VV156 RC4
Integrals

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Problems in Mid 1

Ex 2.1

$$\lim_{x\to 0^+} [\cos(2x)]^{1/x^2}$$

Problems in Mid 1

Ex 2.1 Solution

Note that

$$\ln\left([\cos(2x)]^{1/x^2}\right) = \frac{\ln[\cos(2x)]}{x^2}$$

which is of the form $0/0$ as $x \to 0^+$. Applying L'Hospital's rule yields

$$\lim_{x\to 0^+} \frac{\ln[\cos(2x)]}{x^2} = \lim_{x\to 0^+} \frac{-2\sin(2x)/\cos(2x)}{2x} = \lim_{x\to 0^+} \frac{-\tan(2x)}{x}$$

which is again of the form $0/0$. Apply L'Hospital's rule again, then

$$\lim_{x\to 0^+} \frac{-\tan(2x)}{x} = \frac{-2\sec^2(2y)}{1} = -2$$

Therefore

$$\lim_{x\to 0^+} [\cos(2x)]^{1/x^2} = \exp\left(\lim_{x\to 0^+} \frac{\ln[\cos(2x)]}{x^2}\right) = e^{-2}$$

Problems in Mid 1

Ex 7

Given $f : \mathbb{R} \to \mathbb{R}$, with

$$f(x) = \begin{cases} x^4 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(i) (10 points) Calculate $f'(x)$.
(ii) (5 points) Calculate $f''(0)$.

Problems in Mid 1

Ex 7.1 Solution

When $x = 0$, by definition of derivative,

$$f'(0) = \lim_{h\to 0} \frac{h^4 \sin\left(\frac{1}{h}\right)}{h} = \lim_{h\to 0} h^3 \sin\left(\frac{1}{h}\right) = 0$$

When $x \neq 0$

$$\begin{aligned} f'(x) &= 4x^3 \sin\left(\frac{1}{x}\right) + x^4 \cos\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) \\ &= 4x^3 \sin\left(\frac{1}{x}\right) - x^2 \cos\left(\frac{1}{x}\right) \end{aligned}$$

Ex 7.2 Solution

By definition of the derivative,

$$
\begin{aligned}
f''(0) &= \lim_{h\to 0} \frac{4h^3 \sin\left(\frac{1}{h}\right) - h^2 \cos\left(\frac{1}{h}\right)}{h} \\
&= \lim_{h\to 0} \left[4h^2 \sin\left(\frac{1}{h}\right) - h\cos\left(\frac{1}{h}\right)\right] \\
&= 0
\end{aligned}
$$

Problems in Mid 1

Ex 8

Suppose that f satisfies the equation

$$f(x+y) = f(x) + f(y) + x^2y + xy^2$$

for all $x, y \in \mathbb{R}$ Suppose further that

$$\lim_{x \to 0} \frac{f(x)}{x} = 1$$

(i) (5 points) Calculate $f'(0)$.
(ii) (5 points) Calculate $f'(x)$.

Problems in Mid 1

Ex 8.1 Solution

Let $x = y = 0$, then we have $f(0) = 2f(0) + 0$, hence $f(0) = 0$. Thus by definition of derivative at 0 ,

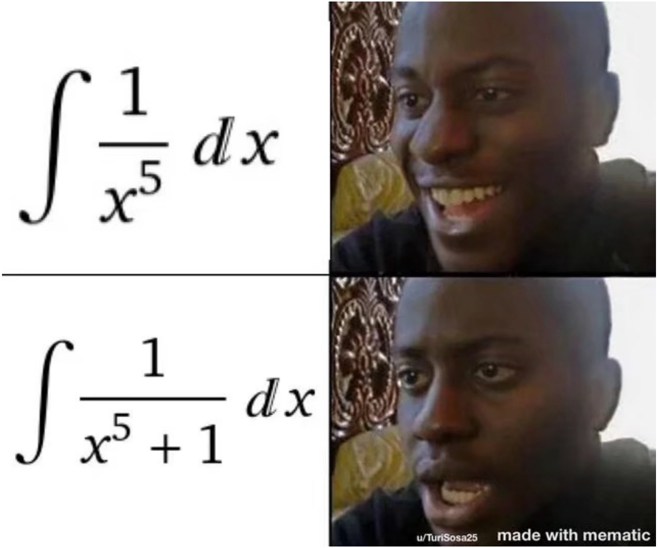
$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h} = 1$$

