VV156 Honors Calculus II
Fall 2021 HW7 Solutions
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Exercise 7.1

i) The function $f(x) = 1/\sqrt[5]{x} = x^{-1/5}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies. $\int_1^\infty x^{-1/5}dx = \lim_{t\to\infty}\int_1^t x^{-1/5}dx = \lim_{t\to\infty}\left[\frac{5}{4}x^{4/5}\right]_1^t = \lim_{t\to\infty}\left(\frac{5}{4}t^{4/5} - \frac{5}{4}\right) = \infty$, so $\sum_{n=1}^\infty 1/\sqrt[5]{n}$ diverges.

ii) The function $f(x) = \frac{1}{(2x+1)^3}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies. $\int_1^\infty \frac{1}{(2x+1)^3}dx = \lim_{t\to\infty}\int_1^t \frac{1}{(2x+1)^3}dx = \lim_{t\to\infty}\left[-\frac{1}{4}\frac{1}{(2x+1)^2}\right]_1^t = \lim_{t\to\infty}\left(-\frac{1}{4(2t+1)^2} + \frac{1}{36}\right) = \frac{1}{36}$. Since this improper integral is convergent, the series $\sum_{n=1}^\infty \frac{1}{(2n+1)^3}$ is also convergent by the Integral Test.

iii) The function $f(x) = \frac{x}{x^2+1}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies. $\int_1^\infty \frac{x}{x^2+1}dx = \lim_{t\to\infty}\int_1^t \frac{x}{x^2+1}dx = \lim_{t\to\infty}\left[\frac{1}{2}\ln(x^2+1)\right]_1^t = \frac{1}{2}\lim_{t\to\infty}\left[\ln(t^2+1) - \ln 2\right] = \infty$. Since this improper integral is divergent, the series $\sum_{n=1}^\infty \frac{n}{n^2+1}$ is also divergent by the Integral Test.

iv) The function $f(x) = x^2e^{-x^3}$ is continuous, positive, and decreasing ($\star$) on $[1, \infty)$, so the Integral Test applies. $\int_1^\infty x^2e^{-x^3}dx = \lim_{t\to\infty}\int_1^t x^2e^{-x^3}dx = \lim_{t\to\infty}\left[-\frac{1}{3}e^{-x^3}\right]_1^t = -\frac{1}{3}\lim_{t\to\infty}\left(e^{-t^3} - e^{-1}\right) = -\frac{1}{3}\left(0 - \frac{1}{e}\right) = \frac{1}{3e}$. Since this improper integral is convergent, the series $\sum_{n=1}^\infty n^2e^{-n^3}$ is also convergent by the Integral Test. $f'(x) = x^2e^{-x^3}(-3x^2) + e^{-x^3}(2x) = xe^{-x^3}(-3x^3+2) = \frac{x(2-3x^3)}{e^{x^3}} < 0$ for $x > 1$

Exercise 7.2 Clearly, if $p < 0$ then the series diverges, since $\lim_{n\to\infty}\frac{1}{n^p\ln n} = \infty$. If $0 \le p \le 1$, then $n^p\ln n \le n\ln n \Rightarrow \frac{1}{n^p\ln n} \ge \frac{1}{n\ln n}$ and $\sum_{n=2}^\infty \frac{1}{n\ln n}$ diverges (Exercise 11.3.21), so $\sum_{n=2}^\infty \frac{1}{n^p\ln n}$ diverges. If $p > 1$, use the Limit Comparison Test with $a_n = \frac{1}{n^p\ln n}$ and $b_n = \frac{1}{n^p} \cdot \sum_{n=2}^\infty b_n$ converges, and $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1}{\ln n} = 0$, so $\sum_{n=2}^\infty \frac{1}{n^p\ln n}$ also converges. (Or use the Comparison Test, since $n^p\ln n > n^p$ for $n > e$.) In summary, the series converges if and only if $p > 1$.

Exercise 7.3 Since $\sum a_n$ converges, $\lim_{n\to\infty} a_n = 0$, so there exists N such that $|a_n - 0| < 1$ for all $n > N \Rightarrow 0 \le a_n < 1$ for all $n > N \Rightarrow 0 \le a_n^2 \le a_n$. Since $\sum a_n$ converges, so does $\sum a_n^2$ by the Comparison Test.

