VV156 RC3
Derivatives

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1. Remember this!
2. Differentiation Rules
3. Application questions
4. Application of Derivatives
5. Q&A

Differentiation Formulas

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$(cf)' = cf'$$

$$(f+g)' = f' + g'$$

$$(f-g)' = f' - g'$$

$$(fg)' = fg' + gf'$$

$$(\frac{f}{g})' = \frac{gf' - fg'}{g^2}$$

Differentiation Formulas

$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

Differentiation Formulas

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$$

Differentiation Formulas

$$\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x \ln a}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

Question type in Exam

Multiple choice question

About 20 points, the correct options are 0-4.

Calculating question

Show your steps.

Application Problems

Show your steps.

Answer format

Sample

The Riemann zeta function is defined by:

$$\xi(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

and is used to study the distribution of prime numbers. What's the domain of ξ ?

Answer format

Sample

Standard Answer: The domain is the set of x such that the series is convergence.

When $x = 0$, the series diverges.

When $x < 0, \lim_{n\to\infty} \frac{1}{n^x} = \infty$, the series diverges.

When $0 < x < 1, \int_1^\infty n^{-x} dn = \frac{n^{1-x}}{1-x}\Big|_1^\infty$. Since $1 - x > 0$, the improper integral diverges, thus the series diverges.

When $x = 1, \int_1^\infty n^{-1} dn = \ln n \|_1^\infty \to \infty$. The series diverges.

When $x > 1, \int_1^\infty n^{-x} dn = \frac{n^{1-x}}{1-x}\Big|_1^\infty = \frac{1}{x-1}$. Hence the series converges.

Therefore, we conclude that the domain is $x > 1$.

Answer format

Sample

Interpretation of domain ————————1 point

When $x = 0$ ————————1 point

When $x < 0$ ————————2 points

$-\lim_{x\to\infty} 1/n^x = \infty$ (1 point)

- Diverge (1 point)

When $0 < x < 1$ ————————2 points

- Calculate the integral - Integral diverges, so series diverges.

When $x = 1$ ————————2 points

- One point for integral, one for conclusion When $x > 1$ ————————2 points

- One point for integral, one for conclusion The final answer $x > 1$ ————————1 point

1 Remember this!

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Exercise 1

Calculate the Derivative

1. $$f(x) = \frac{1 - xe^x}{x + e^x}$$

2. $$f(x) = xe^x \csc x$$

3. $$f(x) = x \ln x - x$$

Exercise 1

Solutions

1.

$$f(x) = \frac{1 - xe^x}{x + e^x} \quad \overset{QR}{\Longrightarrow}$$

$$f'(x) = \frac{(x + e^x)(-xe^x)' - (1 - xe^x)(1 + e^x)}{(x + e^x)^2}$$

$$\overset{PR}{\Longrightarrow} \quad f'(x) = \frac{(x + e^x)[-(xe^x + e^x \cdot 1)] - (1 + e^x - xe^x - xe^{2x})}{(x + e^x)^2}$$

$$= \frac{-x^2e^x - xe^x - xe^{2x} - e^{2x} - 1 - e^x + xe^x + xe^{2x}}{(x + e^x)^2}$$

$$= \frac{-x^2e^x - e^{2x} - e^x - 1}{(x + e^x)^2}$$

Exercise 1

Solutions

2.

$$\begin{aligned}(fgh)' = [(fg)h]' = (fg)'h + (fg)h' &= \left(f'g + fg'\right)h + (fg)h' \\ &= f'gh + fg'h + fgh'\end{aligned}$$

$$\begin{aligned}f'(x) &= (x)'e^x \csc x + x\left(e^x\right)' \csc x + xe^x(\csc x)' \\ &= 1e^x \csc x + xe^x \csc x + xe^x(-\cot x \csc x) \\ &= e^x \csc x(1 + x - x\cot x)\end{aligned}$$

