3 Applications of Differentiation

3.1 Maximum and Minimum Values

Maximum and Minimum Values

Some of the most important applications of differential calculus are *optimization problems*, in which we are required to find the optimal (best) way of doing something. These can be done by finding the maximum or minimum values of a function.

Let's first explain exactly what we mean by maximum and minimum values.

We see that the highest point on the graph of the function f shown in Figure 1 is the point (3, 5).

In other words, the largest value of f is $f(3) = 5$. Likewise, the smallest value is $f(6) = 2$.

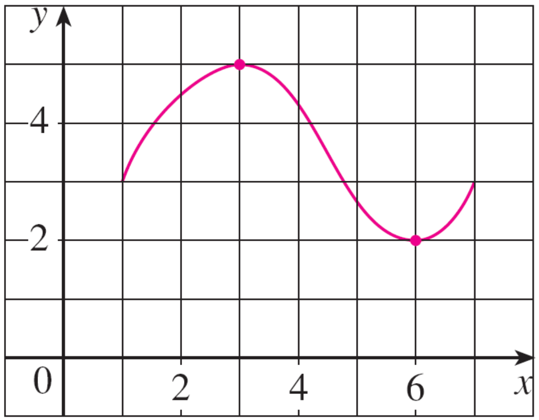


Figure 1

Maximum and Minimum Values

We say that $f(3) = 5$ is the *absolute maximum* of f and $f(6) = 2$ is the *absolute minimum*. In general, we use the following definition.

1 Definition Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geqslant f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leqslant f(x)$ for all x in D.

An absolute maximum or minimum is sometimes called a **global** maximum or minimum.

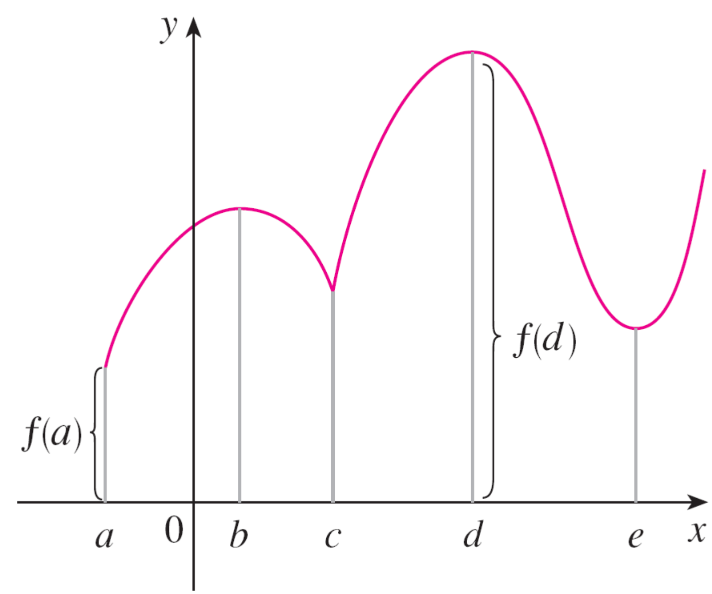
The maximum and minimum values of f are called **extreme values** of f.

Maximum and Minimum Values

Figure 2 shows the graph of a function f with absolute maximum at d and absolute minimum at a.

Note that $(d, f(d))$ is the highest point on the graph and $(a, f(a))$ is the lowest point.

In Figure 2, if we consider only values of x near b [for instance, if we restrict our attention to the interval (a, c)], then $f(b)$ is the largest of those values of $f(x)$ and is called a *local maximum value* of f.



Abs min $f(a)$, abs max $f(d)$
loc min $f(c)$, $f(e)$, loc max $f(b)$, $f(d)$

Figure 2

Maximum and Minimum Values

Likewise, $f(c)$ is called a *local minimum value* of f because $f(c) \leq f(x)$ for x near c [in the interval (b, d), for instance].

The function f also has a local minimum at e. In general, we have the following definition.

$\boxed{2}$ **Definition** The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geqslant f(x)$ when x is near c.
- **local minimum** value of f if $f(c) \leqslant f(x)$ when x is near c.

In Definition 2 (and elsewhere), if we say that something is true **near** c, we mean that it is true on some open interval containing c.

Maximum and Minimum Values

For instance, in Figure 3 we see that $f(4) = 5$ is a local minimum because it's the smallest value of f on the interval I.

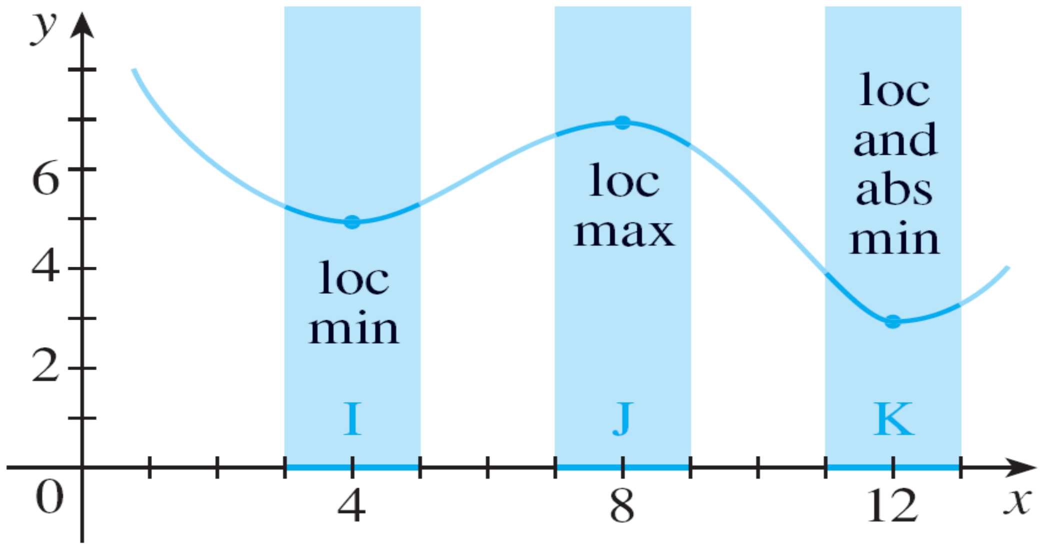


Figure 3

Maximum and Minimum Values

It's not the absolute minimum because $f(x)$ takes smaller values when x is near 12 (in the interval K, for instance).

In fact $f(12) = 3$ is both a local minimum and the absolute minimum.

Similarly, $f(8) = 7$ is a local maximum, but not the absolute maximum because f takes larger values near 1.

Example 1

The function $f(x) = \cos x$ takes on its (local and absolute) maximum value of 1 infinitely many times, since $\cos 2n\pi = 1$ for any integer n and $-1 \leq \cos x \leq 1$ for all x.

Likewise, $\cos(2n + 1)\pi = -1$ is its minimum value, where n is any integer.

Maximum and Minimum Values

The following theorem gives conditions under which a function is guaranteed to possess extreme values.

$\boxed{3}$ **The Extreme Value Theorem** If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Maximum and Minimum Values

The Extreme Value Theorem is illustrated in Figure 7.

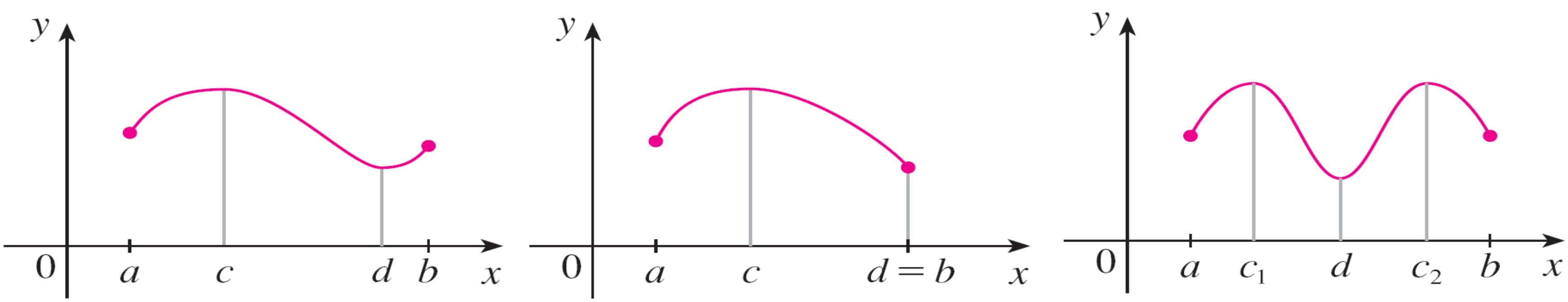


Figure 7

Note that an extreme value can be taken on more than once.

Maximum and Minimum Values

Figures 8 and 9 show that a function need not possess extreme values if either hypothesis (continuity or closed interval) is omitted from the Extreme Value Theorem.

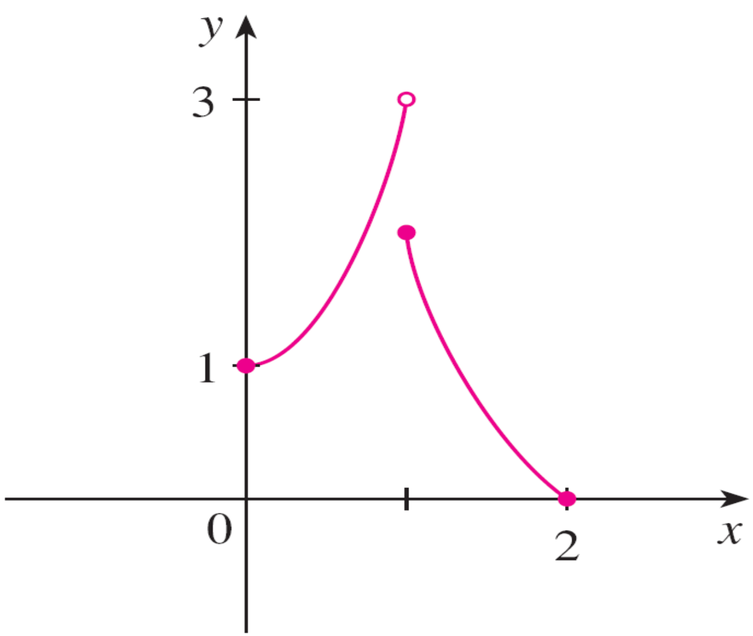


Figure 8 This function has minimum value $f(2) = 0$, but no maximum value.

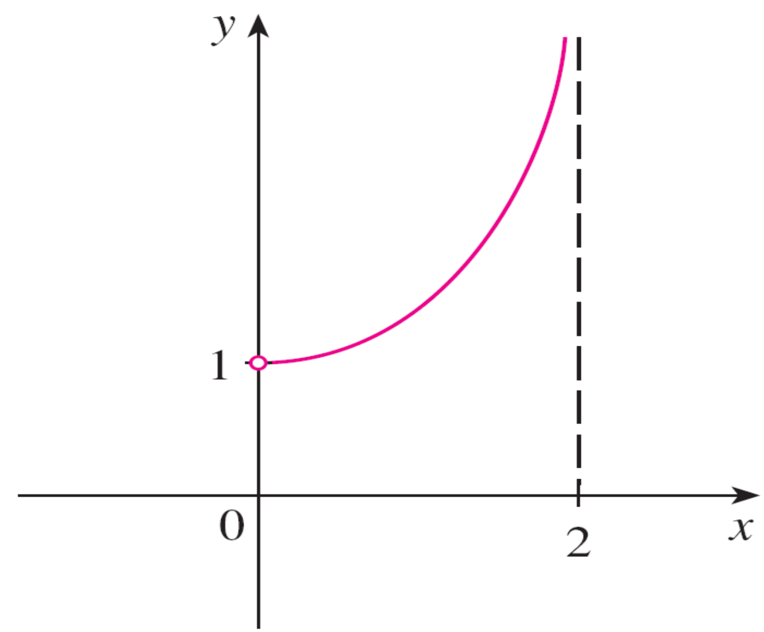


Figure 9 This continuous function g has no maximum or minimum.

Maximum and Minimum Values

The function *f* whose graph is shown in Figure 8 is defined on the closed interval [0, 2] but has no maximum value. [Notice that the range of *f* is [0, 3). The function takes on values arbitrarily close to 3, but never actually attains the value 3.]

This does not contradict the Extreme Value Theorem because *f* is not continuous.

Maximum and Minimum Values

The function g shown in Figure 9 is continuous on the open interval (0, 2) but has neither a maximum nor a minimum value. [The range of g is $(1, \infty)$. The function takes on arbitrarily large values.]

This does not contradict the Extreme Value Theorem because the interval (0, 2) is not closed.

Maximum and Minimum Values

The Extreme Value Theorem says that a continuous function on a closed interval has a maximum value and a minimum value, but it does not tell us how to find these extreme values. We start by looking for local extreme values.

Figure 10 shows the graph of a function *f* with a local maximum at *c* and a local minimum at *d*.

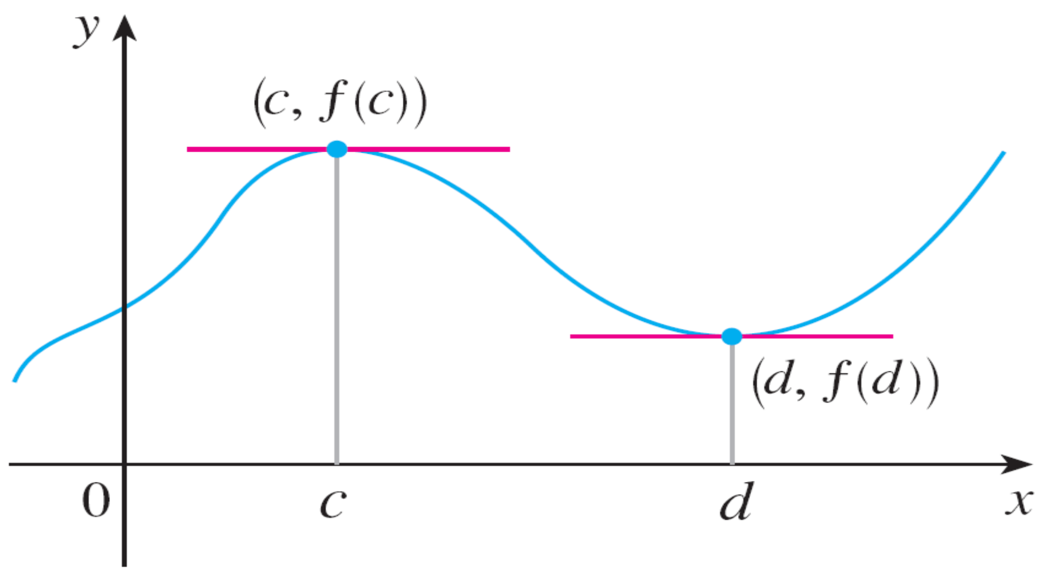


Figure 10

Maximum and Minimum Values

It appears that at the maximum and minimum points the tangent lines are horizontal and therefore each has slope 0.

We know that the derivative is the slope of the tangent line, so it appears that $f'(c) = 0$ and $f'(d) = 0$. The following theorem says that this is always true for differentiable functions.

$\boxed{4}$ **Fermat's Theorem** If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Example 5

If $f(x) = x^3$, then $f'(x) = 3x^2$, so $f'(0) = 0$.

But f has no maximum or minimum at 0, as you can see from its graph in Figure 11.

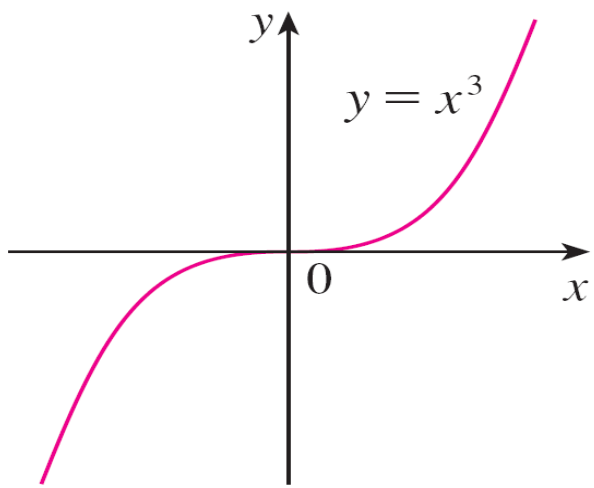


Figure 11

If $f(x) = x^3$, then $f'(0) = 0$ but f has no maximum or minimum.

Example 5

The fact that $f'(0) = 0$ simply means that the curve $y = x^3$ has a horizontal tangent at (0, 0).

Instead of having a maximum or minimum at (0, 0), the curve crosses its horizontal tangent there.

Example 6

The function $f(x) = |x|$ has its (local and absolute) minimum value at 0, but that value can't be found by setting $f'(x) = 0$ because, $f'(0)$ does not exist. (see Figure 12)

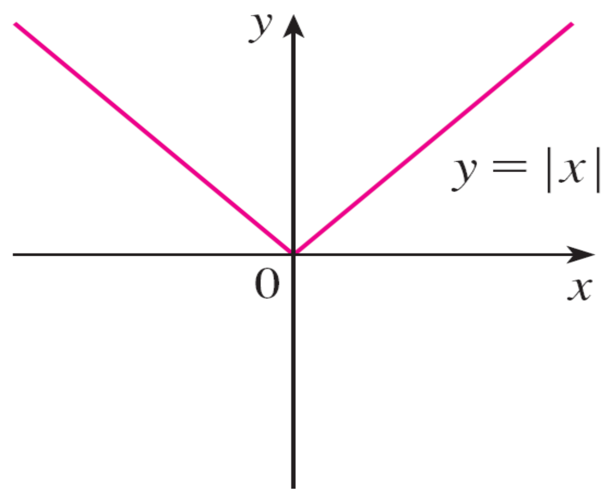


Figure 12

If $f(x) = |x|$, then $f(0) = 0$ is a minimum value, but $f'(0)$ does not exist.

