VV156 RC2
Limits and Derivatives

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1. Introduction
2. Limits
3. Continuity
4. Derivatives
5. Appendix
6. Q&A

About Homework 1

Ex 1.1

f	g	$f+g$	$f \cdot g$
even	even	even	even
even	odd	**depends**	odd
odd	even	**depends**	odd
odd	odd	odd	even

About Homework 1

Ex 1.5

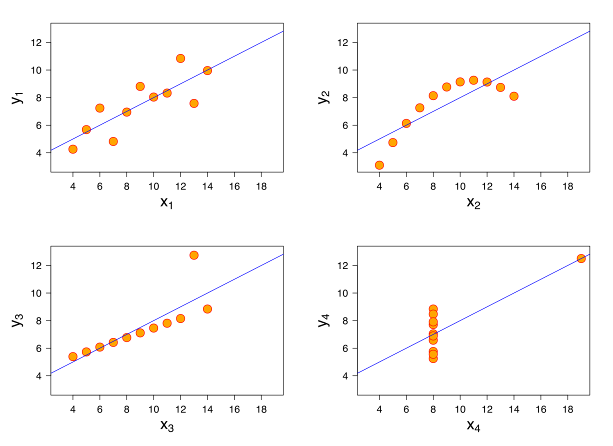

Figure: Ex 1.5

Greek alphabet

Greek alphabet

Command	Cap	Low	Command	Cap	Low
alpha	A	α	beta	B	β
gamma	Γ	γ	delta	Δ	δ
epsilon	E	ϵ, ε	zeta	Z	ζ
eta	H	η	theta	Θ	θ, ϑ
iota	I	ι	kappa	K	κ
lambda	Λ	λ	mu	M	μ
nu	N	ν	omicron	O	o
xi	Ξ	ξ	pi	Π	π, ϖ
rho	P	ρ, ϱ	sigma	Σ	σ, ς
tau	T	τ	upsilon	Υ	v
phi	Φ	ϕ, φ	chi	X	χ
psi	Ψ	ψ	omega	Ω	ω

Limits

$\varepsilon - \delta$ definition

Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the limit of $f(x)$ as x approaches $\boldsymbol{a}$ is $\boldsymbol{L}$, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$

Limits

Left-Hand Limit

$$\lim_{x \to a^-} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that if $a - \delta < x < a \quad \text{then} \quad |f(x) - L| < \varepsilon$

Right-Hand Limit

$$\lim_{x \to a^+} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that if $a < x < a + \delta \quad \text{then} \quad |f(x) - L| < \varepsilon$

Exercise 1 (Advanced)

$\varepsilon - \delta$ definition

Prove the statement using the ε, δ definition of a limit.

1.
$$\lim_{x \to a} c = c$$

2.
$$\lim_{x \to 0} x^3 = 0$$

Exercise 1 (Advanced)

Solutions

1.Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|c - c| < \varepsilon$. But $|c - c| = 0$, so this will be true no matter what δ we pick.

2.Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 0| < \delta$, then $\left|x^3 - 0\right| < \varepsilon \Leftrightarrow |x|^3 < \varepsilon \Leftrightarrow |x| < \sqrt[3]{\varepsilon}$. Take $\delta = \sqrt[3]{\varepsilon}$ Then $0 < |x - 0| < \delta \Rightarrow \left|x^3 - 0\right| < \delta^3 = \varepsilon$. Thus, $\lim_{x \to 0} x^3 = 0$ by the definition of a limit.

Limit laws

Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x)$$

exist. Then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$
4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{if } \lim_{x\to a} g(x) \neq 0$

Limit laws

6. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$ where n is a positive integer
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$ where n is a positive integer
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

(If n is even, we assume that $a > 0$.)

11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer

[If n is even, we assume that $\lim_{x\to a} f(x) > 0$.]

Other Important Theorems

Left and Right Limits Theorem

$\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$

Theorem

If $f(x) \leqslant g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x\to a} f(x) \leqslant \lim_{x\to a} g(x)$$

Sandwich Theorem

If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L$$

Exercise 2

Calculate the Limit

1.
$$\lim_{x \to 4} \frac{x^2 - 4x}{x^2 - 3x - 4}$$

2.
$$\lim_{x \to -1} \frac{x^2 - 4x}{x^2 - 3x - 4}$$

3.
$$\lim_{h \to 0} \frac{(2+h)^3 - 8}{h}$$

4.
$$\lim_{x \to 0} \frac{\sqrt[n]{x+1} - 1}{x}$$

Exercise 2

Solutions

1. $\lim_{x\to 4} \frac{x^2-4x}{x^2-3x-4} = \lim_{x\to 4} \frac{x(x-4)}{(x-4)(x+1)} = \lim_{x\to 4} \frac{x}{x+1} = \frac{4}{4+1} = \frac{4}{5}$
2. $\lim_{x\to -1} \frac{x^2-4x}{x^2-3x-4}$ does not exist since $x^2-3x-4 \to 0$ but $x^2-4x \to 5$ as $x \to -1$.
3. $\lim_{h\to 0} \frac{(2+h)^3-8}{h} =$
$\lim_{h\to 0} \frac{\left(8+12h+6h^2+h^3\right)-8}{h} = \lim_{h\to 0} \frac{12h+6h^2+h^3}{h}$
$= \lim_{h\to 0} \left(12+6h+h^2\right) = 12+0+0 = 12$