Exercise 7.4

n	S_n	$\mathscr{S}(S_n)$	$\mathscr{S}^{\circ 2}(S_n)$	$\mathscr{S}^{\circ 3}(S_n)$
0	1	–	–	–
1	0.666666668	0.791666668	–	–
2	0.866666668	0.783333333	0.785526315	–
3	0.723809525	0.786309525	0.785362555	0.78539984
4	0.834920635	0.784920635	0.78541083	0.785397715
5	0.744011545	0.78567821	0.785392823	0.785398308
6	0.82093462	0.785220335	0.78540071	0.78539811
7	0.754267955	0.785517955	0.78539683	0.785398185
8	0.813091483	0.785313705	0.785398915	0.785398153
9	0.760459905	0.785459905	0.785397715	0.785398168
10	0.808078953	0.78535168	0.785398443	0.78539816
11	0.764600693	0.785434025	0.78539798	0.785398165
12	0.804600693	0.785369923	0.785398285	0.785398163

$\mathscr{S}^{\circ 3}(S_3) = 0.78539984$

Exercise 7.5

i)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{5^{n+1}}\cdot\frac{5^n}{n}\right| = \lim_{n\to\infty}\left|\frac{1}{5}\cdot\frac{n+1}{n}\right| = \frac{1}{5}\lim_{n\to\infty}\frac{1+1/n}{1} = \frac{1}{5}(1) = \frac{1}{5} < 1,$$

so it is absolutely convergent by the Ratio Test.

ii) $b_n = \frac{n}{n^2+4} > 0$ for $n \geq 1, \{b_n\}$ is decreasing for $n \geq 2$, and $\lim_{n\to\infty} b_n = 0$, so $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+4}$ converges by the Alternating Series Test. To determine absolute convergence, choose $a_n = \frac{1}{n}$ to get $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1/n}{n/(n^2+4)} = \lim_{n\to\infty}\frac{n^2+4}{n^2} = \lim_{n\to\infty}\frac{1+4/n^2}{1} = 1 > 0$, so $\sum_{n=1}^{\infty}\frac{n}{n^2+4}$ diverges by the Limit Comparison Test with the harmonic series. Thus, the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+4}$ is conditionally convergent.

iii) $\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}$ converges by the Alternating Series Test since $\lim_{n\to\infty}\frac{1}{\ln n} = 0$ and $\left\{\frac{1}{\ln n}\right\}$ is decreasing. Now $\ln n < n$, so $\frac{1}{\ln n} > \frac{1}{n}$, and since $\sum_{n=2}^{\infty}\frac{1}{n}$ is the divergent (partial) harmonic series, $\sum_{n=2}^{\infty}\frac{1}{\ln n}$ diverges by the Comparison Test. Thus, $\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}$ is conditionally convergent.

iv)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{[2(n+1)]!}{[(n+1)!]^2}\cdot\frac{(n!)^2}{(2n)!}\right| = \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \lim_{n\to\infty}\frac{(2+2/n)(2+1/n)}{(1+1/n)(1+1/n)}$$

$$= \frac{2\cdot 2}{1\cdot 1} = 4 > 1,$$

so the series $\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}$ diverges by the Ratio Test.

Exercise 7.6

i) If $a_n = (-1)^n n x^n$, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)x^{n+1}}{(-1)^n n x^n}\right| = \lim_{n\to\infty}\left|(-1)\frac{n+1}{n}x\right| = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)|x|\right] = |x|$. By the Ratio Test, the series $\sum_{n=1}^{\infty}(-1)^n n x^n$ converges when $|x| < 1$, so the radius of convergence $R = 1$. Now we'll check the endpoints, that is, $x = \pm 1$. Both series $\sum_{n=1}^{\infty}(-1)^n n(\pm 1)^n = \sum_{n=1}^{\infty}(\mp 1)^n n$ diverge by the Test for Divergence since $\lim_{n\to\infty}|(\mp 1)^n n| = \infty$. Thus, the interval of convergence is $I = (-1, 1)$.