Maximum and Minimum Values

Examples 5 and 6 show that we must be careful when using Fermat's Theorem. Example 5 demonstrates that even when $f'(c) = 0$, f doesn't necessarily have a maximum or minimum at c. (In other words, the converse of Fermat's Theorem is false in general.)

Furthermore, there may be an extreme value even when $f'(c)$ does not exist (as in Example 6).

Maximum and Minimum Values

Fermat's Theorem does suggest that we should at least *start* looking for extreme values of f at the numbers c where $f'(c) = 0$ or where $f'(c)$ does not exist. Such numbers are given a special name.

6 Definition A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

In terms of critical numbers, Fermat's Theorem can be rephrased as follows.

7 If f has a local maximum or minimum at c, then c is a critical number of f.

Maximum and Minimum Values

To find an absolute maximum or minimum of a continuous function on a closed interval, we note that either it is local or it occurs at an endpoint of the interval.

Thus the following three-step procedure always works.

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

3.2 The Mean Value Theorem

The Mean Value Theorem

We will see that many of the results depend on one central fact, which is called the Mean Value Theorem. But to arrive at the Mean Value Theorem we first need the following result.

Rolle's Theorem Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

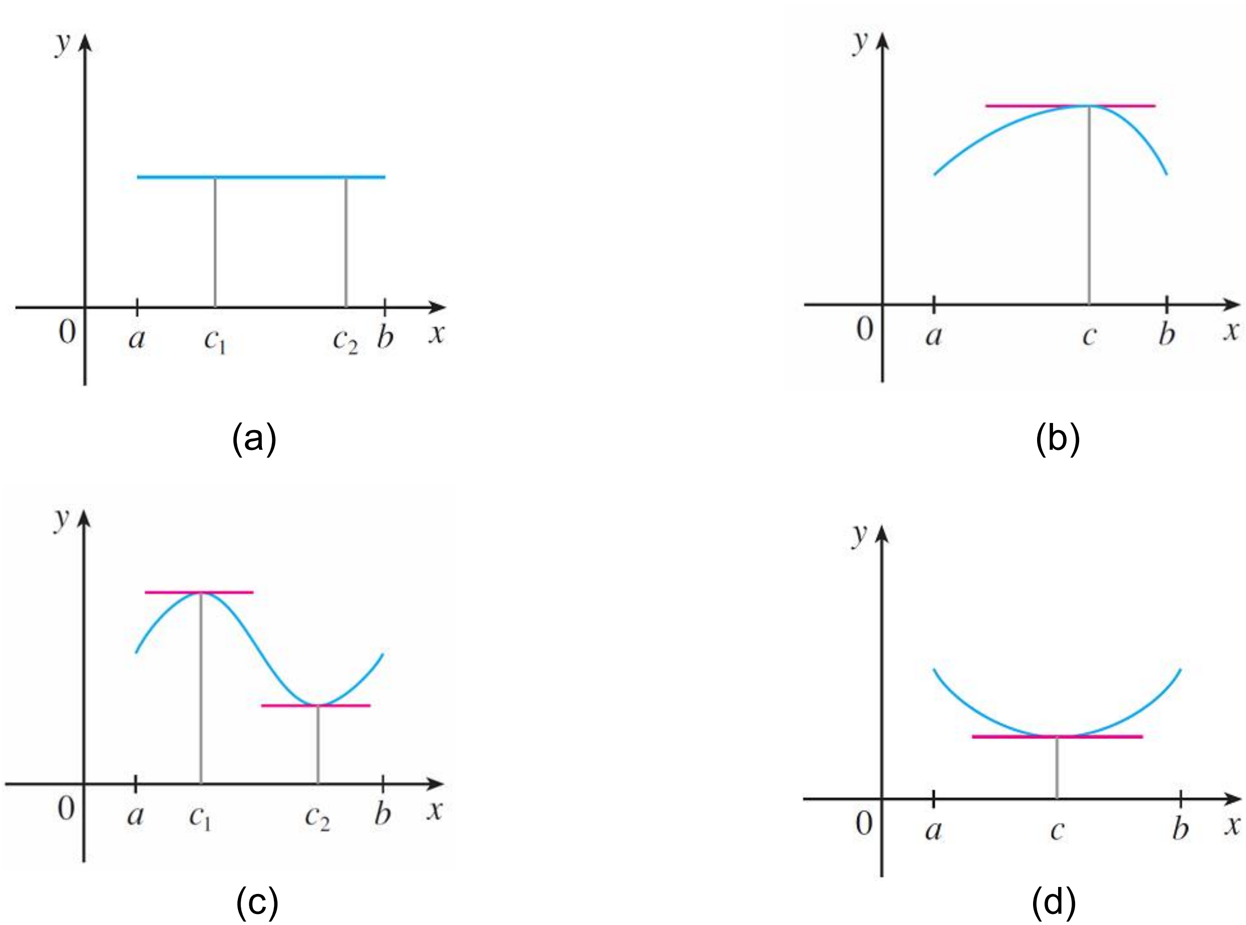
Then there is a number c in (a, b) such that $f'(c) = 0$.

The Mean Value Theorem

Before giving the proof let’s take a look at the graphs of some typical functions that satisfy the three hypotheses.

The Mean Value Theorem

Figure 1 shows the graphs of four such functions.



(a) (b) (c) (d)

Figure 1

The Mean Value Theorem

In each case it appears that there is at least one point (c, f(c)) on the graph where the tangent is horizontal and therefore $f'(c) = 0$.

Thus Rolle's Theorem is plausible.

Example 2

Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

Solution:

First we use the Intermediate Value Theorem to show that a root exists. Let $f(x) = x^3 + x - 1$.

Then $f(0) = -1 < 0$ and $f(1) = 1 > 0$. Since f is a polynomial, it is continuous, so the Intermediate Value Theorem states that there is a number c between 0 and 1 such that $f(c) = 0$.

Thus the given equation has a root.

Example 2 – *Solution*

cont'd

To show that the equation has no other real root, we use Rolle's Theorem and argue by contradiction.

Suppose that it had two roots a and b.

Then $f(a) = 0 = f(b)$ and, since f is a polynomial, it is differentiable on (a, b) and continuous on $[a, b]$.

Thus, by Rolle's Theorem, there is a number c between a and b such that $f'(c) = 0$.

Example 2 – *Solution*

cont'd

But

$$f'(x) = 3x^2 + 1 \geq 1 \qquad \text{for all } x$$

(since $x^2 \geq 0$) so $f'(x)$ can never be 0. This gives a contradiction.

Therefore the equation can't have two real roots.

The Mean Value Theorem

Our main use of Rolle's Theorem is in proving the following important theorem, which was first stated by another French mathematician, Joseph-Louis Lagrange.

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$\boxed{1} \qquad f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$\boxed{2} \qquad f(b) - f(a) = f'(c)(b - a)$$

The Mean Value Theorem

Before proving this theorem, we can see that it is reasonable by interpreting it geometrically. Figures 3 and 4 show the points *A*(*a*, *f*(*a*)) and *B*(*b*, *f*(*b*)) on the graphs of two differentiable functions.

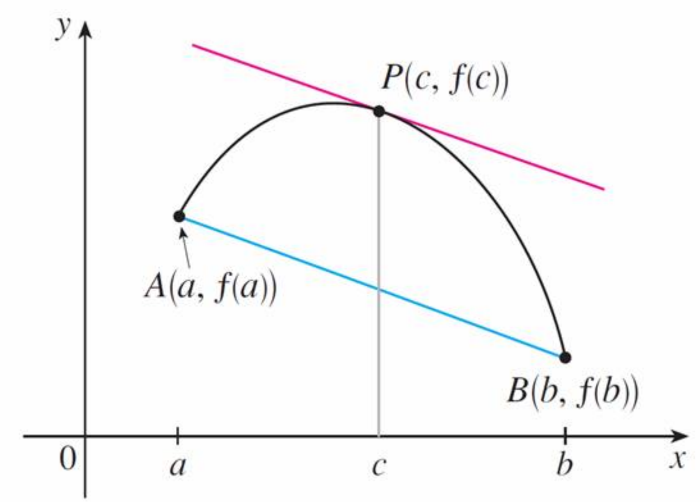


Figure 3

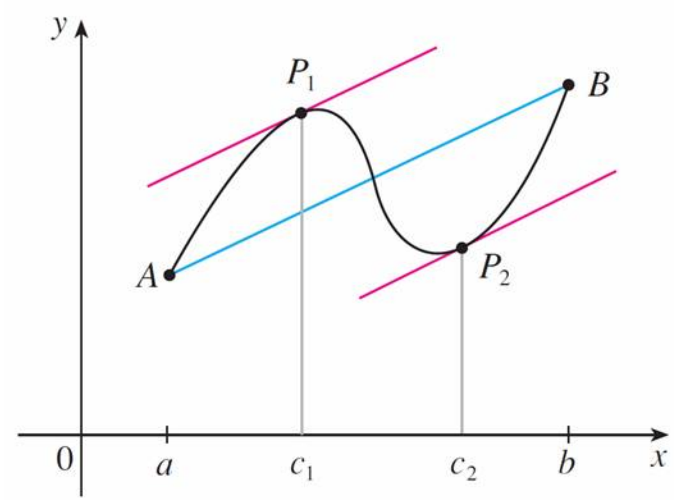


Figure 4

The Mean Value Theorem

The slope of the secant line *AB* is

$$\boxed{3} \qquad m_{AB} = \frac{f(b) - f(a)}{b - a}$$

which is the same expression as on the right side of Equation 1.

The Mean Value Theorem

Since $f'(c)$ is the slope of the tangent line at the point $(c, f(c))$, the Mean Value Theorem, in the form given by Equation 1, says that there is at least one point $P(c, f(c))$ on the graph where the slope of the tangent line is the same as the slope of the secant line AB.

In other words, there is a point P where the tangent line is parallel to the secant line AB.

Example 3

To illustrate the Mean Value Theorem with a specific function, let's consider

$$f(x) = x^3 - x, \; a = 0, \; b = 2.$$

Since f is a polynomial, it is continuous and differentiable for all x, so it is certainly continuous on $[0, 2]$ and differentiable on $(0, 2)$.

Therefore, by the Mean Value Theorem, there is a number c in $(0, 2)$ such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

Example 3

Now $f(2) = 6$,

$f(0) = 0$, and

$f'(x) = 3x^2 - 1$, so this equation becomes

$$6 = (3c^2 - 1)2$$

$$= 6c^2 - 2$$

which gives $c^2 = \frac{4}{3}$, that is, $c = \pm 2/\sqrt{3}$. But c must lie in (0, 2), so $c = 2/\sqrt{3}$.

Example 3

Figure 6 illustrates this calculation:

The tangent line at this value of *c* is parallel to the secant line *OB*.

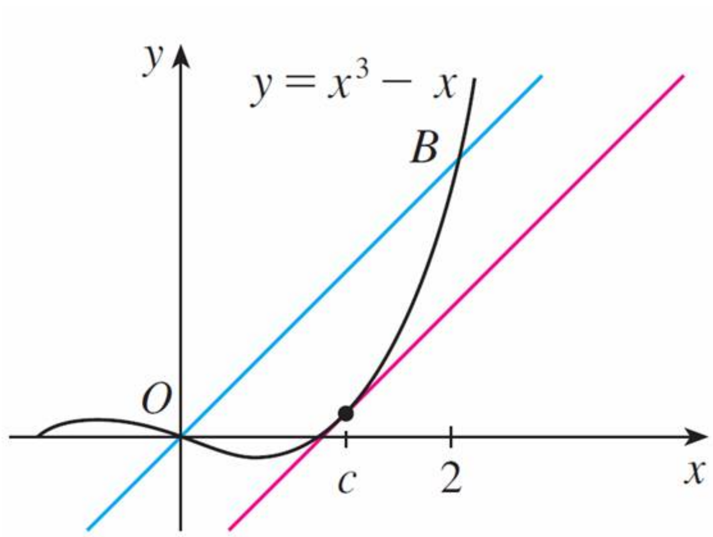


Figure 6

Example 5

Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ possibly be?

Solution:

We are given that f is differentiable (and therefore continuous) everywhere. In particular, we can apply the Mean Value Theorem on the interval [0, 2]. There exists a number c such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

Example 5 – *Solution*

so $$f(2) = f(0) + 2f'(c) = -3 + 2f'(c)$$

We are given that $f'(x) \leq 5$ for all x, so in particular we know that $f'(c) \leq 5$. Multiplying both sides of this inequality by 2, we have $2f'(c) \leq 10$, so

$$f(2) = -3 + 2f'(c) \leq -3 + 10 = 7$$

The largest possible value for $f(2)$ is 7.

The Mean Value Theorem

The Mean Value Theorem can be used to establish some of the basic facts of differential calculus.

One of these basic facts is the following theorem.

5 Theorem If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

7 Corollary If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

The Mean Value Theorem

Note:

Care must be taken in applying Theorem 5. Let

$$f(x) = \frac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

The domain of f is $D = \{x \mid x \neq 0\}$ and $f'(x) = 0$ for all x in D. But f is obviously not a constant function.

This does not contradict Theorem 5 because D is not an interval. Notice that f is constant on the interval $(0, \infty)$ and also on the interval $(-\infty, 0)$.

3.3 How Derivatives Affect the Shape of a Graph

What Does f' Say About f ?

What Does f' Say About f ?

To see how the derivative of f can tell us where a function is increasing or decreasing, look at Figure 1.

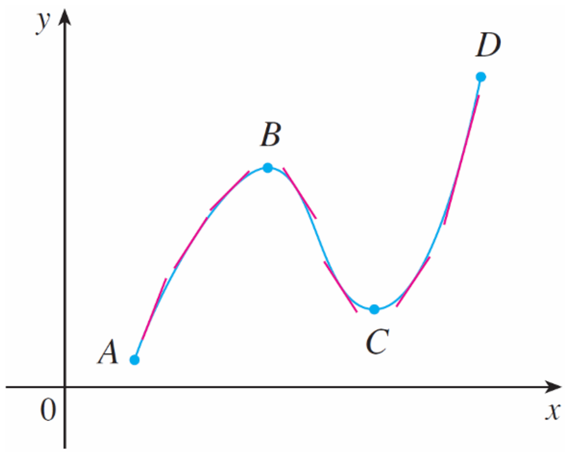


Figure 1

What Does f' Say About f ?

Between *A* and *B* and between *C* and *D*, the tangent lines have positive slope and so $f'(x) > 0$. Between *B* and *C* the tangent lines have negative slope and so $f'(x) < 0$. Thus it appears that f increases when $f'(x)$ is positive and decreases when $f'(x)$ is negative.

To prove that this is always the case, we use the Mean Value Theorem.

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Example 1

Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

Solution:

$$f'(x) = 12x^3 - 12x^2 - 24x$$

$$= 12x(x - 2)(x + 1)$$

To use the I/D Test we have to know where $f'(x) > 0$ and where $f'(x) < 0$. This depends on the signs of the three factors of $f'(x)$, namely, $12x$, $x - 2$, and $x + 1$.

Example 1 – *Solution*

cont'd

We divide the real line into intervals whose endpoints are the critical numbers –1, 0 and 2 and arrange our work in a chart. A plus sign indicates that the given expression is positive, and a minus sign indicates that it is negative.

The last column of the chart gives the conclusion based on the I/D Test. For instance, $f'(x) < 0$ for $0 < x < 2$, so f is decreasing on (0, 2). (It would also be true to say that f is decreasing on the closed interval [0, 2].)

Example 1 – *Solution*

cont'd

Interval	$12x$	$x - 2$	$x + 1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$-$	$-$	$+$	$+$	increasing on $(-1, 0)$
$0 < x < 2$	$+$	$-$	$+$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	$+$	increasing on $(2, \infty)$

The graph of *f* shown in Figure 2 confirms the information in the chart.

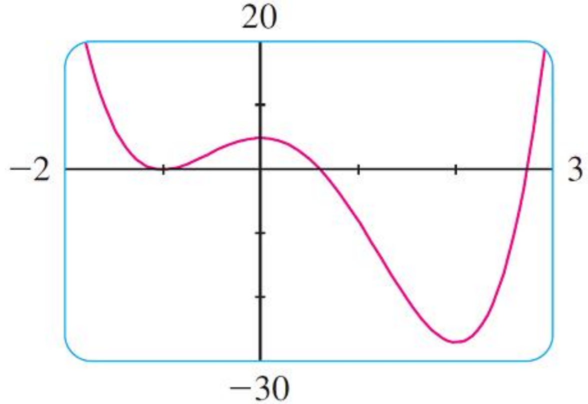


Figure 2

What Does f' Say About f ?

You can see from Figure 2 that $f(0) = 5$ is a local maximum value of f because f increases on $(–1, 0)$ and decreases on $(0, 2)$.

Or, in terms of derivatives, $f'(x) > 0$ for $–1 < x < 0$ and $f'(x) < 0$ for $0 < x < 2$. In other words, the sign of $f'(x)$ changes from positive to negative at 0. This observation is the basis of the following test.

What Does f' Say About f ?

