2 Derivatives

2.1 Derivatives and Rates of Change

Derivatives and Rates of Change

The problem of finding the tangent line to a curve and the problem of finding the velocity of an object both involve finding the same type of limit.

This special type of limit is called a *derivative* and we will see that it can be interpreted as a rate of change in any of the sciences or engineering.

Tangents

Tangents

If a curve C has equation $y = f(x)$ and we want to find the tangent line to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Then we let Q approach P along the curve C by letting x approach a.

Tangents

If m_{PQ} approaches a number m, then we define the *tangent* t to be the line through P with slope m. (This amounts to saying that the tangent line is the limiting position of the secant line PQ as Q approaches P. See Figure 1.)

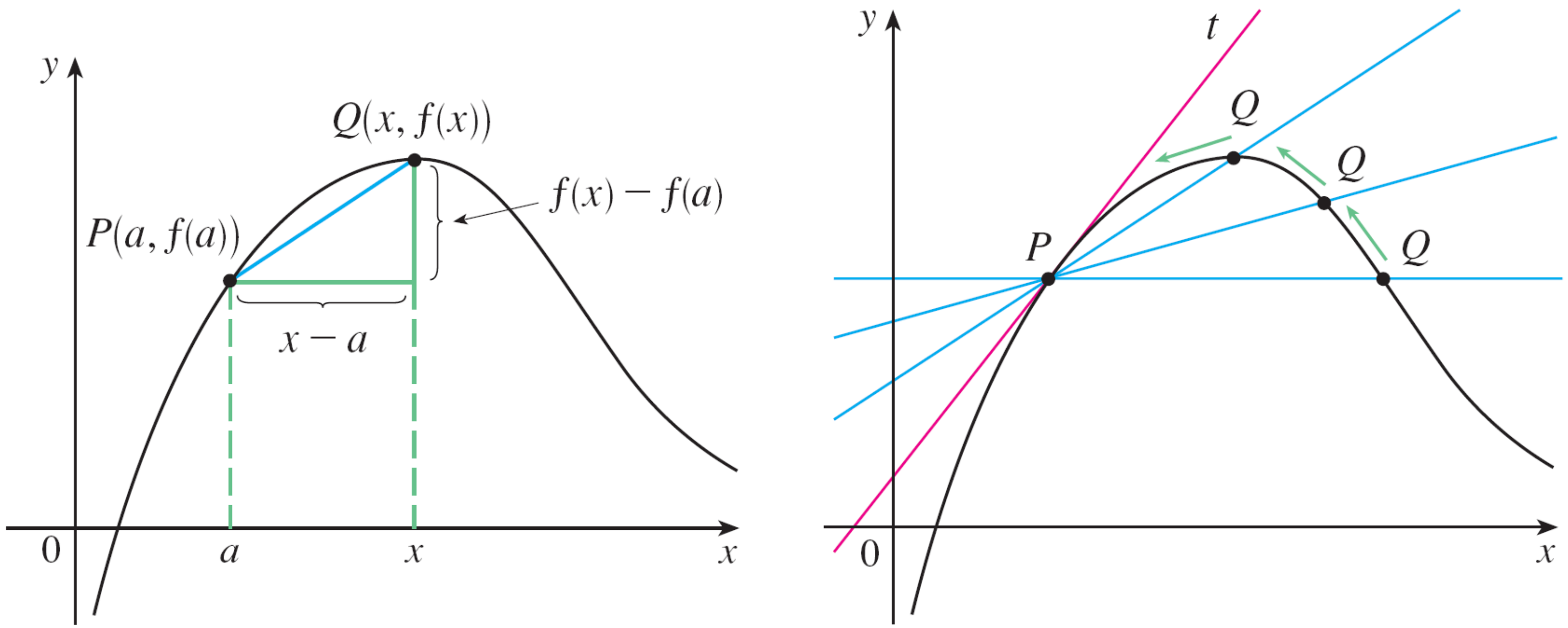


Figure 1

Tangents

$\boxed{1}$ **Definition** The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

Example 1

Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Solution:

Here we have $a = 1$ and $f(x) = x^2$, so the slope is

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1}$$

$$= \lim_{x \to 1} (x + 1)$$

Example 1 – *Solution*

cont'd

$$= 1 + 1$$

$$= 2$$

Using the point-slope form of the equation of a line, we find that an equation of the tangent line at (1, 1) is

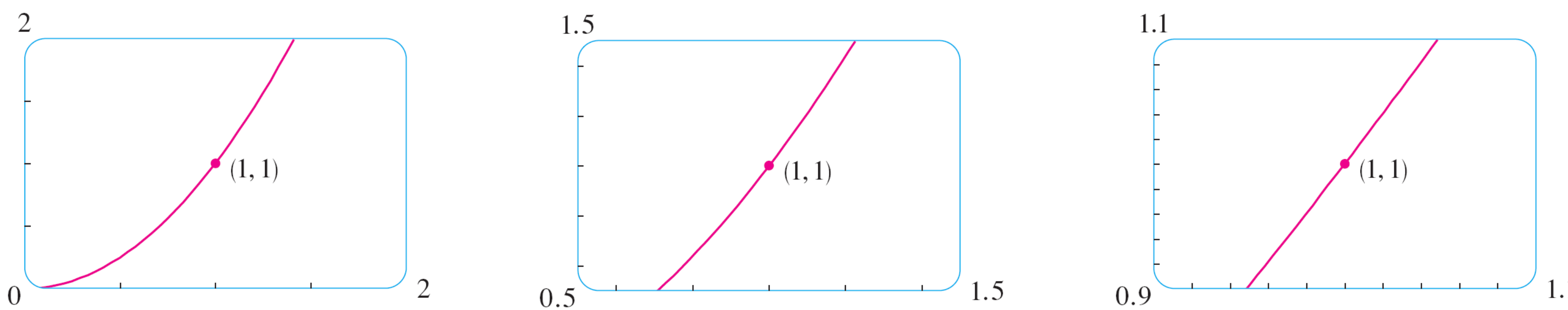
$y - 1 = 2(x - 1)$ or $y = 2x - 1$

Tangents

We sometimes refer to the slope of the tangent line to a curve at a point as the **slope of the curve** at the point.

The idea is that if we zoom in far enough toward the point, the curve looks almost like a straight line.

Figure 2 illustrates this procedure for the curve $y = x^2$ in Example 1.



Zooming in toward the point (1, 1) on the parabola $y = x^2$

Figure 2

Tangents

The more we zoom in, the more the parabola looks like a line.

In other words, the curve becomes almost indistinguishable from its tangent line.

There is another expression for the slope of a tangent line that is sometimes easier to use.

Tangents

If $h = x - a$, then $x = a + h$ and so the slope of the secant line PQ is

$$m_{PQ} = \frac{f(a+h) - f(a)}{h}$$

(See Figure 3 where the case $h > 0$ is illustrated and Q is to the right of P. If it happened that $h < 0$, however, Q would be to the left of P.)

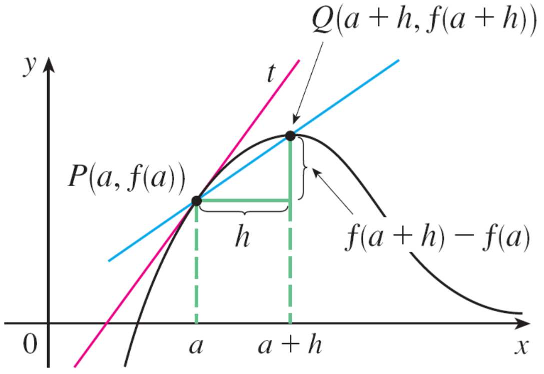


Figure 3

Tangents

Notice that as x approaches a, h approaches 0 (because $h = x - a$) and so the expression for the slope of the tangent line in Definition 1 becomes

2

$$\boxed{m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

Velocities

Velocities

In general, suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement (directed distance) of the object from the origin at time t.

The function f that describes the motion is called the **position function** of the object. In the time interval from $t = a$ to $t = a + h$ the change in position is $f(a + h) - f(a)$. (See Figure 5.)

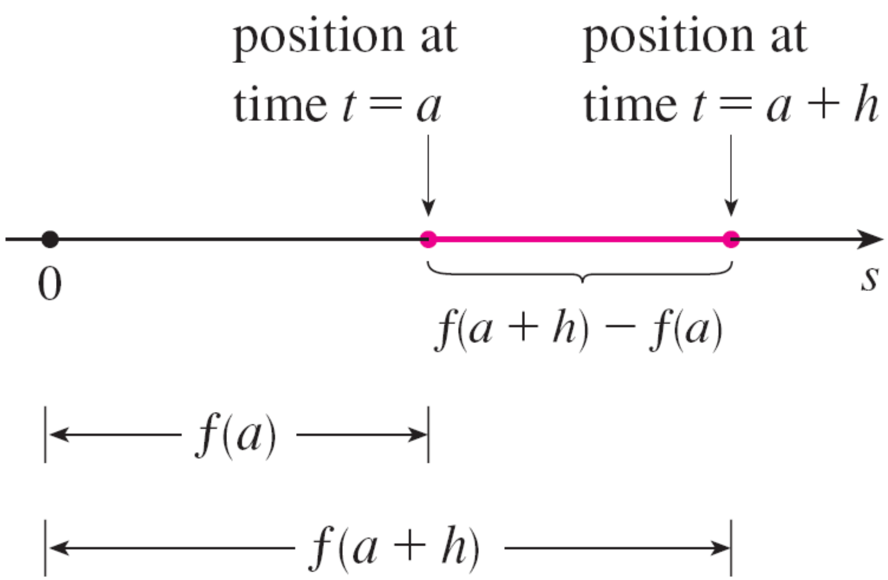


Figure 5

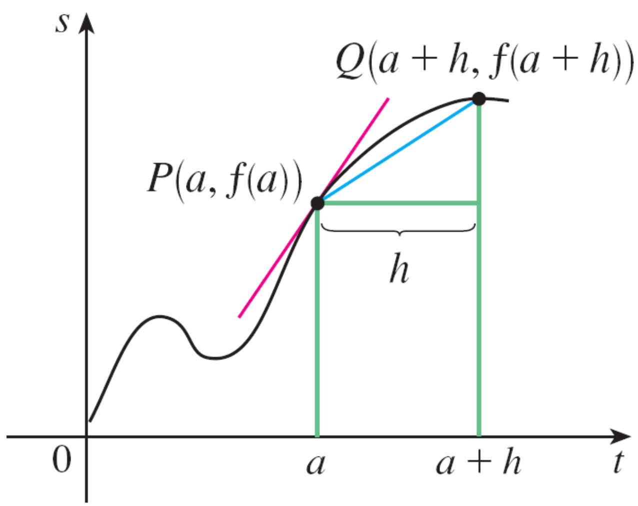
Velocities

The average velocity over this time interval is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a + h) - f(a)}{h}$$

Velocities

which is the same as the slope of the secant line *PQ* in Figure 6.



$$m_{PQ} = \frac{f(a+h) - f(a)}{h}$$
$$= \text{average velocity}$$

Figure 6

Velocities

Now suppose we compute the average velocities over shorter and shorter time intervals [*a*, *a* + *h*].

In other words, we let *h* approach 0. As in the example of the falling ball, we define the **velocity** (or **instantaneous velocity**) *v*(*a*) at time *t* = *a* to be the limit of these average velocities:

3

$$v(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

This means that the velocity at time *t* = *a* is equal to the slope of the tangent line at *P*.

Example 3

Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground.

(a) What is the velocity of the ball after 5 seconds?

(b) How fast is the ball traveling when it hits the ground?

Solution:

We will need to find the velocity both when $t = 5$ and when the ball hits the ground, so it's efficient to start by finding the velocity at a general time $t = a$.

Example 3 – *Solution*

cont'd

Using the equation of motion $s = f(t) = 4.9t^2$, we have

$$
\begin{aligned}
v(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h \to 0} \frac{4.9(a+h)^2 - 4.9a^2}{h} \\
&= \lim_{h \to 0} \frac{4.9(a^2 + 2ah + h^2 - a^2)}{h} \\
&= \lim_{h \to 0} \frac{4.9(2ah + h^2)}{h} \\
&= \lim_{h \to 0} 4.9(2a + h) \\
&= 9.8a
\end{aligned}
$$

Example 3 – *Solution* cont'd

(a) The velocity after 5 s is $v(5) = (9.8)(5) = 49$ m/s.

(b) Since the observation deck is 450 m above the ground, the ball will hit the ground at the time t_1 when $s(t_1) = 450$, that is,

$$4.9t_1^2 = 450$$

This gives

$$t_1^2 = \frac{450}{4.9} \qquad \text{and} \qquad t_1 = \sqrt{\frac{450}{4.9}} \approx 9.6 \text{ s}$$

The velocity of the ball as it hits the ground is therefore

$$v(t_1) = 9.8t_1 = 9.8\sqrt{\frac{450}{4.9}} \approx 94 \text{ m/s}$$

Derivatives

Derivatives

We have seen that the same type of limit arises in finding the slope of a tangent line (Equation 2) or the velocity of an object (Equation 3).

In fact, limits of the form

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

arise whenever we calculate a rate of change in any of the sciences or engineering, such as a rate of reaction in chemistry or a marginal cost in economics.

Since this type of limit occurs so widely, it is given a special name and notation.

Derivatives

4 Definition The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

if this limit exists.

If we write $x = a + h$, then we have $h = x - a$ and h approaches 0 if and only if x approaches a. Therefore an equivalent way of stating the definition of the derivative, as we saw in finding tangent lines, is

5

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Example 4

Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a.

Solution:

From Definition 4 we have

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{[(a + h)^2 - 8(a + h) + 9] - [a^2 - 8a + 9]}{h}$$

$$= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h}$$

Example 4 – *Solution*

cont'd

$$= \lim_{h \to 0} \frac{2ah + h^2 - 8h}{h}$$

$$= \lim_{h \to 0} (2a + h - 8)$$

$$= 2a - 8$$

Derivatives

We defined the tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ to be the line that passes through P and has slope m given by Equation 1 or 2.

Since, by Definition 4, this is the same as the derivative $f'(a)$, we can now say the following.

The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a.

Derivatives

If we use the point-slope form of the equation of a line, we can write an equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$:

$$y - f(a) = f'(a)(x - a)$$

Rates of Change

Rates of Change

Suppose y is a quantity that depends on another quantity x. Thus y is a function of x and we write $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the **increment** of x) is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

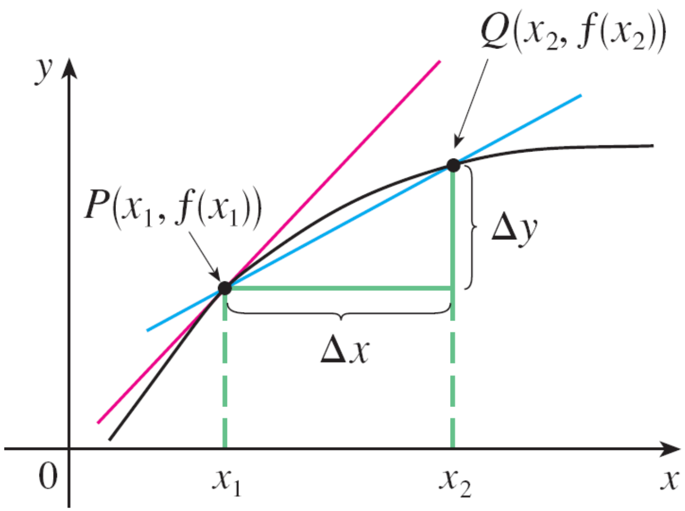
$$\Delta y = f(x_2) - f(x_1)$$

Rates of Change

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the **average rate of change of *y* with respect to *x*** over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line PQ in Figure 8.



average rate of change = m_{PQ}

instantaneous rate of change = slope of tangent at P

Figure 8

Rates of Change

By analogy with velocity, we consider the average rate of change over smaller and smaller intervals by letting x_2 approach x_1 and therefore letting Δx approach 0.

The limit of these average rates of change is called the (**instantaneous**) **rate of change of *y* with respect to *x*** at $x = x_1$, which is interpreted as the slope of the tangent to the curve $y = f(x)$ at $P(x_1, f(x_1))$:

$$\boxed{6} \qquad \text{instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

We recognize this limit as being the derivative $f'(x_1)$.

Rates of Change

We know that one interpretation of the derivative $f'(a)$ is as the slope of the tangent line to the curve $y = f(x)$ when $x = a$. We now have a second interpretation:

> The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

The connection with the first interpretation is that if we sketch the curve $y = f(x)$, then the instantaneous rate of change is the slope of the tangent to this curve at the point where $x = a$.