3.

$$f(x) = x\ln x - x \Rightarrow f'(x) = x \cdot \frac{1}{x} + (\ln x) \cdot 1 - 1 = 1 + \ln x - 1 = \ln x$$

Chain Rule

Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Chain Rule

The Power Rule Combined with the Chain Rule

If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}\left(u^n\right) = nu^{n-1}\frac{du}{dx}$$

Alternatively,

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

Exercise 2

Calculate the Derivative

1. $$F(t) = (3t-1)^4(2t+1)^{-3}$$

2. $$y = \left(\frac{x^2+1}{x^2-1}\right)^3$$

3. $$y = \sqrt{1+2e^{3x}}$$

Exercise 2

Solutions

1.

$$F(t) = (3t-1)^4(2t+1)^{-3} \Rightarrow$$

$$F'(t) = (3t-1)^4(-3)(2t+1)^{-4}(2) + (2t+1)^{-3} \cdot 4(3t-1)^3(3)$$

$$= 6(3t-1)^3(2t+1)^{-4}[-(3t-1)+2(2t+1)]$$

$$= 6(3t-1)^3(2t+1)^{-4}(t+3)$$

Exercise 2

Solutions

2.

$$
\begin{aligned}
y' &= 3\left(\frac{x^2+1}{x^2-1}\right)^2 \cdot \frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right) \\
&= 3\left(\frac{x^2+1}{x^2-1}\right)^2 \cdot \frac{\left(x^2-1\right)(2x)-\left(x^2+1\right)(2x)}{\left(x^2-1\right)^2} \\
&= 3\left(\frac{x^2+1}{x^2-1}\right)^2 \cdot \frac{2x\left[x^2-1-\left(x^2+1\right)\right]}{\left(x^2-1\right)^2} \\
&= 3\left(\frac{x^2+1}{x^2-1}\right)^2 \cdot \frac{2x(-2)}{\left(x^2-1\right)^2} \\
&= \frac{-12x\left(x^2+1\right)^2}{\left(x^2-1\right)^4}
\end{aligned}
$$

Exercise 2

Solutions

3.

$$
\begin{aligned}
y &= \sqrt{1+2e^{3x}} \Rightarrow \\
y' &= \frac{1}{2}\left(1+2e^{3x}\right)^{-1/2} \frac{d}{dx}\left(1+2e^{3x}\right) \\
&= \frac{1}{2\sqrt{1+2e^{3x}}}\left(2e^{3x}\cdot 3\right) \\
&= \frac{3e^{3x}}{\sqrt{1+2e^{3x}}}
\end{aligned}
$$

Implicit Differentiation

Find y' if $\sin(x+y) = y^2 \cos x$

Differentiating implicitly with respect to x and remembering that y is a function of x, we get

$$\cos(x+y) \cdot (1+y') = y^2(-\sin x) + (\cos x)\,(2yy')$$

(Note that we have used the Chain Rule on the left side and the Product Rule and Chain Rule on the right side.) If we collect the terms that involve y', we get

$$\cos(x+y) + y^2 \sin x = (2y\cos x)y' - \cos(x+y) \cdot y'$$

So

$$y' = \frac{y^2 \sin x + \cos(x+y)}{2y\cos x - \cos(x+y)}$$

Exercise 3

Implicit Differentiation

Find y'' if $x^4 + y^4 = 16$

Exercise 3

Solutions

Differentiating the equation implicitly with respect to x, we get

$$4x^3 + 4y^3y' = 0$$

Solving for y' gives 3

$$y' = -\frac{x^3}{y^3}$$

To find y'' we differentiate this expression for y' using the Quotient Rule and remembering that y is a function of x :

$$\begin{aligned} y'' &= \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) = -\frac{y^3(d/dx)\left(x^3\right) - x^3(d/dx)\left(y^3\right)}{(y^3)^2} \\ &= -\frac{y^3 \cdot 3x^2 - x^3\left(3y^2y'\right)}{y^6} \end{aligned}$$