ii) If $a_n = \frac{(-1)^n x^n}{n^2}$, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(-1)^n x^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)xn^2}{(n+1)^2}\right| = \lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^2|x|\right] = 1^2\cdot|x| = |x|$ When $x = 1$, the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ converges by the Alternating Series Test. When $x = -1$, the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges since it is a p-series with $p = 2 > 1$. Thus, the interval of convergence is $[-1, 1]$.

iii) If $a_n = (-1)^n\frac{x^n}{4^n \ln n}$, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{4^{n+1}\ln(n+1)}\cdot\frac{4^n\ln n}{x^n}\right| = \frac{|x|}{4}\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)} = \frac{|x|}{4}\cdot 1$ [by l'Hospital's Rule] $= \frac{|x|}{4}$. By the Ratio Test, the series converges when $\frac{|x|}{4} < 1 \Leftrightarrow\Leftrightarrow |x| < 4$, so $R = 4$. When $x = -4$, $\sum_{n=2}^{\infty}(-1)^n\frac{x^n}{4^n\ln n} = \sum_{n=2}^{\infty}\frac{[(-1)(-4)]^n}{4^n\ln n} = \sum_{n=2}^{\infty}\frac{1}{\ln n}$. Since $\ln n < n$ for $n \geq 2$, $\frac{1}{\ln n} > \frac{1}{n}$ and $\sum_{n=2}^{\infty}\frac{1}{n}$ is the divergent harmonic series (without the $n = 1$ term), $\sum_{n=2}^{\infty}\frac{1}{\ln n}$ is divergent by the Comparison Test. When $x = 4$, $\sum_{n=2}^{\infty}(-1)^n\frac{x^n}{4^n\ln n} = \sum_{n=2}^{\infty}(-1)^n\frac{1}{\ln n}$, which converges by the Alternating Series Test. Thus, $I = (-4, 4]$.

iv) If $a_n = \frac{x^{2n}}{n(\ln n)^2}$, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{(n+1)[\ln(n+1)]^2}\cdot\frac{n(\ln n)^2}{x^{2n}}\right| = |x^2|\lim_{n\to\infty}\frac{n(\ln n)^2}{(n+1)[\ln(n+1)]^2} = x^2$ By the Ratio Test, the series $\sum_{n=2}^{\infty}\frac{x^{2n}}{n(\ln n)^2}$ converges when $x^2 < 1 \Leftrightarrow |x| < 1$, so $R = 1$. When $x = \pm 1, x^{2n} = 1$, the series $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ converges by the Integral Test (see Exercise 11.3.22). Thus, the interval of convergence is $I = [-1, 1]$.

Exercise 7.7

i) $f(x) = \frac{3}{x^2-x-2} = \frac{3}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} \Rightarrow 3 = A(x+1) + B(x-2)$. Let $x = 2$ to get $A = 1$ and $x = -1$ to get $B = -1$. Thus

$$\frac{3}{x^2-x-2} = \frac{1}{x-2} - \frac{1}{x+1} = \frac{1}{-2}\left(\frac{1}{1-(x/2)}\right) - \frac{1}{1-(-x)} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n - \sum_{n=0}^{\infty}(-x)^n$$
$$= \sum_{n=0}^{\infty}\left[-\frac{1}{2}\left(\frac{1}{2}\right)^n - 1(-1)^n\right]x^n = \sum_{n=0}^{\infty}\left[(-1)^{n+1} - \frac{1}{2^{n+1}}\right]x^n$$

We represented f as the sum of two geometric series; the first converges for $x \in (-2, 2)$ and the second converges for $(-1, 1)$. Thus, the sum converges for $x \in (-1, 1) = I$.