Exercise 2

Solutions

4.Setting $t = \sqrt[n]{1+x}$ then your term is equivalent to

$$\frac{t-1}{t^n-1}$$

and then use that

$$t^n - 1 = (t-1)\left(t^{n-1} + t^{n-2} + \cdots + t^2 + t + 1\right)$$

you get

$$\frac{t-1}{(t-1)\left(t^{n-1} + t^{n-2} + \cdots + t^2 + t + 1\right)} =$$

$$\frac{1}{t^{n-1} + t^{n-2} + \cdots + t^2 + t + 1}$$

for t tends to 1 and the last term tends to $\frac{1}{n}$

Exercise 3 (Advanced)

Calculate the Limit

1. $$\lim_{h \to 0} \frac{(3+h)^{-1} - 3^{-1}}{h}$$

2. $$\lim_{x \to -4} \frac{\sqrt{x^2+9} - 5}{x+4}$$

3. $$\lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

Exercise 3 (Advanced)

Solutions

1.

$$\lim_{h\to 0}\frac{(3+h)^{-1}-3^{-1}}{h}=\lim_{h\to 0}\frac{\frac{1}{3+h}-\frac{1}{3}}{h}=\lim_{h\to 0}\frac{3-(3+h)}{h(3+h)3}$$

$$=\lim_{h\to 0}\frac{-h}{h(3+h)3}=\lim_{h\to 0}\left[-\frac{1}{3(3+h)}\right]$$

$$=-\frac{1}{\lim_{h\to 0}[3(3+h)]}=-\frac{1}{3(3+0)}=-\frac{1}{9}$$

Exercise 3 (Advanced)

Solutions

2.

$$\lim_{x\to -4} \frac{\sqrt{x^2+9}-5}{x+4} = \lim_{x\to -4} \frac{\left(\sqrt{x^2+9}-5\right)\left(\sqrt{x^2+9}+5\right)}{(x+4)\left(\sqrt{x^2+9}+5\right)}$$

$$= \lim_{x\to -4} \frac{(x^2+9)-25}{(x+4)\left(\sqrt{x^2+9}+5\right)}$$

$$= \lim_{x\to -4} \frac{x^2-16}{(x+4)\left(\sqrt{x^2+9}+5\right)} = \lim_{x\to -4} \frac{(x+4)(x-4)}{(x+4)\left(\sqrt{x^2+9}+5\right)}$$

$$= \lim_{x\to -4} \frac{x-4}{\sqrt{x^2+9}+5} = \frac{-4-4}{\sqrt{16+9}+5} = \frac{-8}{5+5} = -\frac{4}{5}$$

Exercise 3 (Advanced)

Solutions

3.

$$\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}=\lim_{h\to 0}\frac{\frac{x^2-(x+h)^2}{(x+h)^2x^2}}{h}=\lim_{h\to 0}\frac{x^2-\left(x^2+2xh+h^2\right)}{hx^2(x+h)^2}$$

$$=\lim_{h\to 0}\frac{-h(2x+h)}{hx^2(x+h)^2}=\lim_{h\to 0}\frac{-(2x+h)}{x^2(x+h)^2}=\frac{-2x}{x^2\cdot x^2}=-\frac{2}{x^3}$$

Exercise 4

Prove the statement

$$\text{Prove that } \lim_{x \to 0} x^4 \cos \frac{2}{x} = 0$$

Exercise 4

Solution

$-1 \leq \cos(2/x) \leq 1 \Rightarrow -x^4 \leq x^4 \cos(2/x) \leq x^4$. Since $\lim_{x\to 0}\left(-x^4\right) = 0$ and $\lim_{x\to 0} x^4 = 0$, we have $\lim_{x\to 0}\left[x^4 \cos(2/x)\right] = 0$ by the Squeeze Theorem.

