7.1 Integration by Parts

Integration by Parts

Every differentiation rule has a corresponding integration rule. For instance, the Substitution Rule for integration corresponds to the Chain Rule for differentiation. The rule that corresponds to the Product Rule for differentiation is called the rule for *integration by parts.*

The Product Rule states that if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Integration by Parts

In the notation for indefinite integrals this equation becomes

$$\int [f(x)g'(x) + g(x)f'(x)]\, dx = f(x)g(x)$$

or
$$\int f(x)g'(x)\, dx + \int g(x)f'(x)\, dx = f(x)g(x)$$

We can rearrange this equation as

$$\boxed{1} \qquad \boxed{\int f(x)g'(x)\, dx = f(x)g(x) - \int g(x)f'(x)\, dx}$$

Formula 1 is called the **formula for integration by parts**.

Integration by Parts

It is perhaps easier to remember in the following notation.

Let $u = f(x)$ and $v = g(x)$. Then the differentials are $du = f'(x)\,dx$ and $dv = g'(x)\,dx$, so, by the Substitution Rule, the formula for integration by parts becomes

2

$$\int u\,dv = uv - \int v\,du$$

Example 1

Find $\int x \sin x \, dx$.

Solution Using Formula 1:

Suppose we choose $f(x) = x$ and $g'(x) = \sin x$. Then $f'(x) = 1$ and $g(x) = -\cos x$. (For g we can choose *any* antiderivative of g'.) Thus, using Formula 1, we have

$$\begin{aligned}
\int x \sin x \, dx &= f(x)g(x) - \int g(x)f'(x) \, dx \\
&= x(-\cos x) - \int (-\cos x) \, dx \\
&= -x \cos x + \int \cos x \, dx \\
&= -x \cos x + \sin x + C
\end{aligned}$$

Example 1 – *Solution*

cont'd

It's wise to check the answer by differentiating it. If we do so, we get $x \sin x$, as expected.

Solution Using Formula 2:

Let

$$u = x \qquad\qquad dv = \sin x \, dx$$

Then

$$du = dx \qquad\qquad v = -\cos x$$

and so

$$\int x \sin x \, dx = \int \overbrace{x}^{u} \, \overbrace{\sin x \, dx}^{dv}$$

Example 1 – *Solution*

cont'd

$$= \underbrace{x}_{u}\,\underbrace{(-\cos x)}_{v} - \int \underbrace{(-\cos x)}_{v}\,\underbrace{dx}_{du}$$

$$= -x \cos x + \int \cos x \, dx$$

$$= -x \cos x + \sin x + C$$

Integration by Parts

If we combine the formula for integration by parts with Part 2 of Fundamental Theorem of Calculus, we can evaluate definite integrals by parts.

Evaluating both sides of Formula 1 between *a* and *b*, assuming f' and g' are continuous, and using the Fundamental Theorem, we obtain

$$\boxed{6} \qquad \int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b g(x)f'(x)\,dx$$

7.2 Trigonometric Integrals

Trigonometric Integrals

In this section we use trigonometric identities to integrate certain combinations of trigonometric functions.

We start with powers of sine and cosine.

Example 2

Find $\int \sin^5 x \cos^2 x \, dx$.

Solution:

We could convert $\cos^2 x$ to $1 - \sin^2 x$, but we would be left with an expression in terms of $\sin x$ with no extra $\cos x$ factor.

Instead, we separate a single sine factor and rewrite the remaining $\sin^4 x$ factor in terms of $\cos x$:

$$\sin^5 x \cos^2 x = (\sin^2 x)^2 \cos^2 x \sin x$$

$$= (1 - \cos^2 x)^2 \cos^2 x \sin x$$

Example 2 – *Solution*

cont'd

Substituting $u = \cos x$, we have $du = -\sin x\, dx$ and so

$$\int \sin^5 x \cos^2 x\, dx = \int (\sin^2 x)^2 \cos^2 x \sin x\, dx$$

$$= \int (1 - \cos^2 x)^2 \cos^2 x \sin x\, dx$$

$$= \int (1 - u^2)^2\, u^2\, (-du) = -\int (u^2 - 2u^4 + u^6)du$$

$$= -\left(\frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7}\right) + C$$

$$= -\tfrac{1}{3}\cos^3 x + \tfrac{2}{5}\cos^5 x - \tfrac{1}{7}\cos^7 x + C$$

Example 3

Evaluate $\int_0^{\pi} \sin^2 x\, dx.$

Solution:

If we write $\sin^2 x = 1 - \cos^2 x$, the integral is no simpler to evaluate. Using the half-angle formula for $\sin^2 x$, however, we have

$$\int_0^{\pi} \sin^2 x\, dx = \tfrac{1}{2}\int_0^{\pi} (1 - \cos 2x)\, dx$$

$$= \left[\tfrac{1}{2}\left(x - \tfrac{1}{2}\sin 2x\right)\right]_0^{\pi}$$

Example 3 – *Solution*

cont'd

$$= \tfrac{1}{2}\left(\pi - \tfrac{1}{2}\sin 2\pi\right) - \tfrac{1}{2}\left(0 - \tfrac{1}{2}\sin 0\right)$$

$$= \tfrac{1}{2}\pi$$

Notice that we mentally made the substitution $u = 2x$ when integrating $\cos 2x$.

Trigonometric Integrals

To summarize, we list guidelines to follow when evaluating integrals of the form $\int \sin^m x \cos^n x \, dx$, where $m \geq 0$ and $n \geq 0$ are integers.

Trigonometric Integrals

Strategy for Evaluating $\int \sin^m x \cos^n x\, dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x\, dx = \int \sin^m x\, (\cos^2 x)^k \cos x\, dx$$

$$= \int \sin^m x\, (1 - \sin^2 x)^k \cos x\, dx$$

Then substitute $u = \sin x$.

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x\, dx = \int (\sin^2 x)^k \cos^n x\, \sin x\, dx$$

$$= \int (1 - \cos^2 x)^k \cos^n x\, \sin x\, dx$$

Then substitute $u = \cos x$. [Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \tfrac{1}{2}\sin 2x$$

Trigonometric Integrals

We can use a similar strategy to evaluate integrals of the form $\int \tan^m x \sec^n x\, dx$.

Since $(d/dx) \tan x = \sec^2 x$, we can separate a $\sec^2 x$ factor and convert the remaining (even) power of secant to an expression involving tangent using the identity $\sec^2 x = 1 + \tan^2 x$.

Or, since $(d/dx) \sec x = \sec x \tan x$, we can separate a $\sec x \tan x$ factor and convert the remaining (even) power of tangent to secant.

Example 5

Evaluate $\int \tan^6 x \sec^4 x \, dx$.

Solution:

If we separate one $\sec^2 x$ factor, we can express the remaining $\sec^2 x$ factor in terms of tangent using the identity $\sec^2 x = 1 + \tan^2 x$.

We can then evaluate the integral by substituting $u = \tan x$ so that $du = \sec^2 x \, dx$:

$$\int \tan^6 x \sec^4 x \, dx = \int \tan^6 x \sec^2 x \sec^2 x \, dx$$

Example 5 – *Solution*

cont'd

$$= \int \tan^6 x\,(1 + \tan^2 x)\sec^2 x\,dx$$

$$= \int u^6(1 + u^2)du = \int (u^6 + u^8)du$$

$$= \frac{u^7}{7} + \frac{u^9}{9} + C$$

$$= \tfrac{1}{7}\tan^7 x + \tfrac{1}{9}\tan^9 x + C$$

Trigonometric Integrals

The preceding examples demonstrate strategies for evaluating integrals of the form $\int \tan^m x \sec^n x \, dx$ for two cases, which we summarize here.

Strategy for Evaluating $\int \tan^m x \sec^n x \, dx$

(a) If the power of secant is even ($n = 2k$, $k \geqslant 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \sec^{2k} x \, dx = \int \tan^m x \, (\sec^2 x)^{k-1} \sec^2 x \, dx$$

$$= \int \tan^m x \, (1 + \tan^2 x)^{k-1} \sec^2 x \, dx$$

Then substitute $u = \tan x$.

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2k+1} x \sec^n x \, dx = \int (\tan^2 x)^k \sec^{n-1} x \, \sec x \, \tan x \, dx$$

$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \, \sec x \, \tan x \, dx$$

Then substitute $u = \sec x$.

Trigonometric Integrals

For other cases, the guidelines are not as clear-cut. We may need to use identities, integration by parts, and occasionally a little ingenuity.

We will sometimes need to be able to integrate tan *x* by using the formula given below:

$$\boxed{\int \tan x \, dx = \ln |\sec x| + C}$$

Trigonometric Integrals

We will also need the indefinite integral of secant:

1 $$\boxed{\int \sec x \, dx = \ln |\sec x + \tan x| + C}$$

We could verify Formula 1 by differentiating the right side, or as follows. First we multiply numerator and denominator by sec x + tan x:

$$\int \sec x \, dx = \int \sec x \, \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$$

Trigonometric Integrals

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$

If we substitute $u = \sec x + \tan x$, then
$du = (\sec x \tan x + \sec^2 x)dx$, so the integral becomes

$$\int (1/u)\, du = \ln |u| + C.$$

Thus we have

$$\int \sec x\, dx = \ln |\sec x + \tan x| + C$$

Example 7

Find $\int \tan^3 x \, dx$.

Solution:

Here only $\tan x$ occurs, so we use $\tan^2 x = \sec^2 x - 1$ to rewrite a $\tan^2 x$ factor in terms of $\sec^2 x$:

$$\int \tan^3 x \, dx = \int \tan x \tan^2 x \, dx$$

$$= \int \tan x \, (\sec^2 x - 1) \, dx$$

$$= \int \tan x \sec^2 x \, dx - \int \tan x \, dx$$

Example 7 – *Solution*

cont'd

$$= \frac{\tan^2 x}{2} - \ln|\sec x| + C$$

In the first integral we mentally substituted $u = \tan x$ so that $du = \sec^2 x \, dx$.

Trigonometric Integrals

Finally, we can make use of another set of trigonometric identities:

2 To evaluate the integrals (a) $\int \sin mx \cos nx \, dx$, (b) $\int \sin mx \sin nx \, dx$, or (c) $\int \cos mx \cos nx \, dx$, use the corresponding identity:

$$\text{(a)} \quad \sin A \cos B = \tfrac{1}{2}[\sin(A - B) + \sin(A + B)]$$

$$\text{(b)} \quad \sin A \sin B = \tfrac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\text{(c)} \quad \cos A \cos B = \tfrac{1}{2}[\cos(A - B) + \cos(A + B)]$$

Example 9

Evaluate $\int \sin 4x \cos 5x \, dx$.

Solution:

This integral could be evaluated using integration by parts, but it's easier to use the identity in Equation 2(a) as follows:

$$\int \sin 4x \cos 5x \, dx = \int \tfrac{1}{2}[\sin(-x) + \sin 9x] \, dx$$

$$= \tfrac{1}{2}\int (-\sin x + \sin 9x) \, dx$$

$$= \tfrac{1}{2}(\cos x - \tfrac{1}{9}\cos 9x) + C$$

7.3 Trigonometric Substitution

Trigonometric Substitution

In finding the area of a circle or an ellipse, an integral of the form $\int \sqrt{a^2 - x^2}\, dx$ arises, where $a > 0$.

If it were $\int x\sqrt{a^2 - x^2}\, dx$, the substitution $u = a^2 - x^2$ would be effective but, as it stands, $\int \sqrt{a^2 - x^2}\, dx$ is more difficult.

