VV156 Honors Calculus II
Fall 2021 — HW6 Solutions

December 1, 2021

Exercise 6.1

i)

$$x = 1 - t^2, y = t - 2, -2 \le t \le 2$$

t	-2	-1	0	1	2
x	-3	0	1	0	-3
y	-4	-3	-2	-1	0

$y = t - 2 \Rightarrow t = y + 2$, so $x = 1 - t^2 = 1 - (y+2)^2 \Rightarrow x = -(y+2)^2 + 1$, or $x = -y^2 - 4y - 3$, with $-4 \le y \le 0$

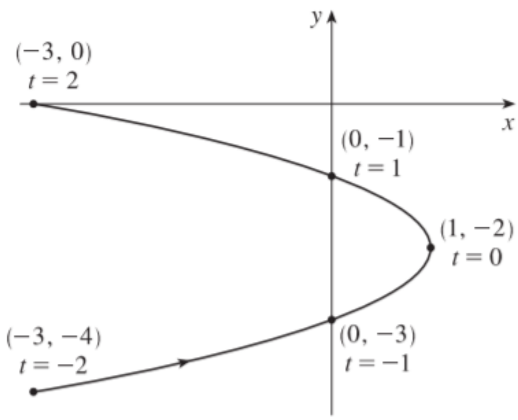

Figure 1: problem 6.1.1

ii)

$$x = t - 1, y = t^3 + 1, -2 \le t \le 2$$

t	-2	-1	0	1	2
x	-3	-2	-1	0	1
y	-7	0	1	2	9

$x = t-1 \quad \Rightarrow \quad t = x+1$, so $y = t^3+1 \quad \Rightarrow \quad y = (x+1)^3+1$, or $y = x^3+3x^2+3x+2$, with $-3 \le x \le 1$

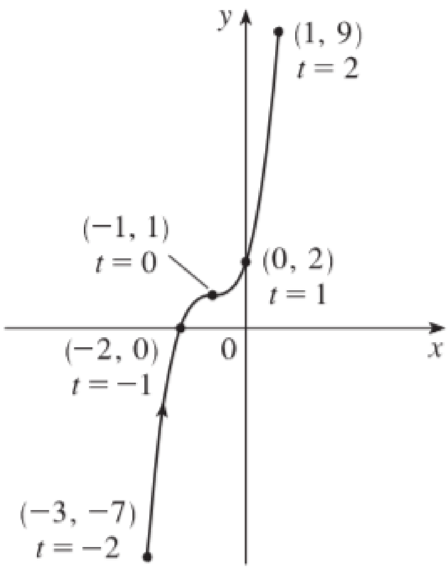

Figure 2: problem 6.1.2

iii) $x = \sin t, y = \csc t, 0 < t < \frac{\pi}{2} \cdot y = \csc t = \frac{1}{\sin t} = \frac{1}{x}$. For $0 < t < \frac{\pi}{2}$, we have $0 < x < 1$ and $y > 1$. Thus, the curve is the portion of the hyperbola $y = 1/x$ with $y > 1$.

