VV156 Honors Calculus II
Fall 2021 — HW5 Solutions

November 17, 2021

Exercise 5.1

i)

$$\begin{aligned} A &= \int_0^{2\pi} [(2-\cos x) - \cos x] dx \\ &= \int_0^{2\pi} (2 - 2\cos x) dx \\ &= [2x - 2\sin x]_0^{2\pi} \\ &= (4\pi - 0) - 0 = 4\pi \end{aligned}$$

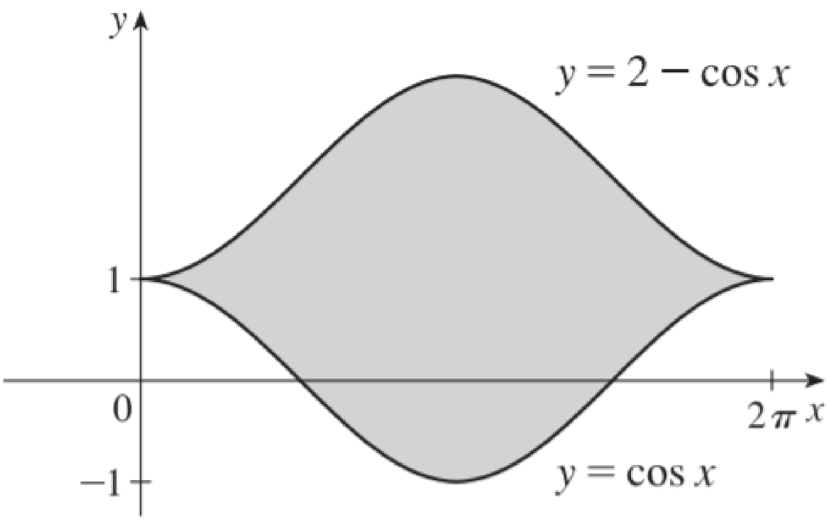

Figure 1: problem 5.1.1

ii)

$$2y^2 = 4 + y^2 \quad \Leftrightarrow \quad y^2 = 4 \quad \Leftrightarrow \quad y = \pm 2, \text{ so}$$

$$\begin{aligned} A &= \int_{-2}^{2} \left[\left(4+y^2\right) - 2y^2\right] dy \\ &= 2\int_0^2 \left(4 - y^2\right) dy \quad [\text{ by symmetry }] \\ &= 2\left[4y - \frac{1}{3}y^3\right]_0^2 = 2\left(8 - \frac{8}{3}\right) = \frac{32}{3} \end{aligned}$$

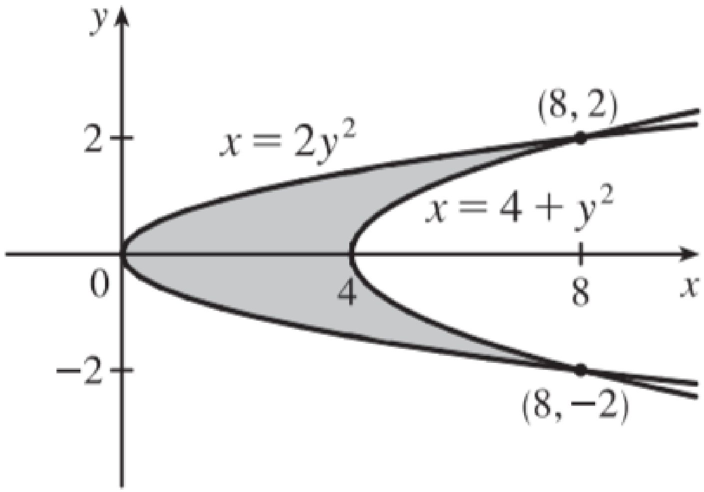

Figure 2: problem 5.1.2

Exercise 5.2

i) The curves intersect when $\sin x = \cos 2x$ (on $[0, \pi/2]$) $\Leftrightarrow$ $\sin x = 1 - 2\sin^2 x$ $\Leftrightarrow$ $2\sin^2 x + \sin x - 1 = 0$ $\Leftrightarrow$

