearrangements

By a **rearrangement** of an infinite series $\Sigma\, a_n$ we mean a series obtained by simply changing the order of the terms. For instance, a rearrangement of $\Sigma\, a_n$ could start as follows:

$$a_1 + a_2 + a_5 + a_3 + a_4 + a_{15} + a_6 + a_7 + a_{20} + \ldots$$

It turns out that

if $\Sigma\, a_n$ is absolutely convergent series with sum $s$,
then any rearrangement of $\Sigma\, a_n$ has the same sum $s$.

# Rearrangements

However, any conditionally convergent series can be rearranged to give a different sum. To illustrate this fact let's consider the alternating harmonic series

$$\boxed{6} \quad 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \tfrac{1}{7} - \tfrac{1}{8} + \cdots = \ln 2$$

If we multiply this series by $\frac{1}{2}$, we get

$$\tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{6} - \tfrac{1}{8} + \cdots = \tfrac{1}{2}\ln 2$$

# Rearrangements

Inserting zeros between the terms of this series, we have

$$\boxed{7} \quad 0 + \tfrac{1}{2} + 0 - \tfrac{1}{4} + 0 + \tfrac{1}{6} + 0 - \tfrac{1}{8} + \cdots = \tfrac{1}{2}\ln 2$$

Now we add the series in Equations 6 and 7:

$$\boxed{8} \quad 1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \cdots = \tfrac{3}{2}\ln 2$$

# Rearrangements

Notice that the series in $\boxed{8}$ contains the same terms as in $\boxed{6}$, but rearranged so that one negative term occurs after each pair of positive terms. The sums of these series, however, are different. In fact, Riemann proved that

> if $\Sigma\, a_n$ is a conditionally convergent series and $r$ is any real number whatsoever, then there is a rearrangement of $\Sigma\, a_n$ that has a sum equal to $r$.

# 11.7 Strategy for Testing Series

# Strategy for Testing Series

We now have several ways of testing a series for convergence or divergence; the problem is to decide which test to use on which series. In this respect, testing series is similar to integrating functions.

Again there are no hard and fast rules about which test to apply to a given series, but you may find the following advice of some use.

It is not wise to apply a list of the tests in a specific order until one finally works. That would be a waste of time and effort. Instead, as with integration, the main strategy is to classify the series according to its *form.*

# Strategy for Testing Series

**1.** If the series is of the form $\Sigma\ 1/n^p$, it is a $p$-series, which we know to be convergent if $p > 1$ and divergent if $p \leq 1$.

**2.** If the series has the form or $\Sigma\ ar^{n-1}$ or $\Sigma\ ar^n$, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geq 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.

# Strategy for Testing Series

**3.** If the series has a form that is similar to a *p*-series or a geometric series, then one of the comparison tests should be considered. In particular, if $a_n$ is a rational function or an algebraic function of $n$ (involving roots of polynomials), then the series should be compared with a *p*-series.

The comparison tests apply only to series with positive terms, but if $\Sigma a_n$ has some negative terms, then we terms, then we can apply the Comparison Test to $\Sigma\ |a_n|$ and test for absolute convergence.

# Strategy for Testing Series

**4.** If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then the Test for Divergence should be used.

**5.** If the series is of the form $\Sigma(-1)^{n-1}b_n$ or $\Sigma(-1)^n b_n$, then the Alternating Series Test is an obvious possibility.

**6.** Series that involve factorials or other products (including a constant raised to the *n*th power) are often conveniently tested using the Ratio Test. Bear in mind that $|a_{n+1}/a_n| \to 1 \text{ as } n \to \infty$ for all *p*-series and therefore all rational or algebraic functions of *n*. Thus the Ratio Test should not be used for such series.

# Strategy for Testing Series

**7.** If $a_n$ is of the form $(b_n)^n$, then the Root Test may be useful.

**8.** If $a_n = f(n)$, where $\int_1^{\infty} f(x)\,dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

In the following examples we don’t work out all the details but simply indicate which tests should be used.

# Example 1

$$\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$$

Since as $a_n \to \frac{1}{2} \neq$ as $n \to \infty$ , we should use the Test for Divergence.

# Example 2

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^3 + 1}}{3n^3 + 4n^2 + 2}$$

Since $a_n$ is an algebraic function of, we compare the given series with a *p*-series.

The comparison series for the Limit Comparison Test is $\Sigma\, b_n$, where

$$b_n = \frac{\sqrt{n^3}}{3n^3} = \frac{n^{3/2}}{3n^3} = \frac{1}{3n^{3/2}}$$

# Example 4

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4 + 1}$$

Since the series is alternating, we use the Alternating Series Test.

# Example 5

$$\sum_{k=1}^{\infty} \frac{2^k}{k!}$$

Since the series involves $k!$, we use the Ratio Test.

# Example 6

$$\sum_{n=1}^{\infty} \frac{1}{2 + 3^n}$$

Since the series is closely related to the geometric series $\Sigma\, 1/3^n$, we use the Comparison Test.

# 11.8 Power Series

# Power Series

A **power series** is a series of the form

$$\boxed{1} \qquad \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where *x* is a variable and the $c_n$'s are constants called the **coefficients** of the series.

For each fixed *x*, the series (1) is a series of constants that we can test for convergence or divergence.

