Vv156 Honors Calculus II

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Part XI

Sequences and Series

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Overview

A ***sequence*** is an infinite list of numbers listed in a definite order, e.g.,

$$a_1, a_2, a_3, a_4, \dots, a_n, \dots$$

Definition

A ***sequence*** is a **function** whose domain is the set of positive integers $\mathbb{N} \setminus \{0\}$, or natural numbers $\mathbb{N}$, e.g.,

$$\begin{aligned} f : \mathbb{N} &\to \mathbb{R} \\ n &\mapsto a_n \end{aligned}$$

Often denoted by $\{a_n\}$, $\{a_n\}_{n=1}^{\infty}$, (a_n), or $(a_n)_{n=1}^{\infty}$. Note that $f \in \mathbb{R}^{\mathbb{N}}$.

Example

- $\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n}{n+1}, \dots\right)$
- $\left(-\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \dots, \frac{(-1)^n(n+1)}{3^n}, \dots\right)$
- $\left(1, -1, 1, -1, \dots, (-1)^n, \dots\right)$

Convergence and Divergence

Definition

A sequence (a_n) has the ***limit*** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.
If $\lim_{n\to\infty} a_n$ exists, we say that the sequence ***converges*** (or is ***convergent***).
Otherwise, we say the sequence ***diverges*** (or is ***divergent***).

Example

- $(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots)$. $\lim_{n\to\infty} \frac{1}{2^n} = 0$. The sequence converges to 0.
- $(2, 4, 8, 16, \dots)$. $\lim_{n\to\infty} 2^n = \infty$. The sequence "diverges to ∞".
- $(1, -1, 1, -1, \dots)$. $\lim_{n\to\infty} (-1)^{n+1}$ does not exist. The sequence diverges.
- $(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \dots)$. $\lim_{n\to\infty} \frac{(-1)^{n+1}}{n} = 0$. The sequence converges to 0.

Convergence and Divergence

Theorem

Let (a_n), (b_n), and (c_n) be real sequences such that $a_n \leq b_n \leq c_n$ for all $n > N$, where N is some positive integer. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L \Rightarrow \lim_{n\to\infty} b_n = L.$$

Example

Compute $\displaystyle\lim_{n\to\infty} \frac{\sin n}{n}$.

Basic Properties

Theorem

Let $\lim_{n\to\infty} a_n = A$ *and* $\lim_{n\to\infty} b_n = B$*, where* $A, B \in \mathbb{R}$*. Then*

1. $\lim_{n\to\infty}(a_n \pm b_n) = A + B$
2. $\lim_{n\to\infty}(a_n b_n) = AB$
3. $\lim_{n\to\infty}(ka_n) = kA$ *for any constant* k*.*
4. $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$ *if* $B \neq 0$*.*
5. $\lim_{n\to\infty}(a_n^p) = A^p$ *if* $p > 0$ *and* $a_n > 0$*.*

Basic Properties

Theorem

*Let f be a **continuous** function defined for all $x \geq N$ for some positive integer N. Suppose that (a_n) is a real sequence such that $a_n = f(n)$ for $n \geq N$. then*

$$\lim_{x\to\infty} f(x) = L \Rightarrow \lim_{n\to\infty} a_n = L$$

Example

- Calculate $\lim_{n\to\infty} \frac{1-n}{n^2}$.

  Consider $f(x) = \frac{1-x}{x^2}$, which is defined for all $x > 1$. Thus

$$\lim_{n\to\infty} \frac{1-n}{n^2} = \lim_{x\to\infty} \frac{1-x}{x^2} = 0$$

Basic Properties

Example (Cont.)

- $\lim_{n\to\infty} \sqrt[n]{n}$

  Consider $f(x) = x^{1/x}$, which defined for all $x \geq 1$, thus

$$\lim_{n\to\infty} n^{1/n} = \lim_{x\to\infty} x^{1/x} = \exp\left[\lim_{x\to\infty} \ln(x^{1/x})\right] = \exp\left(\lim_{x\to\infty} \frac{\ln x}{x}\right)$$
$$= \exp\left(\lim_{x\to\infty} \frac{1/x}{1}\right) = e^0 = 1$$

- $\lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} \frac{(n+1)n!}{n!} = \lim_{n\to\infty} (n+1) = \infty$. (diverges to ∞)

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Overview

A ***series*** is a sum of infinitely many numbers.