ii) $f(x) = \frac{x+2}{2x^2-x-1} = \frac{x+2}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1} \Rightarrow x + 2 = A(x-1) + B(2x+1)$. Let $x = 1$ to get $3 = 3B \Rightarrow B = 1$ and $x = -\frac{1}{2}$ to get $\frac{3}{2} = -\frac{3}{2}A \Rightarrow A = -1$. Thus,

$$\frac{x+2}{2x^2-x-1} = \frac{-1}{2x+1} + \frac{1}{x-1} = -1\left(\frac{1}{1-(-2x)}\right) - 1\left(\frac{1}{1-x}\right) = -\sum_{n=0}^{\infty}(-2x)^n - \sum_{n=0}^{\infty}x^n$$
$$= -\sum_{n=0}^{\infty}\left[(-2)^n + 1\right]x^n$$

We represented f as the sum of two geometric series; the first converges for $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ and the second converges for $(-1, 1)$. Thus, the sum converges for $x \in \left(-\frac{1}{2}, \frac{1}{2}\right) = I$.

Exercise 7.8

i)

$$f(x) = \ln(5-x) = -\int \frac{dx}{5-x} = -\frac{1}{5}\int \frac{dx}{1-x/5} = -\frac{1}{5}\int \left[\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n\right] dx$$

$$= C - \frac{1}{5}\sum_{n=0}^{\infty}\frac{x^{n+1}}{5^n(n+1)} = C - \sum_{n=1}^{\infty}\frac{x^n}{n5^n}$$

Putting $x = 0$, we get $C = \ln 5$. The series converges for $|x/5| < 1 \Leftrightarrow |x| < 5$, so $R = 5$.

ii)

$$f(x) = x^2\tan^{-1}\left(x^3\right) = x^2\sum_{n=0}^{\infty}(-1)^n\frac{\left(x^3\right)^{2n+1}}{2n+1}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+3+2}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+5}}{2n+1}$$

for $|x^3| < 1 \Leftrightarrow |x| < 1$, so $R = 1$.

iii) We know that $\frac{1}{1+4x} = \frac{1}{1-(-4x)} = \sum_{n=0}^{\infty}(-4x)^n$. Differentiating, we get

$$\frac{-4}{(1+4x)^2} = \sum_{n=1}^{\infty}(-4)^n n x^{n-1} = \sum_{n=0}^{\infty}(-4)^{n+1}(n+1)x^n, \text{ so}$$

$$f(x) = \frac{x}{(1+4x)^2} = \frac{-x}{4}\cdot\frac{-4}{(1+4x)^2} = \frac{-x}{4}\sum_{n=0}^{\infty}(-4)^{n+1}(n+1)x^n = \sum_{n=0}^{\infty}(-1)^n4^n(n+1)x^{n+1}$$

for $|-4x| < 1 \quad \Leftrightarrow \quad |x| < \frac{1}{4}$, so $R = \frac{1}{4}$.

iv)

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n, \text{ so}$$

$$\frac{d}{dx}\left(\frac{1}{(1-x)^2}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty}(n+1)x^n\right) \Rightarrow \frac{2}{(1-x)^3} = \sum_{n=1}^{\infty}(n+1)nx^{n-1}. \text{ Thus,}$$

$$f(x) = \frac{x^2-x}{(1-x)^3} = \frac{x^2}{(1-x)^3} - \frac{x}{(1-x)^3} = \frac{x^2}{2}\cdot\frac{2}{(1-x)^3} - \frac{x}{2}\cdot\frac{2}{(1-x)^3}$$

$$= \frac{x^2}{2}\sum_{n=1}^{\infty}(n+1)nx^{n-1} - \frac{x}{2}\sum_{n=1}^{\infty}(n+1)nx^{n-1}$$

$$= \sum_{n=1}^{\infty}\frac{(n+1)n}{2}x^{n+1} - \sum_{n=1}^{\infty}\frac{(n+1)n}{2}x^n$$

$$= \sum_{n=2}^{\infty}\frac{n(n-1)}{2}x^n - \sum_{n=1}^{\infty}\frac{(n+1)n}{2}x^n \quad \text{[make the exponents on } x \text{ equal by changing an index]}$$

$$= \sum_{n=2}^{\infty}\frac{n^2-n}{2}x^n - x - \sum_{n=2}^{\infty}\frac{n^2+n}{2}x^n \quad \text{[make the starting values equal]}$$

$$= -x - \sum_{n=2}^{\infty}nx^n \text{ with } R = 1.$$

Exercise 7.9

i)

n	$f^{(n)}(x)$	$f^{(n)}(-2)$
0	$x - x^3$	6
1	$1 - 3x^2$	-11
2	$-6x$	12
3	-6	-6
4	0	0
5	0	0
$\vdots$	$\vdots$	$\vdots$

