5

Applications of Integration

5.1 Areas Between Curves

Areas Between Curves

Consider the region S that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$, where f and g are continuous functions and $f(x) \geq g(x)$ for all x in $[a, b]$. (See Figure 1.)

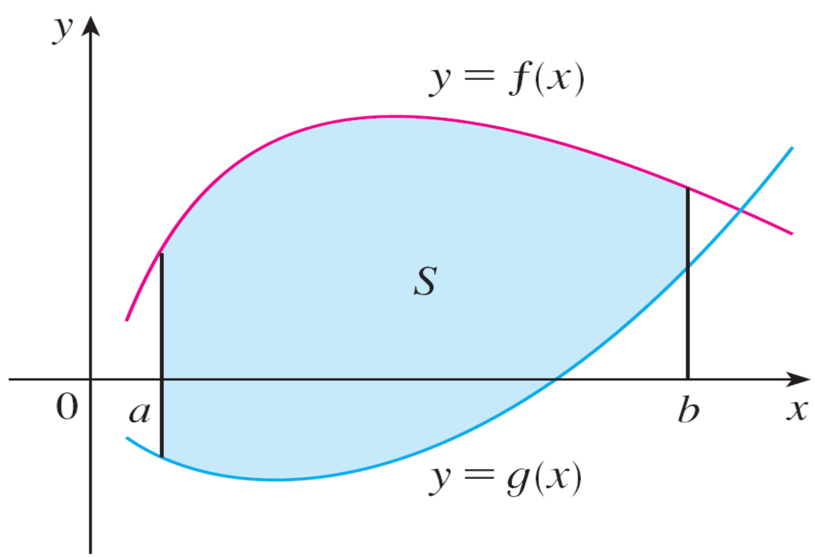


Figure 1

$S = \{(x, y) \mid a \leq x \leq b,\ g(x) \leq y \leq f(x)\}$

Areas Between Curves

We divide S into n strips of equal width and then we approximate the ith strip by a rectangle with base Δx and height $f(x_i^*) - g(x_i^*)$. (See Figure 2. If we like, we could take all of the sample points to be right endpoints, in which case $x_i^* = x_i$.)

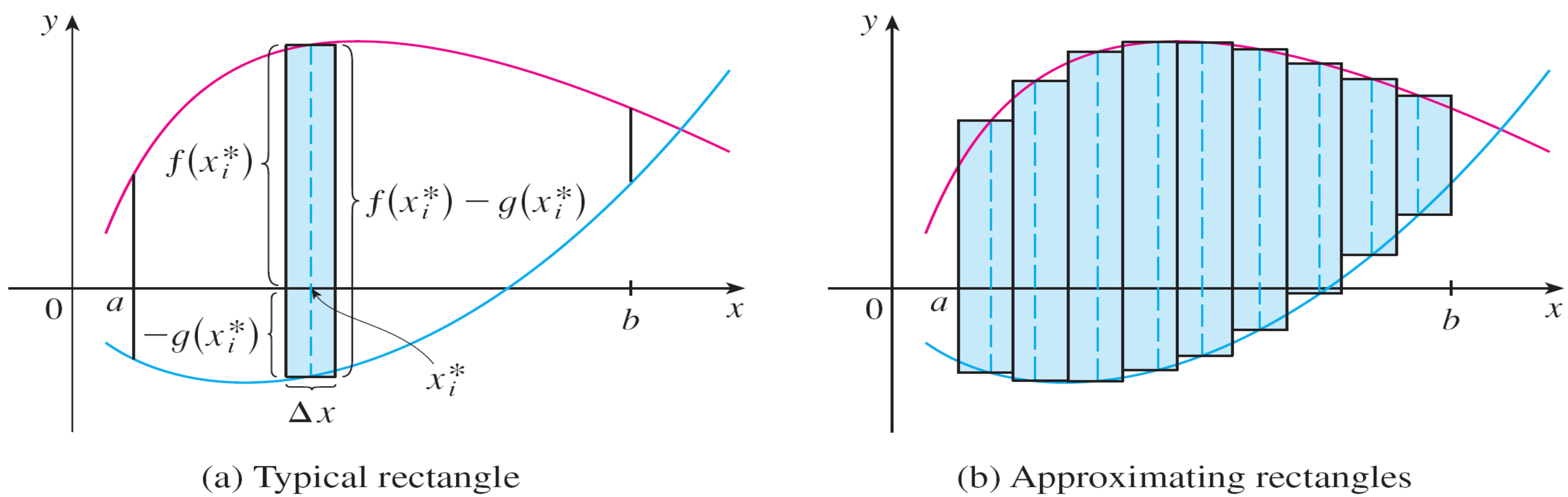


Figure 2

Areas Between Curves

The Riemann sum

$$\sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \, \Delta x$$

is therefore an approximation to what we intuitively think of as the area of *S*.

This approximation appears to become better and better as $n \to \infty$. Therefore we define the **area** *A* of the region *S* as the limiting value of the sum of the areas of these approximating rectangles.

Areas Between Curves

$$\boxed{1} \qquad A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\,\Delta x$$

We recognize the limit in (1) as the definite integral of *f* – *g*.

Therefore we have the following formula for area.

2 The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geqslant g(x)$ for all x in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)]\,dx$$

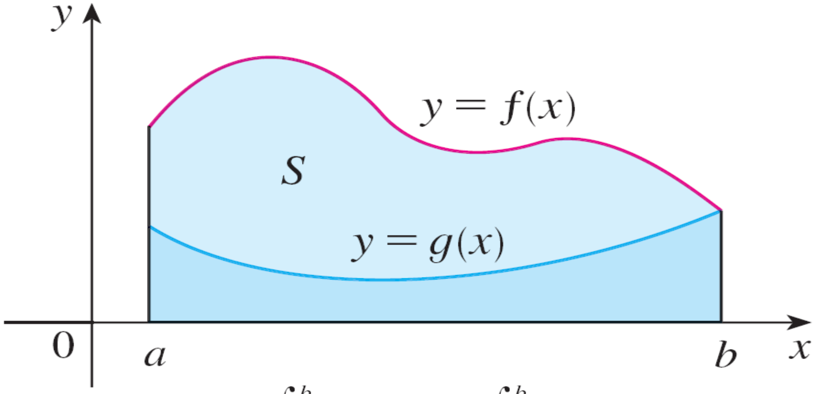
Notice that in the special case where $g(x) = 0$, S is the region under the graph of f and our general definition of area (1) reduces.

Areas Between Curves

In the case where both *f* and *g* are positive, you can see from Figure 3 why (2) is true:

$$A = [\text{area under } y = f(x)] - [\text{area under } y = g(x)]$$

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx$$



$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Figure 3

Example 1

Find the area of the region bounded above by $y = x^2 + 1$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

Solution:

The region is shown in Figure 4. The upper boundary curve is $y = x^2 + 1$ and the lower boundary curve is $y = x$.

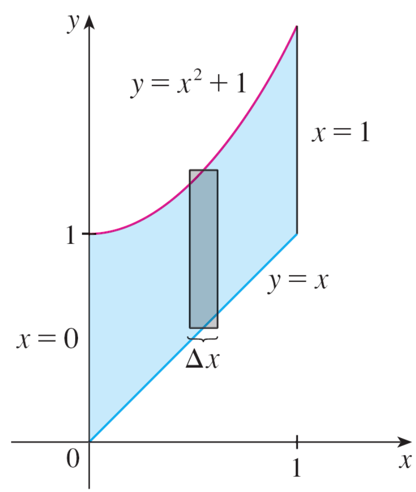


Figure 4

Example 1 – *Solution*

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So we use the area formula (2) with $f(x) = x^2 + 1$, $g(x) = x$, $a = 0$, and $b = 1$:

$$A = \int_0^1 [(x^2 + 1) - x]\,dx$$

$$= \int_0^1 (x^2 - x + 1)\,dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + x \Big]_0^1$$

$$= \frac{1}{3} - \frac{1}{2} + 1$$

$$= \frac{5}{6}$$

Areas Between Curves

In Figure 4 we drew a typical approximating rectangle with width Δx as a reminder of the procedure by which the area is defined in (1).