The First Derivative Test Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

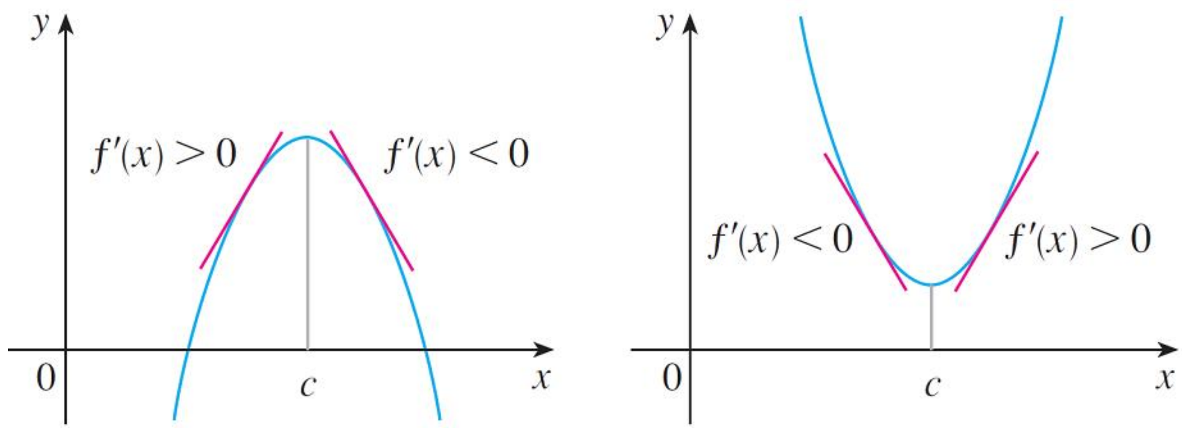
(c) If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

The First Derivative Test is a consequence of the I/D Test. In part (a), for instance, since the sign of $f'(x)$ changes from positive to negative at c, f is increasing to the left of c and decreasing to the right of c.

What Does f' Say About f ?

It follows that f has a local maximum at c.

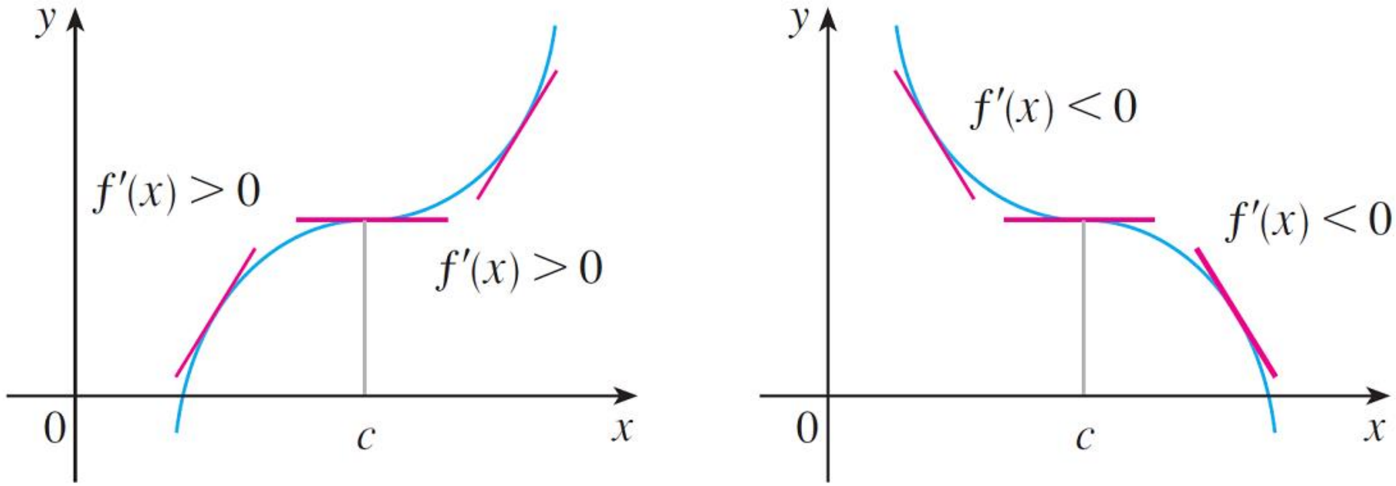
It is easy to remember the First Derivative Test by visualizing diagrams such as those in Figure 3.



(a) Local maximum (b) Local minimum

Figure 3

What Does f' Say About f ?



(c) No maximum or minimum

(d) No maximum or minimum

Figure 3

Example 3

Find the local maximum and minimum values of the function

$$g(x) = x + 2 \sin x \qquad 0 \le x \le 2\pi$$

Solution:

To find the critical numbers of g, we differentiate:

$$g'(x) = 1 + 2 \cos x$$

So $g'(x) = 0$ when $x = -\frac{1}{2}$. The solutions of this equation are $2\pi/3$ and $4\pi/3$.

Example 3 – *Solution*

cont'd

Because g is differentiable everywhere, the only critical numbers are $2\pi/3$ and $4\pi/3$ and so we analyze g in the following table.

Interval	$g'(x) = 1 + 2\cos x$	g
$0 < x < 2\pi/3$	$+$	increasing on $(0, 2\pi/3)$
$2\pi/3 < x < 4\pi/3$	$-$	decreasing on $(2\pi/3, 4\pi/3)$
$4\pi/3 < x < 2\pi$	$+$	increasing on $(4\pi/3, 2\pi)$

Example 3 – *Solution*

cont'd

Because $g'(x)$ changes from positive to negative at $2\pi/3$, the First Derivative Test tells us that there is a local maximum at $2\pi/3$ and the local maximum value is

$$g(2\pi/3) = \frac{2\pi}{3} + 2\sin\frac{2\pi}{3}$$

$$= \frac{2\pi}{3} + 2\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{2\pi}{3} + \sqrt{3}$$

$$\approx 3.83$$

Example 3 – *Solution*

cont'd

Likewise $g'(x)$, changes from negative to positive at $4\pi/3$ and so

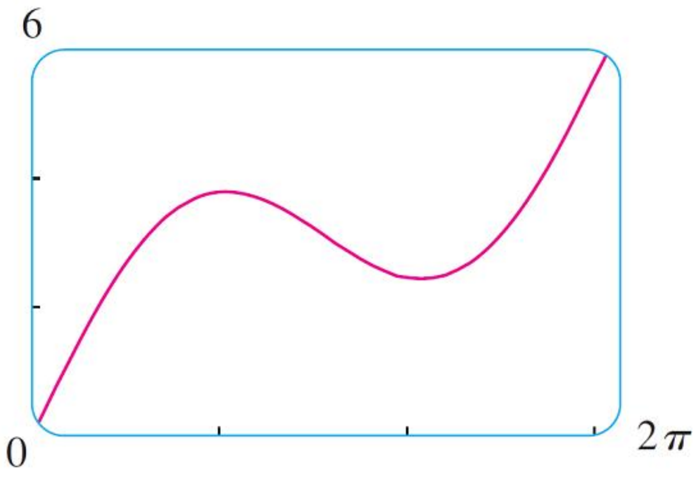
$$
\begin{aligned}
g(4\pi/3) &= \frac{4\pi}{3} + 2\sin\frac{4\pi}{3} \\
&= \frac{4\pi}{3} + 2\left(-\frac{\sqrt{3}}{2}\right) \\
&= \frac{4\pi}{3} - \sqrt{3} \\
&\approx 2.46
\end{aligned}
$$

is a local minimum value.

Example 3 – *Solution*

cont'd

The graph of g in Figure 4 supports our conclusion.



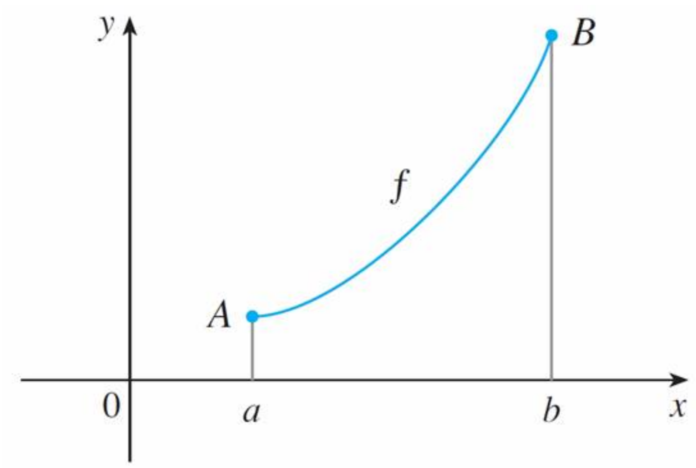
$g(x) = x + 2 \sin x$

Figure 4

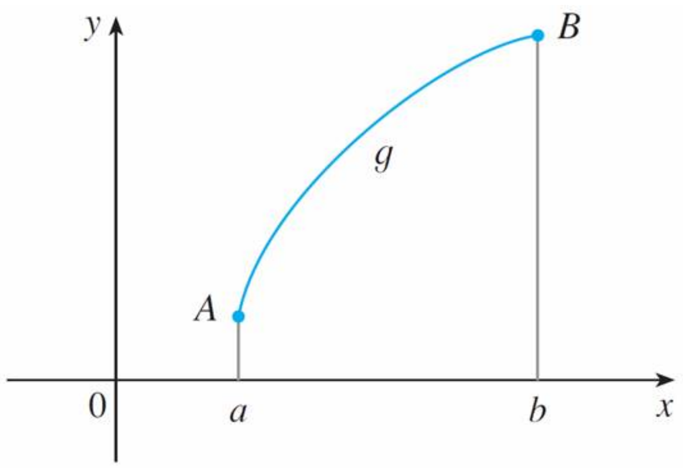
What Does f'' Say About f?

What Does f'' Say About f ?

Figure 5 shows the graphs of two increasing functions on (a, b). Both graphs join point A to point B but they look different because they bend in different directions.



(a)

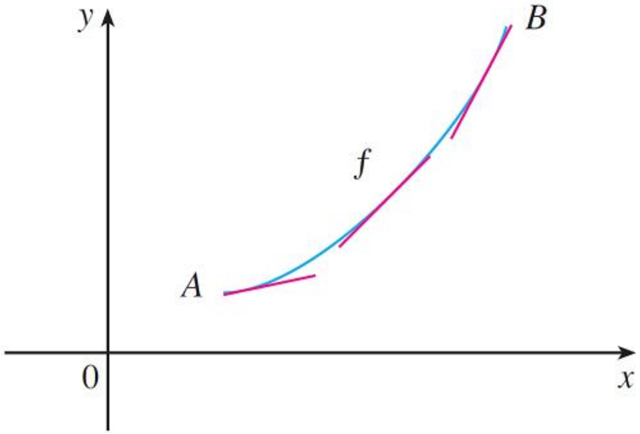


(b)

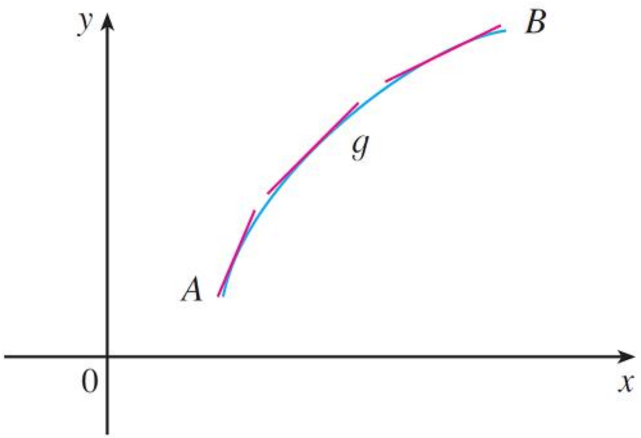
Figure 5

What Does f'' Say About f ?

In Figure 6 tangents to these curves have been drawn at several points. In (a) the curve lies above the tangents and f is called *concave upward* on (a, b). In (b) the curve lies below the tangents and g is called *concave downward* on (a, b).



(a) Concave upward



(b) Concave downward

Figure 6

What Does f'' Say About f ?

Definition If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.

What Does f'' Say About f ?

Figure 7 shows the graph of a function that is concave upward (abbreviated CU) on the intervals (b, c), (d, e), and (e, p) and concave downward (CD) on the intervals (a, b), (c, d), and (p, q).

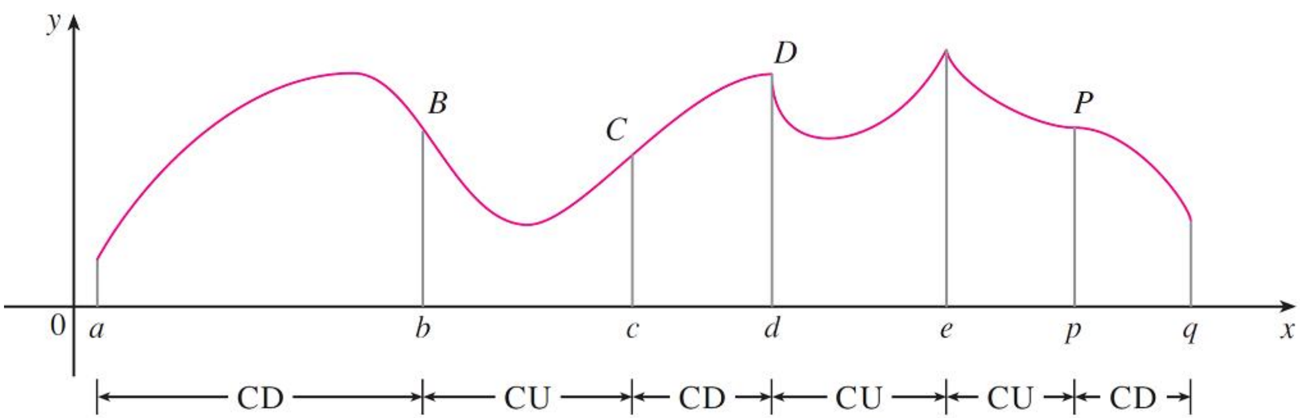


Figure 7

What Does f'' Say About f?

Let's see how the second derivative helps determine the intervals of concavity. Looking at Figure 6(a), you can see that, going from left to right, the slope of the tangent increases.

This means that the derivative f' is an increasing function and therefore its derivative f'' is positive.

Likewise, in Figure 6(b) the slope of the tangent decreases from left to right, so f' decreases and therefore f'' is negative.

What Does f'' Say About f ?

This reasoning can be reversed and suggests that the following theorem is true.

Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Example 4

Figure 8 shows a population graph for Cyprian honeybees raised in an apiary. How does the rate of population increase change over time? When is this rate highest? Over what intervals is P concave upward or concave downward?

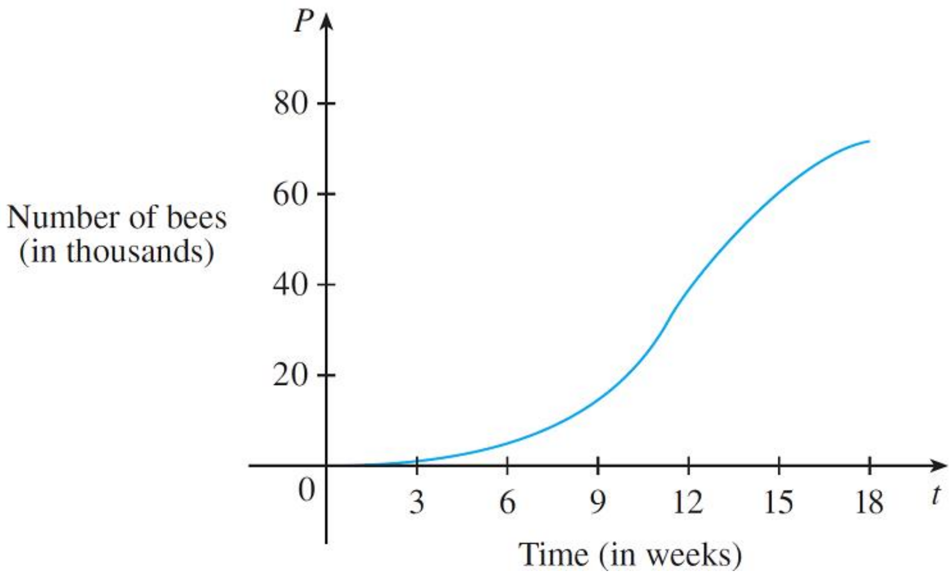


Figure 8

Example 4 – *Solution*

cont'd

By looking at the slope of the curve as t increases, we see that the rate of increase of the population is initially very small, then gets larger until it reaches a maximum at about $t = 12$ weeks, and decreases as the population begins to level off.

As the population approaches its maximum value of about 75,000 (called the *carrying capacity*), the rate of increase, $P'(t)$, approaches 0. The curve appears to be concave upward on (0, 12) and concave downward on (12, 18).

What Does f'' Say About f ?

Definition A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

The Second Derivative Test Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Example 6

Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

Solution:

If

$$f(x) = x^4 - 4x^3,$$

then

$$f'(x) = 4x^3 - 12x^2$$

$$= 4x^2(x - 3)$$

$$f''(x) = 12x^2 - 24x$$

$$= 12x(x - 2)$$

Example 6 – *Solution*

cont'd

To find the critical numbers we set $f'(x) = 0$ and obtain $x = 0$ and $x = 3$. To use the Second Derivative Test we evaluate f'' at these critical numbers:

$$f''(0) = 0 \qquad\qquad f''(3) = 36 > 0$$

Since $f'(3) = 0$ and $f''(3) > 0$, $f(3) = -27$ is a local minimum. Since $f''(0) = 0$, the Second Derivative Test gives no information about the critical number 0.

Example 6 – *Solution*

cont'd

But since $f'(x) < 0$ for $x < 0$ and also for $0 < x < 3$, the First Derivative Test tells us that f does not have a local maximum or minimum at 0.

[In fact, the expression for $f'(x)$ shows that f decreases to the left of 3 and increases to the right of 3.]

