10.1 Curves Defined by Parametric Equations

Curves Defined by Parametric Equations

Imagine that a particle moves along the curve *C* shown in Figure 1. It is impossible to describe *C* by an equation of the form $y = f(x)$ because *C* fails the Vertical Line Test.

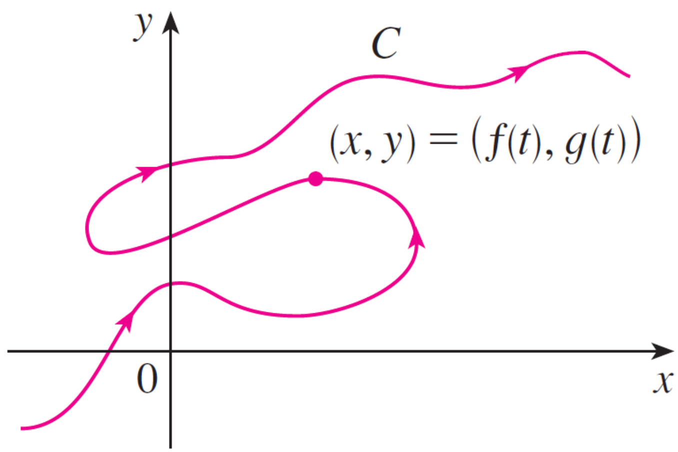


Figure 1

Curves Defined by Parametric Equations

But the *x*- and *y*-coordinates of the particle are functions of time and so we can write $x = f(t)$ and $y = g(t)$. Such a pair of equations is often a convenient way of describing a curve and gives rise to the following definition.

Suppose that *x* and *y* are both given as functions of a third variable *t* (called a **parameter**) by the equations

$$x = f(t) \qquad y = g(t)$$

(called **parametric equations**).

Curves Defined by Parametric Equations

Each value of t determines a point (x, y), which we can plot in a coordinate plane.

As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, which we call a **parametric curve**.

The parameter t does not necessarily represent time and, in fact, we could use a letter other than t for the parameter.

But in many applications of parametric curves, t does denote time and therefore we can interpret $(x, y) = (f(t), g(t))$ as the position of a particle at time t.

Example 1

Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t \qquad\qquad y = t + 1$$

Solution:

Each value of t gives a point on the curve, as shown in the table.

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5

Example 1 – *Solution*

cont'd

For instance, if $t = 0$, then $x = 0$, $y = 1$, and so the corresponding point is (0, 1).

In Figure 2 we plot the points (x, y) determined by several values of the parameter and we join them to produce a curve.

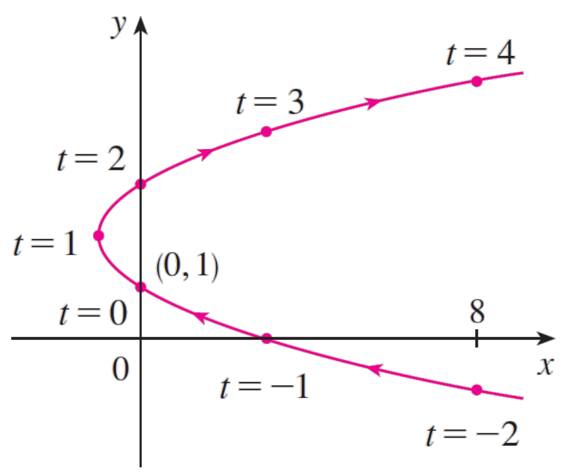


Figure 2

Example 1 – *Solution*

cont'd

A particle whose position is given by the parametric equations moves along the curve in the direction of the arrows as t increases.

Notice that the consecutive points marked on the curve appear at equal time intervals but not at equal distances. That is because the particle slows down and then speeds up as t increases.

It appears from Figure 2 that the curve traced out by the particle may be a parabola.

Example 1 – *Solution*

cont'd

This can be confirmed by eliminating the parameter t as follows. We obtain $t = y - 1$ from the second equation and substitute into the first equation.

This gives

$$x = t^2 - 2t$$

$$= (y - 1)^2 - 2(y - 1)$$

$$= y^2 - 4y + 3$$

and so the curve represented by the given parametric equations is the parabola $x = y^2 - 4y + 3$.

Curves Defined by Parametric Equations

No restriction was placed on the parameter t in Example 1, so we assumed that t could be any real number.

But sometimes we restrict t to lie in a finite interval. For instance, the parametric curve

$$x = t^2 - 2t \qquad y = t + 1 \qquad 0 \le t \le 4$$

Curves Defined by Parametric Equations

Shown in Figure 3 is the part of the parabola in Example 1 that starts at the point (0, 1) and ends at the point (8, 5). The arrowhead indicates the direction in which the curve is traced as *t* increases from 0 to 4.

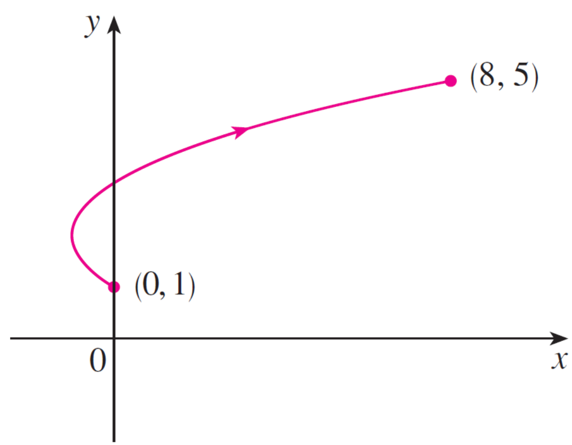


Figure 3

Curves Defined by Parametric Equations

In general, the curve with parametric equations

$$x = f(t) \qquad y = g(t) \qquad a \le t \le b$$

has **initial point** $(f(a), g(a))$ and **terminal point** $(f(b), g(b))$.

Example 2

What curve is represented by the following parametric equations?

$$x = \cos t \qquad y = \sin t \qquad 0 \le t \le 2\pi$$

Solution:

If we plot points, it appears that the curve is a circle. We can confirm this impression by eliminating *t*. Observe that

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

Thus the point (x, y) moves on the unit circle $x^2 + y^2 = 1$.

Example 2 – *Solution*

cont'd

Notice that in this example the parameter *t* can be interpreted as the angle (in radians) shown in Figure 4.

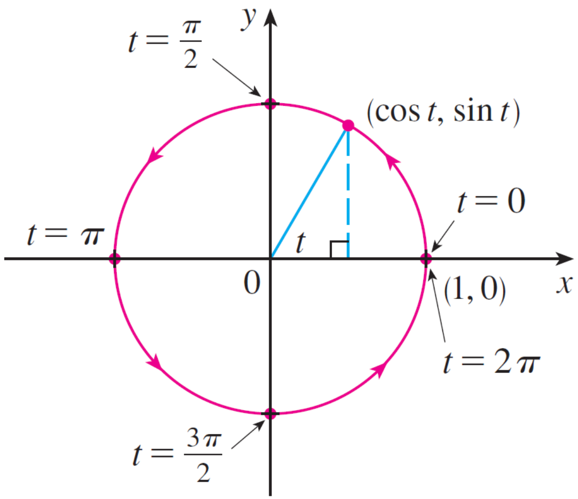


Figure 4

As t increases from 0 to 2π, the point $(x, y) = (\cos t, \sin t)$ moves once around the circle in the counterclockwise direction starting from the point (1, 0).

Graphing Devices

Graphing Devices

Most graphing calculators and computer graphing programs can be used to graph curves defined by parametric equations.

In fact, it's instructive to watch a parametric curve being drawn by a graphing calculator because the points are plotted in order as the corresponding parameter values increase.

Example 6

Use a graphing device to graph the curve $x = y^4 - 3y^2$.

Solution:

If we let the parameter be $t = y$, then we have the equations

$$x = t^4 - 3t^2 \qquad\qquad y = t$$

Example 6 – *Solution*

cont'd

Using these parametric equations to graph the curve, we obtain Figure 9.

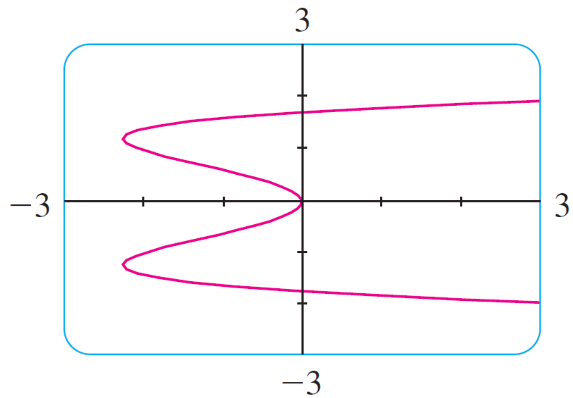


Figure 9

It would be possible to solve the given equation ($x = y^4 - 3y^2$) for y as four functions of x and graph them individually, but the parametric equations provide a much easier method.

Graphing Devices

One of the most important uses of parametric curves is in computer-aided design (CAD).

In the Laboratory Project, we will investigate special parametric curves, called **Bézier curves**, that are used extensively in manufacturing, especially in the automotive industry.

These curves are also employed in specifying the shapes of letters and other symbols in laser printers.

The Cycloid

Example 7

The curve traced out by a point *P* on the circumference of a circle as the circle rolls along a straight line is called a **cycloid** (see Figure 13).

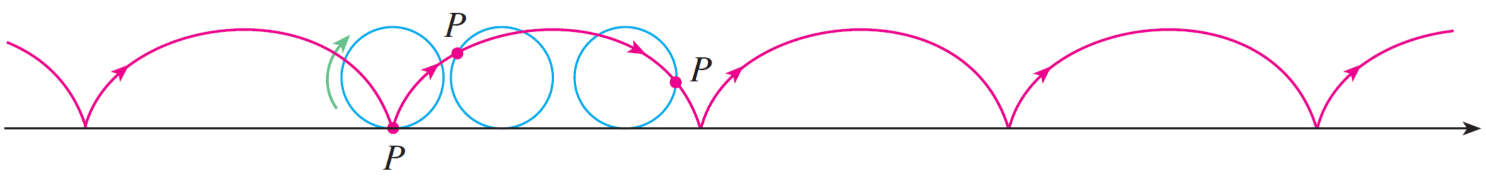


Figure 13

If the circle has radius *r* and rolls along the *x*-axis and if one position of *P* is the origin, find parametric equations for the cycloid.

Example 7 – *Solution*

We choose as parameter the angle of rotation θ of the circle ($\theta = 0$ when P is at the origin). Suppose the circle has rotated through θ radians.

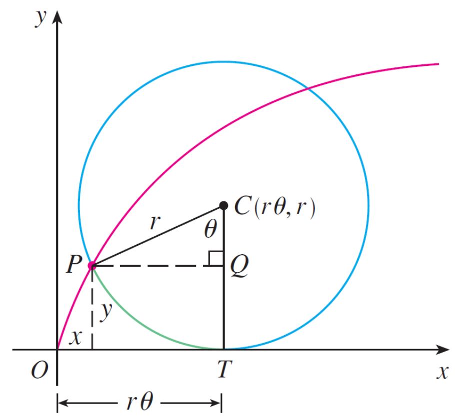


Figure 14

Because the circle has been in contact with the line, we see from Figure 14 that the distance it has rolled from the origin is

$$|OT| = \text{arc } PT = r\theta$$

Example 7 – *Solution*

cont'd

Therefore the center of the circle is $C(r\theta, r)$. Let the coordinates of P be (x, y). Then from Figure 14 we see that

$$x = |OT| - |PQ| = r\theta - r\sin\theta = r(\theta - \sin\theta)$$

$$y = |TC| - |QC| = r - r\cos\theta = r(1 - \cos\theta)$$

Therefore parametric equations of the cycloid are

$$\boxed{1} \qquad x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta) \qquad \theta \in \mathbb{R}$$

Example 7 – *Solution*

cont'd

One arch of the cycloid comes from one rotation of the circle and so is described by $0 \leq \theta \leq 2\pi$.

Although Equations 1 were derived from Figure 14, which illustrates the case where $0 < \theta < \pi/2$, it can be seen that these equations are still valid for other values of θ.

Although it is possible to eliminate the parameter θ from Equations 1, the resulting Cartesian equation in x and y is very complicated and not as convenient to work with as the parametric equations.

The Cycloid

One of the first people to study the cycloid was Galileo, who proposed that bridges be built in the shape of cycloids and who tried to find the area under one arch of a cycloid.

Later this curve arose in connection with the **brachistochrone problem**: Find the curve along which a particle will slide in the shortest time (under the influence of gravity) from a point *A* to a lower point *B* not directly beneath *A*.

The Cycloid

The Swiss mathematician John Bernoulli, who posed this problem in 1696, showed that among all possible curves that join *A* to *B*, as in Figure 15, the particle will take the least time sliding from *A* to *B* if the curve is part of an inverted arch of a cycloid.

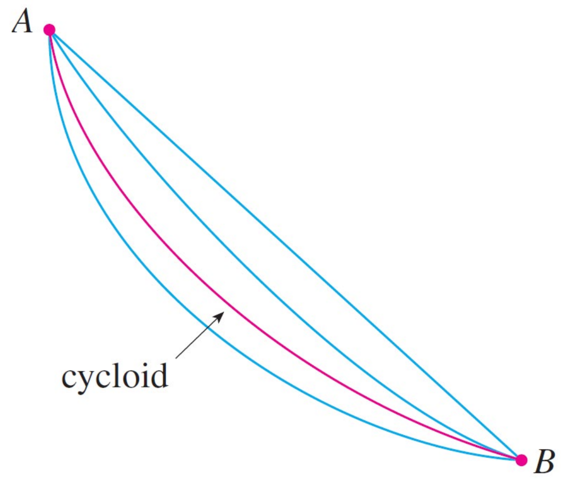


Figure 15

The Cycloid

The Dutch physicist Huygens had already shown that the cycloid is also the solution to the **tautochrone problem**; that is, no matter where a particle *P* is placed on an inverted cycloid, it takes the same time to slide to the bottom (see Figure 16).

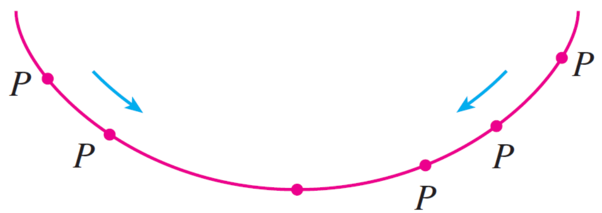


Figure 16

The Cycloid

Huygens proposed that pendulum clocks (which he invented) should swing in cycloidal arcs because then the pendulum would take the same time to make a complete oscillation whether it swings through a wide or a small arc.