Exercise 3

Solutions

If we now substitute Equation 3 into this expression, we get

$$y'' = -\frac{3x^2y^3 - 3x^3y^2\left(-\frac{x^3}{y^3}\right)}{y^6}$$
$$= -\frac{3\left(x^2y^4 + x^6\right)}{y^7} = -\frac{3x^2\left(y^4 + x^4\right)}{y^7}$$

But the values of x and y must satisfy the original equation $x^4 + y^4 = 16$. So the answer simplifies to

$$y'' = -\frac{3x^2(16)}{y^7} = -48\frac{x^2}{y^7}$$

Exercise 4

Implicit Differentiation

The Bessel function of order 0, $y = J(x)$, satisfies the differential equation $xy'' + y' + xy = 0$ for all values of x and its value at 0 is $J(0) = 1$.

(a) Find $J'(0)$.

(b) Use implicit differentiation to find $J''(0)$.

Exercise 4

Solutions

(a) $y = J(x)$ and $xy'' + y' + xy = 0 \Rightarrow xJ''(x) + J'(x) + xJ(x) = 0$.
If $x = 0$, we have $0 + J'(0) + 0 = 0$, so $J'(0) = 0$
(b) Differentiating $xy'' + y' + xy = 0$ implicitly, we get
$xy''' + y'' \cdot 1 + y'' + xy' + y \cdot 1 = 0 \Rightarrow xy''' + 2y'' + xy' + y = 0$, so
$xJ'''(x) + 2J''(x) + xJ'(x) + J(x) = 0$. If $x = 0$, we have
$0 + 2J''(0) + 0 + 1 \quad [J(0) = 1 \text{ is given }] = 0$
$\Rightarrow 2J''(0) = -1 \Rightarrow J''(0) = -\frac{1}{2}$

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Know the words

Base Units

Unit name	Unit symbol	Dimension symbol	Quantity name
second	s	T	time
metre	m	L	length
kilogram	kg	M	mass
ampere	A	I	electric current
kelvin	K	Θ	thermodynamic temperature
mole	mol	N	amount of substance
candela	cd	J	luminous intensity

Know the words

Derived Units

Name	Symbol	Quantity	In SI base units
radian	rad	plane angle	$\mathrm{m/m}$
hertz	Hz	frequency	s^{-1}
newton	N	force, weight	$\mathrm{kg\cdot m\cdot s^{-2}}$
pascal	Pa	pressure, stress	$\mathrm{kg\cdot m^{-1}\cdot s^{-2}}$
joule	J	energy, work, heat	$\mathrm{kg\cdot m^{2}\cdot s^{-2}}$
watt	W	power	$\mathrm{kg\cdot m^{2}\cdot s^{-3}}$
coulomb	C	electric charge	s. A
volt	V	emf	$\mathrm{kg\cdot m^{2}\cdot s^{-3}\cdot A^{-1}}$
farad	F	capacitance	$\mathrm{kg^{-1}\cdot m^{-2}\cdot s^{4}\cdot A^{2}}$
ohm	Ω	resistance	$\mathrm{kg\cdot m^{2}\cdot s^{-3}\cdot A^{-2}}$
siemens	S	conductance	$\mathrm{kg^{-1}\cdot m^{-2}\cdot s^{3}\cdot A^{2}}$
henry	H	inductance	$\mathrm{kg\cdot m^{2}\cdot s^{-2}\cdot A^{-2}}$
lumina	lm	luminous flux	$\mathrm{cd\cdot sr}$

Exercise 5

Growth rate model

In a fish farm, a population of fish is introduced into a pond and harvested regularly. A model for the rate of change of the fish population is given by the equation

$$\frac{dP}{dt} = r_0 \left(1 - \frac{P(t)}{P_c}\right) P(t) - \beta P(t)$$

where r_0 is the birth rate of the fish, P_c is the maximum population that the pond can sustain (called the carrying capacity), and β is the percentage of the population that is harvested.