Trigonometric Substitution

If we change the variable from x to θ by the substitution $x = a \sin \theta$, then the identity $1 - \sin^2\theta = \cos^2\theta$ allows us to get rid of the root sign because

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2\theta}$$

$$= \sqrt{a^2(1 - \sin^2\theta)}$$

$$= \sqrt{a^2 \cos^2\theta}$$

$$= a\left|\cos \theta\right|$$

Trigonometric Substitution

Notice the difference between the substitution $u = a^2 - x^2$ (in which the new variable is a function of the old one) and the substitution $x = a \sin\theta$ (the old variable is a function of the new one).

In general, we can make a substitution of the form $x = g(t)$ by using the Substitution Rule in reverse.

To make our calculations simpler, we assume that g has an inverse function; that is, g is one-to-one.

Trigonometric Substitution

In this case, if we replace *u* by *x* and *x* by *t* in the Substitution Rule, we obtain

$$\int f(x)\,dx = \int f(g(t))g'(t)\,dt$$

This kind of substitution is called *inverse substitution*.

We can make the inverse substitution $x = a \sin \theta$ provided that it defines a one-to-one function.

Trigonometric Substitution

This can be accomplished by restricting θ to lie in the interval $[-\pi/2, \pi/2]$.

In the following table we list trigonometric substitutions that are effective for the given radical expressions because of the specified trigonometric identities.

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Trigonometric Substitution

In each case the restriction on θ is imposed to ensure that the function that defines the substitution is one-to-one.

Example 1

Evaluate $\int \frac{\sqrt{9 - x^2}}{x^2}\, dx.$

Solution:

Let $x = 3 \sin \theta$, where $-\pi/2 \leq \theta \leq \pi/2$. Then $dx = 3 \cos \theta \, d\theta$ and

$$\begin{aligned} \sqrt{9 - x^2} &= \sqrt{9 - 9 \sin^2\theta} \\ &= \sqrt{9 \cos^2\theta} \\ &= 3\,|\cos \theta| \\ &= 3 \cos \theta \end{aligned}$$

(Note that $\cos \theta \geq 0$ because $-\pi/2 \leq \theta \leq \pi/2$.)

Example 1 – *Solution*

cont'd

Thus the Inverse Substitution Rule gives

$$\int \frac{\sqrt{9 - x^2}}{x^2}\,dx = \int \frac{3\cos\theta}{9\sin^2\theta}\,3\cos\theta\,d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \int \cot^2\theta\,d\theta$$

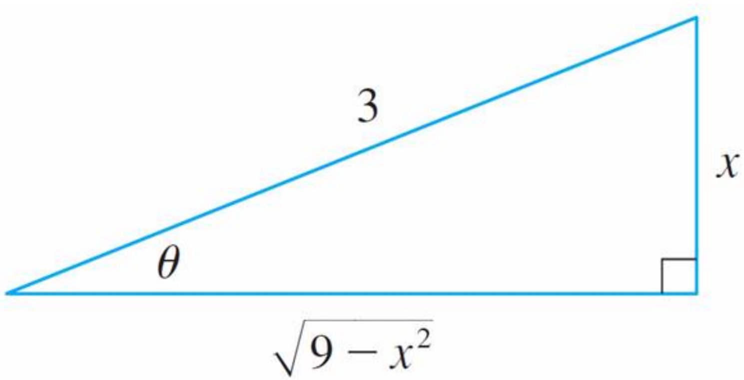
$$= \int (\csc^2\theta - 1)\,d\theta$$

$$= -\cot\theta - \theta + C$$

Example 1 – *Solution*

cont'd

Since this is an indefinite integral, we must return to the original variable x. This can be done either by using trigonometric identities to express cot θ in terms of sin $\theta = x/3$ or by drawing a diagram, as in Figure 1, where θ is interpreted as an angle of a right triangle.



$\sin\theta = \dfrac{x}{3}$

Figure 1

Example 1 – *Solution*

cont'd

Since sin $\theta = x/3$, we label the opposite side and the hypotenuse as having lengths x and 3.

Then the Pythagorean Theorem gives the length of the adjacent side as $\sqrt{9 - x^2}$, so we can simply read the value of cot θ from the figure:

$$\cot\theta = \frac{\sqrt{9 - x^2}}{x}$$

(Although $\theta > 0$ in the diagram, this expression for cot θ is valid even when $\theta < 0$.)

Example 1 – *Solution*

cont'd

Since sin $\theta = x/3$, we have $\theta = \sin^{-1}(x/3)$ and so

$$\int \frac{\sqrt{9 - x^2}}{x^2}\,dx = -\frac{\sqrt{9 - x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

Example 2

Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solution:

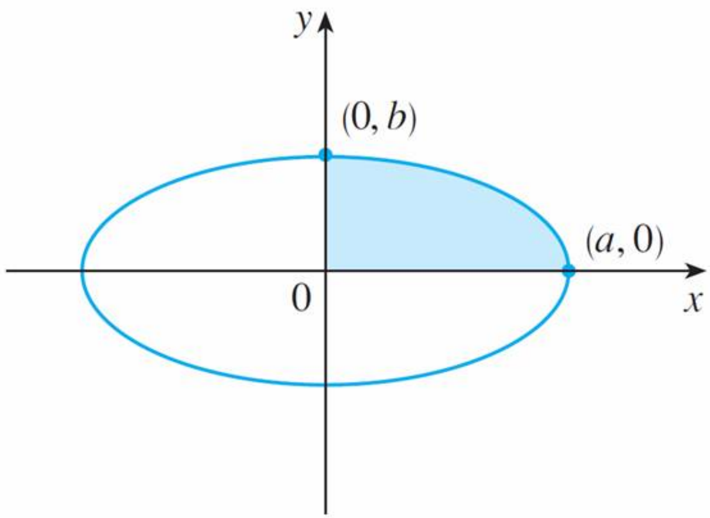
Solving the equation of the ellipse for *y*, we get

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2} \qquad \text{or} \qquad y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$

Example 2 – *Solution*

cont'd

Because the ellipse is symmetric with respect to both axes, the total area A is four times the area in the first quadrant (see Figure 2).



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Figure 2

Example 2 – *Solution*

cont'd

The part of the ellipse in the first quadrant is given by the function

$$y = \frac{b}{a}\sqrt{a^2 - x^2} \qquad 0 \leqslant x \leqslant a$$

and so

$$\tfrac{1}{4}A = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx$$

To evaluate this integral we substitute $x = a \sin \theta$.
Then $dx = a \cos \theta \, d\theta$.

Example 2 – *Solution*

cont'd

To change the limits of integration we note that when $x = 0$, $\sin\theta = 0$, so $\theta = 0$; when $x = a$, $\sin\theta = 1$, so $\theta = \pi/2$. Also

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta}$$

$$= \sqrt{a^2\cos^2\theta}$$

$$= a\left|\cos\theta\right|$$

$$= a\cos\theta$$

Example 2 – *Solution*

cont'd

Since $0 \le \theta \le \pi/2$. Therefore

$$A = 4\frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$

$$= 4\frac{b}{a}\int_0^{\pi/2} a\cos\theta \cdot a\cos\theta\,d\theta$$

$$= 4ab\int_0^{\pi/2} \cos^2\theta\,d\theta$$

$$= 4ab\int_0^{\pi/2} \tfrac{1}{2}(1 + \cos 2\theta)\,d\theta$$

Example 2 – *Solution*

cont'd

$$= 2ab\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$

$$= 2ab\left(\frac{\pi}{2} + 0 - 0\right)$$

$$= \pi ab$$

We have shown that the area of an ellipse with semiaxes a and b is πab.

In particular, taking $a = b = r$, we have proved the famous formula that the area of a circle with radius r is πr^2.

Trigonometric Substitution

Note:

Since the integral in Example 2 was a definite integral, we changed the limits of integration and did not have to convert back to the original variable x.

Example 3

Find $\int \frac{1}{x^2\sqrt{x^2 + 4}}\,dx.$

Solution:

Let $x = 2\tan\theta$, $-\pi/2 < \theta < \pi/2$. Then $dx = 2\sec^2\theta\, d\theta$ and

$$\sqrt{x^2 + 4} = \sqrt{4(\tan^2\theta + 1)} = \sqrt{4\sec^2\theta}$$

$$= 2|\sec\theta|$$

$$= 2\sec\theta$$

Example 3 – *Solution*

cont'd

Thus we have

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2\sec^2\theta\, d\theta}{4\tan^2\theta \cdot 2\sec\theta}$$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

To evaluate this trigonometric integral we put everything in terms of sin θ and cos θ:

$$\frac{\sec\theta}{\tan^2\theta} = \frac{1}{\cos\theta} \cdot \frac{\cos^2\theta}{\sin^2\theta}$$

Example 3 – *Solution*

cont'd

$$= \frac{\cos\theta}{\sin^2\theta}$$

Therefore, making the substitution $u = \sin\theta$, we have

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$$= \frac{1}{4}\int \frac{du}{u^2}$$

$$= \frac{1}{4}\left(-\frac{1}{u}\right) + C$$

Example 3 – *Solution*

cont'd

$$= -\frac{1}{4 \sin \theta} + C$$

$$= -\frac{\csc \theta}{4} + C$$

Example 3 – *Solution*

cont'd

We use Figure 3 to determine that csc $\theta = \sqrt{x^2 + 4}/x$ and so

$$\int \frac{dx}{x^2\sqrt{x^2 + 4}} = -\frac{\sqrt{x^2 + 4}}{4x} + C$$

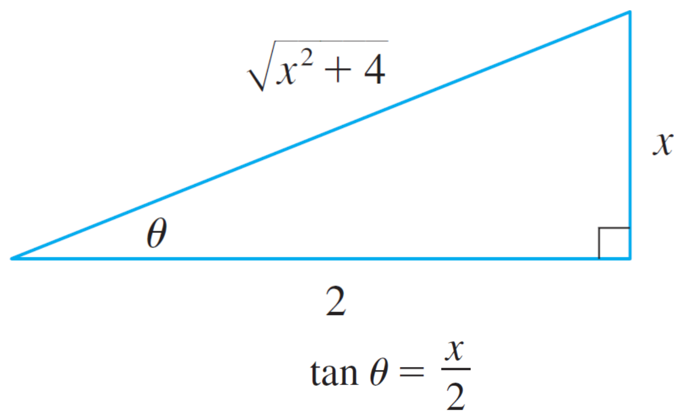


Figure 3

Example 5

Evaluate $\int \frac{dx}{\sqrt{x^2 - a^2}}$, where $a > 0$.

Solution 1:

We let $x = a \sec \theta$, where $0 < \theta < \pi/2$ or $\pi < \theta < 3\pi/2$.