Families of Parametric Curves

Example 8

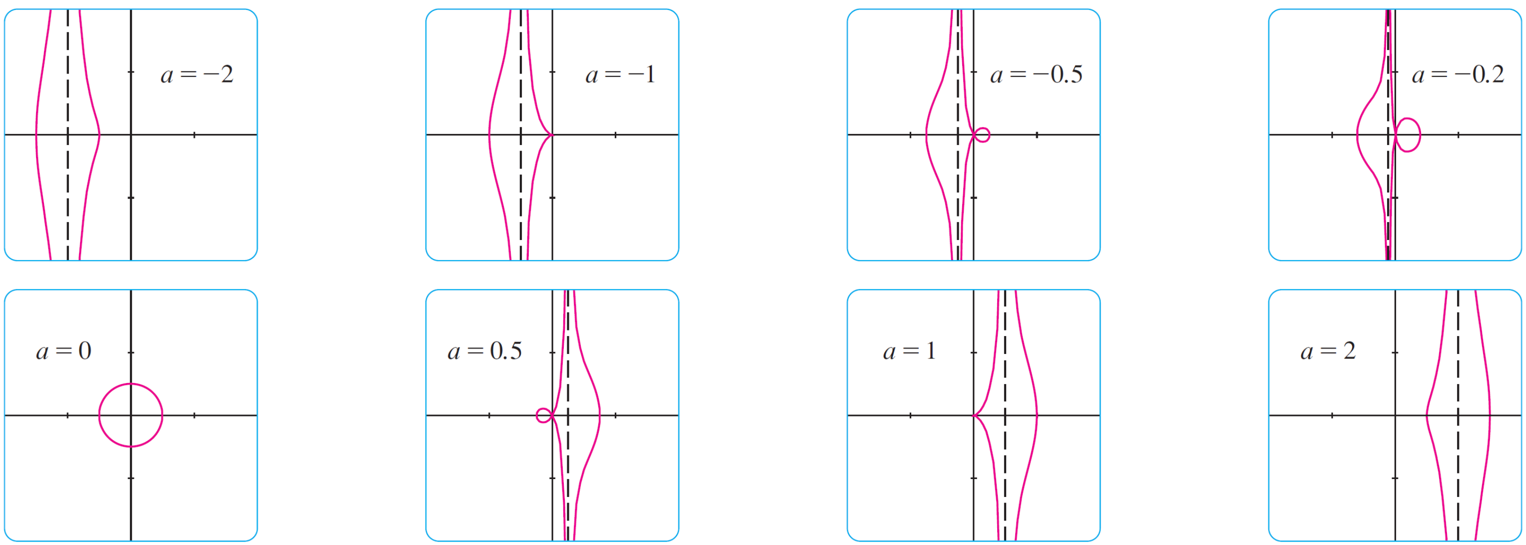
Investigate the family of curves with parametric equations

$$x = a + \cos t \qquad\qquad y = a \tan t + \sin t$$

What do these curves have in common? How does the shape change as a increases?

Example 8 – *Solution*

We use a graphing device to produce the graphs for the cases a = –2, –1, –0.5, –0.2, 0, 0.5, 1 and 2 shown in Figure 17.



Members of the family $x = a + \cos t$, $y = a \tan t + \sin t$, all graphed in the viewing rectangle [–4, 4] by [–4, 4]

Figure 17

Example 8 – *Solution*

cont'd

Notice that all of these curves (except the case $a = 0$) have two branches, and both branches approach the vertical asymptote $x = a$ as x approaches a from the left or right.

When a < –1, both branches are smooth; but when a reaches –1, the right branch acquires a sharp point, called a *cusp*.

For a between –1 and 0 the cusp turns into a loop, which becomes larger as a approaches 0. When $a = 0$, both branches come together and form a circle.

Example 8 – *Solution*

cont'd

For a between 0 and 1, the left branch has a loop, which shrinks to become a cusp when $a = 1$.

For $a > 1$, the branches become smooth again, and as a increases further, they become less curved. Notice that the curves with a positive are reflections about the y-axis of the corresponding curves with a negative.

These curves are called **conchoids of Nicomedes** after the ancient Greek scholar Nicomedes. He called them conchoids because the shape of their outer branches resembles that of a conch shell or mussel shell.

10.2 Calculus with Parametric Curves

Tangents

Tangents

Suppose *f* and *g* are differentiable functions and we want to find the tangent line at a point on the curve where *y* is also a differentiable function of *x*.

Then the Chain Rule gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Tangents

If $dx/dt \neq 0$, we can solve for dy/dx:

$$\boxed{1} \qquad \frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} \qquad \text{if} \quad \frac{dx}{dt} \neq 0$$

Equation 1 (which you can remember by thinking of canceling the *dt*'s) enables us to find the slope dy/dx of the tangent to a parametric curve without having to eliminate the parameter t.

Tangents

We see from $\boxed{1}$ that the curve has a horizontal tangent when $dy/dt = 0$ (provided that $dx/dt \neq 0$) and it has a vertical tangent when $dx/dt = 0$ (provided that $dy/dt \neq 0$).

This information is useful for sketching parametric curves.

It is also useful to consider d^2y/dx^2. This can be found by replacing y by dy/dx in Equation 1:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}$$

Example 1

A curve C is defined by the parametric equations $x = t^2$, $y = t^3 - 3t$.

(a) Show that C has two tangents at the point (3, 0) and find their equations.

(b) Find the points on C where the tangent is horizontal or vertical.

(c) Determine where the curve is concave upward or downward.

(d) Sketch the curve.

Example 1(a) – *Solution*

Notice that $y = t^3 - 3t = t(t^2 - 3) = 0$ when $t = 0$ or $t = \pm\sqrt{3}$. Therefore the point (3, 0) on C arises from two values of the parameter, $t = \sqrt{3}$ and $t = -\sqrt{3}$.

This indicates that C crosses itself at (3, 0).

Example 1(a) – *Solution*

cont'd

Since

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t}$$

$$= \frac{3}{2}\left(t - \frac{1}{t}\right)$$

the slope of the tangent when $t = \pm\sqrt{3}$ is $dy/dx = \pm 6/\left(2\sqrt{3}\right) = \pm\sqrt{3}$, so the equations of the tangents at (3, 0) are

$$y = \sqrt{3}\,(x - 3) \qquad \text{and} \qquad y = -\sqrt{3}\,(x - 3)$$

Example 1(b) – *Solution*

cont'd

C has a horizontal tangent when $dy/dx = 0$, that is, when $dy/dt = 0$ and $dx/dt \neq 0$. Since $dy/dt = 3t^2 - 3$, this happens when $t^2 = 1$, that is, $t = \pm 1$.

The corresponding points on C are $(1, -2)$ and $(1, 2)$.

C has a vertical tangent when $dx/dt = 2t = 0$, that is, $t = 0$. (Note that $dy/dt \neq 0$ there.)

The corresponding point on C is $(0, 0)$.

Example 1(c) – *Solution*

cont'd

To determine concavity we calculate the second derivative:

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}} = \frac{\dfrac{3}{2}\left(1 + \dfrac{1}{t^2}\right)}{2t}$$

$$= \frac{3(t^2 + 1)}{4t^3}$$

Thus the curve is concave upward when $t > 0$ and concave downward when $t < 0$.

Example 1(d) – *Solution*

cont'd

Using the information from parts (b) and (c), we sketch C in Figure 1.

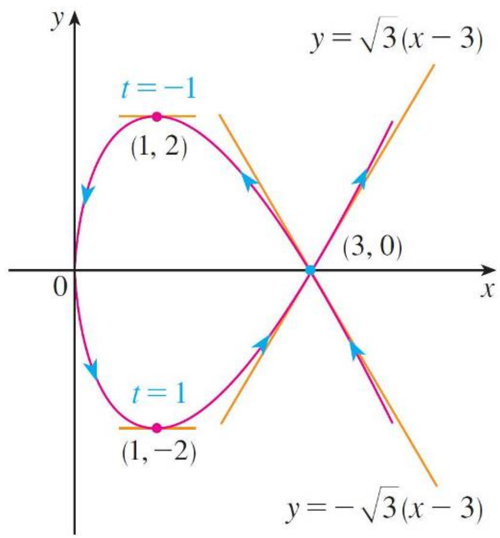


Figure 1

Areas

Areas

We know that the area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x)\,dx$, where $F(x) \geq 0$.

If the curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t)f'(t)\,dt \qquad \left[\text{or} \quad \int_\beta^\alpha g(t)f'(t)\,dt\right]$$

Example 3

Find the area under one arch of the cycloid

$$x = r(\theta - \sin\ \theta) \qquad y = r(1 - \cos\ \theta)$$

(See Figure 3.)

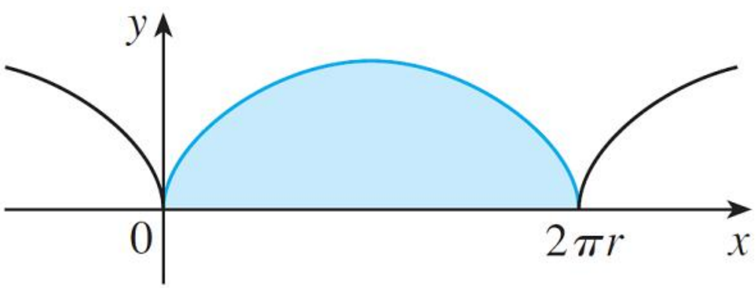


Figure 3

Example 3 – *Solution*

One arch of the cycloid is given by $0 \le \theta \le 2\pi$.

Using the Substitution Rule with $y = r(1 - \cos\theta)$ and $dx = r(1 - \cos\theta)\, d\theta$, we have

$$A = \int_0^{2\pi r} y\, dx = \int_0^{2\pi} r(1 - \cos\theta)\, r(1 - \cos\theta)\, d\theta$$

$$= r^2 \int_0^{2\pi} (1 - \cos\theta)^2\, d\theta$$

$$= r^2 \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)\, d\theta$$

Example 3 – *Solution*

cont'd

$$= r^2 \int_0^{2\pi} \left[1 - 2\cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\right] d\theta$$

$$= r^2\left[\tfrac{3}{2}\theta - 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}$$

$$= r^2\left(\tfrac{3}{2} \cdot 2\pi\right) = 3\pi r^2$$

Arc Length

Arc Length

We already know how to find the length L of a curve C given in the form $y = F(x)$, $a \leq x \leq b$.

If F' is continuous, then

$$\boxed{2} \qquad L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Suppose that C can also be described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, where $dx/dt = f'(t) > 0$.

Arc Length

This means that C is traversed once, from left to right, as t increases from α to β and $f(\alpha) = a$, $f(\beta) = b$.

Putting Formula 1 into Formula 2 and using the Substitution Rule, we obtain

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt}\, dt$$

Since $dx/dt > 0$, we have

$$\boxed{3} \qquad L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Arc Length

Even if C can't be expressed in the form $y = F(x)$, Formula 3 is still valid but we obtain it by polygonal approximations.

We divide the parameter interval $[\alpha, \beta]$ into n subintervals of equal width Δt.

If $t_0, t_1, t_2, \ldots, t_n$ are the endpoints of these subintervals, then $x_i = f(t_i)$ and $y_i = g(t_i)$ are the coordinates of points $P_i(x_i, y_i)$ that lie on C and the polygon with vertices $P_0, P_1, \ldots, P_n$ approximates C.

Arc Length

(See Figure 4.)

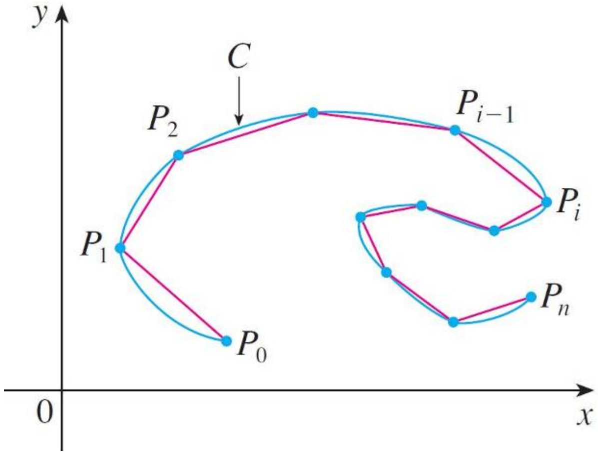


Figure 4

We define the length L of C to be the limit of the lengths of these approximating polygons as $n \to \infty$:

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

Arc Length

The Mean Value Theorem, when applied to f on the interval $[t_{i-1}, t_i]$, gives a number t_i^* in (t_{i-1}, t_i) such that

$$f(t_i) - f(t_{i-1}) = f'(t_i^*)\,(t_i - t_{i-1})$$

If we let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$, this equation becomes

$$\Delta x_i = f'(t_i^*)\,\Delta t$$

Similarly, when applied to g, the Mean Value Theorem gives a number t_i^{**} in (t_{i-1}, t_i) such that

$$\Delta y_i = g'(t_i^{**})\,\Delta t$$

Arc Length

Therefore

$$|P_{i-1}P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{[f'(t_i^*)\,\Delta t]^2 + [g'(t_i^{**})\,\Delta t]^2}$$

$$= \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2}\;\Delta t$$

and so

$$\boxed{4} \qquad L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2}\;\Delta t$$

Arc Length

The sum in $\boxed{4}$ resembles a Riemann sum for the function $\sqrt{[f'(t)]^2 + [g'(t)]^2}$ but it is not exactly a Riemann sum because $t_i^* \neq t_i^{**}$ in general.

Nevertheless, if f' and g' are continuous, it can be shown that the limit in $\boxed{4}$ is the same as if t_i^* and t_i^{**} were equal, namely,

$$L = \int_\alpha^\beta \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

Arc Length

Thus, using Leibniz notation, we have the following result, which has the same form as Formula 3.

5 Theorem If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Notice that the formula in Theorem 5 is consistent with the general formulas $L = \int ds$ and $(ds)^2 = (dx)^2 + (dy)^2$.

Arc Length

Notice that the integral gives twice the arc length of the circle because as t increases from 0 to 2π, the point ($\sin 2t$, $\cos 2t$) traverses the circle twice.

In general, when finding the length of a curve C from a parametric representation, we have to be careful to ensure that C is traversed only once as t increases from α to β.

Example 5

Find the length of one arch of the cycloid $x = r(\theta - \sin\,\theta)$, $y = r(1 - \cos\,\theta)$.

Solution:

From Example 3 we see that one arch is described by the parameter interval $0 \le \theta \le 2\pi$.