$$(2\sin x - 1)(\sin x + 1) = 0 \Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$$

$$\begin{aligned}
A &= \int_0^{\pi/2} |\sin x - \cos 2x| dx \\
&= \int_0^{\pi/6} (\cos 2x - \sin x) dx + \int_{\pi/6}^{\pi/2} (\sin x - \cos 2x) dx \\
&= \left[\frac{1}{2}\sin 2x + \cos x\right]_0^{\pi/6} + \left[-\cos x - \frac{1}{2}\sin 2x\right]_{\pi/6}^{\pi/2} \\
&= \left(\frac{1}{4}\sqrt{3} + \frac{1}{2}\sqrt{3}\right) - (0 + 1) + (0 - 0) - \left(-\frac{1}{2}\sqrt{3} - \frac{1}{4}\sqrt{3}\right) \\
&= \frac{3}{2}\sqrt{3} - 1
\end{aligned}$$

ii)

$$\begin{aligned}
A &= \int_{-1}^{1} |3^x - 2^x|\, dx = \int_{-1}^{0} (2^x - 3^x)\, dx + \int_0^1 (3^x - 2^x)\, dx \\
&= \left[\frac{2^x}{\ln^2} - \frac{3^x}{\ln^3}\right]_{-1}^{0} + \left[\frac{3^x}{\ln 3} - \frac{2^x}{\ln 2}\right]_0^1 \\
&= \left(\frac{1}{\ln 2} - \frac{1}{\ln 3}\right) - \left(\frac{1}{2\ln 2} - \frac{1}{3\ln 3}\right) + \left(\frac{3}{\ln 3} - \frac{2}{\ln 2}\right) - \left(\frac{1}{\ln 3} - \frac{1}{\ln 2}\right) \\
&= \frac{2 - 1 - 4 + 2}{2\ln 2} + \frac{-3 + 1 + 9 - 3}{3\ln 3} = \frac{4}{3\ln 3} - \frac{1}{2\ln 2}
\end{aligned}$$