Ex 8.2 Solution

By definition of derivative at x, we have

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x) + f(h) + x^2h + xh^2 - f(x)}{h} \\ &= \lim_{h \to 0} \left(\frac{f(h)}{h} + x^2 + xh \right) = 1 + x^2 \end{aligned}$$

Integral memes



Integral memes

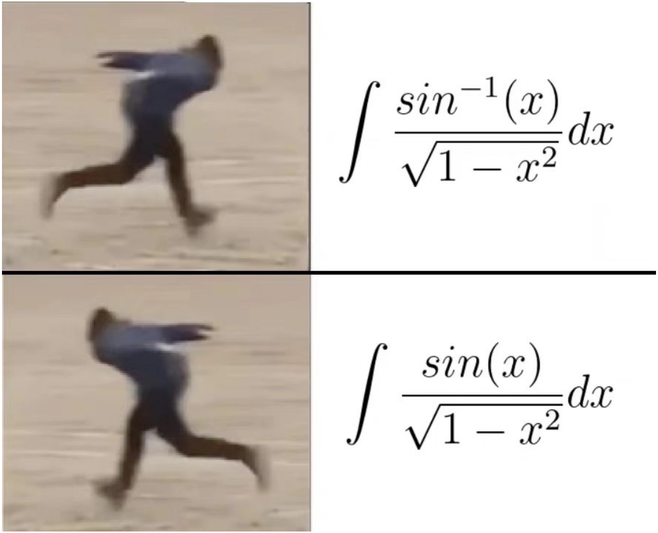
$$
\begin{aligned}
\int \frac{1}{1+x}\,dx &= \int \left(\frac{1}{x} + \frac{1}{1}\right) dx \\
&= \int \frac{1}{x}\,dx + \int \frac{1}{1}\,dx \\
&= \log(x) + \log(1) \\
&= \log(x+1) + C.
\end{aligned}
$$

Integral memes

$$\int \ln(x)\,dx$$

$$\int \frac{1}{\ln(x)}\,dx$$

Integral memes



Integral memes

“你应该尊重其他人的观点！”

他们的观点：

$$\sum_{k=0}^{\infty}\int_0^{\infty}\frac{(-x)^k}{k!}\,dx=\int_0^{\infty}\sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\,dx$$
$$=\int_0^{\infty}e^{-x}\,dx$$
$$=1$$

Integral memes

Antiderivatives

Definition

A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

Theorem

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Antiderivative Table

Function	Antiderivative	Function	Antiderivative
$cf(x)$	$cF(x)$	$\sec^2 x$	$\tan x$
$f(x)+g(x)$	$F(x)+G(x)$	$\sec x \tan x$	$\sec x$
$x^n (n \neq -1)$	$\frac{x^{n+1}}{n+1}$	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
$\frac{1}{x}$	$\ln \lvert x \rvert$	$\frac{1}{1+x^2}$	$\tan^{-1} x$
e^x	e^x	$\cosh x$	$\sinh x$
$\cos x$	$\sin x$	$\sinh x$	$\cosh x$
$\sin x$	$-\cos x$		

Darboux integral (Optional)

A partition of an interval $[a, b]$ is a finite sequence of values x_i such that

$$a = x_0 < x_1 < \cdots < x_n = b$$

Each interval $[x_{i-1}, x_j]$ is called a subinterval of the partition. Let $f. \; [a, b] \to \mathbf{R}$ be a bounded function, and let

$$P = (x_0, \ldots, x_n)$$

be a partition of $[a, b]$. Let

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$$

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$$

Darboux integral (Optional)

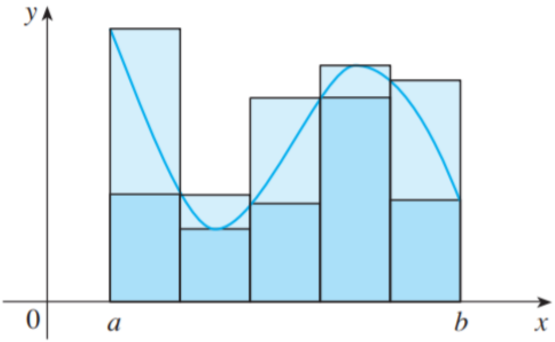
The upper Darboux sum of f with respect to P is

$$U_{f,P} = \sum_{i=1}^{n} (x_i - x_{i-1})\, M_i$$

The lower Darboux sum of f with respect to P is

$$L_{f,P} = \sum_{i=1}^{n} (x_i - x_{i-1})\, m_i$$

Darboux integral (Optional)



Definite integral

If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b-a)/n$. We let $x_0(=a), x_1, x_2, \ldots, x_n(=b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any sample points in these subintervals, so x_i^* lies in the i th subinterval $[x_{i-1}, x_i]$. Then the definite integral of $\boldsymbol{f}$ from $\boldsymbol{a}$ to $\boldsymbol{b}$ is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is integrable on $[a, b]$.