Since

$$\frac{dx}{d\theta} = r(1 - \cos\theta) \qquad \text{and} \qquad \frac{dy}{d\theta} = r\sin\theta$$

Example 5 – *Solution*

cont'd

We have

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2(1 - \cos\theta)^2 + r^2\sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)}\, d\theta$$

$$= r\int_0^{2\pi} \sqrt{2(1 - \cos\theta)}\, d\theta$$

Example 5 – *Solution*

cont'd

To evaluate this integral we use the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ with $\theta = 2x$, which gives $1 - \cos\theta = 2\sin^2(\theta/2)$.

Since $0 \le \theta \le 2\pi$, we have $0 \le \theta/2 \le \pi$ and so $\sin(\theta/2) \ge 0$.

Therefore

$$\sqrt{2(1 - \cos\theta)} = \sqrt{4\sin^2(\theta/2)}$$

$$= 2\left|\sin(\theta/2)\right|$$

$$= 2\sin(\theta/2)$$

Example 5 – *Solution*

cont'd

And so

$$L = 2r \int_0^{2\pi} \sin(\theta/2)\, d\theta$$

$$= 2r\Big[-2\cos(\theta/2)\Big]_0^{2\pi}$$

$$= 2r[2 + 2]$$

$$= 8r$$

Surface Area

Surface Area

If the curve given by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, is rotated about the x-axis, where f', g' are continuous and $g(t) \geq 0$, then the area of the resulting surface is given by

$$\boxed{6} \qquad S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The general symbolic formulas $S = \int 2\pi y\, ds$ and $S = \int 2\pi x\, ds$ are still valid, but for parametric curves we use

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example 6

Show that the surface area of a sphere of radius r is $4\pi r^2$.

Solution:

The sphere is obtained by rotating the semicircle

$$x = r\cos t \qquad y = r\sin t \qquad 0 \le t \le \pi$$

about the x-axis.

Therefore, from Formula 6, we get

$$S = \int_0^{\pi} 2\pi r \sin t \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt$$

Example 6 – *Solution*

cont'd

$$= 2\pi \int_0^\pi r \sin t \sqrt{r^2(\sin^2 t + \cos^2 t)}\, dt$$

$$= 2\pi \int_0^\pi r \sin t \cdot r\, dt$$

$$= 2\pi r^2 \int_0^\pi \sin t\, dt$$

$$= 2\pi r^2(-\cos t)\Big]_0^\pi$$

$$= 4\pi r^2$$

10.3 Polar Coordinates

Polar Coordinates

A coordinate system represents a point in the plane by an ordered pair of numbers called coordinates. Usually we use Cartesian coordinates, which are directed distances from two perpendicular axes.

Here we describe a coordinate system introduced by Newton, called the **polar coordinate system**, which is more convenient for many purposes.

We choose a point in the plane that is called the **pole** (or origin) and is labeled *O*. Then we draw a ray (half-line) starting at *O* called the **polar axis**.

Polar Coordinates

This axis is usually drawn horizontally to the right and corresponds to the positive x-axis in Cartesian coordinates. If P is any other point in the plane, let r be the distance from O to P and let θ be the angle (usually measured in radians) between the polar axis and the line OP as in Figure 1.

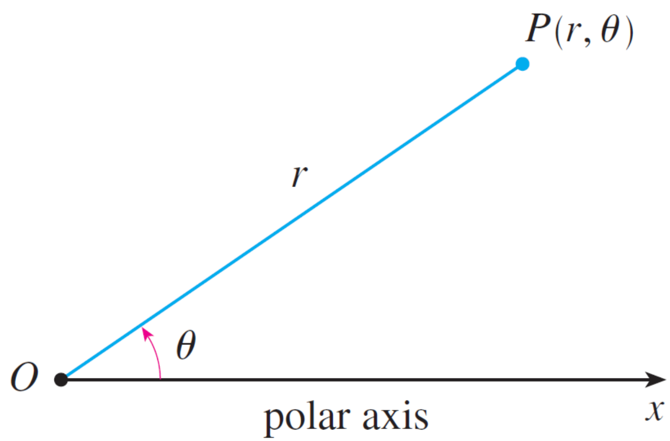


Figure 1

Polar Coordinates

Then the point P is represented by the ordered pair (r, θ) and r, θ are called **polar coordinates** of P.

We use the convention that an angle is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction.

If $P = O$, then $r = 0$ and we agree that $(0, \theta)$ represents the pole for any value of θ.

Polar Coordinates

We extend the meaning of polar coordinates (r, θ) to the case in which r is negative by agreeing that, as in Figure 2, the points ($-r$, θ) and (r, θ) lie on the same line through O and at the same distance $|r|$ from O, but on opposite sides of O.

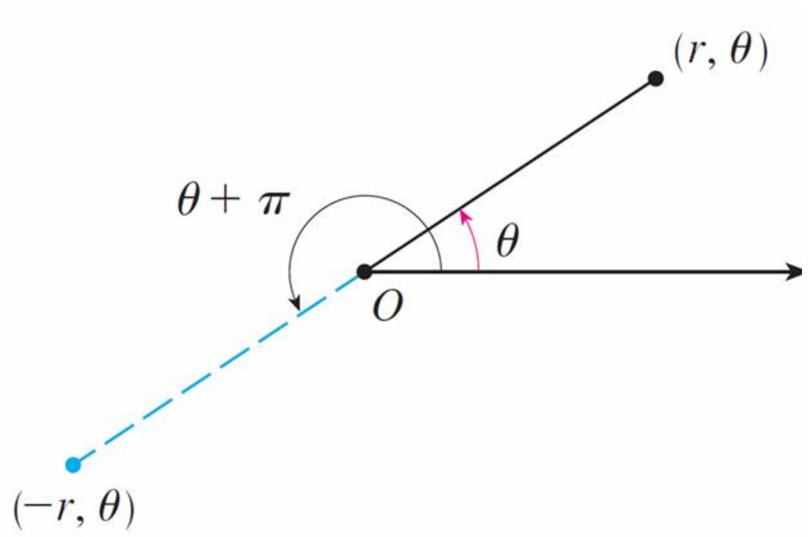


Figure 2

Polar Coordinates

If $r > 0$, the point (r, θ) lies in the same quadrant as θ; if $r < 0$, it lies in the quadrant on the opposite side of the pole. Notice that $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$.

Example 1

Plot the points whose polar coordinates are given.

(a) $(1, 5\pi/4)$ (b) $(2, 3\pi)$ (c) $(2, -2\pi/3)$ (d) $(-3, 3\pi/4)$

Solution:

The points are plotted in Figure 3.

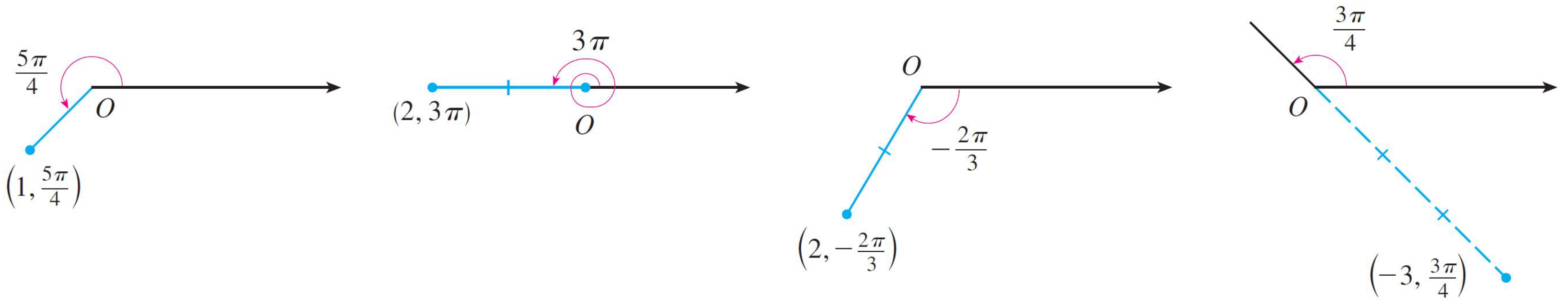


Figure 3

Example 1 – *Solution*

cont'd

In part (d) the point $(–3, 3\pi/4)$ is located three units from the pole in the fourth quadrant because the angle $3\pi/4$ is in the second quadrant and $r = –3$ is negative.

Polar Coordinates

In the Cartesian coordinate system every point has only one representation, but in the polar coordinate system each point has many representations. For instance, the point in $(1, 5\pi/4)$ Example 1(a) could be written as $(1, -3\pi/4)$ or $(1, 13\pi/4)$ or $(-1, \pi/4)$. (See Figure 4.)

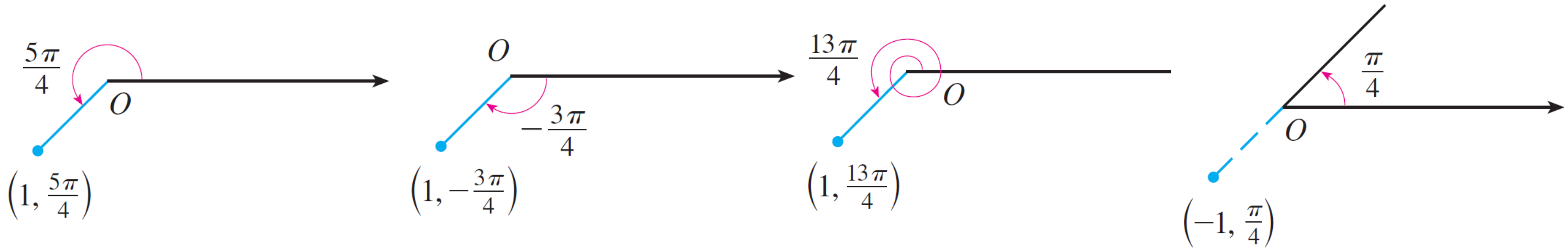


Figure 4

Polar Coordinates

In fact, since a complete counterclockwise rotation is given by an angle 2π, the point represented by polar coordinates (r, θ) is also represented by

$$(r, \theta + 2n\pi) \quad \text{and} \quad (-r, \theta + (2n + 1)\pi)$$

where n is any integer.

Polar Coordinates

The connection between polar and Cartesian coordinates can be seen from Figure 5, in which the pole corresponds to the origin and the polar axis coincides with the positive *x*-axis.

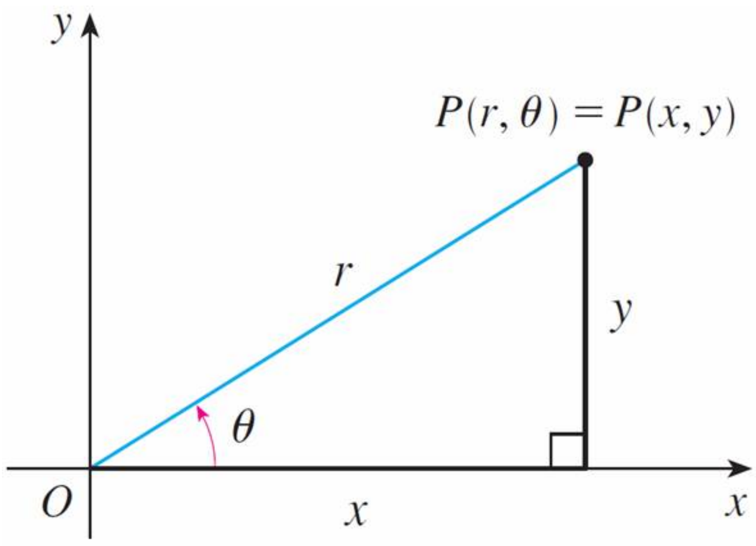


Figure 5

Polar Coordinates

If the point P has Cartesian coordinates (x, y) and polar coordinates (r, θ), then, from the figure, we have

$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r}$$

and so

1

$$\boxed{x = r\cos\theta \qquad y = r\sin\theta}$$

Although Equations 1 were deduced from Figure 5, which illustrates the case where $r > 0$ and $0 < \theta < \pi/2$, these equations are valid for all values of r and θ.

Polar Coordinates

Equations 1 allow us to find the Cartesian coordinates of a point when the polar coordinates are known. To find r and θ when x and y are known, we use the equations

2

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

which can be deduced from Equations 1 or simply read from Figure 5.

Example 2

Convert the point $(2, \pi/3)$ from polar to Cartesian coordinates.

Solution:

Since $r = 2$ and $\theta = \pi/3$, Equations 1 give

$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2\cdot\frac{1}{2} = 1$$

$$y = r\sin\theta = 2\sin\frac{\pi}{3} = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$$

Therefore the point is $(1, \sqrt{3})$ in Cartesian coordinates.

Example 3

Represent the point with Cartesian coordinates (1, –1) in terms of polar coordinates.

Solution:

If we choose r to be positive, then Equations 2 give

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\tan\theta = \frac{y}{x} = -1$$

Since the point (1, –1) lies in the fourth quadrant, we can choose $\theta = -\pi/4$ or $\theta = 7\pi/4$. Thus one possible answer is $(\sqrt{2}, -\pi/4)$; another is $(\sqrt{2}, 7\pi/4)$.

Polar Coordinates

Note:

Equations 2 do not uniquely determine θ when x and y are given because, as θ increases through the interval $0 \leq \theta < 2\pi$, each value of tan θ occurs twice.

Therefore, in converting from Cartesian to polar coordinates, it's not good enough just to find r and θ that satisfy Equations 2.

As in Example 3, we must choose θ so that the point (r, θ) lies in the correct quadrant.

Polar Curves

Polar Curves

The **graph of a polar equation** $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

Example 4

What curve is represented by the polar equation $r = 2$?

Solution:

The curve consists of all points (r, θ) with $r = 2$. Since r represents the distance from the point to the pole, the curve $r = 2$ represents the circle with center O and radius 2. In general, the equation $r = a$ represents a circle with center O and radius $|a|$. (See Figure 6.)

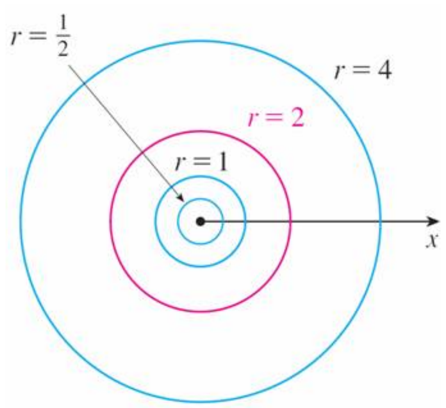


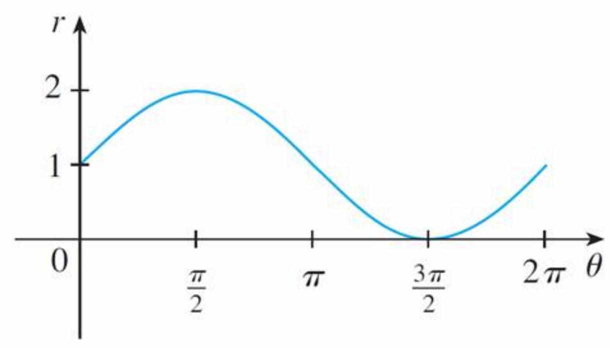
Figure 6

Example 7

Sketch the curve $r = 1 + \sin\theta$.

