Leibniz notation

Notation

$$y' + 3y = 0 \iff \frac{dy}{dx} + 3y = 0, \quad \frac{d^2y}{dx^2} + y = 0$$

$$\dot{x} + 6x = 0$$

"x dot"

Newton's notation

Comet

$$\ddot{x} + 6x = 0 \iff \ddot{x}(t) + 6x(t) = 0$$

second derivative w.r.t time

$$\ddot{x} + 7x = e^{-t}$$

Let's solve differential equations of the form

$$y' + p(x)\,y = g(x), \qquad y = y(x)$$

e.g.,

$$r' = -r \iff \frac{r'}{r} = -1$$

- Linear: If y_1 & y_2 are sol'ns to the homogeneous equation $y' + p(x) = 0$
  $\lambda_1 y_1 + \lambda_2 y_2$ is also a sol'n, λ_1, λ_2 const to the homogeneous equation

$$\left.\begin{aligned} y_1' + p(x) y_1 &= 0 \\ y_2' + p(x) y_2 &= 0 \end{aligned}\right\} \begin{array}{|l} \times \lambda_1 \\ \times \lambda_2 \end{array}$$

$$\begin{cases} * \; y_1, y_2 \text{ sol'n} \Rightarrow y_1 + y_2 \text{ sol'n} \\ * \; y \text{ sol'n} \Rightarrow \lambda y \text{ sol'n} \end{cases}$$

$$\Rightarrow (\lambda_1 y_1 + \lambda_2 y_2)' + p(x)(\lambda_1 y_1 + \lambda_2 y_2) = 0$$

- Inhomogeneous / Nonhomogeneous
  b/c the RHS is $\neq 0$

- variable coefficients
  b/c $p(x), q(x)$ not necessarily const.

- First order
  b/c highest order of derivative of y is 1
  i.e., y'

LHS is a linea combination of y & y', i.e., $ay'+by$

$$y' + \boxed{p(x)}\, y = g(x)$$

Use "Integrating factor" $e^{\int p(x)dx} = \exp\left(\int p(x)dx\right)$

Multiply both sides by $e^{\int p(x)dx}$

$$\underbrace{e^{\int p(x)dx}\left(y' + p(x)y\right)}_{\left[e^{\int p(x)dx}\cdot y\right]'} = g(x)\, e^{\int p(x)dx}$$

linear: no term like y^2, $(y')^2$, $y'y$, $\sin(y)$ ...

$$\Rightarrow \quad e^{\int p(x)dx}\, y = \int g(x)\, e^{\int p(x)dx}\, dx + C$$

$$\Rightarrow \quad \boxed{y(x) = e^{-\int p(x)dx} \int g(x)\, e^{\int p(x)dx}\, dx + C e^{-\int p(x)dx}}$$

$p(x) = -\frac{2}{x}$ $\quad q(x) = 1$

Ex $\quad y' - \frac{2}{x} y = 1$

Integrating factor $\quad e^{\int -\frac{2}{x}dx}$

$$= e^{-2\ln x} = e^{\ln(x^{-2})}$$

$$= (x^{-2})^{\ln e} = 1/x^2$$

Multiply both sides by $\frac{1}{x^2}$

$$\underbrace{\frac{1}{x^2} y' - \frac{2y}{x^3}}_{\left(\frac{y}{x^2}\right)' = \frac{d}{dx}\left(\frac{y}{x^2}\right)} = \frac{1}{x^2}$$

$$\Rightarrow \quad y/x^2 = \int \frac{1}{x^2} = -\frac{1}{x} + C$$

$$\Rightarrow \quad y = -x + Cx^2, \quad C \text{ const.}$$

$$y' - \frac{2}{x} y = 1$$

$$y' = -1 + 2Cx$$

$$-\frac{2}{x} y = -\frac{2}{x}(-x + Cx^2)$$

$$= 2 - 2Cx$$

$+ \quad = 1$

$$y' + 2y = e^{-2x}$$

Integrating factor $e^{\int 2\,dx} = e^{2x}$

$$\Rightarrow \quad \underbrace{e^{2x}(y' + 2y)} = e^{-2x} \cdot e^{2x} = 1$$

$$\Rightarrow \quad (ye^{2x})' = 1$$

$$\Rightarrow \quad ye^{2x} = x + C$$

$$\Rightarrow \quad y = xe^{-2x} + Ce^{-2x}$$

C const.