In general, when we set up an integral for an area, it's helpful to sketch the region to identify the top curve y_T, the bottom curve y_B, and a typical approximating rectangle as in Figure 5.

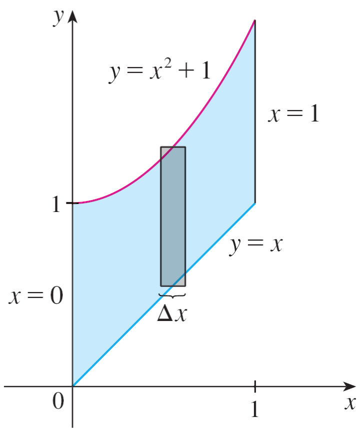


Figure 4

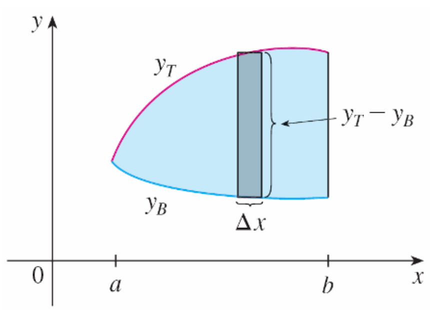


Figure 5

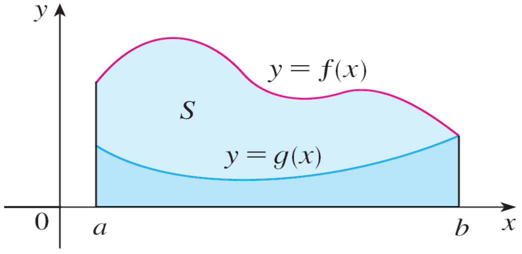
Areas Between Curves

Then the area of a typical rectangle is $(y_T - y_B)\,\Delta x$ and the equation

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} (y_T - y_B)\,\Delta x = \int_a^b (y_T - y_B)\,dx$$

summarizes the procedure of adding (in a limiting sense) the areas of all the typical rectangles.

Notice that in Figure 5 the left-hand boundary reduces to a point, whereas in Figure 3 the right-hand boundary reduces to a point.



$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Figure 3

Areas Between Curves

If we are asked to find the area between the curves $y = f(x)$ and $y = g(x)$ where $f(x) \geq g(x)$ for some values x of but $g(x) \geq f(x)$ for other values of x, then we split

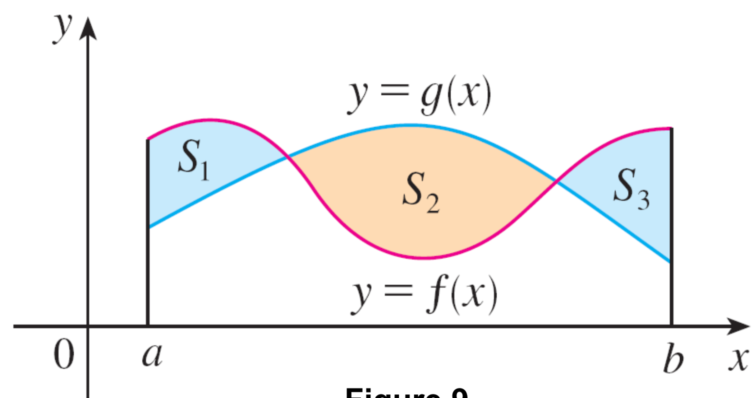


Figure 9

the given region S into several regions S_1, S_2 , . . . with areas A_1, A_2 , . . . as shown in Figure 9. We then define the area of the region S to be the sum of the areas of the smaller regions S_1, S_2 , . . . that is $A = A_1 + A_2 + \ldots$ Since

$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) & \text{when } f(x) \geq g(x) \\ g(x) - f(x) & \text{when } g(x) \geq f(x) \end{cases}$$

Areas Between Curves

we have the following expression for *A*.

$\boxed{3}$ The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is

$$A = \int_a^b |f(x) - g(x)|\, dx$$

When evaluating the integral in (3), however, we must still split it into integrals corresponding to A_1, A_2,……

Example 5

Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$

Solution:

The points of intersection occur when $\sin x = \cos x$, that is, when $x = \pi/4$ (since $0 \le x \le \pi/2$). The region is sketched in Figure 10. Observe that $\cos x \ge \sin x$ when $0 \le x \le \pi/4$ but $\sin x \ge \cos x$ when $\pi/4 \le x \le \pi/2$.

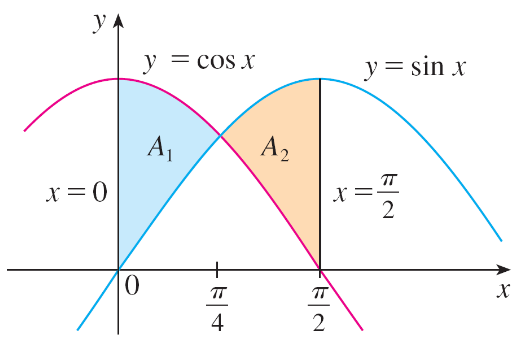


Figure 10

Example 5 – *Solution*

cont'd

Therefore the required area is

$$A = \int_0^{\pi/2} |\cos x - \sin x|\, dx = A_1 + A_2$$

$$= \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx$$

$$= \Big[\sin x + \cos x\Big]_0^{\pi/4} + \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi/2}$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1\right) + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$= 2\sqrt{2} - 2$$

Example 5 – *Solution*

cont'd

In this particular example we could have saved some work by noticing that the region is symmetric about $x = \pi/4$ and so

$$A = 2A_1 = 2\int_0^{\pi/4} (\cos x - \sin x)\, dx$$

Areas Between Curves

Some regions are best treated by regarding x as a function of y. If a region is bounded by curves with equations

$x = f(y)$, $x = g(y)$, $y = c$, and $y = d$, where f and g are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$ (see Figure 11), then its area is

$$A = \int_c^d [f(y) - g(y)]\,dy$$

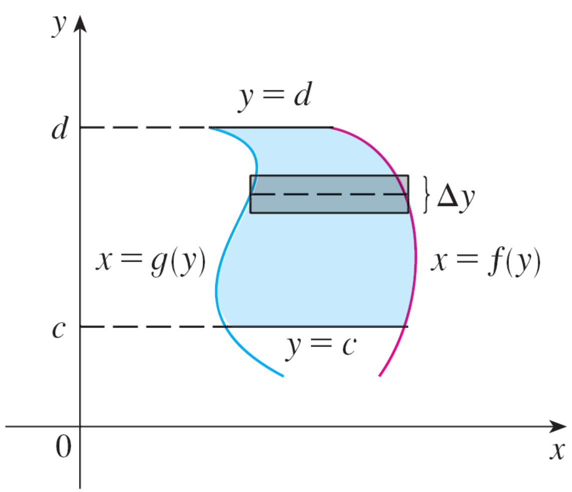


Figure 11

Areas Between Curves

If we write x_R for the right boundary and x_L for the left boundary, then, as Figure 12 illustrates, we have