Solution:

We first sketch the graph of $r = 1 + \sin\theta$ in *Cartesian* coordinates in Figure 10 by shifting the sine curve up one unit.



$r = 1 + \sin\theta$ in Cartesian coordinates, $0 \le \theta \le 2\pi$

Figure 10

Example 7 – *Solution*

cont'd

This enables us to read at a glance the values of r that correspond to increasing values of θ. For instance, we see that as θ increases from 0 to $\pi/2$, r (the distance from O) increases from 1 to 2, so we sketch the corresponding part of the polar curve in Figure 11(a).

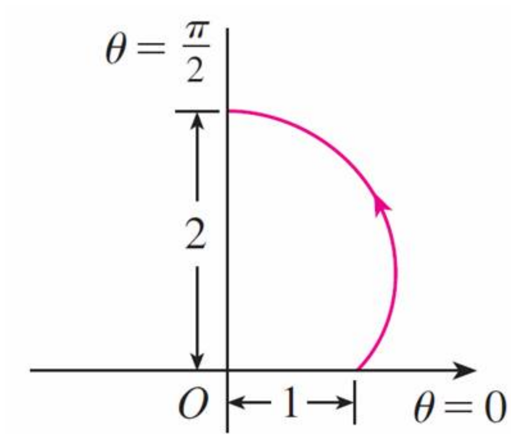


Figure 11(a)

Example 7 – *Solution*

cont'd

As θ increases from $\pi/2$ to π, Figure 10 shows that r decreases from 2 to 1, so we sketch the next part of the curve as in Figure 11(b).

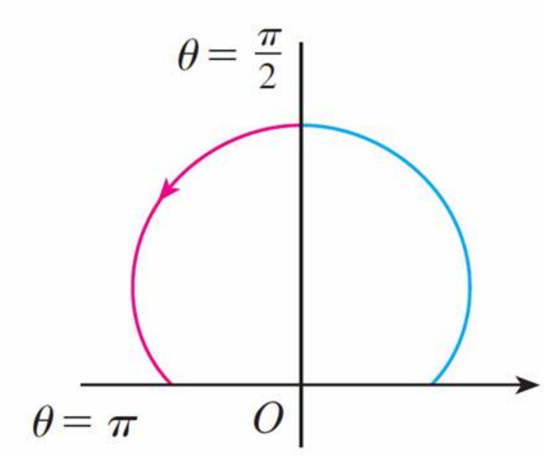
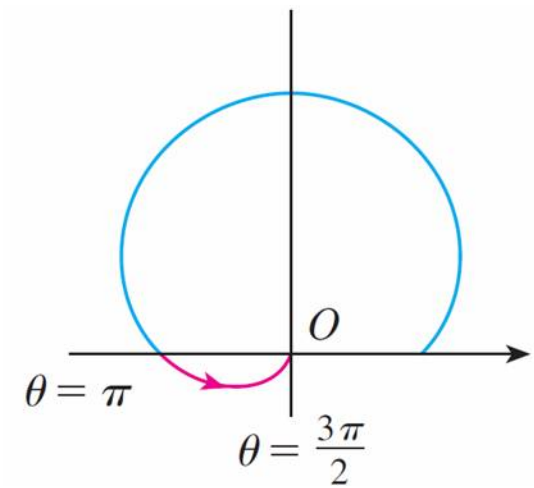


Figure 11(b)

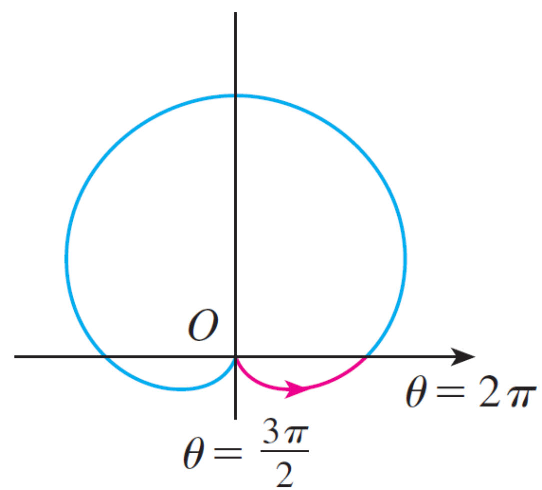
Example 7 – *Solution*

cont'd

As θ increases from π to $3\pi/2$, r decreases from 1 to 0 as shown in part (c). Finally, as θ increases from $3\pi/2$ to 2π, r increases from 0 to 1 as shown in part (d). If we let θ increase beyond 2π or decrease beyond 0, we would simply retrace our path.



(c)



(d)

Figure 11

Example 7 – *Solution*

cont'd

Putting together the parts of the curve from Figure 11(a)–(d), we sketch the complete curve in part (e). It is called a **cardioid** because it's shaped like a heart.

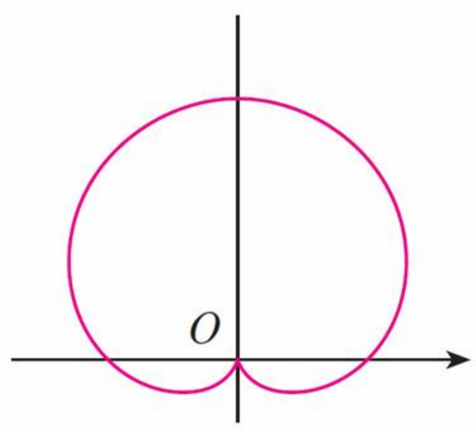


Figure 11(e)

Polar Curves

The remainder of the curve is drawn in a similar fashion, with the arrows and numbers indicating the order in which the portions are traced out. The resulting curve has four loops and is called a **four-leaved rose**.

Symmetry

Symmetry

When we sketch polar curves it is sometimes helpful to take advantage of symmetry. The following three rules are explained by Figure 14.

(a) If a polar equation is unchanged when θ is replaced by $-\theta$, the curve is symmetric about the polar axis.

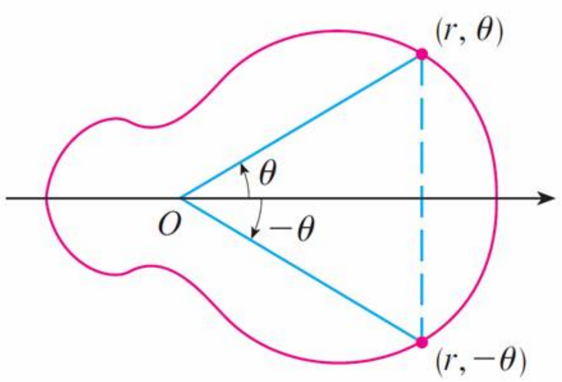


Figure 14(a)

Symmetry

(b) If the equation is unchanged when r is replaced by $-r$, or when θ is replaced by $\theta + \pi$, the curve is symmetric about the pole. (This means that the curve remains unchanged if we rotate it through 180° about the origin.)

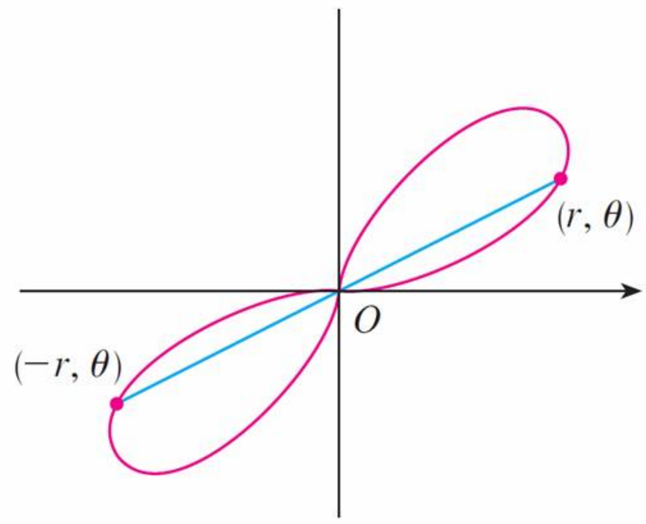


Figure 14(b)

Symmetry

(c) If the equation is unchanged when θ is replaced by $\pi - \theta$, the curve is symmetric about the vertical line $\theta = \pi/2$.

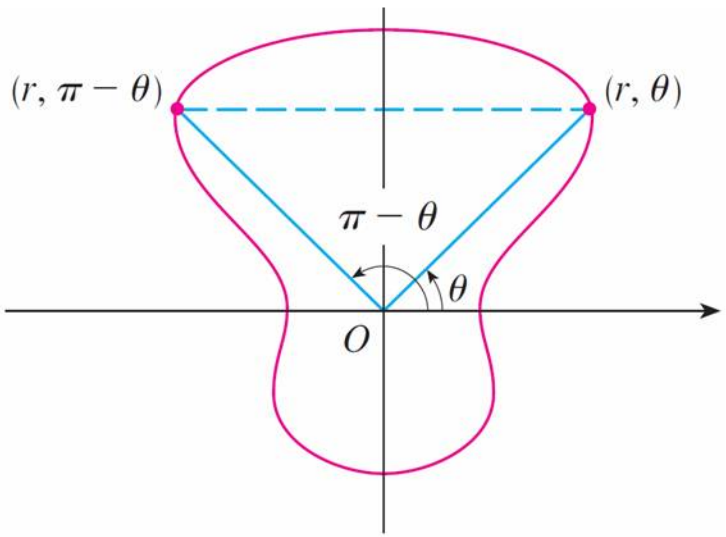


Figure 14(c)

Symmetry

The curves sketched in Examples 6 and 8 are symmetric about the polar axis, since $\cos(-\theta) = \cos\theta$.

The curves in Examples 7 and 8 are symmetric about $\theta = \pi/2$ because $\sin(\pi - \theta) = \sin\theta$ and $\cos 2(\pi - \theta) = \cos 2\theta$.

The four-leaved rose is also symmetric about the pole. These symmetry properties could have been used in sketching the curves.

Tangents to Polar Curves

Tangents to Polar Curves

To find a tangent line to a polar curve $r = f(\theta)$, we regard θ as a parameter and write its parametric equations as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Then, using the method for finding slopes of parametric curves and the Product Rule, we have

$$\boxed{3} \qquad \frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Tangents to Polar Curves

We locate horizontal tangents by finding the points where $dy/d\theta = 0$ (provided that $dy/d\theta \neq 0$). Likewise, we locate vertical tangents at the points where $dy/d\theta = 0$ (provided that $dy/d\theta \neq 0$). Notice that if we are looking for tangent lines at the pole, then $r = 0$ and Equation 3 simplifies to

$$\frac{dy}{dx} = \tan\theta \qquad \text{if} \quad \frac{dr}{d\theta} \neq 0$$

Example 9

(a) For the cardioid $r = 1 + \sin\,\theta$, find the slope of the tangent line when $\theta = \pi/3$.

(b) Find the points on the cardioid where the tangent line is horizontal or vertical.

Solution:

Using Equation 3 with $r = 1 + \sin\,\theta$, we have

$$\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\cos\theta\,\sin\theta + (1 + \sin\theta)\cos\theta}{\cos\theta\,\cos\theta - (1 + \sin\theta)\sin\theta}$$

Example 9 – *Solution*

cont'd

$$= \frac{\cos\theta\,(1 + 2\sin\theta)}{1 - 2\sin^2\theta - \sin\theta} = \frac{\cos\theta\,(1 + 2\sin\theta)}{(1 + \sin\theta)(1 - 2\sin\theta)}$$

(a) The slope of the tangent at the point where $\theta = \pi/3$ is

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/3} = \frac{\cos(\pi/3)(1 + 2\sin(\pi/3))}{(1 + \sin(\pi/3))(1 - 2\sin(\pi/3))}$$

$$= \frac{\frac{1}{2}\left(1 + \sqrt{3}\right)}{(1 + \sqrt{3}/2)\left(1 - \sqrt{3}\right)}$$

Example 9 – *Solution*

cont'd

$$= \frac{1 + \sqrt{3}}{(2 + \sqrt{3})(1 - \sqrt{3})}$$

$$= \frac{1 + \sqrt{3}}{-1 - \sqrt{3}}$$

$$= -1$$

(b) Observe that

$$\frac{dy}{d\theta} = \cos\theta\,(1 + 2\sin\theta) = 0 \quad \text{when } \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

Example 9 – *Solution*

cont'd

$$\frac{dx}{d\theta} = (1 + \sin\theta)(1 - 2\sin\theta) = 0 \qquad \text{when } \theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

Therefore there are horizontal tangents at the points $(2, \pi/2), \left(\frac{1}{2}, 7\pi/6\right), \left(\frac{1}{2}, 11\pi/6\right)$ and vertical tangents at $\left(\frac{3}{2}, \pi/6\right)$ and $\left(\frac{3}{2}, 5\pi/6\right)$. When $\theta = 3\pi/2$, both *dy*/*d*θ and *dx*/*d*θ are 0, so we must be careful. Using l'Hospital's Rule, we have

$$\lim_{\theta \to (3\pi/2)^-} \frac{dy}{dx} = \left(\lim_{\theta \to (3\pi/2)^-} \frac{1 + 2\sin\theta}{1 - 2\sin\theta}\right)\left(\lim_{\theta \to (3\pi/2)^-} \frac{\cos\theta}{1 + \sin\theta}\right)$$

Example 9 – *Solution*

cont'd

$$= -\frac{1}{3} \lim_{\theta \to (3\pi/2)^-} \frac{\cos\theta}{1 + \sin\theta}$$

$$= -\frac{1}{3} \lim_{\theta \to (3\pi/2)^-} \frac{-\sin\theta}{\cos\theta}$$

$$= \infty$$

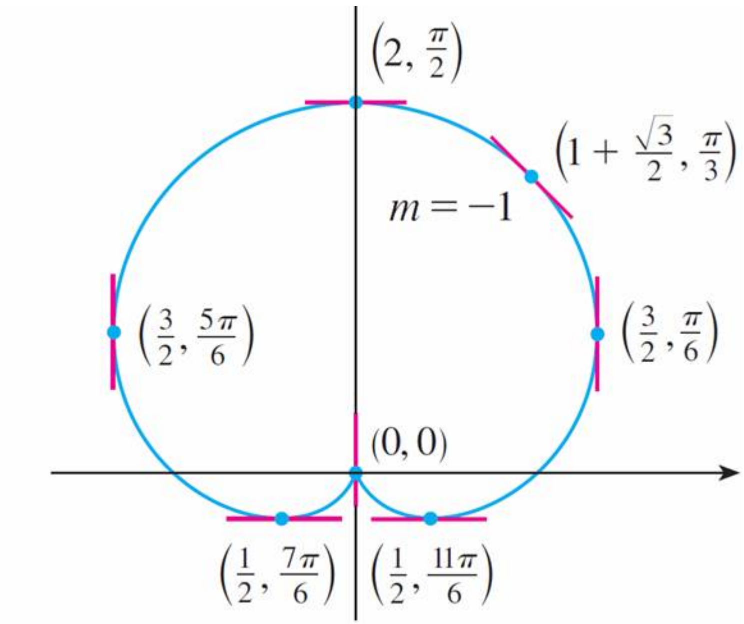
By symmetry,

$$\lim_{\theta \to (3\pi/2)^+} \frac{dy}{dx} = -\infty$$

Example 9 – *Solution*

cont'd

Thus there is a vertical tangent line at the pole (see Figure 15).