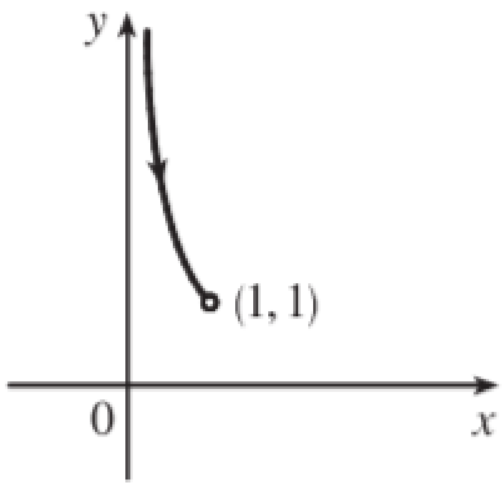

Figure 3: problem 6.1.3

iv) $x = \tan^2 \theta, y = \sec \theta, -\pi/2 < \theta < \pi/2$ $1 + \tan^2 \theta = \sec^2 \theta \Rightarrow 1 + x = y^2 \Rightarrow x = y^2 - 1$. For $-\pi/2 < \theta \leq 0$, we have $x \geq 0$ and $y \geq 1$. For $0 < \theta < \pi/2$, we have $0 < x$ and $1 < y$. Thus, the curve is the portion of the parabola $x = y^2 - 1$ in the first quadrant. As θ increases from $-\pi/2$ to 0, the point (x, y) approaches $(0, 1)$ along the parabola. As θ increases from 0 to $\pi/2$, the point (x, y) retreats from $(0, 1)$ along the parabola.

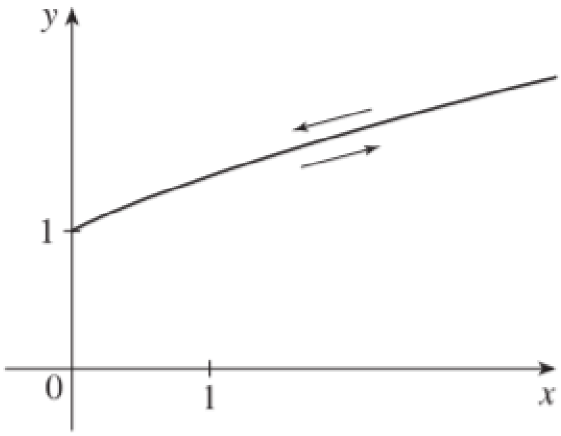

Figure 4: problem 6.1.4

Exercise 6.2

i) $x = 2\sin t, y = 3\cos t, 0 < t < 2\pi$
$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3\sin t}{2\cos t} = -\frac{3}{2}\tan t$, so $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{-\frac{3}{2}\sec^2 t}{2\cos t} = -\frac{3}{4}\sec^3 t$
The curve is CU when $\sec^3 t < 0 \Rightarrow \sec t < 0 \Rightarrow \cos t < 0 \Rightarrow \frac{\pi}{2} < t < \frac{3\pi}{2}$

ii) $x = \cos 2t, \quad y = \cos t, \quad 0 < t < \pi$
$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin t}{-2\sin 2t} = \frac{\sin t}{2\cdot 2\sin t\cos t} = \frac{1}{4\cos t} = \frac{1}{4}\sec t$, so $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{\frac{1}{4}\sec t\tan t}{-4\sin t\cos t} = -\frac{1}{16}\sec^3 t$
The curve is CU when $\sec^3 t < 0 \Rightarrow \sec t < 0 \Rightarrow \cos t < 0 \Rightarrow \frac{\pi}{2} < t < \pi$

Exercise 6.3

i)

$$\text{By symmetry, } A = 4\int_0^a y dx = 4\int_{\pi/2}^0 a\sin^3\theta\left(-3a\cos^2\theta\sin\theta\right)d\theta = 12a^2\int_0^{\pi/2}\sin^4\theta\cos^2\theta d\theta$$

$$\int\sin^4\theta\cos^2\theta d\theta = \int\sin^2\theta\left(\tfrac{1}{4}\sin^2 2\theta\right)d\theta = \tfrac{1}{8}\int(1-\cos 2\theta)\sin^2 2\theta d\theta$$
$$= \tfrac{1}{8}\int\left[\tfrac{1}{2}(1-\cos 4\theta) - \sin^2 2\theta\cos 2\theta\right]d\theta = \tfrac{1}{16}\theta - \tfrac{1}{64}\sin 4\theta - \tfrac{1}{48}\sin^3 2\theta + C$$

$$\int_0^{\pi/2}\sin^4\theta\cos^2\theta d\theta = \left[\frac{1}{16}\theta - \frac{1}{64}\sin 4\theta - \frac{1}{48}\sin^3 2\theta\right]_0^{\pi/2} = \frac{\pi}{32}. \text{ Thus, } A = 12a^2\left(\frac{\pi}{32}\right) = \frac{3}{8}\pi a^2$$