Continuity

Definition

A function f is continuous at a number $\boldsymbol{a}$ if

$$\lim_{x \to a} f(x) = f(a)$$

Notice that Definition 1 implicitly requires three things if f is continuous at a :

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

Continuity

Important Theorems

If f and g are continuous at a and c is a constant, then the following functions are also continuous at a :

1. $f+g$
2. $f-g$
3. cf
4. fg
5. $\frac{f}{g}$ if $g(a) \neq 0$

Important Theorems

The following types of functions are continuous at every number in their domains:

polynomials	rational functions
root functions	trigonometric functions

Discontinuity

Category

1. Removable
2. Infinite discontinuity
3. Jump discontinuities

The Intermediate Value Theorem

The Intermediate Value Theorem

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

Exercise 5

The Intermediate Value Theorem

Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

$$\sqrt[3]{x} = 1 - x, \quad (0, 1)$$

Exercise 5

Solution

$f(x) = \sqrt[3]{x} + x - 1$ is continuous on the interval $[0, 1]$, $f(0) = -1$, and $f(1) = 1$. Since $-1 < 0 < 1$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\sqrt[3]{x} + x - 1 = 0$, or $\sqrt[3]{x} = 1 - x$, in the interval $(0, 1)$

Exercise 6

Discontinuity

Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither? Sketch the graph of f.

$$f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ e^x & \text{if } 0 \leqslant x \leqslant 1 \\ 2-x & \text{if } x > 1 \end{cases}$$

Exercise 6

Solution

f is continuous on $(-\infty, 0)$ and $(1, \infty)$ since on each of these intervals it is a polynomial; it is continuous on $(0, 1)$ since it is an exponential. Now $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x+2) = 2$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^x = 1$, so f is discontinuous at 0. Since $f(0) = 1, f$ is continuous from the right at 0. Also $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} e^x = e$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(2-x) = 1$, so f is discontinuous at 1 . Since $f(1) = e, f$ is continuous from the left at 1 .

Exercise 6

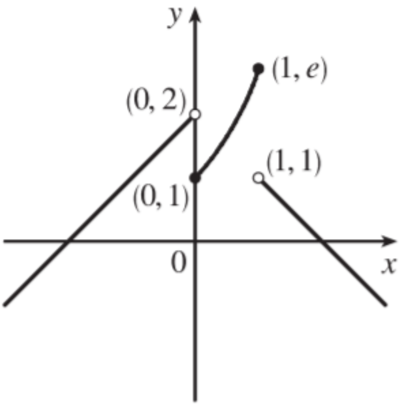

Figure: Ex 6

Derivatives

Tangent Line

The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

Definition

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$\text{instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Derivatives

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The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

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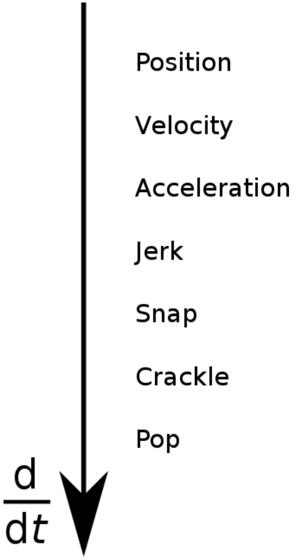
Derivatives



Derivatives

Notations

Newton:

$$\dot{y}$$

Leibniz:

$$\frac{dy}{dx}$$

Lagrange:

$$f'(x)$$

Jacobi: (Partial Derivatives)

$$\frac{\partial f}{\partial x}$$

Derivatives

Differentiable

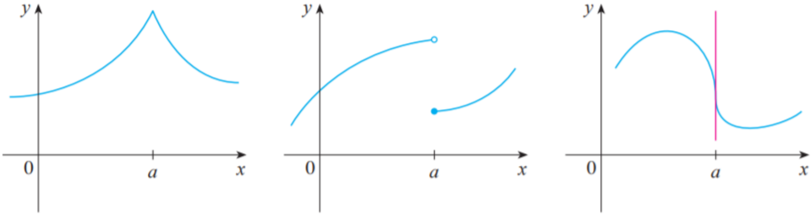
A function f is differentiable at $\boldsymbol{a}$ if $f'(a)$ exists. It is differentiable on an open interval (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Differentiable and Continuity

If f is differentiable at a, then f is continuous at a.
NOTE The converse of Theorem is false; that is, there are functions that are continuous but not differentiable. For instance, the function $f(x) = |x|$ is continuous at 0 because

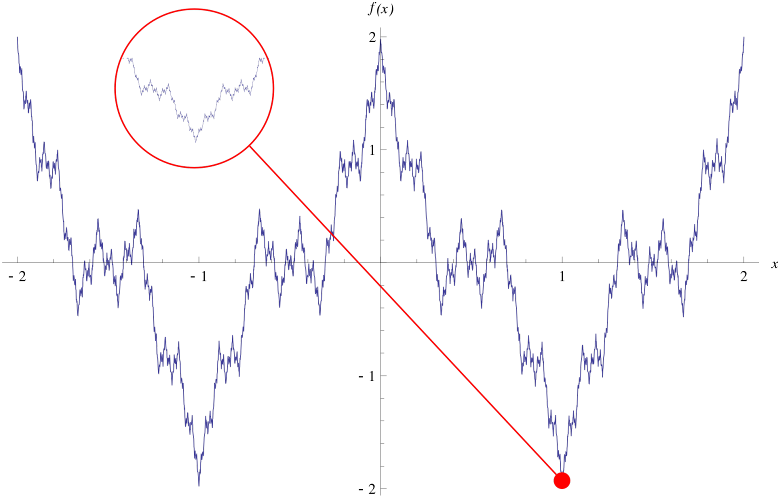
$$\lim_{x \to 0} f(x) = \lim_{x \to 0} |x| = 0 = f(0)$$