Since $f''(x) = 0$ when $x = 0$ or 2, we divide the real line into intervals with these numbers as endpoints and complete the following chart.

Interval	$f''(x) = 12x(x - 2)$	Concavity
$(-\infty, 0)$	$+$	upward
$(0, 2)$	$-$	downward
$(2, \infty)$	$+$	upward

Example 6 – *Solution*

cont'd

The point (0, 0) is an inflection point since the curve changes from concave upward to concave downward there. Also (2, –16) is an inflection point since the curve changes from concave downward to concave upward there.

Using the local minimum, the intervals of concavity, and the inflection points, we sketch the curve in Figure 11.

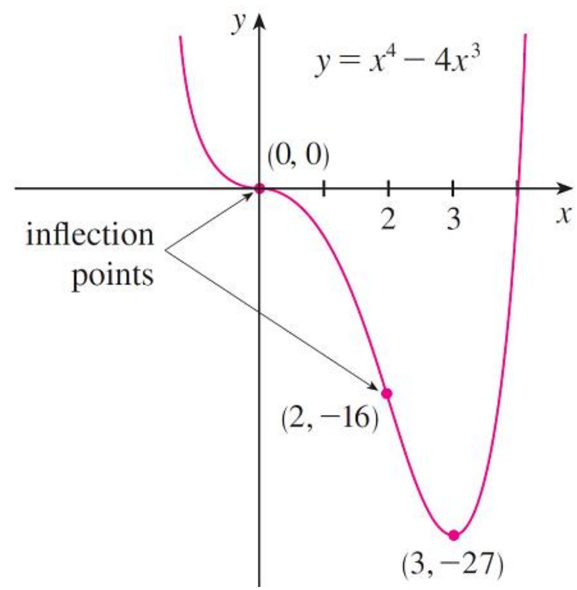


Figure 11

What Does f'' Say About f ?

Note:
The Second Derivative Test is inconclusive when $f''(c) = 0$. In other words, at such a point there might be a maximum, there might be a minimum, or there might be neither (as in Example 6).

This test also fails when $f''(c)$ does not exist. In such cases the First Derivative Test must be used. In fact, even when both tests apply, the First Derivative Test is often the easier one to use.

Example 7

Sketch the graph of the function $f(x) = x^{2/3}(6 - x)^{1/3}$

Solution:

Calculation of the first two derivatives gives

$$f'(x) = \frac{4 - x}{x^{1/3}(6 - x)^{2/3}} \qquad f''(x) = \frac{-8}{x^{4/3}(6 - x)^{5/3}}$$

Since $f'(x) = 0$ when $x = 4$ and $f'(x)$ does not exist when $x = 0$ or $x = 6$, the critical numbers are 0, 4 and 6.

Interval	$4 - x$	$x^{1/3}$	$(6 - x)^{2/3}$	$f'(x)$	f
$x < 0$	+	−	+	−	decreasing on $(-\infty, 0)$
$0 < x < 4$	+	+	+	+	increasing on $(0, 4)$
$4 < x < 6$	−	+	+	−	decreasing on $(4, 6)$
$x > 6$	−	+	+	−	decreasing on $(6, \infty)$

Example 7 – *Solution*

cont'd

To find the local extreme values we use the First Derivative Test. Since f' changes from negative to positive at 0, $f(0) = 0$ is a local minimum. Since f' changes from positive to negative at 4, $f(4) = 2^{5/3}$ is a local maximum.

The sign of f' does not change at 6, so there is no minimum or maximum there.

(The Second Derivative Test could be used at 4 but not at 0 or 6 since f'' does not exist at either of these numbers.)

Example 7 – *Solution*

cont'd

Looking at the expression $f''(x)$ for and noting that $x^{4/3} \geq 0$ for all x, we have $f''(x) < 0$ for $x < 0$ and for and $0 < x < 6$ for $x > 6$.

So f is concave downward on $(-\infty, 0)$ and $(0, 6)$ concave upward on $(6, \infty)$, and the only inflection point is $(6, 0)$. The graph is sketched in Figure 12.

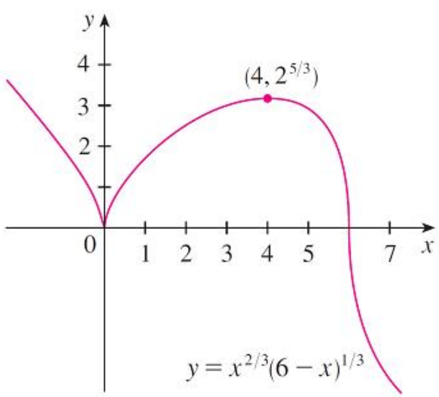


Figure 12

Example 7 – *Solution*

cont'd

Note that the curve has vertical tangents at (0,0) and (6,0) because $|f'(x)| \to \infty$ as $x \to 0$ and as $x \to 6$.

3.4 Limits at Infinity; Horizontal Asymptotes

Limits at Infinity; Horizontal Asymptotes

In this section we let *x* become arbitrarily large (positive or negative) and see what happens to *y*. We will find it very useful to consider this so-called *end behavior* when sketching graphs.

Let's begin by investigating the behavior of the function *f* defined by

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

as *x* becomes large.

Limits at Infinity; Horizontal Asymptotes

The table at the left gives values of this function correct to six decimal places, and the graph of *f* has been drawn by a computer in Figure 1.

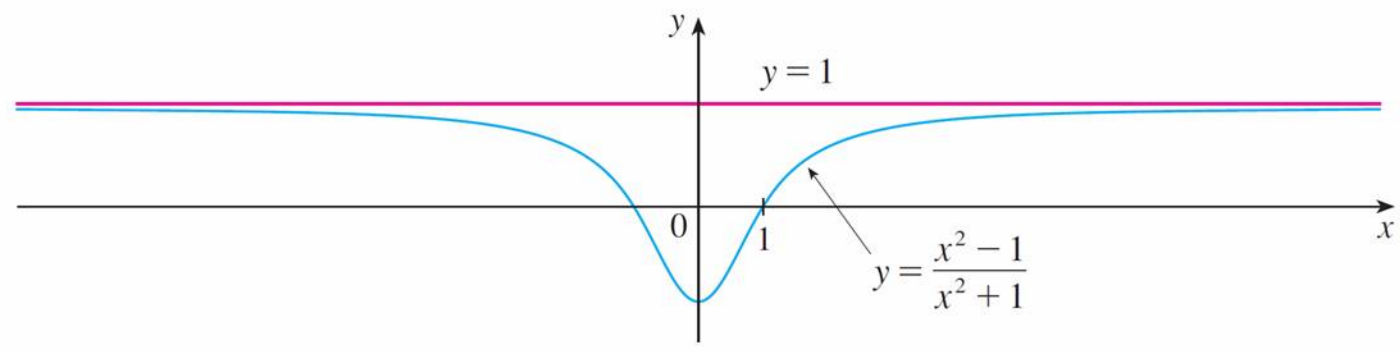


Figure 1

Limits at Infinity; Horizontal Asymptotes

As x grows larger and larger you can see that the values of $f(x)$ get closer and closer to 1. In fact, it seems that we can make the values of $f(x)$ as close as we like to 1 by taking x sufficiently large.

This situation is expressed symbolically by writing

$$\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

Limits at Infinity; Horizontal Asymptotes

In general, we use the notation

$$\lim_{x \to \infty} f(x) = L$$

to indicate that the values of $f(x)$ approach L as x becomes larger and larger.

$\boxed{1}$ **Definition** Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

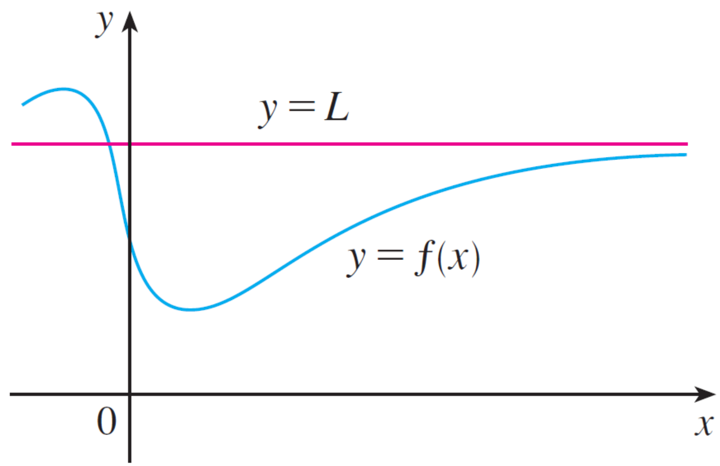
Limits at Infinity; Horizontal Asymptotes

Another notation for $\lim_{x \to \infty} f(x) = L$ is

$$f(x) \to L \quad \text{as} \quad x \to \infty$$

Limits at Infinity; Horizontal Asymptotes

Geometric illustrations of Definition 1 are shown in Figure 2.

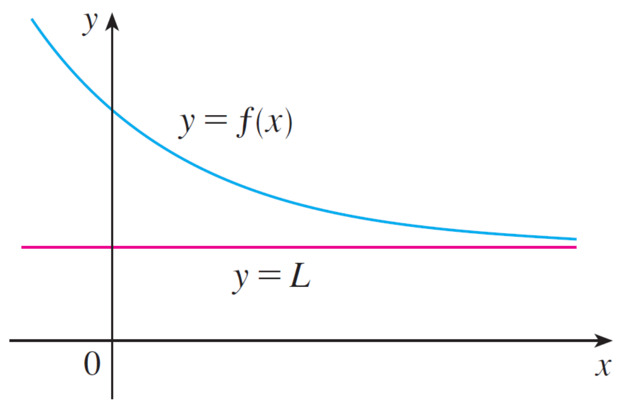


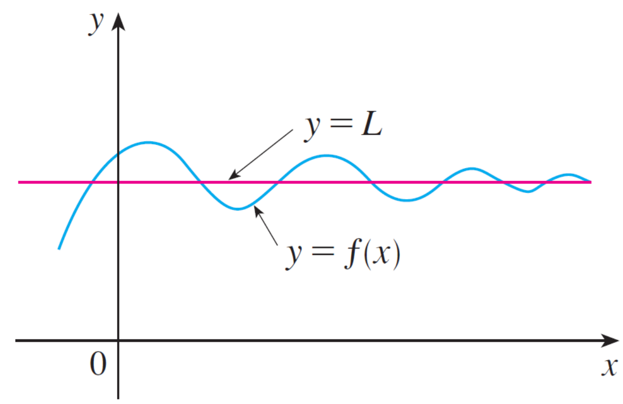
Examples illustrating $\lim_{x\to\infty} f(x) = L$

Figure 2

Limits at Infinity; Horizontal Asymptotes

Notice that there are many ways for the graph of *f* to approach the line *y* = *L* (which is called a *horizontal asymptote*) as we look to the far right of each graph.





Limits at Infinity; Horizontal Asymptotes

Referring back to Figure 1, we see that for numerically large negative values of x, the values of $f(x)$ are close to 1.

By letting x decrease through negative values without bound, we can make $f(x)$ as close to 1 as we like.

This is expressed by writing

$$\lim_{x \to -\infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

Limits at Infinity; Horizontal Asymptotes

The general definition is as follows.

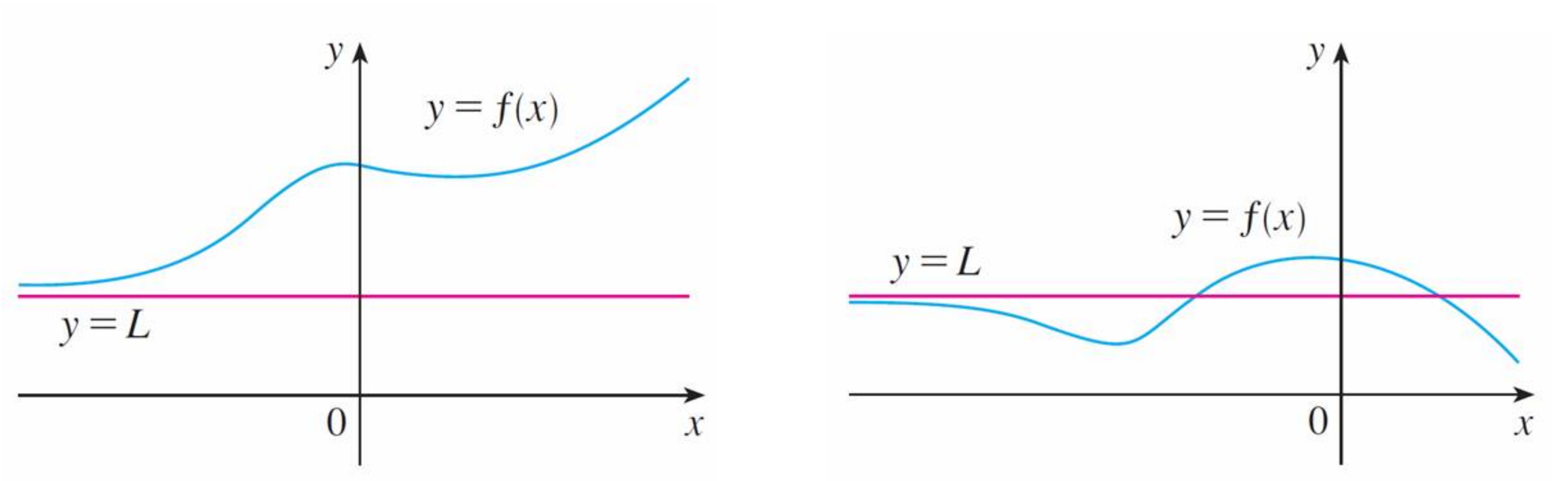
2 Definition Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

Limits at Infinity; Horizontal Asymptotes

Definition 2 is illustrated in Figure 3. Notice that the graph approaches the line $y = L$ as we look to the far left of each graph.



Examples illustrating $\lim_{x \to \infty} f(x) = L$

Figure 3

Limits at Infinity; Horizontal Asymptotes

3 Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$

Limits at Infinity; Horizontal Asymptotes

The curve $y = f(x)$ sketched in Figure 4 has both $y = -1$ and $y = 2$ as horizontal asymptotes because

$$\lim_{x \to \infty} f(x) = -1 \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = 2$$

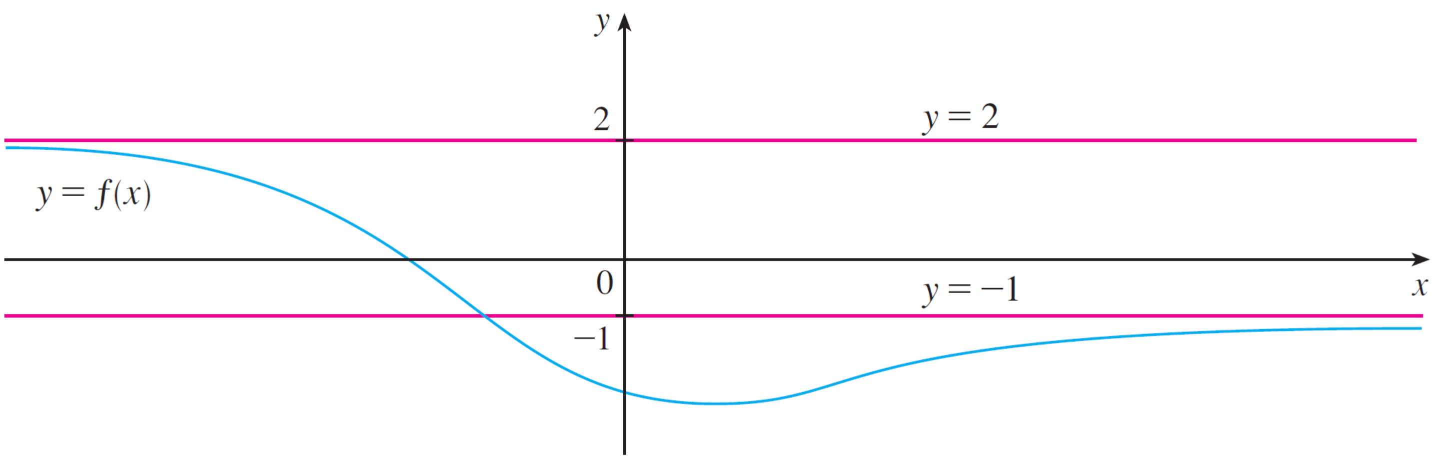


Figure 4

Example 2

Find $\lim_{x \to \infty} \frac{1}{x}$ and $\lim_{x \to -\infty} \frac{1}{x}$.

Solution:

Observe that when is large, 1/*x* is small. For instance,

$$\frac{1}{100} = 0.01 \qquad \frac{1}{10{,}000} = 0.0001 \qquad \frac{1}{1{,}000{,}000} = 0.000001$$

In fact, by taking *x* large enough, we can make 1/*x* as close to 0 as we please.

Example 2 – *Solution*

cont'd

Therefore, according to Definition 1, we have

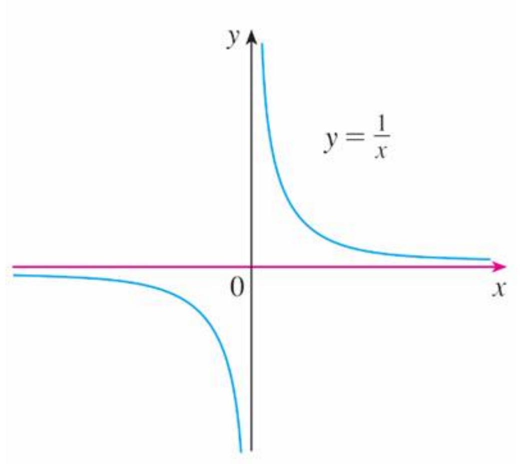
$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Similar reasoning shows that when x is large negative, $1/x$ is small negative, so we also have

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

Limits at Infinity; Horizontal Asymptotes

It follows that the line $y = 0$ (the x-axis) is a horizontal asymptote of the curve $y = 1/x$. (This is an equilateral hyperbola; see Figure 6.)



$$\lim_{x \to \infty} \frac{1}{x} = 0, \quad \lim_{x \to -\infty} \frac{1}{x} = 0$$

Figure 6

Limits at Infinity; Horizontal Asymptotes

4 Theorem If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

Example 3

Evaluate $\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$ and indicate which properties of limits are used at each stage.