$$A = \int_c^d (x_R - x_L)\, dy$$

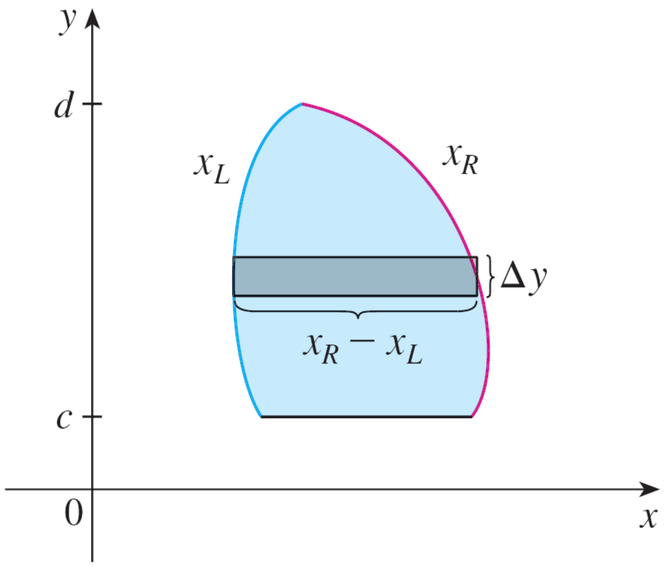


Figure 12

5.2 Volumes

Volumes

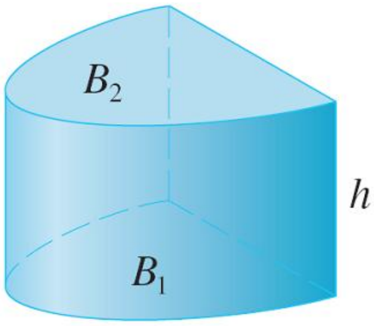
In trying to find the volume of a solid we face the same type of problem as in finding areas.

We have an intuitive idea of what volume means, but we must make this idea precise by using calculus to give an exact definition of volume.

Volumes

We start with a simple type of solid called a **cylinder** (or, more precisely, a *right cylinder*).

As illustrated in Figure 1(a), a cylinder is bounded by a plane region B_1, called the **base**, and a congruent region B_2 in a parallel plane.



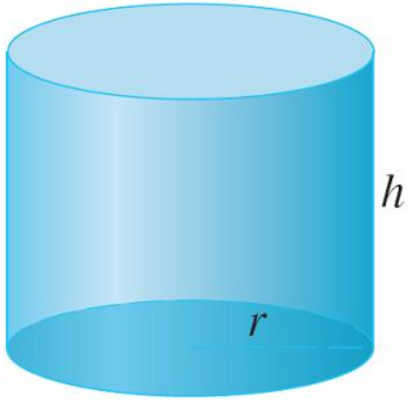
(a) Cylinder $V = Ah$

Figure 1(a)

The cylinder consists of all points on line segments that are perpendicular to the base and join B_1 to B_2. If the area of the base is A and the height of the cylinder (the distance from B_1 to B_2) is h, then the volume V of the cylinder is defined as $V = Ah$

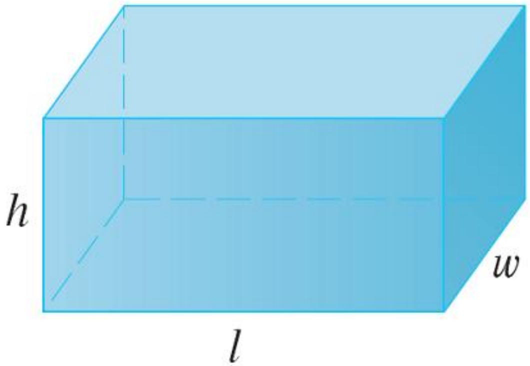
Volumes

In particular, if the base is a circle with radius r, then the cylinder is a circular cylinder with volume $V = \pi r^2 h$ [see Figure 1(b)], and if the base is a rectangle with length l and width w, then the cylinder is a rectangular box (also called a *rectangular parallelepiped*) with volume $V = lwh$ [see Figure 1(c)].



(b) Circular cylinder $V = \pi r^2 h$

Figure 1(b)



(c) Rectangular box $V = lwh$

Figure 1(c)

Volumes

For a solid *S* that isn't a cylinder we first "cut" *S* into pieces and approximate each piece by a cylinder. We estimate the volume of *S* by adding the volumes of the cylinders. We arrive at the exact volume of *S* through a limiting process in which the number of pieces becomes large.

We start by intersecting *S* with a plane and obtaining a plane region that is called a **cross-section** of *S*.

Volumes

Let $A(x)$ be the area of the cross-section of S in a plane P_x perpendicular to the x-axis and passing through the point x, where $a \le x \le b$. (See Figure 2. Think of slicing S with a knife through x and computing the area of this slice.)

The cross-sectional area $A(x)$ will vary as x increases from a to b.

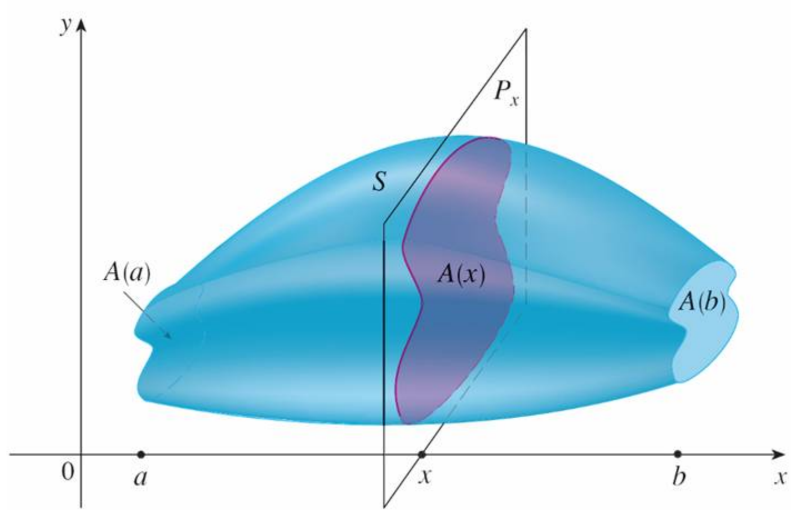


Figure 2

Volumes

Let's divide S into n "slabs" of equal width Δx by using the planes P_{x_1}, P_{x_2}, . . . to slice the solid. (Think of slicing a loaf of bread.)

If we choose sample points x_i^* in $[x_{i-1}, x_i]$, we can approximate the ith slab S_i (the part of S that lies between the planes $P_{x_{i-1}}$ and P_{x_i}) by a cylinder with base area $A(x_i^*)$ and "height" Δx. (See Figure 3.)