A power series may converge for some values of *x* and diverge for other values of *x*.

# Power Series

The sum of the series is a function

$$f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n + \cdots$$

whose domain is the set of all $x$ for which the series converges. Notice that $f$ resembles a polynomial. The only difference is that $f$ has infinitely many terms.

For instance, if we take $c_n = 1$ for all $n$, the power series becomes the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

which converges when $-1 < x < 1$ and diverges when $|x| \geq 1$.

# Power Series

More generally, a series of the form

$$\boxed{2} \qquad \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called a **power series in** $(x - a)$ or a **power series centered at** $\boldsymbol{a}$ or a **power series about** $\boldsymbol{a}$.

Notice that in writing out the term corresponding to $n = 0$ in Equations 1 and 2 we have adopted the convention that $(x - a)^0 = 1$ even when $x = a$.

Notice also that when $x = a$ all of the terms are 0 for $n \geq 1$ and so the power series (2) always converges when $x = a$.

# Example 1

For what values of $x$ is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

Solution:

We use the Ratio Test. If we let $a_n$, as usual, denote the $n$th term of the series, then $a_n = n!x^n$. If $x \neq 0$, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right|$$

$$= \lim_{n\to\infty}(n+1)|x|$$

$$= \infty$$

By the Ratio Test, the series diverges when $x \neq 0$. Thus the given series converges only when $x = 0$.

# Power Series

We will see that the main use of a power series is that it provides a way to represent some of the most important functions that arise in mathematics, physics, and chemistry.

In particular, the sum of the power series,

$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$, is called a **Bessel function.**

Recall that the sum of a series is equal to the limit of the sequence of partial sums. So when we define the Bessel function as the sum of a series we mean that, for every real number *x*, $J_0(x) = \lim_{n \to \infty} s_n(x)$ where $s_n(x) = \sum_{i=0}^{n} \frac{(-1)^i x^{2i}}{2^{2i}(i!)^2}$

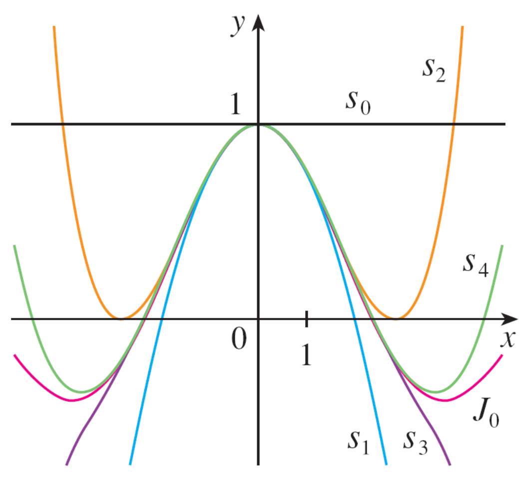
# Power Series

The first few partial sums are

$$s_0(x) = 1 \qquad s_1(x) = 1 - \frac{x^2}{4} \qquad s_2(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64}$$

$$s_3(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} \qquad s_4(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \frac{x^8}{147{,}456}$$

Figure 1 shows the graphs of these partial sums, which are polynomials. They are all approximations to the function $J_0$, but notice that the approximations become better when more terms are included.




Partial sums of the Bessel function $J_0$

**Figure 1**

# Power Series

Figure 2 shows a more complete graph of the Bessel function.

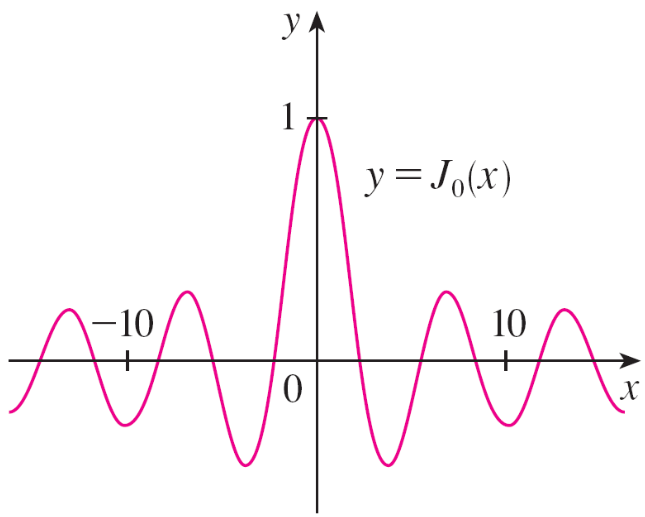



**Figure 2**

For the power series that we have looked at so far, the set of values of *x* for which the series is convergent has always turned out to be an interval.

# Power Series

The following theorem says that this is true in general.

**3 Theorem** For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all $x$.

(iii) There is a positive number $R$ such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

The number *R* in case (iii) is called the **radius of convergence** of the power series. By convention, the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii).

# Power Series

The **interval of convergence** of a power series is the interval that consists of all values of *x* for which the series converges.

In case (i) the interval consists of just a single point *a*.

In case (ii) the interval is $(-\infty, \infty)$.

In case (iii) note that the inequality $|x - a| < R$ can be rewritten as $a - R < x < a + R$.