Solution:

As x becomes large, both numerator and denominator become large, so it isn't obvious what happens to their ratio. We need to do some preliminary algebra.

To evaluate the limit at infinity of any rational function, we first divide both the numerator and denominator by the highest power of x that occurs in the denominator. (We may assume that $x \neq 0$, since we are interested only in large values of x.)

Example 3 – *Solution*

cont'd

In this case the highest power of x in the denominator is x^2, so we have

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to\infty} \frac{\dfrac{3x^2 - x - 2}{x^2}}{\dfrac{5x^2 + 4x + 1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{3 - \dfrac{1}{x} - \dfrac{2}{x^2}}{5 + \dfrac{4}{x} + \dfrac{1}{x^2}}$$

Example 3 – *Solution*

cont'd

$$= \frac{\lim_{x\to\infty}\left(3 - \frac{1}{x} - \frac{2}{x^2}\right)}{\lim_{x\to\infty}\left(5 + \frac{4}{x} + \frac{1}{x^2}\right)}$$

(by Limit Law 5)

$$= \frac{\lim_{x\to\infty} 3 - \lim_{x\to\infty} \frac{1}{x} - 2 \lim_{x\to\infty} \frac{1}{x^2}}{\lim_{x\to\infty} 5 + 4 \lim_{x\to\infty} \frac{1}{x} + \lim_{x\to\infty} \frac{1}{x^2}}$$

(by 1, 2, and 3)

$$= \frac{3 - 0 - 0}{5 + 0 + 0}$$

(by 7 and Theorem 4)

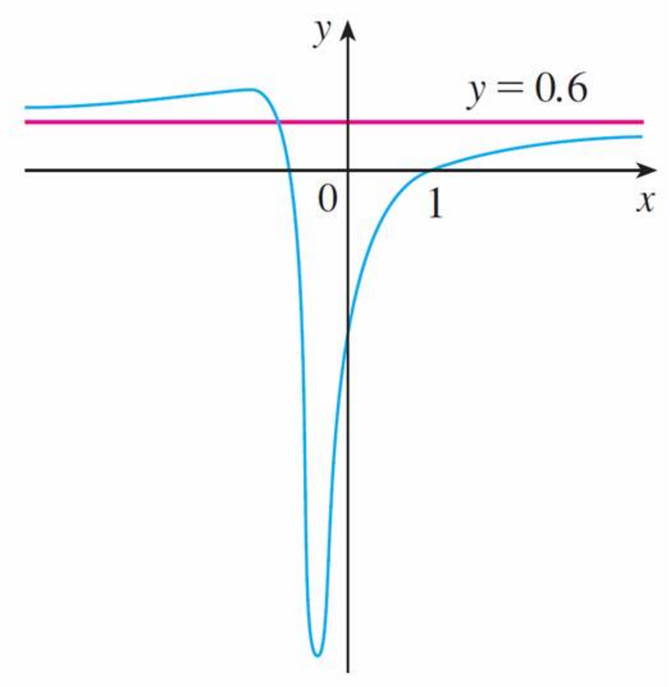
$$= \frac{3}{5}$$

Example 3 – *Solution*

cont'd

A similar calculation shows that the limit as $x \to -\infty$ is also $\frac{3}{5}$.

Figure 7 illustrates the results of these calculations by showing how the graph of the given rational function approaches the horizontal asymptote $y = \frac{3}{5}$.



$$y = \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

Figure 7

Example 4

Find the horizontal and vertical asymptotes of the graph of the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

Solution:

Dividing both numerator and denominator by *x* and using the properties of limits, we have

$$\lim_{x\to\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \lim_{x\to\infty} \frac{\sqrt{2 + \dfrac{1}{x^2}}}{3 - \dfrac{5}{x}}$$

(since $\sqrt{x^2}$ – x for $x > 0$)

Example 4 – *Solution*

cont'd

$$= \frac{\lim_{x\to\infty} \sqrt{2 + \dfrac{1}{x^2}}}{\lim_{x\to\infty} \left(3 - \dfrac{5}{x}\right)}$$

$$= \frac{\sqrt{\lim_{x\to\infty} 2 + \lim_{x\to\infty} \dfrac{1}{x^2}}}{\lim_{x\to\infty} 3 - 5 \lim_{x\to\infty} \dfrac{1}{x}}$$

$$= \frac{\sqrt{2 + 0}}{3 - 5 \cdot 0}$$

Example 4 – *Solution*

cont'd

$$= \frac{\sqrt{2}}{3}$$

Therefore the line $y = \sqrt{2}/3$ is a horizontal asymptote of the graph of f.

Example 4 – *Solution*

cont'd

In computing the limit as $x \to -\infty$, we must remember that for $x < 0$, we have $\sqrt{x^2} = |x| = -x$.

So when we divide the numerator by x, for $x < 0$ we get

$$\frac{1}{x}\sqrt{2x^2 + 1} = -\frac{1}{\sqrt{x^2}}\sqrt{2x^2 + 1}$$

$$= -\sqrt{2 + \frac{1}{x^2}}$$

Example 4 – *Solution*

cont'd

Therefore

$$\lim_{x \to -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \lim_{x \to -\infty} \frac{-\sqrt{2 + \dfrac{1}{x^2}}}{3 - \dfrac{5}{x}}$$

$$= \frac{-\sqrt{2 + \displaystyle\lim_{x \to -\infty} \frac{1}{x^2}}}{3 - 5 \displaystyle\lim_{x \to -\infty} \frac{1}{x}}$$

$$= -\frac{\sqrt{2}}{3}$$

Example 4 – *Solution*

cont'd

Thus the line $y = -\sqrt{2}/3$ is also a horizontal asymptote.

A vertical asymptote is likely to occur when the denominator, $3x - 5$, is 0, that is, when $x = \frac{5}{3}$.

If x is close to $\frac{5}{3}$ and $x > \frac{5}{3}$, then the denominator is close to 0 and $3x - 5$ is positive. The numerator $\sqrt{2x^2 + 1}$ is always positive, so $f(x)$ is positive.

Example 4 – *Solution*

cont'd

Therefore

$$\lim_{x \to (5/3)^+} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \infty$$

If is close to $\frac{5}{3}$ but $x < \frac{5}{3}$, then $3x - 5 < 0$ and so $f(x)$ is large negative.

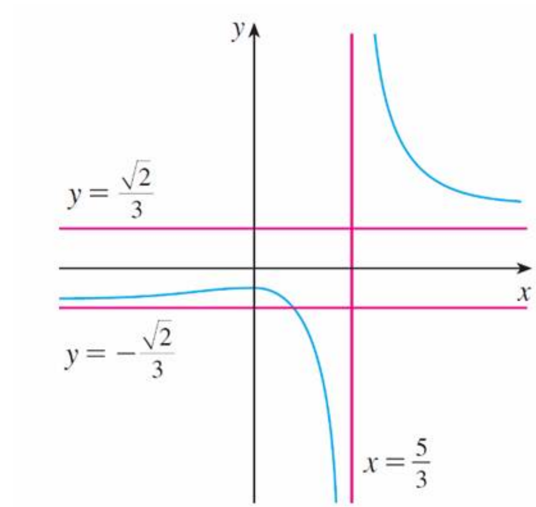
Example 4 – *Solution*

cont'd

Thus

$$\lim_{x \to (5/3)^-} \frac{\sqrt{2x^2 + 1}}{3x - 5} = -\infty$$

The vertical asymptote is $x = \frac{5}{3}$. All three asymptotes are shown in Figure 8.



$$y = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

Figure 8

Infinite Limits at Infinity

Infinite Limits at Infinity

The notation

$$\lim_{x \to \infty} f(x) = \infty$$

is used to indicate that the values of *f*(*x*) become large as *x* becomes large. Similar meanings are attached to the following symbols:

$$\lim_{x \to -\infty} f(x) = \infty \qquad \lim_{x \to \infty} f(x) = -\infty \qquad \lim_{x \to -\infty} f(x) = -\infty$$

Example 8

Find $\lim_{x \to \infty} x^3$ and $\lim_{x \to -\infty} x^3$.

Solution:

When becomes large, x^3 also becomes large.

For instance,

$$10^3 = 1000 \qquad 100^3 = 1,000,000 \qquad 1000^3 = 1,000,000,000$$

In fact, we can make x^3 as big as we like by taking x large enough. Therefore we can write

$$\lim_{x \to \infty} x^3 = \infty$$

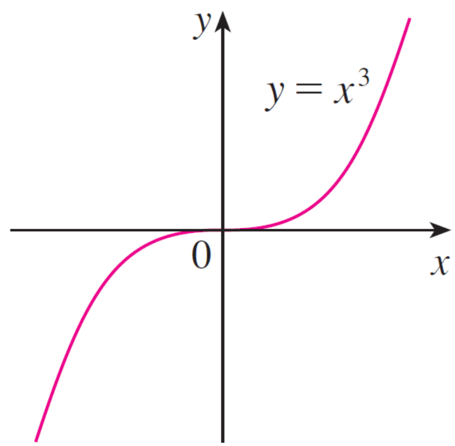
Example 8 – *Solution*

cont'd

Similarly, when x is large negative, so is x^3. Thus

$$\lim_{x \to -\infty} x^3 = -\infty$$

These limit statements can also be seen from the graph of $y = x^3$ in Figure 10.



$$\lim_{x \to \infty} x^3 = \infty, \quad \lim_{x \to -\infty} x^3 = -\infty$$

Figure 10

Precise Definitions

Precise Definitions

Definition 1 can be stated precisely as follows.

$\boxed{5}$ **Definition** Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

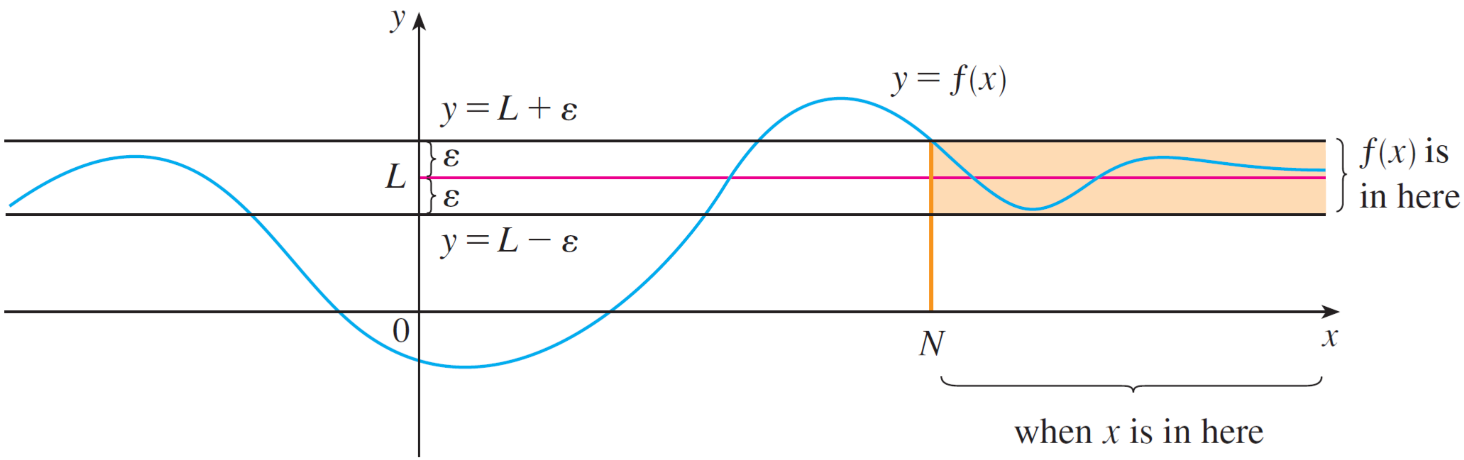
means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \quad x > N \quad \text{then} \quad |f(x) - L| < \varepsilon$$

In words, this says that the values of $f(x)$ can be made arbitrarily close to L (within a distance ε, where ε is any positive number) by taking x sufficiently large (larger than N, where depends on ε).

Precise Definitions

Graphically it says that by choosing x large enough (larger than some number N) we can make the graph of f lie between the given horizontal lines $y = L – \varepsilon$ and $y = L + \varepsilon$ as in Figure 12.

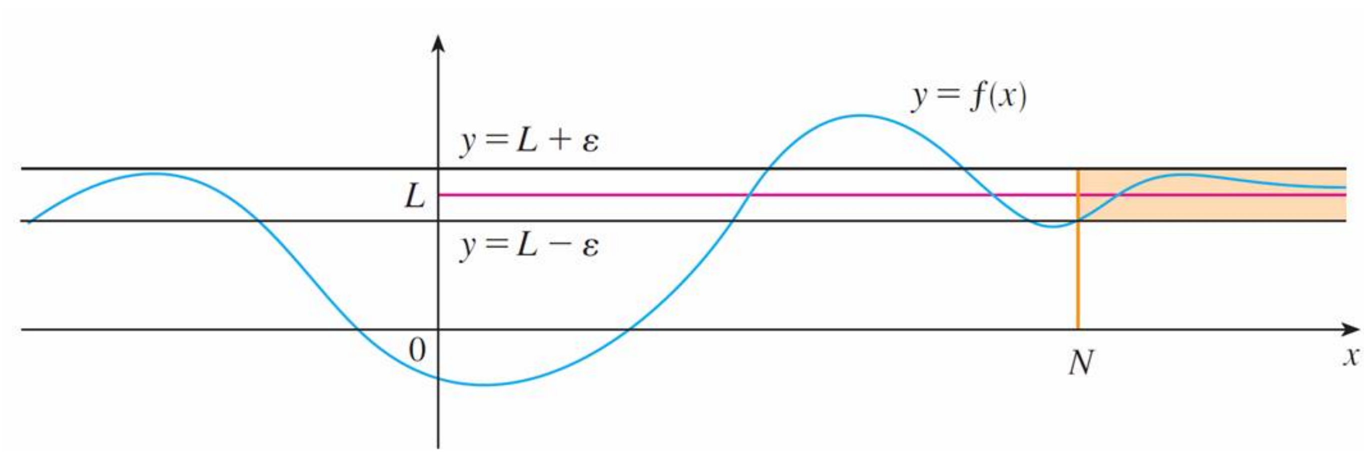


$$\lim_{x \to \infty} f(x) = L$$

Figure 12

Precise Definitions

This must be true no matter how small we choose *ε*. Figure 13 shows that if a smaller value of *ε* is chosen, then a larger value of *N* may be required.



$$\lim_{x \to \infty} f(x) = L$$

Figure 13

Precise Definitions

$\boxed{6}$ **Definition** Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \quad x < N \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

Example 13

Use Definition 5 to prove that $\lim_{x \to \infty} \frac{1}{x} = 0$.

Solution:

Given $\varepsilon > 0$, we want to find N such that

$$\text{if} \quad x > N \text{ then } \left| \frac{1}{x} - 0 \right| < \varepsilon$$

In computing the limit we may assume that $x > 0$.

Then

$$1/x < \varepsilon \iff x > 1/\varepsilon.$$

Example 13 – *Solution*

cont'd

Let's choose N = 1/*ε*. So

If $x > N = \dfrac{1}{\varepsilon}$ then $\left|\dfrac{1}{x} - 0\right| = \dfrac{1}{x} < \varepsilon$

Therefore, by Definition 5,

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Example 13 – *Solution*

cont'd

Figure 16 illustrates the proof by showing some values of ε and the corresponding values of N.