Rates of Change

This means that when the derivative is large (and therefore the curve is steep, as at the point P in Figure 9), the y-values change rapidly.

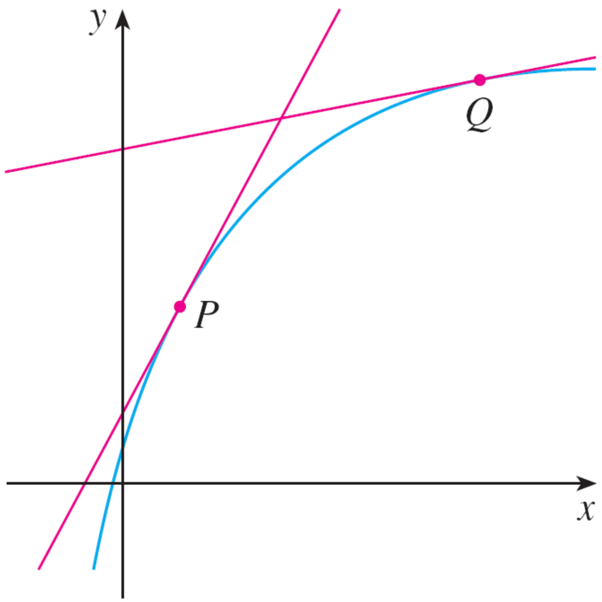


Figure 9

The y-values are changing rapidly at P and slowly at Q.

Rates of Change

When the derivative is small, the curve is relatively flat (as at point Q) and the y-values change slowly.

In particular, if $s = f(t)$ is the position function of a particle that moves along a straight line, then $f'(a)$ is the rate of change of the displacement s with respect to the time t.

In other words, $f'(a)$ *is the velocity of the particle at time* $t = a$.

The **speed** of the particle is the absolute value of the velocity, that is, $|f'(a)|$.

Example 6

A manufacturer produces bolts of a fabric with a fixed width. The cost of producing x yards of this fabric is $C = f(x)$ dollars.

(a) What is the meaning of the derivative $f'(x)$? What are its units?

(b) In practical terms, what does it mean to say that $f'(1000) = 9$?

(c) Which do you think is greater, $f'(50)$ or $f'(500)$? What about $f'(5000)$?

Example 6(a) – *Solution*

The derivative $f'(x)$ is the instantaneous rate of change of C with respect to x; that is, $f'(x)$ means the rate of change of the production cost with respect to the number of yards produced.

Because

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x}$$

the units for $f'(x)$ are the same as the units for the difference quotient $\Delta C/\Delta x$.

Since ΔC is measured in dollars and Δx in yards, it follows that the units for $f'(x)$ are dollars per yard.

Example 6(b) – *Solution*

cont'd

The statement that $f'(1000) = 9$ means that, after 1000 yards of fabric have been manufactured, the rate at which the production cost is increasing is \$9/yard.
(When $x = 1000$, C is increasing 9 times as fast as x.)

Since $\Delta x = 1$ is small compared with $x = 1000$, we could use the approximation

$$f'(1000) \approx \frac{\Delta C}{\Delta x} = \frac{\Delta C}{1} = \Delta C$$

and say that the cost of manufacturing the 1000th yard (or the 1001st) is about \$9.

Example 6(c) – *Solution*

cont'd

The rate at which the production cost is increasing (per yard) is probably lower when $x = 500$ than when $x = 50$ (the cost of making the 500th yard is less than the cost of the 50th yard) because of economies of scale. (The manufacturer makes more efficient use of the fixed costs of production.)

So

$$f'(50) > f'(500)$$

Example 6(c) – *Solution*

cont'd

But, as production expands, the resulting large-scale operation might become inefficient and there might be overtime costs.

Thus it is possible that the rate of increase of costs will eventually start to rise.

So it may happen that

$$f'(5000) > f'(500)$$

2.2 The Derivative as a Function

The Derivative as a Function

We have considered the derivative of a function *f* at a fixed number *a*:

1 $$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Here we change our point of view and let the number a vary. If we replace a in Equation 1 by a variable x, we obtain

2 $$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}}$$

The Derivative as a Function

Given any number x for which this limit exists, we assign to x the number $f'(x)$. So we can regard f' as a new function, called the **derivative of *f*** and defined by Equation 2.

We know that the value of f' at x, $f'(x)$, can be interpreted geometrically as the slope of the tangent line to the graph of f at the point $(x, f(x))$.

The function f' is called the derivative of f because it has been "derived" from f by the limiting operation in Equation 2. The domain of f' is the set $\{x \mid f'(x) \text{ exists}\}$ and may be smaller than the domain of f.

Example 1 – *Derivative of a Function given by a Graph*

The graph of a function f is given in Figure 1. Use it to sketch the graph of the derivative f'.

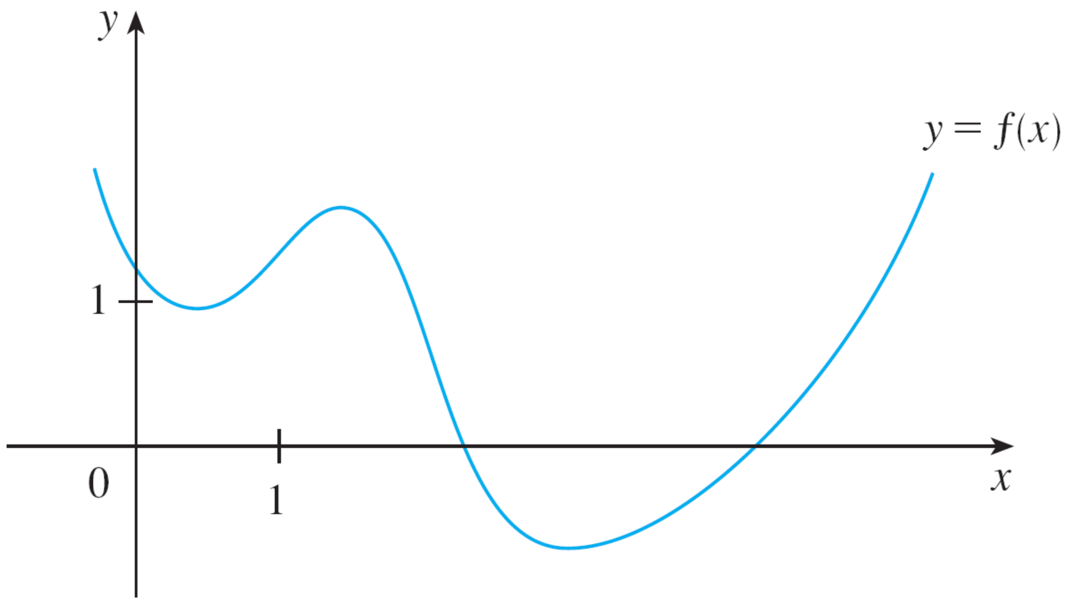


Figure 1

Example 1 – *Solution*

We can estimate the value of the derivative at any value of x by drawing the tangent at the point $(x, f(x))$ and estimating its slope. For instance, for $x = 5$ we draw the tangent at P in Figure 2(a) and estimate its slope to be about $\frac{3}{2}$, so $f'(5) \approx 1.5$.

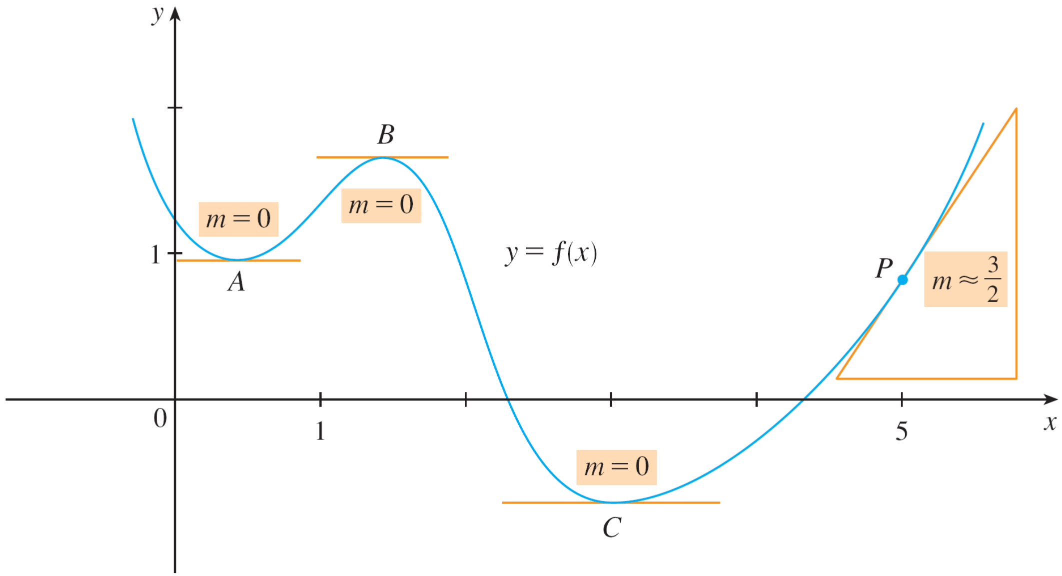


Figure 2(a)

Example 1 – *Solution*

cont'd

This allows us to plot the point $P'(5, 1.5)$ on the graph of f' directly beneath P. Repeating this procedure at several points, we get the graph shown in Figure 2(b).

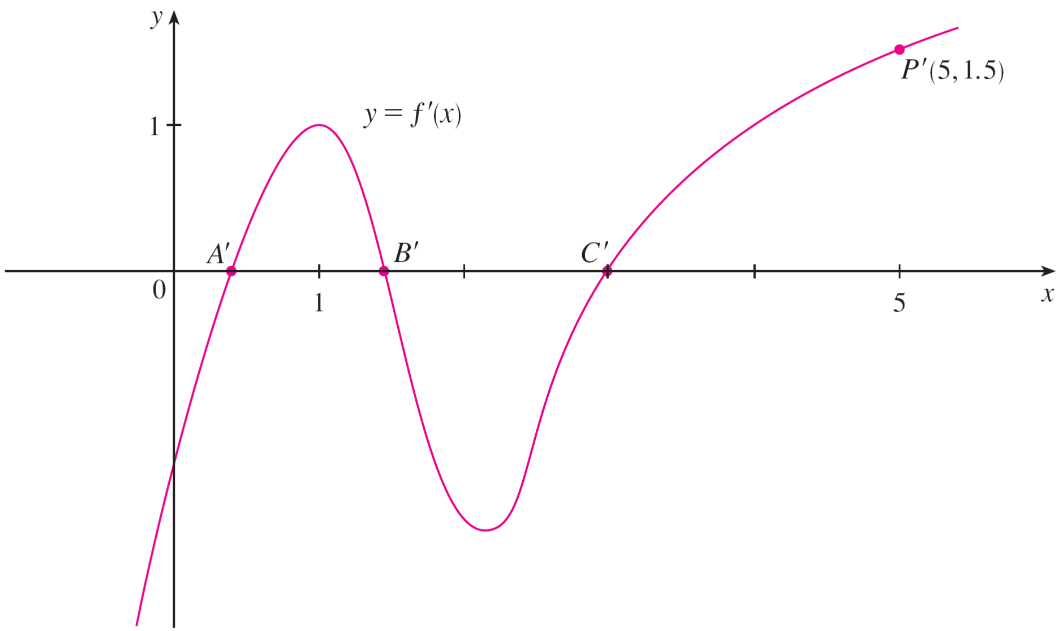


Figure 2(b)

Example 1 – *Solution*

cont'd

Notice that the tangents at A, B, and C are horizontal, so the derivative is 0 there and the graph of f' crosses the x-axis at the points A', B', and C', directly beneath A, B, and C.

Between A and B the tangents have positive slope, so $f'(x)$ is positive there. But between B and C the tangents have negative slope, so $f'(x)$ is negative there.

Other Notations

Other Notations

If we use the traditional notation $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

The symbols D and d/dx are called **differentiation operators** because they indicate the operation of **differentiation**, which is the process of calculating a derivative.

Other Notations

The symbol *dy*/*dx*, which was introduced by Leibniz, should not be regarded as a ratio (for the time being); it is simply a synonym for $f'(x)$. Nonetheless, it is a very useful and suggestive notation, especially when used in conjunction with increment notation.

We can rewrite the definition of derivative in Leibniz notation in the form

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Other Notations

If we want to indicate the value of a derivative dy/dx in Leibniz notation at a specific number a, we use the notation

$$\left.\frac{dy}{dx}\right|_{x=a} \qquad \text{or} \qquad \left.\frac{dy}{dx}\right]_{x=a}$$

which is a synonym for $f'(a)$.

$\boxed{3}$ **Definition** A function f is **differentiable at *a*** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Example 5

Where is the function $f(x) = |x|$ differentiable?

Solution:

If $x > 0$, then $|x| = x$ and we can choose h small enough that $x + h > 0$ and hence $|x + h| = x + h$.

Therefore, for $x > 0$, we have

$$f'(x) = \lim_{h \to 0} \frac{|x + h| - |x|}{h} = \lim_{h \to 0} \frac{(x + h) - x}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

and so f is differentiable for any $x > 0$.

Example 5 – *Solution*

cont'd

Similarly, for $x < 0$ we have $|x| = -x$ and h can be chosen small enough that $x + h < 0$ and so $|x + h| = -(x + h)$.

Therefore, for $x < 0$,

$$f'(x) = \lim_{h \to 0} \frac{|x + h| - |x|}{h}$$

$$= \lim_{h \to 0} \frac{-(x + h) - (-x)}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} (-1) = -1$$

and so f is differentiable for any $x < 0$.

Example 5 – *Solution*

cont'd

For $x = 0$ we have to investigate

$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|0 + h| - |0|}{h} \qquad \text{(if it exists)}$$

Let's compute the left and right limits separately:

$$\lim_{h \to 0^+} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

and

$$\lim_{h \to 0^-} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1$$

Example 5 – *Solution*

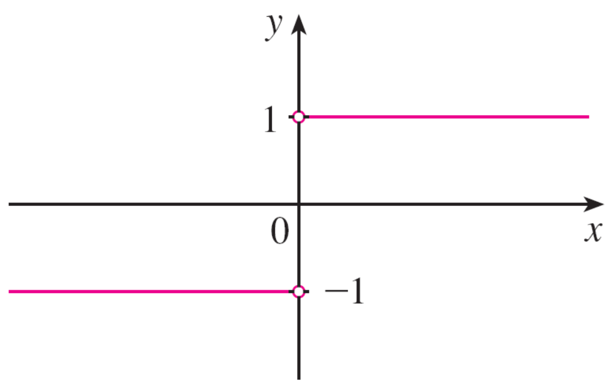
cont'd

Since these limits are different, $f'(0)$ does not exist. Thus f is differentiable at all x except 0.

A formula for f' is given by

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

and its graph is shown in Figure 5(b).



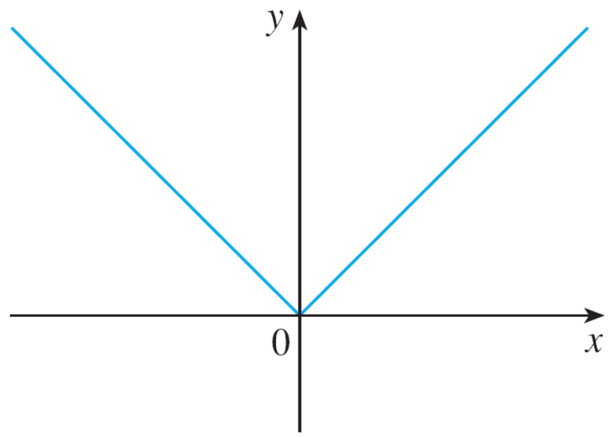
$y = f'(x)$

Figure 5(b)

Example 5 – *Solution*

cont'd

The fact that $f'(0)$ does not exist is reflected geometrically in the fact that the curve $y = |x|$ does not have a tangent line at (0, 0). [See Figure 5(a).]



$y = f(x) = |x|$

Figure 5(a)

Other Notations

Both continuity and differentiability are desirable properties for a function to have. The following theorem shows how these properties are related.

> **4 Theorem** If f is differentiable at a, then f is continuous at a.