ii) TBD

Exercise 6.4

i) $\phi = \tan^{-1}\left(\frac{dy}{dx}\right) \Rightarrow \frac{d\phi}{dt} = \frac{d}{dt}\tan^{-1}\left(\frac{dy}{dx}\right) = \frac{1}{1+(dy/dx)^2}\left[\frac{d}{dt}\left(\frac{dy}{dx}\right)\right]$. But $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dot{y}}{\dot{x}} \Rightarrow \frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right) = \frac{\ddot{y}\dot{x}-\ddot{x}\dot{y}}{\dot{x}^2} \Rightarrow \frac{d\phi}{dt} = \frac{1}{1+(\dot{y}/\dot{x})^2}\left(\frac{\ddot{y}\dot{x}-\ddot{x}\dot{y}}{\dot{x}^2}\right) = \frac{\dot{x}\ddot{y}-\ddot{x}\dot{y}}{\dot{x}^2+\dot{y}^2}$. Using the Chain Rule, and the fact that $s = \int_0^t\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt \Rightarrow \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = \left(\dot{x}^2+\dot{y}^2\right)^{1/2}$, we have that $\frac{d\phi}{ds} = \frac{d\phi/dt}{ds/dt} = \left(\frac{\dot{x}\ddot{y}-\ddot{x}\dot{y}}{\dot{x}^2+\dot{y}^2}\right)\frac{1}{(\dot{x}^2+\dot{y}^2)^{1/2}} = \frac{\dot{x}\ddot{y}-\ddot{x}\dot{y}}{(\dot{x}^2+\dot{y}^2)^{3/2}}$. So $\kappa = \left|\frac{d\phi}{ds}\right| = \left|\frac{\dot{x}\ddot{y}-\ddot{x}\dot{y}}{(\dot{x}^2+\dot{y}^2)^{3/2}}\right| = \frac{|\dot{x}\ddot{y}-\ddot{x}\dot{y}|}{(\dot{x}^2+\dot{y}^2)^{3/2}}$

ii) $x=x$ and $y=f(x) \Rightarrow \dot{x}=1, \ddot{x}=0$ and $\dot{y}=\frac{dy}{dx}, \ddot{y}=\frac{d^2y}{dx^2}$ So $\kappa=\frac{\left|1\cdot\left(d^2y/dx^2\right)-0\cdot(dy/dx)\right|}{[1+(dy/dx)^2]^{3/2}}=\frac{\left|d^2y/dx^2\right|}{[1+(dy/dx)^2]^{3/2}}$.

Exercise 6.5

i)

$$r=1+\cos\theta \Rightarrow x=r\cos\theta=\cos\theta(1+\cos\theta), y=r\sin\theta=\sin\theta(1+\cos\theta)\Rightarrow$$
$$\frac{dy}{d\theta}=(1+\cos\theta)\cos\theta-\sin^2\theta=2\cos^2\theta+\cos\theta-1=(2\cos\theta-1)(\cos\theta+1)=0$$
$$\Rightarrow \cos\theta=\frac{1}{2} \text{ or } -1$$
$$\theta=\frac{\pi}{3},\pi, \text{ or } \frac{5\pi}{3} \Rightarrow \text{ horizontal tangent at } \left(\frac{3}{2},\frac{\pi}{3}\right),(0,\pi), \text{ and } \left(\frac{3}{2},\frac{5\pi}{3}\right)$$
$$\frac{dx}{d\theta}=-(1+\cos\theta)\sin\theta-\cos\theta\sin\theta=-\sin\theta(1+2\cos\theta)=0\Rightarrow\sin\theta=0 \text{ or } \cos\theta=-\frac{1}{2}\Rightarrow$$
$$\theta=0,\pi,\frac{2\pi}{3}, \text{ or } \frac{4\pi}{3} \Rightarrow \text{ vertical tangent at } (2,0),\left(\frac{1}{2},\frac{2\pi}{3}\right), \text{ and } \left(\frac{1}{2},\frac{4\pi}{3}\right)$$

