8.1 Arc Length

Arc Length

What do we mean by the length of a curve? We might think of fitting a piece of string to the curve in Figure 1 and then measuring the string against a ruler. But that might be difficult to do with much accuracy if we have a complicated curve.

We need a precise definition for the length of an arc of a curve, in the same spirit as the definitions we developed for the concepts of area and volume.

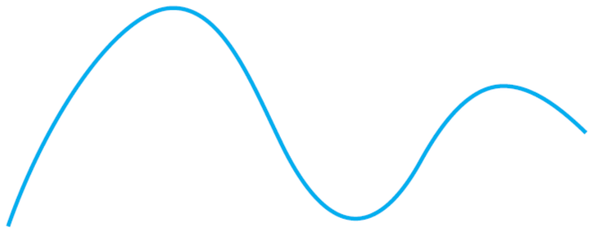


Figure 1

Arc Length

If the curve is a polygon, we can easily find its length; we just add the lengths of the line segments that form the polygon. (We can use the distance formula to find the distance between the endpoints of each segment.) We are going to define the length of a general curve by first approximating it by a polygon and then taking a limit as the number of segments of the polygon is increased. This process is familiar for the case of a circle, where the circumference is the limit of lengths of inscribed polygons (see Figure 2).

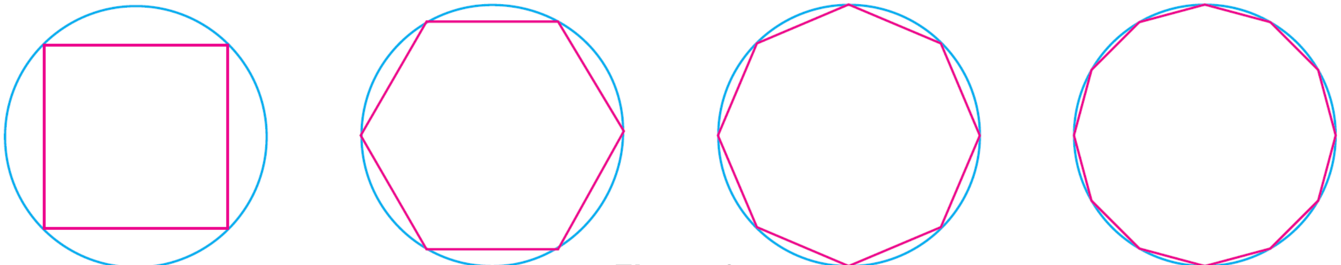


Figure 2

Arc Length

Suppose that a curve C is defined by the equation $y = f(x)$ where f is continuous and $a \leq x \leq b$.

We obtain a polygonal approximation to C by dividing the interval $[a, b]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx. If $y_i = f(x_i)$, then the point $P_i(x_i, y_i)$ lies on C and the polygon with vertices $P_0, P_1, \ldots, P_n$, illustrated in Figure 3, is an approximation to C.

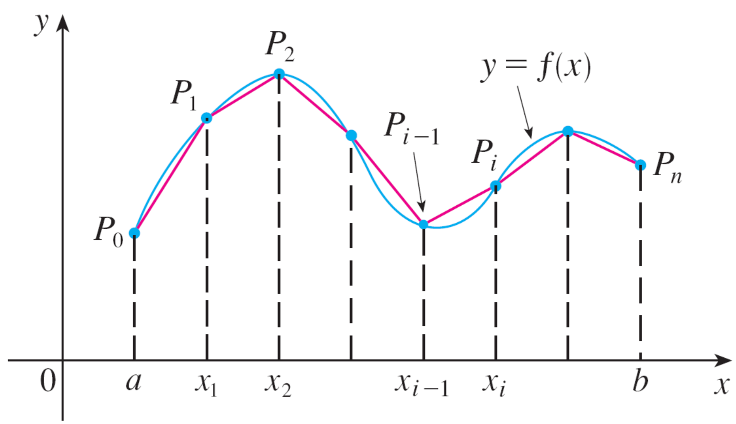


Figure 3

Arc Length

The length L of C is approximately the length of this polygon and the approximation gets better as we let n increase. (See Figure 4, where the arc of the curve between P_{i-1} and P_i has been magnified and approximations with successively smaller values of Δx are shown.)

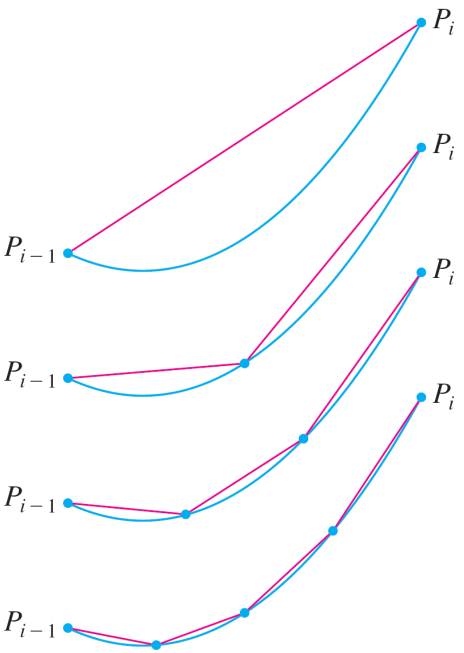


Figure 4

Therefore we define the **length** L of the curve C with equation, $y = f(x)$, $a \leq x \leq b$ as the limit of the lengths of these inscribed polygons (if the limit exists):

Arc Length

$$\boxed{1} \qquad L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

Notice that the procedure for defining arc length is very similar to the procedure we used for defining area and volume: We divided the curve into a large number of small parts. We then found the approximate lengths of the small parts and added them. Finally, we took the limit as $n \to \infty$

Arc Length

The definition of arc length given by Equation 1 is not very convenient for computational purposes, but we can derive an integral formula for L in the case where f has a continuous derivative. [Such a function is called **smooth** because a small change in x produces a small change in $f'(x)$.]

If we let $\Delta y_i = y_i - y_{i-1}$, then

$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$

Arc Length

By applying the Mean Value Theorem to f on the interval, $[x_i - x_{i-1}]$, we find that there is a number x_i^* between x_{i-1} and x_i such that

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$$

that is $\qquad \Delta y_i = f'(x_i^*)\,\Delta x$

Thus we have

$$|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (\Delta y_i)^2} = \sqrt{(\Delta x)^2 + [f'(x_i^*)\,\Delta x]^2}$$

$$= \sqrt{1 + [f'(x_i^*)]^2}\,\sqrt{(\Delta x)^2} = \sqrt{1 + [f'(x_i^*)]^2}\,\Delta x \qquad \text{(since } \Delta x > 0\text{)}$$

Arc Length

Therefore, by Definition 1,

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1}P_i| = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x$$

We recognize this expression as being equal to

$$\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

by the definition of a definite integral. This integral exists because the function $g(x) = \sqrt{1 + [f'(x)]^2}$ is continuous. Thus we have proved the following theorem:

Arc Length

2 The Arc Length Formula If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \leqslant x \leqslant b$, is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

If we use Leibniz notation for derivatives, we can write the arc length formula as follows:

3

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Example 1

Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points (1, 1) and (4, 8). (See Figure 5.)

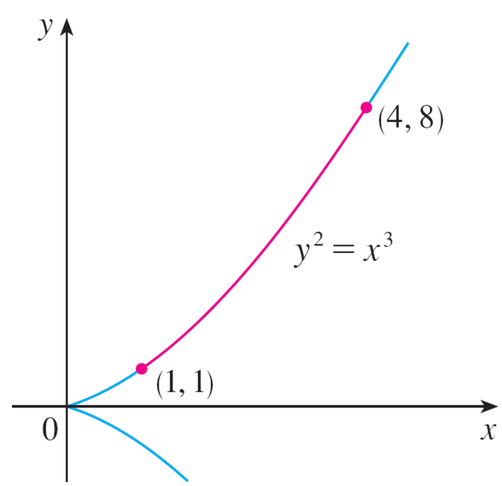


Figure 5

Example 1 – *Solution*

For the top half of the curve we have

$$y = x^{3/2} \qquad \frac{dy}{dx} = \tfrac{3}{2}x^{1/2}$$

So the arc length formula gives

$$L = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^4 \sqrt{1 + \tfrac{9}{4}x}\, dx$$

If we substitute $u = 1 + \tfrac{9}{4}x$, then $du = \tfrac{9}{4}\, dx$.

When $x = 1$, $u = \tfrac{13}{4}$; when $x = 4$, $u = 10$.

Example 1 – *Solution*

cont'd

Therefore

$$L = \frac{4}{9}\int_{13/4}^{10} \sqrt{u}\, du = \frac{4}{9} \cdot \frac{2}{3} u^{3/2}\Big]_{13/4}^{10}$$

$$= \frac{8}{27}\left[10^{3/2} - \left(\tfrac{13}{4}\right)^{3/2}\right]$$

$$= \frac{1}{27}\left(80\sqrt{10} - 13\sqrt{13}\right)$$

Arc Length

If a curve has the equation $x = g(y)$, $c \le y \le d$, and $g'(y)$ is continuous, then by interchanging the roles of x and y in Formula 2 or Equation 3, we obtain the following formula for its length:

$$\boxed{4} \qquad L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

The Arc Length Function

The Arc Length Function

We will find it useful to have a function that measures the arc length of a curve from a particular starting point to any other point on the curve. Thus if a smooth curve C has the equation ,$y = f(x)$, $a \leq x \leq b$ let $s(x)$ be the distance along C from the initial point $P_0(a, f(a))$ to the point $Q(x, f(x))$. Then s is a function, called the **arc length function**, and, by Formula 2,

$$\boxed{5} \qquad s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$$

The Arc Length Function

(We have replaced the variable of integration by so that does not have two meanings.) We can use Part 1 of the Fundamental Theorem of Calculus to differentiate Equation 5 (since the integrand is continuous):

$$\boxed{6} \qquad \frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Equation 6 shows that the rate of change of *s* with respect to *x* is always at least 1 and is equal to 1 when $f'(x)$, the slope of the curve, is 0.

The Arc Length Function

The differential of arc length is

$$\boxed{7} \qquad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

and this equation is sometimes written in the symmetric form

$$\boxed{8} \qquad (ds)^2 = (dx)^2 + (dy)^2$$

The geometric interpretation of Equation 8 is shown in Figure 7. It can be used as a mnemonic device for remembering both of the Formulas 3 and 4.

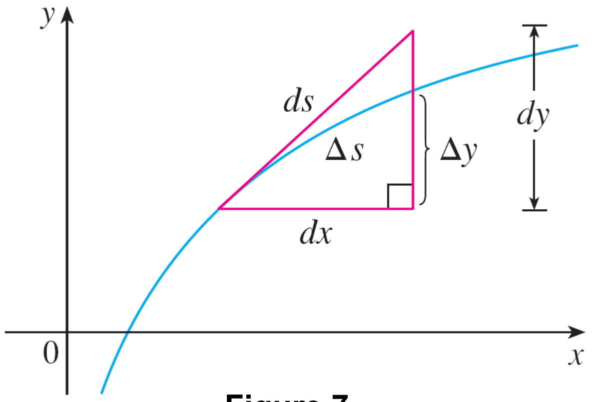


Figure 7

The Arc Length Function

If we write $L = \int ds$, then from Equation 8 either we can solve to get (7), which gives (3), or we can solve to get

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Example 4

Find the arc length function for the curve $y = x^2 - \frac{1}{8} \ln x$ taking $P_0(1, 1)$ as the starting point.