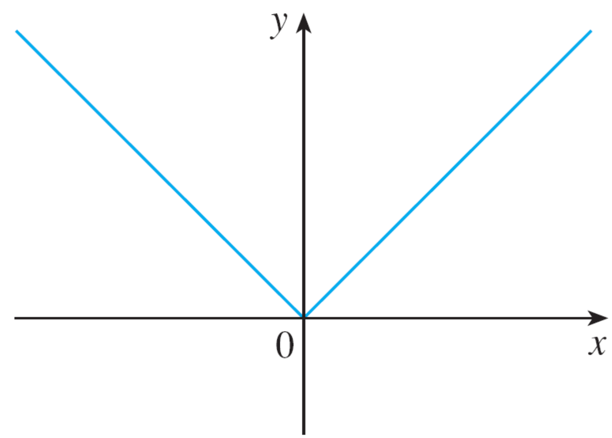
The converse of Theorem 4 is false; that is, there are functions that are continuous but not differentiable.

How Can a Function Fail to Be Differentiable?

How Can a Function Fail to Be Differentiable?

We saw that the function $y = |x|$ in Example 5 is not differentiable at 0 and Figure 5(a) shows that its graph changes direction abruptly when $x = 0$.

In general, if the graph of a function f has a "corner" or "kink" in it, then the graph of f has no tangent at this point and f is not differentiable there. [In trying to compute $f'(a)$, we find that the left and right limits are different.]



$y = f(x) = |x|$

Figure 5(a)

How Can a Function Fail to Be Differentiable?

Theorem 4 gives another way for a function not to have a derivative. It says that if f is not continuous at a, then f is not differentiable at a. So at any discontinuity (for instance, a jump discontinuity) f fails to be differentiable.

A third possibility is that the curve has a **vertical tangent line** when $x = a$; that is, f is continuous at a and

$$\lim_{x \to a} |f'(x)| = \infty$$

How Can a Function Fail to Be Differentiable?

This means that the tangent lines become steeper and steeper as $x \longrightarrow a$. Figure 6 shows one way that this can happen; Figure 7(c) shows another.

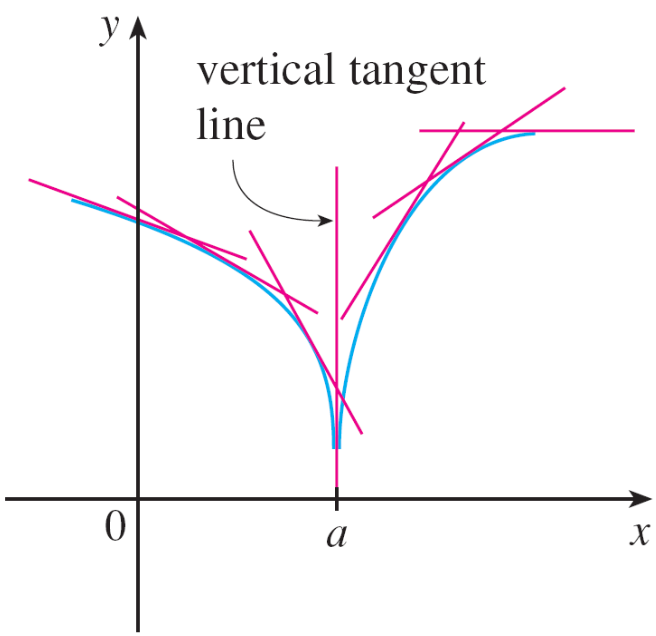


Figure 6

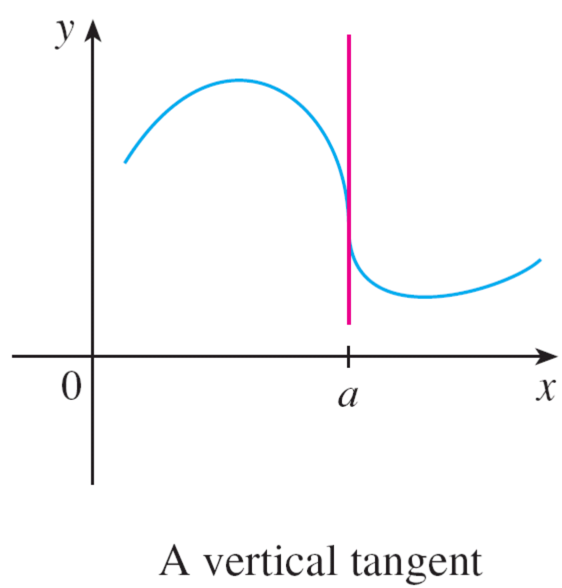
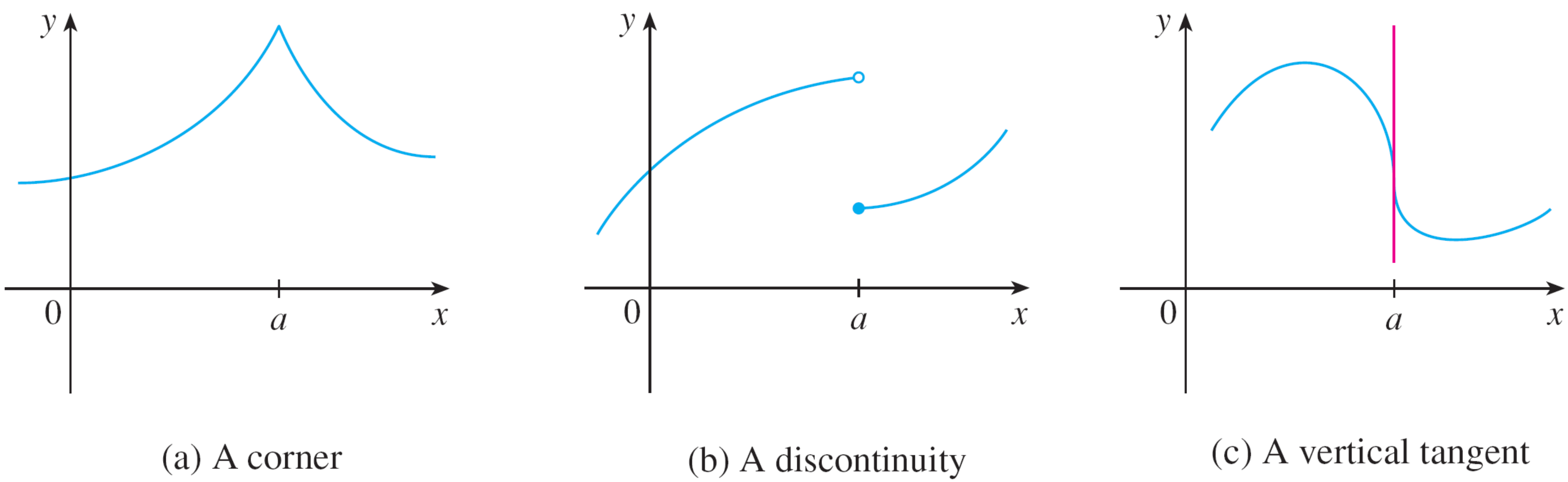


Figure 7(c)

How Can a Function Fail to Be Differentiable?

Figure 7 illustrates the three possibilities that we have discussed.



Three ways for *f* not to be differentiable at *a*

Figure 7

Higher Derivatives

Higher Derivatives

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the **second derivative** of f because it is the derivative of the derivative of f.

Using Leibniz notation, we write the second derivative of $y = f(x)$ as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Example 6

If $f(x) = x^3 - x$, find and interpret $f''(x)$.

Solution:

The first derivative of $f(x) = x^3 - x$ is $f'(x) = 3x^2 - 1$.

So the second derivative is

$$f''(x) = (f')'(x)$$

$$= \lim_{h \to 0} \frac{f'(x + h) - f'(x)}{h}$$

$$= \lim_{h \to 0} \frac{[3(x + h)^2 - 1] - [3x^2 - 1]}{h}$$

Example 6 – *Solution*

cont'd

$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 1 - 3x^2 + 1}{h}$$

$$= \lim_{h \to 0} (6x + 3h)$$

$$= 6x$$

The graphs of *f*, *f*′, and *f*″ are shown in Figure 10.

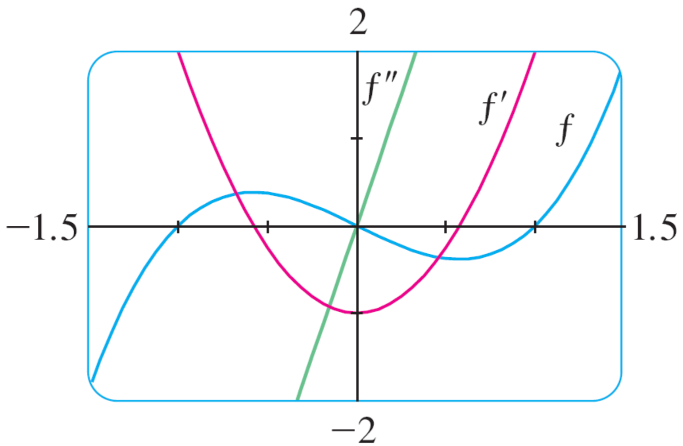


Figure 10

Example 6 – *Solution*

cont'd

We can interpret $f''(x)$ as the slope of the curve $y = f'(x)$ at the point $(x, f'(x))$. In other words, it is the rate of change of the slope of the original curve $y = f(x)$.

Notice from Figure 10 that $f''(x)$ is negative when $y = f'(x)$ has negative slope and positive when $y = f'(x)$ has positive slope. So the graphs Serve as a check on our calculations.

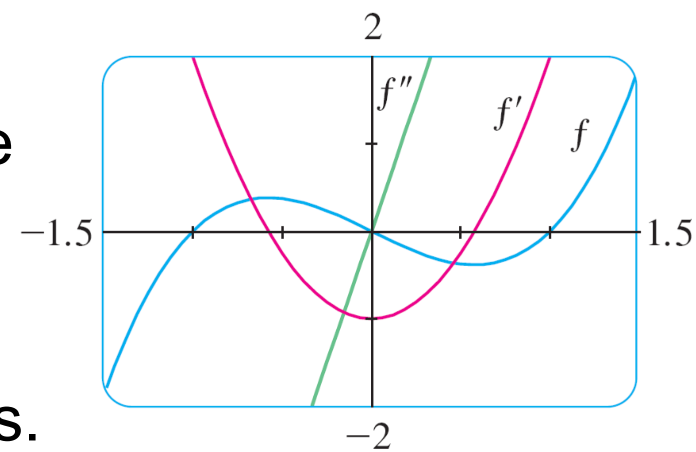


Figure 10

Higher Derivatives

In general, we can interpret a second derivative as a rate of change of a rate of change. The most familiar example of this is *acceleration,* which we define as follows.

If $s = s(t)$ is the position function of an object that moves in a straight line, we know that its first derivative represents the velocity $v(t)$ of the object as a function of time:

$$v(t) = s'(t) = \frac{ds}{dt}$$

Higher Derivatives

The instantaneous rate of change of velocity with respect to time is called the **acceleration** $a(t)$ of the object. Thus the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

$$a(t) = v'(t) = s''(t)$$

or, in Leibniz notation,

$$a = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$

Higher Derivatives

The **third derivative** f''' is the derivative of the second derivative: $f''' = (f'')'$. So $f'''(x)$ can be interpreted as the slope of the curve $y = f''(x)$ or as the rate of change of $f''(x)$.

If $y = f(x)$, then alternative notations for the third derivative are

$$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

Higher Derivatives

The process can be continued. The fourth derivative f'''' is usually denoted by $f^{(4)}$.

In general, the nth derivative of f is denoted by $f^{(n)}$ and is obtained from f by differentiating n times.

If $y = f(x)$, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

Higher Derivatives

We can also interpret the third derivative physically in the case where the function is the position function $s = s(t)$ of an object that moves along a straight line.

Because $s''' = (s'')' = a'$, the third derivative of the position function is the derivative of the acceleration function and is called the **jerk**:

$$j = \frac{da}{dt} = \frac{d^3s}{dt^3}$$

Higher Derivatives

Thus the jerk j is the rate of change of acceleration.

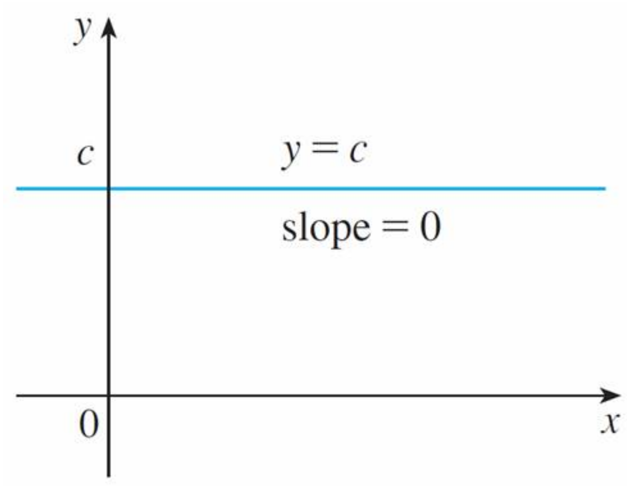
It is aptly named because a large jerk means a sudden change in acceleration, which causes an abrupt movement in a vehicle.

2.3 Differentiation Formulas

Differentiation Formulas

Let's start with the simplest of all functions, the constant function $f(x) = c$. The graph of this function is the horizontal line $y = c$, which has slope 0, so we must have $f'(x) = 0$.

See Figure 1.



The graph of $f(x) = c$ is the line $y = c$, so $f'(x) = 0$.

Figure 1

Differentiation Formulas

A formal proof, from the definition of a derivative, is also easy:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0$$

In Leibniz notation, we write this rule as follows.

Derivative of a Constant Function

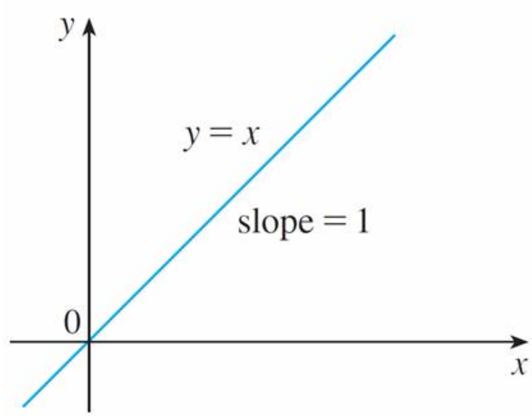
$$\frac{d}{dx}(c) = 0$$

Power Functions

Power Functions

We next look at the functions $f(x) = x^n$, where n is a positive integer. If $n = 1$, the graph of $f(x) = x$ is the line $y = x$, which has slope 1.

(See Figure 2.)



The graph of $f(x) = x$ is the line $y = x$, so $f'(x) = 1$.

Figure 2

Power Functions

So

1 $$\frac{d}{dx}(x) = 1$$

We have already investigated the cases $n = 2$ and $n = 3$. In fact, we found that

2 $$\frac{d}{dx}(x^2) = 2x \qquad \frac{d}{dx}(x^3) = 3x^2$$

Power Functions

For n = 4 we find the derivative of $f(x) = x^4$ as follows:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)^4 - x^4}{h}$$

$$= \lim_{h \to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h}$$

$$= \lim_{h \to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h}$$

$$= \lim_{h \to 0} (4x^3 + 6x^2h + 4xh^2 + h^3) = 4x^3$$

Power Functions

Thus

$$\boxed{3} \qquad \frac{d}{dx}(x^4) = 4x^3$$

Comparing the equations in $\boxed{1}$, $\boxed{2}$, and $\boxed{3}$, we see a pattern emerging. It seems to be a reasonable guess that, when n is a positive integer, $(d/dx)(x^n) = nx^{n-1}$.

This turns out to be true. We prove it in two ways; the second proof uses the Binomial Theorem.

The Power Rule If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Example 1

(a) If $f(x) = x^6$, then $f'(x) = 6x^5$.

(b) If $y = x^{1000}$, then $y' = 1000x^{999}$.

(c) If $y = t^4$, then $\dfrac{dy}{dt} = 4t^3$.

(d) $\dfrac{d}{dr}(r^3) = 3r^2$

New Derivatives from Old

New Derivatives from Old

When new functions are formed from old functions by addition, subtraction, or multiplication by a constant, their derivatives can be calculated in terms of derivatives of the old functions.

