VV156 Honors Calculus II
Fall 2021 — HW3 Solutions

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Exercise 3.1

i) For $x > 0, |x| = x$, and $y = f(x) = \frac{x}{\sqrt{2-x^2}} \Rightarrow$

$$f'(x) = \frac{\sqrt{2-x^2}(1) - x\left(\frac{1}{2}\right)(2-x^2)^{-1/2}(-2x)}{\left(\sqrt{2-x^2}\right)^2} \cdot \frac{(2-x^2)^{1/2}}{(2-x^2)^{1/2}}$$
$$= \frac{(2-x^2)+x^2}{(2-x^2)^{3/2}} = \frac{2}{(2-x^2)^{3/2}}$$

So at $(1,1)$, the slope of the tangent line is $f'(1) = 2$ and its equation is $y-1 = 2(x-1)$ or $y = 2x-1$

ii) Plot

Exercise 3.2

i) If f is even, then $f(x) = f(-x)$. Using the Chain Rule to differentiate this equation, we get $f'(x) = f'(-x)\frac{d}{dx}(-x) = -f'(-x)$. Thus, $f'(-x) = -f'(x)$, so f' is odd.

ii) If f is odd, then $f(x) = -f(-x)$. Differentiating this equation, we get $f'(x) = -f'(-x)(-1) = f'(-x)$, so f' is even.

Exercise 3.3

The Chain Rule says that $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$, so

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{du}\frac{du}{dx}\right) = \left[\frac{d}{dx}\left(\frac{dy}{du}\right)\right]\frac{du}{dx} + \frac{dy}{du}\frac{d}{dx}\left(\frac{du}{dx}\right) \quad [\text{ Product Rule }]$$
$$= \left[\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}\right]\frac{du}{dx} + \frac{dy}{du}\frac{d^2u}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2u}{dx^2}$$

Exercise 3.4

i) $x^2+y^2 = (2x^2+2y^2-x)^2 \Rightarrow 2x+2yy' = 2(2x^2+2y^2-x)(4x+4yy'-1)$. When $x = 0$ and $y = \frac{1}{2}$, we have $0+y' = 2\left(\frac{1}{2}\right)(2y'-1) \Rightarrow y' = 2y'-1 \Rightarrow y' = 1$, so an equation of the tangent line is $y - \frac{1}{2} = 1(x-0)$ or $y = x+\frac{1}{2}$.

ii) $x^{2/3}+y^{2/3} = 4 \Rightarrow \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0 \Rightarrow \frac{1}{\sqrt[3]{x}} + \frac{y'}{\sqrt[3]{y}} = 0 \Rightarrow y' = -\frac{\sqrt[3]{y}}{\sqrt[3]{x}}$. When $x = -3\sqrt{3}$ and $y = 1$, we have $y' = -\frac{1}{(-3\sqrt{3})^{1/3}} = -\frac{(-3\sqrt{3})^{2/3}}{-3\sqrt{3}} = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}}$, so an equation of the tangent line is $y-1 = \frac{1}{\sqrt{3}}(x+3\sqrt{3})$ or $y = \frac{1}{\sqrt{3}}x+4$

iii) $2\left(x^2+y^2\right)^2 = 25\left(x^2-y^2\right) \Rightarrow 4\left(x^2+y^2\right)\left(2x+2yy'\right) = 25\left(2x-2yy'\right) \Rightarrow 4\left(x+yy'\right)\left(x^2+y^2\right) = 25\left(x-yy'\right) \quad \Rightarrow \quad 4yy'\left(x^2+y^2\right)+25yy' = 25x-4x\left(x^2+y^2\right) \Rightarrow y' = \frac{25x-4x\left(x^2+y^2\right)}{25y+4y(x^2+y^2)}$.
When $x=3$ and $y=1$, we have $y' = \frac{75-120}{25+40} = -\frac{45}{65} = -\frac{9}{13}$ so an equation of the tangent line is $y-1 = -\frac{9}{13}(x-3)$ or $y = -\frac{9}{13}x + \frac{40}{13}$.