Properties of the Integral

1. $$\int_a^b c\,dx = c(b-a)$$
2. $$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$
3. $$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$
4. $$\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$
5. $$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$$

Properties of the Integral

6. If $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)dx \geqslant 0$.
7. If $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)dx \geqslant \int_a^b g(x)dx$.
8. If $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then

$$m(b-a) \leqslant \int_a^b f(x)dx \leqslant M(b-a)$$

Ex 1

Calculate the definite integral by definition

1. $$\int_{-3}^{0} \left(1+\sqrt{9-x^2}\right) dx$$

2. $$\int_{\pi}^{\pi} \sin^2 x \cos^4 x dx$$

Ex 1

Solution

1. $\int_{-3}^{0}\left(1+\sqrt{9-x^2}\right)dx$ can be interpreted as the area under the graph of $f(x)=1+\sqrt{9-x^2}$ between $x=-3$ and $x=0$. This is equal to one-quarter the area of the circle with radius 3 , plus the area of the rectangle, so

$$\int_{-3}^{0}\left(1+\sqrt{9-x^2}\right)dx=\frac{1}{4}\pi\cdot 3^2+1\cdot 3=3+\frac{9}{4}\pi$$

2.

$$\int_{\pi}^{\pi}\sin^2 x\cos^4 x dx=0$$

since the limits of intergration are equal.

The Fundamental Theorem of Calculus

Suppose f is continuous on $[a, b]$.
1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.
2. $\int_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

Ex 2

Concave function and integral

On what interval is the curve

$$y = \int_0^x \frac{t^2}{t^2 + t + 2} dt$$

concave downward?

Ex 2

Solution

$$y = \int_0^x \frac{t^2}{t^2+t+2}\,dt \Rightarrow y' = \frac{x^2}{x^2+x+2} \Rightarrow$$

$$\begin{aligned} y'' &= \frac{\left(x^2+x+2\right)(2x) - x^2(2x+1)}{(x^2+x+2)^2} \\ &= \frac{2x^3+2x^2+4x-2x^3-x^2}{(x^2+x+2)^2} \\ &= \frac{x^2+4x}{(x^2+x+2)^2} \\ &= \frac{x(x+4)}{(x^2+x+2)^2} \end{aligned}$$

The curve y is concave downward when $y'' < 0$; that is, on the interval $(-4, 0)$.

Indefinite Integrals

$$\int f(x)dx = F(x) \quad \text{means} \quad F'(x) = f(x)$$

$\int cf(x)dx = c\int f(x)dx$

$\int kdx = kx + C$

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$

$\int e^x dx = e^x + C$

$\int \sin x dx = -\cos x + C$

$\int \sec^2 x dx = \tan x + C$

$\int \sec x \tan x dx = \sec x + C$

$\int \frac{1}{x^2+1} dx = \tan^{-1} x + C$

$\int \sinh x dx = \cosh x + C$

$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$

$\int \frac{1}{x} dx = \ln|x| + C$

$\int a^x dx = \frac{a^x}{\ln a} + C$

$\int \cos x dx = \sin x + C$

$\int \csc^2 x dx = -\cot x + C$

$\int \csc x \cot x dx = -\csc x + C$

$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

$\int \cosh x dx = \sinh x + C$

Note: The table will be given in the exam, you don't need to recite them.