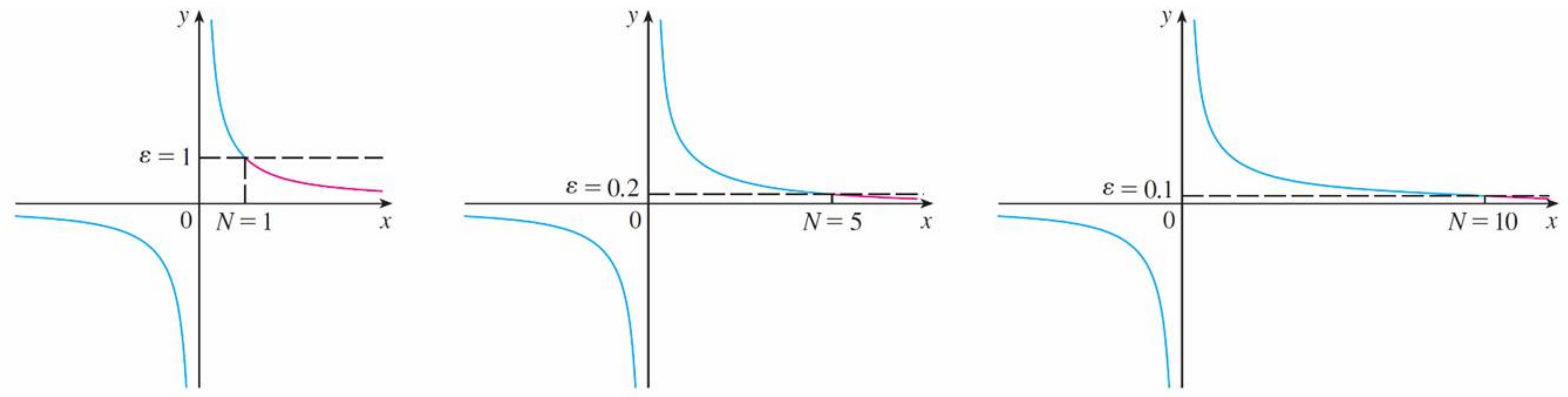
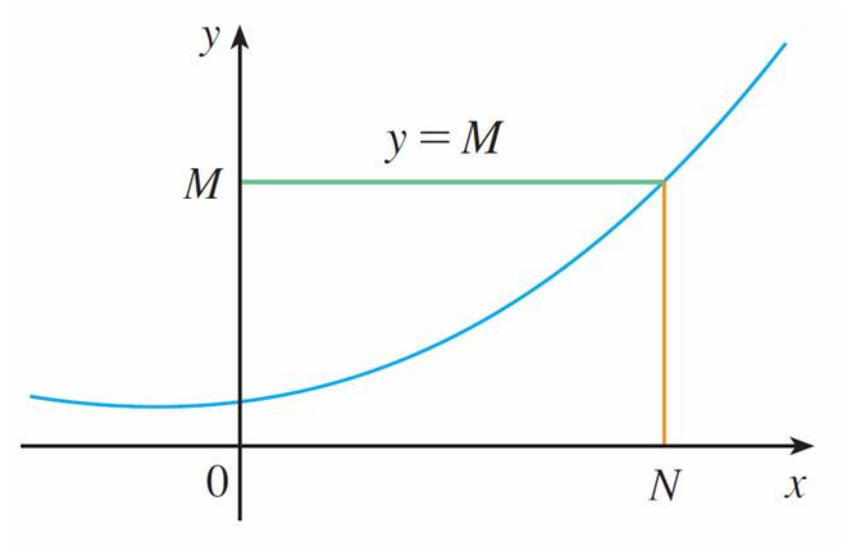


Figure 16

Precise Definitions

Finally we note that an infinite limit at infinity can be defined as follows. The geometric illustration is given in Figure 17.



$$\lim_{x \to \infty} f(x) = \infty$$

Figure 17

Precise Definitions

7 Definition Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = \infty$$

means that for every positive number M there is a corresponding positive number N such that

$$\text{if} \quad x > N \qquad \text{then} \qquad f(x) > M$$

Similar definitions apply when the symbol ∞ is replaced by $-\infty$.

3.5 Summary of Curve Sketching

Guidelines for Sketching a Curve

Guidelines for Sketching a Curve

The following checklist is intended as a guide to sketching a curve $y = f(x)$ by hand. Not every item is relevant to every function. (For instance, a given curve might not have an asymptote or possess symmetry.)

But the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function.

A. Domain It's often useful to start by determining the domain D of f, that is, the set of values of for which $f(x)$ is defined.

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B. Intercepts The *y*-intercept is $f(0)$ and this tells us where the curve intersects the *y*-axis. To find the *x*-intercepts, we set $y = 0$ and solve for *x*. (You can omit this step if the equation is difficult to solve.)

C. Symmetry

(i) If $f(-x) = f(x)$ for all *x* in *D*, that is, the equation of the curve is unchanged when *x* is replaced by $-x$, then *f* is an **even function** and the curve is symmetric about the *y*-axis.

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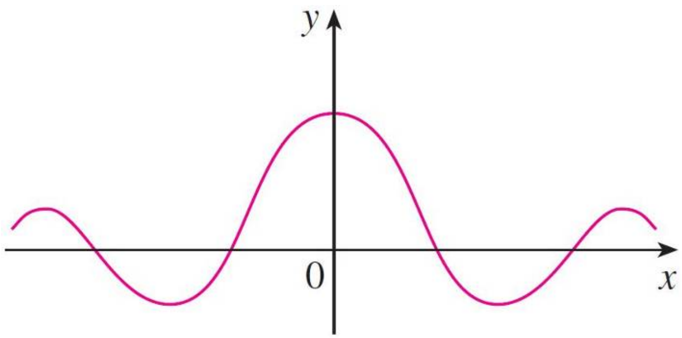
This means that our work is cut in half. If we know what the curve looks like for $x \geq 0$, then we need only reflect about the y-axis to obtain the complete curve [see Figure 3(a)].



(a) Even function: reflectional symmetry

Figure 3

Here are some examples: $y = x^2$, $y = x^4$, $y = |x|$, and $y = \cos x$.

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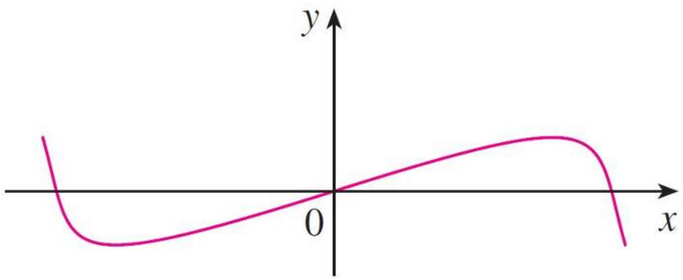
(ii) If $f(-x) = -f(x)$ If for all in x in D, then f is an **odd function** and the curve is symmetric about the origin.

Again we can obtain the complete curve if we know what it looks like for $x \geq 0$.

[Rotate 180° about the origin; see Figure 3(b).]



(b) Odd function: rotational symmetry

Figure 3

Some simple examples of odd functions are $y = x$, $y = x^3$, $y = x^5$, and $y = \sin x$.

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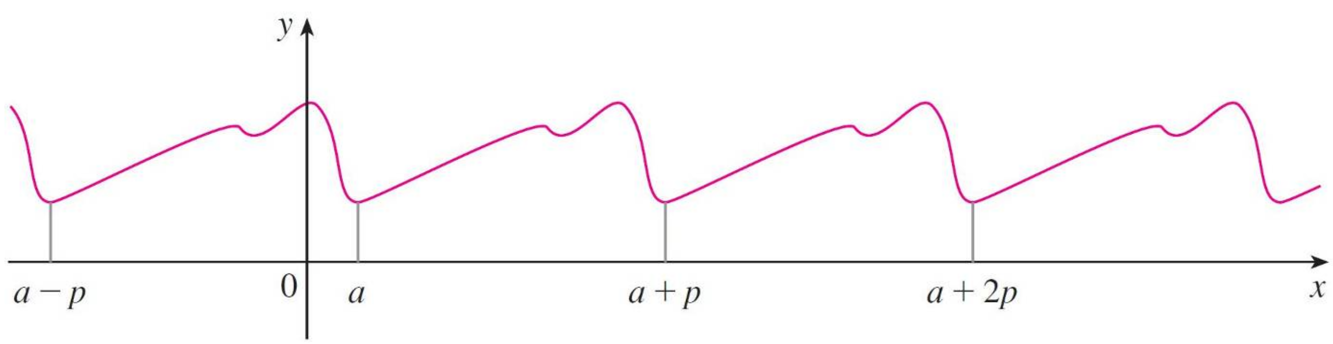
(iii) If $f(x + p) = f(x)$ for all x in D, where p is a positive constant, then f is called a **periodic function** and the smallest such number p is called the **period.**

For instance, $y = \sin x$ has period 2π and $y = \tan x$ has period π. If we know what the graph looks like in an interval of length p, then we can use translation to sketch the entire graph (see Figure 4).



Periodic function: translational symmetry

Figure 4

Guidelines for Sketching a Curve

D. Asymptotes

(i) *Horizontal Asymptotes.* If either $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to \infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$.

If it turns out that $\lim_{x \to \infty} f(x) = \infty$ (or $-\infty$), then we do not have an asymptote to the right, but that is still useful information for sketching the curve.

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(ii) *Vertical Asymptotes.* The line *x* = *a* is a vertical asymptote if at least one of the following statements is true:

$$\boxed{1} \qquad \lim_{x \to a^+} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty$$

$$\lim_{x \to a^+} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty$$

(For rational functions you can locate the vertical asymptotes by equating the denominator to 0 after canceling any common factors. But for other functions this method does not apply.)

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Furthermore, in sketching the curve it is very useful to know exactly which of the statements in $\boxed{1}$ is true.

If $f(a)$ is not defined but a is an endpoint of the domain of f, then you should compute $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$, whether or not this limit is infinite.

(iii) *Slant Asymptotes.*

Guidelines for Sketching a Curve

E. Intervals of Increase or Decrease Use the I/D Test. Compute $f'(x)$ and find the intervals on which $f'(x)$ is positive (f is increasing) and the intervals on which $f'(x)$ is negative (f is decreasing).

F. Local Maximum and Minimum Values Find the critical numbers of f [the numbers c where $f'(c) = 0$ or $f'(c)$ does not exist].

Then use the First Derivative Test. If f' changes from positive to negative at a critical number c, then $f(c)$ is a local maximum.

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If f' changes from negative to positive at c, then $f(c)$ is a local minimum. Although it is usually preferable to use the First Derivative Test, you can use the Second Derivative Test if $f'(c) = 0$ and $f''(c) \neq 0$.

Then $f''(c) > 0$ implies that $f(c)$ is a local minimum, whereas $f''(c) < 0$ implies that $f(c)$ is a local maximum.

G. Concavity and Points of Inflection Compute $f''(x)$ and use the Concavity Test. The curve is concave upward where $f''(x)$ **>** 0 and concave downward where $f''(x)$ **<** 0. Inflection points occur where the direction of concavity changes.

Guidelines for Sketching a Curve

H. Sketch the Curve Using the information in items A–G, draw the graph. Sketch the asymptotes as dashed lines. Plot the intercepts, maximum and minimum points, and inflection points.

Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes.

If additional accuracy is desired near any point, you can compute the value of the derivative there. The tangent indicates the direction in which the curve proceeds.

Example 1

Use the guidelines to sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$.

A. The domain is

$$\{x \mid x^2 - 1 \neq 0\} = \{x \mid x \neq \pm 1\}$$

$$= (-\infty, -1) \cup (-1, 1), \cup (1, \infty)$$

B. The *x*- and *y*-intercepts are both 0.

C. Since $f(-x) = f(x)$, the function f is even. The curve is symmetric about the *y*-axis.

Example 1

cont'd

D. $\displaystyle \lim_{x\to\pm\infty} \frac{2x^2}{x^2-1} = \lim_{x\to\pm\infty} \frac{2}{1-1/x^2} = 2$

Therefore the line $y = 2$ is a horizontal asymptote.

Since the denominator is 0 when $x = \pm 1$, we compute the following limits:

$$\lim_{x\to 1^+} \frac{2x^2}{x^2-1} = \infty \qquad \lim_{x\to 1^-} \frac{2x^2}{x^2-1} = -\infty$$

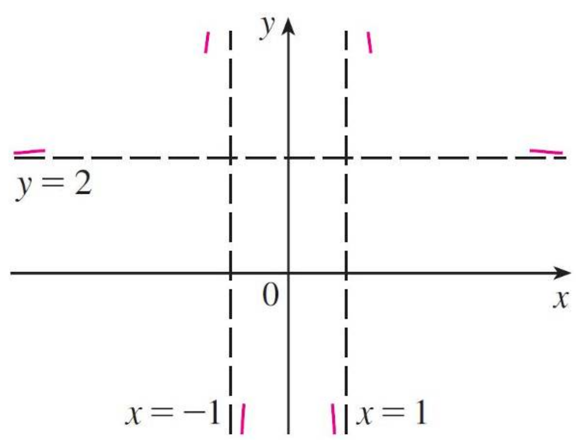
$$\lim_{x\to -1^+} \frac{2x^2}{x^2-1} = -\infty \qquad \lim_{x\to -1^-} \frac{2x^2}{x^2-1} = \infty$$

Example 1

cont'd

Therefore the lines $x = 1$ and $x = -1$ are vertical asymptotes.

This information about limits and asymptotes enables us to draw the preliminary sketch in Figure 5, showing the parts of the curve near the asymptotes.



Preliminary sketch

Figure 5

Example 1

cont'd

E. $f'(x) = \dfrac{4x(x^2 - 1) - 2x^2 \cdot 2x}{(x^2 - 1)^2} = \dfrac{-4x}{(x^2 - 1)^2}$

Since $f'(x) > 0$ when $x < 0$ ($x \neq -1$) and $f'(x) < 0$ when $x > 0$ ($x \neq 1$), f is increasing on $(-\infty, -1)$ and $(-1, 0)$ and decreasing on $(0, 1)$ and $(1, \infty)$.

F. The only critical number is $x = 0$.

Since f' changes from positive to negative at 0, $f(0) = 0$ is a local maximum by the First Derivative Test.

Example 1

cont'd

G. $f''(x) = \dfrac{-4(x^2 - 1)^2 + 4x \cdot 2(x^2 - 1)2x}{(x^2 - 1)^4} = \dfrac{12x^2 + 4}{(x^2 - 1)^3}$

Since $12x^2 + 4 > 0$ for all x, we have

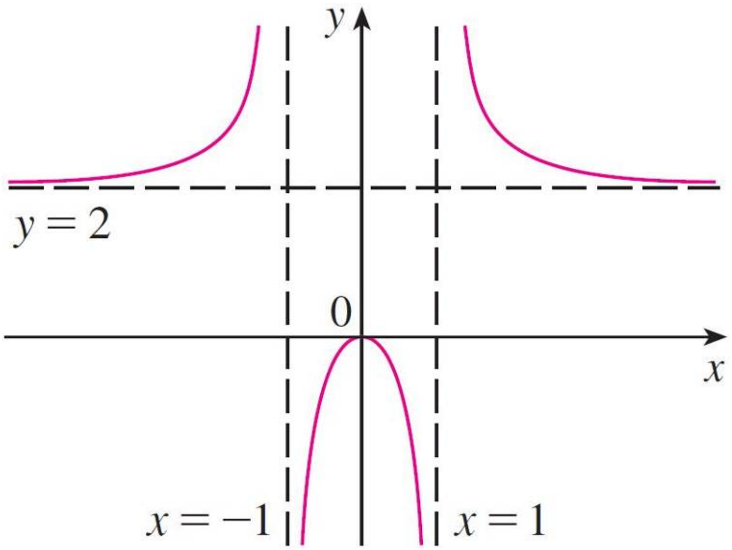
$$f''(x) > 0 \iff x^2 - 1 > 0 \iff |x| > 1$$

and $f''(x) < 0 \iff |x| < 1$. Thus the curve is concave upward on the intervals $(-\infty, -1)$ and $(1, \infty)$ and concave downward on $(-1, 1)$. It has no point of inflection since 1 and -1 are not in the domain of f.

Example 1

cont'd

H. Using the information in E–G, we finish the sketch in Figure 6.



Finished sketch of $y = \dfrac{2x^2}{x^2 - 1}$

Figure 6

Slant Asymptotes

Slant Asymptotes

Some curves have asymptotes that are *oblique,* that is, neither horizontal nor vertical. If

$$\lim_{x \to \infty} [f(x) - (mx + b)] = 0$$

then the line $y = mx + b$ is called a **slant asymptote** because the vertical distance between the curve $y = f(x)$ and the line $y = mx + b$ approaches 0, as in Figure 10.

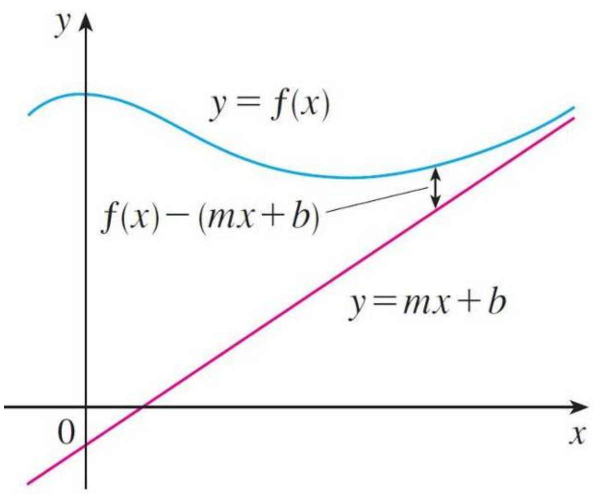


Figure 10

(A similar situation exists if we let $x \to -\infty$.)

Slant Asymptotes

For rational functions, slant asymptotes occur when the degree of the numerator is one more than the degree of the denominator.

In such a case the equation of the slant asymptote can be found by long division as in the next example.

Example 4

Sketch the graph of $f(x) = \dfrac{x^3}{x^2 + 1}$.

A. The domain is $\mathbb{R} = (-\infty, \infty)$.

B. The *x*- and *y*-intercepts are both 0.

C. Since $f(-x) = -f(x)$, f is odd and its graph is symmetric about the origin.

D. Since $x^2 + 1$ is never 0, there is no vertical asymptote. Since $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$, there is no horizontal asymptote.

Example 4

cont'd

But long division gives

$$f(x) = \frac{x^3}{x^2 + 1} = x - \frac{x}{x^2 + 1}$$

$$f(x) - x = -\frac{x}{x^2 + 1}$$

$$= -\frac{\frac{1}{x}}{1 + \frac{1}{x^2}} \to 0 \qquad \text{as} \quad x \to \pm\infty$$

So the line $y = x$ is a slant asymptote.

Example 4

cont'd

E. $f'(x) = \dfrac{3x^2(x^2 + 1) - x^3 \cdot 2x}{(x^2 + 1)^2}$

$$= \frac{x^2(x^2 + 3)}{(x^2 + 1)^2}$$

Since $f'(x) > 0$ for all x (except 0), f is increasing on $(-\infty, \infty)$.