In particular, the following formula says that *the derivative of a constant times a function is the constant times the derivative of the function.*

The Constant Multiple Rule If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

Example 2

(a) $\dfrac{d}{dx}(3x^4) = 3\dfrac{d}{dx}(x^4)$

$= 3(4x^3)$

$= 12x^3$

(b) $\dfrac{d}{dx}(-x) = \dfrac{d}{dx}[(-1)x]$

$= (-1)\dfrac{d}{dx}(x)$

$= -1(1)$

$= -1$

New Derivatives from Old

The next rule tells us that *the derivative of a sum of functions is the sum of the derivatives.*

The Sum Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

The Sum Rule can be extended to the sum of any number of functions. For instance, using this theorem twice, we get

$$(f + g + h)' = [(f + g) + h]' = (f + g)' + h' = f' + g' + h'$$

New Derivatives from Old

By writing $f - g$ as $f + (-1)g$ and applying the Sum Rule and the Constant Multiple Rule, we get the following formula.

The Difference Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

The Constant Multiple Rule, the Sum Rule, and the Difference Rule can be combined with the Power Rule to differentiate any polynomial, as the following examples demonstrate.

Example 3

$$\frac{d}{dx}(x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5)$$

$$= \frac{d}{dx}(x^8) + 12\frac{d}{dx}(x^5) - 4\frac{d}{dx}(x^4) + 10\frac{d}{dx}(x^3) - 6\frac{d}{dx}(x) + \frac{d}{dx}(5)$$

$$= 8x^7 + 12(5x^4) - 4(4x^3) + 10(3x^2) - 6(1) + 0$$

$$= 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6$$

New Derivatives from Old

Next we need a formula for the derivative of a product of two functions. By analogy with the Sum and Difference Rules, one might be tempted to guess, as Leibniz did three centuries ago, that the derivative of a product is the product of the derivatives.

We can see, however, that this guess is wrong by looking at a particular example. Let $f(x) = x$ and $g(x) = x^2$. Then the Power Rule gives $f'(x) = 1$ and $g'(x) = 2x$. But $(fg)(x) = x^3$, so

$$(fg)'(x) = 3x^2.$$

Thus $(fg)' \neq f'g'$.

New Derivatives from Old

The correct formula was discovered by Leibniz and is called the Product Rule.

The Product Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

In words, the Product Rule says that *the derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function*.

Example 6

Find $F'(x)$ if $F(x) = (6x^3)(7x^4)$.

Solution:

By the Product Rule, we have

$$F'(x) = (6x^3)\frac{d}{dx}(7x^4) + (7x^4)\frac{d}{dx}(6x^3)$$

$$= (6x^3)(28x^3) + (7x^4)(18x^2)$$

$$= 168x^6 + 126x^6$$

$$= 294x^6$$

New Derivatives from Old

The Quotient Rule If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\dfrac{d}{dx}[f(x)] - f(x)\dfrac{d}{dx}[g(x)]}{[g(x)]^2}$$

In words, the Quotient Rule says that the *derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

Example 8

Let $y = \dfrac{x^2 + x - 2}{x^3 + 6}$. Then

$$y' = \frac{(x^3 + 6)\dfrac{d}{dx}(x^2 + x - 2) - (x^2 + x - 2)\dfrac{d}{dx}(x^3 + 6)}{(x^3 + 6)^2}$$

$$= \frac{(x^3 + 6)(2x + 1) - (x^2 + x - 2)(3x^2)}{(x^3 + 6)^2}$$

$$= \frac{(2x^4 + x^3 + 12x + 6) - (3x^4 + 3x^3 - 6x^2)}{(x^3 + 6)^2}$$

$$= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3 + 6)^2}$$

New Derivatives from Old

Note:
Don't use the Quotient Rule *every* time you see a quotient.

Sometimes it's easier to rewrite a quotient first to put it in a form that is simpler for the purpose of differentiation.

For instance, although it is possible to differentiate the function

$$F(x) = \frac{3x^2 + 2\sqrt{x}}{x}$$

using the Quotient Rule.

New Derivatives from Old

It is much easier to perform the division first and write the function as

$$F(x) = 3x + 2x^{-1/2}$$

before differentiating.

General Power Functions

General Power Functions

The Quotient Rule can be used to extend the Power Rule to the case where the exponent is a negative integer.

If n is a positive integer, then

$$\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$$

Example 9

(a) If $y = \dfrac{1}{x}$, then $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^{-1})$

$$= -x^{-2}$$

$$= -\frac{1}{x^2}$$

(b) $\dfrac{d}{dt}\left(\dfrac{6}{t^3}\right) = 6\dfrac{d}{dt}(t^{-3})$

$$= 6(-3)t^{-4}$$

$$= -\frac{18}{t^4}$$

General Power Functions

The Power Rule (General Version) If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Example 11

Differentiate the function $f(t) = \sqrt{t}\,(a + bt)$.

Solution 1:

Using the Product Rule, we have

$$f'(t) = \sqrt{t}\,\frac{d}{dt}(a + bt) + (a + bt)\frac{d}{dt}\left(\sqrt{t}\right)$$

$$= \sqrt{t} \cdot b + (a + bt) \cdot \tfrac{1}{2}t^{-1/2}$$

$$= b\sqrt{t} + \frac{a + bt}{2\sqrt{t}} = \frac{a + 3bt}{2\sqrt{t}}$$

Example 11 – *Solution 2*

cont'd

If we first use the laws of exponents to rewrite $f(t)$, then we can proceed directly without using the Product Rule.

$$f(t) = a\sqrt{t} + bt\sqrt{t} = at^{1/2} + bt^{3/2}$$

$$f'(t) = \tfrac{1}{2}at^{-1/2} + \tfrac{3}{2}bt^{1/2}$$

General Power Functions

The differentiation rules enable us to find tangent lines without having to resort to the definition of a derivative.

They also enable us to find *normal lines*.

The **normal line** to a curve C at point P is the line through P that is perpendicular to the tangent line at P.

Example 12

Find equations of the tangent line and normal line to the curve

$y = \sqrt{x}/(1 + x^2)$ at the point $(1, \frac{1}{2})$.

Solution:

According to the Quotient Rule, we have

$$\frac{dy}{dx} = \frac{(1 + x^2)\dfrac{d}{dx}\left(\sqrt{x}\right) - \sqrt{x}\,\dfrac{d}{dx}(1 + x^2)}{(1 + x^2)^2}$$

$$= \frac{(1 + x^2)\dfrac{1}{2\sqrt{x}} - \sqrt{x}\,(2x)}{(1 + x^2)^2}$$

Example 12 – *Solution*

cont'd

$$= \frac{(1 + x^2) - 4x^2}{2\sqrt{x}\,(1 + x^2)^2} = \frac{1 - 3x^2}{2\sqrt{x}\,(1 + x^2)^2}$$

So the slope of the tangent line at $(1, \frac{1}{2})$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{1 - 3 \cdot 1^2}{2\sqrt{1}\,(1 + 1^2)^2} = -\frac{1}{4}$$

We use the point-slope form to write an equation of the tangent line at $(1, \frac{1}{2})$:

$$y - \tfrac{1}{2} = -\tfrac{1}{4}(x - 1) \quad \text{or} \quad y = -\tfrac{1}{4}x + \tfrac{3}{4}$$

Example 12 – *Solution*

cont'd

The slope of the normal line at $(1, \frac{1}{2})$ is the negative reciprocal of $-\frac{1}{4}$, namely 4, so an equation is

$$y - \frac{1}{2} = 4(x - 1) \qquad \text{or} \qquad y = 4x - \frac{7}{2}$$

The curve and its tangent and normal lines are graphed in Figure 5.

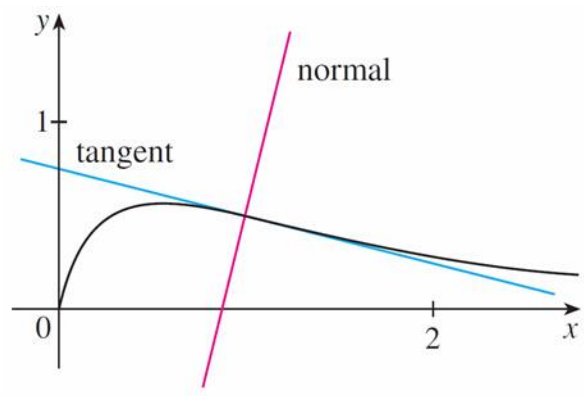


Figure 5

General Power Functions

We summarize the differentiation formulas we have learned so far as follows.

Table of Differentiation Formulas

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x^n) = nx^{n-1}$$

$$(cf)' = cf' \qquad (f+g)' = f' + g' \qquad (f-g)' = f' - g'$$

$$(fg)' = fg' + gf' \qquad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

2.4 Derivatives of Trigonometric Functions

Derivatives of Trigonometric Functions

In particular, it is important to remember that when we talk about the function f defined for all real numbers x by

$$f(x) = \sin x$$

it is understood that $\sin x$ means the sine of the angle whose *radian* measure is x. A similar convention holds for the other trigonometric functions cos, tan, csc, sec, and cot.

All of the trigonometric functions are continuous at every number in their domains.

Derivatives of Trigonometric Functions

If we sketch the graph of the function $f(x) = \sin x$ and use the interpretation of $f'(x)$ as the slope of the tangent to the sine curve in order to sketch the graph of f', then it looks as if the graph of f' may be the same as the cosine curve. (See Figure 1).

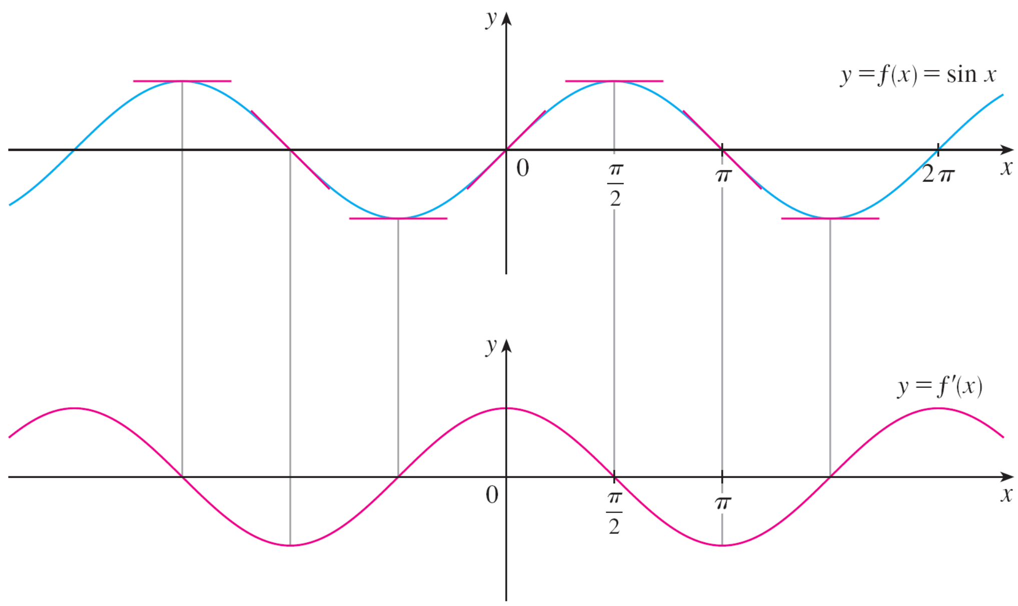


Figure 1

Derivatives of Trigonometric Functions

Let's try to confirm our guess that if $f(x) = \sin x$, then $f'(x) = \cos x$. From the definition of a derivative, we have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right]$$

Derivatives of Trigonometric Functions

$$= \lim_{h \to 0} \left[\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right]$$

$\boxed{1}$ $$= \lim_{h \to 0} \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \lim_{h \to 0} \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

Two of these four limits are easy to evaluate. Since we regard *x* as a constant when computing a limit as $h \to 0$, we have

$$\lim_{h \to 0} \sin x = \sin x \qquad \text{and} \qquad \lim_{h \to 0} \cos x = \cos x$$

Derivatives of Trigonometric Functions

The limit of (sin h)/h is not so obvious. We made the guess, on the basis of numerical and graphical evidence, that

$$\boxed{2} \qquad \boxed{\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1}$$

Derivatives of Trigonometric Functions

We now use a geometric argument to prove Equation 2. Assume first that θ lies between 0 and $\pi/2$. Figure 2(a) shows a sector of a circle with center O, central angle θ, and radius 1.

BC is drawn perpendicular to OA.
By the definition of radian measure, we have arc $AB = \theta$.
Also $|BC| = |OB| \sin\theta = \sin\theta$.

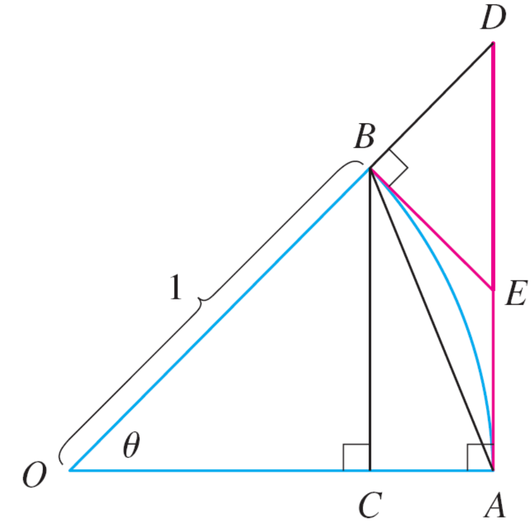


Figure 2(a)

Derivatives of Trigonometric Functions

From the diagram we see that

$$|BC| < |AB| < \text{arc } AB$$

Therefore $\sin\theta < \theta$ so $\dfrac{\sin\theta}{\theta} < 1$

Let the tangent lines at *A* and *B* intersect at *E*. You can see from Figure 2(b) that the circumference of a circle is smaller than the length of a circumscribed polygon, and so arc $AB < |AE| + |EB|$.

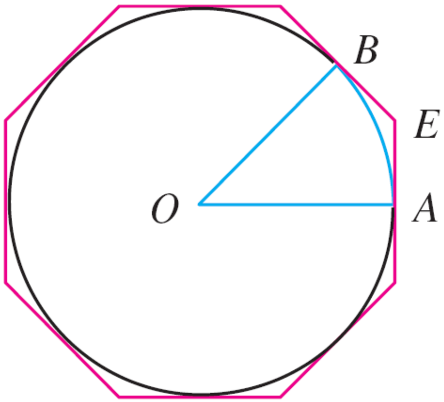


Figure 2(b)

Derivatives of Trigonometric Functions

Thus

$$\begin{aligned}\theta = \text{arc } AB &< |AE| + |EB| \\ &< |AE| + |ED| \\ &= |AD| = |OA| \tan \theta \\ &= \tan \theta\end{aligned}$$

Therefore we have

$$\theta < \frac{\sin \theta}{\cos \theta}$$

so

$$\cos \theta < \frac{\sin \theta}{\theta} < 1$$

Derivatives of Trigonometric Functions

We know that $\lim_{\theta \to 0} 1 = 1$ and $\lim_{\theta \to 0} \cos \theta = 1$, so by the Squeeze Theorem, we have

$$\lim_{\theta \to 0^+} \frac{\sin \theta}{\theta} = 1$$

But the function $(\sin \theta)/\theta$ is an even function, so its right and left limits must be equal. Hence, we have

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

so we have proved Equation 2.

Derivatives of Trigonometric Functions

We can deduce the value of the remaining limit in $\boxed{1}$ as follows:

$$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = \lim_{\theta \to 0} \left(\frac{\cos \theta - 1}{\theta} \cdot \frac{\cos \theta + 1}{\cos \theta + 1} \right) = \lim_{\theta \to 0} \frac{\cos^2 \theta - 1}{\theta (\cos \theta + 1)}$$

$$= \lim_{\theta \to 0} \frac{-\sin^2 \theta}{\theta (\cos \theta + 1)} = -\lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta} \cdot \frac{\sin \theta}{\cos \theta + 1} \right)$$

$$= -\lim_{\theta \to 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \to 0} \frac{\sin \theta}{\cos \theta + 1}$$

$$= -1 \cdot \left(\frac{0}{1 + 1} \right) = 0$$

(by Equation 2)

Derivatives of Trigonometric Functions

$\boxed{3}$

$$\boxed{\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0}$$

If we now put the limits $\boxed{2}$ and $\boxed{3}$ in $\boxed{1}$, we get

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \lim_{h \to 0} \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h} \\ &= (\sin x) \cdot 0 + (\cos x) \cdot 1 = \cos x \end{aligned}$$

Derivatives of Trigonometric Functions

So we have proved the formula for the derivative of the sine function:

$$\boxed{4} \qquad \frac{d}{dx}(\sin x) = \cos x$$

Example 1

Differentiate $y = x^2 \sin x$.