Then $dx = a \sec \theta \tan \theta \, d\theta$ and

$$\begin{aligned} \sqrt{x^2 - a^2} &= \sqrt{a^2(\sec^2\theta - 1)} \\ &= \sqrt{a^2 \tan^2\theta} \\ &= a\left|\tan \theta\right| \\ &= a \tan \theta \end{aligned}$$

Example 5 – *Solution* 1

cont'd

Therefore

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a \sec \theta \tan \theta}{a \tan \theta}\, d\theta$$

$$= \int \sec \theta \, d\theta$$

$$= \ln \left| \sec \theta + \tan \theta \right| + C$$

Example 5 – *Solution* 1

cont'd

The triangle in Figure 4 gives $\tan\theta = \sqrt{x^2 - a^2}/a$, so we have

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right| + C$$

$$= \ln\left|x + \sqrt{x^2 - a^2}\right| - \ln a + C$$

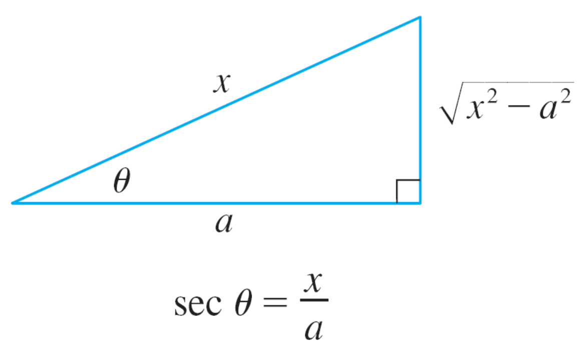


Figure 4

Example 5 – *Solution* 1

cont'd

Writing $C_1 = C - \ln a$, we have

$$\boxed{1} \qquad \int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| x + \sqrt{x^2 - a^2} \right| + C_1$$

Example 5 – *Solution 2*

cont'd

For $x > 0$ the hyperbolic substitution $x = a \cosh t$ can also be used.

Using the identity $\cosh^2 y - \sinh^2 y = 1$, we have

$$\sqrt{x^2 - a^2} = \sqrt{a^2(\cosh^2 t - 1)}$$

$$= \sqrt{a^2 \sinh^2 t}$$

$$= a \sinh t$$

Example 5 – *Solution 2*

cont'd

Since $dx = a \sinh t\, dt$, we obtain

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a \sinh t\, dt}{a \sinh t}$$

$$= \int dt$$

$$= t + C$$

Since $\cosh t = x/a$, we have $t = \cosh^{-1}(x/a)$ and

$$\boxed{2} \qquad \int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C$$

Trigonometric Substitution

Note:
As Example 5 illustrates, hyperbolic substitutions can be used in place of trigonometric substitutions and sometimes they lead to simpler answers.

But we usually use trigonometric substitutions because trigonometric identities are more familiar than hyperbolic identities.

Example 6

Find $\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2 + 9)^{3/2}}\, dx.$

Solution:

First we note that $(4x^2 + 9)^{3/2} = \left(\sqrt{4x^2 + 9}\right)^3$ so trigonometric substitution is appropriate.

Although $\sqrt{4x^2 + 9}$ is not quite one of the expressions in the table of trigonometric substitutions, it becomes one of them if we make the preliminary substitution $u = 2x$.

Example 6 – *Solution*

cont'd

When we combine this with the tangent substitution, we have $x = \frac{3}{2}\tan\theta$, which gives $dx = \frac{3}{2}\sec^2\theta\, d\theta$ and

$$\sqrt{4x^2 + 9} = \sqrt{9\tan^2\theta + 9}$$

$$= 3\sec\theta$$

When $x = 0$, $\tan\theta = 0$, so $\theta = 0$; when $x = 3\sqrt{3}/2$, $\tan\theta = \sqrt{3}$, so $\theta = \pi/3$.

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2 + 9)^{3/2}}\, dx = \int_0^{\pi/3} \frac{\frac{27}{8}\tan^3\theta}{27\sec^3\theta}\, \frac{3}{2}\sec^2\theta\, d\theta$$

Example 6 – *Solution*

cont'd

$$= \frac{3}{16} \int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta}\, d\theta$$

$$= \frac{3}{16} \int_0^{\pi/3} \frac{\sin^3\theta}{\cos^2\theta}\, d\theta$$

$$= \frac{3}{16} \int_0^{\pi/3} \frac{1 - \cos^2\theta}{\cos^2\theta} \sin\theta\, d\theta$$

Now we substitute $u = \cos\theta$ so that $du = -\sin\theta\, d\theta$.
When $\theta = 0$, $u = 1$; when $\theta = \pi/3$, $u = \frac{1}{2}$.

Example 6 – *Solution*

cont'd

Therefore

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx = -\tfrac{3}{16}\int_1^{1/2} \frac{1-u^2}{u^2}\,du$$

$$= \tfrac{3}{16}\int_1^{1/2} (1-u^{-2})\,du$$

$$= \tfrac{3}{16}\left[u + \frac{1}{u}\right]_1^{1/2}$$

$$= \tfrac{3}{16}\left[\left(\tfrac{1}{2}+2\right) - (1+1)\right]$$

$$= \frac{3}{32}$$

7.4 Integration of Rational Functions by Partial Fractions

Integration of Rational Functions by Partial Fractions

In this section we show how to integrate any rational function (a ratio of polynomials) by expressing it as a sum of simpler fractions, called *partial fractions*, that we already know how to integrate.

To illustrate the method, observe that by taking the fractions $2/(x - 1)$ and $1/(x + 2)$ to a common denominator we obtain

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2}$$

Integration of Rational Functions by Partial Fractions

If we now reverse the procedure, we see how to integrate the function on the right side of this equation:

$$\int \frac{x+5}{x^2+x-2}\,dx = \int \left(\frac{2}{x-1} - \frac{1}{x+2} \right) dx$$

$$= 2 \ln |x - 1| - \ln |x + 2| + C$$

Integration of Rational Functions by Partial Fractions

To see how the method of partial fractions works in general, let's consider a rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

where *P* and *Q* are polynomials. It's possible to express *f* as a sum of simpler fractions provided that the degree of *P* is less than the degree of *Q*. Such a rational function is called *proper*.

Integration of Rational Functions by Partial Fractions

Recall that if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$, then the degree of P is n and we write $\deg(P) = n$.

If f is *improper*, that is, $\deg(P) \geq \deg(Q)$, then we must take the preliminary step of dividing Q into P (by long division) until a remainder $R(x)$ is obtained such that $\deg(R) < \deg(Q)$.

Integration of Rational Functions by Partial Fractions

The division statement is

$$\boxed{1} \qquad f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where *S* and *R* are also polynomials.

As the next example illustrates, sometimes this preliminary step is all that is required.

Example 1

Find $\int \frac{x^3 + x}{x - 1}\,dx.$

Solution:

Since the degree of the numerator is greater than the degree of the denominator, we first perform the long division.

This enables us to write

$$\int \frac{x^3 + x}{x - 1}\,dx = \int \left(x^2 + x + 2 + \frac{2}{x - 1} \right) dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x - 1| + C$$

Integration of Rational Functions by Partial Fractions

The next step is to factor the denominator $Q(x)$ as far as possible.

It can be shown that any polynomial Q can be factored as a product of linear factors (of the form $ax + b$) and irreducible quadratic factors (of the form $ax^2 + bx + c$, where $b^2 - 4ac < 0$).

For instance, if $Q(x) = x^4 - 16$, we could factor it as

$$Q(x) = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4)$$

The third step is to express the proper rational function $R(x)/Q(x)$(from Equation 1) as a sum of **partial fractions** of the form

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^j}$$

A theorem in algebra guarantees that it is always possible to do this. We explain the details for the four cases that occur.

Integration of Rational Functions by Partial Fractions

Case I The denominator $Q(x)$ is a product of distinct linear factors.

This means that we can write

$$Q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_kx + b_k)$$

where no factor is repeated (and no factor is a constant multiple of another).

Integration of Rational Functions by Partial Fractions

In this case the partial fraction theorem states that there exist constants $A_1, A_2, \ldots, A_k$ such that

$$\boxed{2} \qquad \frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

These constants can be determined as in the next example.

Example 2

Evaluate $\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\, dx.$

Solution:

Since the degree of the numerator is less than the degree of the denominator, we don't need to divide.

We factor the denominator as

$$2x^3 + 3x^2 - 2x = x(2x^2 + 3x - 2)$$

$$= x(2x - 1)(x + 2)$$

Example 2 – *Solution*

cont'd

Since the denominator has three distinct linear factors, the partial fraction decomposition of the integrand $\boxed{2}$ has the form

$$\boxed{3} \qquad \frac{x^2 + 2x - 1}{x(2x - 1)(x + 2)} = \frac{A}{x} + \frac{B}{2x - 1} + \frac{C}{x + 2}$$

To determine the values of A, B, and C, we multiply both sides of this equation by the product of the denominators, $x(2x - 1)(x + 2)$, obtaining

$$\boxed{4} \qquad x^2 + 2x - 1 = A(2x - 1)(x + 2) + Bx(x + 2) + Cx(2x - 1)$$

Example 2 – *Solution*

cont'd

Expanding the right side of Equation 4 and writing it in the standard form for polynomials, we get

$$\boxed{5} \quad x^2 + 2x - 1 = (2A + B + 2C)x^2 + (3A + 2B - C)x - 2A$$

The polynomials in Equation 5 are identical, so their coefficients must be equal. The coefficient of x^2 on the right side, $2A + B + 2C$, must equal the coefficient of x^2 on the left side—namely, 1.

Likewise, the coefficients of x are equal and the constant terms are equal.

Example 2 – *Solution*

cont'd

This gives the following system of equations for *A*, *B*, and *C*:

$$2A + B + 2C = 1$$

$$3A + 2B - C = 2$$

$$-2A = -1$$

Solving, we get, $A = \frac{1}{2}$, $B = \frac{1}{5}$, and $C = -\frac{1}{10}$, and so

$$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\,dx = \int \left[\frac{1}{2}\frac{1}{x} + \frac{1}{5}\frac{1}{2x-1} - \frac{1}{10}\frac{1}{x+2}\right] dx$$

Example 2 – *Solution*

cont'd

$$= \tfrac{1}{2}\ln|x| + \tfrac{1}{10}\ln|2x - 1| - \tfrac{1}{10}\ln|x + 2| + K$$

In integrating the middle term we have made the mental substitution $u = 2x - 1$, which gives $du = 2\,dx$ and $dx = \frac{1}{2}\,du$.

Note:
We can use an alternative method to find the coefficients *A*, *B* and *C* in Example 2. Equation 4 is an identity; it is true for every value of *x*. Let's choose values of *x* that simplify the equation.

If we put $x = 0$ in Equation 4, then the second and third terms on the right side vanish and the equation then becomes $-2A = -1$, or $A = \frac{1}{2}$.

Likewise, $x = \frac{1}{2}$ gives $5B/4 = \frac{1}{4}$ and $x = -2$ gives $10C = -1$, so $B = \frac{1}{5}$ and $C = -\frac{1}{10}$.

Integration of Rational Functions by Partial Fractions

Case II: $Q(x)$ is a product of linear factors, some of which are repeated.

Suppose the first linear factor $(a_1x + b_1)$ is repeated r times; that is, $(a_1x + b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x + b_1)$ in Equation 2, we would use

$$\boxed{7} \qquad \frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}$$

Integration of Rational Functions by Partial Fractions

By way of illustration, we could write

$$\frac{x^3 - x + 1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

but we prefer to work out in detail a simpler example.

Example 4

Find $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx.$

Solution:

The first step is to divide. The result of long division is

$$\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \frac{4x}{x^3 - x^2 - x + 1}$$

Example 4 – *Solution*

cont'd

The second step is to factor the denominator $Q(x) = x^3 - x^2 - x + 1$.