F. Although $f'(0) = 0$, f' does not change sign at 0, so there is no local maximum or minimum.

Example 4

cont'd

G. $f''(x) = \dfrac{(4x^3 + 6x)(x^2 + 1)^2 - (x^4 + 3x^2) \cdot 2(x^2 + 1)2x}{(x^2 + 1)^4} = \dfrac{2x(3 - x^2)}{(x^2 + 1)^3}$

Since $f''(x) = 0$ when $x = 0$ or $x = \pm\sqrt{3}$, we set up the following chart:

Interval	x	$3 - x^2$	$(x^2 + 1)^3$	$f''(x)$	f
$x < -\sqrt{3}$	$-$	$-$	$+$	$+$	CU on $(-\infty, -\sqrt{3})$
$-\sqrt{3} < x < 0$	$-$	$+$	$+$	$-$	CD on $(-\sqrt{3}, 0)$
$0 < x < \sqrt{3}$	$+$	$+$	$+$	$+$	CU on $(0, \sqrt{3})$
$x > \sqrt{3}$	$+$	$-$	$+$	$-$	CD on $(\sqrt{3}, \infty)$

The points of inflection are $\left(-\sqrt{3}, -\frac{3}{4}\sqrt{3}\right)$, (0, 0), and $\left(\sqrt{3}, \frac{3}{4}\sqrt{3}\right)$.

Example 4

cont'd

H. The graph of *f* is sketched in Figure 11.

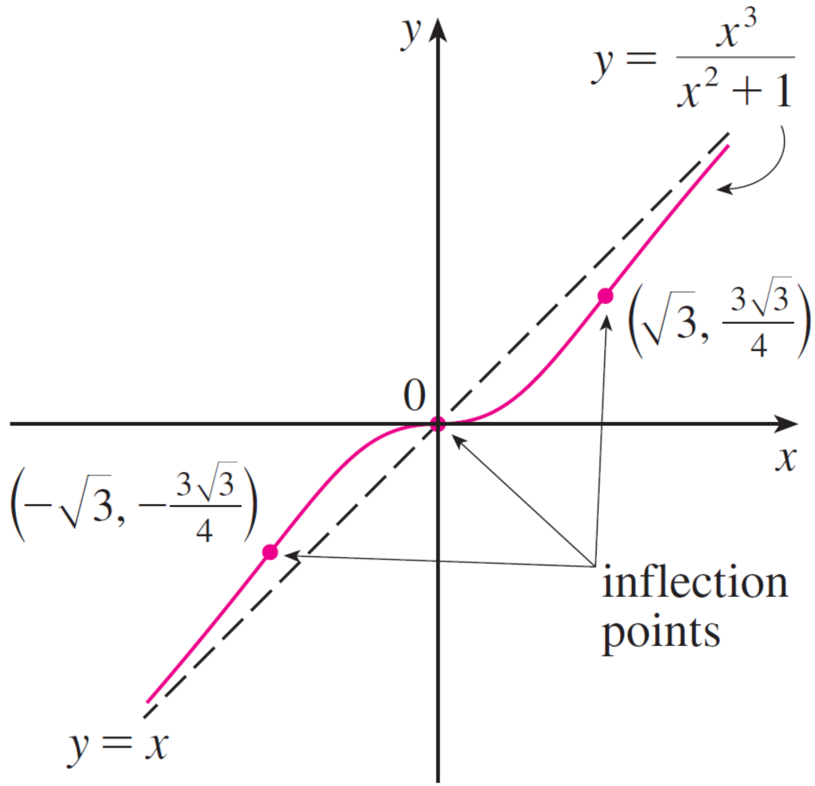


Figure 11

3.6 Graphing with Calculus and Calculators

Graphing with Calculus *and* Calculators

In this section we *start* with a graph produced by a graphing calculator or computer and then we refine it.

We use calculus to make sure that we reveal all the important aspects of the curve.

And with the use of graphing devices we can tackle curves that would be far too complicated to consider without technology. The theme is the *interaction* between calculus and calculators.

Example 1

Graph the polynomial $f(x) = 2x^6 + 3x^5 + 3x^3 - 2x^2$. Use the graphs of f' and f'' to estimate all maximum and minimum points and intervals of concavity.

Solution:

If we specify a domain but not a range, many graphing devices will deduce a suitable range from the values computed.

Figure 1 shows the plot from one such device if we specify that $-5 \leq x \leq 5$.

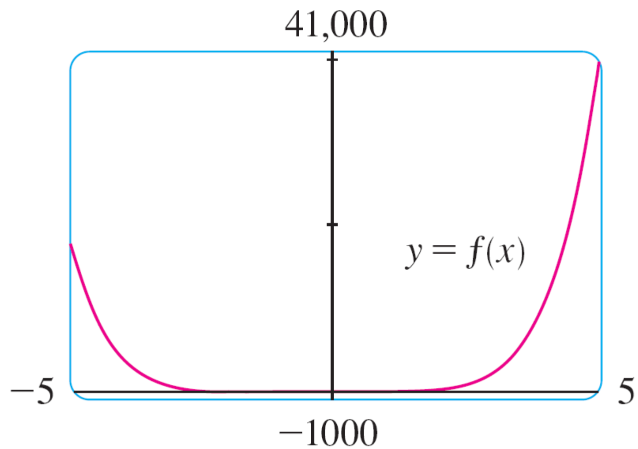


Figure 1

Example 1 – *Solution*

cont'd

Although this viewing rectangle is useful for showing that the asymptotic behavior (or end behavior) is the same as for $y = 2x^6$, it is obviously hiding some finer detail.

So we change to the viewing rectangle [–3, 2] by [–50, 100] shown in Figure 2.

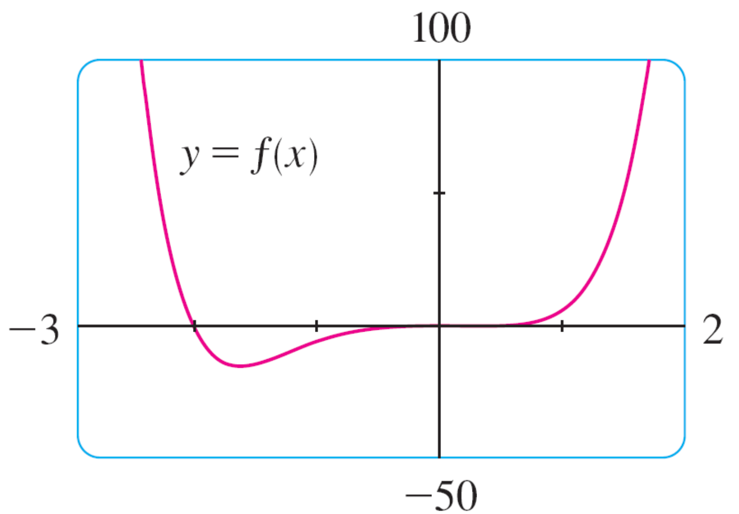


Figure 2

Example 1 – *Solution*

cont'd

From this graph it appears that there is an absolute minimum value of about –15.33 when $x \approx -1.62$ (by using the cursor) and f is decreasing on $(-\infty, -1.62)$ and increasing on $(-1.62, \infty)$.

Also there appears to be a horizontal tangent at the origin and inflection points when $x = 0$ and when x is somewhere between –2 and –1.

Now let's try to confirm these impressions using calculus. We differentiate and get

$$f'(x) = 12x^5 + 15x^4 + 9x^2 - 4x$$

$$f''(x) = 60x^4 + 60x^3 + 18x - 4$$

Example 1 – *Solution*

cont'd

When we graph f' in Figure 3 we see that $f'(x)$ changes from negative to positive when $x \approx -1.62$; this confirms (by the First Derivative Test) the minimum value that we found earlier. But, perhaps to our surprise, we also notice that $f'(x)$ changes from positive to negative when $x = 0$ and from negative to positive when $x \approx 0.35$.

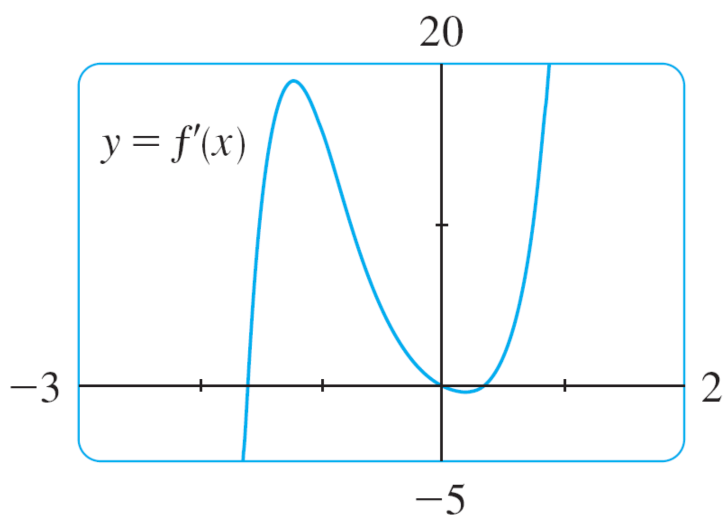


Figure 3

Example 1 – *Solution*

cont'd

This means that *f* has a local maximum at 0 and a local minimum when $x \approx 0.35$, but these were hidden in Figure 2. Indeed, if we now zoom in toward the origin in Figure 4, we see what we missed before: a local maximum value of 0 when $x = 0$ and a local minimum value of about –0.1 when $x \approx 0.35$.

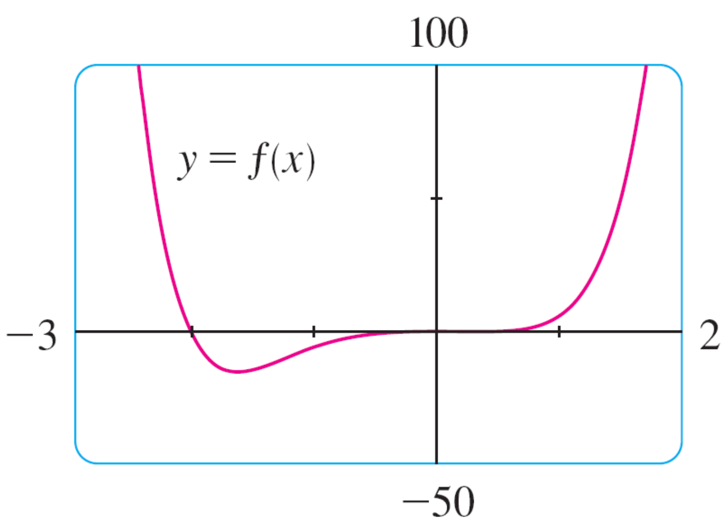


Figure 2

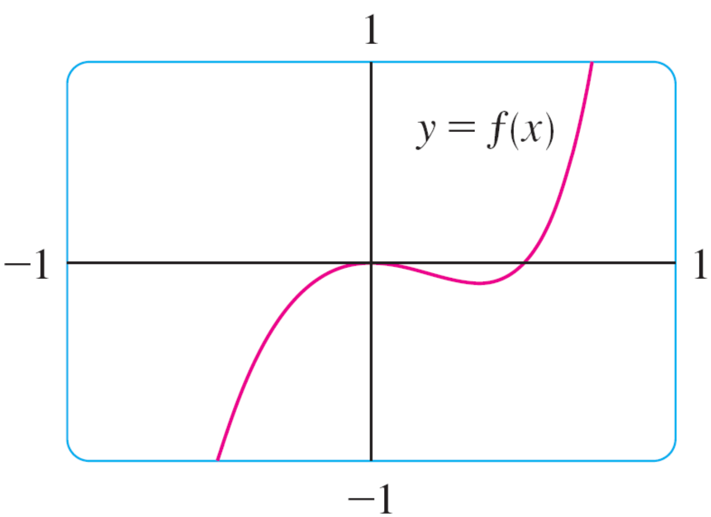


Figure 4

Example 1 – *Solution*

cont'd

What about concavity and inflection points?

From Figures 2 and 4 there appear to be inflection points when x is a little to the left of –1 and when x is a little to the right of 0. But it's difficult to determine inflection points from the graph of f, so we graph the second derivative f'' in Figure 5.

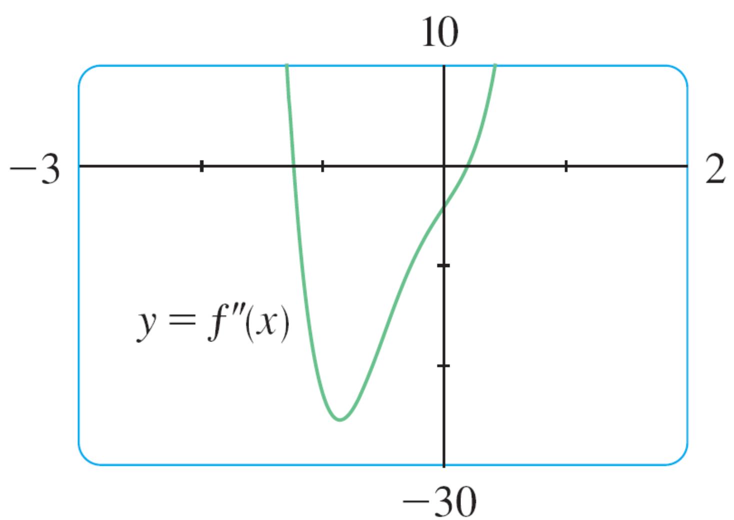


Figure 5

Example 1 – *Solution*

cont'd

We see that f'' changes from positive to negative when $x \approx -1.23$ and from negative to positive when $x \approx 0.19$.

So, correct to two decimal places, f is concave upward on $(-\infty, -1.23)$ and $(0.19, \infty)$ and concave downward on $(-1.23, 0.19)$.

The inflection points are $(-1.23, -10.18)$ and $(0.19, -0.05)$.

We have discovered that no single graph reveals all the important features of this polynomial. But Figures 2 and 4, when taken together, do provide an accurate picture.

3.7 Optimization Problems

Optimization Problems

In solving such practical problems the greatest challenge is often to convert the word problem into a mathematical optimization problem by setting up the function that is to be maximized or minimized.

Let's recall the problem-solving principles.

Optimization Problems

Steps in Solving Optimization Problems

1. **Understand the Problem** The first step is to read the problem carefully until it is clearly understood. Ask yourself: What is the unknown? What are the given quantities? What are the given conditions?
2. **Draw a Diagram** In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.
3. **Introduce Notation** Assign a symbol to the quantity that is to be maximized or minimized (let's call it Q for now). Also select symbols $(a, b, c, \ldots, x, y)$ for other unknown quantities and label the diagram with these symbols. It may help to use initials as suggestive symbols—for example, A for area, h for height, t for time.
4. Express Q in terms of some of the other symbols from Step 3.
5. If Q has been expressed as a function of more than one variable in Step 4, use the given information to find relationships (in the form of equations) among these variables. Then use these equations to eliminate all but one of the variables in the expression for Q. Thus Q will be expressed as a function of *one* variable x, say, $Q = f(x)$. Write the domain of this function.
6. Use the methods of Sections 3.1 and 3.3 to find the *absolute* maximum or minimum value of f. In particular, if the domain of f is a closed interval, then the Closed Interval Method in Section 3.1 can be used.

Example 1

A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

Example 1 – *Solution*

In order to get a feeling for what is happening in this problem, let's experiment with some special cases. Figure 1 (not to scale) shows three possible ways of laying out the 2400 ft of fencing.

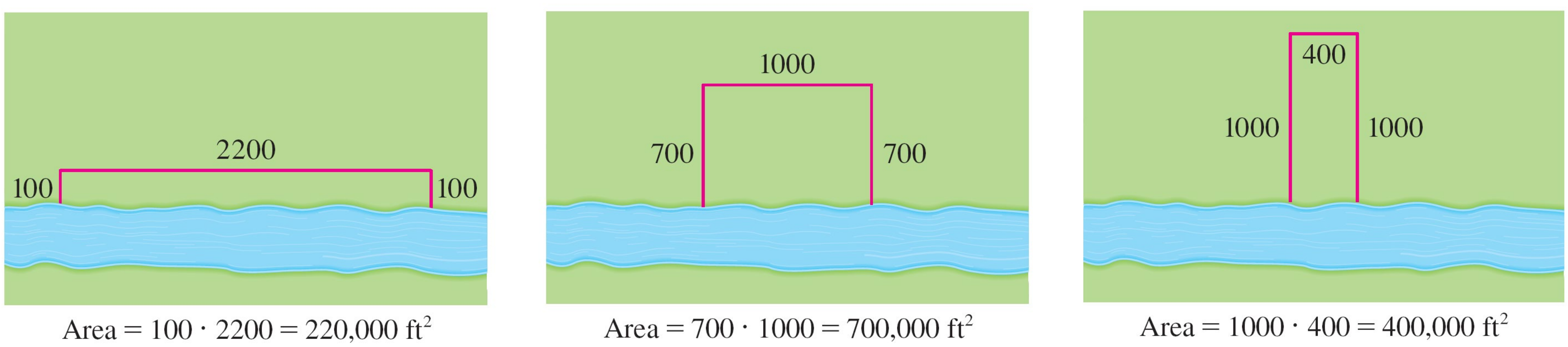


Figure 1

Example 1 – *Solution*

cont'd

We see that when we try shallow, wide fields or deep, narrow fields, we get relatively small areas. It seems plausible that there is some intermediate configuration that produces the largest area.

Figure 2 illustrates the general case. We wish to maximize the area A of the rectangle.