$f^{(n)}(x) = 0$ for $n \geq 4$, so f has a finite series expansion about $a = -2$.

$$\begin{aligned} f(x) = x - x^3 &= \sum_{n=0}^{3} \frac{f^{(n)}(-2)}{n!}(x+2)^n \\ &= \frac{6}{0!}(x+2)^0 + \frac{-11}{1!}(x+2)^1 + \frac{12}{2!}(x+2)^2 + \frac{-6}{3!}(x+2)^3 \\ &= 6 - 11(x+2) + 6(x+2)^2 - (x+2)^3 \end{aligned}$$

A finite series converges for all x, so $R = \infty$.

ii)

n	$f^{(n)}(x)$	$f^{(n)}(-3)$
0	$1/x$	$-1/3$
1	$-1/x^2$	$-1/3^2$
2	$2/x^3$	$-2/3^3$
3	$-6/x^4$	$-6/3^4$
4	$24/x^5$	$-24/3^5$
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned} f(x) = \frac{1}{x} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(-3)}{n!}(x+3)^n \\ =& \frac{-1/3}{0!}(x+3)^0 + \frac{-1/3^2}{1!}(x+3)^1 + \frac{-2/3^3}{2!}(x+3)^2 \\ &+ \frac{-6/3^4}{3!}(x+3)^3 + \frac{-24/3^5}{4!}(x+3)^4 + \cdots \\ =& \sum_{n=0}^{\infty} \frac{-n!/3^{n+1}}{n!}(x+3)^n = -\sum_{n=0}^{\infty} \frac{(x+3)^n}{3^{n+1}} \end{aligned}$$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x+3)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{(x+3)^n}\right| = \lim_{n\to\infty} \frac{|x+3|}{3} = \frac{|x+3|}{3} < 1$ for convergence, so $|x+3| < 3$ and $R = 3$.

iii)

n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$
0	$\sin x$	1
1	$\cos x$	0
2	$-\sin x$	-1
3	$-\cos x$	0
4	$\sin x$	1
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned} f(x) = \sin x &= \sum_{k=0}^{\infty} \frac{f^{(k)}(\pi/2)}{k!}\left(x - \frac{\pi}{2}\right)^k \\ &= 1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} - \frac{(x-\pi/2)^6}{6!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(x-\pi/2)^{2n}}{(2n)!} \end{aligned}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{|x-\pi/2|^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{|x-\pi/2|^{2n}}\right] = \lim_{n\to\infty}\frac{|x-\pi/2|^2}{(2n+2)(2n+1)} = 0 < 1$$

for all x, so $R = \infty$

iv)

n	$f^{(n)}(x)$	$f^{(n)}(16)$
0	$\sqrt{x}$	4
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{2}\cdot\frac{1}{4}$
2	$-\frac{1}{4}x^{-3/2}$	$-\frac{1}{4}\cdot\frac{1}{4^3}$
3	$\frac{3}{8}x^{-5/2}$	$\frac{3}{8}\cdot\frac{1}{4^5}$
4	$-\frac{15}{16}x^{-7/2}$	$-\frac{15}{16}\cdot\frac{1}{4^7}$
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned} f(x) = \sqrt{x} &= \sum_{n=0}^{\infty}\frac{f^{(n)}(16)}{n!}(x-16)^n \\ &= \frac{4}{0!}(x-16)^0 + \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{1!}(x-16)^1 - \frac{1}{4}\cdot\frac{1}{4^3}\cdot\frac{1}{2!}(x-16)^2 \\ &+ \frac{3}{8}\cdot\frac{1}{4^5}\cdot\frac{1}{3!}(x-16)^3 - \frac{15}{16}\cdot\frac{1}{4^7}\cdot\frac{1}{4!}(x-16)^4 + \cdots \\ &= 4 + \frac{1}{8}(x-16) + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n 4^{2n-1} n!}(x-16)^n \\ &= 4 + \frac{1}{8}(x-16) + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^{5n-2} n!}(x-16)^n \end{aligned}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n 1\cdot 3\cdot 5\cdots(2n-1)(x-16)^{n+1}}{2^{5n+3}(n+1)!}\cdot\frac{2^{5n-2}n!}{(-1)^{n-1}1\cdot 3\cdot 5\cdot\cdots(2n-3)(x-16)^n}\right|$$
$$= \lim_{n\to\infty}\frac{(2n-1)|x-16|}{2^5(n+1)} = \frac{|x-16|}{32}\lim_{n\to\infty}\frac{2-1/n}{1+1/n} = \frac{|x-16|}{32}\cdot 2$$
$$= \frac{|x-16|}{16} < 1 \quad \text{for convergence, so } |x-16| < 16 \text{ and } R = 16.$$