Tangent lines for $r = 1 + \sin\theta$

Figure 15

Tangents to Polar Curves

Note:

Instead of having to remember Equation 3, we could employ the method used to derive it. For instance, in Example 9 we could have written

$x = r \cos \theta = (1 + \sin \theta) \cos \theta = \cos \theta + \frac{1}{2} \sin 2\theta$

$y = r \sin \theta = (1 + \sin \theta) \sin \theta = \sin \theta + \sin^2 \theta$

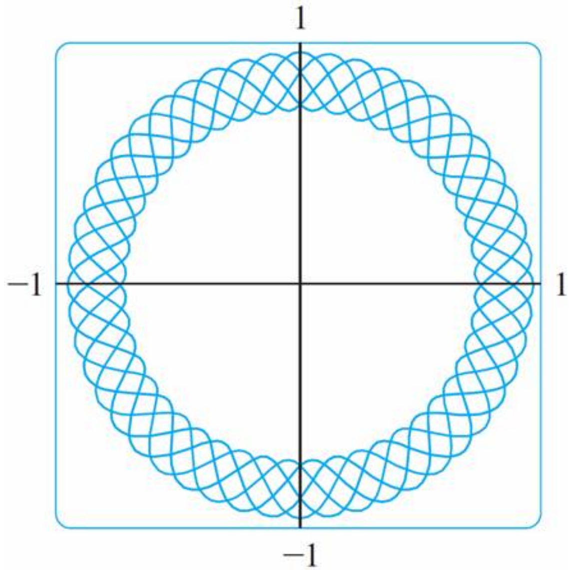
Then we have

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos \theta + 2 \sin \theta \cos \theta}{-\sin \theta + \cos 2\theta} = \frac{\cos \theta + \sin 2\theta}{-\sin \theta + \cos 2\theta}$$

Graphing Polar Curves with Graphing Devices

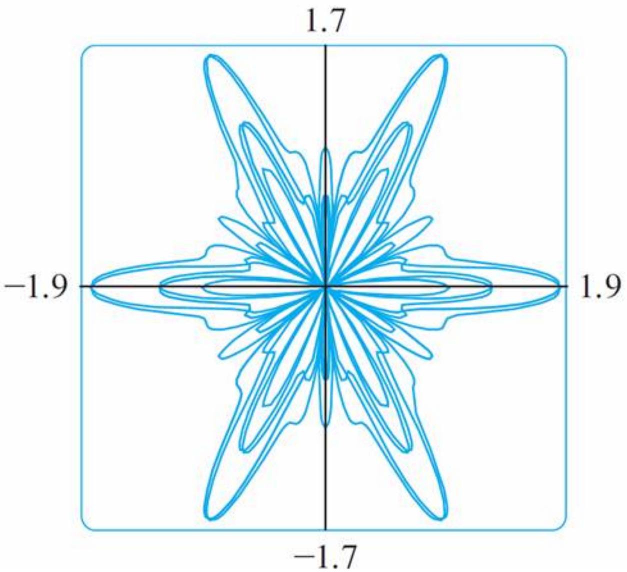
Graphing Polar Curves with Graphing Devices

Although it’s useful to be able to sketch simple polar curves by hand, we need to use a graphing calculator or computer when we are faced with a curve as complicated as the ones shown in Figures 16 and 17.



$r = \sin^2(2.4\theta) + \cos^4(2.4\theta)$

Figure 16



$r = \sin^2(1.2\theta) + \cos^3(6\theta)$

Figure 17

Graphing Polar Curves with Graphing Devices

Some graphing devices have commands that enable us to graph polar curves directly. With other machines we need to convert to parametric equations first. In this case we take the polar equation $r = f(\theta)$ and write its parametric equations as

$$x = r \cos \theta = f(\theta) \cos \theta \qquad y = r \sin \theta = f(\theta) \sin \theta$$

Some machines require that the parameter be called t rather than θ.

Example 10

Graph the curve $r = \sin(8\theta/5)$.

Solution:

Let's assume that our graphing device doesn't have a built-in polar graphing command.

In this case we need to work with the corresponding parametric equations, which are

$$x = r\cos\theta = \sin(8\theta/5)\cos\theta \qquad x = r\sin\theta = \sin(8\theta/5)\sin\theta$$

Example 10 – *Solution*

cont'd

In any case we need to determine the domain for θ. So we ask ourselves: How many complete rotations are required until the curve starts to repeat itself? If the answer is n, then

$$\sin \frac{8(\theta + 2n\pi)}{5} = \sin\left(\frac{8\theta}{5} + \frac{16n\pi}{5}\right)$$

$$= \sin \frac{8\theta}{5}$$

and so we require that $16n\pi/5$ be an even multiple of π. This will first occur when $n = 5$. Therefore we will graph the entire curve if we specify that $0 \leq \theta \leq 10\pi$.

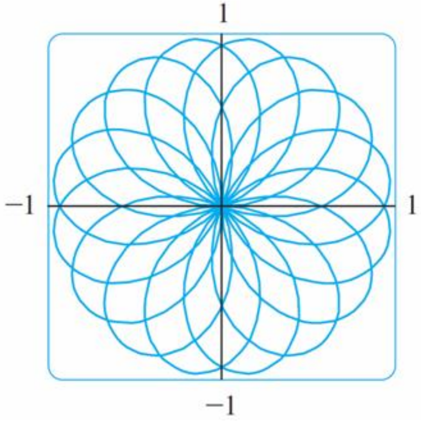
Example 10 – *Solution*

cont'd

Switching from θ to t, we have the equations

$$x = \sin(8t/5)\cos t \qquad y = \sin(8t/5)\sin t \qquad 0 \le t \le 10\pi$$

and Figure 18 shows the resulting curve. Notice that this rose has 16 loops.



$r = \sin(8\theta/5)$

Figure 18

10.4

Areas and Lengths in Polar Coordinates

Areas and Lengths in Polar Coordinates

In this section we develop the formula for the area of a region whose boundary is given by a polar equation. We need to use the formula for the area of a sector of a circle:

$$\boxed{1} \qquad A = \tfrac{1}{2} r^2 \theta$$

where, as in Figure 1, r is the radius and θ is the radian measure of the central angle.

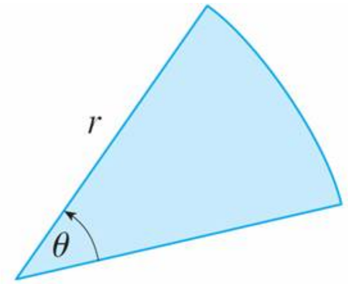


Figure 1

Areas and Lengths in Polar Coordinates

Formula 1 follows from the fact that the area of a sector is proportional to its central angle:

$$A = (\theta/2\pi)\pi r^2 = \tfrac{1}{2}r^2\theta.$$

Areas and Lengths in Polar Coordinates

Let $\mathscr{R}$ be the region, illustrated in Figure 2, bounded by the polar curve $r = f(\theta)$ and by the rays $\theta = a$ and $\theta = b$, where f is a positive continuous function and where $0 < b - a \leq 2\pi$.

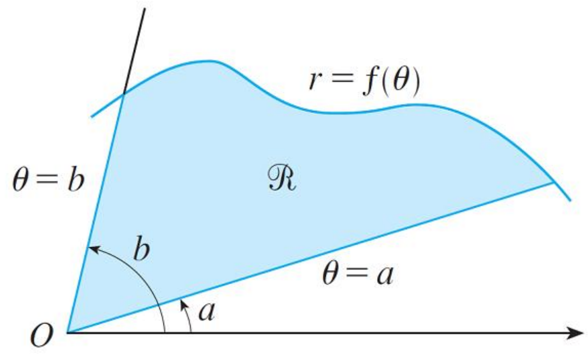


Figure 2

We divide the interval $[a, b]$ into subintervals with endpoints $\theta_0, \theta_1, \theta_2, \ldots, \theta_n$ and equal width $\Delta\theta$.

Areas and Lengths in Polar Coordinates

The rays $\theta = \theta_i$ then divide $\mathcal{R}$ into n smaller regions with central angle $\Delta\theta = \theta_i - \theta_{i-1}$. If we choose θ_i^* in the ith subinterval $[\theta_{i-1}, \theta_i]$, then the area ΔA_i of the ith region is approximated by the area of the sector of a circle with central angle $\Delta\theta$ and radius $f(\theta_i^*)$. (See Figure 3.)

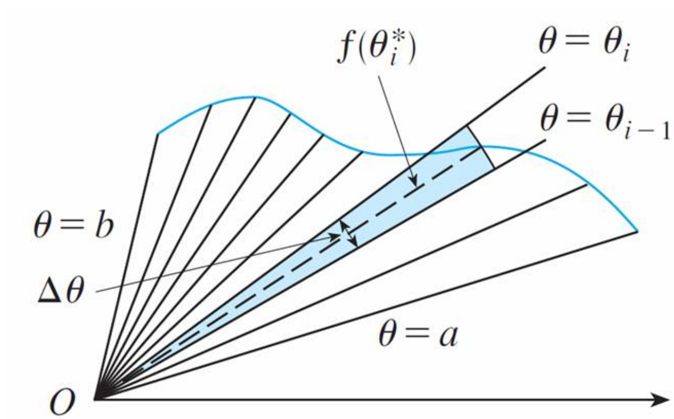


Figure 3

Areas and Lengths in Polar Coordinates

Thus from Formula 1 we have

$$\Delta A_i \approx \tfrac{1}{2}[f(\theta_i^*)]^2 \,\Delta\theta$$

and so an approximation to the total area A of $\mathscr{R}$ is

$$\boxed{2} \qquad A \approx \sum_{i=1}^{n} \tfrac{1}{2}[f(\theta_i^*)]^2 \,\Delta\theta$$

It appears from Figure 3 that the approximation in $\boxed{2}$ improves as $n \to \infty$.

Areas and Lengths in Polar Coordinates

But the sums in $\boxed{2}$ are Riemann sums for the function $g(\theta) = \frac{1}{2}[f(\theta)]^2$, so

$$\lim_{n \to \infty} \sum_{i=1}^{n} \tfrac{1}{2}[f(\theta_i^*)]^2 \, \Delta\theta = \int_a^b \tfrac{1}{2}[f(\theta)]^2 \, d\theta$$

It therefore appears plausible that the formula for the area A of the polar region $\mathcal{R}$ is

$$\boxed{3} \qquad \boxed{A = \int_a^b \tfrac{1}{2}[f(\theta)]^2 \, d\theta}$$

Areas and Lengths in Polar Coordinates

Formula 3 is often written as

4

$$A = \int_a^b \tfrac{1}{2} r^2 \, d\theta$$

with the understanding that $r = f(\theta)$. Note the similarity between Formulas 1 and 4.

When we apply Formula 3 or 4 it is helpful to think of the area as being swept out by a rotating ray through O that starts with angle a and ends with angle b.

Example 1

Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

Solution:

Notice from Figure 4 that the region enclosed by the right loop is swept out by a ray that rotates from $\theta = -\pi/4$ to $\theta = \pi/4$.

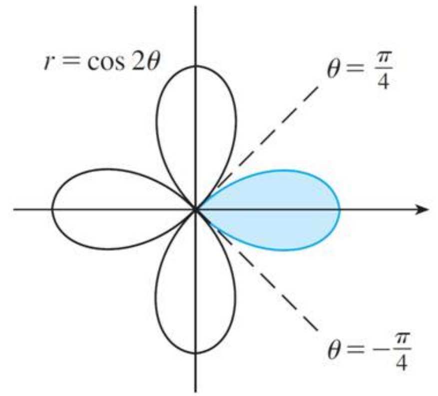


Figure 4

Example 1 – *Solution*

cont'd

Therefore Formula 4 gives

$$A = \int_{-\pi/4}^{\pi/4} \tfrac{1}{2} r^2 \, d\theta$$

$$= \tfrac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2 2\theta \, d\theta$$

$$= \int_{0}^{\pi/4} \cos^2 2\theta \, d\theta$$

Example 1 – *Solution*

cont'd

$$A = \int_0^{\pi/4} \tfrac{1}{2}(1 + \cos 4\theta)\, d\theta$$

$$= \tfrac{1}{2}\Big[\theta + \tfrac{1}{4}\sin 4\theta\Big]_0^{\pi/4}$$

$$= \frac{\pi}{8}$$

Arc Length

Arc Length

To find the length of a polar curve $r = f(\theta)$, $a \leq \theta \leq b$, we regard θ as a parameter and write the parametric equations of the curve as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Using the Product Rule and differentiating with respect to θ, we obtain

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

Arc Length

so, using $\cos^2\theta + \sin^2\theta = 1$, we have

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\,\frac{dr}{d\theta}\cos\theta\,\sin\theta + r^2\sin^2\theta$$

$$+ \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r\,\frac{dr}{d\theta}\sin\theta\,\cos\theta + r^2\cos^2\theta$$

$$= \left(\frac{dr}{d\theta}\right)^2 + r^2$$

Arc Length

Assuming that f' is continuous, we can write the arc length as

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Therefore the length of a curve with polar equation $r = f(\theta)$, $a \leq \theta \leq b$, is

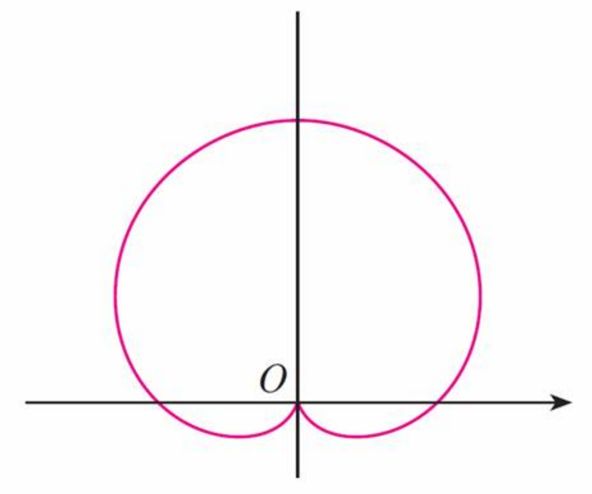
$$\boxed{5} \qquad \boxed{L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$$

Example 4

Find the length of the cardioid $r = 1 + \sin \theta$.