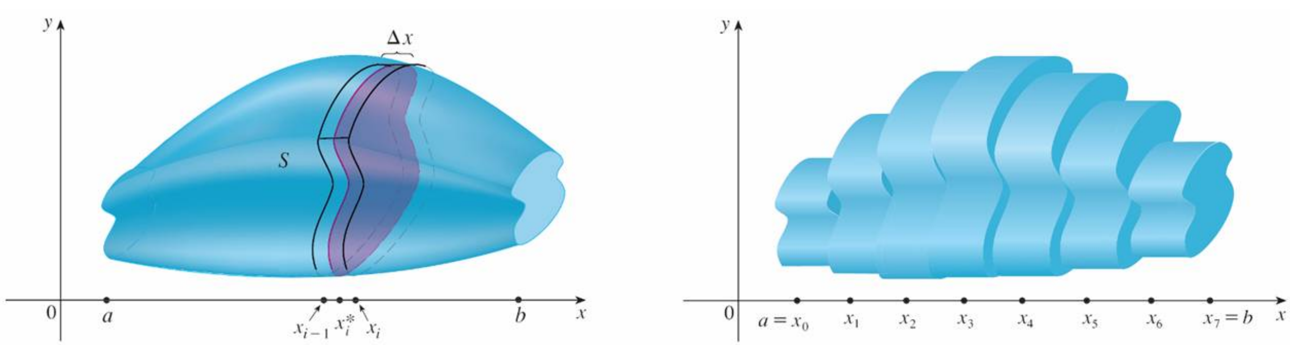


Figure 3

Volumes

The volume of this cylinder is $A(x_i^*)\,\Delta x$, so an approximation to our intuitive conception of the volume of the ith slab S_i is

$$V(S_i) \approx A(x_i^*)\,\Delta x$$

Adding the volumes of these slabs, we get an approximation to the total volume (that is, what we think of intuitively as the volume):

$$V \approx \sum_{i=1}^{n} A(x_i^*)\,\Delta x$$

Volumes

This approximation appears to become better and better as $n \to \infty$. (Think of the slices as becoming thinner and thinner.)

Therefore we *define* the volume as the limit of these sums as $n \to \infty$.

But we recognize the limit of Riemann sums as a definite integral and so we have the following definition.

Definition of Volume Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the **volume** of S is

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*)\, \Delta x = \int_a^b A(x)\, dx$$

Volumes

When we use the volume formula $V = \int_a^b A(x)\,dx$, it is important to remember that $A(x)$ is the area of a moving cross-section obtained by slicing through x perpendicular to the x-axis.

Notice that, for a cylinder, the cross-sectional area is constant: $A(x) = A$ for all x. So our definition of volume gives $V = \int_a^b A\,dx = A(b - a)$; this agrees with the formula $V = Ah$.

Example 1

Show that the volume of a sphere of radius *r* is $V = \frac{4}{3}\pi r^3$.

Solution:

If we place the sphere so that its center is at the origin (see Figure 4), then the plane P_x intersects the sphere in a circle whose radius (from the Pythagorean Theorem) is

$y = \sqrt{r^2 - x^2}$.

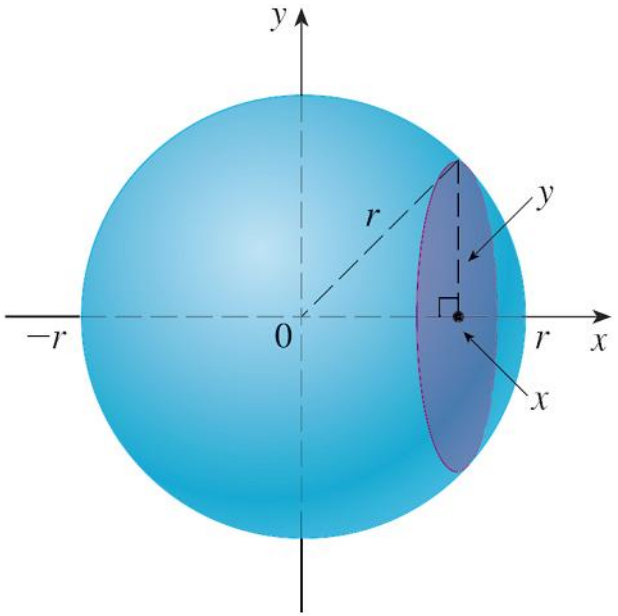


Figure 4

So the cross-sectional area is

$$A(x) = \pi y^2 = \pi(r^2 - x^2)$$

Example 1 – *Solution*

cont'd

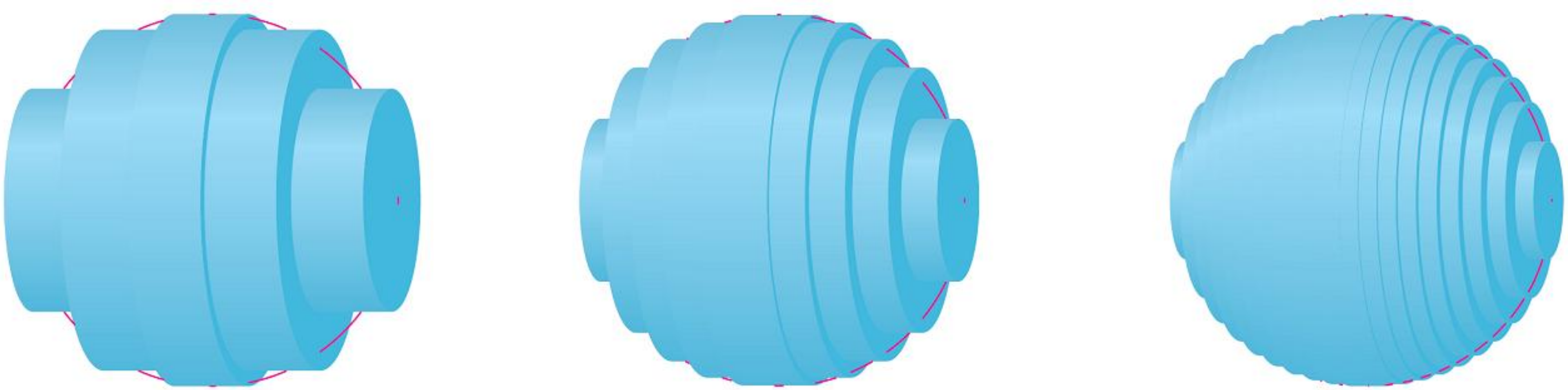
Using the definition of volume with $a = -r$ and $b = r$, we have

$$
\begin{aligned}
V &= \int_{-r}^{r} A(x)\,dx \\
&= \int_{-r}^{r} \pi(r^2 - x^2)\,dx \\
&= 2\pi \int_{0}^{r} (r^2 - x^2)\,dx \qquad \text{(The integrand is even.)} \\
&= 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r \\
&= 2\pi \left(r^3 - \frac{r^3}{3} \right) \\
&= \tfrac{4}{3}\pi r^3
\end{aligned}
$$

Volumes

Figure 5 illustrates the definition of volume when the solid is a sphere with radius $r = 1$.

From the result of Example 1, we know that the volume of the sphere is $\frac{4}{3}\pi$, which is approximately 4.18879.



(a) Using 5 disks, $V \approx 4.2726$ (b) Using 10 disks, $V \approx 4.2097$ (c) Using 20 disks, $V \approx 4.1940$

Figure 5

Approximating the volume of a sphere with radius 1

Volumes

Here the slabs are circular cylinders, or *disks*, and the three parts of Figure 5 show the geometric interpretations of the Riemann sums

$$\sum_{i=1}^{n} A(\bar{x}_i)\,\Delta x = \sum_{i=1}^{n} \pi(1^2 - \bar{x}_i^2)\,\Delta x$$

when n = 5, 10, and 20 if we choose the sample points x_i^* to be the midpoints $\bar{x}_i$.

Notice that as we increase the number of approximating cylinders, the corresponding Riemann sums become closer to the true volume.