# Power Series

When $x$ is an *endpoint* of the interval, that is, $x = a \pm R$, anything can happen—the series might converge at one or both endpoints or it might diverge at both endpoints.

Thus in case (iii) there are four possibilities for the interval of convergence:

$(a - R, a + R)$ $(a - R, a + R]$ $[a - R, a + R)$ $[a - R, a + R]$

The situation is illustrated in Figure 3.

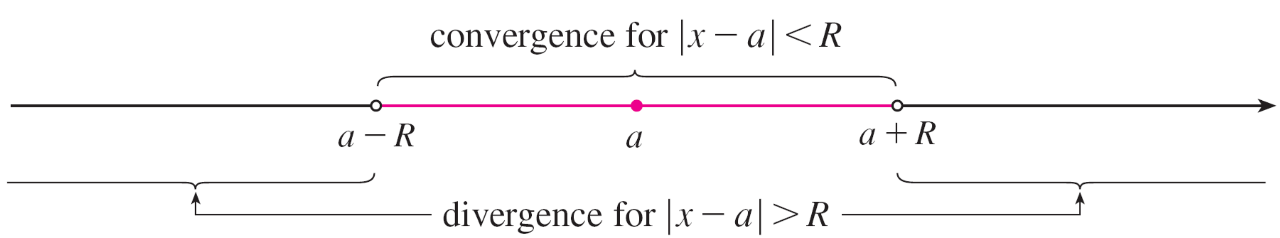



**Figure 3**

# Power Series

We summarize here the radius and interval of convergence for each of the examples already considered in this section.

| | Series | Radius of convergence | Interval of convergence |
|---|---|---|---|
| Geometric series | $\sum_{n=0}^{\infty} x^n$ | $R = 1$ | $(-1, 1)$ |
| Example 1 | $\sum_{n=0}^{\infty} n!\, x^n$ | $R = 0$ | $\{0\}$ |
| Example 2 | $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ | $R = 1$ | $[2, 4)$ |
| Example 3 | $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ | $R = \infty$ | $(-\infty, \infty)$ |

# 11.9 Representations of Functions as Power Series

# Representations of Functions as Power Series

We start with an equation that we have seen before:

1 $$\boxed{\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1}$$

We have obtained this equation by observing that the series is a geometric series with *a* = 1 and *r* = *x*.

But here our point of view is different. We now regard Equation 1 as expressing the function $f(x) = 1/(1 - x)$ as a sum of a power series.

# Example 1

Express $1/(1 + x^2)$ as the sum of a power series and find the interval of convergence.

Solution:

Replacing $x$ by $–x^2$ in Equation 1, we have

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

$$= \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$= 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

Because this is a geometric series, it converges when $|–x^2| < 1$, that is, $x^2 < 1$, or $|x| < 1$.

# Example 1 – *Solution*

cont'd

Therefore the interval of convergence is (–1, 1).

(Of course, we could have determined the radius of convergence by applying the Ratio Test, but that much work is unnecessary here.)

# Differentiation and Integration of Power Series

# Differentiation and Integration of Power Series

The sum of a power series is a function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ whose domain is the interval of convergence of the series.

We would like to be able to differentiate and integrate such functions, and the following theorem says that we can do so by differentiating or integrating each individual term in the series, just as we would for a polynomial.

This is called **term-by-term differentiation and integration**.

# Differentiation and Integration of Power Series

**2 Theorem** If the power series $\sum c_n(x - a)^n$ has radius of convergence $R > 0$, then the function $f$ defined by

$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x - a)^n$$

is differentiable (and therefore continuous) on the interval $(a - R, a + R)$ and

(i) $$f'(x) = c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x - a)^{n-1}$$

(ii) $$\int f(x)\,dx = C + c_0(x - a) + c_1\frac{(x - a)^2}{2} + c_2\frac{(x - a)^3}{3} + \cdots$$

$$= C + \sum_{n=0}^{\infty} c_n\frac{(x - a)^{n+1}}{n + 1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both $R$.

# Example 4

We have seen the Bessel function

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

is defined for all $x$.

Thus, by Theorem 2, $J_0$ is differentiable for all $x$ and its derivative is found by term-by-term differentiation as follows:

$$J_0'(x) = \sum_{n=0}^{\infty} \frac{d}{dx} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^{2n}(n!)^2}$$

11.10

# Taylor and Maclaurin Series

# Taylor and Maclaurin Series

We start by supposing that $f$ is any function that can be represented by a power series

$\boxed{1}$ $f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + c_4(x - a)^4 + \cdots \quad |x - a| < R$

Let's try to determine what the coefficients $c_n$ must be in terms of $f$.