Definition

A ***series*** is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots ,$$

where a_1, a_2, ... are the terms. We say that a_n is the *n*-th or general term of the series.

Example

- $\frac{1}{3} = 0.3333\ldots = 0.3 + 0.03 + 0.003 + \cdots = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots$
- $\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$

Geometric Series

Definition

A series is ***geometric*** if each term (except for the first) can be obtained from the previous by multiplying it by a constant r, called the ***ratio*** of the series.

Theorem

A geometric series

$$\sum_{n=1}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = ar^0 + ar^1 + ar^2 + ar^3 + \cdots$$

where $a \neq 0$ is a fixed constant, is

1. *convergent if $|r| < 1$, in which case $\displaystyle\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$,*
2. *divergent if $|r| \geq 1$.*

Example of Geometric Series

1. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{10^n} = \frac{3/10^1}{1 - 1/10} = \frac{1}{3}.$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1/2}{1 - 1/2} = 1.$
3. $\displaystyle\sum_{n=1}^{\infty} 2^n$ diverges. (b/c $r = 2 \geq 1$)
4. $\displaystyle\sum_{n=1}^{\infty} \left(-\frac{1}{5}\right)^n = \frac{-1/5}{1 - (-1/5)} = -\frac{1}{6}.$

Non-Geometric Series

- $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \cdots$
- $\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$

Theorem

If $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

Corollary ("Divergence Test")

If $\lim_{n\to\infty} a_n \neq 0$ *or does not exist* (*i.e.,* $a_n \not\to 0$ *as* $n \to \infty$), *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

- $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges. (b/c $\frac{n}{n+1} \to 1$ as $n \to \infty$)
- The harmonic series: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges. (but **NOT** due to the divergence test.)

Partial Sums

Definition

A ***partial sum*** of a series is the sum of all terms up to a_k,

$$s_k = a_1 + a_2 + \cdots + a_{k-1} + a_k = \sum_{n=1}^{k} a_n$$

Theorem/Definition

The series $\sum_{n=1}^{\infty} a_n = s$ iff the sequence of partial sums (s_k) converges to s.

Example

- $\sum_{n=0}^{\infty}(-1)^n$, with $(s_k) = (1, 0, 1, 0, \dots)$
- $\sum_{n=1}^{\infty} \frac{1}{2^n}$, with $(s_k) = (\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots)$
- A "telescoping series" $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$, with $(s_k) = (\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots)$

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Overview

Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

The converse of this statement fails: a series may diverge even though $a_n \to 0$ as $n \to \infty$.

Example

Harmonic series: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges while $\lim_{n\to\infty} \frac{1}{n} = 0$.

Integral Test

Integral Test

Suppose f is a continuous, positive, decreasing function for all $x \geq N$, where N is a positive integer. If $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n < \infty \Leftrightarrow \int_N^{\infty} f(x)\,dx < \infty$$

Sketch of proof.

Note that

$$\sum_{n=N+1}^{\infty} f(n) \cdot 1 \leq \int_N^{\infty} f(x)\,dx \leq \sum_{n=N}^{\infty} f(n) \cdot 1$$

that is,

$$\sum_{n=N+1}^{\infty} a_n \leq \int_N^{\infty} f(x)\,dx \leq \sum_{n=N}^{\infty} a_n$$

□

Integral Test

Example

Harmonic Series: $\sum_{n=1}^{\infty} \frac{1}{n}$.

Consider $f(x) = \frac{1}{x}$, which is continuous, positive, and decreasing for all $x \geq 1$. Note that

$$\int_1^{\infty} \frac{1}{x}\, dx \text{ diverges}$$

hence the harmonic series diverges.

p-series

Definition

A ***p-series*** is a series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where p is a fixed real number.