Volumes

The solids as in Example 1 are all called **solids of revolution** because they are obtained by revolving a region about a line. In general, we calculate the volume of a solid of revolution by using the basic defining formula

$$V = \int_a^b A(x)\, dx \qquad \text{or} \qquad V = \int_c^d A(y)\, dy$$

and we find the cross-sectional area $A(x)$ or $A(y)$ in one of the following ways:

If the cross-section is a disk, we find the radius of the disk (in terms of x or y) and use

$$A = \pi(\text{radius})^2$$

Volumes

If the cross-section is a washer, we find the inner radius r_{in} and outer radius r_{out} from a sketch (as in Figure 10) and compute the area of the washer by subtracting the area of the inner disk from the area of the outer disk:

$$A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$$

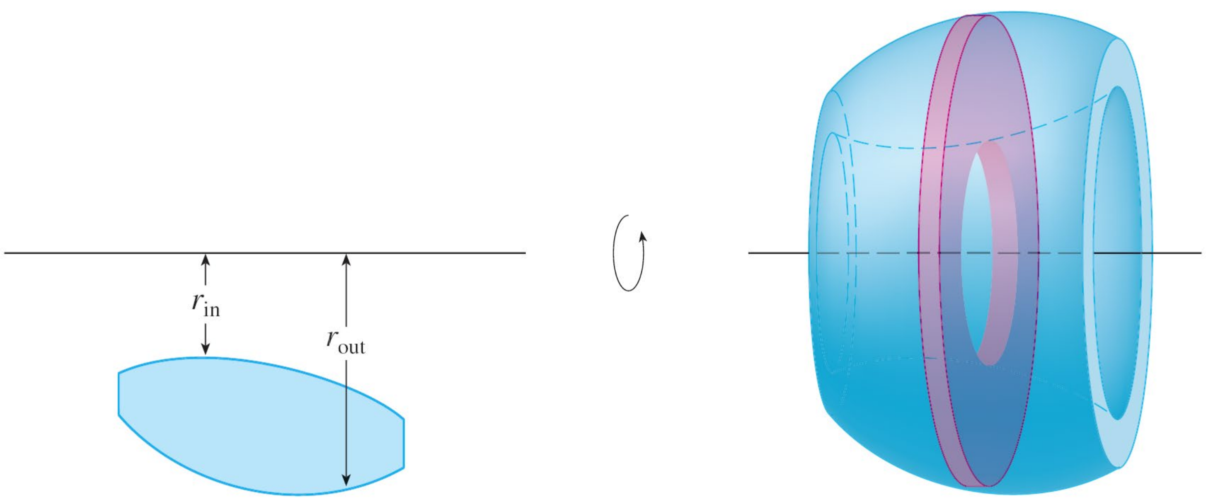


Figure 10

5.3 Volumes by Cylindrical Shells

Volumes by Cylindrical Shells

Let's consider the problem of finding the volume of the solid obtained by rotating about the *y*-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$. (See Figure 1.)

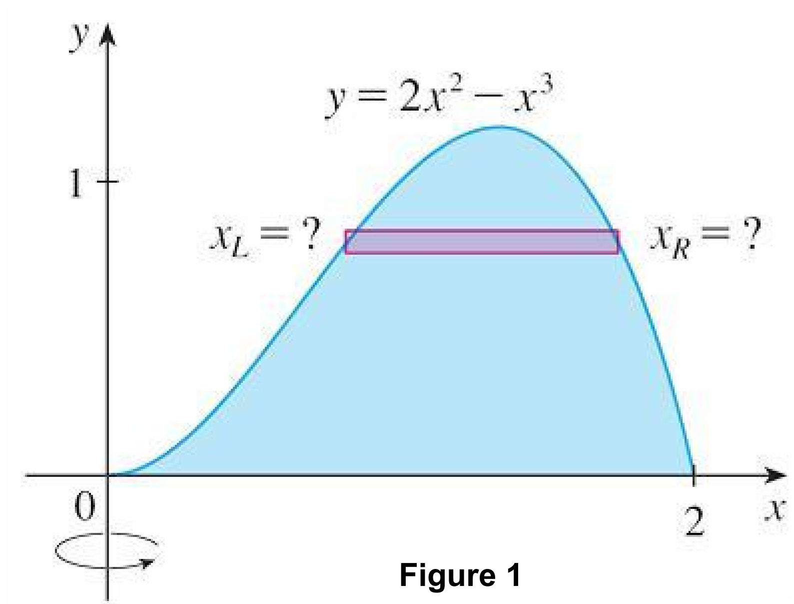


Figure 1

If we slice perpendicular to the *y*-axis, we get a washer.

Volumes by Cylindrical Shells

But to compute the inner radius and the outer radius of the washer, we'd have to solve the cubic equation $y = 2x^2 - x^3$ for x in terms of y; that's not easy.

Fortunately, there is a method, called the **method of cylindrical shells**, that is easier to use in such a case. Figure 2 shows a cylindrical shell with inner radius r_1, outer radius r_2, and height h.

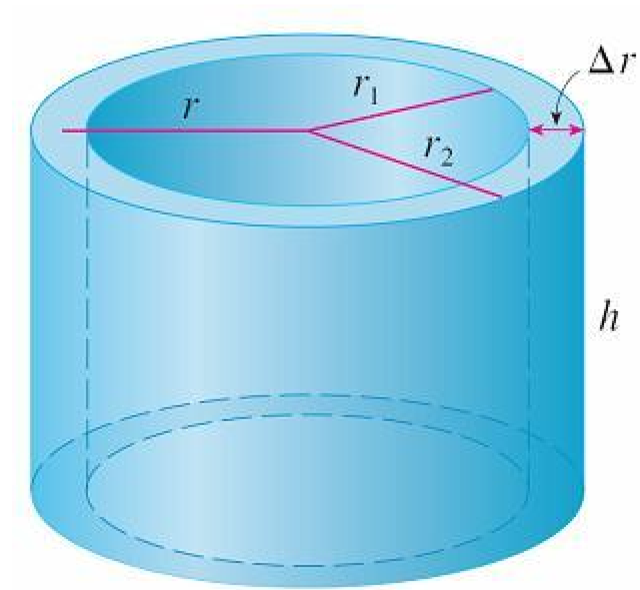


Figure 2

Volumes by Cylindrical Shells

Its volume V is calculated by subtracting the volume V_1 of the inner cylinder from the volume V_2 of the outer cylinder:

$$V = V_2 - V_1$$

$$= \pi r_2^2 h - \pi r_1^2 h = \pi(r_2^2 - r_1^2)h$$

$$= \pi(r_2 + r_1)(r_2 - r_1)h$$

$$= 2\pi \frac{r_2 + r_1}{2} h(r_2 - r_1)$$

Volumes by Cylindrical Shells

If we let $\Delta r = r_2 - r_1$ (the thickness of the shell) and $r = \frac{1}{2}(r_2 + r_1)$ (the average radius of the shell), then this formula for the volume of a cylindrical shell becomes

$$V = 2\pi r h \, \Delta r \tag{1}$$

and it can be remembered as

V = [circumference] [height] [thickness]

Volumes by Cylindrical Shells

Now let S be the solid obtained by rotating about the y-axis the region bounded by $y = f(x)$ [where $f(x) \geq 0$], $y = 0$, $x = a$ and $x = b$, where $b > a \geq 0$. (See Figure 3.)