Solution:

If $f(x) = x^2 - \frac{1}{8} \ln x$, then

$$f'(x) = 2x - \frac{1}{8x}$$

$$1 + [f'(x)]^2 = 1 + \left(2x - \frac{1}{8x}\right)^2 = 1 + 4x^2 - \frac{1}{2} + \frac{1}{64x^2}$$

$$= 4x^2 + \frac{1}{2} + \frac{1}{64x^2} = \left(2x + \frac{1}{8x}\right)^2$$

$$\sqrt{1 + [f'(x)]^2} = 2x + \frac{1}{8x}$$

Example 4 – *Solution*

cont'd

Thus the arc length function is given by

$$s(x) = \int_1^x \sqrt{1 + [f'(t)]^2}\, dt$$

$$= \int_1^x \left(2t + \frac{1}{8t}\right) dt = t^2 + \tfrac{1}{8} \ln t \Big]_1^x$$

$$= x^2 + \tfrac{1}{8} \ln x - 1$$

For instance, the arc length along the curve from (1, 1) to (3, *f*(3)) is

$$s(3) = 3^2 + \tfrac{1}{8} \ln 3 - 1 = 8 + \frac{\ln 3}{8} \approx 8.1373$$

8.2 Area of a Surface of Revolution

Area of a Surface of Revolution

A surface of revolution is formed when a curve is rotated about a line. Such a surface is the lateral boundary of a solid of revolution.

We want to define the area of a surface of revolution in such a way that it corresponds to our intuition. If the surface area is A, we can imagine that painting the surface would require the same amount of paint as does a flat region with area A.

Area of a Surface of Revolution

Let's start with some simple surfaces. The lateral surface area of a circular cylinder with radius r and height h is taken to be $A = 2\pi rh$ because we can imagine cutting the cylinder and unrolling it (as in Figure 1) to obtain a rectangle with dimensions $2\pi r$ and h.

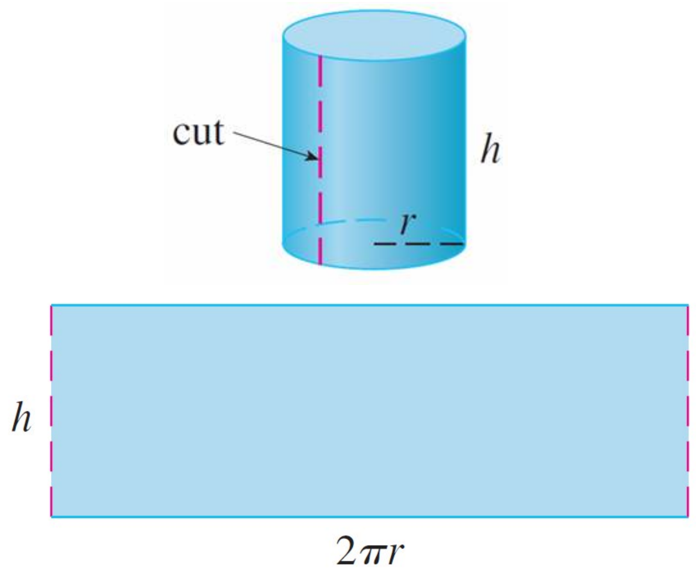


Figure 1

Area of a Surface of Revolution

Likewise, we can take a circular cone with base radius r and slant height l, cut it along the dashed line in Figure 2, and flatten it to form a sector of a circle with radius l and central angle $\theta = 2\pi r/l$.

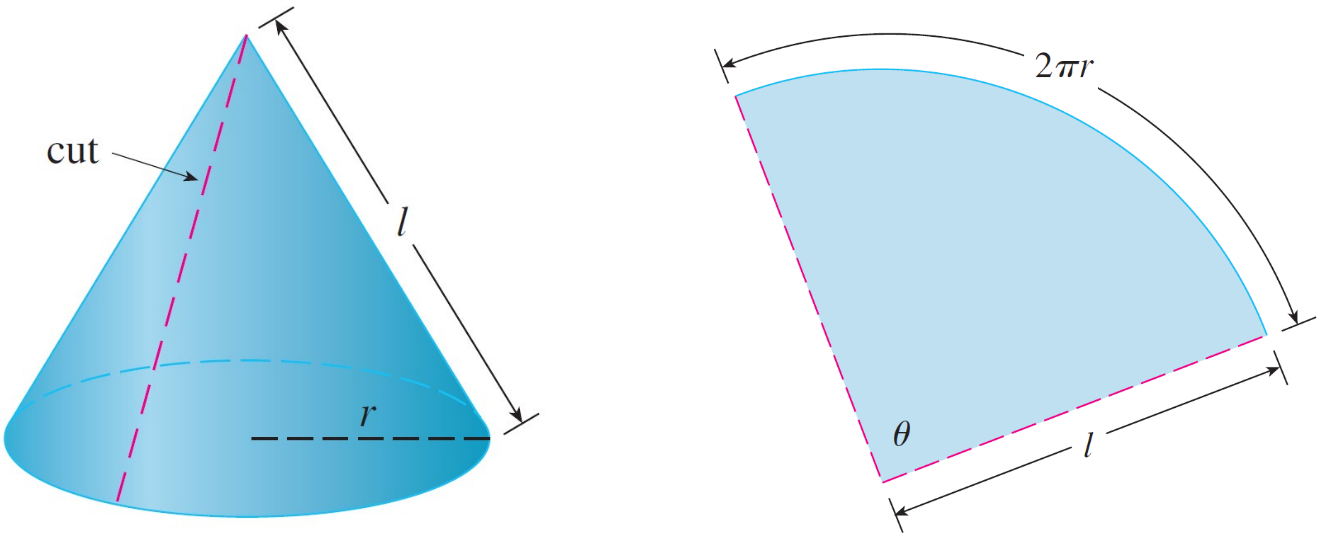


Figure 2

Area of a Surface of Revolution

We know that, in general, the area of a sector of a circle with radius l and angle θ is $\frac{1}{2}l^2\theta$ and so in this case the area is

$$A = \frac{1}{2}l^2\theta = \frac{1}{2}l^2\left(\frac{2\pi r}{l}\right) = \pi r l$$

Therefore we define the lateral surface area of a cone to be $A = \pi r l$.

What about more complicated surfaces of revolution? If we follow the strategy we used with arc length, we can approximate the original curve by a polygon.

Area of a Surface of Revolution

When this polygon is rotated about an axis, it creates a simpler surface whose surface area approximates the actual surface area.

By taking a limit, we can determine the exact surface area.

The approximating surface, then, consists of a number of *bands,* each formed by rotating a line segment about an axis.

Area of a Surface of Revolution

To find the surface area, each of these bands can be considered a portion of a circular cone, as shown in Figure 3.

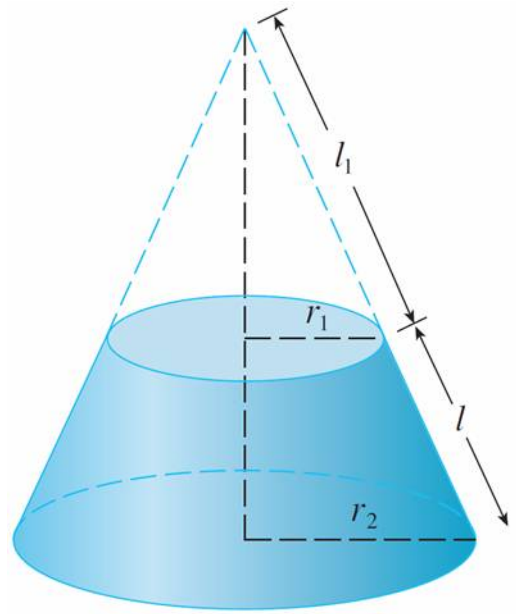


Figure 3

Area of a Surface of Revolution

The area of the band (or frustum of a cone) with slant height l and upper and lower radii r_1 and r_2 is found by subtracting the areas of two cones:

$$\boxed{1} \qquad A = \pi r_2(l_1 + l) - \pi r_1 l_1 = \pi[(r_2 - r_1)l_1 + r_2 l]$$

From similar triangles we have

$$\frac{l_1}{r_1} = \frac{l_1 + l}{r_2}$$

which gives

$$r_2 l_1 = r_1 l_1 + r_1 l \qquad \text{or} \qquad (r_2 - r_1)l_1 = r_1 l$$

Area of a Surface of Revolution

Putting this in Equation 1, we get

$$A = \pi (r_1 l + r_2 l)$$

or

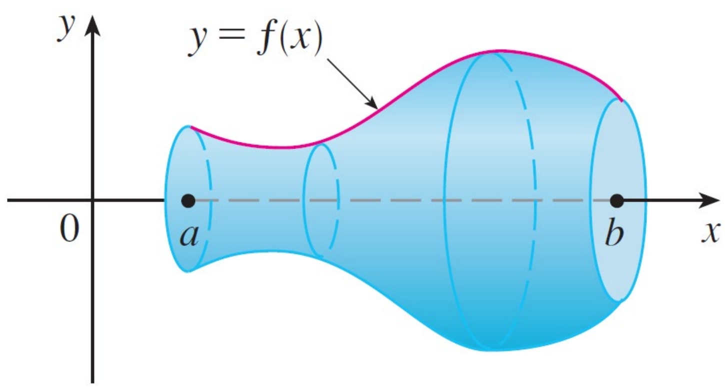
2

$$\boxed{A = 2\pi r l}$$

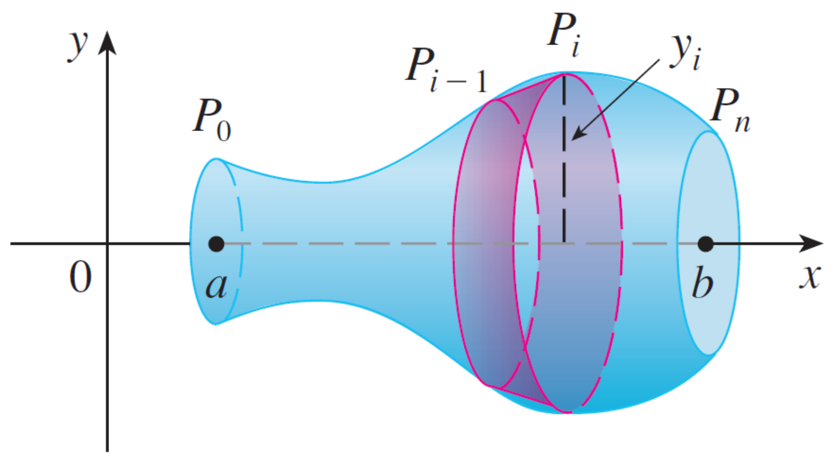
where $r = \frac{1}{2}(r_1 + r_2)$ is the average radius of the band.

Area of a Surface of Revolution

Now we apply this formula to our strategy. Consider the surface shown in Figure 4, which is obtained by rotating the curve $y = f(x)$, $a \le x \le b$, about the x-axis, where f is positive and has a continuous derivative.



(a) Surface of revolution



(b) Approximating band

Figure 4

Area of a Surface of Revolution

In order to define its surface area, we divide the interval $[a, b]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx.

If $y_i = f(x_i)$, then the point $P_i(x_i, y_i)$ lies on the curve.