To begin, notice that if we put $x = a$ in Equation 1, then all terms after the first one are 0 and we get

$$f(a) = c_0$$

# Taylor and Maclaurin Series

We can differentiate the series in Equation 1 term by term:

$$\boxed{2} \qquad f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots \quad |x-a| < \boldsymbol{R}$$

and substitution of $x = a$ in Equation 2 gives

$$f'(a) = c_1$$

Now we differentiate both sides of Equation 2 and obtain

$$\boxed{3} \qquad f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a)^2 + \cdots \quad |x-a| < \boldsymbol{R}$$

Again we put $x = a$ in Equation 3. The result is

$$f''(a) = 2c_2$$

# Taylor and Maclaurin Series

Let's apply the procedure one more time. Differentiation of the series in Equation 3 gives

$$\boxed{4} \quad f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x - a) + 3 \cdot 4 \cdot 5c_5(x - a)^2 + \cdots \quad |x - a| < \boldsymbol{R}$$

and substitution of $x = a$ in Equation 4 gives

$$f'''(a) = 2 \cdot 3c_3 = 3!c_3$$

By now you can see the pattern. If we continue to differentiate and substitute $x = a$, we obtain

$$f^{(n)}(a) = 2 \cdot 3 \cdot 4 \cdot \cdots \cdot nc_n = n!c_n$$

# Taylor and Maclaurin Series

Solving this equation for the *n*th coefficient $c_n$, we get

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This formula remains valid even for $n = 0$ if we adopt the conventions that $0! = 1$ and $f^{(0)} = f$. Thus we have proved the following theorem.

**5 Theorem** If $f$ has a power series representation (expansion) at $a$, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

# Taylor and Maclaurin Series

Substituting this formula for $c_n$ back into the series, we see that *if f* has a power series expansion at *a*, then it must be of the following form.

$$\boxed{6} \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

The series in Equation 6 is called the **Taylor series of the function *f* at *a*** (or **about *a*** or **centered at *a***).

# Taylor and Maclaurin Series

For the special case *a* = 0 the Taylor series becomes

$$\boxed{7} \qquad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots$$

This case arises frequently enough that it is given the special name **Maclaurin series**.

# Example 1

Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

Solution:

If $f(x) = e^x$, then $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = e^0 = 1$ for all $n$. Therefore the Taylor series for $f$ at 0 (that is, the Maclaurin series) is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

# Example 1 – *Solution*

cont'd

To find the radius of convergence we let $a_n = x^n/n!$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0 < 1$$

so, by the Ratio Test, the series converges for all $x$ and the radius of convergence is $R = \infty$ .

# Taylor and Maclaurin Series

The conclusion we can draw from Theorem 5 and Example 1 is that *if* $e^x$ has a power series expansion at 0, then

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

So how can we determine whether $e^x$ *does* have a power series representation?

Let's investigate the more general question: Under what circumstances is a function equal to the sum of its Taylor series? In other words, if $f$ has derivatives of all orders, when is it true that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

# Taylor and Maclaurin Series

As with any convergent series, this means that $f(x)$ is the limit of the sequence of partial sums. In the case of the Taylor series, the partial sums are

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Notice that $T_n$ is a polynomial of degree $n$ called the ***n*th-degree Taylor polynomial of *f* at *a***.

# Taylor and Maclaurin Series

For instance, for the exponential function $f(x) = e^x$, the result of Example 1 shows that the Taylor polynomials at 0 (or Maclaurin polynomials) with $n = 1$, 2, and 3 are

$$T_1(x) = 1 + x \qquad T_2(x) = 1 + x + \frac{x^2}{2!} \qquad T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

The graphs of the exponential function and these three Taylor polynomials are drawn in Figure 1.

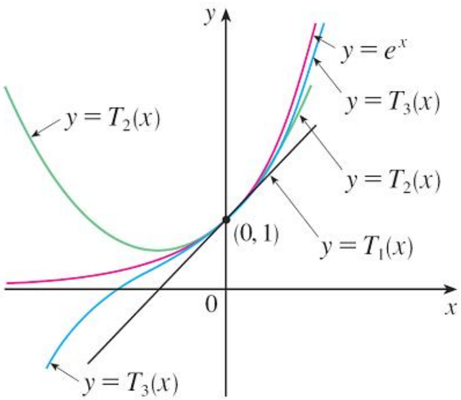



**Figure 1**

As $n$ increases, $T_n(x)$ appears to approach $e^x$ in Figure 1. This suggests that $e^x$ is equal to the sum of its Taylor series.

# Taylor and Maclaurin Series

In general, $f(x)$ is the sum of its Taylor series if

$$f(x) = \lim_{n \to \infty} T_n(x)$$

If we let

$$R_n(x) = f(x) - T_n(x) \quad \text{so that} \quad f(x) = T_n(x) + R_n(x)$$

then $R_n(x)$ is called the **remainder** of the Taylor series. If we can somehow show that $\lim_{n \to \infty} R_n(x) = 0$, then it follows that

$$\lim_{n \to \infty} T_n(x) = \lim_{n \to \infty} [f(x) - R_n(x)] = f(x) - \lim_{n \to \infty} R_n(x) = f(x)$$

# Taylor and Maclaurin Series

We have therefore proved the following.

**8 Theorem** If $f(x) = T_n(x) + R_n(x)$, where $T_n$ is the $n$th-degree Taylor polynomial of $f$ at $a$ and

$$\lim_{n \to \infty} R_n(x) = 0$$

for $|x - a| < R$, then $f$ is equal to the sum of its Taylor series on the interval $|x - a| < R$.

In trying to show that $\lim_{n \to \infty} R_n(x) = 0$ for a specific function $f$, we usually use the following Theorem.