Note that the tangent is horizontal, not vertical when $\theta=\pi$, since $\lim_{\theta\to\pi}\dfrac{dy/d\theta}{dx/d\theta}=0$

ii)

$$r=e^\theta \Rightarrow x=r\cos\theta=e^\theta\cos\theta, y=r\sin\theta=e^\theta\sin\theta\Rightarrow$$
$$\frac{dy}{d\theta}=e^\theta\sin\theta+e^\theta\cos\theta=e^\theta(\sin\theta+\cos\theta)=0\Rightarrow\sin\theta=-\cos\theta\Rightarrow\tan\theta=-1\Rightarrow$$
$$\theta=-\frac{1}{4}\pi+n\pi[n \text{ any integer }]\Rightarrow \text{ horizontal tangents at } \left(e^{\pi(n-1/4)},\pi\left(n-\frac{1}{4}\right)\right)$$
$$\frac{dx}{d\theta}=e^\theta\cos\theta-e^\theta\sin\theta=e^\theta(\cos\theta-\sin\theta)=0\Rightarrow\sin\theta=\cos\theta\Rightarrow\tan\theta=1\Rightarrow$$
$$\theta=\frac{1}{4}\pi+n\pi[n \text{ any integer }]\Rightarrow \quad \text{vertical tangents at } \left(e^{\pi(n+1/4)},\pi\left(n+\frac{1}{4}\right)\right)$$

Exercise 6.6

To determine when the strophoid $r=2\cos\theta-\sec\theta$ passes through the pole, we solve

$$r=0\Rightarrow 2\cos\theta-\frac{1}{\cos\theta}=0\Rightarrow 2\cos^2\theta-1=0\Rightarrow\cos^2\theta=\frac{1}{2}\Rightarrow$$
$$\cos\theta=\pm\frac{1}{\sqrt{2}}\Rightarrow\theta=\frac{\pi}{4} \text{ or } \theta=\frac{3\pi}{4} \text{ for } 0\le\theta\le\pi \text{ with } \theta\neq\frac{\pi}{2}.$$
$$A=2\int_0^{\pi/4}\frac{1}{2}(2\cos\theta-\sec\theta)^2d\theta=\int_0^{\pi/4}\left(4\cos^2\theta-4+\sec^2\theta\right)d\theta$$
$$=\int_0^{\pi/4}\left[4\cdot\frac{1}{2}(1+\cos2\theta)-4+\sec^2\theta\right]d\theta=\int_0^{\pi/4}\left(-2+2\cos2\theta+\sec^2\theta\right)d\theta$$
$$=[-2\theta+\sin2\theta+\tan\theta]_0^{\pi/4}=\left(-\frac{\pi}{2}+1+1\right)-0=2-\frac{\pi}{2}$$

Exercise 6.7

i) $2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$.

$$\begin{aligned} A &= 2\int_0^{\pi/3} \frac{1}{2}\left[(2\cos\theta)^2 - 1^2\right] d\theta = \int_0^{\pi/3} \left(4\cos^2\theta - 1\right) d\theta \\ &= \int_0^{\pi/3} \left\{4\left[\frac{1}{2}(1+\cos 2\theta)\right] - 1\right\} d\theta = \int_0^{\pi/3} (1 + 2\cos 2\theta) d\theta \\ &= [\theta + \sin 2\theta]_0^{\pi/3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2} \end{aligned}$$

ii)

$$\begin{aligned} &1 - \sin\theta = 1 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0 \text{ or } \pi \Rightarrow \\ A &= \int_\pi^{2\pi} \frac{1}{2}\left[(1-\sin\theta)^2 - 1\right] d\theta = \frac{1}{2}\int_\pi^{2\pi} \left(\sin^2\theta - 2\sin\theta\right) d\theta \\ &= \frac{1}{4}\int_\pi^{2\pi} (1 - \cos 2\theta - 4\sin\theta) d\theta = \frac{1}{4}\left[\theta - \frac{1}{2}\sin 2\theta + 4\cos\theta\right]_\pi^{2\pi} \\ &= \frac{1}{4}\pi + 2 \end{aligned}$$