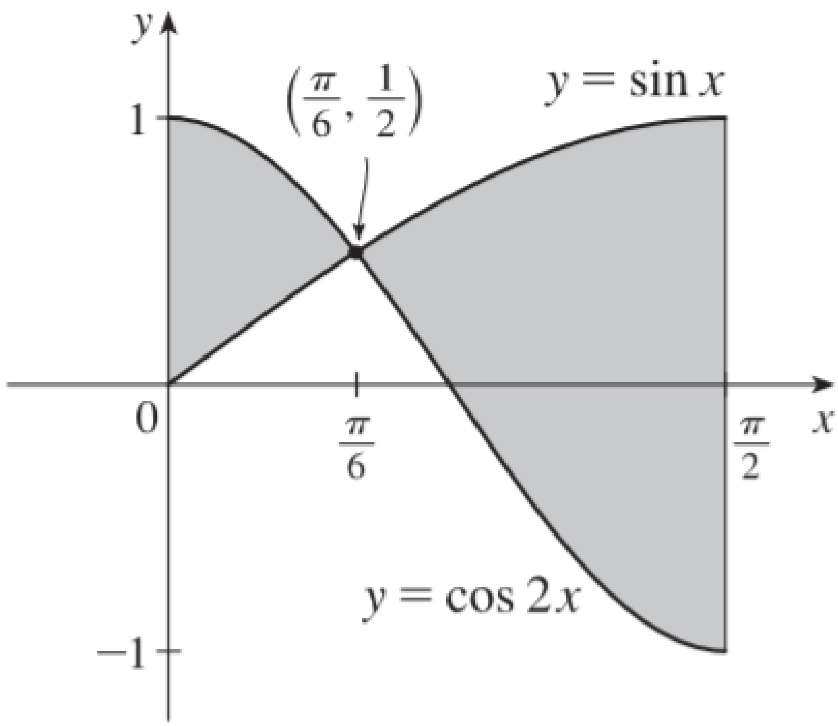

Figure 3: problem 5.2.1

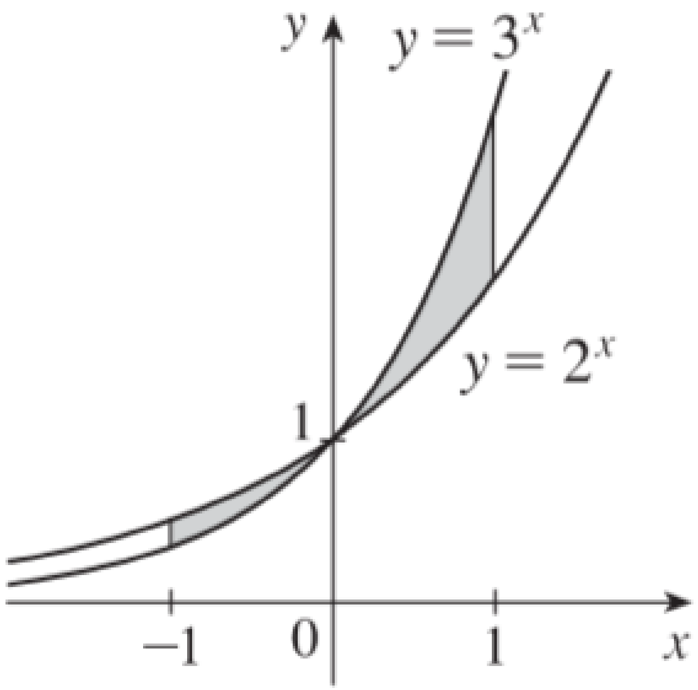

Figure 4: problem 5.2.2

Exercise 5.3
Each cross-section of the solid S in a plane perpendicular to the x-axis is a square (since the edges of the cut lie on the cylinders, which are perpendicular). One-quarter of this square and one-eighth of S are shown. The area of this quarter-square is $|PQ|^2 = r^2 - x^2$. Therefore, $A(x) = 4\left(r^2 - x^2\right)$ and the volume of S is

$$\begin{aligned} V &= \int_{-r}^{r} A(x)dx = 4\int_{-r}^{r}\left(r^2 - x^2\right)dx \\ &= 8\left(r^2 - x^2\right)dx = 8\left[r^2x - \frac{1}{3}x^3\right]_0^r = \frac{16}{3}r^3 \end{aligned}$$

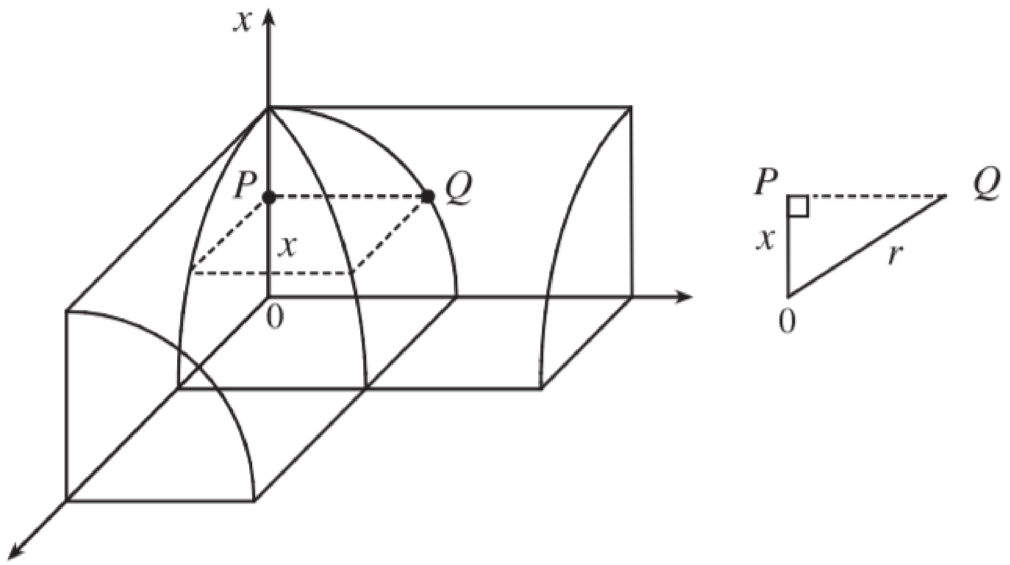

Figure 5: problem 5.3

Exercise 5.4

i)

