$$F'(x) = f'(g(x)) \cdot g'(x)$$

$$\frac{d f(g(x))}{dx} = \left. \frac{d f(y)}{d y} \right|_{y=g(x)} \cdot \frac{d g(x)}{d x}$$

$$y = [g(x)]^2 = g(x) \cdot g(x)$$

$$\frac{dy}{dx} = g'(x) \cdot g(x) + g(x) \cdot g'(x)$$

$$= 2g(x) \cdot g'(x)$$

$$z = [g(x)]^3$$

$$\frac{dz}{dx} = \frac{d\,g(x)}{dx}\, g(x)^2 + \frac{d[g(x)]^2}{dx}\, g(x)$$

$$= g'(x)\, g(x)^2 + 2g(x) \cdot g'(x) \cdot g(x)$$

$$= 3g(x) \cdot g'(x)$$

P-202 Differentiate $y = e^{\sin x}$

$$\boxed{\frac{de^x}{dx} = e^x}$$

$$\frac{dy}{dx} = e^{\sin x} \cdot \cos x$$

Let $u = \sin x$, then $y = e^u$,

$$\frac{dy}{dx} = \left.\frac{dy}{du}\right|_{u=\sin x} \cdot \frac{du}{dx}$$

$$= \left. e^u \right|_{u=\sin x} \cdot (\sin x)'$$

$$= e^{\sin x} \cdot \cos x$$

$\square$

p. 203 $\quad \frac{d}{dx}(a^x) = ?$

Recall $\quad a = \underline{e^{\ln a}} = e^{\log_e a}$

$= a^{\log_e e} = a^1 = a$

$$\boxed{\frac{d}{dx}(a^x)} = \frac{d}{dx}\left[\left(e^{\ln a}\right)^x\right]$$