Since $Q(1) = 0$, we know that $x - 1$ is a factor and we obtain

$$x^3 - x^2 - x + 1 = (x - 1)(x^2 - 1)$$

$$= (x - 1)(x - 1)(x + 1)$$

$$= (x - 1)^2(x + 1)$$

Example 4 – *Solution*

cont'd

Since the linear factor $x – 1$ occurs twice, the partial fraction decomposition is

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

Multiplying by the least common denominator, $(x – 1)^2(x + 1)$, we get

$$\boxed{8} \qquad 4x = A(x – 1)(x + 1) + B(x + 1) + C(x – 1)^2$$

Example 4 – *Solution*

cont'd

$$= (A + C)x^2 + (B - 2C)x + (-A + B + C)$$

Now we equate coefficients:

$$A + C = 0$$

$$B - 2C = 4$$

$$-A + B + C = 0$$

Example 4 – *Solution*

cont'd

Solving, we obtain $A = 1$, $B = 2$, and $C = –1$, so

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx = \int \left[x + 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1} \right] dx$$

$$= \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + K$$

$$= \frac{x^2}{2} + x - \frac{2}{x-1} + \ln\left|\frac{x-1}{x+1}\right| + K$$

Case III: $Q(x)$ contains irreducible quadratic factors, none of which is repeated.

If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then, in addition to the partial fractions in Equations 2 and 7, the expression for $R(x)/Q(x)$ will have a term of the form

$$\boxed{9} \qquad \frac{Ax + B}{ax^2 + bx + c}$$

where A and B are constants to be determined.

Integration of Rational Functions by Partial Fractions

For instance, the function given by
$f(x) = x/[(x - 2)(x^2 + 1)(x^2 + 4)]$ has a partial fraction decomposition of the form

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}$$

The term given in $\boxed{9}$ can be integrated by completing the square (if necessary) and using the formula

10

$$\boxed{\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C}$$

Example 6

Evaluate $\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3}\, dx.$

Solution:

Since the degree of the numerator is *not less than* the degree of the denominator, we first divide and obtain

$$\frac{4x^2 - 3x + 2}{4x^2 - 4x + 3} = 1 + \frac{x - 1}{4x^2 - 4x + 3}$$

Example 6 – *Solution*

cont'd

Notice that the quadratic $4x^2 - 4x + 3$ is irreducible because its discriminant is $b^2 - 4ac = -32 < 0$. This means it can't be factored, so we don't need to use the partial fraction technique.

To integrate the given function we complete the square in the denominator:

$$4x^2 - 4x + 3 = (2x - 1)^2 + 2$$

This suggests that we make the substitution $u = 2x - 1$.

Example 6 – *Solution*

cont'd

Then $du = 2\,dx$ and $x = \frac{1}{2}(u + 1)$, so

$$\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3}\,dx = \int \left(1 + \frac{x - 1}{4x^2 - 4x + 3}\right) dx$$

$$= x + \tfrac{1}{2} \int \frac{\frac{1}{2}(u + 1) - 1}{u^2 + 2}\,du$$

$$= x + \tfrac{1}{4} \int \frac{u - 1}{u^2 + 2}\,du$$

Example 6 – *Solution*

cont'd

$$= x + \tfrac{1}{4}\int \frac{u}{u^2 + 2}\,du - \tfrac{1}{4}\int \frac{1}{u^2 + 2}\,du$$

$$= x + \tfrac{1}{8}\ln(u^2 + 2) - \frac{1}{4}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$

$$= x + \tfrac{1}{8}\ln(4x^2 - 4x + 3) - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{2x - 1}{\sqrt{2}}\right) + C$$

Integration of Rational Functions by Partial Fractions

Note:
Example 6 illustrates the general procedure for integrating a partial fraction of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

where $b^2 - 4ac < 0$

We complete the square in the denominator and then make a substitution that brings the integral into the form

$$\int \frac{Cu + D}{u^2 + a^2}\,du = C\int \frac{u}{u^2 + a^2}\,du + D\int \frac{1}{u^2 + a^2}\,du$$

Then the first integral is a logarithm and the second is expressed in terms of $\tan^{-1}$.

Case IV: $Q(x)$ contains a repeated irreducible quadratic factor.

If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then instead of the single partial fraction $\boxed{9}$, the sum

$$\boxed{11} \qquad \frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

occurs in the partial fraction decomposition of $R(x)/Q(x)$. Each of the terms in $\boxed{11}$ can be integrated by using a substitution or by first completing the square if necessary.

Example 8

Evaluate $\int \frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2} \, dx.$

Solution:

The form of the partial fraction decomposition is

$$\frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}$$

Multiplying by $x(x^2 + 1)^2$, we have

$$-x^3 + 2x^2 - x + 1 = A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x$$

Example 8 – *Solution*

cont'd

$$= A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex$$

$$= (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A$$

If we equate coefficients, we get the system

$A + B = 0 \qquad C = -1 \qquad 2A + B + D = 2 \qquad C + E = -1 \qquad A = 1$

which has the solution $A = 1$, $B = -1$, $C = -1$, $D = 1$ and $E = 0$.

Example 8 – *Solution*

cont'd

Thus

$$\int \frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2}\,dx = \int \left(\frac{1}{x} - \frac{x + 1}{x^2 + 1} + \frac{x}{(x^2 + 1)^2} \right) dx$$

$$= \int \frac{dx}{x} - \int \frac{x}{x^2 + 1}\,dx - \int \frac{dx}{x^2 + 1} + \int \frac{x\,dx}{(x^2 + 1)^2}$$

$$= \ln|x| - \tfrac{1}{2}\ln(x^2 + 1) - \tan^{-1}x - \frac{1}{2(x^2 + 1)} + K$$

Rationalizing Substitutions

Rationalizing Substitutions

Some nonrational functions can be changed into rational functions by means of appropriate substitutions.

In particular, when an integrand contains an expression of the form $\sqrt[n]{g(x)}$, then the substitution $u = \sqrt[n]{g(x)}$ may be effective. Other instances appear in the exercises.

Example 9

Evaluate $\int \frac{\sqrt{x+4}}{x}\, dx.$

Solution:

Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $dx = 2u\, du$. Therefore

$$\int \frac{\sqrt{x+4}}{x}\, dx = \int \frac{u}{u^2-4}\, 2u\, du$$

$$= 2\int \frac{u^2}{u^2-4}\, du$$

$$= 2\int \left(1 + \frac{4}{u^2-4}\right) du$$

Example 9 – *Solution*

cont'd

We can evaluate this integral either by factoring $u^2 - 4$ as $(u - 2)(u + 2)$ and using partial fractions or by using Formula 6 with $a = 2$:

$$\int \frac{\sqrt{x+4}}{x}\,dx = 2\int du + 8\int \frac{du}{u^2 - 4}$$

$$= 2u + 8 \cdot \frac{1}{2 \cdot 2} \ln \left| \frac{u-2}{u+2} \right| + C$$

$$= 2\sqrt{x+4} + 2\ln \left| \frac{\sqrt{x+4} - 2}{\sqrt{x+4} + 2} \right| + C$$

7.5 Strategy for Integration

Strategy for Integration

In this section we present a collection of miscellaneous integrals in random order and the main challenge is to recognize which technique or formula to use.

No hard and fast rules can be given as to which method applies in a given situation, but we give some advice on strategy that you may find useful.

A prerequisite for applying a strategy is a knowledge of the basic integration formulas.

Strategy for Integration

In the table of Integration Formulas we have collected the integrals with several additional formulas that we have learned in this chapter.

Most of them should be memorized. It is useful to know them all, but the ones marked with an asterisk need not be memorized since they are easily derived.

Formula 19 can be avoided by using partial fractions, and trigonometric substitutions can be used in place of Formula 20.

Strategy for Integration

Table of Integration Formulas Constants of integration have been omitted.

1. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$

2. $\displaystyle\int \frac{1}{x}\,dx = \ln|x|$

3. $\displaystyle\int e^x\,dx = e^x$

4. $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a}$

5. $\displaystyle\int \sin x\,dx = -\cos x$

6. $\displaystyle\int \cos x\,dx = \sin x$

7. $\displaystyle\int \sec^2 x\,dx = \tan x$

8. $\displaystyle\int \csc^2 x\,dx = -\cot x$

9. $\displaystyle\int \sec x \tan x\,dx = \sec x$

10. $\displaystyle\int \csc x \cot x\,dx = -\csc x$

11. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x|$

12. $\displaystyle\int \csc x\,dx = \ln|\csc x - \cot x|$

13. $\displaystyle\int \tan x\,dx = \ln|\sec x|$

14. $\displaystyle\int \cot x\,dx = \ln|\sin x|$

15. $\displaystyle\int \sinh x\,dx = \cosh x$

16. $\displaystyle\int \cosh x\,dx = \sinh x$

17. $\displaystyle\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

18. $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right), \quad a > 0$

*19. $\displaystyle\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$

*20. $\displaystyle\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$

Strategy for Integration

1. Simplify the Integrand if Possible

Sometimes the use of algebraic manipulation or trigonometric identities will simplify the integrand and make the method of integration obvious. Here are some examples:

$$\int \sqrt{x}\,(1 + \sqrt{x})\,dx = \int (\sqrt{x} + x)\,dx$$

$$\int \frac{\tan\theta}{\sec^2\theta}\,d\theta = \int \frac{\sin\theta}{\cos\theta}\cos^2\theta\,d\theta$$

$$= \int \sin\theta\,\cos\theta\,d\theta = \tfrac{1}{2}\int \sin 2\theta\,d\theta$$

Strategy for Integration

$$\int (\sin x + \cos x)^2\, dx = \int (\sin^2 x + 2 \sin x \cos x + \cos^2 x)\, dx$$

$$= \int (1 + 2 \sin x \cos x)\, dx$$

2. Look for an Obvious Substitution

Try to find some function $u = g(x)$ in the integrand whose differential $du = g'(x)\, dx$ also occurs, apart from a constant factor. For instance, in the integral

$$\int \frac{x}{x^2 - 1}\, dx$$

we notice that if $u = x^2 - 1$, then $du = 2x\, dx$.

Strategy for Integration

Therefore we use the substitution $u = x^2 - 1$ instead of the method of partial fractions.

3. Classify the Integrand According to Its Form

If Steps 1 and 2 have not led to the solution, then we take a look at the form of the integrand $f(x)$.

(a) *Trigonometric functions.* If $f(x)$ is a product of powers of $\sin x$ and $\cos x$, of $\tan x$ and $\sec x$, or of $\cot x$ and $\csc x$, then we use the substitutions.

(b) *Rational functions.* If f is a rational function, we use the procedure involving partial fractions.

Strategy for Integration

(c) *Integration by parts.* If $f(x)$ is a product of a power of x (or a polynomial) and a transcendental function (such as a trigonometric, exponential, or logarithmic function), then we try integration by parts, choosing u and dv.

(d) *Radicals.* Particular kinds of substitutions are recommended when certain radicals appear.

(i) If $\sqrt{\pm x^2 \pm a^2}$ occurs, we use a trigonometric substitution.

(ii) If $\sqrt[n]{ax + b}$ occurs, we use the rationalizing substitution $u = \sqrt[n]{ax + b}$. More generally, this sometimes works for $\sqrt[n]{g(x)}$.

Strategy for Integration

4. Try Again

If the first three steps have not produced the answer, remember that there are basically only two methods of integration: substitution and parts.

(a) *Try substitution.* Even if no substitution is obvious (Step 2), some inspiration or ingenuity (or even desperation) may suggest an appropriate substitution.

(b) *Try parts.* Although integration by parts is used most of the time on products of the form described in Step 3(c), it is sometimes effective on single functions.