Example

- $\sum_{n=1}^{\infty} \frac{1}{n}$
- $\sum_{n=1}^{\infty} \frac{1}{n^2}$
- $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$
- $\sum_{n=1}^{\infty} n^{1/3}$

Theorem

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

- converges if $p > 1$,
- diverges if $p \leq 1$.

Proof.

Use the integral test. $\square$

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Limit Comparison Test

Limit Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. Let $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = L$. Then

- If $0 < L < \infty$, then both series converge or both diverge.
- If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
- If $L = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Example

Consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^{3/2}+5}$. Let $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(2n^{3/2}+5)}{1/n^{3/2}} = \lim_{n\to\infty} \frac{n^{3/2}}{2n^{3/2}+5} = \frac{1}{2}$$

By limit comparison test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^{3/2}+5}$ is convergent.

Limit Comparison Test

How to choose $\sum b_n$

To find b_n, simplify the ratio of the largest power of n in the numerator and the largest power of n in the denominator.

Example

- $\sum a_n = \sum_{n=1}^{\infty} \frac{3n+1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{3n+1}{n^2+2n+1}$. Let $b_n = \frac{n}{n^2} = \frac{1}{n}$. Now

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(3n+1)/(n^2+2n+1)}{1/n} = \lim_{n\to\infty} \frac{3n^2+n}{n^2+2n+1} = 3 > 0$$

  Now since $\sum b_n = \sum \frac{1}{n}$ diverges, $\sum a_n$ also diverges.

- $\sum a_n = \sum_{n=1}^{\infty} \frac{1}{n!}$. Try $b_n = \frac{1}{n^2}$, note that $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/n!}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{n!} = 0$. In fact, $\sum_{n=1}^{\infty} \frac{1}{n!} = e - 1$.

Ratio Test

Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series and let $r = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$. Then

- $\sum a_n$ converges if $r < 1$.
- $\sum a_n$ diverges if $r > 1$ or $r = \infty$.
- inconclusive if $r = 1$.

Example

- $\sum a_n = \sum_{n=1}^{\infty} \frac{1}{n!}$. Note that

$$\lim_{n\to\infty} \frac{1/(n+1)!}{1/n!} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Hence $\sum a_n < \infty$.

Ratio Test

Example

- $\sum a_n = \sum_{n=1}^{\infty} \frac{10^n}{(2n)!}$. Note

$$\lim_{n\to\infty} \frac{10^{n+1}/(2n+2)!}{10^n/(2n)!} = \lim_{n\to\infty} \frac{10^{n+1}(2n)!}{10^n(2n+2)!} = \lim_{n\to\infty} \frac{10}{(2n+1)(2n+2)} = 0$$

Hence $\sum a_n < \infty$.

Root Test

Root Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with nonnegative terms and let $r = \lim_{n\to\infty} \sqrt[n]{a_n}$. then

- $\sum a_n$ converges if $0 \le r < 1$.
- $\sum a_n$ diverges if $r > 1$ or $r = \infty$.
- inconclusive if $r = 1$.

Example

- $\sum a_n = \sum_{n=1}^{\infty} \frac{2^n}{n^{10}}$. Note that

$$r = \lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^{10}}} = \lim_{n\to\infty} \frac{2}{\sqrt[n]{n^{10}}} = \lim_{n\to\infty} \frac{2}{n^{10/n}} = 2$$

Hence $\sum a_n$ diverges.

Root Test

Example

- $\sum a_n = 1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \cdots$
  - From ratio test: $(a_{n+1}/a_n) = (1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, \ldots)$ diverges.
  - From root test: note that $a_n \leq \frac{1}{2^{n/2-1}}$, we have

$$\lim_{n\to\infty} \sqrt[n]{a_n} \leq \lim_{n\to\infty} \sqrt[n]{\frac{1}{2^{n/2-1}}} = \frac{1}{\sqrt{2}} < 1$$

  Hence $\sum a_n$ is convergent.