Ex 3

Evaluate the integral

1. $$\int (u+4)(2u+1)du$$

2. $$\int \left(x^2 + 1 + \frac{1}{x^2+1} \right) dx$$

3. $$\int (\theta - \csc\theta \cot\theta) d\theta$$

Ex 3

Solution

1.
$$\int (u+4)(2u+1)du = \int \left(2u^2+9u+4\right) du$$
$$= 2\frac{u^3}{3} + 9\frac{u^2}{2} + 4u + C$$
$$= \frac{2}{3}u^3 + \frac{9}{2}u^2 + 4u + C$$

2.
$$\int \left(x^2 + 1 + \frac{1}{x^2+1}\right) dx = \frac{x^3}{3} + x + \tan^{-1} x + C$$

3.
$$\int (\theta - \csc\theta\cot\theta)d\theta = \frac{1}{2}\theta^2 + \csc\theta + C$$

The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

The Substitution Rule: Example

$$\text{Calculate } \int \tan x dx$$

First we write tangent in terms of sine and cosine:

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx$$

This suggests that we should substitute
$u = \cos x$, since $then du = -\sin x dx$ and so $\sin x dx = -du$

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{1}{u} du$$
$$= -\ln|u| + C = -\ln|\cos x| + C$$

Since $-\ln|\cos x| = \ln\left(|\cos x|^{-1}\right) = \ln(1/|\cos x|) = \ln|\sec x|$, the result of Example can also be written as

$$\int \tan x dx = \ln|\sec x| + C$$

Integrals of Symmetric Functions

Suppose f is continuous on $[-a, a]$.
(a) If f is even $[f(-x) = f(x)]$, then

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

(b) If f is odd $[f(-x) = -f(x)]$, then

$$\int_{-a}^{a} f(x)dx = 0$$

Ex 4 (Practice these problems as many as possible!)

Evaluate the integral

1.
$$\int \frac{1+x}{1+x^2}dx$$

2.
$$\int x(2x+5)^8 dx$$

3.
$$\int_0^{\pi/2} \cos x \sin(\sin x) dx$$

Ex 4 (Practice these problems as many as possible!)

Solution

1.Let $u = 1 + x^2$. Then $du = 2xdx$, so

$$\begin{aligned}
\int \frac{1+x}{1+x^2}dx &= \int \frac{1}{1+x^2}dx + \int \frac{x}{1+x^2}dx \\
&= \tan^{-1} x + \int \frac{\frac{1}{2}du}{u} \\
&= \tan^{-1} x + \frac{1}{2}\ln|u| + C \\
&= \tan^{-1} x + \frac{1}{2}\ln\left|1+x^2\right| + C \\
&= \tan^{-1} x + \frac{1}{2}\ln\left(1+x^2\right) + C \quad \left[\text{ since } 1+x^2 > 0\right]
\end{aligned}$$

Ex 4 (Practice these problems as many as possible!)

Solution

2.Let $u = 2x + 5$. Then $du = 2dx$ and $x = \frac{1}{2}(u-5)$, so

$$
\begin{aligned}
\int x(2x+5)^8 dx &= \int \frac{1}{2}(u-5)u^8 \left(\frac{1}{2}du\right) \\
&= \frac{1}{4}\int \left(u^9 - 5u^8\right) du \\
&= \frac{1}{4}\left(\frac{1}{10}u^{10} - \frac{5}{9}u^9\right) + C \\
&= \frac{1}{40}(2x+5)^{10} - \frac{5}{36}(2x+5)^9 + C
\end{aligned}
$$

Ex 4 (Practice these problems as many as possible!)

Solution

3.Let $u = \sin x$, so $du = \cos x dx$. When $x = 0, u = 0$; when $x = \frac{\pi}{2}, u = 1$. Thus,

$$\begin{aligned}\int_0^{\pi/2} \cos x \sin(\sin x) dx &= \int_0^1 \sin u du \\ &= [-\cos u]_0^1 \\ &= -(\cos 1 - 1) \\ &= 1 - \cos 1\end{aligned}$$

Integration by Parts

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

$$\int udv = uv - \int vdu$$

Integration by Parts

Example

$$\int \frac{xe^{2x}}{(1+2x)^2}\,dx$$

Integration by Parts

Example Solution

Let $u = xe^{2x}$, $dv = \frac{1}{(1+2x)^2}dx \Rightarrow du = \left(x \cdot 2e^{2x} + e^{2x} \cdot 1\right) dx = e^{2x}(2x+1)dx$, $v = -\frac{1}{2(1+2x)}$ Then

$$\begin{aligned}
\int \frac{xe^{2x}}{(1+2x)^2}dx &= -\frac{xe^{2x}}{2(1+2x)} + \frac{1}{2}\int \frac{e^{2x}(2x+1)}{1+2x}dx \\
&= -\frac{xe^{2x}}{2(1+2x)} + \frac{1}{2}\int e^{2x}dx \\
&= -\frac{xe^{2x}}{2(1+2x)} + \frac{1}{4}e^{2x} + C
\end{aligned}$$