Strategy for Integration

(c) *Manipulate the integrand.* Algebraic manipulations (perhaps rationalizing the denominator or using trigonometric identities) may be useful in transforming the integral into an easier form. These manipulations may be more substantial than in Step 1 and may involve some ingenuity. Here is an example:

$$\int \frac{dx}{1 - \cos x} = \int \frac{1}{1 - \cos x} \cdot \frac{1 + \cos x}{1 + \cos x}\, dx = \int \frac{1 + \cos x}{1 - \cos^2 x}\, dx$$

$$= \int \frac{1 + \cos x}{\sin^2 x}\, dx = \int \left(\csc^2 x + \frac{\cos x}{\sin^2 x} \right) dx$$

Strategy for Integration

(d) *Relate the problem to previous problems.* When you have built up some experience in integration, you may be able to use a method on a given integral that is similar to a method you have already used on a previous integral. Or you may even be able to express the given integral in terms of a previous one.

Strategy for Integration

For instance, $\int \tan^2 x \sec x \, dx$ is a challenging integral, but if we make use of the identity $\tan^2 x = \sec^2 x - 1$, we can write

$$\int \tan^2 x \sec x \, dx = \int \sec^3 x \, dx - \int \sec x \, dx$$

and if $\int \sec^3 x \, dx$ has previously been evaluated, then that calculation can be used in the present problem.

Strategy for Integration

(e) *Use several methods.* Sometimes two or three methods are required to evaluate an integral. The evaluation could involve several successive substitutions of different types, or it might combine integration by parts with one or more substitutions.

Example 1

$$\int \frac{\tan^3 x}{\cos^3 x}\,dx$$

In Step 1 we rewrite the integral:

$$\int \frac{\tan^3 x}{\cos^3 x}\,dx = \int \tan^3 x \; \sec^3 x \, dx$$

The integral is now of the form $\int \tan^m x \sec^n x \, dx$ with m odd.

Alternatively, if in Step 1 we had written

$$\int \frac{\tan^3 x}{\cos^3 x}\,dx = \int \frac{\sin^3 x}{\cos^3 x}\,\frac{1}{\cos^3 x}\,dx$$

Example 1

cont'd

$$= \int \frac{\sin^3 x}{\cos^6 x}\, dx$$

then we could have continued as follows with the substitution *u* = cos *x*:

$$\int \frac{\sin^3 x}{\cos^6 x}\, dx = \int \frac{1 - \cos^2 x}{\cos^6 x} \sin x \, dx$$

$$= \int \frac{1 - u^2}{u^6} (-du)$$

$$= \int \frac{u^2 - 1}{u^6}\, du$$

$$= \int (u^{-4} - u^{-6})\, du$$

Can We Integrate All Continuous Functions?

Can We Integrate All Continuous Functions?

The functions that we have been studied here are called **elementary functions**.

These are the polynomials, rational functions, power functions (x^a), exponential functions (a^x), logarithmic functions, trigonometric and inverse trigonometric functions, hyperbolic and inverse hyperbolic functions, and all functions that can be obtained from these by the five operations of addition, subtraction, multiplication, division, and composition.

Can We Integrate All Continuous Functions?

For instance, the function

$$f(x) = \sqrt{\frac{x^2 - 1}{x^3 + 2x - 1}} + \ln(\cosh x) - xe^{\sin 2x}$$

is an elementary function.

If f is an elementary function, then f' is an elementary function but $\int f(x)\,dx$ need not be an elementary function. Consider $f(x) = e^{x^2}$.

Can We Integrate All Continuous Functions?

Since *f* is continuous, its integral exists, and if we define the function *F* by

$$F(x) = \int_0^x e^{t^2}\, dt$$

then we know from Part 1 of the Fundamental Theorem of Calculus that

$$F'(x) = e^{x^2}$$

Thus $f(x) = e^{x^2}$ has an antiderivative *F*, but it has been proved that *F* is not an elementary function.

This means that no matter how hard we try, we will never succeed in evaluating $\int e^{x^2}\, dx$ in terms of the functions we know.

Can We Integrate All Continuous Functions?

The same can be said of the following integrals:

$$\int \frac{e^x}{x}\,dx \qquad \int \sin(x^2)\,dx \qquad \int \cos(e^x)\,dx$$

$$\int \sqrt{x^3+1}\,dx \qquad \int \frac{1}{\ln x}\,dx \qquad \int \frac{\sin x}{x}\,dx$$

In fact, the majority of elementary functions don’t have elementary antiderivatives.

You may be assured, though, that the integrals in the following exercises are all elementary functions.

7.6 Integration Using Tables and Computer Algebra Systems

Tables of Integrals

Tables of Integrals

Tables of indefinite integrals are very useful when we are confronted by an integral that is difficult to evaluate by hand and we don’t have access to a computer algebra system.

Usually we need to use the Substitution Rule or algebraic manipulation to transform a given integral into one of the forms in the table.

Example 1

The region bounded by the curves $y = \arctan x$, $y = 0$, and $x = 1$ is rotated about the y-axis. Find the volume of the resulting solid.

Solution:

Using the method of cylindrical shells, we see that the volume is

$$V = \int_0^1 2\pi x \arctan x \, dx$$

Example 1 – *Solution*

cont'd

In the section of the Table of Integrals titled *Inverse Trigonometric Forms* we locate Formula 92:

$$\int u \tan^{-1}u \, du = \frac{u^2 + 1}{2} \tan^{-1}u - \frac{u}{2} + C$$

Thus the volume is

$$V = 2\pi \int_0^1 x \tan^{-1}x \, dx = 2\pi \left[\frac{x^2 + 1}{2} \tan^{-1}x - \frac{x}{2} \right]_0^1$$

$$= \pi\left[(x^2 + 1)\tan^{-1}x - x\right]_0^1 = \pi(2 \tan^{-1}1 - 1)$$

$$= \pi[2(\pi/4) - 1] = \tfrac{1}{2}\pi^2 - \pi$$

Computer Algebra Systems

Computer Algebra Systems

Computers are particularly good at matching patterns. And just as we used substitutions in conjunction with tables, a CAS can perform substitutions that transform a given integral into one that occurs in its stored formulas.

So it isn't surprising that computer algebra systems excel at integration.

To begin, let's see what happens when we ask a machine to integrate the relatively simple function $y = 1/(3x - 2)$.

Computer Algebra Systems

Using the substitution $u = 3x - 2$, an easy calculation by hand gives

$$\int \frac{1}{3x - 2}\, dx = \tfrac{1}{3} \ln|3x - 2| + C$$

whereas Derive, Mathematica, and Maple all return the answer

$$\tfrac{1}{3} \ln(3x - 2)$$

The first thing to notice is that computer algebra systems omit the constant of integration.

Computer Algebra Systems

In other words, they produce a *particular* antiderivative, not the most general one.

Therefore, when making use of a machine integration, we might have to add a constant.

Second, the absolute value signs are omitted in the machine answer. That is fine if our problem is concerned only with values of *x* greater than $\frac{2}{3}$.

But if we are interested in other values of *x*, then we need to insert the absolute value symbol.

Example 5

Use a computer algebra system to find $\int x\sqrt{x^2 + 2x + 4}\, dx$.

Solution:

Maple responds with the answer

$$\tfrac{1}{3}(x^2 + 2x + 4)^{3/2} - \tfrac{1}{4}(2x + 2)\sqrt{x^2 + 2x + 4} - \frac{3}{2}\operatorname{arcsinh}\frac{\sqrt{3}}{3}(1 + x)$$

The third term can be rewritten using the identity

$$\operatorname{arcsinh} x = \ln(x + \sqrt{x^2 + 1})$$

Example 5 – *Solution*

cont'd

Thus

$$\operatorname{arcsinh}\frac{\sqrt{3}}{3}(1+x) = \ln\left[\frac{\sqrt{3}}{3}(1+x) + \sqrt{\tfrac{1}{3}(1+x)^2+1}\,\right]$$

$$= \ln\frac{1}{\sqrt{3}}\left[1+x+\sqrt{(1+x)^2+3}\,\right]$$

$$= \ln\frac{1}{\sqrt{3}} + \ln\left(x+1+\sqrt{x^2+2x+4}\,\right)$$

The resulting extra term $-\frac{3}{2}$ ln(1/ $\sqrt{3}$) can be absorbed into the constant of integration.

Example 5 – *Solution*

cont'd

Mathematica gives the answer

$$\left(\frac{5}{6} + \frac{x}{6} + \frac{x^2}{3}\right)\sqrt{x^2 + 2x + 4} - \frac{3}{2}\operatorname{arcsinh}\left(\frac{1 + x}{\sqrt{3}}\right)$$

Mathematica combined the first two terms of the Maple result into a single term by factoring.

Derive gives the answer

$$\tfrac{1}{6}\sqrt{x^2 + 2x + 4}\,(2x^2 + x + 5) - \tfrac{3}{2}\ln\left(\sqrt{x^2 + 2x + 4} + x + 1\right)$$

7.7 Approximate Integration

Approximate Integration

There are two situations in which it is impossible to find the exact value of a definite integral.

The first situation arises from the fact that in order to evaluate $\int_a^b f(x)\,dx$ Fundamental Theorem of Calculus we need to know an antiderivative of *f*.

Sometimes, however, it is difficult, or even impossible, to find an antiderivative. For example, it is impossible to evaluate the following integrals exactly:

$$\int_0^1 e^{x^2}dx \qquad \int_{-1}^1 \sqrt{1 + x^3}\,dx$$

Approximate Integration

The second situation arises when the function is determined from a scientific experiment through instrument readings or collected data.

There may be no formula for the function.

In both cases we need to find approximate values of definite integrals. We already know one such method.

Approximate Integration

Recall that the definite integral is defined as a limit of Riemann sums, so any Riemann sum could be used as an approximation to the integral: If we divide $[a, b]$ into n subintervals of equal length $\Delta x = (b - a)/n$, then we have

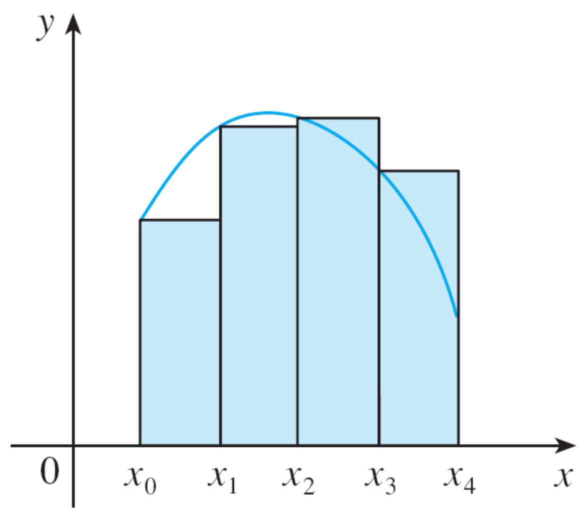
$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

where x_i^* is any point in the ith subinterval $[x_{i-1}, x_i]$. If x_i^* is chosen to be the left endpoint of the interval, then $x_i^* = x_{i-1}$ and we have

1 $$\int_a^b f(x)\,dx \approx L_n = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

Approximate Integration

If $f(x) \geq 0$, then the integral represents an area and (1) represents an approximation of this area by the rectangles shown in Figure 1(a).