Solution:
Using the Product Rule and Formula 4, we have

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2)$$

$$= x^2 \cos x + 2x \sin x$$

Derivatives of Trigonometric Functions

Using the same methods as in the proof of Formula 4, one can prove that

5 $$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

The tangent function can also be differentiated by using the definition of a derivative, but it is easier to use the Quotient Rule together with Formulas 4 and 5:

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

Derivatives of Trigonometric Functions

$$= \frac{\cos x \dfrac{d}{dx}(\sin x) - \sin x \dfrac{d}{dx}(\mathrm{co}}{\cos^2 x}$$

$$= \frac{\cos x \cdot \cos x - \sin x\,(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

Derivatives of Trigonometric Functions

6

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

The derivatives of the remaining trigonometric functions, csc, sec, and cot, can also be found easily using the Quotient Rule.

Derivatives of Trigonometric Functions

We collect all the differentiation formulas for trigonometric functions in the following table. Remember that they are valid only when *x* is measured in radians.

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Derivatives of Trigonometric Functions

Trigonometric functions are often used in modeling real-world phenomena. In particular, vibrations, waves, elastic motions, and other quantities that vary in a periodic manner can be described using trigonometric functions. In the following example we discuss an instance of simple harmonic motion.

Example 3

An object at the end of a vertical spring is stretched 4 cm beyond its rest position and released at time t = 0. (See Figure 5 and note that the downward direction is positive.)

Its position at time t is

$$s = f(t) = 4 \cos t$$

Find the velocity and acceleration at time t and use them to analyze the motion of the object.

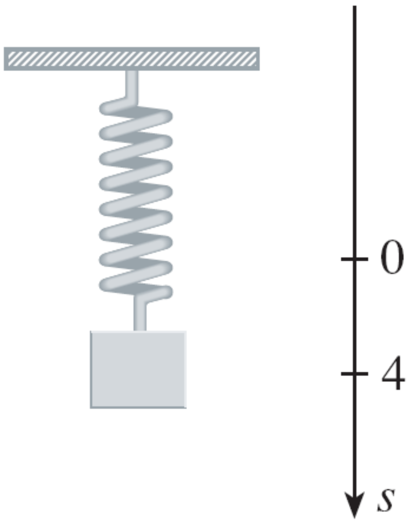


Figure 5

Example 3 – *Solution*

The velocity and acceleration are

$$
\begin{aligned}
v &= \frac{ds}{dt} \\
&= \frac{d}{dt}(4\cos t) \\
&= 4\frac{d}{dt}(\cos t) \\
&= -4\sin t
\end{aligned}
$$

Example 3 – *Solution*

cont'd

$$a = \frac{dv}{dt}$$

$$= \frac{d}{dt}(-4 \sin t)$$

$$= -4 \frac{d}{dt}(\sin t)$$

$$= -4 \cos t$$

The object oscillates from the lowest point (s = 4 cm) to the highest point (s = –4 cm). The period of the oscillation is 2π, the period of cos t.

Example 3 – *Solution*

cont'd

The speed is $|v| = 4\,|\sin t|$, which is greatest when $|\sin t| = 1$, that is, when $\cos t = 0$.

So the object moves fastest as it passes through its equilibrium position ($s = 0$). Its speed is 0 when $\sin t = 0$, that is, at the high and low points.

The acceleration $a = -4 \cos t = 0$ when $s = 0$. It has greatest magnitude at the high and low points. See the graphs in Figure 6.

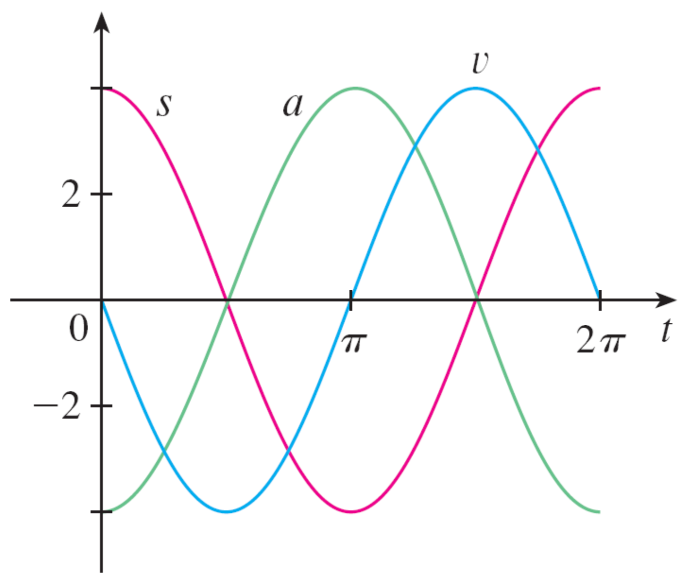


Figure 6

2.5 The Chain Rule

The Chain Rule

Suppose you are asked to differentiate the function

$$F(x) = \sqrt{x^2 + 1}$$

The differentiation formulas you learned in the previous sections of this chapter do not enable you to calculate $F'(x)$.

Observe that F is a composite function. In fact, if we let $y = f(u) = \sqrt{u}$ and let $u = g(x) = x^2 + 1$, then we can write $y = F(x) = f(g(x))$, that is, $F = f \circ g$.

We know how to differentiate both f and g, so it would be useful to have a rule that tells us how to find the derivative of $F = f \circ g$ in terms of the derivatives of f and g.

The Chain Rule

It turns out that the derivative of the composite function $f \circ g$ is the product of the derivatives of f and g. This fact is one of the most important of the differentiation rules and is called the *Chain Rule.*

It seems plausible if we interpret derivatives as rates of change. Regard du/dx as the rate of change of u with respect to x, dy/du as the rate of change of y with respect to u, and dy/dx as the rate of change of y with respect to x. If u changes twice as fast as x and y changes three times as fast as u, then it seems reasonable that y changes six times as fast as x, and so we expect that

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

The Chain Rule

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

The Chain Rule

The Chain Rule can be written either in the prime notation

$$\boxed{2} \qquad (f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

or, if $y = f(u)$ and $u = g(x)$, in Leibniz notation:

$$\boxed{3} \qquad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Equation 3 is easy to remember because if dy/du and du/dx were quotients, then we could cancel du.

Remember, however, that du has not been defined and du/dx should not be thought of as an actual quotient.

Example 1

Find $F'(x)$ if $F(x) = \sqrt{x^2 + 1}$.

Solution 1:

(Using Equation 2): We have expressed F as
$F(x) = (f \circ g)(x) = f(g(x))$ where $f(u) = \sqrt{u}$ and $g(x) = x^2 + 1$.

Since

$$f'(u) = \tfrac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}} \quad \text{and} \quad g'(x) = 2x$$

we have $F'(x) = f'(g(x)) \cdot g'(x)$

$$= \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x = \frac{x}{\sqrt{x^2 + 1}}$$

Example 1 – *Solution*

cont'd

(Using Equation 3): If we let $u = x^2 + 1$ and $y = \sqrt{u}$, then

$$
\begin{aligned}
F'(x) &= \frac{dy}{du}\frac{du}{dx} \\
&= \frac{1}{2\sqrt{u}}(2x) \\
&= \frac{1}{2\sqrt{x^2 + 1}}(2x) \\
&= \frac{x}{\sqrt{x^2 + 1}}
\end{aligned}
$$

The Chain Rule

When using Formula 3 we should bear in mind that *dy*/*dx* refers to the derivative of *y* when *y* is considered as a function of *x* (called the *derivative of y with respect to x*), whereas *dy*/*du* refers to the derivative of *y* when considered as a function of *u* (the derivative of *y* with respect to *u*). For instance, in Example 1, *y* can be considered as a function of *x* ($y = \sqrt{x^2 + 1}$) and also as a function of *u* ($y = \sqrt{u}$).

Note that

$$\frac{dy}{dx} = F'(x) = \frac{x}{\sqrt{x^2 + 1}} \quad \text{whereas} \quad \frac{dy}{du} = f'(u) = \frac{1}{2\sqrt{u}}$$

The Chain Rule

In general, if $y = \sin u$, where u is a differentiable function of x, then, by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \cos u \frac{du}{dx}$$

Thus

$$\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$$

In a similar fashion, all of the formulas for differentiating trigonometric functions can be combined with the Chain Rule.

The Chain Rule

Let's make explicit the special case of the Chain Rule where the outer function *f* is a power function.

If $y = [g(x)]^n$, then we can write $y = f(u) = u^n$ where $u = g(x)$. By using the Chain Rule and then the Power Rule, we get

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = nu^{n-1}\frac{du}{dx} = n[g(x)]^{n-1}g'(x)$$

4 The Power Rule Combined with the Chain Rule If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Alternatively, $$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

Example 3

Differentiate $y = (x^3 - 1)^{100}$.

Solution:

Taking $u = g(x) = x^3 - 1$ and $n = 100$ in $\boxed{4}$, we have

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 1)^{100}$$

$$= 100(x^3 - 1)^{99}\frac{d}{dx}(x^3 - 1)$$

$$= 100(x^3 - 1)^{99} \cdot 3x^2$$

$$= 300x^2(x^3 - 1)^{99}$$

How to Prove the Chain Rule

How to Prove the Chain Rule

Recall that if $y = f(x)$ and x changes from a to $a + \Delta x$, we defined the increment of y as

$$\Delta y = f(a + \Delta x) - f(a)$$

According to the definition of a derivative, we have

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(a)$$

So if we denote by ε the difference between the difference quotient and the derivative, we obtain

$$\lim_{\Delta x \to 0} \varepsilon = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x} - f'(a) \right) = f'(a) - f'(a) = 0$$

How to Prove the Chain Rule

But

$$\varepsilon = \frac{\Delta y}{\Delta x} - f'(a) \qquad \Rightarrow \qquad \Delta y = f'(a)\,\Delta x + \varepsilon\,\Delta x$$

If we define ε to be 0 when $\Delta x = 0$, then ε becomes a continuous function of Δx. Thus, for a differentiable function f, we can write

$$\boxed{5} \qquad \Delta y = f'(a)\,\Delta x + \varepsilon\,\Delta x \quad \text{where} \quad \varepsilon \to 0 \text{ as } \Delta x \to 0$$

and ε is a continuous function of Δx. This property of differentiable functions is what enables us to prove the Chain Rule.

2.6 Implicit Differentiation

Implicit Differentiation

The functions that we have met so far can be described by expressing one variable explicitly in terms of another variable—for example,

$$y = \sqrt{x^3 + 1} \quad \text{or} \quad y = x \sin x$$

or, in general, $y = f(x)$.

Some functions, however, are defined implicitly by a relation between x and y such as

$$\boxed{1} \qquad x^2 + y^2 = 25$$

or

$$\boxed{2} \qquad x^3 + y^3 = 6xy$$

Implicit Differentiation

In some cases it is possible to solve such an equation for *y* as an explicit function (or several functions) of *x*.

For instance, if we solve Equation 1 for *y*, we get $y = \pm\sqrt{25 - x^2}$, so two of the functions determined by the implicit Equation 1 are $f(x) = \sqrt{25 - x^2}$ and $g(x) = -\sqrt{25 - x^2}$.

The graphs of *f* and *g* are the upper and lower semicircles of the circle $x^2 + y^2 = 25$. (See Figure 1.)

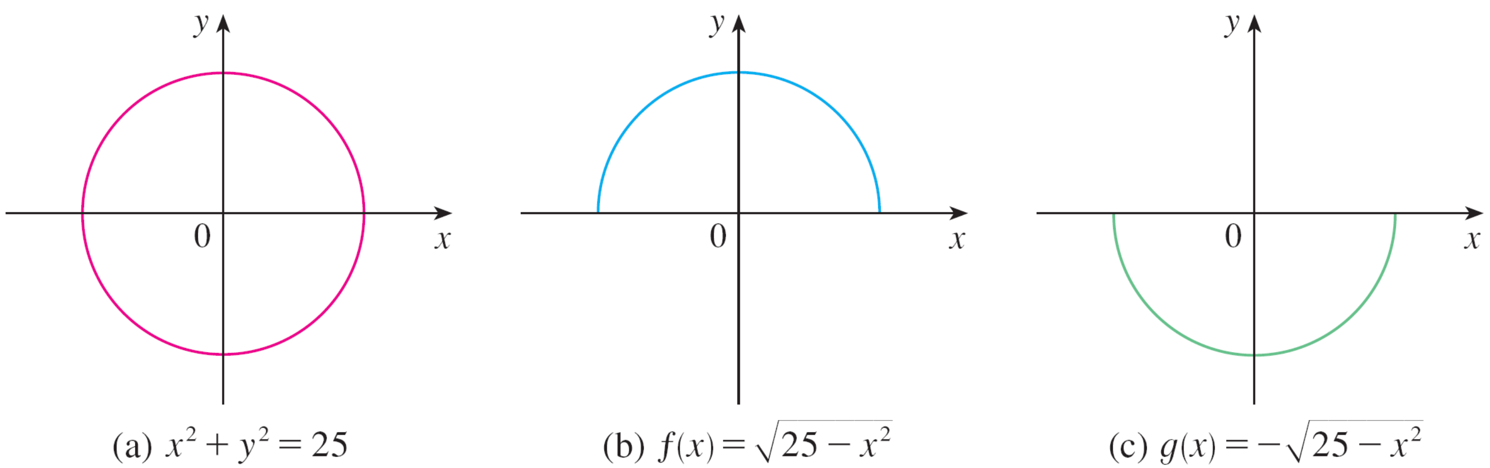
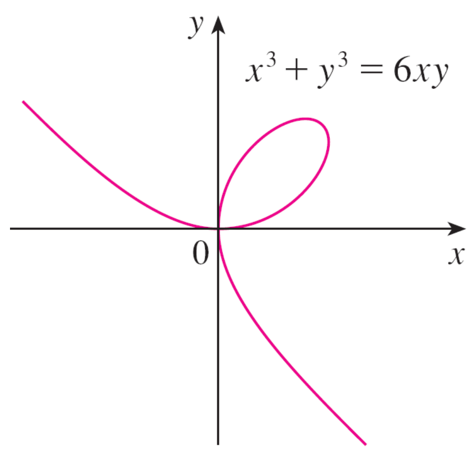


Figure 1

Implicit Differentiation

It's not easy to solve Equation 2 for *y* explicitly as a function of *x* by hand. (A computer algebra system has no trouble, but the expressions it obtains are very complicated.)

Nonetheless, (2) is the equation of a curve called the **folium of Descartes** shown in Figure 2 and it implicitly defines *y* as several functions of *x*.

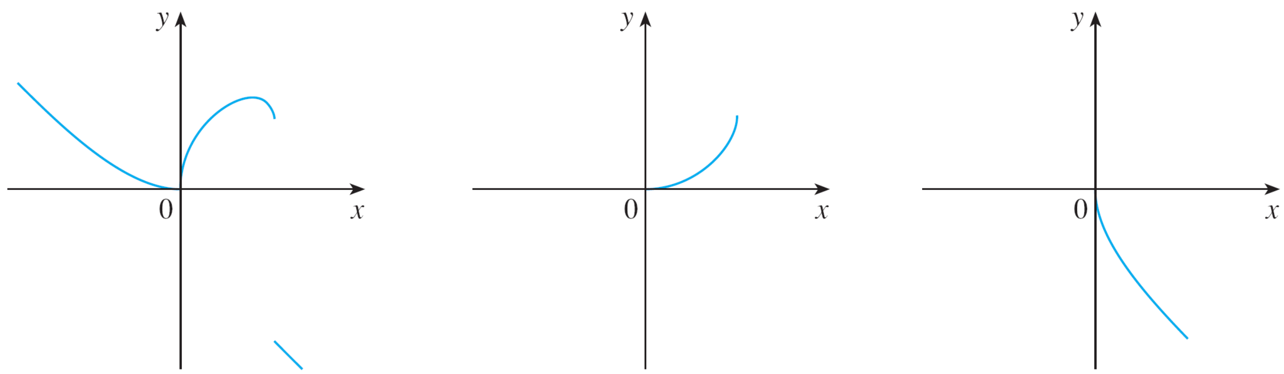


The folium of Descartes

Figure 2

Implicit Differentiation

The graphs of three such functions are shown in Figure 3.



Graphs of three functions defined by the folium of Descartes

Figure 3

When we say that *f* is a function defined implicitly by Equation 2, we mean that the equation

$$x^3 + [f(x)]^3 = 6x f(x)$$

is true for all values of *x* in the domain of *f*.

Implicit Differentiation

Fortunately, we don't need to solve an equation for y in terms of x in order to find the derivative of y. Instead we can use the method of **implicit differentiation**.

This consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for y'.

In the examples and exercises of this section it is always assumed that the given equation determines y implicitly as a differentiable function of x so that the method of implicit differentiation can be applied.