The part of the surface between x_{i-1} and x_i is approximated by taking the line segment $P_{i-1}P_i$ and rotating it about the x-axis.

Area of a Surface of Revolution

The result is a band with slant height $l = |P_{i-1}P_i|$ and average radius $r = \frac{1}{2}(y_{i-1} + y_i)$ so, by Formula 2, its surface area is

$$2\pi \frac{y_{i-1} + y_i}{2} |P_{i-1}P_i|$$

We have

$$|P_{i-1}P_i| = \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x$$

where x_i^* is some number in $[x_{i-1}, x_i]$.

Area of a Surface of Revolution

When Δx is small, we have $y_i = f(x_i) \approx f(x_i^*)$ and also $Y_{i-1} = f(x_{i-1}) \approx f(x_i^*)$, since f is continuous.

Therefore

$$2\pi \frac{y_{i-1} + y_i}{2} |P_{i-1}P_i| \approx 2\pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x$$

and so an approximation to what we think of as the area of the complete surface of revolution is

$$\boxed{3} \qquad \sum_{i=1}^{n} 2\pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x$$

Area of a Surface of Revolution

This approximation appears to become better as $n \to \infty$ and, recognizing as $\boxed{3}$ Riemann sum for the function $g(x) = 2\pi f(x) \sqrt{1 + [f'(x)]^2}$, we have

$$\lim_{n \to \infty} \sum_{i=1}^{n} 2\pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2} \, \Delta x = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} \, dx$$

Therefore, in the case where *f* is positive and has a continuous derivative, we define the **surface area** of the surface obtained by rotating the curve $y = f(x)$, $a \leq x \leq b$, about the x-axis as

$$\boxed{4} \qquad \boxed{S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} \, dx}$$

Area of a Surface of Revolution

With the Leibniz notation for derivatives, this formula becomes

5

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

If the curve is described as $x = g(y)$, $c \leq y \leq d$, then the formula for surface area becomes

6

$$S = \int_c^d 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Area of a Surface of Revolution

Now both Formulas 5 and 6 can be summarized symbolically, using the notation for arc length, as

7

$$S = \int 2\pi y \, ds$$

For rotation about the *y*-axis, the surface area formula becomes

8

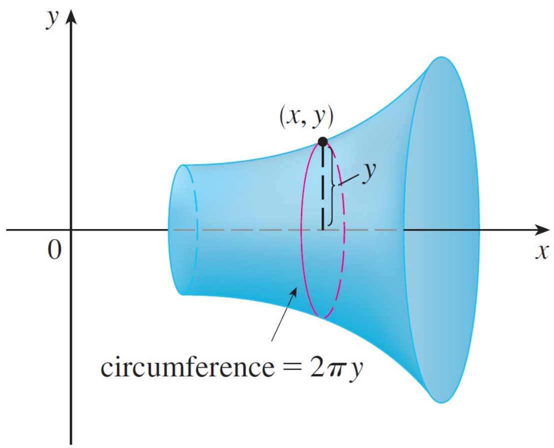
$$S = \int 2\pi x \, ds$$

where, as before, we can use either

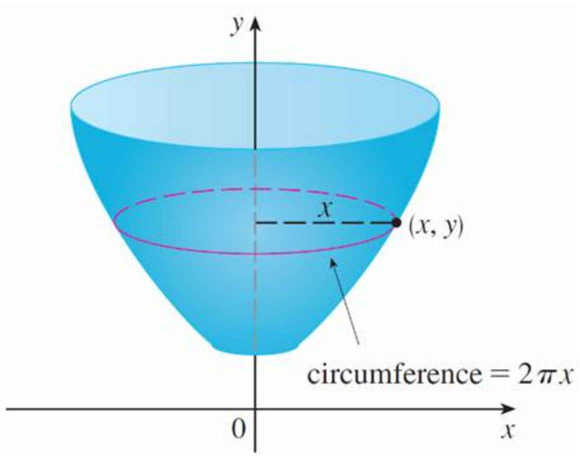
$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{or} \qquad ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Area of a Surface of Revolution

These formulas can be remembered by thinking of $2\pi y$ or $2\pi x$ as the circumference of a circle traced out by the point (x, y) on the curve as it is rotated about the x-axis or y-axis, respectively (see Figure 5).



(a) Rotation about x-axis: $S = \int 2\pi y \, ds$



(b) Rotation about y-axis: $S = \int 2\pi x \, ds$

Figure 5

Example 1

The curve $y = \sqrt{4 - x^2}$, $-1 \le x \le 1$, is an arc of the circle $x^2 + y^2 = 4$.

Find the area of the surface obtained by rotating this arc about the x-axis. (The surface is a portion of a sphere of radius 2. See Figure 6.)

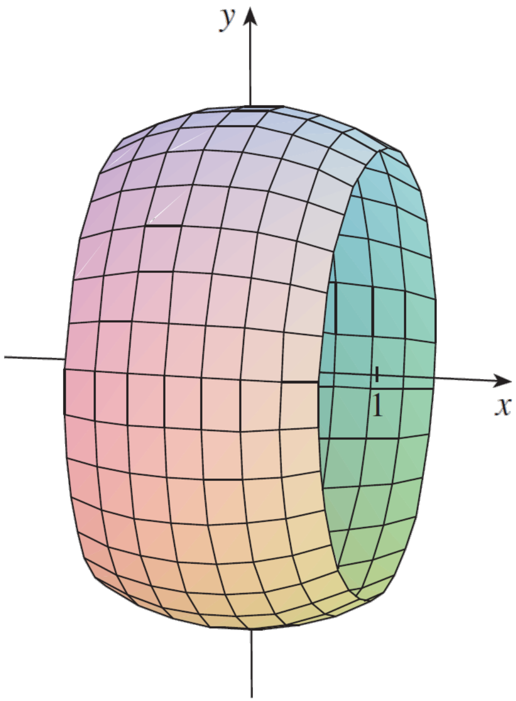


Figure 6

Example 1 – *Solution*

We have

$$\frac{dy}{dx} = \tfrac{1}{2}(4 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{4 - x^2}}$$

and so, by Formula 5, the surface area is

$$S = \int_{-1}^{1} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_{-1}^{1} \sqrt{4 - x^2}\, \sqrt{1 + \frac{x^2}{4 - x^2}}\, dx$$

Example 1 – *Solution*

cont'd

$$= 2\pi \int_{-1}^{1} \sqrt{4 - x^2}\, \frac{2}{\sqrt{4 - x^2}}\, dx$$

$$= 4\pi \int_{-1}^{1} 1\, dx$$

$$= 4\pi(2)$$

$$= 8\pi$$

Example 3

Find the area of the surface generated by rotating the curve $y = e^x$, $0 \le x \le 1$, about the x-axis.

Solution:

Using Formula 5 with

$$y = e^x \qquad \text{and} \qquad \frac{dy}{dx} = e^x$$

we have

$$S = \int_0^1 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_0^1 e^x \sqrt{1 + e^{2x}}\, dx$$

Example 3 – *Solution*

cont'd

$$= 2\pi \int_1^e \sqrt{1 + u^2}\, du$$ (where $u = e^x$)

$$= 2\pi \int_{\pi/4}^{\alpha} \sec^3\theta\, d\theta$$ (where $u = \tan\theta$ and $\alpha = \tan^{-1} e$)

$$= 2\pi \cdot \tfrac{1}{2}\Big[\sec\theta \tan\theta + \ln\left|\sec\theta + \tan\theta\right|\Big]_{\pi/4}^{\alpha}$$

$$= \pi[\sec\alpha \tan\alpha + \ln(\sec\alpha + \tan\alpha) - \sqrt{2} - \ln(\sqrt{2} + 1)]$$

Example 3 – *Solution*

cont'd

Since tan $\alpha = e$, we have $\sec^2\alpha = 1 + \tan^2\alpha = 1 + e^2$ and

$$S = \pi\left[e + \sqrt{1 + e^2} + \ln\left(e + \sqrt{1 + e^2}\right) - \sqrt{2} - \ln(\sqrt{2} + 1)\right]$$

8.3 Applications to Physics and Engineering

Applications to Physics and Engineering

Among the many applications of integral calculus to physics and engineering, we consider two here: force due to water pressure, and centers of mass.

Hydrostatic Pressure and Force

Hydrostatic Pressure and Force

Deep-sea divers realize that water pressure increases as they dive deeper. This is because the weight of the water above them increases.

In general, suppose that a thin horizontal plate with area *A* square meters is submerged in a fluid of density ρ kilograms per cubic meter at a depth *d* meters below the surface of the fluid as in Figure 1.

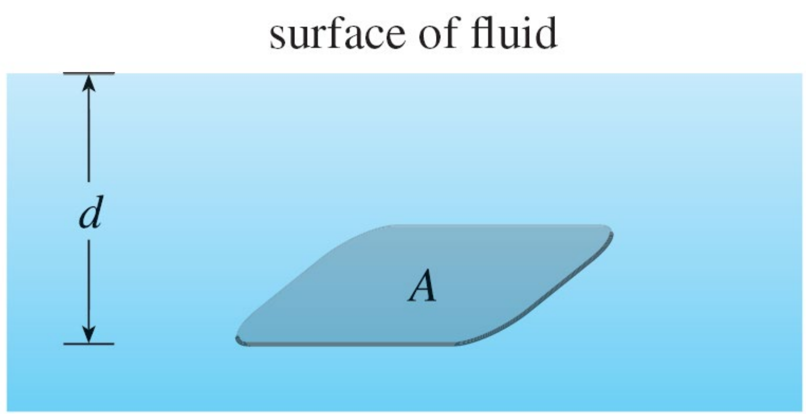


Figure 1

Hydrostatic Pressure and Force

The fluid directly above the plate has volume $V = Ad$, so its mass is $m = \rho V = \rho Ad$. The force exerted by the fluid on the plate is therefore

$$F = mg = \rho gAd$$

where g is the acceleration due to gravity. The **pressure** P on the plate is defined to be the force per unit area:

$$P = \frac{F}{A} = \rho gd$$

The SI unit for measuring pressure is newtons per square meter, which is called a pascal (abbreviation: $1 \text{ N/m}^2 = 1 \text{ Pa}$). Since this is a small unit, the kilopascal (kPa) is often used.

Hydrostatic Pressure and Force

For instance, because the density of water is $\rho = 1000$ kg/m^3, the pressure at the bottom of a swimming pool 2 m deep is

$$P = \rho g d = 1000 \text{ kg/m}^3 \times 9.8 \text{ m/s}^2 \times 2 \text{ m}$$

$$= 19{,}600 \text{ Pa} = 19.6 \text{ kPa}$$

An important principle of fluid pressure is the experimentally verified fact that *at any point in a liquid the pressure is the same in all directions.* (A diver feels the same pressure on nose and both ears.)

Hydrostatic Pressure and Force

Thus the pressure in *any* direction at a depth d in a fluid with mass density ρ is given by

$$P = \rho g d = \delta d \qquad \boxed{1}$$

This helps us determine the hydrostatic force against a *vertical* plate or wall or dam in a fluid.

This is not a straightforward problem because the pressure is not constant but increases as the depth increases.

Example 1

A dam has the shape of the trapezoid shown in Figure 2. The height is 20 m and the width is 50 m at the top and 30 m at the bottom. Find the force on the dam due to hydrostatic pressure if the water level is 4 m from the top of the dam.