$$\begin{aligned} V &= 2\pi\int_0^8 [y(\sqrt[3]{y} - 0)]dy \\ &= 2\pi\int_0^8 y^{4/3}dy = 2\pi\left[\frac{3}{7}y^{7/3}\right]_0^8 \\ &= \frac{6\pi}{7}\left(8^{7/3}\right) = \frac{6\pi}{7}\left(2^7\right) = \frac{768}{7}\pi \end{aligned}$$

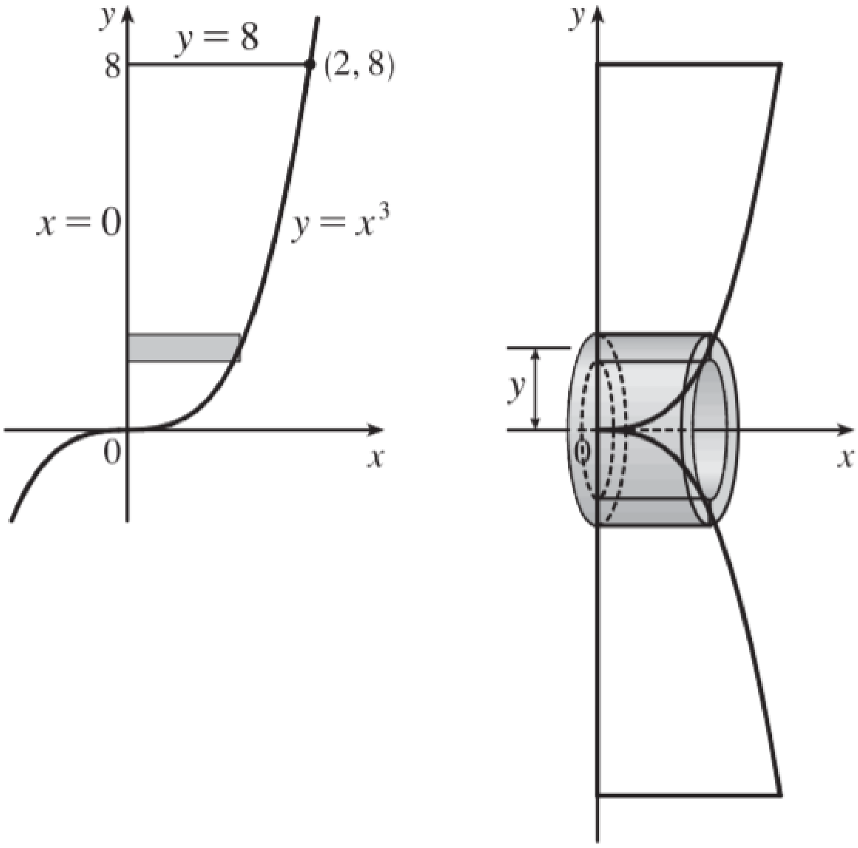

Figure 6: problem 5.4.1

ii)

$$\begin{aligned}
V &= 2\pi \int_0^4 \left[y\left(4y^2 - y^3\right)\right] dy \\
&= 2\pi \int_0^4 \left(4y^3 - y^4\right) dy \\
&= 2\pi \left[y^4 - \frac{1}{5}y^5\right]_0^4 = 2\pi \left(256 - \frac{1024}{5}\right) \\
&= 2\pi \left(\frac{256}{5}\right) = \frac{512}{5}\pi
\end{aligned}$$