(a) What value of dP/dt corresponds to a stable population?

(b) If the pond can sustain 10,000 fish, the birth rate is 5%, and the harvesting rate is 4%, find the stable population level.

(c) What happens if β is raised to 5% ?

Exercise 5

Solutions

(a) If $dP/dt = 0$, the population is stable (it is constant).

(b) $\frac{dP}{dt} = 0 \Rightarrow \beta P = r_0 \left(1 - \frac{P}{P_c}\right) P \Rightarrow \frac{\beta}{r_0} = 1 - \frac{P}{P_c} \Rightarrow \frac{P}{P_c} = 1 - \frac{\beta}{r_0} \Rightarrow P = P_c \left(1 - \frac{\beta}{r_0}\right)$. If $P_c = 10,000$, $r_0 = 5\% = 0.05$, and $\beta = 4\% = 0.04$, then $P = 10,000 \left(1 - \frac{4}{5}\right) = 2000$.

(c) If $\beta = 0.05$, then $P = 10,000 \left(1 - \frac{5}{5}\right) = 0$. There is no stable population.

Exercise 6

Boyle's Law

Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P and volume V satisfy the equation $PV = C$, where C is a constant. Suppose that at a certain instant the volume is 600 cm^3, the pressure is 150kPa, and the pressure is increasing at a rate of 20$\mathrm{kPa}/\mathrm{min}$. At what rate is the volume decreasing at this instant?

Exercise 6

Solutions

Differentiating both sides of $PV = C$ with respect to t and using the Product Rule gives us $P\frac{dV}{dt} + V\frac{dP}{dt} = 0 \Rightarrow \frac{dV}{dt} = -\frac{V}{P}\frac{dP}{dt}$. When $V = 600$, $P = 150$ and $\frac{dP}{dt} = 20$, so we have $\frac{dV}{dt} = -\frac{600}{150}(20) = -80$. Thus, the volume is decreasing at a rate of 80 $\mathrm{cm}^3/\mathrm{min}$.

Linear Approximation

1.
$$f(x) \approx f(a) + f'(a)(x-a)$$
is called the linear approximation or tangent line approximation of f at a. The linear function whose graph is this tangent line, that is,

2.
$$L(x) = f(a) + f'(a)(x-a)$$
is called the linearization of f at a.

Exercise 7

Linear Approximation

$$1. f(x) = \sin x, \quad a = \pi/6$$

$$2. f(x) = x^{3/4}, \quad a = 16$$

Exercise 7

Solutions

1.$f(x) = \sin x \Rightarrow f'(x) = \cos x$, so $f\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and $f'\left(\frac{\pi}{6}\right) = \frac{1}{2}\sqrt{3}$.
Thus, $L(x) = f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) = \frac{1}{2} + \frac{1}{2}\sqrt{3}\left(x - \frac{\pi}{6}\right) =$
$\frac{1}{2}\sqrt{3}x + \frac{1}{2} - \frac{1}{12}\sqrt{3}\pi$
2.$f(x) = x^{3/4} \Rightarrow f'(x) = \frac{3}{4}x^{-1/4}$, so $f(16) = 8$ and $f'(16) = \frac{3}{8}$
Thus, $L(x) = f(16) + f'(16)(x - 16) = 8 + \frac{3}{8}(x - 16) = \frac{3}{8}x + 2$.