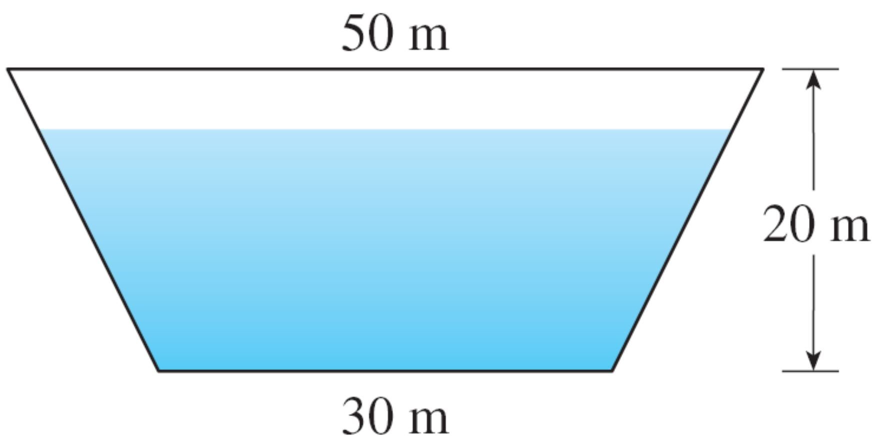


Figure 2

Example 1 – *Solution*

We choose a vertical *x*-axis with origin at the surface of the water as in Figure 3(a).

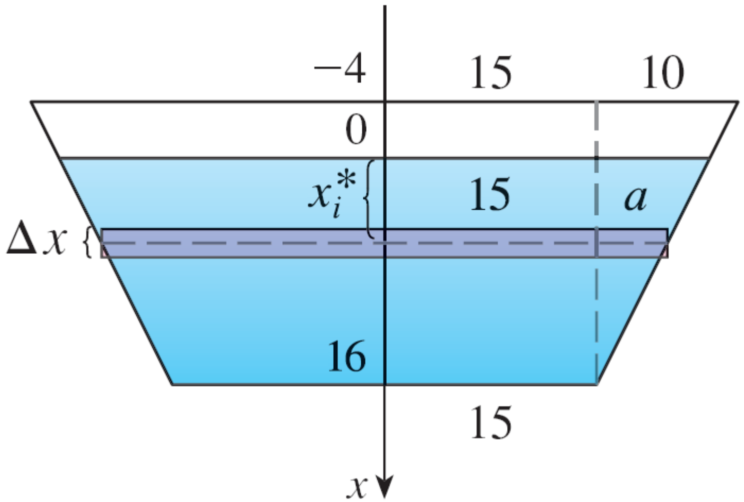


Figure 3(a)

The depth of the water is 16 m, so we divide the interval [0, 16] into subintervals of equal length with endpoints x_i and we choose $x_i^* \in [x_{i-1}, x_i]$.

Example 1 – *Solution*

cont'd

The ith horizontal strip of the dam is approximated by a rectangle with height Δx and width w_i, where, from similar triangles in Figure 3(b),

$$\frac{a}{16 - x_i^*} = \frac{10}{20}$$

or $a = \dfrac{16 - x_i^*}{2} = 8 - \dfrac{x_i^*}{2}$

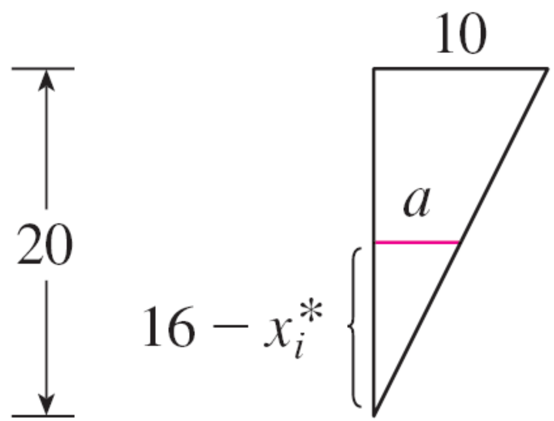


Figure 3(b)

and so $$w_i = 2(15 + a)$$

$$= 2(15 + 8 - \tfrac{1}{2}x_i^*)$$

$$= 46 - x_i^*$$

Example 1 – *Solution*

cont'd

If A_i is the area of the ith strip, then

$$A_i \approx w_i \,\Delta x$$
$$= (46 - x_i^*) \,\Delta x$$

If Δx is small, then the pressure P_i on the ith strip is almost constant and we can use Equation 1 to write

$$P_i \approx 1000 g x_i^*$$

The hydrostatic force F_i acting on the ith strip is the product of the pressure and the area:

$$F_i = P_i A_i$$
$$\approx 1000 g x_i^* (46 - x_i^*) \,\Delta x$$

Example 1 – *Solution*

cont'd

Adding these forces and taking the limit as $n \to \infty$, we obtain the total hydrostatic force on the dam:

$$
\begin{aligned}
F &= \lim_{n\to\infty} \sum_{i=1}^{n} 1000gx_i^*(46 - x_i^*)\,\Delta x \\
&= \int_0^{16} 1000gx(46 - x)\,dx \\
&= 1000(9.8)\int_0^{16} (46x - x^2)\,dx \\
&= 9800\left[23x^2 - \frac{x^3}{3}\right]_0^{16} \\
&\approx 4.43 \times 10^7\ \mathrm{N}
\end{aligned}
$$

Moments and Centers of Mass

Moments and Centers of Mass

Our main objective here is to find the point *P* on which a thin plate of any given shape balances horizontally as in Figure 8.

This point is called the **center of mass** (or center of gravity) of the plate.

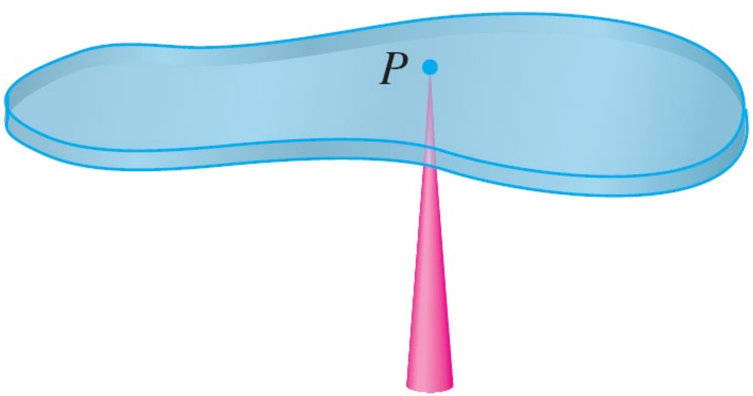


Figure 5

Moments and Centers of Mass

We first consider the simpler situation illustrated in Figure 6, where two masses m_1 and m_2 are attached to a rod of negligible mass on opposite sides of a fulcrum and at distances d_1 and d_2 from the fulcrum.

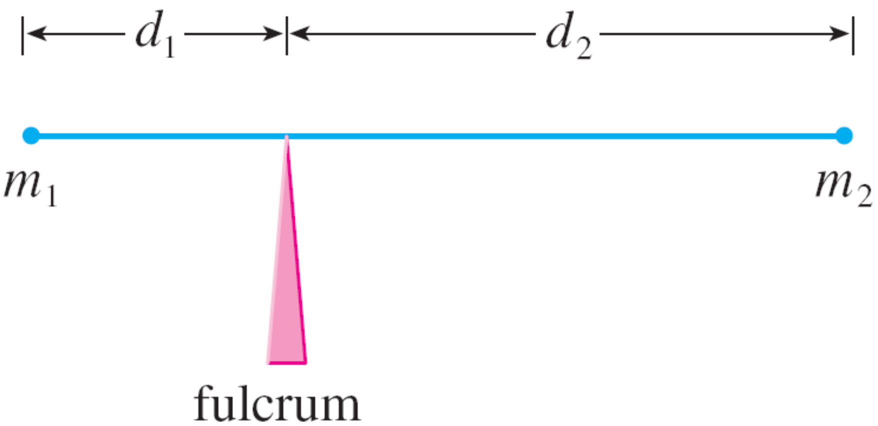


Figure 6

The rod will balance if

$$m_1 d_1 = m_2 d_2 \qquad \boxed{2}$$

Moments and Centers of Mass

This is an experimental fact discovered by Archimedes and called the Law of the Lever. (Think of a lighter person balancing a heavier one on a seesaw by sitting farther away from the center.)

Now suppose that the rod lies along the x-axis with m_1 at x_1 and m_2 at x_2 and the center of mass at $\overline{x}$.

Moments and Centers of Mass

If we compare Figures 6 and 7, we see that $d_1 = \bar{x} - x_1$ and $d_2 = x_2 - \bar{x}$ and so Equation 2 gives

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

$$m_1\bar{x} + m_2\bar{x} = m_1x_1 + m_2x_2$$

$$\boxed{3} \qquad \bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

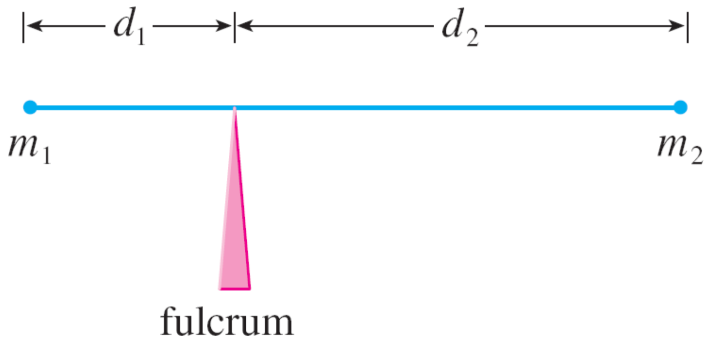


Figure 6

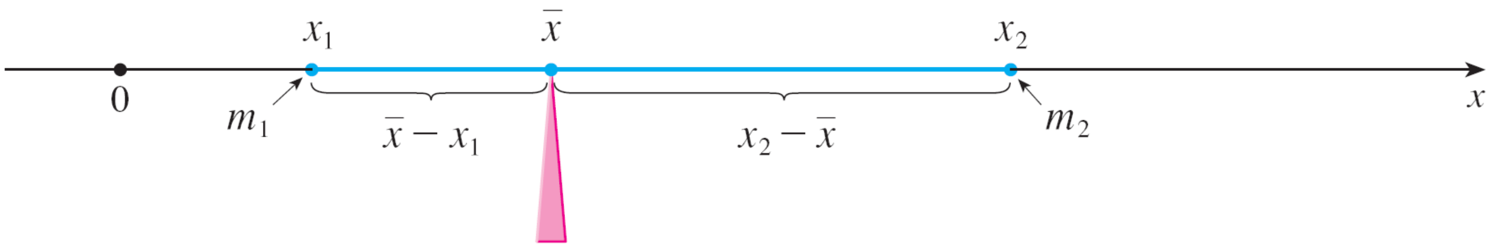


Figure 7

Moments and Centers of Mass

The numbers m_1x_1 and m_2x_2 are called the **moments** of the masses m_1 and m_2 (with respect to the origin), and Equation 3 says that the center of mass $\bar{x}$ is obtained by adding the moments of the masses and dividing by the total mass $m = m_1 + m_2$.