Exercise 7.10

i)

Linear equation with constant coefficients $\mathrm{a}_0 t^{(\mathrm{n})} + \mathrm{a}_1 t^{(\mathrm{n}-1)} + \ldots + \mathrm{a}_{\mathrm{n}-1} t' + \mathrm{a}_\mathrm{n} t = f(t)$
Let's compose the characteristic equation $\mathrm{a}_0\lambda^\mathrm{n} + \mathrm{a}_1\lambda^{\mathrm{n}-1} + \ldots + \mathrm{a}_{\mathrm{n}-1}\lambda + \mathrm{a}_\mathrm{n} = 0:$

$$\lambda^2 + 4\lambda + 5 = 0$$

Find roots $\lambda_1 \ldots \lambda_\mathrm{n}$, where $\mathrm{k} -$ multiplicity of the root, $\tau -$ summand for the root :

$$\lambda^2 + 4\lambda + 5 \to \lambda_{1,2} = \pm i - 2$$

General solution there is a sum of summands of the form:

$$t = \sum P_{\mathrm{k}-1}(t)e^{\alpha t}\sin\beta t + Q_{\mathrm{k}-1}(t)e^{\alpha t}\cos\beta t$$

Particular solution for the right side $f_1 + \ldots + f_p = \dfrac{t\cos(t)}{e^{2t}} + e^{5t}$

equal to the sum of particular solutions for the right-hand sides $f_1, \ldots, f_p = e^{5t_0}, \dfrac{t_1\cos(t_1)}{e^{2t_1}}$

For the right side:

A particular solution is sought in the form:

$$x_\mathrm{i} = t^s e^{\alpha t}\left(R_\mathrm{m}(t)\cos\beta t + T_\mathrm{m}(t)\sin\beta t\right)$$

Private solution for e^{5t_0} : $\alpha + \beta i = 5 \to s = 0$ $x = Ae^{5t_0}$ $\downarrow ?\{1\}$ Calculate derivatives:

$$x' = 5Ae^{5t_0}$$
$$x'' = 25Ae^{5t_0}$$

Substitute in original equation:

$$50Ae^{5t_0} = e^{5t_0}$$

Find coefficients:

$$50A = 1 \to A = \frac{1}{50}$$

Substitute in $\{1\}$:

$$x = \frac{e^{5t_0}}{50}$$

Private solution for $\dfrac{t_1\cos(t_1)}{e^{2t_1}}$:

$$\alpha + \beta i = i - 2 \to s = 1$$

$$x = \frac{t_1\left((Ct_1 + D)\sin(t_1) + (At_1 + B)\cos(t_1)\right)}{e^{2t_1}}$$

Calculate derivatives:

$x' = -\frac{\left((2C+A)t_1^2+(2D-2C+B)t_1-D\right)\sin(t_1)+\left((2A-C)t_1^2+(-D+2B-2A)t_1-B\right)\cos(t_1)}{e^{2t_1}}$

$x'' = \frac{\left((3C+4A)t_1^2+(3D-8C+4B-4A)t_1-4D+2C-2B\right)\sin(t_1)+\left((3A-4C)t_1^2+(-4D+4C+3B-8A)t_1+2D-4B+2A\right)\cos(t_1)}{e^{2t_1}}$