Second-order const-coefficients linear homogeneous ordinary differential equations

$$y'' + p y' + q y = 0 \qquad y(0)=1, \; y'(0) = 2$$

First solve the quadratic equation $(p, q \in \mathbb{R})$
characteristic eqn / eigen eqn.

$$\lambda^2 + p\lambda + q = 0 \qquad (\text{try } e^{\lambda x})$$

$\Delta = p^2 - 4q$, discriminant

$\Delta > 0$ • $\lambda = \lambda_1, \lambda_2$, $\lambda_1 \neq \lambda_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}, \quad C_1, C_2 \text{ consts.}$$

$\Delta = 0$ • $\lambda = \lambda_1 = \lambda_2$

$$y(x) = C_1 e^{\lambda x} + C_2 e^{\lambda x} \cdot x$$

$$= e^{\lambda x}(C_1 + C_2 x) \qquad C_1, C_2 \text{ consts.}$$

$\Delta < 0$ • $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \overline{\lambda_1} = \alpha - i\beta$, $\alpha, \beta \in \mathbb{R}$
$\beta \neq 0$

$$y(x) = C_1 e^{\alpha x} \cos\beta x + C_2 e^{\alpha x} \sin\beta x$$

$$= e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$$

C_1, C_2 const

- $y'' - (\lambda_1 + \lambda_2) y' + \lambda_1 \lambda_2 y = 0, \quad \lambda_1, \lambda_2 \in \mathbb{R}$

$$\underline{\lambda_1 \neq \lambda_2}$$

$$(\boxed{y' - \lambda_1 y})' - \lambda_2 (\boxed{y' - \lambda_1 y}) = 0$$

$$\Rightarrow \quad y' - \lambda_1 y = c_2 e^{\lambda_2 x} \qquad (1)$$

$$(y' - \lambda_2 y)' - \lambda_1 (y' - \lambda_2 y) = 0$$

$$\Rightarrow \quad y' - \lambda_2 y = c_1 e^{\lambda_1 x} \qquad (2)$$

Take (2) − (1)

$$(\lambda_1 - \lambda_2) y = c_1 e^{\lambda_1 x} - c_2 e^{\lambda_2 x}$$

$$\Rightarrow \quad y = \underbrace{\frac{c_1}{\lambda_1 - \lambda_2}}_{\tilde{c}_1} e^{\lambda_1 x} + \underbrace{\frac{-c_2}{\lambda_1 - \lambda_2}}_{\tilde{c}_2} e^{\lambda_2 x}$$

$$\Rightarrow \quad y = \tilde{c}_1 e^{\lambda_1 x} + \tilde{c}_2 e^{\lambda_2 x}$$

$\tilde{c}_1, \tilde{c}_2$ const.

$$y'' - 2\lambda y' + \lambda^2 y = 0, \quad \lambda \in \mathbb{R}$$

$$\Leftrightarrow \quad (y' - \lambda y)' - \lambda (y' - \lambda y) = 0$$

$$\Rightarrow \quad y' - \lambda y = c_1 e^{\lambda x}$$

Integrating factor given by

$$e^{\int (-\lambda) dx} = e^{-\lambda x}$$

Then $$e^{-\lambda x}(y' - \lambda y) = c_1 e^{\lambda x} \cdot e^{-\lambda x}$$

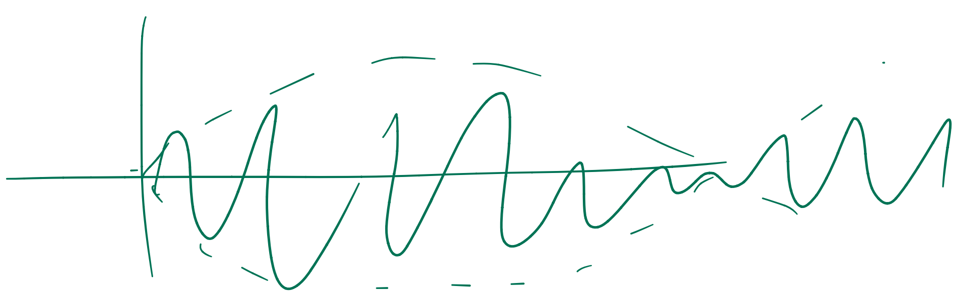
$$\Rightarrow \quad (e^{-\lambda x} y)' = c_1$$