Moments and Centers of Mass

In general, if we have a system of n particles with masses $m_1, m_2, \ldots, m_n$ located at the points $x_1, x_2, \ldots, x_n$ on the x-axis, it can be shown similarly that the center of mass of the system is located at

$$\boxed{4} \qquad \bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i x_i}{m}$$

where $m = \Sigma\, m_i$ is the total mass of the system, and the sum of the individual moments

$$M = \sum_{i=1}^{n} m_i x_i$$

is called the **moment of the system about the origin**.

Moments and Centers of Mass

Then Equation 4 could be rewritten as $m\bar{x} = M$, which says that if the total mass were considered as being concentrated at the center of mass $\bar{x}$, then its moment would be the same as the moment of the system.

Now we consider a system of n particles with masses $m_1, m_2, \ldots, m_n$ located at the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in the xy-plane as shown in Figure 8.

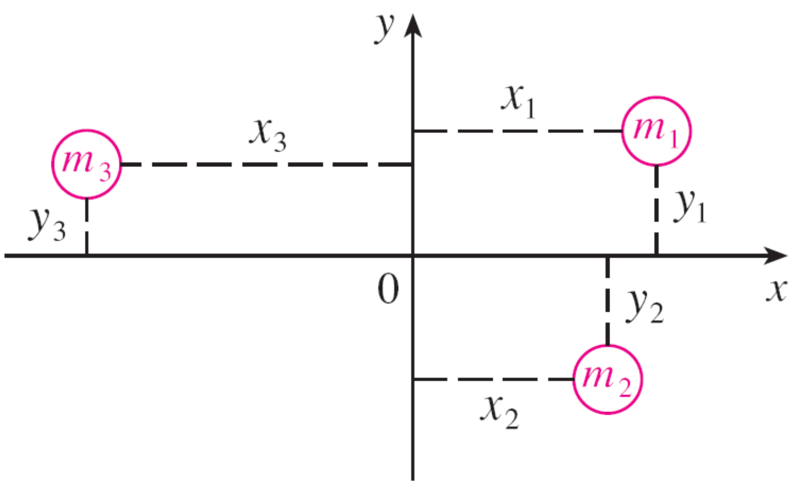


Figure 8

Moments and Centers of Mass

By analogy with the one-dimensional case, we define the **moment of the system about the *y*-axis** to be

$$\boxed{5} \qquad M_y = \sum_{i=1}^{n} m_i x_i$$

and the **moment of the system about the *x*-axis** as

$$\boxed{6} \qquad M_x = \sum_{i=1}^{n} m_i y_i$$

Then M_y measures the tendency of the system to rotate about the y-axis and M_x measures the tendency to rotate about the x-axis.

Moments and Centers of Mass

As in the one-dimensional case, the coordinates $(\bar{x}, \bar{y})$ of the center of mass are given in terms of the moments by the formulas

$$\boxed{7} \qquad \bar{x} = \frac{M_y}{m} \qquad \bar{y} = \frac{M_x}{m}$$

where $m = \Sigma\, m_i$ is the total mass. Since $m\bar{x} = M_y$ and $m\bar{y} = M_x$, the center of mass $(\bar{x}, \bar{y})$ is the point where a single particle of mass m would have the same moments as the system.

Example 3

Find the moments and center of mass of the system of objects that have masses 3, 4, and 8 at the points (–1, 1), (2, –1), and (3, 2), respectively.

Solution:

We use Equations 5 and 6 to compute the moments:

$$M_y = 3(-1) + 4(2) + 8(3) = 29$$

$$M_x = 3(1) + 4(-1) + 8(2) = 15$$

Since $m = 3 + 4 + 8 = 15$, we use Equations 11 to obtain

$$\bar{x} = \frac{M_y}{m} = \frac{29}{15} \qquad \bar{y} = \frac{M_x}{m} = \frac{15}{15} = 1$$

Example 3 – *Solution*

cont'd

Thus the center of mass is $\left(1\frac{14}{15}, 1\right)$. (See Figure 9.)

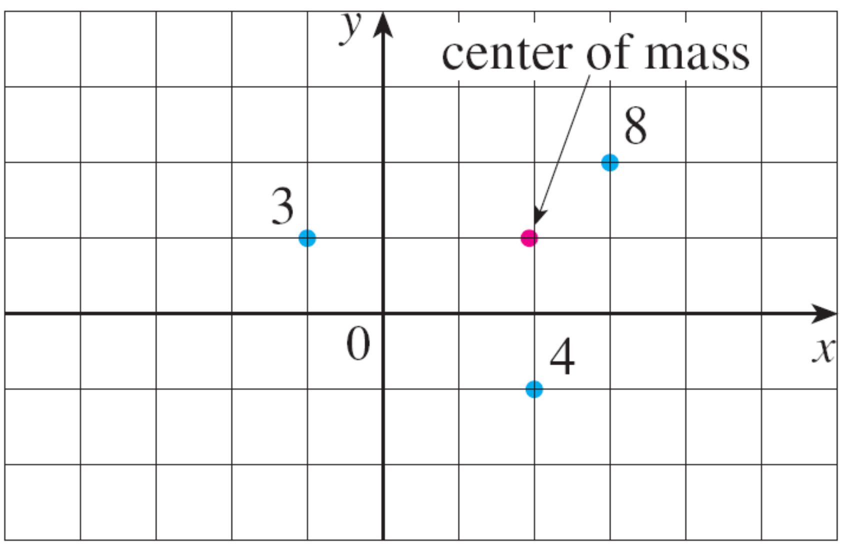


Figure 9

Moments and Centers of Mass

Next we consider a flat plate (called a *lamina*) with uniform density ρ that occupies a region $\mathcal{R}$ of the plane.

We wish to locate the center of mass of the plate, which is called the **centroid** of $\mathcal{R}$.

In doing so we use the following physical principles: The **symmetry principle** says that if $\mathcal{R}$ is symmetric about a line *l*, then the centroid of $\mathcal{R}$ lies on *l*. (If $\mathcal{R}$ is reflected about *l*, then $\mathcal{R}$ remains the same so its centroid remains fixed. But the only fixed points lie on *l*.)

Thus the centroid of a rectangle is its center.

Moments and Centers of Mass

Moments should be defined so that if the entire mass of a region is concentrated at the center of mass, then its moments remain unchanged.

Also, the moment of the union of two nonoverlapping regions should be the sum of the moments of the individual regions.

Suppose that the region $\mathcal{R}$ is of the type shown in Figure 10(a); that is, $\mathcal{R}$ lies between the lines $x = a$ and $x = b$, above the x-axis, and beneath the graph of f, where f is a continuous function.

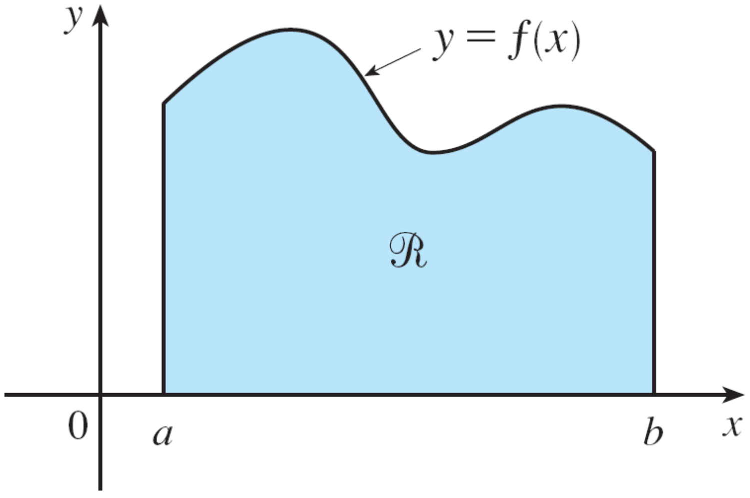


Figure 10(a)

Moments and Centers of Mass

We divide the interval $[a, b]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx. We choose the sample point x_i^* to be the midpoint $\overline{x}_i$ of the ith subinterval, that is, $\overline{x}_i = (x_{i-1} + x_i)/2$.

This determines the polygonal approximation to $\mathcal{R}$ shown in Figure 10(b).

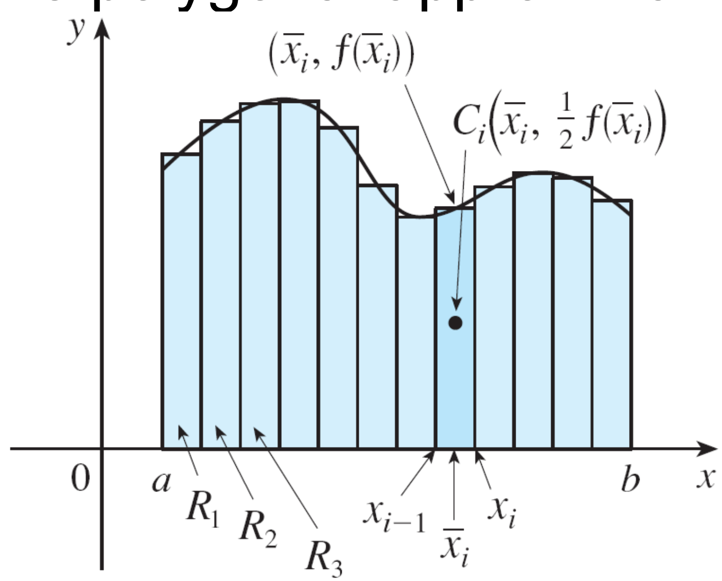


Figure 10(b)

Moments and Centers of Mass

The centroid of the *i*th approximating rectangle R_i is its center $C_i\left(\bar{x}_i, \frac{1}{2}f(\bar{x}_i)\right)$. Its area is $f(\bar{x}_i)\,\Delta x$, so its mass is

$$\rho f(\bar{x}_i)\,\Delta x$$

The moment of R_i about the *y*-axis is the product of its mass and the distance from C_i to the *y*-axis, which is $\bar{x}_i$. Thus

$$M_y(R_i) = [\rho f(\bar{x}_i)\,\Delta x]\,\bar{x}_i = \rho\bar{x}_i f(\bar{x}_i)\,\Delta x$$

Adding these moments, we obtain the moment of the polygonal approximation to $\mathfrak{R}$, and then by taking the limit as $n \to \infty$ we obtain the moment of $\mathfrak{R}$ itself about the *y*-axis:

Moments and Centers of Mass

$$M_y = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \overline{x}_i f(\overline{x}_i)\,\Delta x = \rho \int_a^b x f(x)\,dx$$

In a similar fashion we compute the moment of R_i about the x-axis as the product of its mass and the distance from C_i to the x-axis:

$$M_x(R_i) = [\rho f(\overline{x}_i)\,\Delta x]\,\tfrac{1}{2} f(\overline{x}_i) = \rho \cdot \tfrac{1}{2}[f(\overline{x}_i)]^2\,\Delta x$$

Again we add these moments and take the limit to obtain the moment of $\mathcal{R}$ about the x-axis:

$$M_x = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \cdot \tfrac{1}{2}[f(\overline{x}_i)]^2\,\Delta x = \rho \int_a^b \tfrac{1}{2}[f(x)]^2\,dx$$

Moments and Centers of Mass

Just as for systems of particles, the center of mass of the plate is defined so that $m\bar{x} = M_y$ and $m\bar{y} = M_x$. But the mass of the plate is the product of its density and its area:

$$m = \rho A = \rho \int_a^b f(x)\,dx$$

and so

$$\bar{x} = \frac{M_y}{m} = \frac{\rho \int_a^b x f(x)\,dx}{\rho \int_a^b f(x)\,dx} = \frac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\rho \int_a^b \frac{1}{2}[f(x)]^2\,dx}{\rho \int_a^b f(x)\,dx} = \frac{\int_a^b \frac{1}{2}[f(x)]^2\,dx}{\int_a^b f(x)\,dx}$$

Moments and Centers of Mass

Notice the cancellation of the ρ's. The location of the center of mass is independent of the density.