Example 1

(a) If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.

(b) Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point (3, 4).

Solution 1:

(a) Differentiate both sides of the equation $x^2 + y^2 = 25$:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

Example 1 – *Solution*

cont'd

Remembering that *y* is a function of *x* and using the Chain Rule, we have

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx}$$

$$= 2y\frac{dy}{dx}$$

Thus

$$2x + 2y\frac{dy}{dx} = 0$$

Now we solve this equation for *dy*/*dx*:

$$\frac{dy}{dx} = -\frac{x}{y}$$

Example 1 – *Solution*

cont'd

(b) At the point (3, 4) we have $x = 3$ and $y = 4$, so

$$\frac{dy}{dx} = -\frac{3}{4}$$

An equation of the tangent to the circle at (3, 4) is therefore

$$y - 4 = -\tfrac{3}{4}(x - 3) \qquad \text{or} \qquad 3x + 4y = 25$$

Solution 2:

(b) Solving the equation $x^2 + y^2 = 25$, we get $y = \pm\sqrt{25 - x^2}$.

The point (3, 4) lies on the upper semicircle $y = \sqrt{25 - x^2}$ and so we consider the function $f(x) = \sqrt{25 - x^2}$.

Example 1 – *Solution*

cont'd

Differentiating *f* using the Chain Rule, we have

$$f'(x) = \tfrac{1}{2}(25 - x^2)^{-1/2}\frac{d}{dx}(25 - x^2)$$

$$= \tfrac{1}{2}(25 - x^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{25 - x^2}}$$

So

$$f'(3) = -\frac{3}{\sqrt{25 - 3^2}} = -\frac{3}{4}$$

and, as in Solution 1, an equation of the tangent is $3x + 4y = 25$.

2.7 Rates of Change in the Natural and Social Sciences

Rates of Change in the Natural and Social Sciences

We know that if $y = f(x)$, then the derivative dy/dx can be interpreted as the rate of change of y with respect to x.

If x changes from x_1 to x_2, then the change in x is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

Rates of Change in the Natural and Social Sciences

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is the **average rate of change of *y* with respect to *x*** over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line *PQ* in Figure 1.

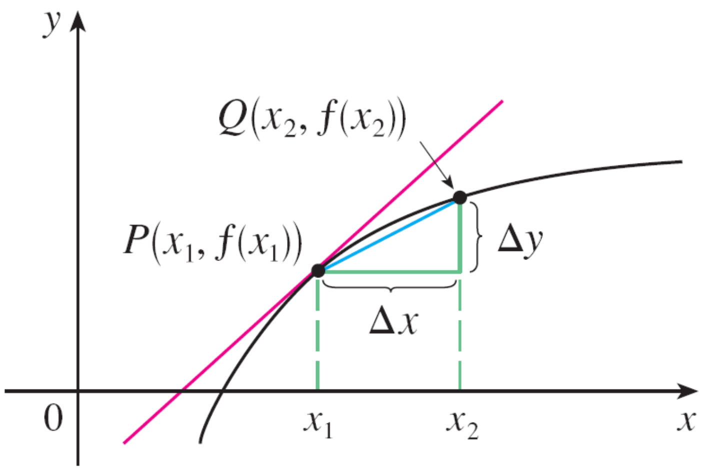


Figure 1

m_{PQ} = average rate of change
$m = f'(x_1)$ = instantaneous rate of change

Rates of Change in the Natural and Social Sciences

Its limit as $\Delta x \to 0$ is the derivative $f'(x_1)$, which can therefore be interpreted as the **instantaneous rate of change of *y* with respect to *x*** or the slope of the tangent line at $P(x_1, f(x_1))$.

Using Leibniz notation, we write the process in the form

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Physics

Physics

If $s = f(t)$ is the position function of a particle that is moving in a straight line, then $\Delta s/\Delta t$ represents the average velocity over a time period Δt, *and* $v = ds/dt$ represents the instantaneous **velocity** (the rate of change of displacement with respect to time).

The instantaneous rate of change of velocity with respect to time is **acceleration**: $a(t) = v'(t) = s''(t)$.

Now that we know the differentiation formulas, we are able to solve problems involving the motion of objects more easily.

Example 1

The position of a particle is given by the equation

$$s = f(t) = t^3 - 6t^2 + 9t$$

where t is measured in seconds and s in meters.

(a) Find the velocity at time t.

(b) What is the velocity after 2 s? After 4 s?

(c) When is the particle at rest?

(d) When is the particle moving forward (that is, in the positive direction)?

(e) Draw a diagram to represent the motion of the particle.

(f) Find the total distance traveled by the particle during the first five seconds.

(g) Find the acceleration at time t and after 4 s.

Example 1

(h) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 5$.

(i) When is the particle speeding up? When is it slowing down?

Solution:

(a) The velocity function is the derivative of the position function.

$$s = f(t) = t^3 - 6t^2 + 9t$$

$$v(t) = \frac{ds}{dt} = 3t^2 - 12t + 9$$

Example 1 – *Solution*

cont'd

(b) The velocity after 2 s means the instantaneous velocity when $t = 2$, that is,

$$v(2) = \left.\frac{ds}{dt}\right|_{t=2} = 3(2)^2 - 12(2) + 9$$

$$= -3 \text{ m/s}$$

The velocity after 4 s is

$$v(4) = 3(4)^2 - 12(4) + 9$$

$$= 9 \text{ m/s}$$

Example 1 – *Solution*

(c) The particle is at rest when $v(t) = 0$, that is,

$$3t^2 - 12t + 9 = 3(t^2 - 4t + 3)$$

$$= 3(t - 1)(t - 3)$$

$$= 0$$

and this is true when $t = 1$ or $t = 3$.

Thus the particle is at rest after 1 s and after 3 s.

Example 1 – *Solution*

cont'd

(d) The particle moves in the positive direction when $v(t) > 0$, that is,

$$3t^2 - 12t + 9 = 3(t - 1)(t - 3) > 0$$

This inequality is true when both factors are positive ($t > 3$) or when both factors are negative ($t < 1$).

Thus the particle moves in the positive direction in the time intervals $t < 1$ and $t > 3$.

It moves backward (in the negative direction) when $1 < t < 3$.

Example 1 – *Solution*

cont'd

(e) Using the information from part (d) we make a schematic sketch in Figure 2 of the motion of the particle back and forth along a line (the *s*-axis).

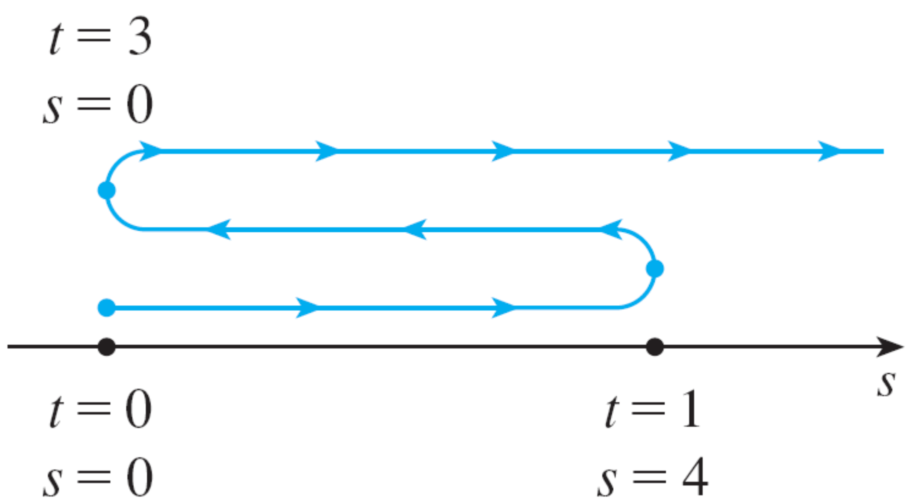


Figure 2

Example 1 – *Solution*

cont'd

(f) Because of what we learned in parts (d) and (e), we need to calculate the distances traveled during the time intervals [0, 1], [1, 3], and [3, 5] separately.

The distance traveled in the first second is

$$|f(1) - f(0)| = |4 - 0| = 4 \text{ m}$$

From $t = 1$ to $t = 3$ the distance traveled is

$$|f(3) - f(1)| = |0 - 4| = 4 \text{ m}$$

From $t = 3$ to $t = 5$ the distance traveled is

$$|f(5) - f(3)| = |20 - 0| = 20 \text{ m}$$

The total distance is 4 + 4 + 20 = 28 m.

Example 1 – *Solution*

cont'd

(g) The acceleration is the derivative of the velocity function:

$$a(t) = \frac{d^2s}{dt^2}$$

$$= \frac{dv}{dt}$$

$$= 6t - 12$$

$$a(4) = 6(4) - 12$$

$$= 12 \text{ m/s}^2$$

Example 1 – *Solution*

cont'd

(h) Figure 3 shows the graphs of *s*, *v*, and *a*.

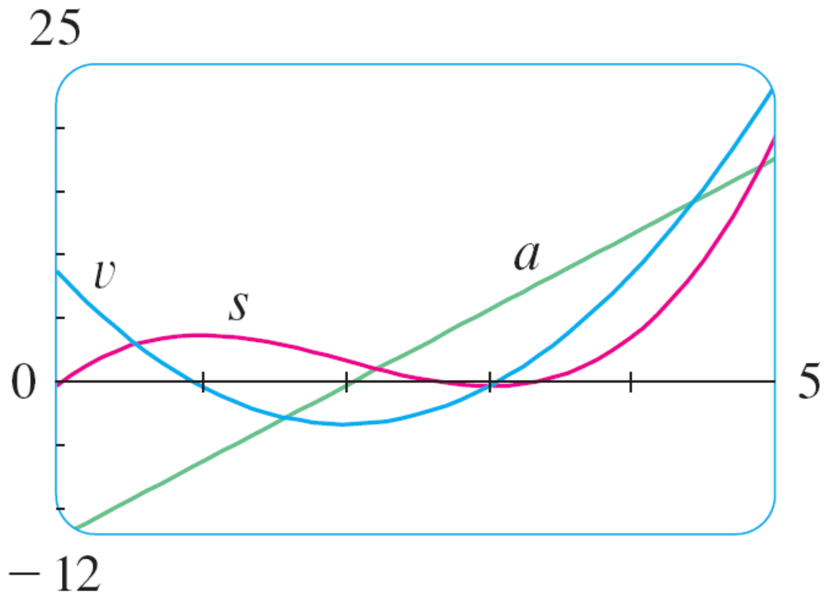


Figure 3

Example 1 – *Solution*

cont'd

(i) The particle speeds up when the velocity is positive and increasing (v and a are both positive) and also when the velocity is negative and decreasing (v and a are both negative).

In other words, the particle speeds up when the velocity and acceleration have the same sign. (The particle is pushed in the same direction it is moving.)

From Figure 3 we see that this happens when $1 < t < 2$ and when $t > 3$.

Example 1 – *Solution*

cont'd

The particle slows down when v and a have opposite signs, that is, when $0 \le t < 1$ and when $2 < t < 3$.

Figure 4 summarizes the motion of the particle.

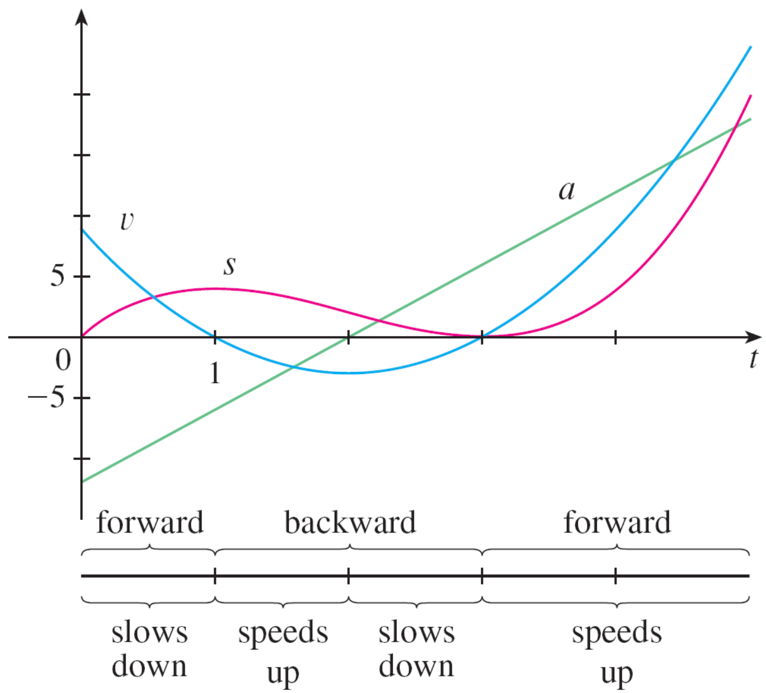


Figure 4

Example 2

If a rod or piece of wire is homogeneous, then its linear density is uniform and is defined as the mass per unit length ($\rho = m/l$) and measured in kilograms per meter.

Suppose, however, that the rod is not homogeneous but that its mass measured from its left end to a point x is $m = f(x)$, as shown in Figure 5.

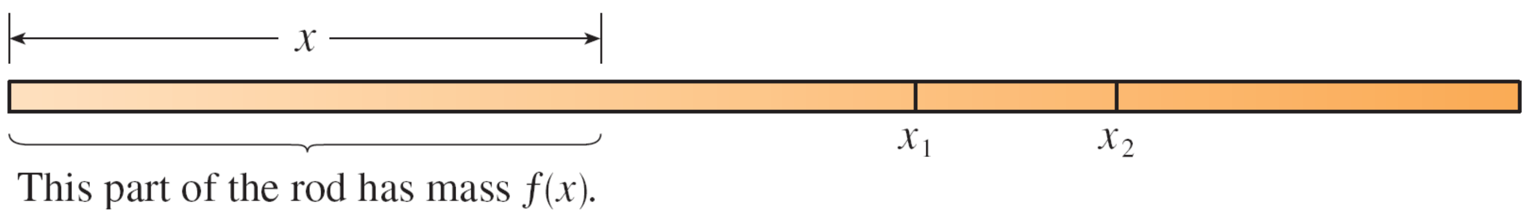


Figure 5

Example 2

cont'd

The mass of the part of the rod that lies between $x = x_1$ and $x = x_2$ is given by $\Delta m = f(x_2) - f(x_1)$, so the average density of that part of the rod is

$$\text{average density} = \frac{\Delta m}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

If we now let $\Delta x \to 0$ (that is, $x_2 \to x_1$), we are computing the average density over smaller and smaller intervals.

The **linear density** ρ at x_1 is the limit of these average densities as $\Delta x \to 0$; that is, the linear density is the rate of change of mass with respect to length.

Example 2

cont'd

Symbolically,

$$\rho = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta x} = \frac{dm}{dx}$$

Thus the linear density of the rod is the derivative of mass with respect to length.

For instance, if *m* = *f*(*x*) = $\sqrt{x}$, where *x* is measured in meters and *m* in kilograms, then the average density of the part of the rod given by 1 ≤ *x* ≤ 1.2 is

$$\frac{\Delta m}{\Delta x} = \frac{f(1.2) - f(1)}{1.2 - 1} = \frac{\sqrt{1.2} - 1}{0.2} \approx 0.48 \text{ kg/m}$$

while the density right at *x* = 1 is

$$\rho = \left.\frac{dm}{dx}\right|_{x=1} = \left.\frac{1}{2\sqrt{x}}\right|_{x=1} = 0.50 \text{ kg/m}$$

Example 3

A current exists whenever electric charges move. Figure 6 shows part of a wire and electrons moving through a plane surface, shaded red.

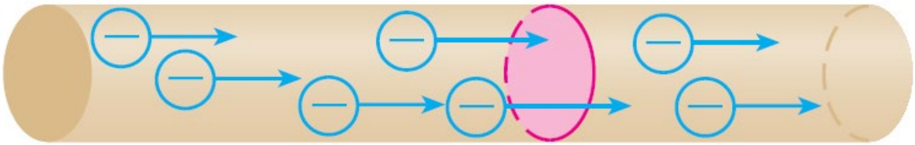


Figure 6

If ΔQ is the net charge that passes through this surface during a time period Δt, then the average current during this time interval is defined as

$$\text{average current} = \frac{\Delta Q}{\Delta t} = \frac{Q_2 - Q_1}{t_2 - t_1}$$

Example 3

cont'd

If we take the limit of this average current over smaller and smaller time intervals, we get what is called the **current *I*** at a given time t_1:

$$I = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}$$

Thus the current is the rate at which charge flows through a surface. It is measured in units of charge per unit time (often coulombs per second, called amperes).