$$= \frac{d}{dx}\left[\left(e^x\right)^{\ln a}\right]$$

$$= \ln a \, \underbrace{(e^x)^{\ln a - 1} \cdot e^x}_{(e^x)^{\ln a} = a^x}$$

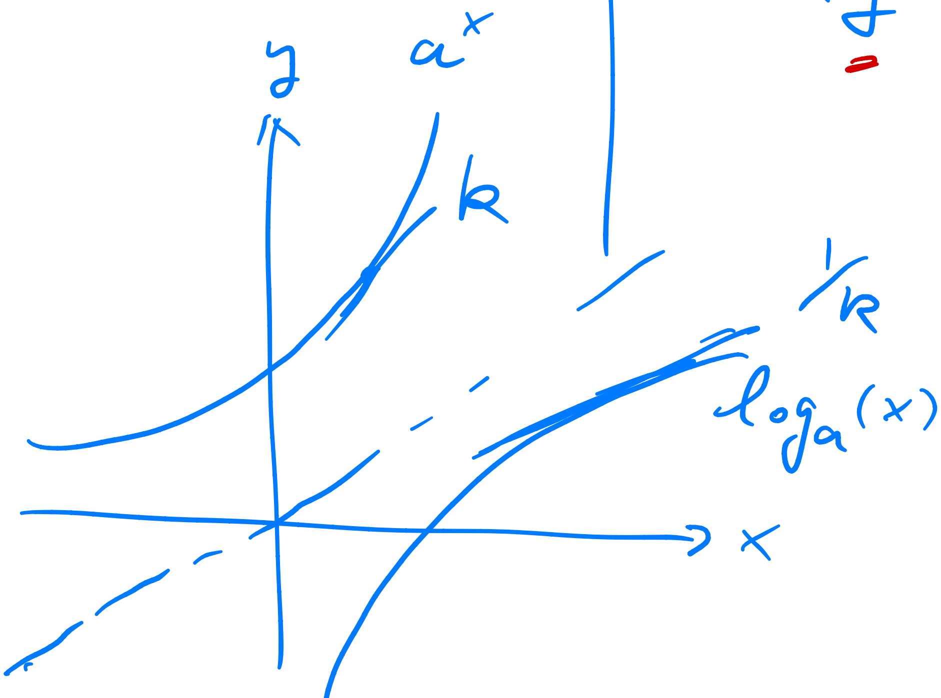
$$\boxed{= \ln(a) \cdot a^x}$$

Let $a = e$

$\Rightarrow (e^x)' = e^x$

$$y = f(x) = a^x$$

$$\frac{df}{dx} = \ln(a) \cdot a^x$$



$$x = f^{-1}(y) = \log_a(y)$$

$$\frac{df^{-1}(y)}{dy} = \frac{1}{\ln(a)\,a^x} \qquad a^x = y$$

$$= \frac{1}{\ln(a)\,y}$$

P. 218

$\frac{df}{dx}$ w/ $f(x) = \sin(\cos(\tan x))$

p. 203

$f = \sin \circ \cos \circ \tan$

$$\frac{df}{dx} = \cos(\cos(\tan x)) \cdot (-\sin(\tan x)) \cdot (\sec^2(x))$$

$= \cdot - \cdot$

$(\tan x)'$

$= \sec^2(x)$

$= \sec(x)^2$

$\sec^{-1}(x)$

$$\frac{1}{f(x)} = [f(x)]^{-1}$$

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

$$x^3 + [f(x)]^3 = 6xf(x)$$

$1^3 + y^3 = 6y$

$\Rightarrow y = y_0$

$$3x^2 + 3f(x)^2 \cdot \underline{f'(x)} = 6f(x) + 6x \cdot \underline{f'(x)}$$

$$\Rightarrow f'(x) = \frac{6f(x) - 3x^2}{3f(x)^2 - 6x}$$

$$= \frac{6y - 3x^2}{3y^2 - 6x} \quad (\text{b/c } y = f(x))$$

$= f(1)$

$$f'(1) = \frac{6y_0 - 3}{3y_0^2 - 6}$$

P.212 Find y' if $\sin(x+y) = y^2 \cos x$

$$\cos(x+y) \cdot (1+y') = 2y \cdot y' \cdot \cos x + y^2(-\sin x)$$

$$y'\left[\cos(x+y) - 2y\cos x\right] = -y^2 \sin x - \cos(x+y)$$

$$\Rightarrow \quad y' = \frac{-y^2\sin x - \cos(x+y)}{\cos(x+y) - 2y\cos x}$$

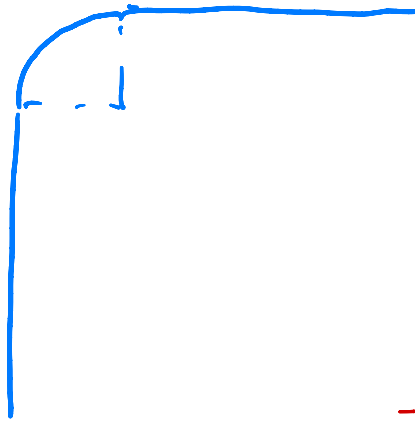
p.213 Find y'' if $x^4 + y^4 = 16$

$$4x^3 + 4y^3 \cdot y' = 0 \Rightarrow \frac{dy}{dx} = y' = -\frac{x^3}{y^3}$$