In summary, the center of mass of the plate (or the centroid of $\mathcal{R}$) is located at the point $(\bar{x}, \bar{y})$, where

8

$$\boxed{\bar{x} = \frac{1}{A}\int_a^b x f(x)\,dx \qquad \bar{y} = \frac{1}{A}\int_a^b \tfrac{1}{2}[f(x)]^2\,dx}$$

Moments and Centers of Mass

If the region $\mathcal{R}$ lies between two curves $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$, as illustrated in Figure 13, then the same sort of argument that led to Formulas 8 can be used to show that the centroid of $\mathcal{R}$ is $(\bar{x}, \bar{y})$, where

9

$$\bar{x} = \frac{1}{A} \int_a^b x[f(x) - g(x)]\, dx$$

$$\bar{y} = \frac{1}{A} \int_a^b \tfrac{1}{2}\{[f(x)]^2 - [g(x)]^2\}\, dx$$

Moments and Centers of Mass

We end this section by showing a surprising connection between centroids and volumes of revolution.

Theorem of Pappus Let $\mathcal{R}$ be a plane region that lies entirely on one side of a line l in the plane. If $\mathcal{R}$ is rotated about l, then the volume of the resulting solid is the product of the area A of $\mathcal{R}$ and the distance d traveled by the centroid of $\mathcal{R}$.

8.4 Applications to Economics and Biology

Applications to Economics and Biology

In this section we consider some applications of integration to economics (consumer surplus) and biology (blood flow, cardiac output).

Consumer Surplus

Consumer Surplus

Recall that the demand function $p(x)$ is the price that a company has to charge in order to sell x units of a commodity.

Usually, selling larger quantities requires lowering prices, so the demand function is a decreasing function. The graph of a typical demand function, called a **demand curve**, is shown in Figure 1.

If X is the amount of the commodity that is currently available, then $P = p(X)$ is the current selling price.

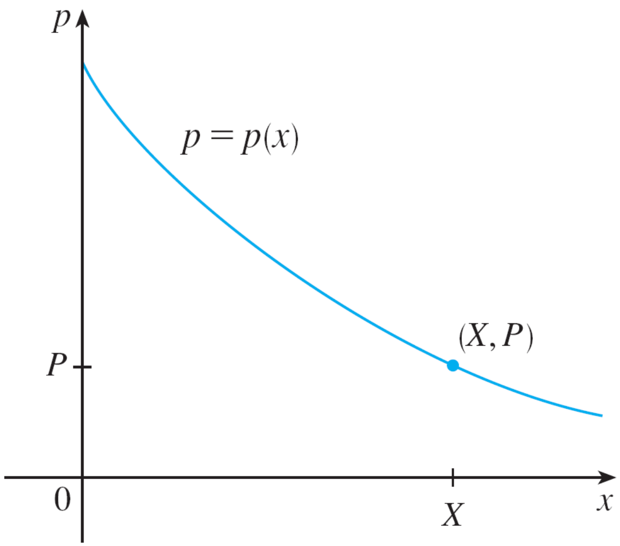


Figure 1

A typical demand curve

Consumer Surplus

We divide the interval [0, X] into n subintervals, each of length $\Delta x = X/n$, and let $x_i^* = x_i$ be the right endpoint of the ith subinterval, as in Figure 2.

If, after the first x_{i-1} units were sold, a total of only x_i units had been available and the price per unit had been set at $p(x_i)$ dollars, then the additional Δx units could have been sold (but no more).

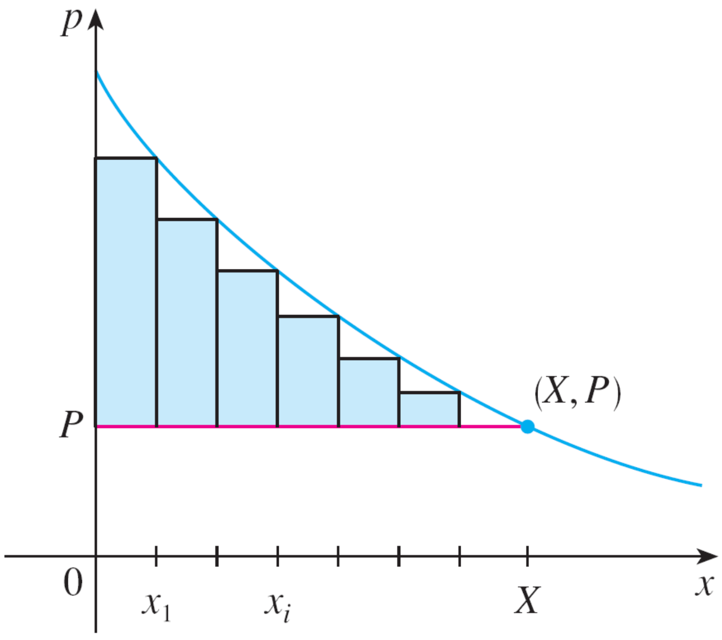


Figure 2

Consumer Surplus

The consumers who would have paid $p(x_i)$ dollars placed a high value on the product; they would have paid what it was worth to them.

So, in paying only P dollars they have saved an amount of

$$(\text{savings per unit})\ (\text{number of units}) = [p(x_i) - P]\ \Delta x$$

Consumer Surplus

Considering similar groups of willing consumers for each of the subintervals and adding the savings, we get the total savings:

$$\sum_{i=1}^{n} [p(x_i) - P] \, \Delta x$$

(This sum corresponds to the area enclosed by the rectangles in Figure 2.)

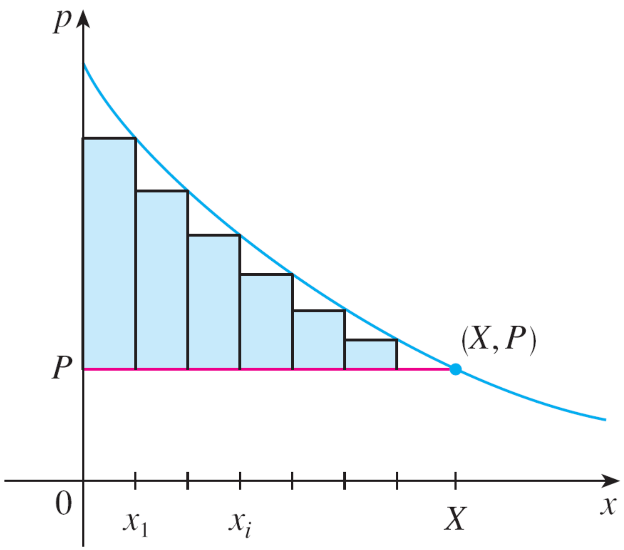


Figure 2

Consumer Surplus

If we let $n \to \infty$, this Riemann sum approaches the integral

$$\boxed{1} \qquad \int_0^X [p(x) - P]\, dx$$

which economists call the **consumer surplus** for the commodity.

The consumer surplus represents the amount of money saved by consumers in purchasing the commodity at price P, corresponding to an amount demanded of X.

Consumer Surplus

Figure 3 shows the interpretation of the consumer surplus as the area under the demand curve and above the line $p = P$.

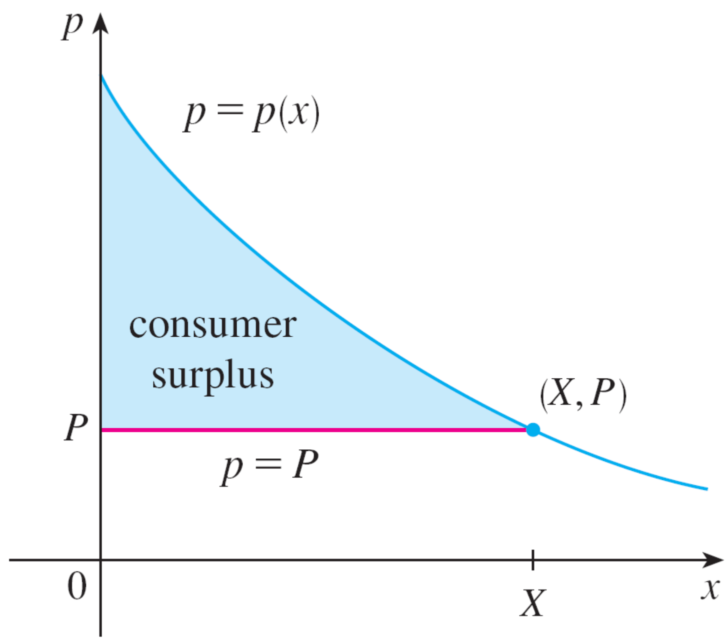


Figure 3

Example 1

The demand for a product, in dollars, is

$$p = 1200 - 0.2x - 0.0001x^2$$

Find the consumer surplus when the sales level is 500.

Solution:

Since the number of products sold is $X = 500$, the corresponding price is

$$\begin{aligned} P &= 1200 - (0.2)(500) - (0.0001)(500)^2 \\ &= 1075 \end{aligned}$$

Example 1 – *Solution*

cont'd

Therefore, from Definition 1, the consumer surplus is

$$\int_0^{500} [p(x) - P]\, dx = \int_0^{500} (1200 - 0.2x - 0.0001x^2 - 1075)dx$$

$$= \int_0^{500} (125 - 0.2x - 0.0001x^2)\, dx$$

$$= 125x - 0.1x^2 - (0.0001)\left(\frac{x^3}{3}\right)\Bigg]_0^{500}$$

$$= (125)(500) - (0.1)(500)^2 - \frac{(0.0001)(500)^3}{3}$$

$$= \$33{,}333.33$$

Blood Flow

Blood Flow

We have discussed the law of laminar flow:

$$v(r) = \frac{P}{4\eta l}(R^2 - r^2)$$

which gives the velocity v of blood that flows along a blood vessel with radius R and length l at a distance r from the central axis, where P is the pressure difference between the ends of the vessel and η is the viscosity of the blood.

Now, in order to compute the rate of blood flow, or *flux* (volume per unit time), we consider smaller, equally spaced radii $r_1, r_2, \ldots$.

Blood Flow

The approximate area of the ring (or washer) with inner radius r_{i-1} and outer radius r_i is

$$2\pi r_i \,\Delta r \qquad \text{where} \qquad \Delta r = r_i - r_{i-1}$$

(See Figure 4.)

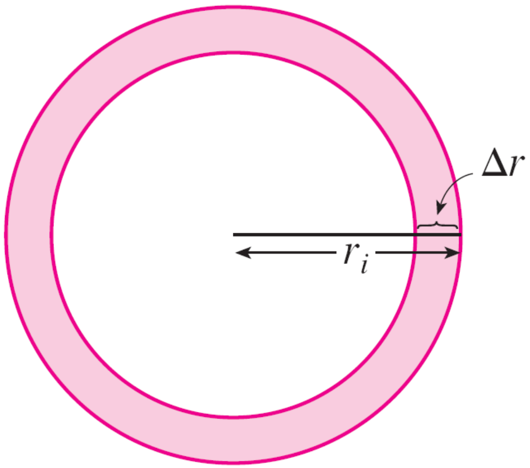


Figure 4

Blood Flow

If Δr is small, then the velocity is almost constant throughout this ring and can be approximated by $v(r_i)$.