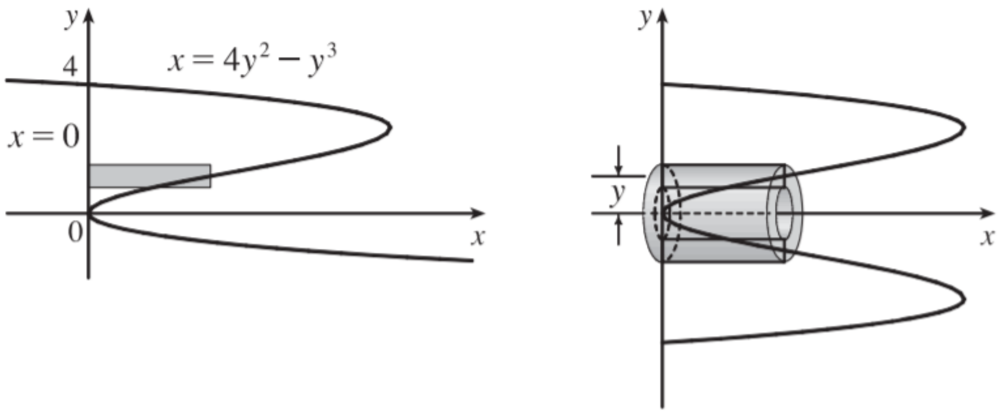

Figure 7: problem 5.4.2

iii)

The shell has radius $2 - x$, circumference $2\pi(2 - x)$, and height x^4.

$$\begin{aligned}
V &= \int_0^1 2\pi(2 - x)x^4 dx \\
&= 2\pi \int_0^1 \left(2x^4 - x^5\right) dx \\
&= 2\pi \left[\frac{2}{5}x^5 - \frac{1}{6}x^6\right]_0^1 \\
&= 2\pi \left[\left(\frac{2}{5} - \frac{1}{6}\right) - 0\right] = 2\pi \left(\frac{7}{30}\right) = \frac{7}{15}\pi
\end{aligned}$$

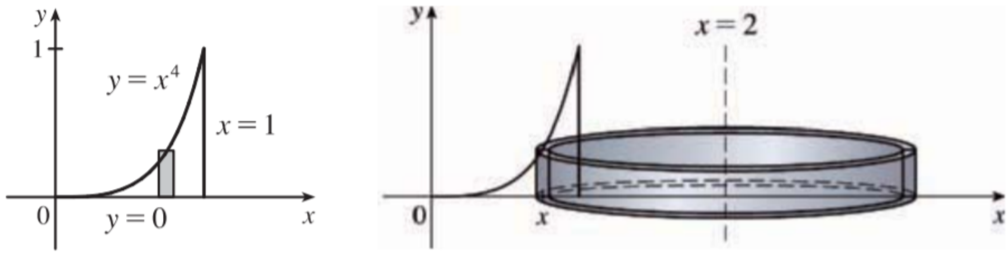

Figure 8: problem 5.4.3

Exercise 5.5

i) f is continuous on $[1, 3]$, so by the Mean Value Theorem for Integrals there exists a number c in $[1, 3]$ such that $\int_1^3 f(x)dx = f(c)(3-1) \Rightarrow 8 = 2f(c)$; that is, there is a number c such that $f(c) = \frac{8}{2} = 4$.

ii) The requirement is that $\frac{1}{b-0}\int_0^b f(x)dx = 3$. The LHS of this equation is equal to $\frac{1}{b}\int_0^b (2 + 6x - 3x^2)\,dx = \frac{1}{b}\left[2x + 3x^2 - x^3\right]_0^b = 2 + 3b - b^2$, so we solve the equation $2 + 3b - b^2 = 3 \Leftrightarrow b^2 - 3b + 1 = 0 \quad \Leftrightarrow \quad b = \frac{3 \pm \sqrt{(-3)^2 - 4\cdot 1\cdot 1}}{2\cdot 1} = \frac{3\pm\sqrt{5}}{2}$. Both roots are valid since they are positive.