Left endpoint approximation

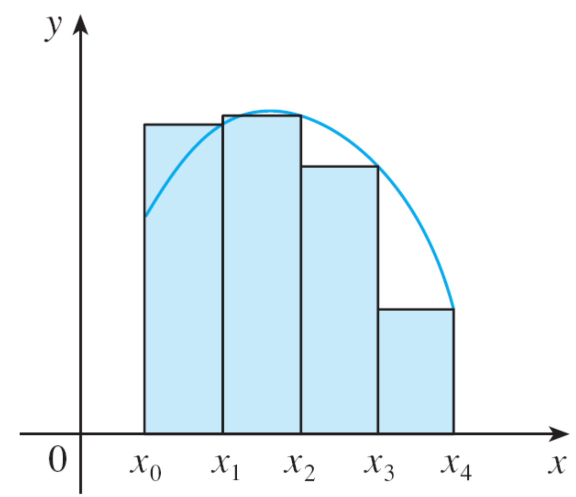
Figure 1(a)

Approximate Integration

If we choose x_i^* to be the right endpoint, then $x_i^* = x_i$ and we have

$$\boxed{2} \qquad \int_a^b f(x)\,dx \approx R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$$

[See Figure 1(b).]



Right endpoint approximation

Figure 1(b)

The approximations L_n and R_n defined by Equations 1 and 2 are called the **left endpoint approximation** and **right endpoint approximation**, respectively.

Approximate Integration

We also considered the case where x_i^* is chosen to be the midpoint $\overline{x}_i$ of the subinterval $[x_{i-1}, x_i]$. Figure 1(c) shows the midpoint approximation M_n, which appears to be better than either L_n or R_n.

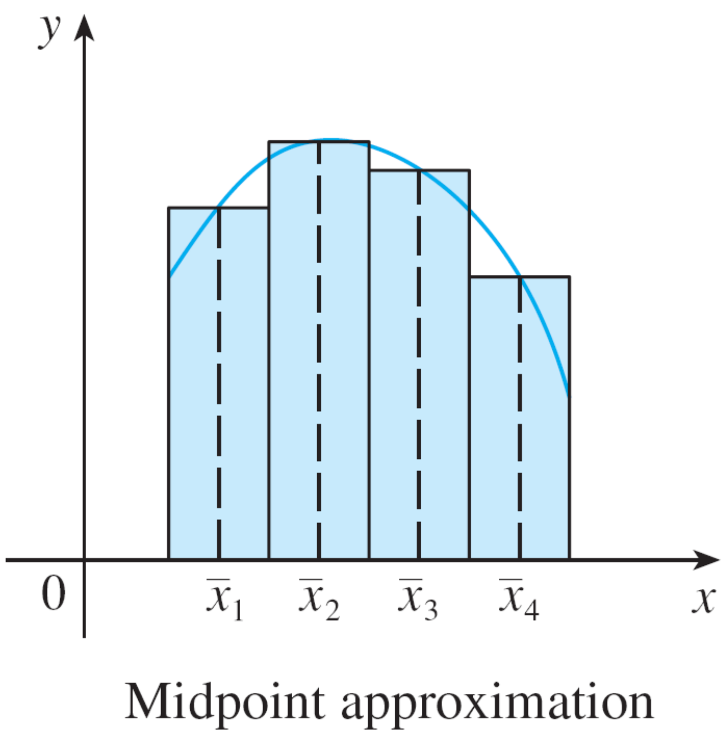


Figure 1(c)

Approximate Integration

Midpoint Rule

$$\int_a^b f(x)\,dx \approx M_n = \Delta x\,[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)]$$

where $\Delta x = \dfrac{b-a}{n}$

and $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$

Approximate Integration

Another approximation, called the Trapezoidal Rule, results from averaging the approximations in Equations 1 and 2:

$$\int_a^b f(x)\,dx \approx \frac{1}{2}\left[\sum_{i=1}^{n} f(x_{i-1})\,\Delta x + \sum_{i=1}^{n} f(x_i)\,\Delta x\right] = \frac{\Delta x}{2}\left[\sum_{i=1}^{n}\big(f(x_{i-1}) + f(x_i)\big)\right]$$

$$= \frac{\Delta x}{2}\left[\big(f(x_0) + f(x_1)\big) + \big(f(x_1) + f(x_2)\big) + \cdots + \big(f(x_{n-1}) + f(x_n)\big)\right]$$

$$= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

Trapezoidal Rule

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

where $\Delta x = (b - a)/n$ and $x_i = a + i\,\Delta x$.

Approximate Integration

The reason for the name Trapezoidal Rule can be seen from Figure 2, which illustrates the case with $f(x) \geq 0$ and $n = 4$.

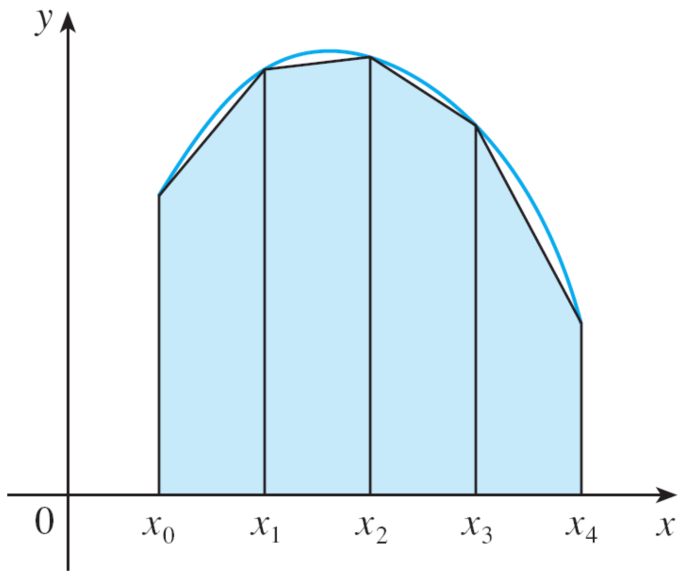


Figure 2

Trapezoidal approximation

Approximate Integration

The area of the trapezoid that lies above the *i*th subinterval is

$$\Delta x \left(\frac{f(x_{i-1}) + f(x_i)}{2} \right) = \frac{\Delta x}{2} [f(x_{i-1}) + f(x_i)]$$

and if we add the areas of all these trapezoids, we get the right side of the Trapezoidal Rule.

Example 1

Use (a) the Trapezoidal Rule and (b) the Midpoint Rule with n = 5 to approximate the integral $\int_1^2 (1/x)\,dx$.

Solution:

(a) With n = 5, a = 1 and b = 2, we have
Δx = (2 – 1)/5 = 0.2,

and so the Trapezoidal Rule gives

$$\int_1^2 \frac{1}{x}\,dx \approx T_5 = \frac{0.2}{2}\left[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)\right]$$

$$= 0.1\left(\frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2}\right)$$

≈ 0.695635

Example 1 – *Solution*

cont'd

This approximation is illustrated in Figure 3.

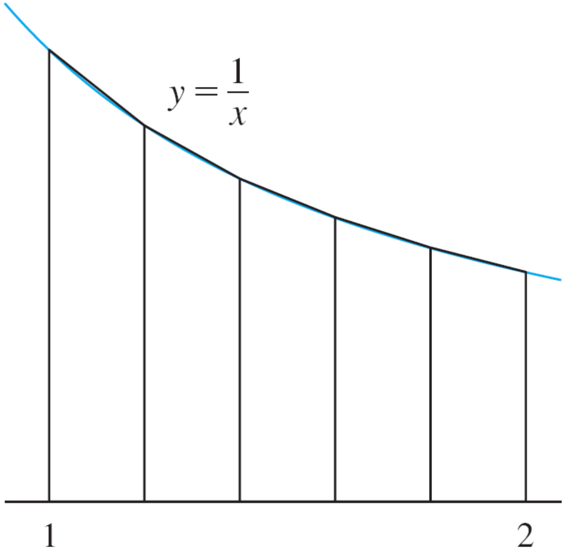


Figure 3

Example 1 – *Solution*

cont'd

(b) The midpoints of the five subintervals are 1.1, 1.3, 1.5, 1.7, and 1.9, so the Midpoint Rule gives

$$\int_1^2 \frac{1}{x}\,dx \approx \Delta x\,[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)]$$

$$= \frac{1}{5}\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right)$$

≈ 0.691908

This approximation is illustrated in Figure 4.

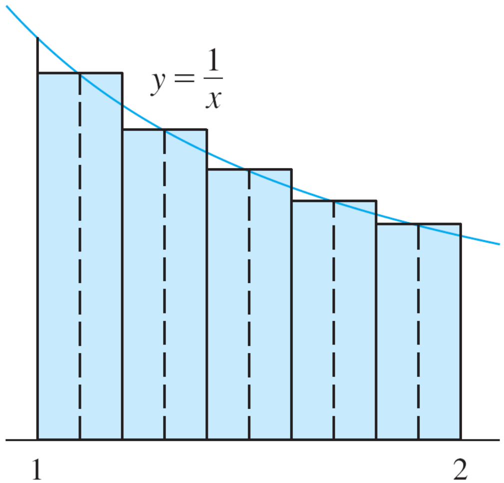


Figure 4

Approximate Integration

In Example 1 we deliberately chose an integral whose value can be computed explicitly so that we can see how accurate the Trapezoidal and Midpoint Rules are.

By the fundamental Theorem of Calculus,

$$\int_1^2 \frac{1}{x}\,dx = \ln x\Big]_1^2 = \ln 2 = 0.693147\ldots$$

The **error** in using an approximation is defined to be the amount that needs to be added to the approximation to make it exact.

Approximate Integration

From the values in Example 1 we see that the errors in the Trapezoidal and Midpoint Rule approximations for $n = 5$ are

$$E_T \approx -0.002488 \qquad \text{and} \qquad E_M \approx 0.001239$$

In general, we have

$$E_T = \int_a^b f(x)\,dx - T_n \qquad \text{and} \qquad E_M = \int_a^b f(x)\,dx - M_n$$

Approximate Integration

The following tables show the results of calculations similar to those in Example 1, but for n = 5, 10, and 20 and for the left and right endpoint approximations as well as the Trapezoidal and Midpoint Rules.

Approximations to $\int_1^2 \frac{1}{x}\,dx$

n	L_n	R_n	T_n	M_n
5	0.745635	0.645635	0.695635	0.691908
10	0.718771	0.668771	0.693771	0.692835
20	0.705803	0.680803	0.693303	0.693069

Corresponding errors

n	E_L	E_R	E_T	E_M
5	−0.052488	0.047512	−0.002488	0.001239
10	−0.025624	0.024376	−0.000624	0.000312
20	−0.012656	0.012344	−0.000156	0.000078

Approximate Integration

We can make several observations from these tables:

1. In all of the methods we get more accurate approximations when we increase the value of n. (But very large values of n result in so many arithmetic operations that we have to beware of accumulated round-off error.)

2. The errors in the left and right endpoint approximations are opposite in sign and appear to decrease by a factor of about 2 when we double the value of n.

Approximate Integration

3. The Trapezoidal and Midpoint Rules are much more accurate than the endpoint approximations.

4. The errors in the Trapezoidal and Midpoint Rules are opposite in sign and appear to decrease by a factor of about 4 when we double the value of n.

5. The size of the error in the Midpoint Rule is about half the size of the error in the Trapezoidal Rule.

Approximate Integration

$\boxed{3}$ **Error Bounds** Suppose $|f''(x)| \leqslant K$ for $a \leqslant x \leqslant b$. If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then

$$|E_T| \leqslant \frac{K(b-a)^3}{12n^2} \qquad \text{and} \qquad |E_M| \leqslant \frac{K(b-a)^3}{24n^2}$$

Let's apply this error estimate to the Trapezoidal Rule approximation in Example 1.

If $f(x) = 1/x$, then $f'(x) = -1/x^2$ and $f''(x) = 2/x^3$.