Not Differentiable

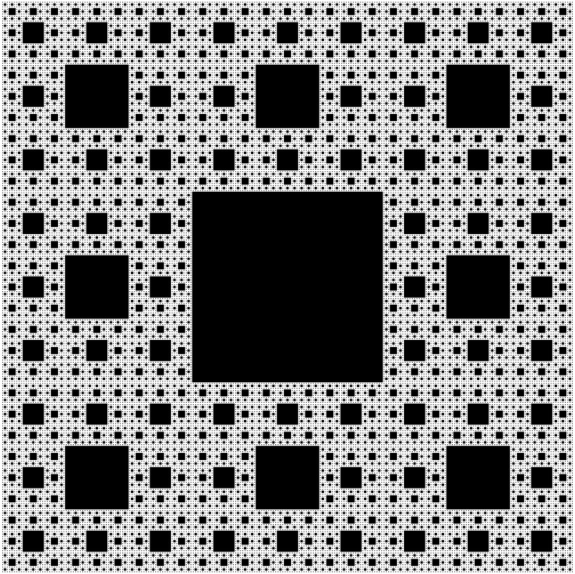


(a) A corner (b) A discontinuity (c) A vertical tangent

Weierstraß-Function



Fractal



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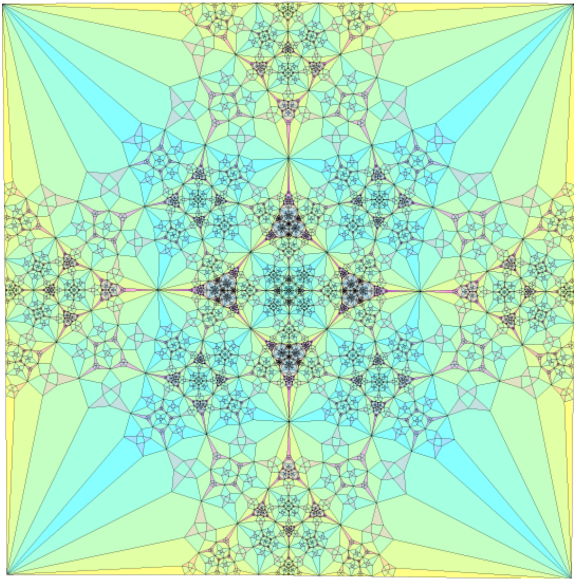
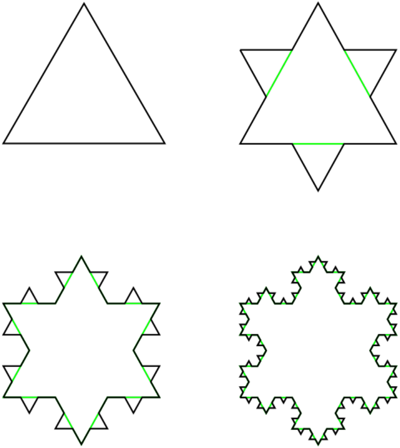


Fractal



Differentiation Formulas

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$(cf)' = cf'$$

$$(f+g)' = f' + g'$$

$$(f-g)' = f' - g'$$

$$(fg)' = fg' + gf'$$

$$(\frac{f}{g})' = \frac{gf' - fg'}{g^2}$$

List of Limits

1. $\lim_{x\to a} c = c$
2. $\lim_{x\to a} x^n = a^n$
3. $\lim_{x\to 0} \frac{\sin x}{x} = 1$
4. $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$
5. $\lim_{x\to 0} \frac{\arcsin x}{x} = 1$
6. $\lim_{x\to 0} \frac{e^x-1}{x} = 1$
7. $\lim_{x\to 0^+} x^x = 1$
8. $\lim_{x\to 0^+} x \ln x = 0$
9. $\lim_{x\to \infty} \frac{\ln x}{x} = 0$
10. $\lim_{x\to \infty} x^{\frac{1}{x}} = 1$
11. $\lim_{x\to \pm\infty} \left(1+\frac{1}{x}\right)^x = e$
12. $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$
13. $\lim_{x\to 0}(1+\sin x)^{\frac{1}{x}} = e$
14. $\lim_{x\to \infty} \left(x-\sqrt{x^2-a^2}\right) = 0$

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