Find coefficients:

$$\begin{cases} 4C = 1 \\ -4A = 0 \\ 2C - 2B = 0 \\ 2D + 2A = 0 \end{cases} = \begin{cases} A = 0 \\ B = \frac{1}{4} \\ C = \frac{1}{4} \\ D = 0 \end{cases}$$

Substitute in $\{2\}$:

$$x = \frac{t_1\left(\frac{t_1\sin(t_1)}{4} + \frac{\cos(t_1)}{4}\right)}{e^{2t_1}}$$

Solve equation: $x = \text{General solution} + \text{Private solution} = \bar{x} + x_0 + x_1$

$$x = \frac{(t^2 + C_1)\sin(t) + (t + 4C_2)\cos(t)}{4e^{2t}} + \frac{e^{5t}}{50}$$

ii)

Linear equation with constant coefficients $a_0 t^{(\mathrm{n})} + a_1 t^{(\mathrm{n}-1)} + \ldots + a_{\mathrm{n}-1}t' + a_{\mathrm{n}}t = f(t)$

Let's compose the characteristic equation $a_0\lambda^{\mathrm{n}} + a_1\lambda^{\mathrm{n}-1} + \ldots + a_{\mathrm{n}-1}\lambda + a_{\mathrm{n}} = 0$:

$\lambda^2 + 4\lambda + 4 = 0 \to (\lambda + 2)^2 = 0$

Find roots $\lambda_1 \ldots \lambda_{\mathrm{n}}$, where $k-$ multiplicity of the root, $\tau-$ summand for the root :

$(\lambda + 2)^2 \to \lambda_{1,2} = -2 \quad k = 2 \quad \tau : \frac{C_1 t + C_2}{e^{2t}}$

General solution there is a sum of summands of the form:

$$t = \sum P_{\mathrm{k}-1}(t)e^{\alpha t}\sin\beta t + Q_{\mathrm{k}-1}(t)e^{\alpha t}\cos\beta t$$

where $\lambda = \alpha \pm \beta i$ and $P_{\mathrm{k}-1}(t), Q_{\mathrm{k}-1}(t) \to C_1 + \ldots + C_{\mathrm{k}}t^{\mathrm{k}-1}$ General solution:

$$\bar{x} = \frac{C_1 t + C_2}{e^{2t}}$$

Method of undefined coefficients search for a particular solution

For the right side:

A particular solution is sought in the form:

$$x_{\mathrm{i}} = t^{\mathrm{s}}e^{\alpha t}\left(R_{\mathrm{m}}(t)\cos\beta t + T_{\mathrm{m}}(t)\sin\beta t\right)$$

where $\mathrm{s} = 0$, if $\alpha + \beta i$ not a root of the char. equation and $\mathrm{s} = \mathrm{k} - \mathrm{if\,rot}\,(\lambda_1 \ldots \lambda_{\mathrm{n}})$

$$\text{Private solution for } \frac{t_0^2}{e^2 t_0}:$$

$$\alpha + \beta i = -2 \to \text{s} = 2$$

$$x = \frac{t_0^2\left(At_0^2 + Bt_0 + C\right)}{e^{2t_0}}$$

Calculate derivatives:

$$x'' = \frac{4At_0^4 + (4B - 16A)t_0^3 + (4C - 12B + 12A)t_0^2 + (6B - 8C)t_0 + 2C}{e^{2t_0}}$$

Substitute in original equation:

$$\frac{12At_0^2 + 6Bt_0 + 2C}{e^2 t_0} = \frac{t_0^2}{e^2 t_0}$$

Find coefficients:

$$\begin{cases} 12A = 1 \\ 6B = 0 \\ 2C = 0 \end{cases} = \begin{cases} A = \frac{1}{12} \\ B = 0 \\ C = 0 \end{cases}$$

Substitute in $\{1\}$:

$$x = \frac{t_0^4}{12e^{2t_0}}$$

Solve equation: $x = \text{General solution} + \text{Private solution} = \bar{x} + x_0$

$$x = \frac{t^4}{12e^{2t}} + \frac{C_1 t + C_2}{e^{2t}}$$