Exercise 5.6

i) Take $g(x) = x$ and $g'(x) = 1$

ii) By part (a), $\int_a^b f(x)dx = bf(b) - af(a) - \int_a^b xf'(x)dx$. Now let $y = f(x)$, so that $x = g(y)$ and $dy = f'(x)dx$. Then $\int_a^b xf'(x)dx = \int_{f(a)}^{f(b)} g(y)dy$. The result follows.

iii) Part (b) says that the area of region $ABFC$ is

$$= bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)dy$$

$=$ (area of rectangle $OBFE$)$-$(area of rectangle $OACD$)$-$(area of region $DCFE$)

iv) We have $f(x) = \ln x$, so $f^{-1}(x) = e^x$, and since $g = f^{-1}$, we have $g(y) = e^y$. By part (b),

$$\int_1^e \ln x dx = e\ln e - 1\ln 1 - \int_{\ln 1}^{\ln e} e^y dy = e - \int_0^1 e^y dy = e - \left[e^y\right]_0^1 = e - (e-1) = 1$$

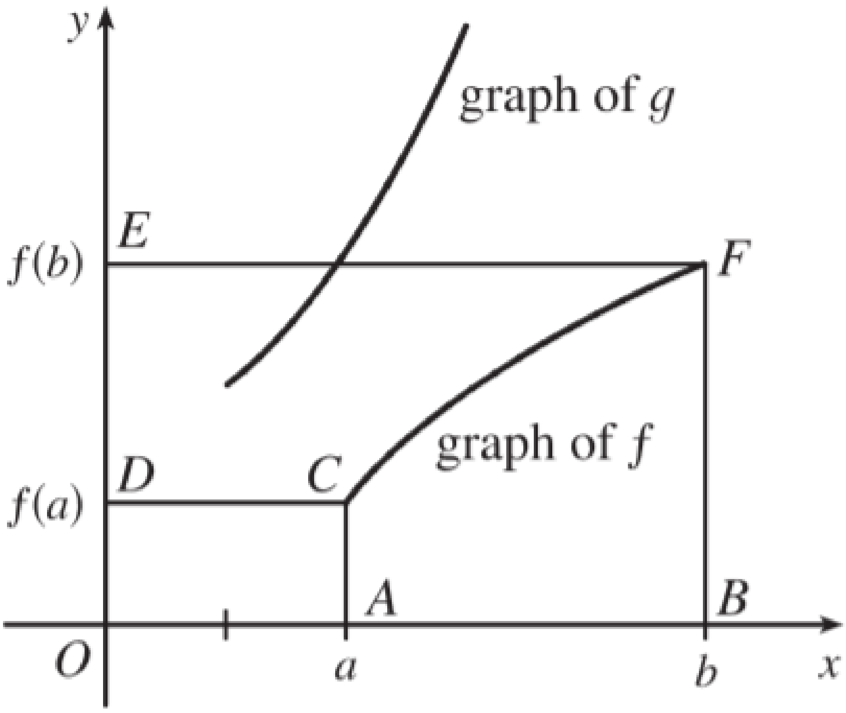

Figure 9: problem 5.6

Exercise 5.7

i) Just note that the integrand is odd $[f(-x) = -f(x)]$. *Or* : If $m \neq n$, calculate

$$\begin{aligned}\int_{-\pi}^{\pi} \sin mx \cos nx dx &= \int_{-\pi}^{\pi} \frac{1}{2}[\sin(m-n)x + \sin(m+n)x]dx \\ &= \frac{1}{2}\left[-\frac{\cos(m-n)x}{m-n} - \frac{\cos(m+n)x}{m+n}\right]_{-\pi}^{\pi} \\ &= 0\end{aligned}$$

If $m = n$, then the first term in each set of brackets is zero.