**9 Taylor's Inequality** If $|f^{(n+1)}(x)| \leqslant M$ for $|x - a| \leqslant d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leqslant \frac{M}{(n+1)!} |x - a|^{n+1} \qquad \text{for } |x - a| \leqslant d$$

# Taylor and Maclaurin Series

To see why this is true for $n = 1$, we assume that $|f''(x)| \leq M$. In particular, we have $f''(x) \leq M$, so for $a \leq x \leq a + d$ we have

$$\int_a^x f''(t)\,dt \leq \int_a^x M\,dt$$

An antiderivative of $f''$ is $f'$, so by Part 2 of the Fundamental Theorem of Calculus, we have

$$f'(x) - f'(a) \leq M(x - a) \quad \text{or} \quad f'(x) \leq f'(a) + M(x - a)$$

Thus

$$\int_a^x f'(t)\,dt \leq \int_a^x [f'(a) + M(t - a)]\,dt$$

$$f(x) - f(a) \leq f'(a)(x - a) + M\frac{(x - a)^2}{2}$$

$$f(x) - f(a) - f'(a)(x - a) \leq \frac{M}{2}(x - a)^2$$

# Taylor and Maclaurin Series

But $R_1(x) = f(x) - T_1(x) = f(x) - f(a) - f'(a)(x - a)$. So

$$R_1(x) \leqslant \frac{M}{2}(x - a)^2$$

A similar argument, using $f''(x) \geq -M$, shows that

$$R_1(x) \geqslant -\frac{M}{2}(x - a)^2$$

So

$$|R_1(x)| \leqslant \frac{M}{2}|x - a|^2$$

# Taylor and Maclaurin Series

Although we have assumed that $x > a$, similar calculations show that this inequality is also true for $x < a$.

This proves Taylor's Inequality for the case where $n = 1$. The result for any $n$ is proved in a similar way by integrating $n + 1$ times.

In applying Theorems 8 and 9 it is often helpful to make use of the following fact.

**10** $$\lim_{n \to \infty} \frac{x^n}{n!} = 0 \qquad \text{for every real number } x$$

This is true because we know from Example 1 that the series $\Sigma\, x^n/n!$ converges for all $x$ and so its $n$th term approaches 0.

# Example 2

Prove that $e^x$ is equal to the sum of its Maclaurin series.

Solution:

If $f(x) = e^x$, then $f^{(n+1)}(x) = e^x$ for all $n$. If $d$ is any positive number and $|x| \leq d$, then $|f^{(n+1)}(x)| = e^x \leq e^d$.

So Taylor's Inequality, with $a = 0$ and $M = e^d$, says that

$$|R_n(x)| \leqslant \frac{e^d}{(n+1)!}|x|^{n+1} \qquad \text{for } |x| \leqslant d$$

# Example 2 – *Solution*

cont'd

Notice that the same constant $M = e^d$ works for every value of $n$. But, from Equation 10, we have

$$\lim_{n \to \infty} \frac{e^d}{(n+1)!} |x|^{n+1} = e^d \lim_{n \to \infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

It follows from the Squeeze Theorem that $\lim_{n \to \infty} |R_n(x)| = 0$ and therefore $\lim_{n \to \infty} R_n(x) = 0$ for all values of $x$.

By Theorem 8, $e^x$ is equal to the sum of its Maclaurin series, that is,

11
$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{for all } x}$$

# Taylor and Maclaurin Series

In particular, if we put $x = 1$ in Equation 11, we obtain the following expression for the number $e$ as a sum of an infinite series:

**12** $$\boxed{e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots}$$

# Example 8

Find the Maclaurin series for $f(x) = (1 + x)^k$, where $k$ is any real number.

Solution:

Arranging our work in columns, we have

$$
\begin{array}{ll}
f(x) = (1 + x)^k & f(0) = 1 \\
f'(x) = k(1 + x)^{k-1} & f'(0) = k \\
f''(x) = k(k-1)(1 + x)^{k-2} & f''(0) = k(k-1) \\
f'''(x) = k(k-1)(k-2)(1 + x)^{k-3} & f'''(0) = k(k-1)(k-2) \\
\vdots & \vdots \\
f^{(n)}(x) = k(k-1)\cdots(k-n+1)(1+x)^{k-n} & f^{(n)}(0) = k(k-1)\cdots(k-n+1)
\end{array}
$$

# Example 8 – *Solution*

cont'd

Therefore the Maclaurin series of $f(x) = (1 + x)^k$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

This series is called the **binomial series**.

# Example 8 – *Solution*

cont'd

If its $n$th term is $a_n$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{k(k-1)\cdots(k-n+1)(k-n)x^{n+1}}{(n+1)!}\cdot\frac{n!}{k(k-1)\cdots(k-n+1)x^n}\right|$$

$$= \frac{|k-n|}{n+1}|x| = \frac{\left|1-\dfrac{k}{n}\right|}{1+\dfrac{1}{n}}|x| \to |x| \qquad \text{as } n \to \infty$$

Thus, by the Ratio Test, the binomial series converges if $|x| < 1$ and diverges if $|x| > 1$.

# Taylor and Maclaurin Series

The traditional notation for the coefficients in the binomial series is

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

and these numbers are called the **binomial coefficients**. The following theorem states that $(1 + x)^k$ is equal to the sum of its Maclaurin series.

It is possible to prove this by showing that the remainder term $R_n(x)$ approaches 0, but that turns out to be quite difficult.