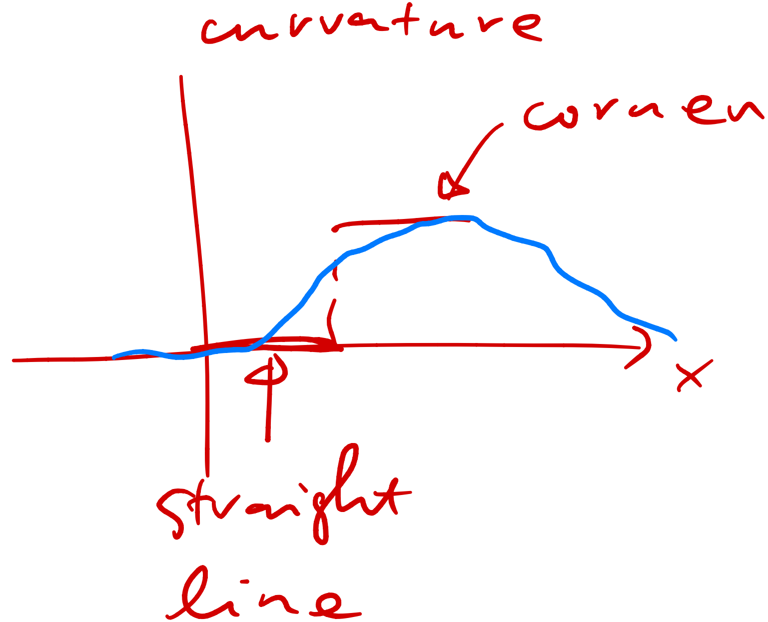
$$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{x^3}{y^3}\right)$$

$$= -\frac{3x^2 \cdot y^3 - x^3 \cdot 3y^2 \cdot y'}{(y^3)^2}$$

$$= -\frac{3x^2 \cdot y^3 - x^3 \cdot 3y^2 \cdot \left(-\frac{x^3}{y^3}\right)}{(y^3)^2}$$



squircle



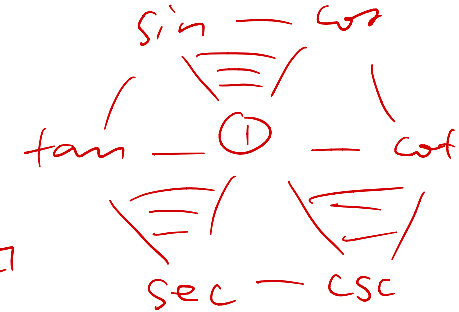
$$\underline{y = f^{-1}(x)} \implies x = \overset{f(y(x))}{\underset{f(f^{-1}(x))}{f(y)}} \overset{\frac{d}{dx}}{\implies} 1 = f'(y)\,\frac{dy}{dx}$$

$$\implies \frac{dy}{dx} = \frac{1}{f'(y)} = \underline{\frac{1}{f'\left(f^{-1}(x)\right)}} = \frac{1}{\frac{dx}{dy}}$$

$$\frac{d\cancel{y}}{d\cancel{x}}\,\frac{d\cancel{x}}{d\cancel{y}} = 1$$

P. 214

$$\sin^{-1}(x) \neq \frac{1}{\sin(x)}$$



- $y = \sin^{-1} x = \arcsin(x) \iff x = \sin(y)$

$$\frac{dy}{dx} = \frac{1}{\sin'(y)} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-\underbrace{\sin(y)}_{=x}{}^2}}$$

$$= \frac{1}{\sqrt{1-x^2}} \quad \text{or} \quad \frac{d\sin^{-1}(x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

- $y = \tan^{-1} x = \arctan(x) \iff x = \tan(y)$

$$y' = \frac{dy}{dx} = \frac{1}{\tan'(y)} = \frac{1}{\sec^2(y)} = \frac{1}{1+\tan^2(y)} = \frac{1}{1+x^2}$$

p.218

- $y = \log_a x = f^{-1}(x)$ w/ $f(y) = a^y$

$$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{a^y \ln(a)} = \frac{1}{a^{\log_a x} \ln a} = \frac{1}{x \ln(a)}$$

If $a = e \approx 2.718\ldots$

$$\frac{d\ln(x)}{dx} = \frac{1}{x \ln(e)} = \frac{1}{x}$$

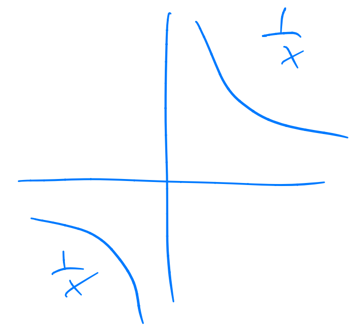
Q: Are there any other function whose derivative is $\frac{1}{x}$?