The answer could be written as $\frac{e^{2x}}{4(2x+1)} + C$

Ex 5

Evaluate the integral

$$\int_0^{\pi} e^{\cos t} \sin 2t dt$$

Ex 5

Solution

Let $x = \cos t$, so that $dx = -\sin t dt$. Thus, $\int_0^{\pi} e^{\cos t} \sin 2t dt = \int_0^{\pi} e^{\cos t}(2 \sin t \cos t) dt = \int_1^{-1} e^x \cdot 2x(-dx) = 2\int_{-1}^{1} xe^x dx$. Now use parts with $u = x$, $dv = e^x dx, du = dx, v = e^x$ to get $2\int_{-1}^{1} xe^x dx = 2\left([xe^x]_{-1}^{1} - \int_{-1}^{1} e^x dx\right) = 2\left(e^1 + e^{-1} - [e^x]_{-1}^{1}\right) = 2\left(e + e^{-1} - \left[e^1 - e^{-1}\right]\right) = 2\left(2e^{-1}\right) = 4/e$

Type 1: Infinite Intervals

(a) If $\int_a^t f(x)dx$ exists for every number $t \geqslant a$, then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)dx$ exists for every number $t \leqslant b$, then

$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are called convergent if the corresponding limit exists and divergent if the limit does not exist.

Type 1: Infinite Intervals

(c) If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent, then we define

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx$$

In part (c) any real number a can be used.

Type 2: Discontinuous Integrands

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

if this limit exists (as a finite number).
(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

if this limit exists (as a finite number).
The improper integral $\int_a^b f(x)dx$ is called convergent if the corresponding limit exists and divergent if the limit does not exist.
(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Comparison Theorem

Suppose that f and g are continuous functions with $f(x) \geqslant g(x) \geqslant 0$ for $x \geqslant a$.
(a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.
(b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

Ex 6

Evaluate the integral

1.
$$\int_e^{\infty} \frac{1}{x(\ln x)^3}\,dx$$

2.
$$\int_0^{3} \frac{dx}{x^2 - 6x + 5}$$

Ex 6

Solution

1.

$$\int_e^\infty \frac{1}{x(\ln x)^3}dx = \lim_{t\to\infty}\int_e^t \frac{1}{x(\ln x)^3}dx$$

$$= \lim_{t\to\infty}\int_1^{\ln t} u^{-3}du \quad \begin{bmatrix} u = \ln x, \\ du = dx/x \end{bmatrix}$$

$$= \lim_{t\to\infty}\left[-\frac{1}{2u^2}\right]_1^{\ln t}$$

$$= \lim_{t\to\infty}\left[-\frac{1}{2(\ln t)^2} + \frac{1}{2}\right]$$

$$= 0 + \frac{1}{2} = \frac{1}{2}. \quad \text{Convergent}$$

Ex 6

Solution

2.

$$I = \int_0^3 \frac{dx}{x^2 - 6x + 5} = \int_0^3 \frac{dx}{(x-1)(x-5)} = I_1 + I_2$$

$$= \int_0^1 \frac{dx}{(x-1)(x-5)} + \int_1^3 \frac{dx}{(x-1)(x-5)}$$

Now

$$\frac{1}{(x-1)(x-5)} = \frac{A}{x-1} + \frac{B}{x-5} \Rightarrow 1 = A(x-5) + B(x-1)$$

Set $x = 5$ to get $1 = 4B$, so $B = \frac{1}{4}$. Set $x = 1$ to get $1 = -4A$, so $A = -\frac{1}{4}$. Thus

Ex 6

Solution

2.

$$
\begin{aligned}
I_1 &= \lim_{t\to 1^-} \int_0^t \left(\frac{-\frac{1}{4}}{x-1} + \frac{\frac{1}{4}}{x-5} \right) dx \\
&= \lim_{t\to 1^-} \left[-\frac{1}{4} \ln|x-1| + \frac{1}{4} \ln|x-5| \right]_0^t \\
&= \lim_{t\to 1^-} \left[\left(-\frac{1}{4} \ln|t-1| + \frac{1}{4} \ln|t-5| \right) \left(-\frac{1}{4} \ln|-1| + \frac{1}{4} \ln|-5| \right) \right] \\
&= \infty, \quad \text{since } \lim_{t\to 1^-} \left(-\frac{1}{4} \ln|t-1| \right) = \infty
\end{aligned}
$$

Since I_1 is divergent, I is divergent.