Chemistry

Example 4

A chemical reaction results in the formation of one or more substances (called *products*) from one or more starting materials (called *reactants*).

For instance, the "equation"

$$2H_2 + O_2 \rightarrow 2H_2O$$

indicates that two molecules of hydrogen and one molecule of oxygen form two molecules of water.

Let's consider the reaction

$$A + B \rightarrow C$$

where A and B are the reactants and C is the product.

Example 4

cont'd

The **concentration** of a reactant A is the number of moles (1 mole = 6.022×10^{23} molecules) per liter and is denoted by [A].

The concentration varies during a reaction, so [A], [B], and [C] are all functions of time (t).

The average rate of reaction of the product C over a time interval $t_1 \leq t \leq t_2$ is

$$\frac{\Delta[\mathrm{C}]}{\Delta t} = \frac{[\mathrm{C}](t_2) - [\mathrm{C}](t_1)}{t_2 - t_1}$$

Example 4

cont'd

But chemists are more interested in the **instantaneous rate of reaction**, which is obtained by taking the limit of the average rate of reaction as the time interval Δt approaches 0:

$$\text{rate of reaction} = \lim_{\Delta t \to 0} \frac{\Delta[\mathrm{C}]}{\Delta t} = \frac{d[\mathrm{C}]}{dt}$$

Since the concentration of the product increases as the reaction proceeds, the derivative $d[\mathrm{C}]/dt$ will be positive, and so the rate of reaction of C is positive.

The concentrations of the reactants, however, decrease during the reaction, so, to make the rates of reaction of A and B positive numbers, we put minus signs in front of the derivatives $d[\mathrm{A}]/dt$ and $d[\mathrm{B}]/dt$.

Example 4

cont'd

Since [A] and [B] each decrease at the same rate that [C] increases, we have

$$\text{rate of reaction} = \frac{d[\mathrm{C}]}{dt} = -\frac{d[\mathrm{A}]}{dt} = -\frac{d[\mathrm{B}]}{dt}$$

More generally, it turns out that for a reaction of the form

$$a\mathrm{A} + b\mathrm{B} \rightarrow c\mathrm{C} + d\mathrm{D}$$

we have

$$-\frac{1}{a}\frac{d[\mathrm{A}]}{dt} = -\frac{1}{b}\frac{d[\mathrm{B}]}{dt} = \frac{1}{c}\frac{d[\mathrm{C}]}{dt} = \frac{1}{d}\frac{d[\mathrm{D}]}{dt}$$

The rate of reaction can be determined from data and graphical methods. In some cases there are explicit formulas for the concentrations as functions of time, which enable us to compute the rate of reaction.

Example 5

One of the quantities of interest in thermodynamics is compressibility. If a given substance is kept at a constant temperature, then its volume V depends on its pressure P. We can consider the rate of change of volume with respect to pressure—namely, the derivative dV/dP. As P increases, V decreases, so $dV/dP < 0$.

The **compressibility** is defined by introducing a minus sign and dividing this derivative by the volume V:

$$\text{isothermal compressibility} = \beta = -\frac{1}{V}\frac{dV}{dP}$$

Example 5

cont'd

Thus β measures how fast, per unit volume, the volume of a substance decreases as the pressure on it increases at constant temperature.

For instance, the volume V (in cubic meters) of a sample of air at 25°C was found to be related to the pressure P (in kilopascals) by the equation

$$V = \frac{5.3}{P}$$

The rate of change of V with respect to P when P = 50 kPa is

$$\left.\frac{dV}{dP}\right|_{P=50} = -\left.\frac{5.3}{P^2}\right|_{P=50}$$

Example 5

cont'd

$$= -\frac{5.3}{2500}$$

= –0.00212 m^3/kPa

The compressibility at that pressure is

$$\beta = -\frac{1}{V}\frac{dV}{dP}\bigg|_{P=50}$$

$$= \frac{0.00212}{\frac{5.3}{50}}$$

= 0.02 (m^3/kPa)/m^3

Biology

Example 6

Let $n = f(t)$ be the number of individuals in an animal or plant population at time t.

The change in the population size between the times $t = t_1$ and $t = t_2$ is $\Delta n = f(t_2) - f(t_1)$, and so the average rate of growth during the time period $t_1 \leq t \leq t_2$ is

$$\text{average rate of growth} = \frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

The **instantaneous rate of growth** is obtained from this average rate of growth by letting the time period Δt approach 0:

$$\text{growth rate} = \lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt}$$

Example 6

cont'd

Strictly speaking, this is not quite accurate because the actual graph of a population function $n = f(t)$ would be a step function that is discontinuous whenever a birth or death occurs and therefore not differentiable.

However, for a large animal or plant population, we can replace the graph by a smooth approximating curve as in Figure 7.

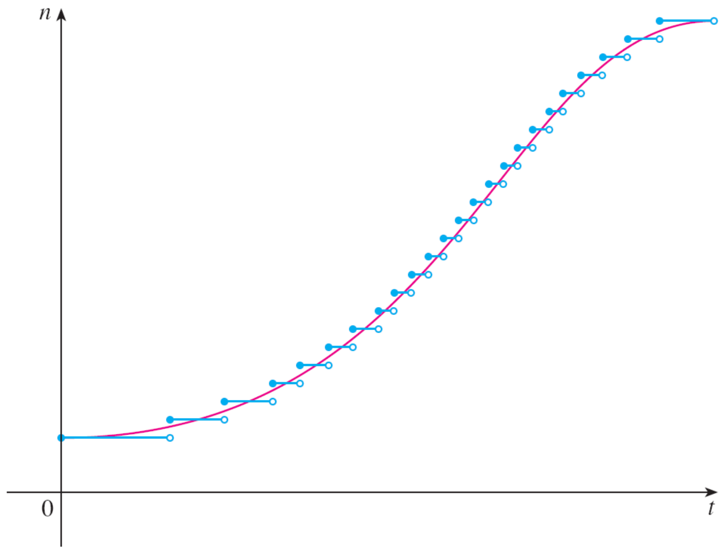


Figure 7

A smooth curve approximating a growth function

Example 6

cont'd

To be more specific, consider a population of bacteria in a homogeneous nutrient medium.

Suppose that by sampling the population at certain intervals it is determined that the population doubles every hour. If the initial population is n_0 and the time t is measured in hours, then

$$f(1) = 2f(0) = 2n_0$$

$$f(2) = 2f(1) = 2^2 n_0$$

$$f(3) = 2f(2) = 2^3 n_0$$

and, in general,

$$f(t) = 2^t n_0$$

Example 6

cont'd

The population function is $n_0 = n_0 2^t$.

We have shown that

$$\frac{d}{dx}(a^x) = a^x \ln a$$

So the rate of growth of the bacteria population at time t is

$$\frac{dn}{dt} = \frac{d}{dt}(n_0 2^t)$$

$$= n_0 2^t \ln 2$$

This is an example of an exponential function.

Example 7

When we consider the flow of blood through a blood vessel, such as a vein or artery, we can model the shape of the blood vessel by a cylindrical tube with radius *R* and length *l* as illustrated in Figure 8.

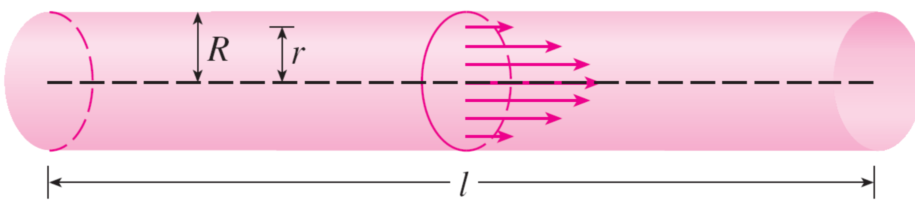


Figure 8
Blood flow in an artery

Because of friction at the walls of the tube, the velocity *v* of the blood is greatest along the central axis of the tube and decreases as the distance *r* from the axis increases until *v* becomes 0 at the wall.

Example 7

cont'd

The relationship between v and r is given by the **law of laminar flow** discovered by the French physician Jean-Louis-Marie Poiseuille in 1840.

This law states that

$$\boxed{1} \qquad v = \frac{P}{4\eta l}(R^2 - r^2)$$

where η is the viscosity of the blood and P is the pressure difference between the ends of the tube.

If P and l are constant, then v is a function of r with domain $[0, R]$.

Example 7

cont'd

The average rate of change of the velocity as we move from $r = r_1$ outward to $r = r_2$ is given by

$$\frac{\Delta v}{\Delta r} = \frac{v(r_2) - v(r_1)}{r_2 - r_1}$$

and if we let $\Delta r \to 0$, we obtain the **velocity gradient**, that is, the instantaneous rate of change of velocity with respect to r:

$$\text{velocity gradient} = \lim_{\Delta r \to 0} \frac{\Delta v}{\Delta r} = \frac{dv}{dr}$$

Using Equation 1, we obtain

$$\frac{dv}{dr} = \frac{P}{4\eta l}(0 - 2r) = -\frac{Pr}{2\eta l}$$

Example 7

cont'd

For one of the smaller human arteries we can take η = 0.027, R = 0.008 cm, l = 2 cm, and P = 4000 dynes/cm^2, which gives

$$v = \frac{4000}{4(0.027)2}(0.000064 - r^2)$$

$$\approx 1.85 \times 10^4(6.4 \times 10^{-5} - r^2)$$

At r = 0.02 cm the blood is flowing at a speed of

$$v(0.002) \approx 1.85 \times 10^4(64 \times 10^{-6} - 4 \times 10^{-6})$$

$$= 1.11 \text{ cm/s}$$

and the velocity gradient at that point is

$$\left.\frac{dv}{dr}\right|_{r=0.002} = -\frac{4000(0.002)}{2(0.027)2} \approx -74 \text{ (cm/s)/cm}$$

Example 7

cont'd

To get a feeling for what this statement means, let's change our units from centimeters to micrometers (1 cm = 10,000 μm). Then the radius of the artery is 80 μm.

The velocity at the central axis is 11,850 μm/s, which decreases to 11,110 μm/s at a distance of r = 20 μm.

The fact that $dv/dr = -74$ (μm/s)/μm means that, when r = 20 μm, the velocity is decreasing at a rate of about 74 μm/s for each micrometer that we proceed away from the center.

Economics

Example 8

Suppose $C(x)$ is the total cost that a company incurs in producing x units of a certain commodity.

The function C is called a **cost function**. If the number of items produced is increased from x_1 to x_2, then the additional cost is $\Delta C = C(x_2) - C(x_1)$, and the average rate of change of the cost is

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1} = \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x}$$

Example 8

cont'd

The limit of this quantity as $\Delta x \to 0$, that is, the instantaneous rate of change of cost with respect to the number of items produced, is called the **marginal cost** by economists:

$$\text{marginal cost} = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx}$$

[Since x often takes on only integer values, it may not make literal sense to let Δx approach 0, but we can always replace $C(x)$ by a smooth approximating function as in Example 6.]

Taking $\Delta x = 1$ and n large (so that Δx is small compared to n), we have

$$C'(n) \approx C(n + 1) - C(n)$$

Example 8

cont'd

Thus the marginal cost of producing n units is approximately equal to the cost of producing one more unit [the $(n + 1)$st unit].

It is often appropriate to represent a total cost function by a polynomial

$$C(x) = a + bx + cx^2 + dx^3$$

where a represents the overhead cost (rent, heat, maintenance) and the other terms represent the cost of raw materials, labor, and so on. (The cost of raw materials may be proportional to x, but labor costs might depend partly on higher powers of x because of overtime costs and inefficiencies involved in large-scale operations.)

Example 8

cont'd

For instance, suppose a company has estimated that the cost (in dollars) of producing x items is

$$C(x) = 10{,}000 + 5x + 0.01x^2$$

Then the marginal cost function is

$$C'(x) = 5 + 0.02x$$

The marginal cost at the production level of 500 items is

$$C'(500) = 5 + 0.02(500)$$

$$= \$15/\text{item}$$

Example 8

cont'd

This gives the rate at which costs are increasing with respect to the production level when $x = 500$ and predicts the cost of the 501st item.

The actual cost of producing the 501st item is

$$C(501) - C(500) = [10{,}000 + 5(501) + 0.01(501)^2] - [10{,}000 + 5(500) + 0.01(500)^2]$$

$$= \$15.01$$

Notice that $C'(500) \approx C(501) - C(500)$.

Other Sciences

Other Sciences

Rates of change occur in all the sciences. A geologist is interested in knowing the rate at which an intruded body of molten rock cools by conduction of heat into surrounding rocks.

An engineer wants to know the rate at which water flows into or out of a reservoir.

An urban geographer is interested in the rate of change of the population density in a city as the distance from the city center increases.

A meteorologist is concerned with the rate of change of atmospheric pressure with respect to height.

A Single Idea, Many Interpretations

A Single Idea, Many Interpretations

Velocity, density, current, power, and temperature gradient in physics; rate of reaction and compressibility in chemistry; rate of growth and blood velocity gradient in biology; marginal cost and marginal profit in economics; rate of heat flow in geology; rate of improvement of performance in psychology; rate of spread of a rumor in sociology—these are all special cases of a single mathematical concept, the derivative.

This is an illustration of the fact that part of the power of mathematics lies in its abstractness.

A Single Idea, Many Interpretations

A single abstract mathematical concept (such as the derivative) can have different interpretations in each of the sciences.

When we develop the properties of the mathematical concept once and for all, we can then turn around and apply these results to all of the sciences.

This is much more efficient than developing properties of special concepts in each separate science.

2.8 Related Rates

Related Rates

If we are pumping air into a balloon, both the volume and the radius of the balloon are increasing and their rates of increase are related to each other.

But it is much easier to measure directly the rate of increase of the volume than the rate of increase of the radius.

In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured).

The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time.

Example 1

Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

Solution:

We start by identifying two things:

the *given information*:

the rate of increase of the volume of air is 100 cm^3/s

and the *unknown*:

the rate of increase of the radius when the diameter is 50 cm

Example 1 – *Solution*

cont'd

In order to express these quantities mathematically, we introduce some suggestive *notation*:

Let V be the volume of the balloon and let r be its radius.

The key thing to remember is that rates of change are derivatives. In this problem, the volume and the radius are both functions of the time t.

The rate of increase of the volume with respect to time is the derivative dV/dt, and the rate of increase of the radius is dr/dt.

Example 1 – *Solution*

cont'd

We can therefore restate the given and the unknown as follows:

Given: $\dfrac{dV}{dt} = 100 \text{ cm}^3/\text{s}$

Unknown: $\dfrac{dr}{dt}$ when $r = 25$ cm

In order to connect dV/dt and dr/dt, we first relate V and r by the formula for the volume of a sphere:

$$V = \tfrac{4}{3}\pi r^3$$

Example 1 – *Solution*

cont'd

In order to use the given information, we differentiate each side of this equation with respect to *t*. To differentiate the right side, we need to use the Chain Rule:

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Now we solve for the unknown quantity:

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

Example 1 – *Solution*

cont'd

If we put $r = 25$ and $dV/dt = 100$ in this equation, we obtain

$$\frac{dr}{dt} = \frac{1}{4\pi(25)^2}100$$

$$= \frac{1}{25\pi}$$

The radius of the balloon is increasing at the rate of $1/(25\pi) \approx 0.0127$ cm/s.

2.9 Linear Approximations and Differentials

Linear Approximations and Differentials

We have seen that a curve lies very close to its tangent line near the point of tangency. In fact, by zooming in toward a point on the graph of a differentiable function, we noticed that the graph looks more and more like its tangent line.

This observation is the basis for a method of finding approximate values of functions.

The idea is that it might be easy to calculate a value $f(a)$ of a function, but difficult (or even impossible) to compute nearby values of f.

Linear Approximations and Differentials

So we settle for the easily computed values of the linear function *L* whose graph is the tangent line of *f* at (*a*, *f*(*a*)). (See Figure 1.)