Approximate Integration

Since $1 \leq x \leq 2$, we have $1/x \leq 1$, so

$$|f''(x)| = \left|\frac{2}{x^3}\right| \leqslant \frac{2}{1^3} = 2$$

Therefore, taking $K = 2$, $a = 1$, $b = 2$, and $n = 5$ in the error estimate (3), we see that

$$|E_T| \leqslant \frac{2(2-1)^3}{12(5)^2} = \frac{1}{150} \approx 0.006667$$

Comparing this error estimate of 0.006667 with the actual error of about 0.002488, we see that it can happen that the actual error is substantially less than the upper bound for the error given by (3).

Simpson’s Rule

Simpson's Rule

Another rule for approximate integration results from using parabolas instead of straight line segments to approximate a curve.

As before, we divide $[a, b]$ into n subintervals of equal length $h = \Delta x = (b - a)/n$, but this time we assume that n is an *even* number.

Simpson's Rule

Then on each consecutive pair of intervals we approximate the curve $y = f(x) \geq 0$ by a parabola as shown in Figure 7.

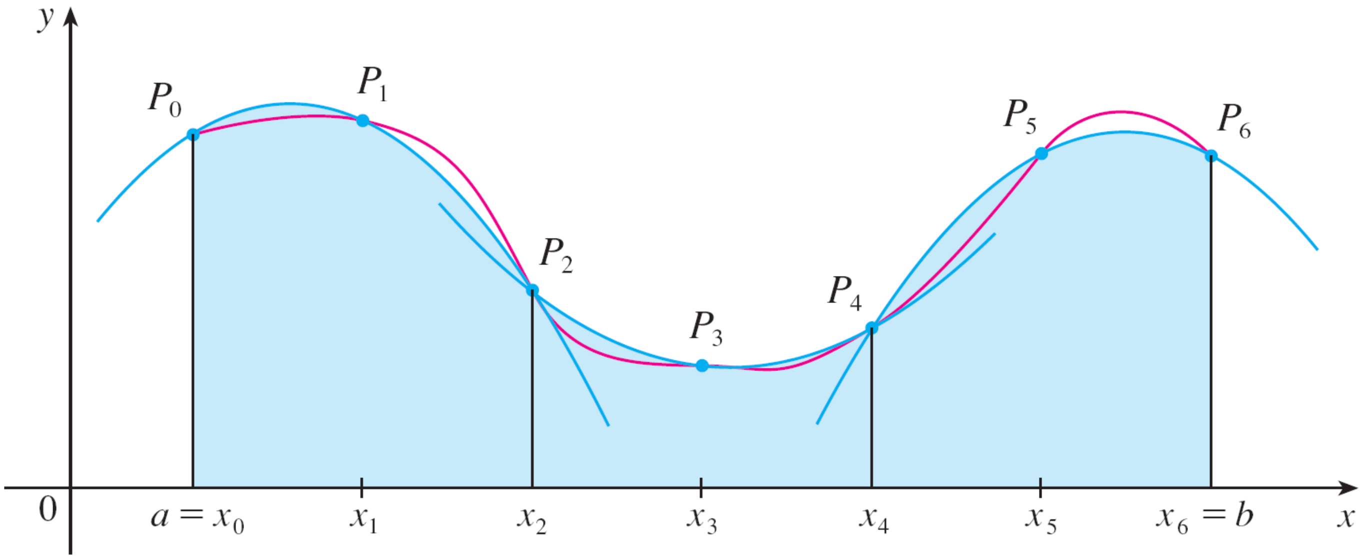


Figure 7

If $y_i = f(x_i)$, then $P_i(x_i, y_i)$ is the point on the curve lying above x_i.

Simpson’s Rule

A typical parabola passes through three consecutive points P_i , P_{i+1}, and P_{i+2}.

To simplify our calculations, we first consider the case where $x_0 = -h$, $x_1 = 0$, and $x_2 = h$. (See Figure 8.)

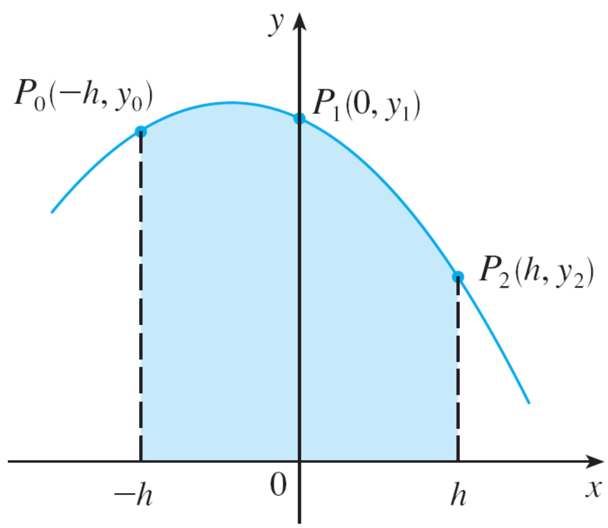


Figure 8

Simpson's Rule

We know that the equation of the parabola through P_0, P_1, and P_2 is of the form $y = Ax^2 + Bx + C$ and so the area under the parabola from $x = -h$ to $x = h$ is

$$\int_{-h}^{h} (Ax^2 + Bx + C)\,dx = 2\int_{0}^{h} (Ax^2 + C)\,dx$$

$$= 2\left[A\frac{x^3}{3} + Cx\right]_0^h$$

$$= 2\left(A\frac{h^3}{3} + Ch\right) = \frac{h}{3}(2Ah^2 + 6C)$$

Simpson's Rule

But, since the parabola passes through $P_0(-h, y_0)$, $P_1(0, y_1)$, and $P_2(h, y_2)$, we have

$$y_0 = A(-h)^2 + B(-h) + C = Ah^2 - Bh + C$$

$$y_1 = C$$

$$y_2 = Ah^2 + Bh + C$$

and therefore $$y_0 + 4y_1 + y_2 = 2Ah^2 + 6C$$

Thus we can rewrite the area under the parabola as

$$\frac{h}{3}(y_0 + 4y_1 + y_2)$$

Simpson's Rule

Now, by shifting this parabola horizontally we do not change the area under it.

This means that the area under the parabola through P_0, P_1, and P_2 from $x = x_0$ to $x = x_2$ in Figure 7 is still

$$\frac{h}{3}(y_0 + 4y_1 + y_2)$$

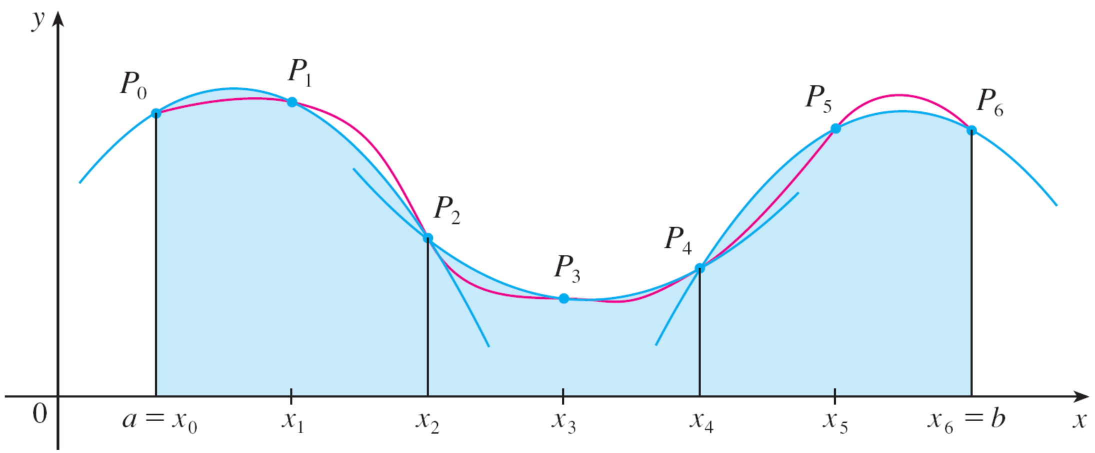


Figure 7

Simpson's Rule

Similarly, the area under the parabola through P_2, P_3, and P_4 from $x = x_2$ to $x = x_4$ is

$$\frac{h}{3}(y_2 + 4y_3 + y_4)$$

If we compute the areas under all the parabolas in this manner and add the results, we get

$$\int_a^b f(x)\,dx \approx \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

$$= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$$

Simpson's Rule

Although we have derived this approximation for the case in which $f(x) \geq 0$, it is a reasonable approximation for any continuous function f and is called Simpson's Rule after the English mathematician Thomas Simpson (1710–1761).

Note the pattern of coefficients:

1, 4, 2, 4, 2, 4, 2, . . . , 4, 2, 4, 1.

Simpson's Rule

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

where n is even and $\Delta x = (b - a)/n$.

Example 4

Use Simpson's Rule with n = 10 to approximate $\int_1^2 (1/x)\,dx$.

Solution:

Putting $f(x) = 1/x$, $n = 10$, and $\Delta x = 0.1$ in Simpson's Rule, we obtain

$$\int_1^2 \frac{1}{x}\,dx \approx S_{10}$$

$$= \frac{\Delta x}{3}[f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + \cdots + 2f(1.8) + 4f(1.9) + f(2)]$$

$$= \frac{0.1}{3}\left(\frac{1}{1} + \frac{4}{1.1} + \frac{2}{1.2} + \frac{4}{1.3} + \frac{2}{1.4} + \frac{4}{1.5} + \frac{2}{1.6} + \frac{4}{1.7} + \frac{2}{1.8} + \frac{4}{1.9} + \frac{1}{2}\right)$$

$$\approx 0.693150$$

Simpson's Rule

The Trapezoidal Rule or Simpson's Rule can still be used to find an approximate value for $\int_a^b y\,dx$, the integral of *y* with respect to *x*.

The table below shows how Simpson's Rule compares with the Midpoint Rule for the integral $\int_1^2 (1/x)\,dx$, whose true value is about 0.69314718.

The second table shows how the error E_s in Simpson's Rule decreases by a factor of about 16 when *n* is doubled.

n	M_n	S_n
4	0.69121989	0.69315453
8	0.69266055	0.69314765
16	0.69302521	0.69314721

n	E_M	E_S
4	0.00192729	−0.00000735
8	0.00048663	−0.00000047
16	0.00012197	−0.00000003

Simpson's Rule

That is consistent with the appearance of n^4 in the denominator of the following error estimate for Simpson's Rule.

It is similar to the estimates given in (3) for the Trapezoidal and Midpoint Rules, but it uses the fourth derivative of *f*.

$\boxed{4}$ **Error Bound for Simpson's Rule** Suppose that $|f^{(4)}(x)| \leqslant K$ for $a \leqslant x \leqslant b$. If E_S is the error involved in using Simpson's Rule, then

$$|E_S| \leqslant \frac{K(b-a)^5}{180n^4}$$

7.8 Improper Integrals

Improper Integrals

In this section we extend the concept of a definite integral to the case where the interval is infinite and also to the case where f has an infinite discontinuity in $[a, b]$. In either case the integral is called an *improper* integral.

Type 1: Infinite Intervals

Type 1: Infinite Intervals

Consider the infinite region S that lies under the curve $y = 1/x^2$, above the x-axis, and to the right of the line $x = 1$.

You might think that, since S is infinite in extent, its area must be infinite, but let's take a closer look.