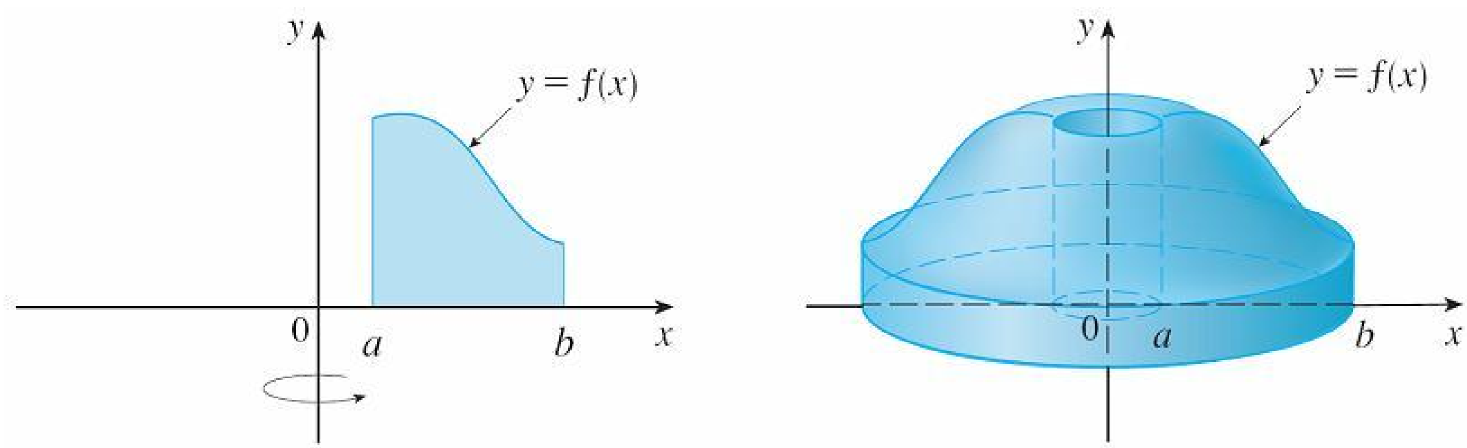


Figure 3

We divide the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width Δx and let $\bar{x}_i$ be the midpoint of the ith subinterval.

Volumes by Cylindrical Shells

If the rectangle with base $[x_{i-1}, x_i]$ and height $f(\bar{x}_i)$ is rotated about the *y*-axis, then the result is a cylindrical shell with average radius $\bar{x}_i$, height $f(\bar{x}_i)$, and thickness Δx (see Figure 4), so by Formula 1 its volume is

$$V_i = (2\pi\bar{x}_i)[f(\bar{x}_i)]\,\Delta x$$

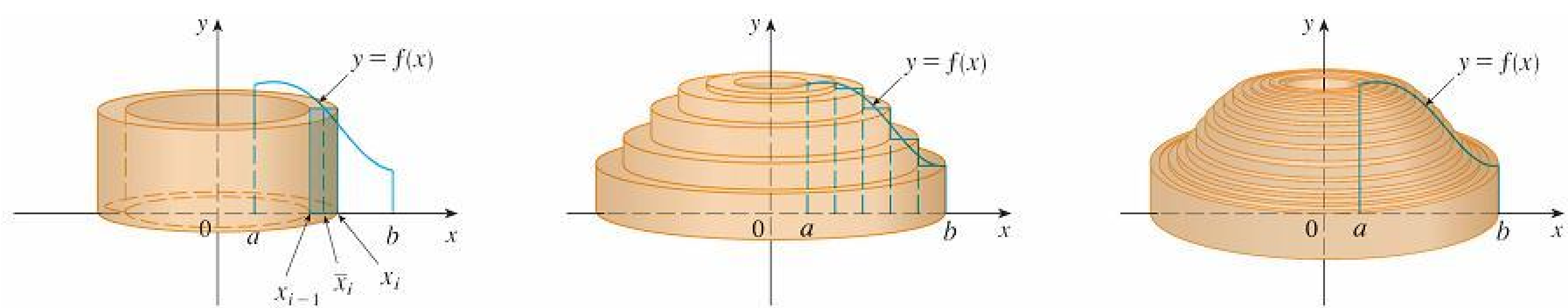


Figure 4

Volumes by Cylindrical Shells

Therefore an approximation to the volume *V* of *S* is given by the sum of the volumes of these shells:

$$V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi \bar{x}_i f(\bar{x}_i)\,\Delta x$$

This approximation appears to become better as $n \to \infty$.

But, from the definition of an integral, we know that

$$\lim_{n \to \infty} \sum_{i=1}^{n} 2\pi \bar{x}_i f(\bar{x}_i)\,\Delta x = \int_a^b 2\pi x f(x)\,dx$$

Volumes by Cylindrical Shells

Thus the following appears plausible:

2 The volume of the solid in Figure 3, obtained by rotating about the y-axis the region under the curve $y = f(x)$ from a to b, is

$$V = \int_a^b 2\pi x f(x)\,dx \qquad \text{where } 0 \leqslant a < b$$

Volumes by Cylindrical Shells

The best way to remember Formula 2 is to think of a typical shell, cut and flattened as in Figure 5, with radius *x*, circumference $2\pi x$, height $f(x)$, and thickness Δx or dx:

$$\int_a^b \underbrace{(2\pi x)}_{\text{circumference}} \underbrace{[f(x)]}_{\text{height}} \underbrace{dx}_{\text{thickness}}$$

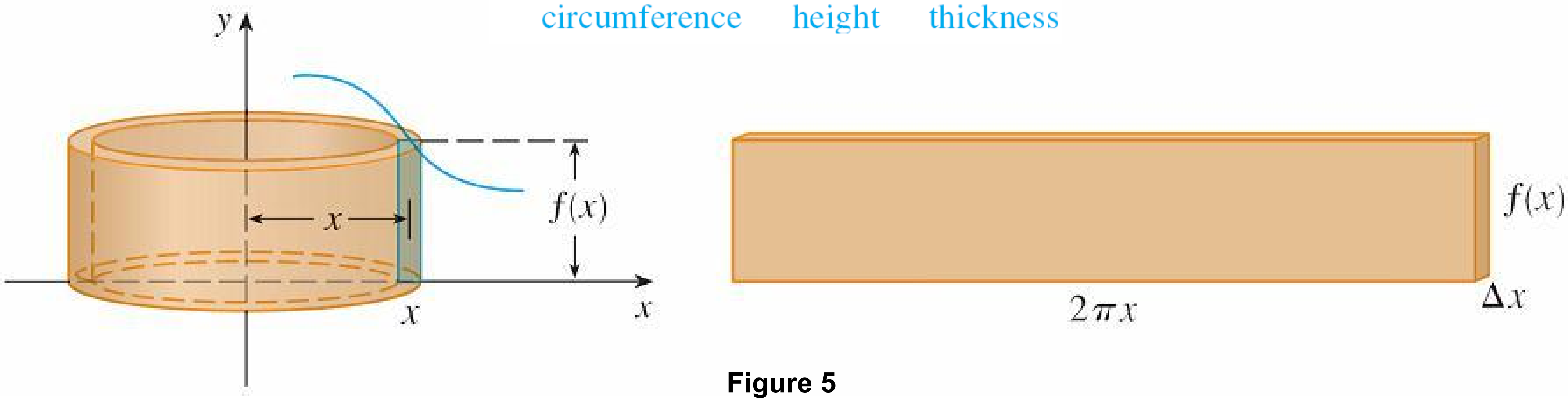


Figure 5

This type of reasoning will be helpful in other situations, such as when we rotate about lines other than the *y*-axis.

Example 1

Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

Solution:

From the sketch in Figure 6 we see that a typical shell has radius x, circumference $2\pi x$, and height $f(x) = 2x^2 - x^3$.