iv) $y^2\left(y^2-4\right) = x^2\left(x^2-5\right) \Rightarrow y^4-4y^2 = x^4-5x^2 \Rightarrow 4y^3y' - 8yy' = 4x^3-10x$. When $x=0$ and $y=-2$, we have $-32y'+16y' = 0 \Rightarrow -16y' = 0 \Rightarrow y' = 0$, so an equation of the tangent line is $y+2 = 0(x-0)$ or $y=-2$

Exercise 3.5

Prove by yourself

Exercise 3.6

i) $y = (\sin x)^{\ln x} \Rightarrow \ln y = \ln(\sin x)^{\ln x} \Rightarrow \ln y = \ln x \cdot \ln\sin x \Rightarrow \frac{1}{y}y' = \ln x \cdot \frac{1}{\sin x} \cdot \cos x + \ln\sin x \cdot \frac{1}{x} \Rightarrow$

$$y' = y\left(\ln x \cdot \frac{\cos x}{\sin x} + \frac{\ln\sin x}{x}\right) \Rightarrow y' = (\sin x)^{\ln x}\left(\ln x \cot x + \frac{\ln\sin x}{x}\right)$$

ii) $y = (\tan x)^{1/x} \Rightarrow \ln y = \ln(\tan x)^{1/x} \Rightarrow \ln y = \frac{1}{x}\ln\tan x \quad \Rightarrow$

$$\frac{1}{y}y' = \frac{1}{x}\cdot\frac{1}{\tan x}\cdot\sec^2 x + \ln\tan x\cdot\left(-\frac{1}{x^2}\right) \Rightarrow y' = y\left(\frac{\sec^2 x}{x\tan x} - \frac{\ln\tan x}{x^2}\right) \Rightarrow$$
$$y' = (\tan x)^{1/x}\left(\frac{\sec^2 x}{x\tan x} - \frac{\ln\tan x}{x^2}\right) \quad \text{or } y' = (\tan x)^{1/x}\cdot\frac{1}{x}\left(\csc x\sec x - \frac{\ln\tan x}{x}\right)$$

Exercise 3.7

$y = \frac{x}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}}\arctan\frac{\sin x}{a+\sqrt{a^2-1}+\cos x}$. Let $k = a+\sqrt{a^2-1}$. Then

$$\begin{aligned}
y' &= \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}}\cdot\frac{1}{1+\sin^2 x/(k+\cos x)^2}\cdot\frac{\cos x(k+\cos x)+\sin^2 x}{(k+\cos x)^2} \\
&= \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}}\cdot\frac{k\cos x+\cos^2 x+\sin^2 x}{(k+\cos x)^2+\sin^2 x} = \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}}\cdot\frac{k\cos x+1}{k^2+2k\cos x+1} \\
&= \frac{k^2+2k\cos x+1-2k\cos x-2}{\sqrt{a^2-1}\,(k^2+2k\cos x+1)} = \frac{k^2-1}{\sqrt{a^2-1}\,(k^2+2k\cos x+1)}
\end{aligned}$$

But $k^2 = 2a^2+2a\sqrt{a^2-1}-1 = 2a\left(a+\sqrt{a^2-1}\right)-1 = 2ak-1$, so $k^2+1 = 2ak$, and $k^2-1 = 2(ak-1)$ So $y' = \frac{2(ak-1)}{\sqrt{a^2-1}(2ak+2k\cos x)} = \frac{ak-1}{\sqrt{a^2-1}k(a+\cos x)}$. But $ak-1 = a^2+a\sqrt{a^2-1}-1 = k\sqrt{a^2-1}$ so $y' = 1/(a+\cos x)$.

Exercise 3.8

Suppose that f has a minimum value at c, so $f(x) \geq f(c)$ for all x near c. Then $g(x) = -f(x) \leq -f(c) = g(c)$ for all x near c, so $g(x)$ has a maximum value at c.