Solution:

The cardioid is shown in Figure 8.



$r = 1 + \sin \theta$

Figure 8

Example 4 – *Solution*

cont'd

$\boxed{5}$

$$\boxed{L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$$

Its full length is given by the parameter interval $0 \leq \theta \leq 2\pi$, so Formula 5 gives

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{(1 + \sin\theta)^2 + \cos^2\theta}\; d\theta$$

Example 4 – *Solution*

cont'd

$$= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\; d\theta$$

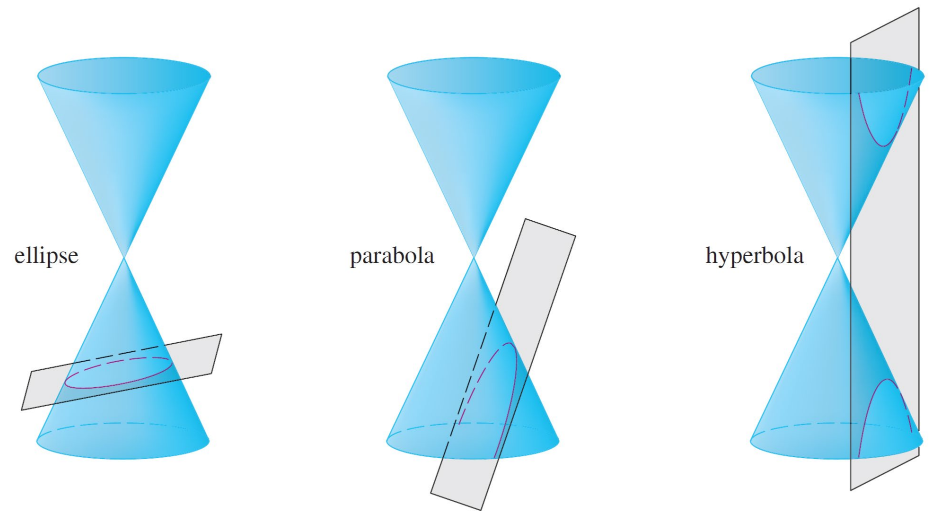
We could evaluate this integral by multiplying and dividing the integrand by $\sqrt{2 - 2\sin\theta}$, or we could use a computer algebra system.

In any event, we find that the length of the cardioid is $L = 8$.

10.5 Conic Sections

Conic Sections

In this section we give geometric definitions of parabolas, ellipses, and hyperbolas and derive their standard equations. They are called **conic sections**, or **conics**, because they result from intersecting a cone with a plane as shown in Figure 1.



Conics

Figure 1

Parabolas

Parabolas

A **parabola** is the set of points in a plane that are equidistant from a fixed point *F* (called the **focus**) and a fixed line (called the **directrix**). This definition is illustrated by Figure 2.

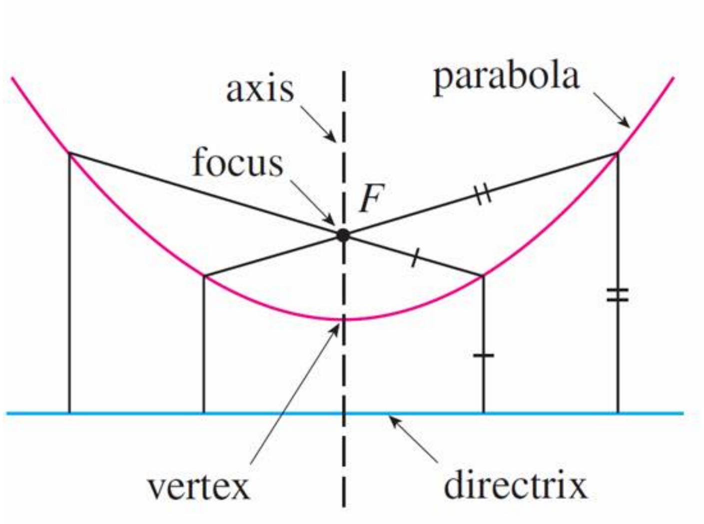


Figure 2

Parabolas

Notice that the point halfway between the focus and the directrix lies on the parabola; it is called the **vertex**.

The line through the focus perpendicular to the directrix is called the **axis** of the parabola.

In the 16th century Galileo showed that the path of a projectile that is shot into the air at an angle to the ground is a parabola. Since then, parabolic shapes have been used in designing automobile headlights, reflecting telescopes, and suspension bridges.

Parabolas

We obtain a particularly simple equation for a parabola if we place its vertex at the origin *O* and its directrix parallel to the *x*-axis as in Figure 3.

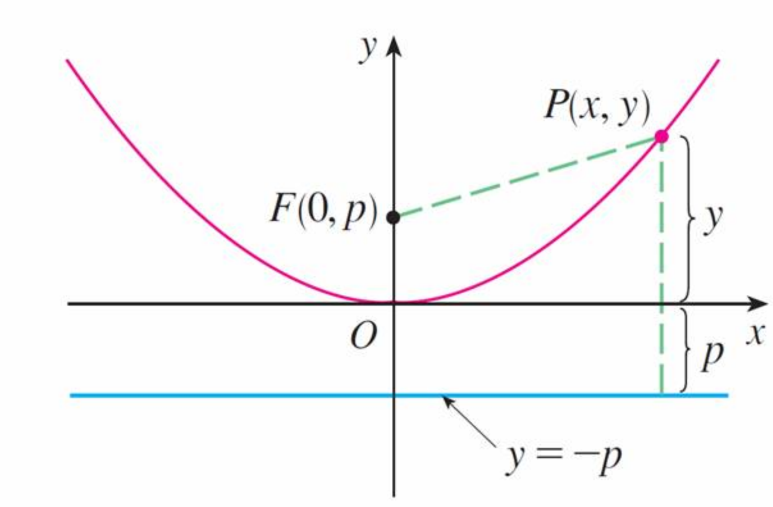


Figure 3

Parabolas

If the focus is the point $(0, p)$, then the directrix has the equation $y = -p$. If $P(x, y)$ is any point on the parabola, then the distance from P to the focus is

$$|PF| = \sqrt{x^2 + (y - p)^2}$$

and the distance from P to the directrix is $|y + p|$. (Figure 3 illustrates the case where $p > 0$.)

Parabolas

The defining property of a parabola is that these distances are equal:

$$\sqrt{x^2 + (y - p)^2} = |y + p|$$

We get an equivalent equation by squaring and simplifying:

$$x^2 + (y - p)^2 = |y + p|^2 = (y + p)^2$$

$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2$$

Parabolas

$$x^2 = 4py$$

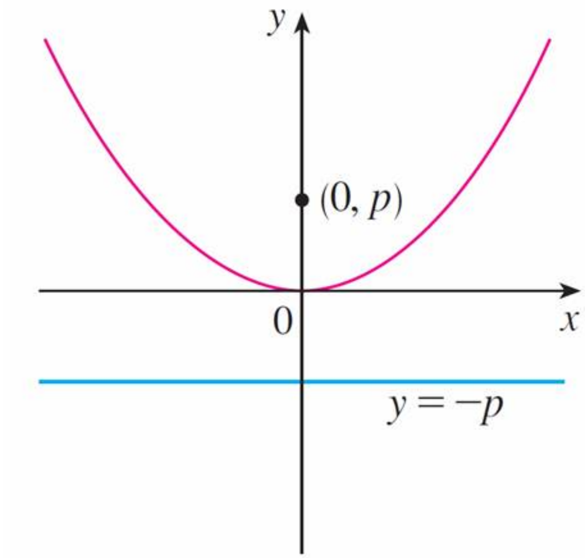
$\boxed{1}$ An equation of the parabola with focus $(0, p)$ and directrix $y = -p$ is

$$x^2 = 4py$$

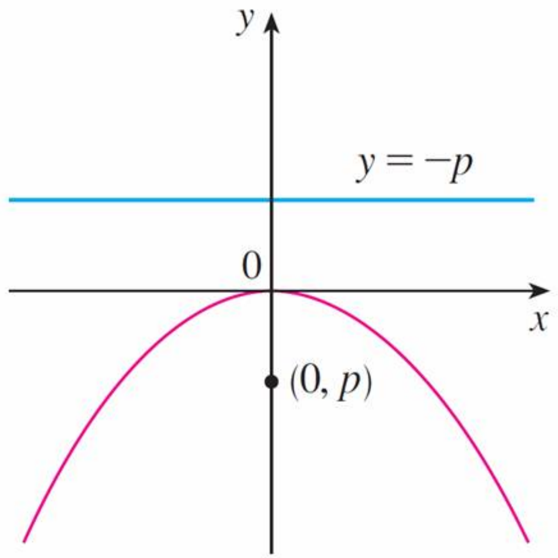
If we write $a = 1/(4p)$, then the standard equation of a parabola $\boxed{1}$ becomes $y = ax^2$.

Parabolas

It opens upward if $p > 0$ and downward if $p < 0$ [see Figure 4, parts (a) and (b)].



(a) $x^2 = 4py$, $p > 0$



(a) $x^2 = 4py$, $p < 0$

Figure 4

Parabolas

The graph is symmetric with respect to the *y*-axis because $\boxed{1}$ is unchanged when is replaced by $-x$.

If we interchange *x* and *y* in $\boxed{1}$, we obtain

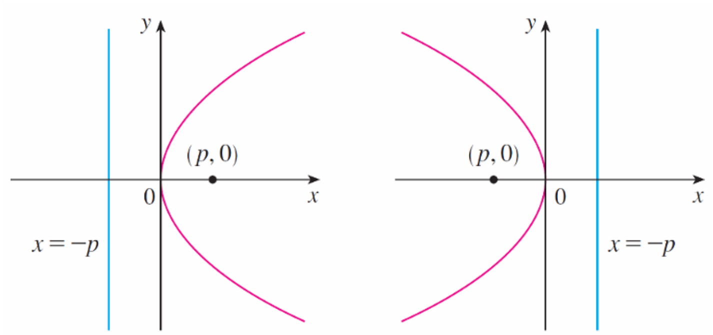
$$\boxed{2} \qquad \boxed{y^2 = 4px}$$

which is an equation of the parabola with focus $(p, 0)$ and directrix $x = -p$.

Parabolas

(Interchanging x and y amounts to reflecting about the diagonal line $y = x$.)

The parabola opens to the right if p > 0 and to the left if p < 0 [see Figure 4, parts (c) and (d)].



(c) $y^2 = 4px$, $p > 0$ (d) $y^2 = 4px$, $p < 0$

Figure 4

Parabolas

In both cases the graph is symmetric with respect to the *x*-axis, which is the axis of the parabola.

Example 1

Find the focus and directrix of the parabola $y^2 + 10x = 0$ and sketch the graph.

Solution:

If we write the equation as $y^2 = -10x$ and compare it with Equation 2, we see that $4p = -10$, so $p = -\frac{5}{2}$.

Thus the focus is $(p, 0) = (-\frac{5}{2}, 0)$ and the directrix is $x = \frac{5}{2}$.

Example 1 – *Solution*

cont'd

The sketch is shown in Figure 5.

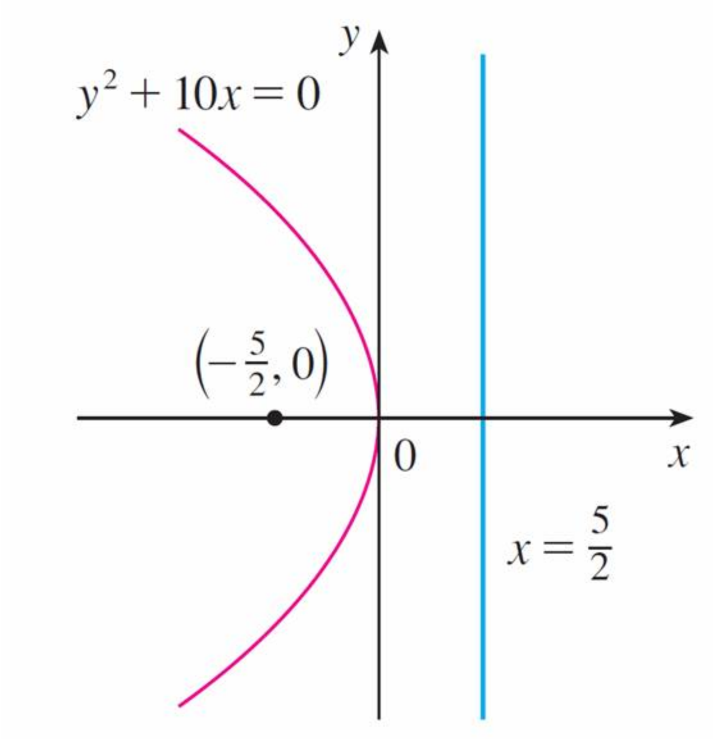


Figure 5

Ellipses

Ellipses

An **ellipse** is the set of points in a plane the sum of whose distances from two fixed points F_1 and F_2 is a constant (see Figure 6).

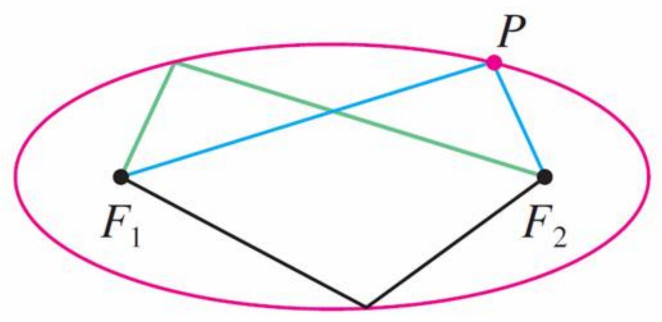


Figure 6

Ellipses

These two fixed points are called the **foci** (plural of **focus**). One of Kepler's laws is that the orbits of the planets in the solar system are ellipses with the sun at one focus.