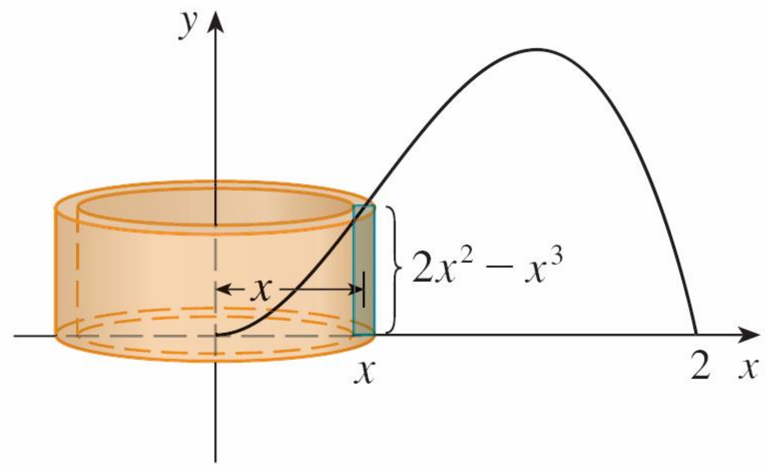


Figure 6

Example 1 – *Solution*

cont'd

So, by the shell method, the volume is

$$V = \int_0^2 (2\pi x)(2x^2 - x^3)\,dx$$

$$= 2\pi \int_0^2 (2x^3 - x^4)\,dx$$

$$= 2\pi\left[\tfrac{1}{2}x^4 - \tfrac{1}{5}x^5\right]_0^2$$

$$= 2\pi\left(8 - \tfrac{32}{5}\right)$$

$$= \tfrac{16}{5}\pi$$

It can be verified that the shell method gives the same answer as slicing.

5.4 Work

Work

The term *work* is used in everyday language to mean the total amount of effort required to perform a task.

In physics it has a technical meaning that depends on the idea of a *force.*

Intuitively, you can think of a force as describing a push or pull on an object—for example, a horizontal push of a book across a table or the downward pull of the earth's gravity on a ball.

Work

In general, if an object moves along a straight line with position function $s(t)$, then the **force** F on the object (in the same direction) is given by Newton's Second Law of Motion as the product of its mass m and its acceleration:

$$\boxed{1} \qquad F = m\frac{d^2s}{dt^2}$$

In the SI metric system, the mass is measured in kilograms (kg), the displacement in meters (m), the time in seconds (s), and the force in newtons (N = kg•m/s^2). Thus a force of 1 N acting on a mass of 1 kg produces an acceleration of 1 m/s^2.

Work

In the US Customary system the fundamental unit is chosen to be the unit of force, which is the pound.

In the case of constant acceleration, the force F is also constant and the work done is defined to be the product of the force F and the distance d that the object moves:

$\boxed{2}$ $$W = Fd \qquad \text{work} = \text{force} \times \text{distance}$$

Work

If F is measured in newtons and d in meters, then the unit for W is a newton-meter, which is called a joule (J).

If F is measured in pounds and d in feet, then the unit for W is a foot-pound (ft-lb), which is about 1.36 J.

Example 1

(a) How much work is done in lifting a 1.2-kg book off the floor to put it on a desk that is 0.7 m high? Use the fact that the acceleration due to gravity is $g = 9.8 \text{ m/s}^2$.

(b) How much work is done in lifting a 20-lb weight 6 ft off the ground?

Example 1(a) – *Solution*

The force exerted is equal and opposite to that exerted by gravity, so Equation 1 gives

$$F = mg = (1.2)(9.8)$$

$$= 11.76 \text{ N}$$

and then Equation 2 gives the work done as

$$W = Fd = (11.76)(0.7)$$

$$\approx 8.2 \text{ J}$$

Example 1(b) – *Solution*

Here the force is given as F = 20 lb, so the work done is

$$W = Fd = 20 \cdot 6$$

$$= 120 \text{ ft-lb}$$

Notice that in part (b), unlike part (a), we did not have to multiply by g because we were given the *weight* (which is a force) and not the mass of the object.

Work

Equation 2 defines work as long as the force is constant, but what happens if the force is variable? Let's suppose that the object moves along the x-axis in the positive direction, from $x = a$ to $x = b$, and at each point x between a and b a force $f(x)$ acts on the object, where f is a continuous function.

We divide the interval $[a, b]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx.

We choose a sample point x_i* in the ith subinterval $[x^{i-1}, x_i]$. Then the force at that point is $f(x_i*)$.

Work

If n is large, then Δx is small, and since f is continuous, the values of f don't change very much over the interval $[x_{i-1}, x_i]$.

In other words, f is almost constant on the interval and so the work W_i that is done in moving the particle from x_{i-1} to x_i is approximately given by Equation2:

$$W_i \approx f(x_i *)\ \Delta x$$

Thus we can approximate the total work by

$$\boxed{3} \qquad W \approx \sum_{i=1}^{n} f(x_i^*)\ \Delta x$$

Work

It seems that this approximation becomes better as we make n larger. Therefore we define the **work done in moving the object from *a* to *b*** as the limit of this quantity as $n \to \infty$.

Since the right side of $\boxed{3}$ is a Riemann sum, we recognize its limit as being a definite integral and so

$$\boxed{4} \qquad \boxed{W = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x = \int_a^b f(x)\, dx}$$

Example 2

When a particle is located a distance x feet from the origin, a force of $x^2 + 2x$ pounds acts on it. How much work is done in moving it from $x = 1$ to $x = 3$?

Solution:

$$W = \int_1^3 (x^2 + 2x)\,dx = \frac{x^3}{3} + x^2 \Big]_1^3$$

$$= \frac{50}{3}$$

The work done is $16\frac{2}{3}$ ft-lb.

Work

In the next example we use a law from physics:

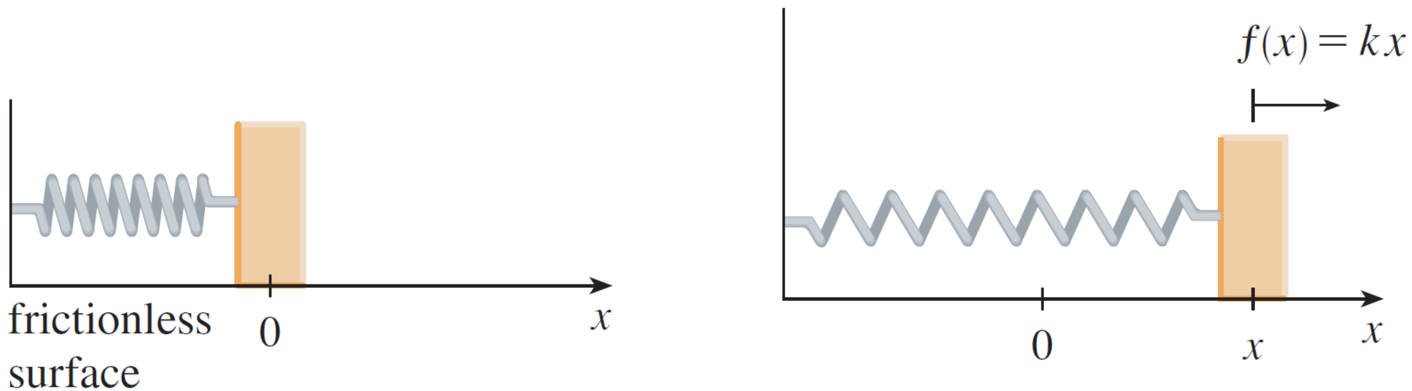
Hooke's Law states that the force required to maintain a spring stretched units beyond its natural length is proportional to *x*:

$$f(x) = kx$$

where *k* is a positive constant (called the **spring constant**).

Work

Hooke's Law holds provided that *x* is not too large (see Figure 1).