Thus the volume of blood per unit time that flows across the ring is approximately

$$(2\pi r_i \, \Delta r) \, v(r_i) = 2\pi r_i \, v(r_i) \, \Delta r$$

and the total volume of blood that flows across a cross-section per unit time is about

$$\sum_{i=1}^{n} 2\pi r_i \, v(r_i) \, \Delta r$$

This approximation is illustrated in Figure 5.

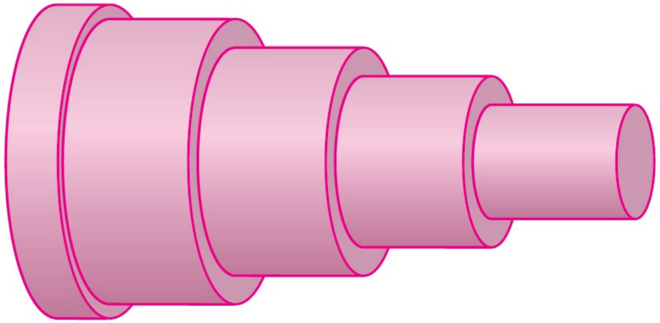


Figure 5

Blood Flow

Notice that the velocity (and hence the volume per unit time) increases toward the center of the blood vessel.

The approximation gets better as *n* increases.

When we take the limit we get the exact value of the **flux** (or *discharge*), which is the volume of blood that passes a cross-section per unit time:

$$F = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi r_i \, v(r_i) \, \Delta r$$

$$= \int_0^R 2\pi r \, v(r) \, dr$$

Blood Flow

$$= \int_0^R 2\pi r \frac{P}{4\eta l}(R^2 - r^2)\,dr$$

$$= \frac{\pi P}{2\eta l}\int_0^R (R^2 r - r^3)\,dr$$

$$= \frac{\pi P}{2\eta l}\left[R^2\frac{r^2}{2} - \frac{r^4}{4}\right]_{r=0}^{r=R}$$

$$= \frac{\pi P}{2\eta l}\left[\frac{R^4}{2} - \frac{R^4}{4}\right]$$

$$= \frac{\pi P R^4}{8\eta l}$$

Blood Flow

The resulting equation

$$\boxed{2} \qquad F = \frac{\pi P R^4}{8 \eta l}$$

is called **Poiseuille's Law**; it shows that the flux is proportional to the fourth power of the radius of the blood vessel.

Cardiac Output

Cardiac Output

Figure 6 shows the human cardiovascular system.

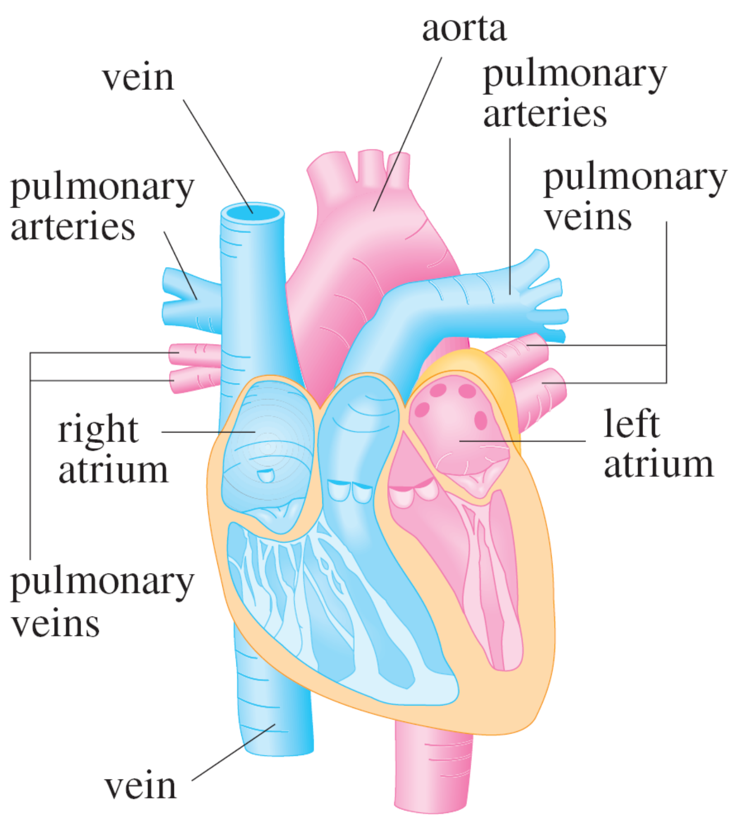


Figure 6

Cardiac Output

Blood returns from the body through the veins, enters the right atrium of the heart, and is pumped to the lungs through the pulmonary arteries for oxygenation.

It then flows back into the left atrium through the pulmonary veins and then out to the rest of the body through the aorta.

The **cardiac output** of the heart is the volume of blood pumped by the heart per unit time, that is, the rate of flow into the aorta.

The *dye dilution method* is used to measure the cardiac output.

Cardiac Output

Dye is injected into the right atrium and flows through the heart into the aorta. A probe inserted into the aorta measures the concentration of the dye leaving the heart at equally spaced times over a time interval $[0, T]$ until the dye has cleared.

Let $c(t)$ be the concentration of the dye at time t. If we divide $[0, T]$ into subintervals of equal length Δt, then the amount of dye that flows past the measuring point during the subinterval from $t = t_{i-1}$ to $t = t_i$ is approximately

$$(\text{concentration})\ (\text{volume}) = c(t_i)\ (F\,\Delta t)$$

where F is the rate of flow that we are trying to determine.

Cardiac Output

Thus the total amount of dye is approximately

$$\sum_{i=1}^{n} c(t_i)F\,\Delta t = F\sum_{i=1}^{n} c(t_i)\,\Delta t$$

and, letting n $\rightarrow \infty$, we find that the amount of dye is

$$A = F\int_0^T c(t)\,dt$$

Thus the cardiac output is given by

$$\boxed{3} \qquad F = \frac{A}{\int_0^T c(t)\,dt}$$

where the amount of dye A is known and the integral can be approximated from the concentration readings.

Example 2

A 5-mg bolus of dye is injected into a right atrium. The concentration of the dye (in milligrams per liter) is measured in the aorta at one-second intervals as shown in the chart. Estimate the cardiac output.

t	$c(t)$	t	$c(t)$
0	0	6	6.1
1	0.4	7	4.0
2	2.8	8	2.3
3	6.5	9	1.1
4	9.8	10	0
5	8.9		

Example 2 – *Solution*

Here $A = 5$, $\Delta t = 1$, and $T = 10$. We use Simpson's Rule to approximate the integral of the concentration:

$$\int_0^{10} c(t)\,dt \approx \frac{1}{3}[0 + 4(0.4) + 2(2.8) + 4(6.5) + 2(9.8) + 4(8.9)$$

$$+ 2(6.1) + 4(4.0) + 2(2.3) + 4(1.1) + 0]$$

$$\approx 41.87$$

Thus Formula 3 gives the cardiac output to be

$$F = \frac{A}{\int_0^{10} c(t)\,dt} \approx \frac{5}{41.87}$$

$$\approx 0.12 \text{ L/s}$$

$$= 7.2 \text{ L/min}$$

8.5 Probability

Probability

Calculus plays a role in the analysis of random behavior. Suppose we consider the cholesterol level of a person chosen at random from a certain age group, or the height of an adult female chosen at random, or the lifetime of a randomly chosen battery of a certain type.

Such quantities are called **continuous random variables** because their values actually range over an interval of real numbers, although they might be measured or recorded only to the nearest integer.

Probability

We might want to know the probability that a blood cholesterol level is greater than 250, or the probability that the height of an adult female is between 60 and 70 inches, or the probability that the battery we are buying lasts between 100 and 200 hours.

If X represents the lifetime of that type of battery, we denote this last probability as follows:

$$P(100 \leq X \leq 200)$$

Probability

According to the frequency interpretation of probability, this number is the long-run proportion of all batteries of the specified type whose lifetimes are between 100 and 200 hours. Since it represents a proportion, the probability naturally falls between 0 and 1.

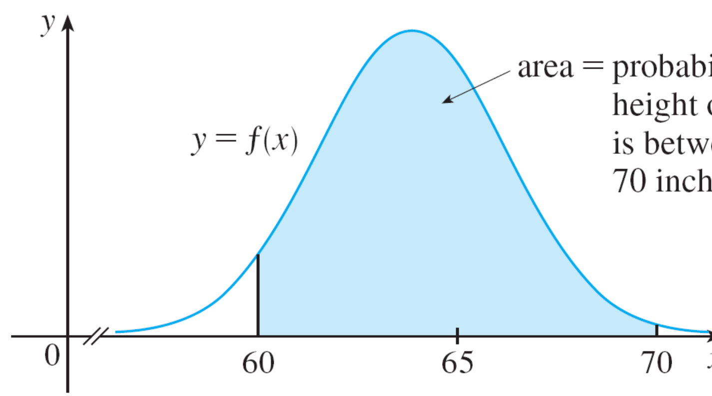
Every continuous random variable *X* has a **probability density function** *f*. This means that the probability that *X* lies between *a* and *b* is found by integrating *f* from *a* to *b*:

$$\boxed{1} \qquad P(a \leqslant X \leqslant b) = \int_a^b f(x)\,dx$$

Probability

For example, Figure 1 shows the graph of a model for the probability density function *f* for a random variable *X* defined to be the height in inches of an adult female in the United States (according to data from the National Health Survey).

The probability that the height of a woman chosen at random from this population is between 60 and 70 inches is equal to the area under the graph of *f* from 60 to 70.



Probability density function for the height of an adult female

Figure 1

Probability

In general, the probability density function *f* of a random variable *X* satisfies the condition $f(x) \geq 0$ for all x.

Because probabilities are measured on a scale from 0 to 1, it follows that

$$\boxed{2} \qquad \int_{-\infty}^{\infty} f(x)\, dx = 1$$

Example 1

Let $f(x) = 0.006x(10 - x)$ for $0 \leq x \leq 10$ and $f(x) = 0$ for all other values of x.

(a) Verify that f is a probability density function.

(b) Find $P(4 \leq X \leq 8)$.

Solution:

(a) For $0 \leq x \leq 10$ we have $0.006x(10 - x) \geq 0$, so $f(x) \geq 0$ for all x. We also need to check that Equation 2 is satisfied:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{10} 0.006x(10 - x)\,dx$$

$$= 0.006 \int_0^{10} (10x - x^2)\,dx$$

Example 1 – *Solution*

cont'd

$$= 0.006\left[5x^2 - \tfrac{1}{3}x^3\right]_0^{10}$$

$$= 0.006\left(500 - \tfrac{1000}{3}\right)$$

$$= 1$$

Therefore *f* is a probability density function.

Example 1 – *Solution*

cont'd

(b) The probability that *X* lies between 4 and 8 is

$$P(4 \leqslant X \leqslant 8) = \int_4^8 f(x)\,dx$$

$$= 0.006 \int_4^8 (10x - x^2)\,dx$$

$$= 0.006\left[5x^2 - \tfrac{1}{3}x^3\right]_4^8$$

$$= 0.544$$

Average Values

Average Values

Suppose you're waiting for a company to answer your phone call and you wonder how long, on average, you can expect to wait.