Exercise 6.8

i) $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_0^\pi \sqrt{(2\cos\theta)^2 + (-2\sin\theta)^2} d\theta$

$$= \int_0^\pi \sqrt{4\left(\cos^2\theta + \sin^2\theta\right)} d\theta = \int_0^\pi \sqrt{4} d\theta = [2\theta]_0^\pi = 2\pi$$

As a check, note that the curve is a circle of radius 1 , so its circumference is $2\pi(1) = 2\pi$.

ii)

$$\begin{aligned} L &= \int_a^b \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_0^{2\pi} \sqrt{\left(5^\theta\right)^2 + \left(5^\theta \ln 5\right)^2} d\theta = \int_0^{2\pi} \sqrt{5^{2\theta}\left[1 + (\ln 5)^2\right]} d\theta \\ &= \sqrt{1 + (\ln 5)^2} \int_0^{2\pi} \sqrt{5^{2\theta}} d\theta = \sqrt{1 + (\ln 5)^2} \int_0^{2\pi} 5^\theta d\theta = \sqrt{1 + (\ln 5)^2} \left[\frac{5^\theta}{\ln 5}\right]_0^{2\pi} \\ &= \sqrt{1 + (\ln 5)^2} \left(\frac{5^{2\pi}}{\ln 5} - \frac{1}{\ln 5}\right) = \frac{\sqrt{1 + (\ln 5)^2}}{\ln 5} \left(5^{2\pi} - 1\right) \end{aligned}$$

iii)

$$\begin{aligned} L &= \int_a^b \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_0^{2\pi} \sqrt{\left(\theta^2\right)^2 + (} \\ &= \int_0^{2\pi} \sqrt{\theta^2\left(\theta^2 + 4\right)} d\theta = \int_0^{2\pi} \theta\sqrt{\theta^2 + 4} d\theta \end{aligned}$$

Now let $u = \theta^2 + 4$, so that $du = 2\theta d\theta \quad \left[\theta d\theta = \frac{1}{2}du\right]$ and

$$\int_0^{2\pi} \theta\sqrt{\theta^2 + 4} d\theta = \int_4^{4\pi^2+4} \frac{1}{2}\sqrt{u} du = \frac{1}{2} \cdot \frac{2}{3}\left[u^{3/2}\right]_4^{4\left(\pi^2+1\right)} = \frac{1}{3}\left[4^{3/2}\left(\pi^2+1\right)^{3/2} - 4^{3/2}\right]$$

$$= \frac{8}{3}\left[\left(\pi^2 + 1\right)^{3/2} - 1\right]$$

iv)

$$
\begin{aligned}
L &= \int_a^b \sqrt{r^2 + (dr/d\theta)^2}d\theta = \int_0^{2\pi} \sqrt{[2(1+\cos\theta)]^2 + (-2\sin\theta)^2}d\theta \\
&= \int_0^{2\pi} \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}d\theta \\
&= \int_0^{2\pi} \sqrt{8 + 8\cos\theta}d\theta = \sqrt{8}\int_0^{2\pi} \sqrt{1+\cos\theta}d\theta = \sqrt{8}\int_0^{2\pi} \sqrt{2\cdot\frac{1}{2}(1+\cos\theta)}d\theta \\
&= \sqrt{8}\int_0^{2\pi} \sqrt{2\cos^2\frac{\theta}{2}}d\theta = \sqrt{8}\sqrt{2}\int_0^{2\pi} \left|\cos\frac{\theta}{2}\right| d\theta = 4\cdot 2\int_0^{\pi} \cos\frac{\theta}{2}d\theta \quad [\text{ by symmetry }] \\
&= 8\left[2\sin\frac{\theta}{2}\right]_0^{\pi} = 8(2) = 16
\end{aligned}
$$