In order to obtain the simplest equation for an ellipse, we place the foci on the *x*-axis at the points $(-c, 0)$ and $(c, 0)$ as in Figure 7 so that the origin is halfway between the foci.

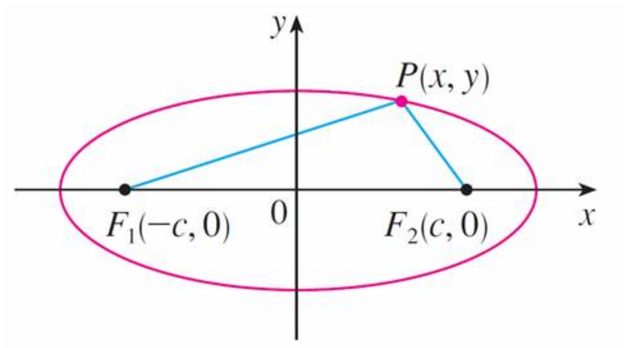


Figure 7

Ellipses

Let the sum of the distances from a point on the ellipse to the foci be 2a > 0. Then $P(x, y)$ is a point on the ellipse when

$$|PF_1| + |PF_2| = 2a$$

that is,

$$\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a$$

or

$$\sqrt{(x - c)^2 + y^2} = 2a - \sqrt{(x + c)^2 + y^2}$$

Ellipses

Squaring both sides, we have

$$x^2 - 2cx + c^2 + y^2$$

$$= 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + x^2 + 2cx + c^2 + y^2$$

which simplifies to

$$a\sqrt{(x + c)^2 + y^2} = a^2 + cx$$

We square again:

$$a^2(x^2 + 2cx + c^2 + y^2) = a^4 + 2a^2cx + c^2x^2$$

which becomes

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

Ellipses

From triangle F_1F_2P in Figure 7 we see that $2c < 2a$, so $c < a$ and therefore $a^2 - c^2 > 0$. For convenience, let $b^2 = a^2 - c^2$.

Then the equation of the ellipse becomes $b^2x^2 + a^2y^2 = a^2b^2$ or, if both sides are divided by a^2b^2,

$$\boxed{3} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Since $b^2 = a^2 - c^2 < a^2$, it follows that $b < a$.

Ellipses

The x-intercepts are found by setting $y = 0$. Then $x^2/a^2 = 1$, or $x^2 = a^2$, so $x = \pm a$.

The corresponding points $(a, 0)$ and $(-a, 0)$ are called the **vertices** of the ellipse and the line segment joining the vertices is called the **major axis**. To find the y-intercepts we set $x = 0$ and obtain $y^2 = b^2$, so $y = \pm b$.

The line segment joining $(0, b)$ and $(0, -b)$ is the **minor axis**.

Ellipses

Equation 3 is unchanged if x is replaced by $–x$ or y is replaced by $–y$, so the ellipse is symmetric about both axes.

Notice that if the foci coincide, then $c = 0$, so $a = b$ and the ellipse becomes a circle with radius $r = a = b$.

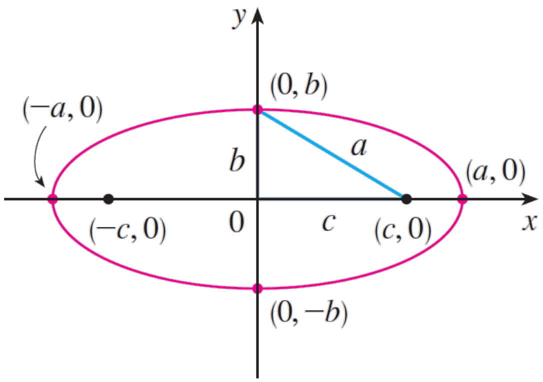
Ellipses

We summarize this discussion as follows (see also Figure 8).

4 The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a \geqslant b > 0$$

has foci $(\pm c, 0)$, where $c^2 = a^2 - b^2$, and vertices $(\pm a, 0)$.

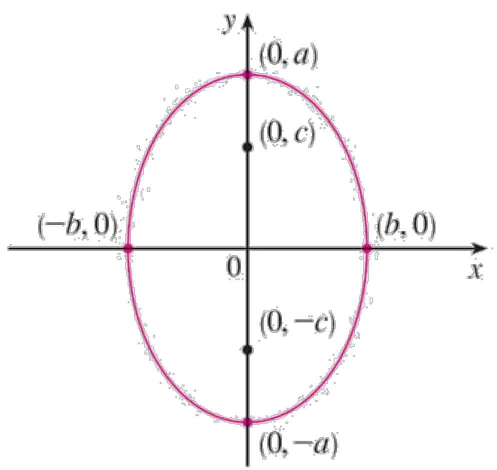


$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad a \geq b$$

Figure 8

Ellipses

If the foci of an ellipse are located on the y-axis at $(0, \pm c)$, then we can find its equation by interchanging x and y in $\boxed{4}$. (See Figure 9.)



$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad , a \geq b$$

Figure 9

Ellipses

5 The ellipse

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad a \geqslant b > 0$$

has foci $(0, \pm c)$, where $c^2 = a^2 - b^2$, and vertices $(0, \pm a)$.

Example 2

Sketch the graph of $9x^2 + 16y^2 = 144$ and locate the foci.

Solution:
Divide both sides of the equation by 144:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

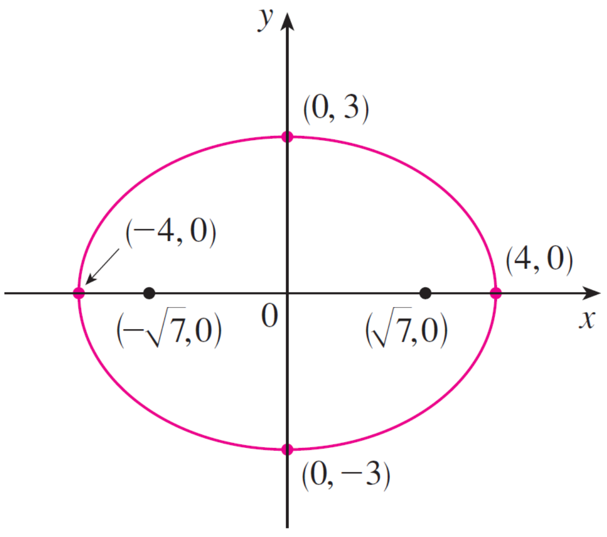
The equation is now in the standard form for an ellipse, so we have $a^2 = 16$, $b^2 = 9$, $a = 4$, and $b = 3$.

The x-intercepts are ± 4 and the y-intercepts are ± 3.

Example 2 – *Solution*

cont'd

Also, $c^2 = a^2 - b^2 = 7$, so $c = \sqrt{7}$ and the foci are $(\pm\sqrt{7}, 0)$. The graph is sketched in Figure 10.



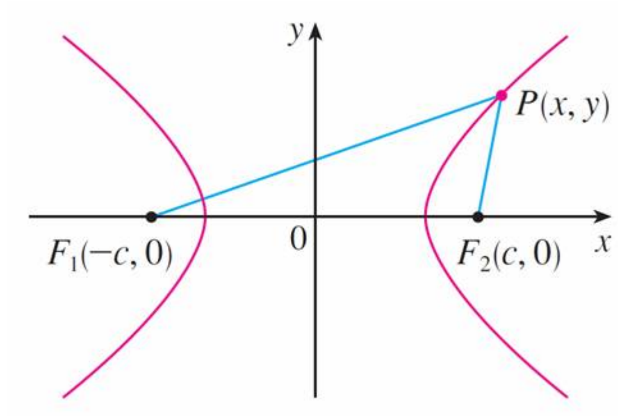
$9x^2 + 16y^2 = 144$

Figure 10

Hyperbolas

Hyperbolas

A **hyperbola** is the set of all points in a plane the difference of whose distances from two fixed points F_1 and F_2 (the foci) is a constant. This definition is illustrated in Figure 11.



P is on the hyperbola when
$|PF_1| - |PF_2| = \pm 2a$.

Figure 11

Hyperbolas

Notice that the definition of a hyperbola is similar to that of an ellipse; the only change is that the sum of distances has become a difference of distances.

In fact, the derivation of the equation of a hyperbola is also similar to the one given earlier for an ellipse.

When the foci are on the *x*-axis at $(\pm c, 0)$ and the difference of distances is $|PF_1| - |PF_2| = \pm 2a$, then the equation of the hyperbola is

$$\boxed{6} \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $c^2 = a^2 + b^2$.

Hyperbolas

Notice that the x-intercepts are again $\pm a$ and the points $(a, 0)$ and $(-a, 0)$ are the **vertices** of the hyperbola.

But if we put $x = 0$ in Equation 6 we get $y^2 = -b^2$, which is impossible, so there is no y-intercept. The hyperbola is symmetric with respect to both axes.

To analyze the hyperbola further, we look at Equation 6 and obtain

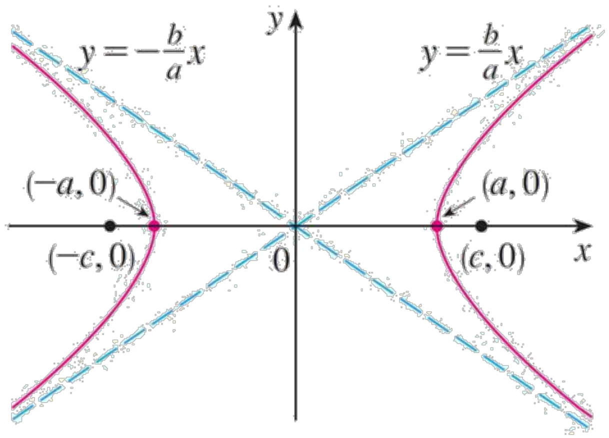
$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geqslant 1$$

This shows that $x^2 \geq a^2$, so $|x| = \sqrt{x^2} \geqslant a.$

Therefore we have $x \geq a$ or $x \leq -a$.

Hyperbolas

This means that the hyperbola consists of two parts, called its *branches.* When we draw a hyperbola it is useful to first draw its **asymptotes**, which are the dashed lines $y = (b/a)x$ and $y = -(b/a)x$ shown in Figure 12.



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Figure 12

Hyperbolas

Both branches of the hyperbola approach the asymptotes; that is, they come arbitrarily close to the asymptotes.

$\boxed{7}$ The hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has foci $(\pm c, 0)$, where $c^2 = a^2 + b^2$, vertices $(\pm a, 0)$, and asymptotes $y = \pm(b/a)x$.

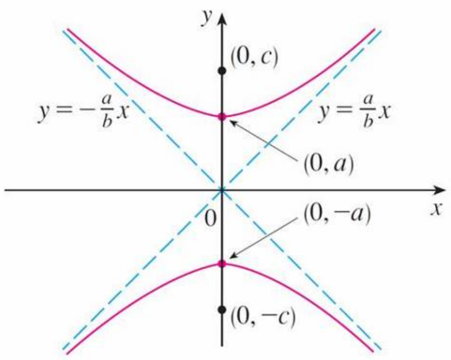
Hyperbolas

If the foci of a hyperbola are on the *y*-axis, then by reversing the roles of *x* and *y* we obtain the following information, which is illustrated in Figure 13.

8 The hyperbola

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

has foci $(0, \pm c)$, where $c^2 = a^2 + b^2$, vertices $(0, \pm a)$, and asymptotes $y = \pm(a/b)x$.



$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Figure 13

Example 4

Find the foci and asymptotes of the hyperbola $9x^2 - 16y^2 = 144$ and sketch its graph.

Solution:

If we divide both sides of the equation by 144, it becomes

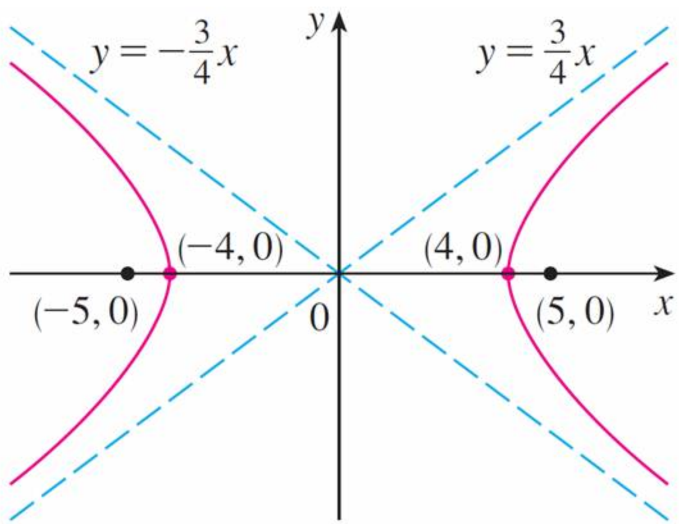
$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

which is of the form given in $\boxed{7}$ with $a = 4$ and $b = 3$.

Example 4 – *Solution*

cont'd

Since $c^2 = 16 + 9 = 25$, the foci are $(\pm 5, 0)$. The asymptotes are the lines $y = \frac{3}{4}x$ and $y = -\frac{3}{4}x$. The graph is shown in Figure 14.



$9x^2 - 16y^2 = 144$

Figure 14

Shifted Conics

Shifted Conics

We shift conics by taking the standard equations $\boxed{1}$, $\boxed{2}$, $\boxed{4}$, $\boxed{5}$, $\boxed{7}$, and $\boxed{8}$ and replacing x and y by $x - h$ and $y - k$.

Example 6

Find an equation of the ellipse with foci (2, –2),(4, –2) and vertices (1, –2),(5, –2).

Solution:

The major axis is the line segment that joins the vertices (1, –2),(5, –2) and has length 4, so $a = 2$. The distance between the foci is 2, so $c = 1$. Thus $b^2 = a^2 - c^2 = 3$. Since the center of the ellipse is (3, –2), we replace x and y in $\boxed{4}$ by $x - 3$ and $y + 2$ to obtain

$$\frac{(x-3)^2}{4} + \frac{(y+2)^2}{3} = 1$$

as the equation of the ellipse.

Example 7

Sketch the conic $9x^2 - 4y^2 - 72x + 8y + 176 = 0$ and find its foci.