- $\sum a_n = \sum_{n=1}^{\infty} \frac{1}{n!}$. Note that $n! > \lceil n/2 \rceil^{\lceil n/2 \rceil}$, hence $(n!)^{1/n} > \sqrt{n/2}$, thus

$$r = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n!}} < \lim_{n\to\infty} \frac{1}{\sqrt{n/2}} = 0$$

  Hence $\sum a_n$ is convergent.

Direct Comparison Test

Direct Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms, and $0 < a_n \leq b_n$ for all n. Then

- $\sum a_n$ converges if $\sum b_n$ converges;
- $\sum b_n$ diverges if $\sum a_n$ diverges.

Example

- $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2}$. Note that $0 < \dfrac{\cos^2 n}{n^2} < \dfrac{1}{n^2}$ for all n, and $\sum \dfrac{1}{n^2} < \infty$. Hence the series is convergent.
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. Divergence follows from

$$\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{k=1}^{\infty} \sum_{j=2^k}^{2^{k+1}-1} \frac{1}{j} > \sum_{k=1}^{\infty} \sum_{j=2^k}^{2^{k+1}-1} \frac{1}{2^{k+1}} = \sum_{k=1}^{\infty} \frac{2^k}{2^{k+1}} = \sum_{k=1}^{\infty} \frac{1}{2} = \infty$$

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Overview

Consider the Leibniz series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \pm \cdots$.

In general, a series of the form $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ with $b_n > 0$ is called an ***alternating series***.

Alternating Series Test

Given $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ with $b_n > 0$. Then the series is convergent if

(1) $\lim_{n\to\infty} b_n = 0$, and

(2) $b_{n+1} \le b_n$ for all $n \ge N$ for some positive integer N.

Remark

Note that (1) and (2) are often summarized as $b_n \downarrow 0$ (or $b_n \searrow 0$) as $n \to \infty$.

Estimating sums

Alternating Series Estimation Theorem

If $s = \sum_{n=1}^{\infty}(-1)^n b_n$ is the sum of an alternating series ($b_n \downarrow 0$ as $n \to \infty$), then the remainder $r_n = s - s_n$ satisfies

$$|s - s_n| = |r_n| \leq b_{n+1}$$

Example

- $\sum a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \pm \cdots$, then

$$\begin{aligned} s_4 = \frac{7}{12} \Rightarrow \left| s - \frac{7}{12} \right| &= |r_4| \leq b_5 = \frac{1}{5} \\ &\Rightarrow -\frac{1}{5} \leq s - \frac{7}{12} \leq \frac{1}{5} \Rightarrow \frac{23}{60} \leq s \leq \frac{47}{60} \end{aligned}$$

that is, roughly, $0.383 \leq s \leq 0.783$.

Types of Convergence

Definition

A series $\sum a_n$ is called

- absolutely convergent if $\sum |a_n|$ is convergent.
- conditionally convergent if $\sum a_n$ is convergent by not absolutely convergent.

Example

- $\sum a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.
- $\sum a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ is absolutely convergent.

Absolute Convergence Test

Theorem

If $\sum a_n$ is absolutely convergent, then it is convergent. That is,

$$\sum |a_n| < \infty \Rightarrow \sum a_n < \infty$$

(*Equivalently, if $\sum a_n$ diverges, then $\sum |a_n|$ diverges.*)

Proof (without Cauchy sequences).

Note that for all k we have $a_n \leq |a_n|$ with $0 \leq |a_n|$, thus $0 \leq a_n + |a_n| \leq 2|a_n|$. Since $\sum a_n$ is absolutely convergent, then by definition $\sum |a_n| < \infty$. Thus $2\sum |a_n| < \infty$, hence $\sum (a_n + |a_n|) < \infty$ by direct comparison test. Now since

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} (a_k + |a_k|) - \sum_{k=1}^{\infty} |a_k|$$

therefore $\sum a_k < \infty$ as the difference of two convergent series. $\square$

Rearrangements

Theorem (Riemann Series Theorem)

If a series $\sum a_n$ is convergent but not absolutely convergent, then for any $\alpha \in \mathbb{R}$ (including $\pm\infty$), there exists a rearrangement of the series that converges to α.