Ex 7

Determine Convergence

$$\int_1^{\infty} \frac{2 + e^{-x}}{x} dx$$

Ex 7

Solution

For $x \geq 1$, $\frac{2+e^{-x}}{x} > \frac{2}{x}$ [since $e^{-x} > 0$] $> \frac{1}{x}$. $\int_1^\infty \frac{1}{x} dx$ is divergent, so $\int_1^\infty \frac{2+e^{-x}}{x} dx$ is divergent by the Comparison Theorem.

Partial Fraction Example

$$\int_1^2 \frac{4y^2 - 7y - 12}{y(y+2)(y-3)}\,dy$$

Example Solution

$$\frac{4y^2 - 7y - 12}{y(y+2)(y-3)} = \frac{A}{y} + \frac{B}{y+2} + \frac{C}{y-3} \Rightarrow 4y^2 - 7y - 12$$
$$= A(y+2)(y-3) + By(y-3) + Cy(y+2). \text{ Setting}$$

$y = 0$ gives $-12 = -6A$, so $A = 2$. Setting $y = -2$ gives $18 = 10B$, so $B = \frac{9}{5}$. Setting $y = 3$ gives $3 = 15C$, so $C = \frac{1}{5}$.

Example Solution

Now

$$
\begin{aligned}
\int_1^2 \frac{4y^2 - 7y - 12}{y(y+2)(y-3)}\,dy &= \int_1^2 \left(\frac{2}{y} + \frac{9/5}{y+2} + \frac{1/5}{y-3} \right) dy \\
&= \left[2\ln|y| + \frac{9}{5}\ln|y+2| + \frac{1}{5}\ln|y-3| \right]_1^2 \\
&= 2\ln 2 + \frac{9}{5}\ln 4 + \frac{1}{5}\ln 1 - 2\ln 1 - \frac{9}{5}\ln 3 - \frac{1}{5}\ln 2 \\
&= 2\ln 2 + \frac{18}{5}\ln 2 - \frac{1}{5}\ln 2 - \frac{9}{5}\ln 3 \\
&= \frac{27}{5}\ln 2 - \frac{9}{5}\ln 3 \\
&= \frac{9}{5}(3\ln 2 - \ln 3) \\
&= \frac{9}{5}\ln\frac{8}{3}
\end{aligned}
$$

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Example

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$$

Example Solution

Let $x = 2\tan\theta, -\pi/2 < \theta < \pi/2$. Then $dx = 2\sec^2\theta d\theta$ and

$$\sqrt{x^2+4} = \sqrt{4\left(\tan^2\theta + 1\right)} = \sqrt{4\sec^2\theta} = 2|\sec\theta| = 2\sec\theta$$

Thus we have

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2\sec^2\theta d\theta}{4\tan^2\theta \cdot 2\sec\theta} = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta} d\theta$$

To evaluate this trigonometric integral we put everything in terms of $\sin\theta$ and $\cos\theta$;

$$\frac{\sec\theta}{\tan^2\theta} = \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta} = \frac{\cos\theta}{\sin^2\theta}$$

Therefore, making the substitution $u = \sin\theta$, we have

Example Solution

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}d\theta = \frac{1}{4}\int \frac{du}{u^2}$$
$$= \frac{1}{4}\left(-\frac{1}{u}\right) + C = -\frac{1}{4\sin\theta} + C$$
$$= -\frac{\csc\theta}{4} + C$$

We determine that $\csc\theta = \sqrt{x^2+4}/x$ and so

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = -\frac{\sqrt{x^2+4}}{4x} + C$$

Trigonometric Integrals: Strategy for Evaluating $\int \sin^m x \cos^n x dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\begin{aligned}\int \sin^m x \cos^{2k+1} x dx &= \int \sin^m x \left(\cos^2 x\right)^k \cos x dx \\ &= \int \sin^m x \left(1 - \sin^2 x\right)^k \cos x dx\end{aligned}$$

Then substitute $u = \sin x$.