**17 The Binomial Series** If $k$ is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

# Taylor and Maclaurin Series

Although the binomial series always converges when $|x| < 1$, the question of whether or not it converges at the endpoints, $\pm 1$, depends on the value of $k$.

It turns out that the series converges at 1 if $-1 < k \leq 0$ and at both endpoints if $k \geq 0$.

Notice that if $k$ is a positive integer and $n > k$, then the expression for $\binom{k}{n}$ contains a factor $(k - k)$, so $\binom{k}{n} = 0$ for $n > k$.

This means that the series terminates and reduces to the ordinary Binomial Theorem when $k$ is a positive integer.

# Taylor and Maclaurin Series

We collect in the following table, for future reference, some important Maclaurin series that we have derived in this section and the preceding one.

| | |
|---|---|
| $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ | $R = 1$ |
| $e^x = \displaystyle\sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$ | $R = \infty$ |
| $\sin x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$ | $R = \infty$ |
| $\cos x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$ | $R = \infty$ |
| $\tan^{-1} x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots$ | $R = 1$ |
| $\ln(1 + x) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^n}{n} = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$ | $R = 1$ |
| $(1 + x)^k = \displaystyle\sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \dfrac{k(k-1)}{2!}x^2 + \dfrac{k(k-1)(k-2)}{3!}x^3 + \cdots$ | $R = 1$ |

Important Maclaurin Series and
Their Radii of Convergence

**Table 1**

# Multiplication and Division of Power Series

# Example 13

Find the first three nonzero terms in the Maclaurin series for (a) $e^x \sin x$ and (b) $\tan x$.

Solution:

(a) Using the Maclaurin series for $e^x$ and $\sin x$ in Table 1, we have

$$e^x \sin x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \cdots\right)$$

# Example 13 – *Solution*

cont'd

We multiply these expressions, collecting like terms just as for polynomials:

$$
\begin{array}{rl}
 & 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots \\
\times & \phantom{1 + {}} x \phantom{{} + \tfrac{1}{2}x^2} - \tfrac{1}{6}x^3 + \cdots \\
\hline
 & x + x^2 + \tfrac{1}{2}x^3 + \tfrac{1}{6}x^4 + \cdots \\
+ & \phantom{x + x^2} - \tfrac{1}{6}x^3 - \tfrac{1}{6}x^4 - \cdots \\
\hline
 & x + x^2 + \tfrac{1}{3}x^3 + \cdots
\end{array}
$$

# Example 13 – *Solution*

cont'd

Thus

$$e^x \sin x = x + x^2 + \tfrac{1}{3}x^3 + \cdots$$

(b) Using the Maclaurin series in Table 1, we have

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots}{1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots}$$

# Example 13 – *Solution*

cont'd

We use a procedure like long division:

$$
\begin{array}{r}
x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots \\
1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots \overline{\big) x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots} \\
x - \frac{1}{2}x^3 + \frac{1}{24}x^5 - \cdots \\
\hline
\frac{1}{3}x^3 - \frac{1}{30}x^5 + \cdots \\
\frac{1}{3}x^3 - \frac{1}{6}x^5 + \cdots \\
\hline
\frac{2}{15}x^5 + \cdots
\end{array}
$$

Thus $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$

# 11.11 Applications of Taylor Polynomials

# Approximating Functions by Polynomials

# Approximating Functions by Polynomials

Suppose that $f(x)$ is equal to the sum of its Taylor series at $a$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

The notation $T_n(x)$ is used to represent the $n$th partial sum of this series and we can call it as it the $n$th-degree Taylor polynomial of $f$ at $a$.

Thus

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} (x - a)^i$$

# Approximating Functions by Polynomials

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Since $f$ is the sum of its Taylor series, we know that $T_n(x) \to f(x)$ as $n \to \infty$ and so $T_n$ can be used as an approximation to $f$:

$$f(x) \approx T_n(x).$$

Notice that the first-degree Taylor polynomial

$$T_1(x) = f(a) + f'(a)(x - a)$$

is the same as the linearization of $f$ at $a$.

# Approximating Functions by Polynomials

Notice also that $T_1$ and its derivative have the same values at $a$ that $f$ and $f'$ have. In general, it can be shown that the derivatives of $T_n$ at $a$ agree with those of $f$ up to and including derivatives of order $n$.

To illustrate these ideas let's take another look at the graphs of $y = e^x$ and its first few Taylor polynomials, as shown in Figure 1.