ii) $\int_{-\pi}^{\pi} \sin mx \sin nx dx = \int_{-\pi}^{\pi} \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]dx$.
If $m \neq n$, this is equal to $\frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0$
If $m = n$, we get $\int_{-\pi}^{\pi} \frac{1}{2}[1 - \cos(m+n)x]dx = \left[\frac{1}{2}x\right]_{-\pi}^{\pi} - \left[\frac{\sin(m+n)x}{2(m+n)}\right]_{-\pi}^{\pi} = \pi - 0 = \pi$

iii) $\int_{-\pi}^{\pi} \cos mx \cos nx dx = \int_{-\pi}^{\pi} \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]dx$
If $m \neq n$, this is equal to $\frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} + \frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0$
If $m = n$, we get $\int_{-\pi}^{\pi} \frac{1}{2}[1 + \cos(m+n)x]dx = \left[\frac{1}{2}x\right]_{-\pi}^{\pi} + \left[\frac{\sin(m+n)x}{2(m+n)}\right]_{-\pi}^{\pi} = \pi + 0 = \pi$

Exercise 5.8

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin mx dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\left(\sum_{n=1}^{m} a_n \sin nx\right)\sin mx\right]dx = \sum_{n=1}^{m}\frac{a_n}{\pi}\int_{-\pi}^{\pi} \sin mx \sin nx dx.$$

By 5.7.2, every term is zero except the m th one, and that term is $\frac{a_m}{\pi} \cdot \pi = a_m$

Exercise 5.9

i)

$$\int \frac{1}{\sqrt{x}(1+x)}dx = 2\arctan\left(\sqrt{x}\right) + C$$
$$\int_1^\infty \frac{1}{\sqrt{x}(1+x)}dx = \frac{\pi}{2}$$

ii)

$$\int_0^1 \frac{\ln x}{1+x^2}dx = -\int_1^\infty \frac{\ln x}{1+x^2}dx$$
$$\int_0^\infty \frac{\ln x}{1+x^2}dx = 0$$

Exercise 5.10

i)

$$y = \ln(\sec x) \Rightarrow \frac{dy}{dx} = \frac{\sec x \tan x}{\sec x} = \tan x \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2 x = \sec^2 x, \text{so}$$
$$L = \int_1^2 \sqrt{\sec^2 x}dx = \int_1^2 |\sec x|dx = \int_1^2 \sec x dx = [\ln(\sec x + \tan x)]_1^2$$
$$= \text{divergent}$$

ii)

$$y = 3 + \frac{1}{2}\cosh 2x \Rightarrow y' = \sinh 2x \Rightarrow 1 + (dy/dx)^2 = 1 + \sinh^2(2x) = \cosh^2(2x). \text{ So}$$
$$L = \int_0^1 \sqrt{\cosh^2(2x)}dx = \int_0^1 \cosh 2x dx = \left[\frac{1}{2}\sinh 2x\right]_0^1 = \frac{1}{2}\sinh 2 - 0 = \frac{1}{2}\sinh 2$$

Exercise 5.11

$$y = \sin^{-1} x + \sqrt{1-x^2} \Rightarrow y' = \frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} = \frac{1-x}{\sqrt{1-x^2}} \Rightarrow$$
$$1 + (y')^2 = 1 + \frac{(1-x)^2}{1-x^2} = \frac{1-x^2+1-2x+x^2}{1-x^2} = \frac{2-2x}{1-x^2} = \frac{2(1-x)}{(1+x)(1-x)} = \frac{2}{1+x} \Rightarrow$$

$\sqrt{1+(y')^2} = \sqrt{\frac{2}{1+x}}$. Thus, the arc length function with starting point $(0, 1)$ is given by

$$s(x) = \int_0^x \sqrt{1+[f'(t)]^2}dt = \int_0^x \sqrt{\frac{2}{1+t}}dt = \sqrt{2}[2\sqrt{1+t}]_0^x = 2\sqrt{2}(\sqrt{1+x} - 1)$$