Trigonometric Integrals: Strategy for Evaluating $\int \sin^m x \cos^n x dx$

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\begin{aligned} \int \sin^{2k+1} x \cos^n x dx &= \int \left(\sin^2 x\right)^k \cos^n x \sin x dx \\ &= \int \left(1 - \cos^2 x\right)^k \cos^n x \sin x dx \end{aligned}$$

Then substitute $u = \cos x$. [Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

Trigonometric Integrals: Strategy for Evaluating $\int \sin^m x \cos^n x dx$

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \frac{1}{2} \sin 2x$$

Trigonometric Integrals: Strategy for Evaluating $\int \tan^m x \sec^n x dx$

(a) If the power of secant is even $(n = 2k, k \geqslant 2)$, save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\begin{aligned}\int \tan^m x \sec^{2k} x dx &= \int \tan^m x \left(\sec^2 x\right)^{k-1} \sec^2 x dx \\ &= \int \tan^m x \left(1 + \tan^2 x\right)^{k-1} \sec^2 x dx\end{aligned}$$

Then substitute $u = \tan x$.

Trigonometric Integrals: Strategy for Evaluating $\int \tan^m x \sec^n x dx$

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\begin{aligned} \int \tan^{2k+1} x \sec^n x dx &= \int \left(\tan^2 x\right)^k \sec^{n-1} x \sec x \tan x dx \\ &= \int \left(\sec^2 x - 1\right)^k \sec^{n-1} x \sec x \tan x dx \end{aligned}$$

Then substitute $u = \sec x$.

Trigonometric Integrals: Strategy for Evaluating $\int \sin mx \cos nx dx$

Product-to-sum	Sum-to-product
$\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}$	$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$
$\cos\alpha\sin\beta = \frac{\sin(\alpha+\beta)-\sin(\alpha-\beta)}{2}$	$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$
$\cos\alpha\cos\beta = \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}$	$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$
$\sin\alpha\sin\beta = -\frac{\cos(\alpha+\beta)-\cos(\alpha-\beta)}{2}$	$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

Ex 8

Evaluate the integral

$$\int_0^{\pi/2} \sin^7\theta \cos^5\theta d\theta$$

Ex 8

Solution

$$\begin{aligned}
\int_0^{\pi/2} \sin^7\theta\cos^5\theta\, d\theta &= \int_0^{\pi/2} \sin^7\theta\cos^4\theta\cos\theta\, d\theta \\
&= \int_0^{\pi/2} \sin^7\theta\left(1-\sin^2\theta\right)^2\cos\theta\, d\theta \\
&\overset{\mathrm{s}}{=} \int_0^1 u^7\left(1-u^2\right)^2 du \\
&= \int_0^1 \left(u^7 - 2u^9 + u^{11}\right) du \\
&= \left[\frac{1}{8}u^8 - \frac{1}{5}u^{10} + \frac{1}{12}u^{12}\right]_0^1 \\
&= \left(\frac{1}{8} - \frac{1}{5} + \frac{1}{12}\right) - 0 = \frac{1}{120}
\end{aligned}$$

Gamma function and Beta function

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$$

or using B notation:

$$B(x, y) = \int_0^{\pi/2} 2\sin^{2x-1}(t)\cos^{2y-1}(t)dt$$

We usually change the trigonometric integrals to Γ function.

Gamma function and Beta function

Definition:

$$\Gamma(z) = \int_0^\infty t^{z-1} \mathrm{e}^{-t} \mathrm{dt}$$

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

Property:

1.

$$\Gamma(n) = (n-1)!$$

(n is a positive integer!)

2.

$$\Gamma(x+1) = x\Gamma(x)$$

3.

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{n!4^n}$$

Gamma function and Beta function

4. Special Points:

$$\Gamma\left(-\frac{3}{2}\right) = \frac{4}{3}\sqrt{\pi}$$

$$\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(1) = 0! = 1$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

$$\Gamma(2) = 1! = 1$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}$$

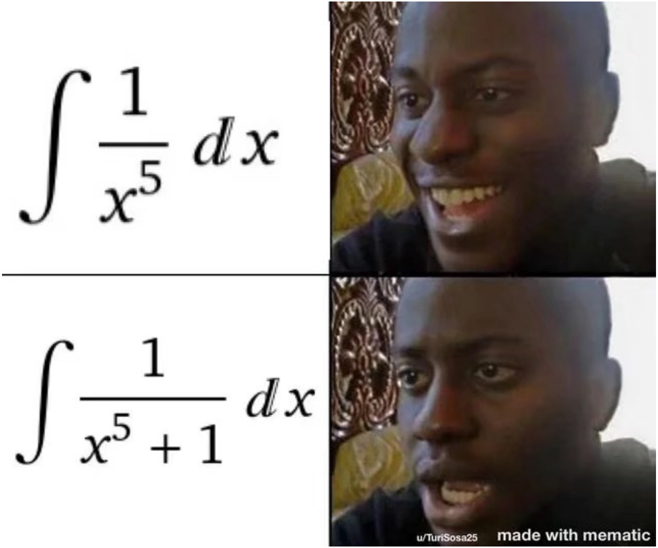
$$\Gamma(4) = 3! = 6$$

Back to Ex 8

$$\int_0^{\frac{\pi}{2}} \sin^7\theta\cos^5\theta \text{ to } \Gamma \text{ function}$$