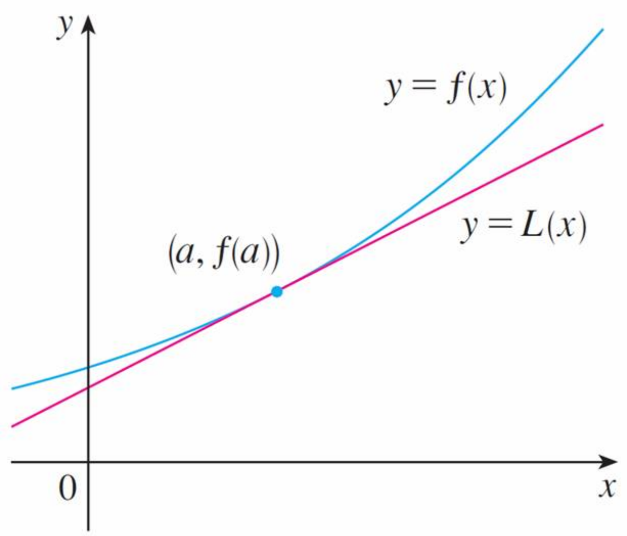


Figure 1

Linear Approximations and Differentials

In other words, we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a. An equation of this tangent line is

$$y = f(a) + f'(a)(x - a)$$

and the approximation

$$\boxed{1} \qquad f(x) \approx f(a) + f'(a)(x - a)$$

is called the **linear approximation** or **tangent line approximation** of f at a.

Linear Approximations and Differentials

The linear function whose graph is this tangent line, that is,

$$\boxed{2} \qquad L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of *f* at *a*.

Example 1

Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

Solution:

The derivative of $f(x) = (x + 3)^{1/2}$ is

$$f'(x) = \tfrac{1}{2}(x+3)^{-1/2}$$

$$= \frac{1}{2\sqrt{x+3}}$$

and so we have $f(1) = 2$ and $f'(1) = \frac{1}{4}$.

Example 1 – *Solution*

cont'd

Putting these values into Equation 2, we see that the linearization is

$$L(x) = f(1) + f'(1)(x - 1)$$

$$= 2 + \tfrac{1}{4}(x - 1)$$

$$= \frac{7}{4} + \frac{x}{4}$$

The corresponding linear approximation $\boxed{1}$ is

$$\sqrt{x + 3} \approx \frac{7}{4} + \frac{x}{4} \qquad \text{(when } x \text{ is near 1)}$$

Example 1 – *Solution*

cont'd

In particular, we have

$$\sqrt{3.98} \approx \frac{7}{4} + \frac{0.98}{4}$$

$$= 1.995$$

and

$$\sqrt{4.05} \approx \frac{7}{4} + \frac{1.05}{4}$$

$$= 2.0125$$

Example 1 – *Solution*

cont'd

The linear approximation is illustrated in Figure 2.

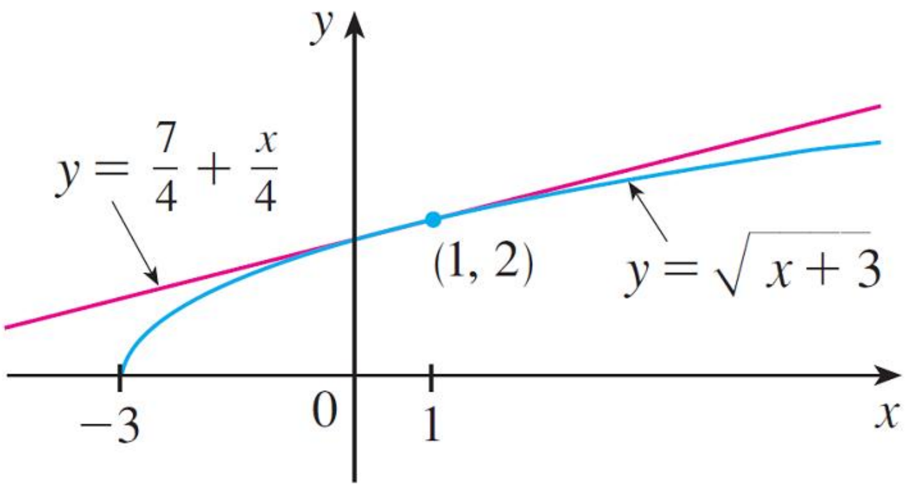


Figure 2

Example 1 – *Solution*

cont'd

We see that, indeed, the tangent line approximation is a good approximation to the given function when x is near 1.

We also see that our approximations are overestimates because the tangent line lies above the curve.

Of course, a calculator could give us approximations for $\sqrt{3.98}$ and $\sqrt{4.05}$, but the linear approximation gives an approximation *over an entire interval.*

Linear Approximations and Differentials

In the following table we compare the estimates from the linear approximation in Example 1 with the true values.

	x	From $L(x)$	Actual value
$\sqrt{3.9}$	0.9	1.975	1.97484176 . . .
$\sqrt{3.98}$	0.98	1.995	1.99499373 . . .
$\sqrt{4}$	1	2	2.00000000 . . .
$\sqrt{4.05}$	1.05	2.0125	2.01246117 . . .
$\sqrt{4.1}$	1.1	2.025	2.02484567 . . .
$\sqrt{5}$	2	2.25	2.23606797 . . .
$\sqrt{6}$	3	2.5	2.44948974 . . .

Linear Approximations and Differentials

Notice from this table, and also from Figure 2, that the tangent line approximation gives good estimates when *x* is close to 1 but the accuracy of the approximation deteriorates when *x* is farther away from 1.

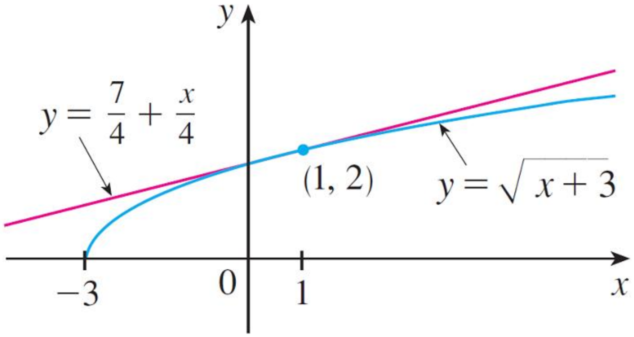


Figure 2

Linear Approximations and Differentials

The next example shows that by using a graphing calculator or computer we can determine an interval throughout which a linear approximation provides a specified accuracy.

Example 2

For what values of *x* is the linear approximation

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$$

accurate to within 0.5? What about accuracy to within 0.1?

Solution:

Accuracy to within 0.5 means that the functions should differ by less than 0.5:

$$\left| \sqrt{x+3} - \left(\frac{7}{4} + \frac{x}{4} \right) \right| < 0.5$$

Example 2 – *Solution*

cont'd

Equivalently, we could write

$$\sqrt{x + 3} - 0.5 < \frac{7}{4} + \frac{x}{4} < \sqrt{x + 3} + 0.5$$

This says that the linear approximation should lie between the curves obtained by shifting the curve $y = \sqrt{x + 3}$ upward and downward by an amount 0.5.

Example 2 – *Solution*

cont'd

Figure 3 shows the tangent line $y = (7 + x)/4$ intersecting the upper curve $y = \sqrt{x+3} + 0.5$ at P and Q.

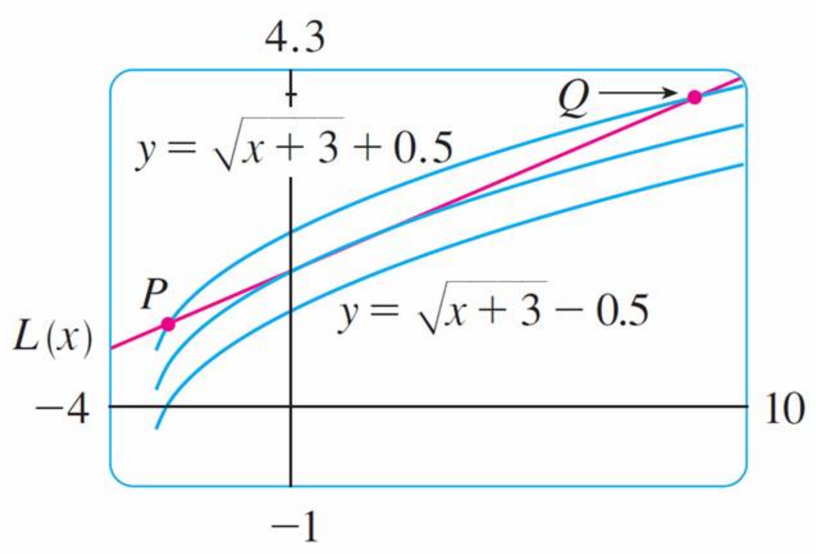


Figure 3

Example 2 – *Solution*

cont'd

Zooming in and using the cursor, we estimate that the x-coordinate of P is about –2.66 and the x-coordinate of Q is about 8.66.

Thus we see from the graph that the approximation

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$$

is accurate to within 0.5 when $-2.6 < x < 8.6$. (We have rounded to be safe.)

Example 2 – *Solution*

cont'd

Similarly, from Figure 4 we see that the approximation is accurate to within 0.1 when –1.1 < x < 3.9.

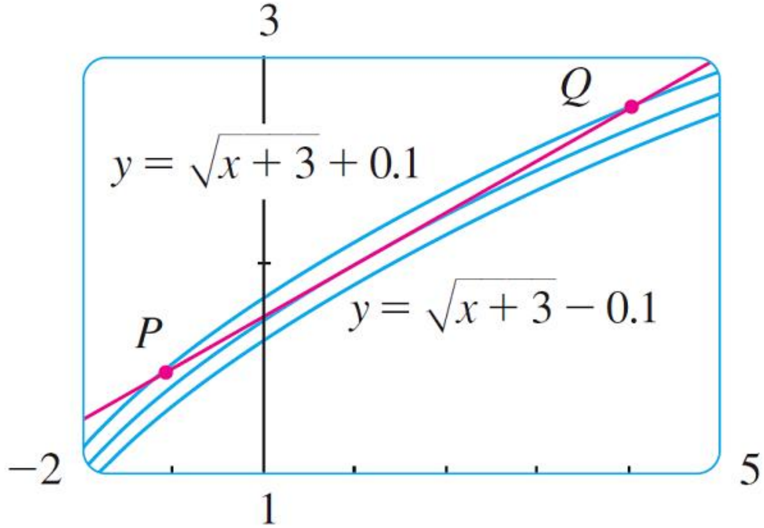


Figure 4

Applications to Physics

Applications to Physics

Linear approximations are often used in physics. In analyzing the consequences of an equation, a physicist sometimes needs to simplify a function by replacing it with its linear approximation.

For instance, in deriving a formula for the period of a pendulum, physics textbooks obtain the expression $a_T = -g \sin \theta$ for tangential acceleration and then replace $\sin \theta$ by θ with the remark that $\sin \theta$ is very close to θ if θ is not too large.

Applications to Physics

You can verify that the linearization of the function $f(x) = \sin x$ at $a = 0$ is $L(x) = x$ and so the linear approximation at 0 is

$$\sin x \approx x$$

So, in effect, the derivation of the formula for the period of a pendulum uses the tangent line approximation for the sine function.

Applications to Physics

Another example occurs in the theory of optics, where light rays that arrive at shallow angles relative to the optical axis are called *paraxial rays.*

In paraxial (or Gaussian) optics, both $\sin \theta$ and $\cos \theta$ are replaced by their linearizations. In other words, the linear approximations

$$\sin \theta \approx \theta \quad \text{and} \quad \cos \theta \approx 1$$

are used because θ is close to 0.

Differentials

Differentials

The ideas behind linear approximations are sometimes formulated in the terminology and notation of *differentials.*

If $y = f(x)$, where f is a differentiable function, then the **differential** dx is an independent variable; that is, dx can be given the value of any real number.

The **differential** dy is then defined in terms of dx by the equation

$$\boxed{3} \qquad dy = f'(x)\, dx$$

Differentials

So dy is a dependent variable; it depends on the values of x and dx.

If dx is given a specific value and x is taken to be some specific number in the domain of f, then the numerical value of dy is determined.

Differentials

The geometric meaning of differentials is shown in Figure 5.

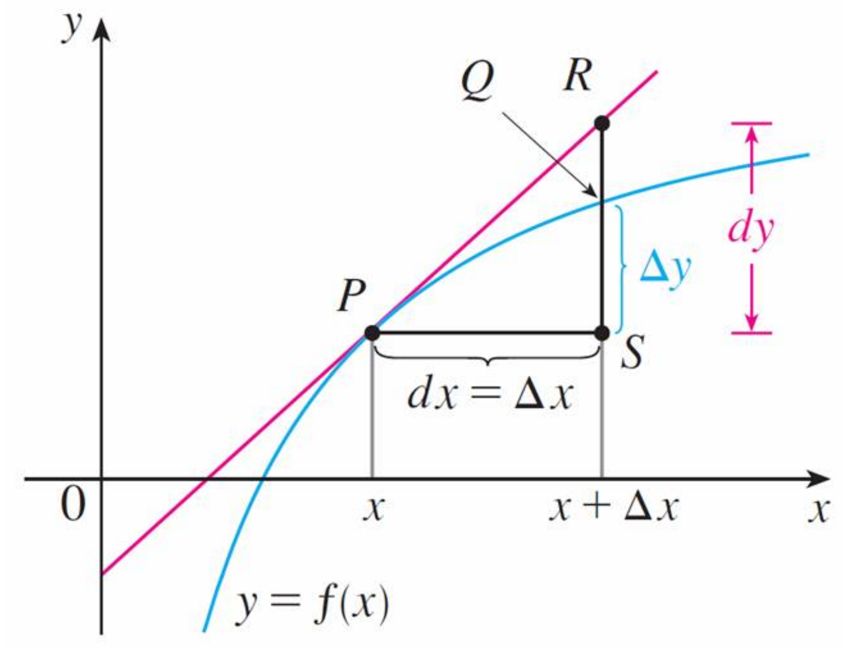


Figure 5

Differentials

Let $P(x, f(x))$ and $Q(x + \Delta x, f(x + \Delta x))$ be points on the graph of f and let $dx = \Delta x$. The corresponding change in y is

$$\Delta y = f(x + \Delta x) - f(x)$$

The slope of the tangent line PR is the derivative $f'(x)$. Thus the directed distance from S to R is $f'(x)\, dx = dy$.

Therefore dy represents the amount that the tangent line rises or falls (the change in the linearization), whereas Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes by an amount dx.

Example 3

Compare the values of Δy and dy if
$y = f(x) = x^3 + x^2 - 2x + 1$ and x changes (a) from 2 to 2.05 and (b) from 2 to 2.01.

Solution:

(a) We have

$$f(2) = 2^3 + 2^2 - 2(2) + 1$$

$$= 9$$

$$f(2.05) = (2.05)^3 + (2.05)^2 - 2(2.05) + 1$$

$$= 9.717625$$

Example 3 – *Solution*

cont'd

$$\Delta y = f(2.05) - f(2)$$

$$= 0.717625$$

In general,

$$dy = f'(x)\,dx$$

$$= (3x^2 + 2x - 2)\,dx$$

When $x = 2$ and $dx = \Delta x = 0.05$, this becomes

$$dy = [3(2)^2 + 2(2) - 2]0.05$$

$$= 0.7$$

Example 3 – *Solution*

cont'd

(b) $f(2.01) = (2.01)^3 + (2.01)^2 - 2(2.01) + 1$

$= 9.140701$

$\Delta y = f(2.01) - f(2)$

$= 0.140701$

When $dx = \Delta x = 0.01$,

$dy = [3(2)^2 + 2(2) - 2]0.01$

$= 0.14$

Differentials

Our final example illustrates the use of differentials in estimating the errors that occur because of approximate measurements.

Example 4

The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

Solution:

If the radius of the sphere is r, then its volume is $V = \frac{4}{3}\pi r^3$. If the error in the measured value of r is denoted by $dr = \Delta r$, then the corresponding error in the calculated value of V is ΔV, which can be approximated by the differential

$$dV = 4\pi r^2\, dr$$

Example 4 – *Solution*

cont'd

When $r = 21$ and $dr = 0.05$, this becomes

$$dV = 4\pi(21)^2 0.05$$

$$\approx 277$$

The maximum error in the calculated volume is about 277 cm^3.

Differentials

Note:

Although the possible error in Example 4 may appear to be rather large, a better picture of the error is given by the **relative error**, which is computed by dividing the error by the total volume:

$$\frac{\Delta V}{V} \approx \frac{dV}{V}$$

$$= \frac{4\pi r^2\,dr}{\frac{4}{3}\pi r^3}$$

$$= 3\,\frac{dr}{r}$$

Differentials

Thus the relative error in the volume is about three times the relative error in the radius.

In Example 4 the relative error in the radius is approximately $dr/r = 0.05/21 \approx 0.0024$ and it produces a relative error of about 0.007 in the volume.

The errors could also be expressed as **percentage errors** of 0.24% in the radius and 0.7% in the volume.