VV156 Honors Calculus II
Fall 2021 — HW4 Solutions

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Exercise 4.1

i) Since $-|f(x)| \leq f(x) \leq |f(x)|$, it follows from Property 7 that

$$-\int_a^b |f(x)|dx \leq \int_a^b f(x)dx \leq \int_a^b |f(x)|dx \Rightarrow \left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$$

Note that the definite integral is a real number, and so the following property applies: $-a \leq b \leq a \Rightarrow |b| \leq a$ for all real numbers b and nonnegative numbers a.

ii)

$$\left|\int_0^{2\pi} f(x)\sin 2x dx\right| \leq \int_0^{2\pi} |f(x)\sin 2x|dx$$

[by part (a)]

$$= \int_0^{2\pi} |f(x)||\sin 2x|dx \leq \int_0^{2\pi} |f(x)|dx$$

by Property 7 , since

$$|\sin 2x| \leq 1 \Rightarrow |f(x)||\sin 2x| \leq |f(x)|$$

Exercise 4.2

i) $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \quad \Rightarrow \quad \int_0^x e^{-t^2}dt = \frac{\sqrt{\pi}}{2}\operatorname{erf}(x)$. By Property 5 of definite integrals in Section 5.2,

$$\int_0^b e^{-t^2}dt = \int_0^a e^{-t^2}dt + \int_a^b e^{-t^2}dt, \text{ so}$$

$$\int_a^b e^{-t^2}dt = \int_0^b e^{-t^2}dt - \int_0^a e^{-t^2}dt = \frac{\sqrt{\pi}}{2}\operatorname{erf}(b) - \frac{\sqrt{\pi}}{2}\operatorname{erf}(a) = \frac{1}{2}\sqrt{\pi}[\operatorname{erf}(b) - \operatorname{erf}(a)]$$

ii) $y = e^{x^2}\operatorname{erf}(x) \Rightarrow y' = 2xe^{x^2}\operatorname{erf}(x) + e^{x^2}\operatorname{erf}'(x) = 2xy + e^{x^2}\cdot\frac{2}{\sqrt{\pi}}e^{-x^2}$ [by FTC1] $= 2xy + \frac{2}{\sqrt{\pi}}$.
The Fundamental Theorem of Calculus [FTC]
Suppose f is continuous on $[a, b]$.
1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.
2. $\int_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

Exercise 4.3

i) Plot

ii) $\mathrm{Si}(x)$ has local maximum values where $\mathrm{Si}'(x)$ changes from positive to negative, passing through 0 . From the Fundamental Theorem we know that $\mathrm{Si}'(x) = \frac{d}{dx}\int_0^x \frac{\sin t}{t}dt = \frac{\sin x}{x}$, so we must have $\sin x = 0$ for a maximum, and for $x > 0$ we must have $x = (2n-1)\pi, n$ any positive integer, for Si' to be changing from positive to negative at x. For $x < 0$, we must have $x = 2n\pi, n$ any positive integer, for a maximum, since the denominator of $\mathrm{Si}'(x)$ is negative for $x < 0$. Thus, the local maxima occur at $x = \pi, -2\pi, 3\pi, -4\pi, 5\pi, -6\pi, \ldots$

iii) To find the first inflection point, we solve $\mathrm{Si}''(x) = \frac{\cos x}{x} - \frac{\sin x}{x^2} = 0$. We can see from the graph that the first inflection point lies somewhere between $x = 3$ and $x = 5$. Using a rootfinder gives the value $x \approx 4.4934$. To find the y-coordinate of the inflection point, we evaluate $\mathrm{Si}(4.4934) \approx 1.6556$. So the coordinates of the first inflection point to the right of the origin are about $(4.4934, 1.6556)$. Alternatively, we could graph $S''(x)$ and estimate the first positive x-value at which it changes sign.

iv) It seems from the graph that the function has horizontal asymptotes at $y \approx 1.5$, with $\lim_{x\to\pm\infty}\mathrm{Si}(x) \approx \pm 1.5$ respectively. Using the limit command, we get $\lim_{x\to\infty}\mathrm{Si}(x) = \frac{\pi}{2}$. Since $\mathrm{Si}(x)$ is an odd function, $\lim_{x\to-\infty}\mathrm{Si}(x) = -\frac{\pi}{2}$. So $\mathrm{Si}(x)$ has the horizontal asymptotes $y = \pm\frac{\pi}{2}$.

v) We use the fsolve command in Maple (or FindRoot in Mathematica) to find that the solution is $x \approx 1.1$.