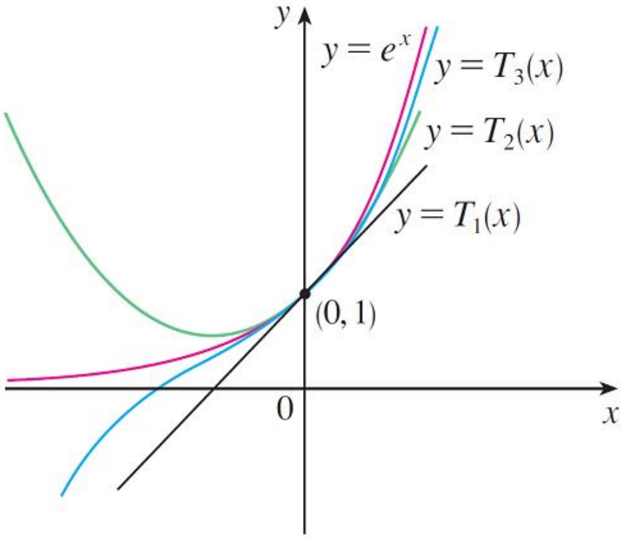



**Figure 1**

# Approximating Functions by Polynomials

The graph of $T_1$ is the tangent line to $y = e^x$ at (0, 1); this tangent line is the best linear approximation to $e^x$ near (0, 1); The graph of $T_2$ is the parabola $y = 1 + x + x^2/2$, and the graph of $T_3$ is the cubic curve $y = 1 + x + x^2/2 + x^3/6$, which is a closer fit to the exponential curve $y = e^x$ than $T_2$.

The next Taylor polynomial $T_4$ would be an even better approximation, and so on.

# Approximating Functions by Polynomials

The values in the table give a numerical demonstration of the convergence of the Taylor polynomials $T_n(x)$ to the function $y = e^x$. We see that when $x = 0.2$ the convergence is very rapid, but when $x = 3$ it is somewhat slower.

In fact, the farther $x$ is from 0, the more slowly $T_n(x)$ converges to $e^x$. When using a Taylor polynomial $T_n$ to approximate a function $f$, we have to ask the questions:
How good an approximation is it?

| | $x = 0.2$ | $x = 3.0$ |
|---|---|---|
| $T_2(x)$ | 1.220000 | 8.500000 |
| $T_4(x)$ | 1.221400 | 16.375000 |
| $T_6(x)$ | 1.221403 | 19.412500 |
| $T_8(x)$ | 1.221403 | 20.009152 |
| $T_{10}(x)$ | 1.221403 | 20.079665 |
| $e^x$ | 1.221403 | 20.085537 |

# Approximating Functions by Polynomials

How large should we take $n$ to be in order to achieve a desired accuracy? To answer these questions we need to look at the absolute value of the remainder:

$$|R_n(x)| = |f(x) - T_n(x)|$$

There are three possible methods for estimating the size of the error:

**1.** If a graphing device is available, we can use it to graph $|R_n(x)|$ and thereby estimate the error.

# Approximating Functions by Polynomials

**2.** If the series happens to be an alternating series, we can use the Alternating Series Estimation Theorem.

**3.** In all cases we can use Taylor's Inequality which says that if $|f^{(n+1)}(x)| \leqslant M$, then

$$|R_n(x)| \leqslant \frac{M}{(n+1)!}|x-a|^{n+1}$$

# Example 1

(a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at *a* = 8.

(b) How accurate is this approximation when $7 \le x \le 9$?

Solution:

(a)

$$f(x) = \sqrt[3]{x} = x^{1/3} \qquad f(8) = 2$$

$$f'(x) = \tfrac{1}{3}x^{-2/3} \qquad f'(8) = \tfrac{1}{12}$$

$$f''(x) = -\tfrac{2}{9}x^{-5/3} \qquad f''(8) = -\tfrac{1}{144}$$

$$f'''(x) = \tfrac{10}{27}x^{-8/3}$$

# Example 1 – *Solution*

cont'd

Thus the second-degree Taylor polynomial is

$$T_2(x) = f(8) + \frac{f'(8)}{1!}(x - 8) + \frac{f''(8)}{2!}(x - 8)^2$$

$$= 2 + \tfrac{1}{12}(x - 8) - \tfrac{1}{288}(x - 8)^2$$

The desired approximation is

$$\sqrt[3]{x} \approx T_2(x)$$

$$= 2 + \tfrac{1}{12}(x - 8) - \tfrac{1}{288}(x - 8)^2$$

# Example 1 – *Solution*

cont'd

(b) The Taylor series is not alternating when $x < 8$, so we can't use the Alternating Series Estimation Theorem in this example.

But we can use Taylor's Inequality with $n = 2$ and $a = 8$:

$$|R_2(x)| \leqslant \frac{M}{3!}|x - 8|^3$$

where $|f'''(x)| \leq M$.

Because $x \geq 7$, we have $x^{8/3} \geq 7^{8/3}$ and so

$$f'''(x) = \frac{10}{27} \cdot \frac{1}{x^{8/3}} \leqslant \frac{10}{27} \cdot \frac{1}{7^{8/3}} < 0.0021$$

# Example 1 – *Solution*

cont'd

Therefore we can take $M = 0.0021$. Also $7 \le x \le 9$, so $-1 \le x - 8 \le 1$ and $|x - 8| \le 1$.

Then Taylor's Inequality gives

$$|R_2(x)| \leqslant \frac{0.0021}{3!} \cdot 1^3 = \frac{0.0021}{6} < 0.0004$$

Thus, if $7 \le x \le 9$, the approximation in part (a) is accurate to within 0.0004.

# Approximating Functions by Polynomials

Let's use a graphing device to check the calculation in Example 1. Figure 2 shows that the graphs of $y = \sqrt[3]{x}$ and $y = T_2(x)$ are very close to each other when $x$ is near 8.