Integrate both sides,

$$\Rightarrow \quad e^{-\lambda x} y = c_1 x + c_2$$

$$\Rightarrow \quad y = e^{\lambda x}(c_1 x + c_2)$$

c_1, c_2 const.



- $y'' - 2\alpha y' + (\alpha^2+\beta^2)y = 0$

$(\lambda_{1,2} = \alpha \pm i\beta,\ \alpha,\beta \in \mathbb{R},\ \beta \neq 0,\ \text{const.})$

The sol'n is give by

$$y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$$

$$c_1, c_2 \in \mathbb{C},\quad \overline{c_1} = c_2$$

Recall Euler's formula $\underline{e^{i\theta} = \cos\theta + i\sin\theta}$

$$\boxed{\begin{aligned} e^{x+iy} &= e^x \cdot e^{iy} \\ &= e^x(\cos y + i\sin y) \end{aligned}}$$

$$\left\{\begin{aligned} & e^{i\pi} = -1 \quad \Big\downarrow \theta=\pi \\ & \underline{e^{i\pi} + 1 = 0} \end{aligned}\right.$$

Let $c_1 = d_1 + id_2$, $d_1, d_2 \in \mathbb{R}$, const.

$$\begin{aligned} y &= (d_1+id_2)\,\underline{e^{\alpha x}}\,(\cos\beta x + i\sin\beta x) \\ &\quad + (d_1 - id_2)\,\underline{e^{\alpha x}}\,(\cos\beta x - i\sin\beta x) \\ &= e^{\alpha x}\Big(2d_1\cos\beta x - 2d_2\sin\beta x \\ &\qquad + \cancel{id_2\cos\beta x} + \cancel{id_1\sin\beta x} \\ &\qquad \cancel{d_1\cos\beta x - d_2\sin\beta x} \\ &\qquad - \cancel{id_2\cos\beta x} - \cancel{id_1\sin\beta x}\Big) \end{aligned}$$

$$= e^{\alpha x}(\underbrace{2d_1}_{\tilde{C}_1}\cos\beta x \underbrace{-2d_2}_{\tilde{C}_2}\sin\beta x)$$

$$\Rightarrow y = e^{\alpha x}(\tilde{C}_1\cos\beta x + \tilde{C}_2\sin\beta x)$$

$\tilde{C}_1, \tilde{C}_2 \in \mathbb{R}$ consts.

In general, try ansatz $\underline{y = e^{\lambda x}}$

to solve $\underline{y'' + py' + qy = 0}$, $p, q \in \mathbb{R}$

thus

$$y'' + py' + qy$$

$$= \lambda^2 e^{\lambda x} + p\lambda e^{\lambda x} + q e^{\lambda x}$$

$$= \underbrace{e^{\lambda x}}_{\neq 0}(\underbrace{\lambda^2 + p\lambda + q}) = 0$$

$$\Rightarrow \lambda^2 + p\lambda + q = 0 \Rightarrow \lambda = \lambda_1, \lambda_2$$

$\Rightarrow e^{\lambda_1 x}$ & $e^{\lambda_2 x}$ are sol'ns

to the ODE (ordinary diff eqn)
(ode)

By linearity ($\lambda_1 \neq \lambda_2$)

$$c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}, \quad c_1, c_2 \text{ arbitrary}$$

is also a solution.