Exercise 3.9

f satisfies the conditions for the Mean Value Theorem, so we use this theorem on the interval $[-b, b] : \frac{f(b)-f(-b)}{b-(-b)} = f'(c)$ for some $c \in (-b, b)$. But since f is odd, $f(-b) = -f(b)$. Substituting this into the above equation, we get $\frac{f(b)+f(b)}{2b} = f'(c) \Rightarrow \frac{f(b)}{b} = f'(c)$

Exercise 3.10

$y = x\sin x \Rightarrow y' = x\cos x + \sin x \Rightarrow y'' = -x\sin x + 2\cos x$. $y'' = 0 \Rightarrow 2\cos x = x\sin x[$ which is $y] \Rightarrow (2\cos x)^2 = (x\sin x)^2 \Rightarrow 4\cos^2 x = x^2\sin^2 x \Rightarrow 4\cos^2 x = x^2(1-\cos^2 x) \Rightarrow 4\cos^2 x + x^2\cos^2 x = x^2 \Rightarrow \cos^2 x(4+x^2) = x^2 \Rightarrow 4\cos^2 x(x^2+4) = 4x^2 \Rightarrow y^2(x^2+4) = 4x^2$ since $y = 2\cos x$ when $y'' = 0$.

Exercise 3.11

The limit, $L = \lim_{x\to\infty}\left[x - x^2\ln\left(\frac{1+x}{x}\right)\right] = \lim_{x\to\infty}\left[x - x^2\ln\left(\frac{1}{x}+1\right)\right]$. Let $t = 1/x$, so as $x \to \infty, t \to 0^+$.

$$L = \lim_{t\to 0^+}\left[\frac{1}{t} - \frac{1}{t^2}\ln(t+1)\right] = \lim_{t\to 0^+}\frac{t-\ln(t+1)}{t^2} \overset{\text{H}}{=} \lim_{t\to 0^+}\frac{1-\frac{1}{t+1}}{2t} = \lim_{t\to 0^+}\frac{t/(t+1)}{2t} = \lim_{t\to 0^+}\frac{1}{2(t+1)} = \frac{1}{2}$$

Note: Starting the solution by factoring out x or x^2 leads to a more complicated solution.

Exercise 3.12

i) For f to be continuous, we need $\lim_{x\to 0} f(x) = f(0) = 1$. We note that for $x \neq 0, \ln f(x) = \ln|x|^x = x\ln|x|$ So $\lim_{x\to 0}\ln f(x) = \lim_{x\to 0} x\ln|x| = \lim_{x\to 0}\frac{\ln|x|}{1/x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{1/x}{-1/x^2} = 0$. Therefore, $\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{\ln f(x)} = e^0 = 1$ So f is continuous at 0 .

ii) To find f', we use logarithmic differentiation: $\ln f(x) = x\ln|x| \Rightarrow \frac{f'(x)}{f(x)} = x\left(\frac{1}{x}\right) + \ln|x| \Rightarrow f'(x) = f(x)(1+\ln|x|) = |x|^x(1+\ln|x|), x \neq 0$. Now $f'(x) \to -\infty$ as $x \to 0$ [since $|x|^x \to 1$ and $(1+\ln|x|) \to -\infty$], so the curve has a vertical tangent at $(0,1)$ and is therefore not differentiable there.

Exercise 3.13

If $B = 0$, the line is vertical and the distance from $x = -\frac{C}{A}$ to (x_1, y_1) is $\left|x_1 + \frac{C}{A}\right| = \frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$, so assume $B \neq 0$. The square of the distance from (x_1, y_1) to the line is $f(x) = (x-x_1)^2 + (y-y_1)^2$ where $Ax+By+C = 0$, so we minimize $f(x) = (x-x_1)^2 + \left(-\frac{A}{B}x - \frac{C}{B} - y_1\right)^2 \Rightarrow f'(x) = 2(x-x_1) + 2\left(-\frac{A}{B}x - \frac{C}{B} - y_1\right)\left(-\frac{A}{B}\right)$ $f'(x) = 0 \Rightarrow x = \frac{B^2x_1 - ABy_1 - AC}{A^2+B^2}$ and this gives a minimum since $f''(x) = 2\left(1+\frac{A^2}{B^2}\right) > 0$. Substituting this value of x into $f(x)$ and simplifying gives $f(x) = \frac{(Ax_1+By_1+C)^2}{A^2+B^2}$, so the minimum distance is $\sqrt{f(x)} = \frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$