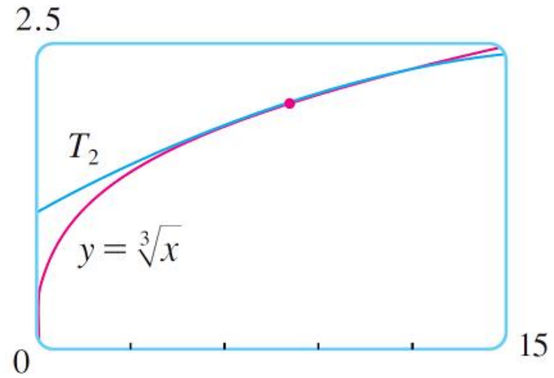



**Figure 2**

# Approximating Functions by Polynomials

Figure 3 shows the graph of $|R_2(x)|$computed from the expression

$$|R_2(x)| = |\sqrt[3]{x} - T_2(x)|$$

We see from the graph that

$$|R_2(x)| < 0.0003$$

when $7 \leq x \leq 9$.

Thus the error estimate from graphical methods is slightly better than the error estimate from Taylor's Inequality in this case.

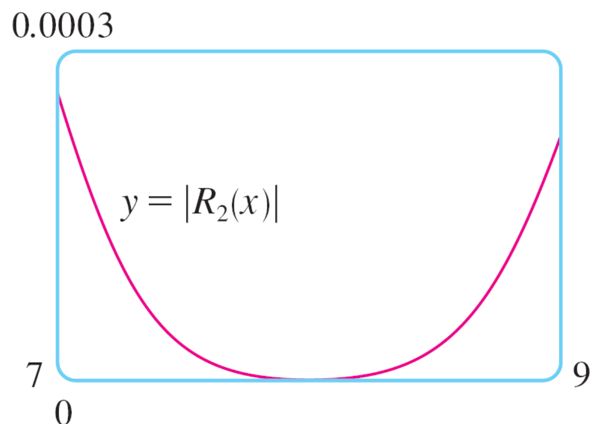



**Figure 3**

# Applications to Physics

# Applications to Physics

Taylor polynomials are also used frequently in physics. In order to gain insight into an equation, a physicist often simplifies a function by considering only the first two or three terms in its Taylor series.

In other words, the physicist uses a Taylor polynomial as an approximation to the function. Taylor's Inequality can then be used to gauge the accuracy of the approximation. The next example shows one way in which this idea is used in special relativity.

# Example 3

In Einstein's theory of special relativity the mass of an object moving with velocity $v$ is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where $m_0$ is the mass of the object when at rest and $c$ is the speed of light. The kinetic energy of the object is the difference between its total energy and its energy at rest:

$$K = mc^2 - m_0c^2$$

# Example 3

cont'd

(a) Show that when $v$ is very small compared with $c$, this expression for $K$ agrees with classical Newtonian physics: $K = \frac{1}{2}m_0v^2$.

(b) Use Taylor's Inequality to estimate the difference in these expressions for $K$ when $|v| \leq 100$ m/s.

Solution:

(a) Using the expressions given for $K$ and $m$, we get

$$K = mc^2 - m_0c^2 = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}} - m_0c^2 = m_0c^2\left[\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right]$$

# Example 3 – *Solution*

cont'd

With $x = -v^2/c^2$, the Maclaurin series for $(1 + x)^{-1/2}$ is most easily computed as a binomial series with $k = -\frac{1}{2}$. (Notice that $|x| < 1$ because $v < c$.) Therefore we have

$$(1 + x)^{-1/2} = 1 - \tfrac{1}{2}x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}x^3 + \cdots$$

$$= 1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 - \tfrac{5}{16}x^3 + \cdots$$

and
$$K = m_0c^2\left[\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right) - 1\right]$$

$$= m_0c^2\left(\frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right)$$

# Example 3 – *Solution*

cont'd

If $v$ is much smaller than $c$, then all terms after the first are very small when compared with the first term. If we omit them, we get

$$K \approx m_0c^2\left(\frac{1}{2}\frac{v^2}{c^2}\right) = \tfrac{1}{2}m_0v^2$$

(b) If $x = -v^2/c^2$, $f(x) = m_0c^2\,[(1 + x)^{-1/2} - 1]$, and $M$ is a number such that $|f''(x)| \leq M$, then we can use Taylor's Inequality to write

$$|R_1(x)| \leqslant \frac{M}{2!}x^2$$

# Example 3 – *Solution*

cont'd

We have $f''(x) = \frac{3}{4} m_0c^2 (1 + x)^{-5/2}$ and we are given that $|v| \leq 100$ m/s, so

$$|f''(x)| = \frac{3m_0c^2}{4(1 - v^2/c^2)^{5/2}} \leqslant \frac{3m_0c^2}{4(1 - 100^2/c^2)^{5/2}} \quad (= M)$$

Thus, with $c = 3 \times 10^8$ m/s,

$$|R_1(x)| \leqslant \frac{1}{2} \cdot \frac{3m_0c^2}{4(1 - 100^2/c^2)^{5/2}} \cdot \frac{100^4}{c^4} < (4.17 \times 10^{-10})m_0$$

So when $|v| \leq 100$ m/s, the magnitude of the error in using the Newtonian expression for kinetic energy is at most $(4.2 \times 10^{-10})m_0$.