Exercise 7

Linear Approximation

$$1. f(x) = \sin x, \quad a = \pi/6$$

$$2. f(x) = x^{3/4}, \quad a = 16$$

Equivalent Infinitesimal

When $x \to 0$

$a^x - 1 \sim x \ln a$

$\arcsin(a)x \sim \sin(a)x \sim (a)x$

$\arctan(a)x \sim \tan(a)x \sim (a)x$

$\ln(1+x) \sim x$

$\sqrt{1+x} - \sqrt{1-x} \sim x$

$(1+ax)^b - 1 \sim abx$

$\sqrt[b]{1+ax} - 1 \sim \frac{a}{b}x$

$1 - \cos x \sim \frac{x^2}{2}$

$x - \ln(1+x) \sim \frac{x^2}{2}$

Equivalent Infinitesimal

When $x \to 0$

$$\tan x - \sin x \sim \frac{x^3}{2}$$

$$\tan x - x \sim \frac{x^3}{3}$$

$$x - \arctan x \sim \frac{x^3}{3}$$

$$x - \sin x \sim \frac{x^3}{6}$$

$$\arcsin x - x \sim \frac{x^3}{6}$$

Equivalent Infinitesimal

Solve the limit

$$\lim_{x\to 0}\frac{\ln(1+4x)}{\sin(3x)} = \lim_{x\to 0}\frac{\ln(1+4x)}{\sin(3x)}\lim_{x\to 0}\frac{4x}{\ln(1+4x)}\lim_{x\to 0}\frac{\sin(3x)}{3x}$$
$$= \lim_{x\to 0}\frac{4x}{3x} = 4/3$$

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Maximum and Minimum

Absolute Maximum and Absolute Minimum

Let c be a number in the domain D of a function f. Then $f(c)$ is the

- absolute maximum value of f on D if $f(c) \geqslant f(x)$ for all x in D.
- absolute minimum value of f on D if $f(c) \leqslant f(x)$ for all x in D.

Local Maximum and Local Minimum

The number $f(c)$ is a

- local maximum value of f if $f(c) \geqslant f(x)$ when x is near c.
- local minimum value of f if $f(c) \leqslant f(x)$ when x is near c.

The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Some Theorems

Fermat's Theorem

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Rolle's Theorem

Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

Some Theorems

Lagrange Mean Value Theorem

Let f be a function that satisfies the following hypotheses:
1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently, $f(b) - f(a) = f'(c)(b - a)$

Some Theorems

Cauchy Mean Value Theorem (Extended Mean Value Theorem)

Let f, g be two functions that satisfy the following hypotheses:
1. f, g is continuous on the closed interval $[a, b]$.
2. f, g is differentiable on the open interval (a, b).
3.$x \in (a, b), g'(x) \neq 0$

Then there is a number c in (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

or, equivalently, $(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$

Exercise 8

Mean Value Theorem

Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

1.
$$f(x) = x^3 - 3x + 2, \quad [-2, 2]$$

2.
$$f(x) = \ln x, \quad [1, 4]$$

Exercise 8

Solutions

1. $f(x) = x^3 - 3x + 2, [-2, 2]$. f is continuous on $[-2, 2]$ and differentiable on $(-2, 2)$ since polynomials are continuous and differentiable on $\mathbb{R}$. $f'(c) = \frac{f(b)-f(a)}{b-a} \Leftrightarrow 3c^2 - 3 = \frac{f(2)-f(-2)}{2-(-2)} = \frac{4-0}{4} = 1 \Leftrightarrow 3c^2 = 4 \Leftrightarrow c^2 = \frac{4}{3} \Leftrightarrow c = \pm\frac{2}{\sqrt{3}}$, which are both in $(-2, 2)$

2. $f(x) = \ln x, [1, 4]$. f is continuous and differentiable on $(0, \infty)$, so f is continuous on $[1, 4]$ and differentiable on $(1, 4)$ $f'(c) = \frac{f(b)-f(a)}{b-a} \Leftrightarrow \frac{1}{c} = \frac{f(4)-f(1)}{4-1} = \frac{\ln 4 - 0}{3} = \frac{\ln 4}{3} \Leftrightarrow c = \frac{3}{\ln 4} \approx 2.16$, which is in $(1, 4)$

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