Acceleration of Convergence

The Shanks Transformation

Let $S_N = \sum_{n=1}^{N} a_n$ be the N-th partial sum of a convergent series $\sum_{n=1}^{\infty} a_n$. Suppose that $S_N = S + AB^N$ with $|B| < 1$, so that $S_N \to S$ as $N \to \infty$. Now we have

$$\begin{aligned} S_{N-1} &= S + AB^{N-1} \\ S_N &= S + AB^N \\ S_{N+1} &= S + AB^{N+1} \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} S_{N-1} - S &= AB^{N-1} \\ S_N - S &= AB^N \\ S_{N+1} - S &= AB^{N+1} \end{aligned}$$

thus

$$\frac{S_N - S}{S_{N-1} - S} = B = \frac{S_{N+1} - S}{S_N - S}$$

thus $S_N^2 - 2SS_N + S^2 = S_{N+1}S_{N-1} - S(S_{N+1} + S_{N-1}) + S^2$, therefore

$$S = \frac{S_N^2 - S_{N+1}S_{N-1}}{2S_N - S_{N+1} - S_{N-1}}$$

Acceleration of Convergence

The Shanks Transformation

We call the nonlinear mapping

$$\mathscr{S}(S_N) = \frac{S_N^2 - S_{N+1}S_{N-1}}{2S_N - S_{N+1} - S_{N-1}}$$

the ***Shanks transformation***.

Acceleration of Convergence

Consider

$$S = 4\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} = 4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right) = \pi$$

Compute the following table.

n	S_n	$\mathscr{S}(S_n)$	$\mathscr{S}^{\circ 2}(S_n)$	$\mathscr{S}^{\circ 3}(S_n)$
0	4.00000000	–	–	–
1	2.66666667	3.16666667	–	–
2	3.46666667	3.13333333	3.14210526	–
3	2.89523810	3.14523810	3.14145022	3.14159936
4	3.33968254	3.13968254	3.14164332	3.14159086
5	2.97604618	3.14271284	3.14157129	3.14159323
6	3.28373848	3.14088134	3.14160284	3.14159244
7	3.01707182	3.14207182	3.14158732	3.14159274
8	3.25236593	3.14125482	3.14159566	3.14159261
9	3.04183962	3.14183962	3.14159086	3.14159267
10	3.23231581	3.14140672	3.14159377	3.14159264
11	3.05840277	3.14173610	3.14159192	3.14159266
12	3.21840277	3.14147969	3.14159314	3.14159265

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Motivation

Recall that we can approximate a function f near the point $x = a$ with the tangent line to the function at $x = a$. For example, near $x = a$,

$$f(x) \approx f(a) + f'(a)(x - a) = f'(a)x + f(a) - af'(a)$$

Tangent Line Approximation

Example

Approximate $f(x) = \cos x$ near $x = 0$ using a ***linear approximation***. Use this to estimate $f(0.05)$, $f(0.4)$, $f(\pi)$.

Note that $f'(0) = -\sin 0 = 0$, we have $f(x) \approx \cos 0 = 1$ near $x = 0$.

x	$\cos x$	linear approximation
0.005	0.998 75	1
0.4	0.921 06	1
π	-1	1

Qudratic Approximation

Example

Approximate $f(x) = \cos x$ near $x = 0$ using a ***quadratic approximation***. Use this to estimate $f(0.05)$, $f(0.4)$, $f(\pi)$.

Note that $f'(0) = -\sin 0 = 0$, we have $f(x) \approx \cos 0 = 1$ near $x = 0$. By symmetry, we choose a quadratic polynomial $1 - cx^2$, with $c > 0$. We choose to match the second derivative, i.e., let $-2c = -\cos 0 = 1$, hence $c = \frac{1}{2}$. Thus we can use $1 - x^2/2$ to approximate $\cos x$ near $x = 0$.

x	$\cos x$	quadratic approximation
0.005	0.998 75	0.998 75
0.4	0.921 06	0.92
π	-1	-3.9348

Next: Try cubic and quartic polynomials.