Solution:

We complete the squares as follows:

$$4(y^2 - 2y) - 9(x^2 - 8x) = 176$$

$$4(y^2 - 2y + 1) - 9(x^2 - 8x + 16) = 176 + 4 - 144$$

$$4(y - 1)^2 - 9(x - 4)^2 = 36$$

Example 7 – *Solution*

cont'd

$$\frac{(y-1)^2}{9} - \frac{(x-4)^2}{4} = 1$$

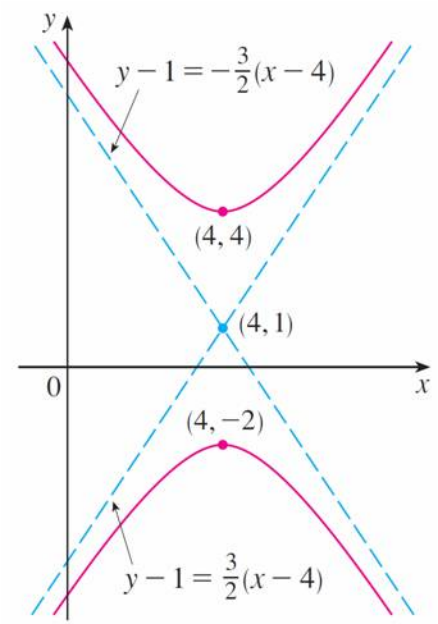
This is in the form $\boxed{8}$ except that x and y are replaced by $x - 4$ and $y - 1$. Thus $a^2 = 9$, $b^2 = 4$, and $c^2 = 13$.

The hyperbola is shifted four units to the right and one unit upward.

Example 7 – *Solution*

cont'd

The foci are $(4, 1 + \sqrt{13})$ and $(4, 1 - \sqrt{13})$ and the vertices are $(4, 4)$ and $(4, -2)$. The asymptotes are $y - 1 = \pm\frac{3}{2}(x - 4)$. The hyperbola is sketched in Figure 15.



$9x^2 - 4y^2 - 72x + 8y + 176 = 0$

Figure 15

10.6 Conic Sections in Polar Coordinates

Conic Sections in Polar Coordinates

In this section we give a more unified treatment of all three types of conic sections in terms of a focus and directrix.

Furthermore, if we place the focus at the origin, then a conic section has a simple polar equation, which provides a convenient description of the motion of planets, satellites, and comets.

Conic Sections in Polar Coordinates

1 Theorem Let F be a fixed point (called the **focus**) and l be a fixed line (called the **directrix**) in a plane. Let e be a fixed positive number (called the **eccentricity**). The set of all points P in the plane such that

$$\frac{|PF|}{|Pl|} = e$$

(that is, the ratio of the distance from F to the distance from l is the constant e) is a conic section. The conic is

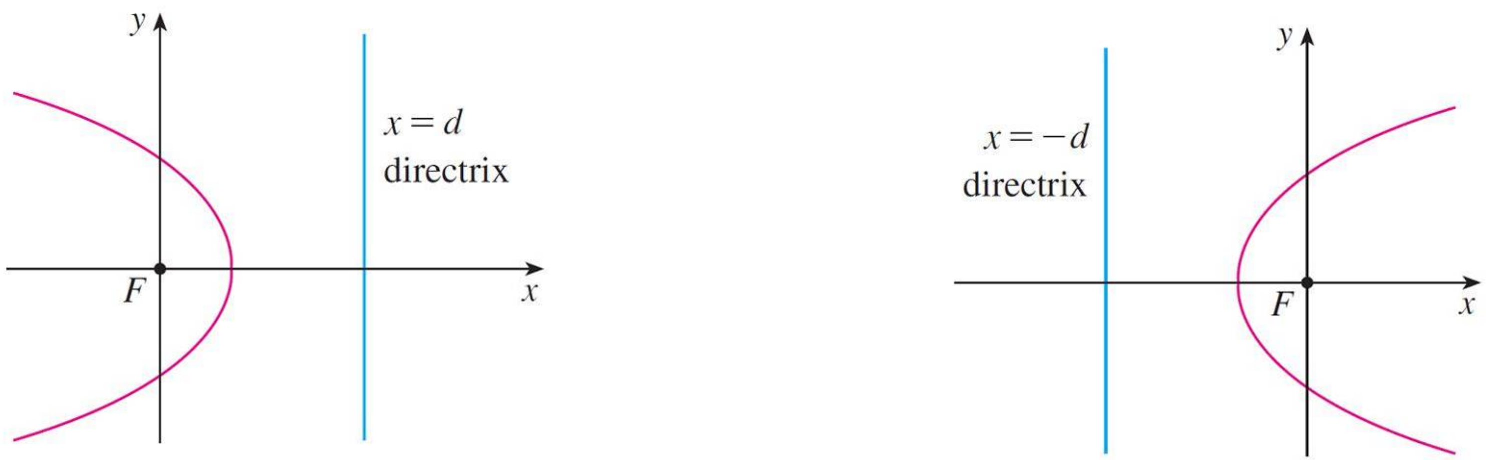
(a) an ellipse if $e < 1$

(b) a parabola if $e = 1$

(c) a hyperbola if $e > 1$

Conic Sections in Polar Coordinates

If the directrix is chosen to be to the left of the focus as $x = -d$, or if the directrix is chosen to be parallel to the polar axis as $y = \pm d$, then the polar equation of the conic is given by the following theorem, which is illustrated by Figure 2.



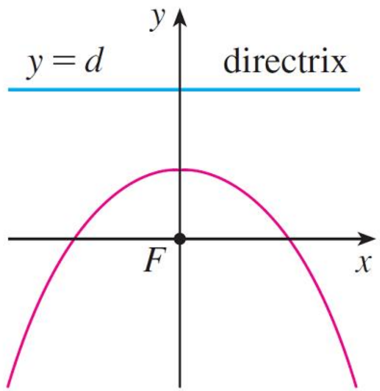
(a) $r = \dfrac{ed}{1 + e\cos\theta}$

(b) $r = \dfrac{ed}{1 - e\cos\theta}$

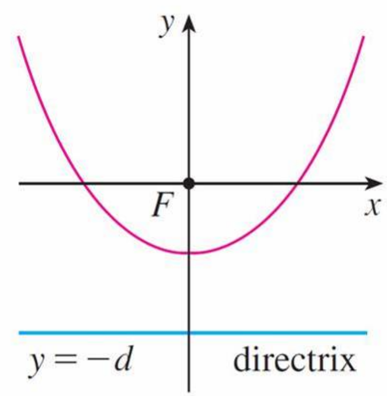
Polar equations of conics

Figure 2

Conic Sections in Polar Coordinates cont'd



(c) $r = \dfrac{ed}{1 + e \sin \theta}$



(d) $r = \dfrac{ed}{1 - e \sin \theta}$

Polar equations of conics

Figure 2

6 Theorem A polar equation of the form

$$r = \frac{ed}{1 \pm e \cos \theta} \qquad \text{or} \qquad r = \frac{ed}{1 \pm e \sin \theta}$$

represents a conic section with eccentricity e. The conic is an ellipse if $e < 1$, a parabola if $e = 1$, or a hyperbola if $e > 1$.

Example 2

A conic is given by the polar equation

$$r = \frac{10}{3 - 2\cos\theta}$$

Find the eccentricity, identify the conic, locate the directrix, and sketch the conic.

Solution:

Dividing numerator and denominator by 3, we write the equation as

$$r = \frac{\frac{10}{3}}{1 - \frac{2}{3}\cos\theta}$$

Example 2 – *Solution*

cont'd

From Theorem 6 we see that this represents an ellipse with $e = \frac{2}{3}$.

6 Theorem A polar equation of the form

$$r = \frac{ed}{1 \pm e \cos\theta} \qquad \text{or} \qquad r = \frac{ed}{1 \pm e \sin\theta}$$

represents a conic section with eccentricity e. The conic is an ellipse if $e < 1$, a parabola if $e = 1$, or a hyperbola if $e > 1$.

Since $ed = \frac{10}{3}$, we have

$$d = \frac{\frac{10}{3}}{e} = \frac{\frac{10}{3}}{\frac{2}{3}} = 5$$

Example 2 – *Solution*

cont'd

so the directrix has Cartesian equation $x = -5$. When $\theta = 0$, $r = 10$; when $\theta = \pi$, $r = 2$. So the vertices have polar coordinates (10, 0) and (2, π). The ellipse is sketched in Figure 3.

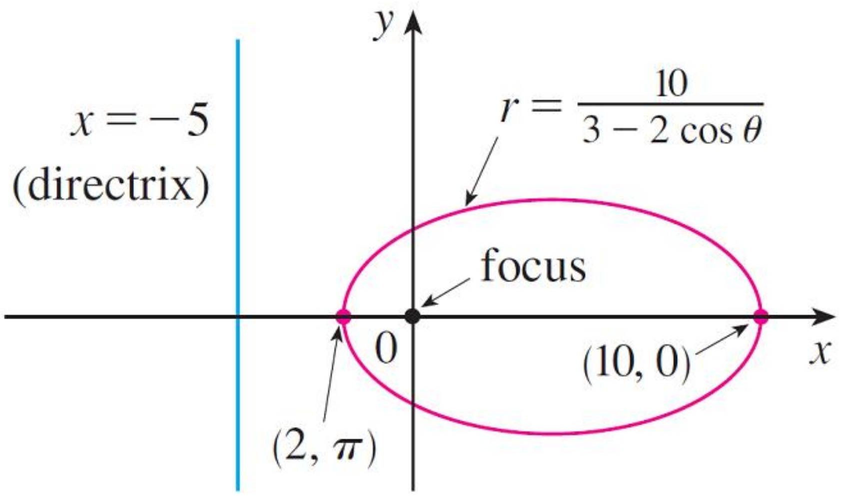


Figure 3

Kepler’s Laws

Kepler's Laws

In 1609 the German mathematician and astronomer Johannes Kepler, on the basis of huge amounts of astronomical data, published the following three laws of planetary motion.

Kepler's Laws

1. A planet revolves around the sun in an elliptical orbit with the sun at one focus.
2. The line joining the sun to a planet sweeps out equal areas in equal times.
3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

Kepler's Laws

Although Kepler formulated his laws in terms of the motion of planets around the sun, they apply equally well to the motion of moons, comets, satellites, and other bodies that orbit subject to a single gravitational force.

Here we use Kepler's First Law, together with the polar equation of an ellipse, to calculate quantities of interest in astronomy.

For purposes of astronomical calculations, it's useful to express the equation of an ellipse in terms of its eccentricity e and its semimajor axis a.

Kepler's Laws

We can write the distance *d* from the focus to the directrix in terms of *a* if we use $\boxed{4}$:

$$a^2 = \frac{e^2 d^2}{(1 - e^2)^2} \quad \Rightarrow \quad d^2 = \frac{a^2(1 - e^2)^2}{e^2} \quad \Rightarrow \quad d = \frac{a(1 - e^2)}{e}$$

So $ed = a(1 - e^2)$. If the directrix is $x = d$, then the polar equation is

$$r = \frac{ed}{1 + e \cos \theta} = \frac{a(1 - e^2)}{1 + e \cos \theta}$$

Kepler's Laws

7 The polar equation of an ellipse with focus at the origin, semimajor axis a, eccentricity e, and directrix $x = d$ can be written in the form

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

The positions of a planet that are closest to and farthest from the sun are called its **perihelion** and **aphelion**, respectively, and correspond to the vertices of the ellipse. (See Figure 7.)

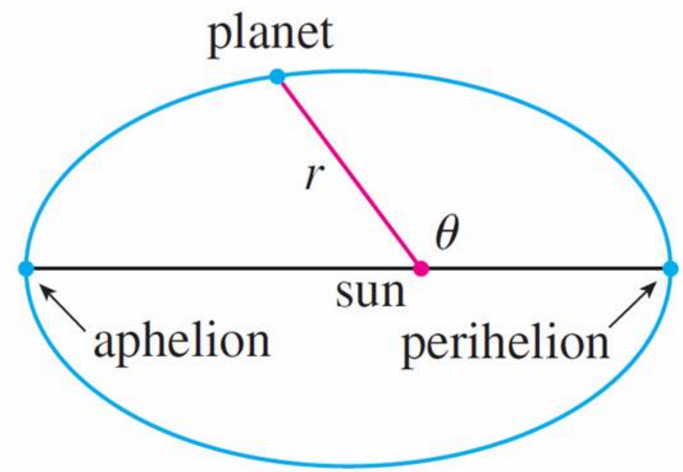


Figure 7

Kepler's Laws

The distances from the sun to the perihelion and aphelion are called the **perihelion distance** and **aphelion distance**, respectively.

In Figure 1 the sun is at the focus *F*, so at perihelion we have $\theta = 0$ and, from Equation 7,

$$r = \frac{a(1 - e^2)}{1 + e \cos 0}$$

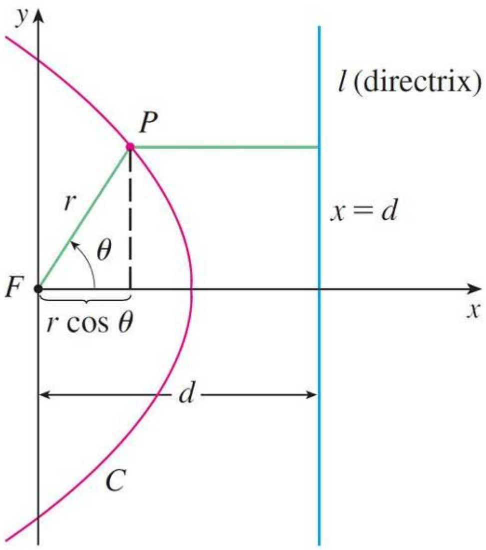


Figure 1

Kepler's Laws

$$= \frac{a(1 - e)(1 + e)}{1 + e}$$

$$= a(1 - e)$$

Similarly, at aphelion $\theta = \pi$ and $r = a(1 + e)$.

8 The perihelion distance from a planet to the sun is $a(1 - e)$ and the aphelion distance is $a(1 + e)$.

Example 5

(a) Find an approximate polar equation for the elliptical orbit of the earth around the sun (at one focus) given that the eccentricity is about 0.017 and the length of the major axis is about 2.99×10^8 km.

(b) Find the distance from the earth to the sun at perihelion and at aphelion.

Solution:

(a) The length of the major axis is $2a = 2.99 \times 10^8$, so $a = 1.495 \times 10^8$.

Example 5 – *Solution*

cont'd

We are given that e = 0.017 and so, from Equation 7, an equation of the earth's orbit around the sun is

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

$$= \frac{(1.495 \times 10^8)[1 - (0.017)^2]}{1 + 0.017\cos\theta}$$

or, approximately,

$$r = \frac{1.49 \times 10^8}{1 + 0.017\cos\theta}$$

Example 5 – *Solution*

cont'd

(b) From $\boxed{8}$, the perihelion distance from the earth to the sun is

$$a(1 - e) \approx (1.495 \times 10^8)(1 - 0.017)$$

$$\approx 1.47 \times 10^8 \text{ km}$$

and the aphelion distance is

$$a(1 + e) \approx (1.495 \times 10^8)(1 + 0.017)$$

$$\approx 1.52 \times 10^8 \text{ km}$$