Exercise 4.4

i)

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^4}{n^5} = \lim_{n\to\infty}\frac{1-0}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^4 = \int_0^1 x^4 dx = \left[\frac{x^5}{5}\right]_0^1 = \frac{1}{5}$$

ii)

$$\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{\frac{1}{n}}+\sqrt{\frac{2}{n}}+\cdots+\sqrt{\frac{n}{n}}\right) = \lim_{n\to\infty}\frac{1-0}{n}\sum_{i=1}^{n}\sqrt{\frac{i}{n}} = \int_0^1 \sqrt{x}dx = \left[\frac{2x^{3/2}}{3}\right]_0^1 = \frac{2}{3} - 0 = \frac{2}{3}$$

Exercise 4.5

$$\begin{aligned}\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt &= \frac{d}{dx}\left[\int_{g(x)}^{a} f(t)dt + \int_a^{h(x)} f(t)dt\right] \quad \text{[where } a \text{ is in the domain of } f\text{]}\\ &= \frac{d}{dx}\left[-\int_a^{g(x)} f(t)dt\right] + \frac{d}{dx}\left[\int_a^{h(x)} f(t)dt\right] = -f(g(x))g'(x) + f(h(x))h'(x)\\ &= f(h(x))h'(x) - f(g(x))g'(x)\end{aligned}$$

Exercise 4.6 Using FTC1, we differentiate both sides of $6 + \int_a^x \frac{f(t)}{t^2}dt = 2\sqrt{x}$ to get $\frac{f(x)}{x^2} = 2\frac{1}{2\sqrt{x}} \Rightarrow f(x) = x^{3/2}$. To find a, we substitute $x = a$ in the original equation to obtain $6 + \int_a^a \frac{f(t)}{t^2}dt = 2\sqrt{a} \Rightarrow 6 + 0 = 2\sqrt{a} \Rightarrow 3 = \sqrt{a} \Rightarrow a = 9$

Exercise 4.7

i) Let $u = \tan x$. Then

$$du = \sec^2 x dx, \text{so} \int e^{\tan x} \sec^2 x dx = \int e^u du = e^u + C = e^{\tan x} + C$$

ii) Let $u = \ln x$. Then

$$du = (1/x)dx, \text{so} \int \frac{\sin(\ln x)}{x} dx = \int \sin u du = -\cos u + C = -\cos(\ln x) + C$$

iii) Let $u = \cot x$. Then $du = -\csc^2 x dx$ and $\csc^2 x dx = -du$, so

$$\int \sqrt{\cot x} \csc^2 x dx = \int \sqrt{u}(-du) = -\frac{u^{3/2}}{3/2} + C = -\frac{2}{3}(\cot x)^{3/2} + C$$

iv) Let $u = \sin^{-1} x$. Then

$$du = \frac{1}{\sqrt{1-x^2}} dx, \text{ so } \int \frac{dx}{\sqrt{1-x^2} \sin^{-1} x} = \int \frac{1}{u} du = \ln |u| + C = \ln \left|\sin^{-1} x\right| + C$$

v) Let $u = 1/x$, so $du = -1/x^2 dx$. When $x = 1, u = 1$; when $x = 2$, $u = \frac{1}{2}$. Thus,

$$\int_1^2 \frac{e^{1/x}}{x^2} dx = \int_1^{1/2} e^u(-du) = -\left[e^u\right]_1^{1/2} = -\left(e^{1/2} - e\right) = e - \sqrt{e}$$

vi)

$$\int_{-\pi/3}^{\pi/3} x^4 \sin x dx = 0 \text{ by Theorem 7(b), since } f(x) = x^4 \sin x \text{ is an odd function}$$

Theorem 7:
Suppose f is continuous on $[-a, a]$.
(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.
(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)dx = 0$.