(a) Natural position of spring

(b) Stretched position of spring

Hooke's Law

Figure 1

Example 3

A force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15 cm. How much work is done in stretching the spring from 15 cm to 18 cm?

Solution:

According to Hooke's Law, the force required to hold the spring stretched x meters beyond its natural length is $f(x) = kx$.

Example 3 – *Solution*

cont'd

When the spring is stretched from 10 cm to 15 cm, the amount stretched is 5 cm = 0.05 m. This means that $f(0.05) = 40$, so

$$0.05k = 40 \qquad k = \frac{40}{0.05} = 800$$

Thus $f(x) = 800x$ and the work done in stretching the spring from 15 cm to 18 cm is

$$W = \int_{0.05}^{0.08} 800x\,dx = 800\,\frac{x^2}{2}\Bigg]_{0.05}^{0.08}$$

$$= 400[(0.08)^2 - 0.05)^2]$$

$$= 1.56 \text{ J}$$

5.5 Average Value of a Function

Average Value of a Function

It is easy to calculate the average value of finitely many numbers $y_1, y_2, \ldots, y_n$:

$$y_{\text{ave}} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

But how do we compute the average temperature during a day if infinitely many temperature readings are possible?

Figure 1 shows the graph of a temperature function $T(t)$, where t is measured in hours and T in °C, and a guess at the average temperature, T_{ave}.

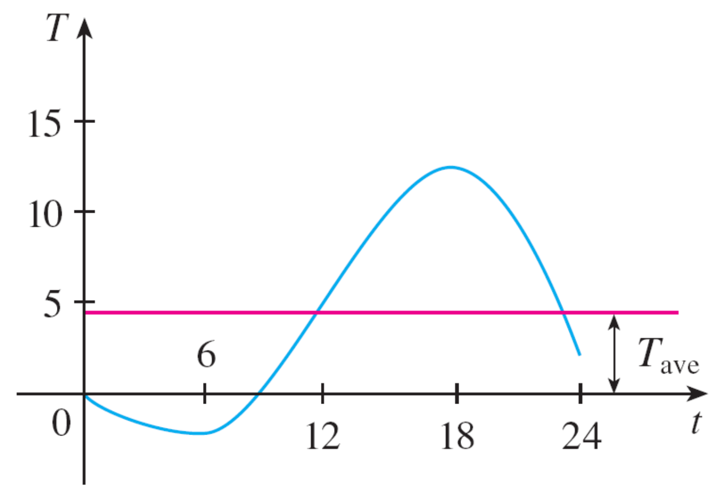


Figure 1

Average Value of a Function

In general, let's try to compute the average value of a function $y = f(x)$, $a \le x \le b$. We start by dividing the interval $[a, b]$ into n equal subintervals, each with length $\Delta x = (b - a)/n$.

Then we choose points $x_1^*, \ldots, x_n^*$ in successive subintervals and calculate the average of the numbers $f(x_1^*), \ldots, f(x_n^*)$:

$$\frac{f(x_1^*) + \cdots + f(x_n^*)}{n}$$

(For example, if f represents a temperature function and $n = 24$, this means that we take temperature readings every hour and then average them.)

Average Value of a Function

Since $\Delta x = (b - a)/n$, we can write $n = (b - a)/\Delta x$ and the average value becomes

$$\frac{f(x_1^*) + \cdots + f(x_n^*)}{\dfrac{b-a}{\Delta x}} = \frac{1}{b-a}\left[f(x_1^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x\right]$$

$$= \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

If we let n increase, we would be computing the average value of a large number of closely spaced values.

Average Value of a Function

The limiting value is

$$\lim_{n\to\infty} \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \frac{1}{b-a} \int_a^b f(x)\,dx$$

by the definition of a definite integral.

Therefore we define the **average value of *f*** on the interval [*a*, *b*] as

$$\boxed{f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)\,dx}$$

Example 1

Find the average value of the function $f(x) = 1 + x^2$ on the interval [–1, 2].

Solution:

With $a = –1$ and $b = 2$ we have

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{2-(-1)}\int_{-1}^{2} (1 + x^2)\,dx$$

$$= \frac{1}{3}\left[x + \frac{x^3}{3}\right]_{-1}^{2}$$

$$= 2$$

Average Value of a Function

If $T(t)$ is the temperature at time t, we might wonder if there is a specific time when the temperature is the same as the average temperature.

For the temperature function graphed in Figure 1, we see that there are two such times—just before noon and just before midnight.

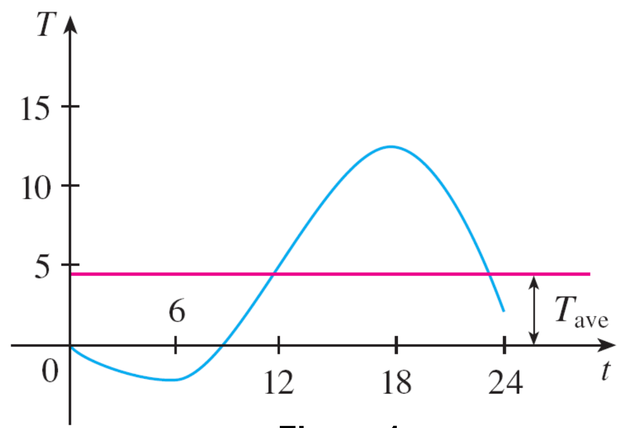


Figure 1

In general, is there a number c at which the value of a function f is exactly equal to the average value of the function, that is, $f(c) = f_{ave}$?

Average Value of a Function

The following theorem says that this is true for continuous functions.

The Mean Value Theorem for Integrals If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

that is,

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

The Mean Value Theorem for Integrals is a consequence of the Mean Value Theorem for derivatives and the Fundamental Theorem of Calculus.

Average Value of a Function

The geometric interpretation of the Mean Value Theorem for Integrals is that, for *positive* functions f, there is a number c such that the rectangle with base $[a, b]$ and height $f(c)$ has the same area as the region under the graph of f from a to b. (See Figure 2.)

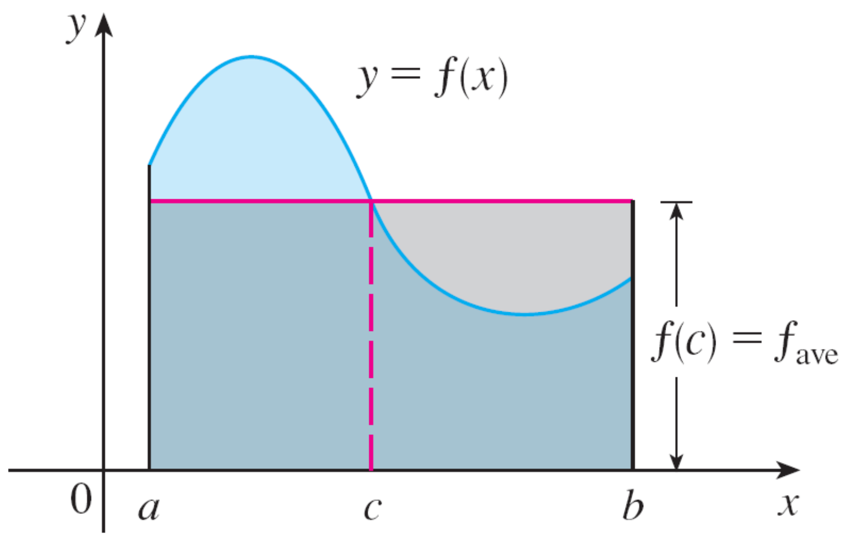


Figure 2