If $\lambda = \lambda_1 = \lambda_2$, try

$$y = c(x) e^{\lambda x}$$

(variation of parameters/constants)

$$\Rightarrow \quad y'' - 2\lambda y' + \lambda^2 y = ?$$

$$y' = e^{\lambda x} \left(\lambda c(x) + c'(x) \right)$$

$$y'' = e^{\lambda x} \left(\lambda^2 c(x) + 2\lambda c'(x) + c''(x) \right)$$

$$y'' + 2\lambda y' + \lambda^2 y$$

$$= e^{\lambda x} \Big(c''(x) + \cancel{2\lambda c'(x)} + \cancel{\lambda^2 c(x)} \underbrace{- \cancel{2\lambda^2 c(x)} - \cancel{2\lambda c'(x)} + \cancel{\lambda^2 c(x)}}_{=0} \Big) = 0$$

$$\Rightarrow \quad c''(x) = 0 \Rightarrow c(x) = ax + b$$

$$\Rightarrow \quad y(x) = c(x)\, e^{\lambda x}$$

$$= (ax+b)\, e^{\lambda x}$$

$a, b \in \mathbb{R}$. const

Inhomogeneous ODE (2nd order const. coeff.)

$$y'' + p y' + q y = e^{\lambda x}$$

Step 1 Find the homogeneous sol'n, i.e., sol'n to $y'' + py' + qy = 0$

call this sol'n y_h

Step 2 Find a particular sol'n, i.e., any sol'n y_p s.t.

$$y_p'' + p y_p' + q y_p = e^{\lambda x}$$

The general sol'n is given by $y_h + y_p$

$\chi(x) = x^2+2x+1$ $\quad \dots = e^{5x}$

$= (x+1)^2$

Ex1 $\quad y'' + 2y' + y = 1e^{0x}$ $\qquad y_p(x) = \dfrac{e^{5x}}{25}$

Step 1 the homo. sol'n y_h is given by

$$y_h = c_1 e^{-x} + c_2 e^{-x} x$$

Step 2 observe that a particular sol'n is given by $y_p = 1$

So the general sol'n is given by

$$y = y_h + y_p$$
$$= c_1 e^{-x} + c_2 e^{-x} x + 1$$

Consider $y'' + py' + qy = \underline{e^{\lambda x}}$, $\lambda \in \mathbb{C}$

Let $\underline{\chi}(x) = x^2 + px + q$

- If $\chi(\lambda) \neq 0$, $\quad y_p(x) = \dfrac{e^{\lambda x}}{\chi(\lambda)}$
- If $\chi(\lambda) = 0$, $\chi'(\lambda) \neq 0$, $\quad y_p(x) = \dfrac{x e^{\lambda x}}{\chi'(\lambda)}$
- If $\chi(\lambda), \chi'(\lambda) = 0$, $\chi''(\lambda) \neq 0$

$$y_p(x) = \frac{x^2 e^{\lambda x}}{\chi''(\lambda)}$$

$$\boxed{y'' + 2y' + 2y = e^{-x}\cos x}$$

$$= \operatorname{Re}\left(e^{(-1+i)x}\right)$$

First solve

$$z'' + 2z' + 2z = e^{(-1+i)x}$$

$$z_p(x) = \frac{x e^{(-1+i)x}}{\chi'(-1+i)}$$

$$= \frac{x e^{(-1+i)x}}{2i}$$

$$= \frac{x e^{-x}(\cos x + i \sin x)}{2i}$$

$$= \underbrace{\frac{x e^{-x}\cos x}{2i}}_{\operatorname{Im}(z_p)} + \underbrace{\frac{x e^{-x}\sin x}{2}}_{\operatorname{Re}(z_p)}$$

$$\chi(x) = x^2 + 2x + 2$$
$$= (x+1)^2 + 1$$

$$\chi(-1+i)$$
$$= (\cancel{-1} + i \cancel{+1})^2 + 1$$
$$= 0$$

$$\chi'(x) = 2x + 2$$

$$\chi'(-1+i)$$
$$= 2(-1+i) + 2$$
$$= -2 + 2i + 2$$
$$= 2i$$

$$\Rightarrow \quad y_p = \operatorname{Re}(z_p) = \frac{1}{2} e^{-x} x \sin(x)$$

In addition, $y_h = e^{-x}(c_1 \cos x + c_2 \sin x)$

$$y = y_h + y_p = e^{-x}(c_1 \cos x + c_2 \sin x) + \frac{1}{2} e^{-x} x \sin(x), \quad c_1, c_2 \text{ const.}$$