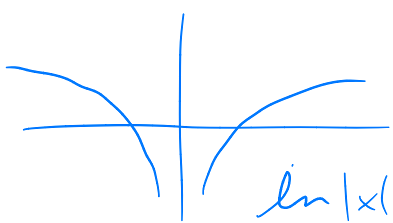
$y = \ln(-x)$, $x < 0$

$$\frac{dy}{dx} = \frac{1}{-x}\underbrace{\frac{d(-x)}{dx}}_{-1} = \frac{1}{x} , \quad x < 0$$

P. 220

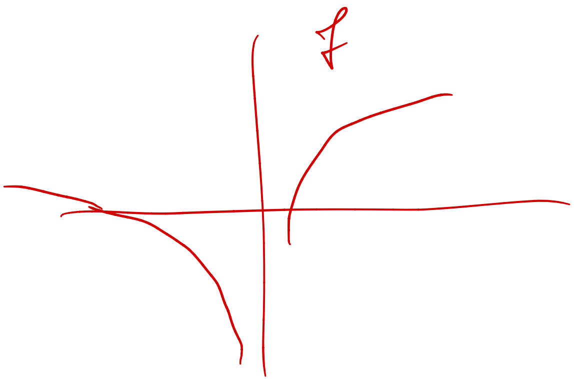
$$\boxed{\frac{d}{dx}\left(\ln|x| + c\right) = \frac{1}{x}}$$

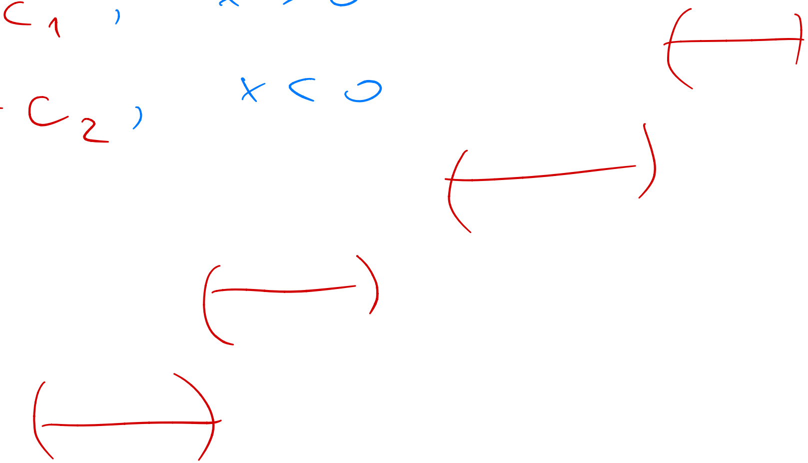




$$\ln|x| = \begin{cases} \ln(x), & x > 0 \\ \ln(-x), & x < 0 \end{cases}$$

$$f(x) = \begin{cases} \ln(x) + c_1, & x > 0 \\ \ln(-x) + c_2, & x < 0 \end{cases}$$