Exercise 5.12

i) $y = x^3 \quad \Rightarrow \quad y' = 3x^2$. So

$$S = \int_0^2 2\pi y\sqrt{1+(y')^2}dx = 2\pi\int_0^2 x^3\sqrt{1+9x^4}dx \quad \left[u = 1+9x^4, du = 36x^3dx\right]$$
$$= \frac{2\pi}{36}\int_1^{145} \sqrt{u}du = \frac{\pi}{18}\left[\frac{2}{3}u^{3/2}\right]_1^{145} = \frac{\pi}{27}(145\sqrt{145} - 1)$$

ii) The curve $9x = y^2 + 18$ is symmetric about the x-axis, so we only use its top half, given by $y = 3\sqrt{x-2}$. $\frac{dy}{dx} = \frac{3}{2\sqrt{x-2}}$, so $1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{9}{4(x-2)}$. Thus,

$$S = \int_2^6 2\pi \cdot 3\sqrt{x-2}\sqrt{1+\frac{9}{4(x-2)}}dx = 6\pi \int_2^6 \sqrt{x-2+\frac{9}{4}}dx = 6\pi \int_2^6 \left(x+\frac{1}{4}\right)^{1/2} dx$$

$$= 6\pi \cdot \frac{2}{3}\left[\left(x+\frac{1}{4}\right)^{3/2}\right]_2^6 = 4\pi\left[\left(\frac{25}{4}\right)^{3/2} - \left(\frac{9}{4}\right)^{3/2}\right] = 4\pi\left(\frac{125}{8} - \frac{27}{8}\right) = 4\pi \cdot \frac{98}{8} = 49\pi$$

Exercise 5.13

i) In general, we must satisfy the two conditions that are mentioned before Example 1− namely, $(\mathbf{1}) f(x) \geq 0$ for all x, and (2) $\int_{-\infty}^{\infty} f(x)dx = 1$. For $0 \leq x \leq 1, f(x) = 30x^2(1-x)^2 \geq 0$ and $f(x) = 0$ for all other values of x, so $f(x) \geq 0$ for all x. Also,

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^1 30x^2(1-x)^2 dx = \int_0^1 30x^2\left(1-2x+x^2\right)dx = \int_0^1 \left(30x^2 - 60x^3 + 30x^4\right)dx$$

$$= \left[10x^3 - 15x^4 + 6x^5\right]_0^1 = 10 - 15 + 6 = 1$$

Therefore, f is a probability density function.

ii) $P\left(X \leq \frac{1}{3}\right) = \int_{-\infty}^{1/3} f(x)dx = \int_0^{1/3} 30x^2(1-x)^2dx = [10x^3 - 15x^4 + 6x^5]_0^{1/3} = \frac{10}{27} - \frac{15}{81} + \frac{6}{243} = \frac{17}{81}$

Exercise 5.14

i) In general, we must satisfy the two conditions that are mentioned before Example 1-namely, (1) $f(x) \geq 0$ for all x, and (2) $\int_{-\infty}^{\infty} f(x)dx = 1$. If $c \geq 0$, then $f(x) \geq 0$, so condition (1) is satisfied. For condition (2), we see that $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{c}{1+x^2}dx$ and

$$\int_0^{\infty} \frac{c}{1+x^2}dx = \lim_{t\to\infty}\int_0^t \frac{c}{1+x^2}dx = c\lim_{t\to\infty}\left[\tan^{-1}x\right]_0^t = c\lim_{t\to\infty}\tan^{-1}t = c\left(\frac{\pi}{2}\right)$$

Similarly, $\int_{-\infty}^0 \frac{c}{1+x^2}dx = c\left(\frac{\pi}{2}\right)$, so $\int_{-\infty}^{\infty} \frac{c}{1+x^2}dx = 2c\left(\frac{\pi}{2}\right) = c\pi$. Since $c\pi$ must equal 1 , we must have $c = 1/\pi$ so that f is a probability density function.

ii) $P(-1 < X < 1) = \int_{-1}^1 \frac{1/\pi}{1+x^2}dx = \frac{2}{\pi}\int_0^1 \frac{1}{1+x^2}dx = \frac{2}{\pi}\left[\tan^{-1}x\right]_0^1 = \frac{2}{\pi}\left(\frac{\pi}{4} - 0\right) = \frac{1}{2}$