vii) Let $u = \ln x$, so $du = \frac{dx}{x}$. When $x = e, u = 1$; when $x = e^4$; $u = 4$. Thus,

$$\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}} = \int_1^4 u^{-1/2} du = 2\left[u^{1/2}\right]_1^4 = 2(2-1) = 2$$

viii) Let $u = 1+\sqrt{x}$, so $du = \frac{1}{2\sqrt{x}} dx \Rightarrow 2\sqrt{x} du = dx \Rightarrow 2(u-1)du = dx$. When $x = 0, u = 1$; when $x = 1$ $u = 2$. Thus,

$$\int_0^1 \frac{dx}{(1+\sqrt{x})^4} = \int_1^2 \frac{1}{u^4} \cdot [2(u-1)du] = 2\int_1^2 \left(\frac{1}{u^3} - \frac{1}{u^4}\right) du = 2\left[-\frac{1}{2u^2} + \frac{1}{3u^3}\right]_1^2$$
$$= 2\left[\left(-\frac{1}{8} + \frac{1}{24}\right) - \left(-\frac{1}{2} + \frac{1}{3}\right)\right] = 2\left(\frac{1}{12}\right) = \frac{1}{6}$$

Exercise 4.8

i) Let $u = -x$. Then $du = -dx$, so

$$\int_a^b f(-x)dx = \int_{-a}^{-b} f(u)(-du) = \int_{-b}^{-a} f(u)du = \int_{-b}^{-a} f(x)dx$$

From the diagram, we see that the equality follows from the fact that we are reflecting the graph of f, and the limits of integration, about the y-axis.

ii) Let $u = x + c$. Then $du = dx$, so

$$\int_a^b f(x+c)dx = \int_{a+c}^{b+c} f(u)du = \int_{a+c}^{b+c} f(x)dx$$

From the diagram, we see that the equality follows from the fact that we are translating the graph of f, and the limits of integration, by a distance c.

iii) Let $u = \pi - x$. Then $du = -dx$. When $x = \pi, u = 0$ and when $x = 0, u = \pi$. So

$$\int_0^\pi xf(\sin x)dx = -\int_\pi^0 (\pi - u)f(\sin(\pi - u))du = \int_0^\pi (\pi - u)f(\sin u)du$$
$$= \pi\int_0^\pi f(\sin u)du - \int_0^\pi uf(\sin u)du = \pi\int_0^\pi f(\sin x)dx - \int_0^\pi xf(\sin x)dx \Rightarrow$$
$$2\int_0^\pi xf(\sin x)dx = \pi\int_0^\pi f(\sin x)dx \Rightarrow \int_0^\pi xf(\sin x)dx = \frac{\pi}{2}\int_0^\pi f(\sin x)dx$$

iv)

$$\int_0^{\pi/2} f(\cos x)dx = \int_0^{\pi/2} f\left[\sin\left(\frac{\pi}{2} - x\right)\right]dx \qquad \left[u = \frac{\pi}{2} - x, du = -dx\right]$$
$$= \int_{\pi/2}^0 f(\sin u)(-du) = \int_0^{\pi/2} f(\sin u)du = \int_0^{\pi/2} f(\sin x)dx$$

Continuity of f is needed in order to apply the substitution rule for definite integrals.

Exercise 4.9

i) $\frac{x\sin x}{1+\cos^2 x} = x\cdot\frac{\sin x}{2-\sin^2 x} = xf(\sin x)$, where $f(t) = \frac{t}{2-t^2}$.

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx = \int_0^\pi xf(\sin x)dx = \frac{\pi}{2}\int_0^\pi f(\sin x)dx = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}dx$$

Let $u = \cos x$. Then $du = -\sin x dx$. When $x = \pi, u = -1$ and when $x = 0, u = 1$. So

$$\frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}dx = -\frac{\pi}{2}\int_1^{-1}\frac{du}{1+u^2} = \frac{\pi}{2}\int_{-1}^1 \frac{du}{1+u^2} = \frac{\pi}{2}\left[\tan^{-1} u\right]_{-1}^1$$
$$= \frac{\pi}{2}\left[\tan^{-1} 1 - \tan^{-1}(-1)\right] = \frac{\pi}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{4}$$

ii) In Exercise 4.8 iv), take $f(x) = x^2$, so $\int_0^{\pi/2}\cos^2 xdx = \int_0^{\pi/2}\sin^2 xdx$. Now

$$\int_0^{\pi/2}\cos^2 xdx + \int_0^{\pi/2}\sin^2 xdx = \int_0^{\pi/2}\left(\cos^2 x + \sin^2 x\right)dx = \int_0^{\pi/2} 1dx = [x]_0^{\pi/2} = \frac{\pi}{2}$$

so $2\int_0^{\pi/2}\cos^2 xdx = \frac{\pi}{2} \Rightarrow \int_0^{\pi/2}\cos^2 xdx = \frac{\pi}{4} \qquad \left[= \int_0^{\pi/2}\sin^2 xdx\right]$