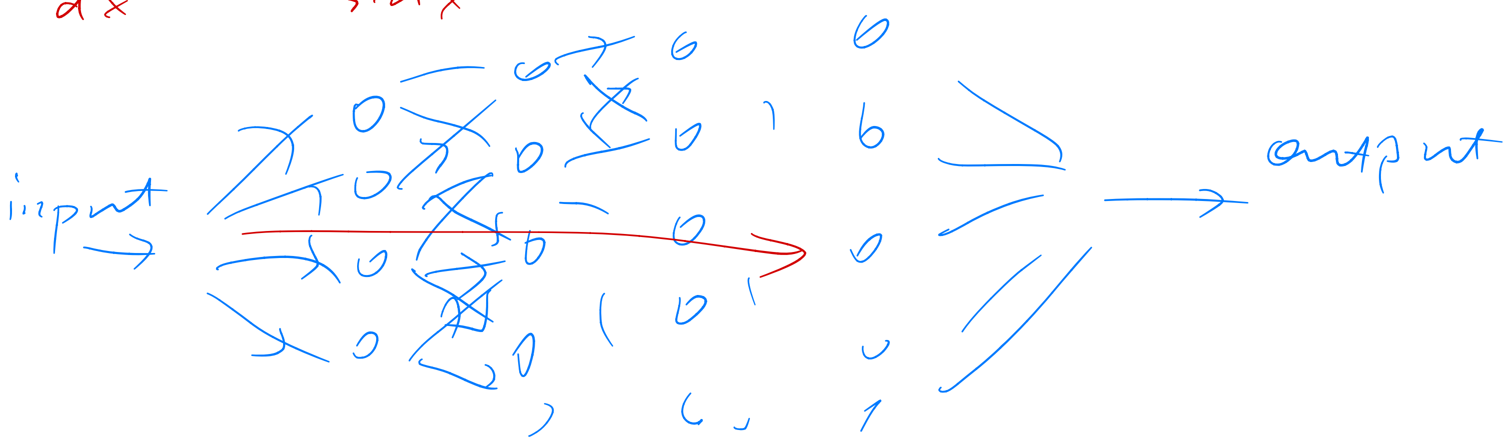


p. 218 Differentiate $y = \ln(x^3+1)$

$$\frac{dy}{dx} = \frac{1}{x^3+1} \cdot 3x^2 = \frac{3x^2}{x^3+1}$$

$$y = \ln(\sin x)$$

$$\frac{dy}{dx} = \frac{1}{\sin x} \cos x = \cot(x)$$



Logarithmic Differentiation P.220

P. 221

- $\frac{d}{dx}(a^b) = 0$ $\quad a, b$ const

- $\frac{d}{dx}[f(x)]^b = b[f(x)]^{b-1} \cdot f'(x)$

- $\frac{d}{dx}[a^{g(x)}] = a^{g(x)} \ln a \cdot g'(x)$

$+$

$\frac{d}{dx}[f(x)g(x)]$

- $$\frac{d}{dx}\left[f(x)^{g(x)}\right] = \underbrace{g(x)\left[f(x)\right]^{g(x)-1} f'(x)}_{\text{pretend } g = \text{const}}$$

$$+ \underbrace{f(x)^{g(x)} \ln f(x) \cdot g'(x)}_{\text{pretend } f = \text{const}}$$

Let $y = f(x)^{g(x)}$ $\qquad \frac{dy}{dx} = ?$

$$\ln y = g(x) \ln f(x)$$

$$\frac{1}{y} \cdot y' = g'(x) \ln f(x) + g(x) \frac{1}{f(x)} \cdot f'(x)$$

$$y' = \left(g'(x) \ln f(x) + g(x) \frac{1}{f(x)} \cdot f'(x) \right) \cdot \underbrace{f(x)^{g(x)}}_{y}$$

$$\frac{d}{dx}(x^x)$$

Let $y = x^x$, then $\ln(y) = x\ln(x)$

take derivative w.r.t. x on both sides

$$\frac{y'}{y} = \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1$$

$$y' = (\ln(x) + 1)\, x^x$$

The number e p.222

$$f(x) = \ln(x) \Rightarrow f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$$

By definition of derivative

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{x \to 0} \frac{\ln(1+x) - \ln(1)}{x} \quad = 0$$

$$= \lim_{x \to 0} \frac{1}{x} \ln(1+x) = \lim_{x \to 0} \ln(1+x)^{1/x} = 1$$

$$e^1 = e^{\lim_{x\to 0} \ln(1+x)^{1/x}} = \lim_{x\to 0} e^{\ln(1+x)^{1/x}}$$

$$\boxed{f(x)' = f(x) \quad f(0) = 1}$$

$$= \boxed{\lim_{x\to 0} (1+x)^{1/x} = e}$$

Let $x = \frac{1}{n}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\boxed{e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$