Let $f(t)$ be the corresponding density function, where t is measured in minutes, and think of a sample of N people who have called this company.

Most likely, none of them had to wait more than an hour, so let's restrict our attention to the interval $0 \leq t \leq 60$.

Average Values

Let's divide that interval into n intervals of length Δt and endpoints 0, t_1, t_2, . . ., t_{60}. (Think of Δt as lasting a minute, or half a minute, or 10 seconds, or even a second.)

The probability that somebody's call gets answered during the time period from t_{i-1} to t_i is the area under the curve $y = f(t)$ from t_{i-1} to t_i, which is approximately equal to $f(\bar{t}_i)\,\Delta t$. (This is the area of the approximating rectangle in Figure 3, where $\bar{t}_i$ is the midpoint of the interval.)

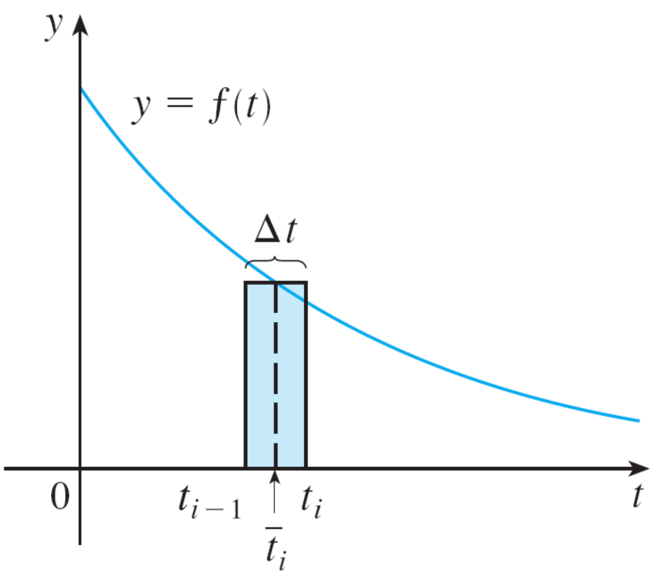


Figure 3

Average Values

Since the long-run proportion of calls that get answered in the time period from t_{i-1} to t_i is $f(\bar{t}_i)\,\Delta t$, we expect that, out of our sample of N callers, the number whose call was answered in that time period is approximately $N\,f(\bar{t}_i)\Delta t$ and the time that each waited is about $\bar{t}_i$.

Therefore the total time they waited is the product of these numbers: approximately $\bar{t}_i\,[N\,f(\bar{t}_i)\,\Delta t]$.

Adding over all such intervals, we get the approximate total of everybody's waiting times:

$$\sum_{i=1}^{n} N\bar{t}_i\,f(\bar{t}_i)\,\Delta t$$

Average Values

If we now divide by the number of callers *N*, we get the approximate *average* waiting time:

$$\sum_{i=1}^{n} \bar{t}_i f(\bar{t}_i)\,\Delta t$$

We recognize this as a Riemann sum for the function $t\,f(t)$. As the time interval shrinks (that is, $\Delta t \to 0$ and $n \to \infty$), this Riemann sum approaches the integral

$$\int_0^{60} t\,f(t)\,dt$$

This integral is called the *mean waiting time.*

Average Values

In general, the **mean** of any probability density function *f* is defined to be

$$\mu = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

The mean can be interpreted as the long-run average value of the random variable *X*. It can also be interpreted as a measure of centrality of the probability density function.

The expression for the mean resembles an integral we have seen before.

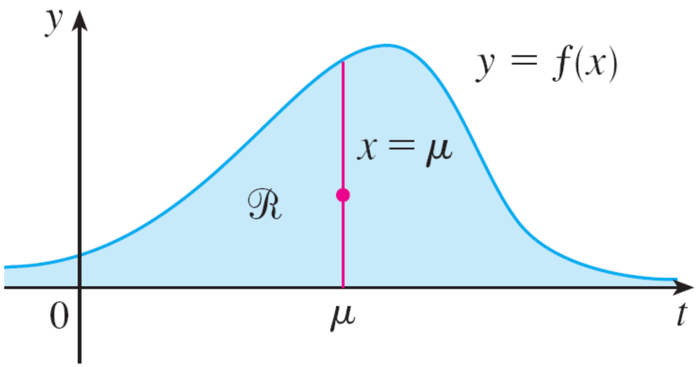
Average Values

If $\mathcal{R}$ is the region that lies under the graph of *f*, we know that the *x*-coordinate of the centroid of $\mathcal{R}$ is

$$\bar{x} = \frac{\int_{-\infty}^{\infty} x\, f(x)\, dx}{\int_{-\infty}^{\infty} f(x)\, dx} = \int_{-\infty}^{\infty} x\, f(x)\, dx = \mu$$

because of Equation 2.

So a thin plate in the shape of $\mathcal{R}$ balances at a point on the vertical line $x = \mu$. (See Figure 4.)



$\mathcal{R}$ balances at a point on the line $x = \mu$

Figure 4

Example 3

Find the mean of the exponential distribution of Example 2:

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ ce^{-ct} & \text{if } t \geq 0 \end{cases}$$

Solution:

According to the definition of a mean, we have

$$\mu = \int_{-\infty}^{\infty} t f(t)\, dt$$

$$= \int_{0}^{\infty} tce^{-ct}\, dt$$

Example 3 – *Solution*

cont'd

To evaluate this integral we use integration by parts, with $u = t$ and $dv = ce^{-ct}\,dt$:

$$\int_0^\infty tce^{-ct}\,dt = \lim_{x \to \infty} \int_0^x tce^{-ct}\,dt$$

$$= \lim_{x \to \infty} \left(-te^{-ct} \Big]_0^x + \int_0^x e^{-ct}\,dt \right)$$

$$= \lim_{x \to \infty} \left(-xe^{-cx} + \frac{1}{c} - \frac{e^{-cx}}{c} \right)$$

$$= \frac{1}{c}$$

Example 3 – *Solution*

cont'd

The mean is $\mu = 1/c$, so we can rewrite the probability density function as

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \mu^{-1}e^{-t/\mu} & \text{if } t \geq 0 \end{cases}$$

Average Values

Another measure of centrality of a probability density function is the *median.*

That is a number *m* such that half the callers have a waiting time less than *m* and the other callers have a waiting time longer than *m*. In general, the **median** of a probability density function is the number *m* such that

$$\int_m^\infty f(x)\,dx = \frac{1}{2}$$

This means that half the area under the graph of *f* lies to the right of *m*.

Normal Distributions

Normal Distributions

Many important random phenomena—such as test scores on aptitude tests, heights and weights of individuals from a homogeneous population, annual rainfall in a given location—are modeled by a **normal distribution**.

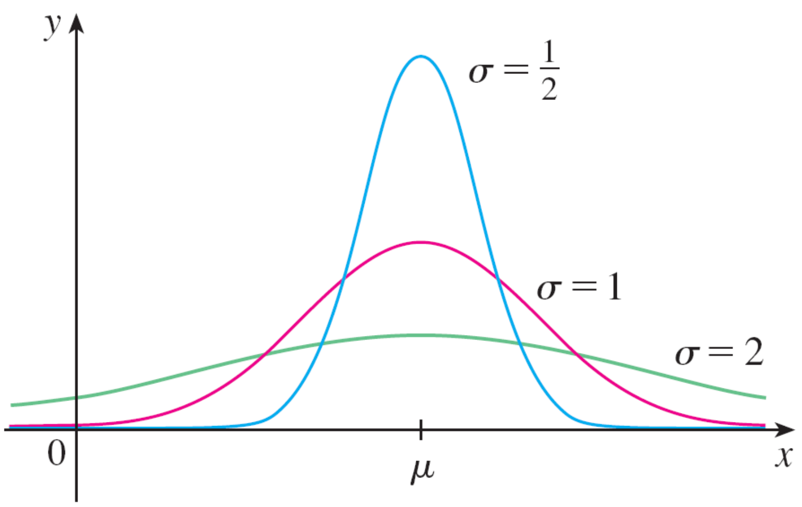
This means that the probability density function of the random variable *X* is a member of the family of functions

$$\boxed{3} \qquad f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)}$$

Normal Distributions

You can verify that the mean for this function is μ. The positive constant σ is called the **standard deviation**; it measures how spread out the values of X are.

From the bell-shaped graphs of members of the family in Figure 5, we see that for small values of σ the values of X are clustered about the mean, whereas for larger values of σ the values of X are more spread out.



Normal distributions

Figure 5

Normal Distributions

Statisticians have methods for using sets of data to estimate μ and σ.

The factor $1/(\sigma\sqrt{2\pi})$ is needed to make *f* a probability density function. In fact, it can be verified using the methods of multivariable calculus that

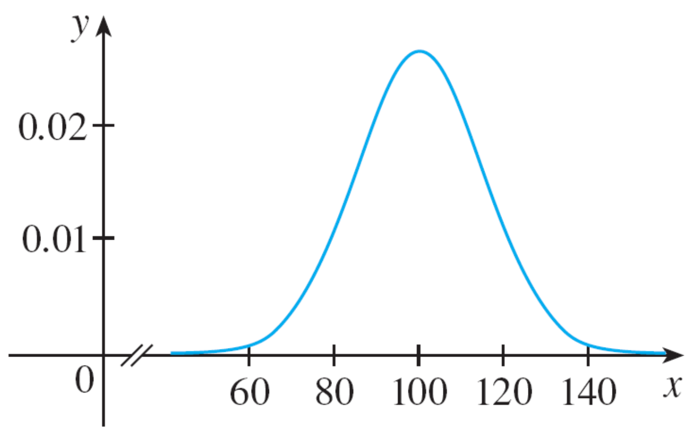
$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} dx = 1$$

Example 5

Intelligence Quotient (IQ) scores are distributed normally with mean 100 and standard deviation 15. (Figure 6 shows the corresponding probability density function.)

(a) What percentage of the population has an IQ score between 85 and 115?

(b) What percentage of the population has an IQ above 140?



Distribution of IQ scores

Figure 6

Example 5(a) – *Solution*

Since IQ scores are normally distributed, we use the probability density function given by Equation 3 with $\mu = 100$ and $\sigma = 15$:

$$P(85 \leqslant X \leqslant 115) = \int_{85}^{115} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/(2\cdot 15^2)}\, dx$$

The function $y = e^{-x^2}$ doesn't have an elementary antiderivative, so we can't evaluate the integral exactly.

Example 5(a) – *Solution*

cont'd

But we can use the numerical integration capability of a calculator or computer (or the Midpoint Rule or Simpson's Rule) to estimate the integral.

Doing so, we find that

$$P(85 \leq X \leq 115) \approx 0.68$$

So about 68% of the population has an IQ between 85 and 115, that is, within one standard deviation of the mean.

Example 5(b) – *Solution*

cont'd

The probability that the IQ score of a person chosen at random is more than 140 is

$$P(X > 140) = \int_{140}^{\infty} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/450}\,dx$$

To avoid the improper integral we could approximate it by the integral from 140 to 200. (It's quite safe to say that people with an IQ over 200 are extremely rare.)

Then

$$P(X > 140) \approx \int_{140}^{200} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/450}\,dx$$

$$\approx 0.0038$$

Therefore about 0.4% of the population has an IQ over 140.