Taylor Polynomials

Key Idea

Incereasing the degree of the approximating polynomial improves the approximation.

These approximating polynomials are called ***Taylor polynomials***.

Definition

The ***Taylor polynomial*** of order n approximating $f(x)$ near $x = a$ (“centered at a” or “about a”) is

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor Polynomials

Example

Compute the 4-th order Taylor polynomial of $f(x) = \cos x$ centered at $x = 0$.

n	$f^{(n)}(x)$	$f^n(0)$	$f^{(n)}(0)/n!$
0	$\cos x$	1	1
1	$-\sin x$	0	0
2	$-\cos x$	-1	$-1/2$
3	$\sin x$	0	0
4	$\cos x$	1	$1/24$

thus $T_4(x) = 1 - \dfrac{1}{2}x + \dfrac{1}{24}x^4$.

x	$\cos x$	quartic approximation
0.005	0.998 75	0.998 75
0.4	0.921 06	0.921 07
π	-1	0.123 91

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Motivation

Example

For any value of x, the following is a geometric series (i.e., $\sum a \cdot r^n$)

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

Note that $r = x$ and $a = 1$. This series converges if $|r| = |x| < 1$, i.e., $-1 < x < 1$.

Definition

Let x be a variable and $a \in \mathbb{R}$ some constant. Then the series

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

is called a ***power series centered at*** a.

In particular, $\sum_{n=0}^{\infty} c_n x^n$ is a ***power series centered at*** 0.

Power Series

Consider a power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

What happens if we plug in $x = a$?

We have

$$\sum_{n=0}^{\infty} c_n 0^n = c_0 0^0$$

By convention, we define $0^0 := 1$.

Convergence of Power Series

Theorem

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, *there are only three possibilites:*

(i) *The series converges only when* $x = a$*;*

(ii) *The series converges for all* x*;*

(iii) *There is a positive number* R *such that the series converges if* $|x-a| < R$*, and diverges if* $|x-a| > R$*.*

The number R in (iii) is called the ***radius of convergence (ROC)*** of the power series. By convention, we have $R = 0$ in (i) and $R = \infty$ in (ii).

Definition

The ***interval of convergence (IOC)*** of a power series is the interval that consists of all values of x for which the series converges. We have in

(i) IOC $= [a, a]$ or $\{a\}$;

(ii) IOC $= (-\infty, \infty)$;

(iii) Four possibilities, IOC $=$

- $(a-R, a+R)$
- $(a-R, a+R]$
- $[a-R, a+R)$
- $[a-R, a+R]$

Convergence of Power Series

Example

Series	ROC	IOC
$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
$\sum_{n=0}^{\infty} n!x^n$	$R = 0$	$\{0\}$
$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$	$R = \infty$	$(-\infty, \infty)$

Inverval of Convergence

Try the ratio test to find the IOC.

Example

Find the IOC of $\displaystyle\sum_{n=1}^{\infty} \frac{n(x+1)^n}{4^{n+1}}$.

Note that

$$\sum_{n=1}^{\infty} \frac{n(x+1)^n}{4^{n+1}} = \sum_{n=1}^{\infty} \frac{n}{4^{n+1}}[x-(-1)]^n$$

Apply the ratio test

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)(x+1)^{n+1}/4^{n+2}}{n(x+1)^n/4^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x+1)}{4n}\right| \\ &= \frac{|x+1|}{4}\lim_{n\to\infty}\left(\frac{n+1}{n}\right) = \frac{|x+1|}{4}\end{aligned}$$

Thus the series converges if $\dfrac{|x+1|}{4} < 1$ and diverges if $\dfrac{|x+1|}{4} > 1$. Therefore the IOC contains $(-5, 3)$. Check -5 and 3 separately.

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Taylor Series

Motivation

Recall the polynomial approximations of functions:

$$nf(x) \approx T_n(x) = \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Letting $n \to \infty$ yields a power series called a ***Taylor series***:

$$\sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

We say that this is the ***series of*** $f(x)$ if it converges to $f(x)$.
If $a = 0$, then a Taylor series is called a ***Maclaurin series***.