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^7\theta\cos^5\theta &= \frac{\Gamma\left(\frac{7+1}{2}\right)\Gamma\left(\frac{5+1}{2}\right)}{2\Gamma\left(\frac{7+5+2}{2}\right)}\\ &= \frac{\Gamma(4)\Gamma(3)}{2\Gamma(6)}\\ &= \frac{3!\times 2!}{2\times 5!}\\ &= \frac{1}{120}\end{aligned}$$

Back to memes



Back to memes

$$\int \frac{1}{x^5} dx = -\frac{1}{4x^4} + C$$

Back to memes

With $\phi_\pm = \frac{1\pm\sqrt{5}}{4}$

$$x^5 + 1 = (1 + x)\left(x^2 - 2\phi_+ x + 1\right)\left(x^2 - 2\phi_- x + 1\right)$$

and

$$\frac{5}{1 + x^5} = \frac{1}{x+1} - \frac{2\phi_+ x - 2}{x^2 - 2\phi_+ x + 1} - \frac{2\phi_- x - 2}{x^2 - 2\phi_- x + 1}$$

The integral for the first term is just $\ln(x+1)$, and for the second and third terms

$$I(x,\phi) = \int \frac{2\phi x - 2}{x^2 - 2\phi x + 1} dx = \int \frac{\phi d\left[(x-\phi)^2\right] - 2\left(1 - \phi^2\right) dx}{(x-\phi)^2 + (1-\phi^2)}$$

$$= \phi \ln\left(x^2 - 2\phi x + 1\right) - 2\sqrt{1-\phi^2}\tan^{-1}\frac{x-\phi}{\sqrt{1-\phi^2}}$$

Back to memes

Thus

$$\int \frac{1}{1+x^5}\,dx = \frac{1}{5}\left[\ln(x+1) - I\left(x,\phi_+\right) - I\left(x,\phi_-\right)\right] + C$$

Back to memes

$$
\begin{aligned}
\int \frac{1}{1+x}\,dx &= \int \left(\frac{1}{x}+\frac{1}{1}\right) dx \\
&= \int \frac{1}{x}\,dx + \int \frac{1}{1}\,dx \\
&= \log(x) + \log(1) \\
&= \log(x+1) + C.
\end{aligned}
$$

Back to memes

$$\int \frac{1}{x+1} dx = \ln|x+1| + C$$

Back to memes

$$\int \ln(x)\,dx \qquad\qquad \int \frac{1}{\ln(x)}\,dx$$

Back to memes

$$\int \ln(x)dx = x\ln(x) - x + C$$

Back to memes

$$
\begin{aligned}
\int \frac{dx}{\ln x} &\overset{\text{let } x=e^t}{=} \int \frac{de^t}{\ln t} \\
&= \int \frac{\lim_{n\to+\infty} \sum_{i=0}^{n} \frac{t^i}{i!}}{t} dt \\
&= \int \left(\frac{1}{t} + 1 + \frac{t^1}{2!} + \frac{t^2}{3!} + \cdots\cdots \right) dt \\
&= \ln|t| + \frac{t^1}{1 \times 1!} + \frac{t^2}{2 \times 2!} + \frac{t^3}{3 \times 3!} + \cdots\cdots + C \\
&= \ln|\ln x| + \lim_{n\to+\infty} \sum_{i=1}^{n} \frac{(\ln x)^i}{i \cdot i!} + C
\end{aligned}
$$

It's not an elementary integral. We can only change it by Liouville's theorem.

Back to memes

$$\int \frac{sin^{-1}(x)}{\sqrt{1-x^2}}dx$$

$$\int \frac{sin(x)}{\sqrt{1-x^2}}dx$$

Back to memes

$$\int \frac{\arcsin(x)}{\sqrt{1-x^2}} dx = \frac{\arcsin^2(x)}{2} + C$$

$$\int \frac{\sin(x)}{\sqrt{1-x^2}} dx = ?\textbf{Cannot solve this}$$

Q&A

Q&A