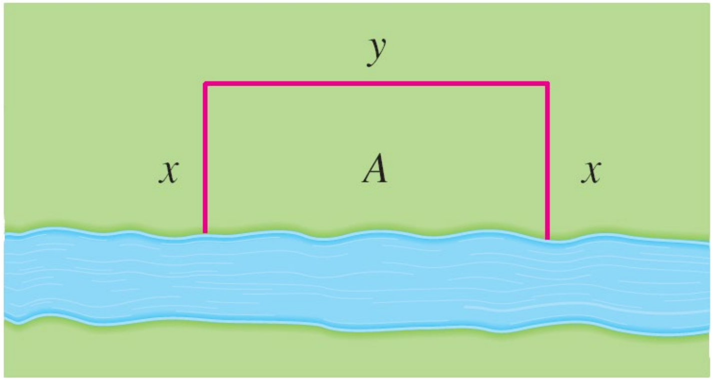


Figure 2

Example 1 – *Solution*

cont'd

Let x and y be the depth and width of the rectangle (in feet). Then we express A in terms of x and y:

$$A = xy$$

We want to express A as a function of just one variable, so we eliminate y by expressing it in terms of x. To do this we use the given information that the total length of the fencing is 2400 ft.

Thus

$$2x + y = 2400$$

From this equation we have $y = 2400 - 2x$, which gives

$$A = x(2400 - 2x) = 2400x - 2x^2$$

Example 1 – *Solution*

cont'd

Note that $x \geq 0$ and $x \leq 1200$ (otherwise $A < 0$). So the function that we wish to maximize is

$$A(x) = 2400x - 2x^2 \qquad 0 \leq x \leq 1200$$

The derivative is $A'(x) = 2400 - 4x$, so to find the critical numbers we solve the equation

$$2400 - 4x = 0$$

which gives $x = 600$.

The maximum value of A must occur either at this critical number or at an endpoint of the interval.

Example 1 – *Solution*

cont'd

Since $A(0) = 0$, $A(600) = 720{,}000$, and $A(1200) = 0$, the Closed Interval Method gives the maximum value as $A(600) = 720{,}000$.

[Alternatively, we could have observed that $A''(x) = -4 < 0$ for all x, so A is always concave downward and the local maximum at $x = 600$ must be an absolute maximum.]

Thus the rectangular field should be 600 ft deep and 1200 ft wide.

Optimization Problems

First Derivative Test for Absolute Extreme Values Suppose that c is a critical number of a continuous function f defined on an interval.

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

Applications to Business and Economics

Applications to Business and Economics

We know that if $C(x)$, the **cost function**, is the cost of producing x units of a certain product, then the **marginal cost** is the rate of change of C with respect to x.

In other words, the marginal cost function is the derivative, $C'(x)$, of the cost function.

Now let's consider marketing. Let $p(x)$ be the price per unit that the company can charge if it sells x units.

Then p is called the **demand function** (or **price function**) and we would expect it to be a decreasing function of x.

Applications to Business and Economics

If x units are sold and the price per unit is $p(x)$, then the total revenue is

$$R(x) = xp(x)$$

and R is called the **revenue function**.

The derivative R' of the revenue function is called the **marginal revenue function** and is the rate of change of revenue with respect to the number of units sold.

Applications to Business and Economics

If x units are sold, then the total profit is

$$P(x) = R(x) - C(x)$$

and P is called the **profit function**.

The **marginal profit function** is P', the derivative of the profit function.

Example 6

A store has been selling 200 Blu-ray disc players a week at \$350 each. A market survey indicates that for each \$10 rebate offered to buyers, the number of units sold will increase by 20 a week. Find the demand function and the revenue function. How large a rebate should the store offer to maximize its revenue?

Solution:

If x is the number of Blu-ray players sold per week, then the weekly increase in sales is $x - 200$.

For each increase of 20 units sold, the price is decreased by \$10.

Example 6 – *Solution*

cont'd

So for each additional unit sold, the decrease in price will be $\frac{1}{20} \times 10$ and the demand function is

$$p(x) = 350 - \frac{10}{20}(x - 200) = 450 - \frac{1}{2}x$$

The revenue function is

$$R(x) = xp(x) = 450x - \frac{1}{2}x^2$$

Since $R'(x) = 450 - x$, we see that $R'(x) = 0$ when $x = 450$. This value of x gives an absolute maximum by the First Derivative Test (or simply by observing that the graph of R is a parabola that opens downward).

Example 6 – *Solution*

cont'd

The corresponding price is

$$p(450) = 450 - \tfrac{1}{2}(450) = 225$$

and the rebate is 350 – 225 = 125.

Therefore, to maximize revenue, the store should offer a rebate of $125.

3.8 Newton’s Method

Newton's Method

Suppose that a car dealer offers to sell you a car for \$18,000 or for payments of \$375 per month for five years. You would like to know what monthly interest rate the dealer is, in effect, charging you.

To find the answer, you have to solve the equation

$$\boxed{1} \qquad 48x(1 + x)^{60} - (1 + x)^{60} + 1 = 0$$

We can find an *approximate* solution to Equation 1 by plotting the left side of the equation.

Newton's Method

Using a graphing device, and after experimenting with viewing rectangles, we produce the graph in Figure 1.

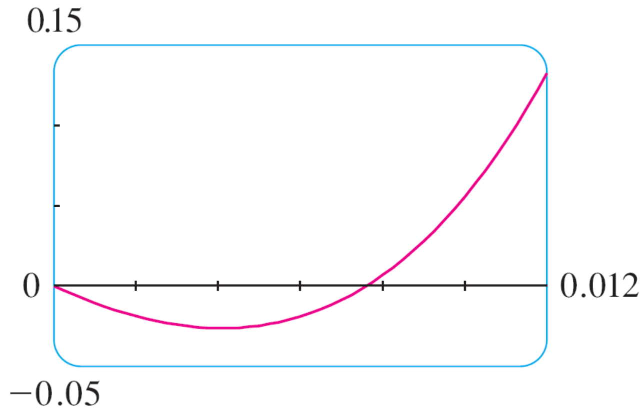


Figure 1

We see that in addition to the solution $x = 0$, which doesn't interest us, there is a solution between 0.007 and 0.008. Zooming in shows that the root is approximately 0.0076. If we need more accuracy we could zoom in repeatedly, but that becomes tiresome.

Newton's Method

A faster alternative is to use a numerical rootfinder on a calculator or computer algebra system. If we do so, we find that the root, correct to nine decimal places, is 0.007628603.

How do those numerical rootfinders work? They use a variety of methods, but most of them make some use of **Newton's method**, also called the **Newton-Raphson method**.

We will explain how this method works, partly to show what happens inside a calculator or computer, and partly as an application of the idea of linear approximation.

Newton's Method

The geometry behind Newton's method is shown in Figure 2, where the root that we are trying to find is labeled *r*.

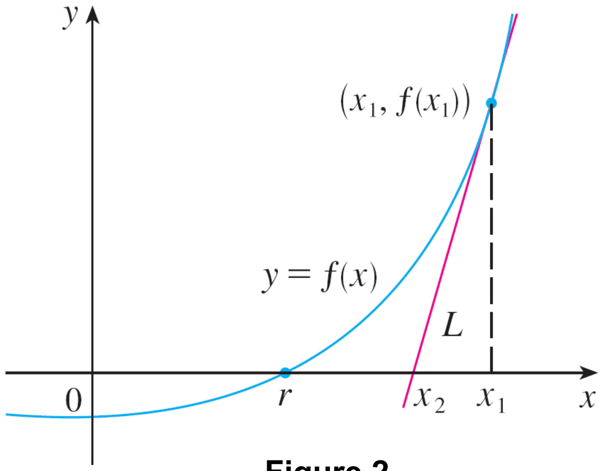


Figure 2

We start with a first approximation x_1, which is obtained by guessing, or from a rough sketch of the graph of *f*, or from a computer-generated graph of *f*.

Newton's Method

Consider the tangent line L to the curve $y = f(x)$ at the point $(x_1, f(x_1))$ and look at the x-intercept of L, labeled x_2.

The idea behind Newton's method is that the tangent line is close to the curve and so its x-intercept, x_2, is close to the x-intercept of the curve (namely, the root r that we are seeking). Because the tangent is a line, we can easily find its x-intercept.

To find a formula for x_2 in terms of x_1 we use the fact that the slope of L is $f'(x_1)$, so its equation is

$$y - f(x_1) = f'(x_1)(x - x_1)$$

Newton’s Method

Since the x-intercept of L is x_2, we set $y = 0$ and obtain

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

If $f'(x_1) \neq 0$, we can solve this equation for x_2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

We use x_2 as a second approximation to r.

Next we repeat this procedure with x_1 replaced by the second approximation x_2, using the tangent line at $(x_2, f(x_2))$.

Newton's Method

This gives a third approximation:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

If we keep repeating this process, we obtain a sequence of approximations x_1, x_2, x_3, x_4, . . . as shown in Figure 3.

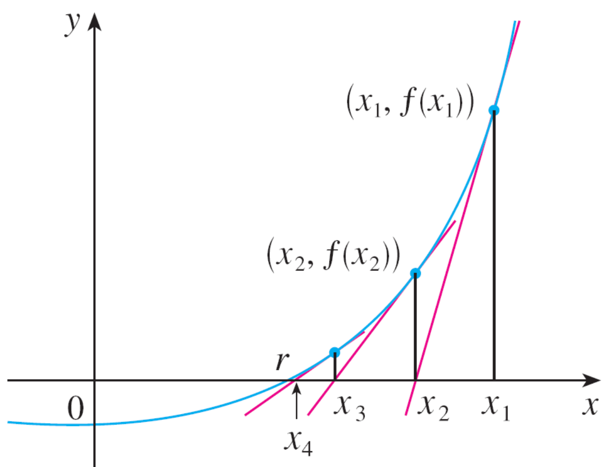


Figure 3

Newton's Method

In general, if the nth approximation is x_n and $f'(x_n) \neq 0$, then the next approximation is given by

$$\boxed{2} \qquad \boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

If the numbers x_n become closer and closer to r as n becomes large, then we say that the sequence *converges* to r and we write

$$\lim_{n \to \infty} x_n = r$$

Example 1

Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation $x^3 - 2x - 5 = 0$.

Solution:

We apply Newton's method with

$$f(x) = x^3 - 2x - 5 \qquad \text{and} \qquad f'(x) = 3x^2 - 2$$

Newton himself used this equation to illustrate his method and he chose $x_1 = 2$ after some experimentation because $f(1) = -6$, $f(2) = -1$, and $f(3) = 16$.

Example 1 – *Solution*

cont'd

Equation 2 becomes

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

With n = 1 we have

$$x_2 = x_1 - \frac{x_1^3 - 2x_1 - 5}{3x_1^2 - 2}$$

$$= 2 - \frac{2^3 - 2(2) - 5}{3(2)^2 - 2}$$

$$= 2.1$$

Example 1 – *Solution*

cont'd

Then with n = 2 we obtain

$$x_3 = x_2 - \frac{x_2^3 - 2x_2 - 5}{3x_2^2 - 2}$$

$$= 2.1 - \frac{(2.1)^3 - 2(2.1) - 5}{3(2.1)^2 - 2}$$

$$\approx 2.0946$$

It turns out that this third approximation $x_3 \approx 2.0946$ is accurate to four decimal places.

3.9 Antiderivatives

Antiderivatives

A physicist who knows the velocity of a particle might wish to know its position at a given time.

An engineer who can measure the variable rate at which water is leaking from a tank wants to know the amount leaked over a certain time period.

A biologist who knows the rate at which a bacteria population is increasing might want to deduce what the size of the population will be at some future time.

Antiderivatives

In each case, the problem is to find a function *F* whose derivative is a known function *f*. If such a function *F* exists, it is called an *antiderivative* of *f*.

Definition A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Antiderivatives

For instance, let $f(x) = x^2$. It isn't difficult to discover an antiderivative of f if we keep the Power Rule in mind. In fact, if $F(x) = \ x^3$, $\frac{1}{3}$ then $F'(x) = x^2 = f(x)$.

But the function $G(x) = \frac{1}{3}x^3 + 100$ also satisfies $G'(x) = x^2$. Therefore both F and G are antiderivatives of f.

Indeed, any function of the form $H(x) = \frac{1}{3}x^3 + C$, where C is a constant, is an antiderivative of f.

Antiderivatives

1 **Theorem** If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

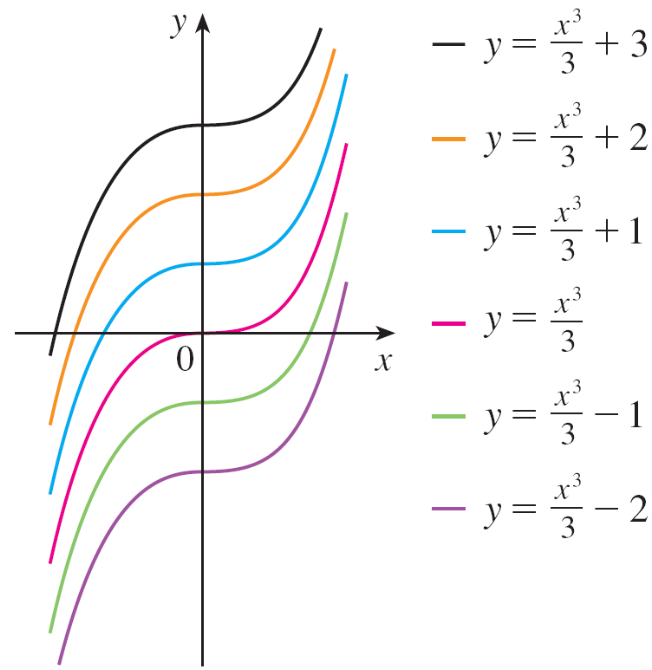
where C is an arbitrary constant.

Going back to the function $f(x) = x^2$, we see that the general antiderivative of f is $x^3/3 + C$.

Antiderivatives

By assigning specific values to the constant *C*, we obtain a family of functions whose graphs are vertical translates of one another (see Figure 1).

This makes sense because each curve must have the same slope at any given value of *x*.



Members of the family of antiderivatives of $f(x) = x^2$

Figure 1

Example 1

Find the most general antiderivative of each of the following functions.

(a) $f(x) = \sin x$ **(b)** $f(x) = x^n$, $n \geq 0$ **(c)** $f(x) = x^{-3}$

Solution:

(a) If $F(x) = -\cos x$, then $F'(x) = \sin x$, so an antiderivative of $\sin x$ is $-\cos x$. By Theorem 1, the most general antiderivative is $G(x) = -\cos x + C$.

Example 1 – *Solution*

cont'd

(b) We use the Power Rule to discover an antiderivative of x^n:

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)x^n}{n+1} = x^n$$

Thus the general antiderivative of $f(x) = x^n$ is

$$F(x) = \frac{x^{n+1}}{n+1} + C$$

This is valid for $n \geq 0$ because $f(x) = x^n$ then is defined on an interval.

Example 1 – *Solution*

cont'd

(c) If we put $n = -3$ in part (b) we get the particular antiderivative $F(x) = x^{-2}/(-2)$ by the same calculation. But notice that $f(x) = x^{-3}$ is not defined at $x = 0$.

Thus Theorem 1 tells us only that the general antiderivative of f is on any interval that does not contain 0. So the general antiderivative of is $f(x) = 1/x^3$ is

$$F(x) = \begin{cases} -\dfrac{1}{2x^2} + C_1 & \text{if } x > 0 \\ -\dfrac{1}{2x^2} + C_2 & \text{if } x < 0 \end{cases}$$

Antiderivatives

As in Example 1, every differentiation formula, when read from right to left, gives rise to an antidifferentiation formula. In Table 2 we list some particular antiderivatives.

2 Table of Antidifferentiation Formulas

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\cos x$	$\sin x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sin x$	$-\cos x$
$x^n \ (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\sec^2 x$	$\tan x$
		$\sec x \tan x$	$\sec x$

To obtain the most general antiderivative from the particular ones in Table 2, we have to add a constant (or constants), as in Example 1.

Antiderivatives

Each formula in the table is true because the derivative of the function in the right column appears in the left column.

In particular, the first formula says that the antiderivative of a constant times a function is the constant times the antiderivative of the function.

The second formula says that the antiderivative of a sum is the sum of the antiderivatives. (We use the notation $F' = f$, $G' = g$.)

Antiderivatives

An equation that involves the derivatives of a function is called a **differential equation**.

The general solution of a differential equation involves an arbitrary constant (or constants).

However, there may be some extra conditions given that will determine the constants and therefore uniquely specify the solution.

Rectilinear Motion

Rectilinear Motion

Antidifferentiation is particularly useful in analyzing the motion of an object moving in a straight line. Recall that if the object has position function $s = f(t)$, then the velocity function is $v(t) = s'(t)$.

This means that the position function is an antiderivative of the velocity function.

Likewise, the acceleration function is $a(t) = v'(t)$, so the velocity function is an antiderivative of the acceleration.

If the acceleration and the initial values $s(0)$ and $v(0)$ are known, then the position function can be found by antidifferentiating twice.

Example 6

A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = –6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

Solution:

Since $v'(t) = a(t) = 6t + 4$, antidifferentiation gives

$$v(t) = 6\frac{t^2}{2} + 4t + C$$

$$= 3t^2 + 4t + C$$

Example 6 – *Solution*

cont'd

Note that $v(0) = C$. But we are given that $v(0) = -6$, so $C = -6$ and

$$v(t) = 3t^2 + 4t - 6$$

Since $v(t) = s'(t)$, s is the antiderivative of v:

$$s(t) = 3\frac{t^3}{3} + 4\frac{t^2}{2} - 6t + D \quad = t^3 + 2t^2 - 6t + D$$

This gives $s(0) = D$. We are given that $s(0) = 9$, so $D = 9$ and the required position function is

$$s(t) = t^3 + 2t^2 - 6t + 9$$