Vv156 Honors Calculus II (Fall 2021)

Assignment 3

Date Due: 22:00 PM, Thursday, Oct. 21, 2021
This assignment has a total of **(32 points)**.
Note: Unless specified otherwise, you must show the details of your work via logical reasoning for each exercise. Simply writing a final result (whether correct or not) will receive **0 point**.

Exercise 3.1 [Ste10, p. 205]

(i) `(1 pt)` The curve $y = |x|/\sqrt{2-x^2}$ is called a *bullet-nose curve*. Find an equation of the tangent line to this curve at the point $(1, 1)$.

(ii) `(1 pt)` Illustrate part (i) by sketch the curve and the tangent line on the same coordinate system.

(2 pts)

Exercise 3.2 [Ste10, p. 208] Use the Chain Rule to prove the following.

(i) `(1 pt)` The derivative of an even function is an odd function.

(ii) `(1 pt)` The derivative of an odd function is an even function.

(2 pts)

Exercise 3.3 [Ste10, p. 208] If $y = f(u)$ and $u = g(x)$, where f and g are twice differentiable functions, show that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 y}{\mathrm{d}u^2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 + \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}$$

(2 pts)

Exercise 3.4 [Ste10, p. 215] Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

(i) `(1 pt)` (cardioid) $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$ at $(0, \frac{1}{2})$.

(ii) `(1 pt)` (astroid) $x^{2/3} + y^{2/3} = 4$ at $(-3\sqrt{3}, 1)$.

(iii) `(1 pt)` (lemniscate) $2(x^2 + y^2)^2 = 25(x^2 - y^2)$ at $(3, 1)$.

(iv) `(1 pt)` (devil's curve) $y^2(y^2 - 4) = x^2(x^2 - 5)$ at $(0, -2)$.

(4 pts)

Exercise 3.5 [Ste10, Sec. 3.11] Given the following hyperbolic functions defined as

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x}$$
$$\operatorname{csch} x = \frac{1}{\sinh x} \qquad \operatorname{sech} x = \frac{1}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

(i) `(3 pts)` Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sinh x) = \cosh x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\cosh x) = \sinh x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\tanh x) = \operatorname{sech}^2 x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{csch} x) = -\operatorname{csch} x \coth x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\coth x) = -\operatorname{csch}^2 x$$

(ii) `(3 pts)` and show that[1]

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\tanh^{-1} x) = \frac{1}{1-x^2}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{x^2+1}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\coth^{-1} x) = \frac{1}{1-x^2}$$

[1] Notice that the formulas for the derivatives of $\tanh^{-1} x$ and $\coth^{-1} x$ appear to be identical. But the domains of these functions have no numbers in common: $\tanh^{-1} x$ is defined for $|x| < 1$, whereas $\coth^{-1} x$ is defined for $|x| > 1$.

(6 pts)

Exercise 3.6 [Ste10, p. 223] Find the derivative of the following functions

(i) `(1 pt)` $y = (\sin x)^{\ln x}$

(ii) `(1 pt)` $y = (\tan x)^{1/x}$

(2 pts)

Exercise 3.7 [Ste10, p. 272] If

$$y = \frac{x}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \arctan \frac{\sin x}{a + \sqrt{a^2-1} + \cos x}$$

Show that $y' = \dfrac{1}{a + \cos x}$.

(2 pts)

Exercise 3.8 [Ste10, p. 282] If f has a local minimum value at c, show that the function $g(x) = -f(x)$ has a local maximum value at c.

(2 pts)

Exercise 3.9 [Ste10, p. 289] Suppose f is an odd function and is differentiable everywhere. Show that for every positive number b, there exists a number $c \in (-b, b)$ such that $f'(c) = f(b)/b$.

(2 pts)

Exercise 3.10 [Ste10, p. 300] Show that the inflection points of the curve $y = x \sin x$ lie on the curve $y^2(x^2+4) = 4x^2$.

(2 pts)

Exercise 3.11 [Ste10, p. 309] Evaluate

$$\lim_{x \to \infty} \left[x - x^2 \ln \left(\frac{1+x}{x} \right) \right]$$

(2 pts)

Exercise 3.12 [Ste10, p. 309] Let

$$f(x) = \begin{cases} |x|^x, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

(i) `(1 pt)` Show that f is continuous at 0.

(ii) `(1 pt)` Calculate $f'(0)$.

(2 pts)

Exercise 3.13 [Ste10, p. 309] Show that the shortest distance from the point (x_1, y_1) to the straight line $Ax + By + C = 0$ is[2]

$$\frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

(2 pts)

References

[Ste10] J. Stewart. *Calculus: Early Transcendentals.* 7th ed. Cengage Learning, 2010 (Cited on pages 1, 2).

[2] of course one approach is to use the Cauchy-Schwarz inequality.