Taylor Series

Example

Find the Maclaurin series of $f(x) = e^x$.

Since $f^{(n)}(x) = e^x$ for all $n \in \mathbb{N}$, thus

$$f(x) = \sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

That is,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

To find the IOC, by the ratio test,

$$\lim_{k\to\infty} \left| \frac{c_{k+1}}{c_k} \right| = \lim_{k\to\infty} \left| \frac{x^{k+1}/(k+1)!}{x^k/k!} \right| = \lim_{n\to\infty} \left| \frac{x}{k+1} \right|$$
$$= |x| \lim_{k\to\infty} \frac{1}{k+1} = 0 \text{ for all } x \in \mathbb{R}$$

Thus the IOC is $(-\infty, \infty)$.

Taylor Series

Example

Find the Maclaurin series of $f(x) = \cos x$.

Note that

$$f^{(n)}(x) = \begin{cases} \cos x, & n = 4k \\ -\sin x, & n = 4k+1 \\ -\cos x, & n = 4k+2 \\ \sin x, & n = 4k+3 \end{cases}, \qquad k \in \mathbb{N}$$

thus

$$f^{(n)}(0) = \begin{cases} 1, & n = 4k \\ -1, & n = 4k+2 \\ 0, & \text{o/w} \end{cases}, \qquad k \in \mathbb{N} \setminus \{0\}$$

Hence

$$\cos x = \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

Taylor Series

Example (Cont.)

To find the IOC, by the ratio test,

$$\lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right| = \lim_{k\to\infty}\left|\frac{(-1)^{k+1}x^{2(k+1)}/(2(k+1))!}{(-1)^k x^{2k}/(2k)!}\right| = \lim_{n\to\infty}\frac{(2k)!}{2k+2}|x|^2$$
$$= |x|^2 \lim_{k\to\infty}\frac{1}{(2k+1)(2k+2)} = 0 \text{ for all } x\in\mathbb{R}$$

Thus the IOC is $(-\infty,\infty)$.

Taylor Series

Example

Find the Maclaurin series of $f(x) = x \cos x$.

Note that

$$\cos x = \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

then

$$x \cos x = x \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k)!}$$

Power Series

Theorem

A real power series may be integrated or differentiated any number of times within the interval of convergence. In particular, a function represented by a power series has derivatives of all orders. Specifically, If the power series $\sum c_n(x-a)^n$ *has radius of convergence* $R > 0$,

(i) $$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}[c_n(x-a)^n] = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

(ii) $$\int\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \int c_n(x-a)^n = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$$

The ***radii of convergence*** *of the power series in Equations* (i) *and* (ii) *are both* R.

Remark

The **radius of convergence** remains the same when a power series is differentiated or integrated, this does not mean that the **interval of convergence** remains the same. For example, consider $f(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2}$, f', and f''.

Power Series

Example

Given power series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

then by differentiating both sides, we get

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + x^3 + \cdots = \sum_{n=1}^{\infty} nx^n = \sum_{n=0}^{\infty} (n+1)x^n$$

Power Series

Example

Given power series

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$$

then by integrating both sides, we get

$$\begin{aligned} \ln(1+x) &= \int \frac{dx}{1+x} = \int (1 - x + x^2 - x^3 + \cdots)\, dx \\ &= C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \\ &= C + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \qquad |x| < 1 \end{aligned}$$

Let $x = 0$, we have $\ln(1+0) = C$, thus $C = 0$.

Power Series

Example

Find a power series representation for $f(x) = \arctan x$.

Observe that $f'(x) = 1/(1+x^2)$. Then

$$\begin{aligned} \arctan x &= \int \frac{dx}{1+x^2} = \int (1 - x^2 + x^4 - x^6 + \cdots)\, dx \\ &= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \end{aligned}$$

Let $x = 0$, then we have $C = \arctan 0 = 0$. Therefore

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$