The area of the part of S that lies to the left of the line $x = t$ (shaded in Figure 1) is

$$A(t) = \int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_1^t$$

$$= 1 - \frac{1}{t}$$

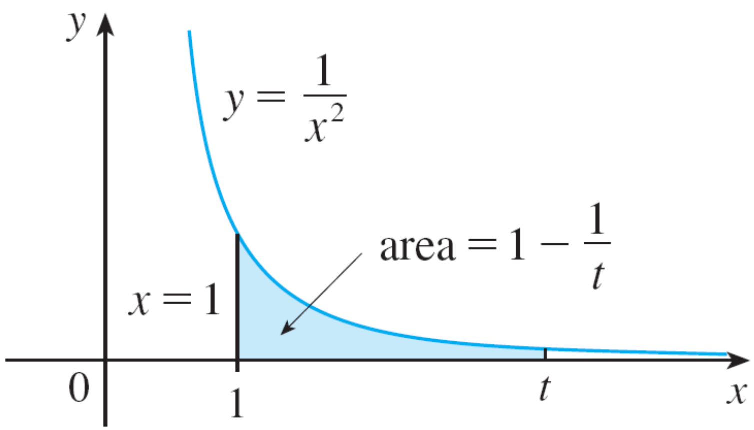


Figure 1

Type 1: Infinite Intervals

Notice that $A(t) < 1$ no matter how large t is chosen. We also observe that

$$\lim_{t \to \infty} A(t) = \lim_{t \to \infty} \left(1 - \frac{1}{t}\right) = 1$$

The area of the shaded region approaches 1 as $t \to \infty$ (see Figure 2), so we say that the area of the infinite region S is equal to 1 and we write

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t \to \infty} \int_1^t \frac{1}{x^2}\,dx = 1$$

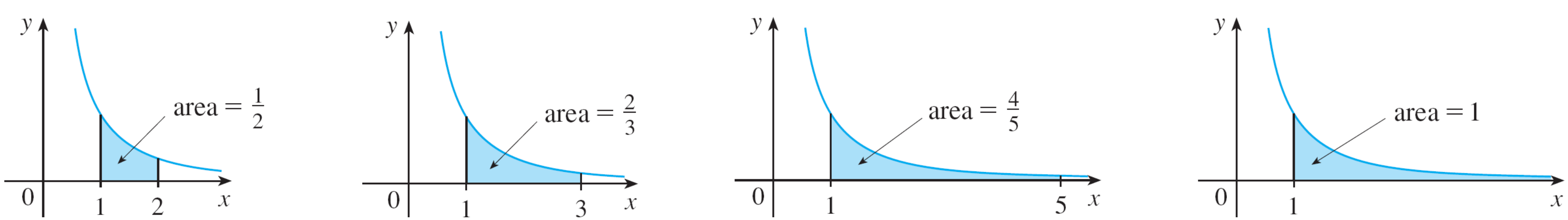


Figure 2

Type 1: Infinite Intervals

Using this example as a guide, we define the integral of *f* (not necessarily a positive function) over an infinite interval as the limit of integrals over finite intervals.

1 Definition of an Improper Integral of Type 1

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geqslant a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leqslant b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

In part (c) any real number a can be used.

Type 1: Infinite Intervals

Any of the improper integrals in Definition 1 can be interpreted as an area provided that *f* is a positive function.

For instance, in case (a) if $f(x) \geq 0$ and the integral $\int_a^\infty f(x)\,dx$ is convergent, then we define the area of the region $S = \{(x, y) \mid x \geq a,\ 0 \leq y \leq f(x)\}$ in Figure 3 to be

$$A(S) = \int_a^\infty f(x)\,dx$$

This is appropriate because $\int_a^\infty f(x)\,dx$ is the limit as $t \to \infty$ of the area under the graph of *f* from *a* to *t*.

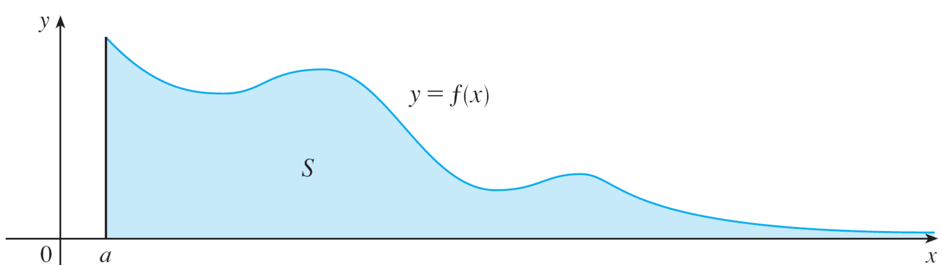


Figure 3

Example 1

Determine whether the integral $\int_1^\infty (1/x)\,dx$ is convergent or divergent.

Solution:

According to part (a) of Definition 1, we have

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty} \ln|x|\Big]_1^t$$

$$= \lim_{t\to\infty} (\ln t - \ln 1)$$

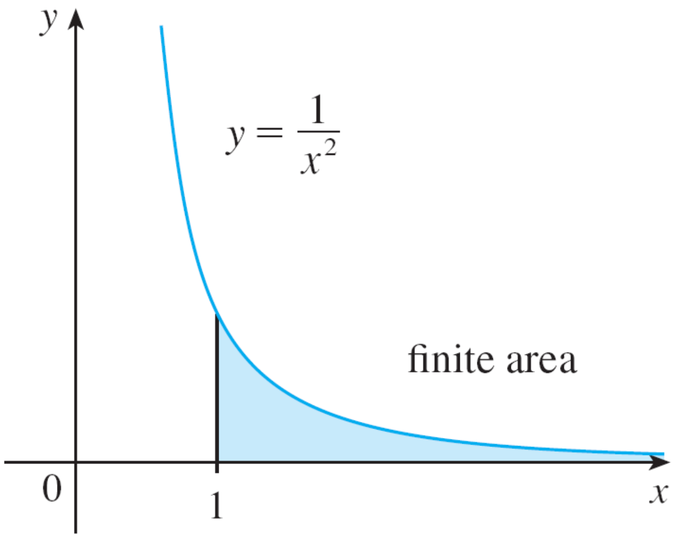
$$= \lim_{t\to\infty} \ln t = \infty$$

The limit does not exist as a finite number and so the improper integral $\int_1^\infty (1/x)\,dx$ is divergent.

Type 1: Infinite Intervals

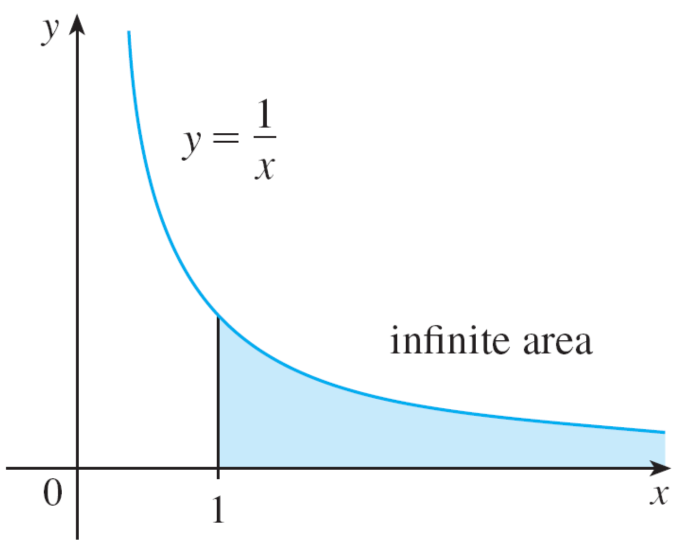
Let's compare the result of Example 1 with the example given at the beginning of this section:

$$\int_1^\infty \frac{1}{x^2}\,dx \quad \text{converges} \qquad\qquad \int_1^\infty \frac{1}{x}\,dx \quad \text{diverges}$$



$\int_1^\infty (1/x^2)\,dx$ converges

Figure 4



$\int_1^\infty (1/x)\,dx$ diverges

Figure 5

Type 1: Infinite Intervals

Geometrically, this says that although the curves $y = 1/x^2$ and $y = 1/x$ look very similar for $x > 0$, the region under $y = 1/x^2$ to the right of $x = 1$ (the shaded region in Figure 4) has finite area whereas the corresponding region under $y = 1/x$ (in Figure 5) has infinite area. Note that both $1/x^2$ and $1/x$ approach 0 as $x \to \infty$ but $1/x^2$ approaches 0 faster than $1/x$. The values of $1/x$ don't decrease fast enough for its integral to have a finite value.

We summarize this as follows:

2 $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

Type 2: Discontinuous Integrands

Type 2: Discontinuous Integrands

Suppose that *f* is a positive continuous function defined on a finite interval [*a*, *b*) but has a vertical asymptote at *b*.

Let *S* be the unbounded region under the graph of *f* and above the *x*-axis between *a* and *b.* (For Type 1 integrals, the regions extended indefinitely in a horizontal direction. Here the region is infinite in a vertical direction.)

The area of the part of *S* between *a* and *t* (the shaded region in Figure 7) is

$$A(t) = \int_a^t f(x)\,dx$$

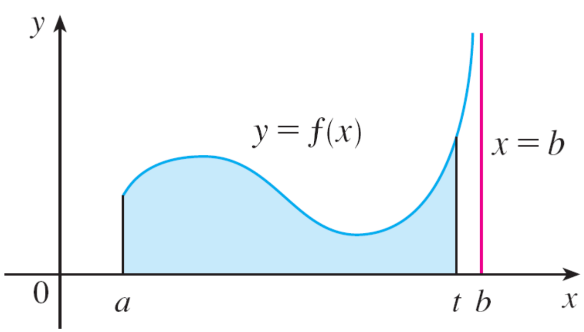


Figure 7

Type 2: Discontinuous Integrands

If it happens that $A(t)$ approaches a definite number A as $t \to b^-$, then we say that the area of the region S is A and we write

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

We use this equation to define an improper integral of Type 2 even when f is not a positive function, no matter what type of discontinuity f has at b.

Type 2: Discontinuous Integrands

3 Definition of an Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Example 5

Find $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx.$

Solution:

We note first that the given integral is improper because $f(x) = 1/\sqrt{x-2}$ has the vertical asymptote $x = 2$.

Since the infinite discontinuity occurs at the left endpoint of [2, 5], we use part (b) of Definition 3:

$$\int_2^5 \frac{dx}{\sqrt{x-2}} = \lim_{t\to 2^+} \int_t^5 \frac{dx}{\sqrt{x-2}}$$

Example 5 – *Solution*

cont'd

$$= \lim_{t \to 2^+} 2\sqrt{x-2}\,\Big]_t^5$$

$$= \lim_{t \to 2^+} 2\left(\sqrt{3} - \sqrt{t-2}\right)$$

$$= 2\sqrt{3}$$

Thus the given improper integral is convergent and, since the integrand is positive, we can interpret the value of the integral as the area of the shaded region in Figure 10.

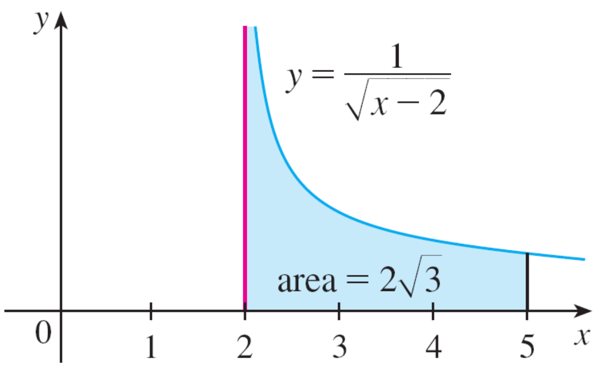


Figure 10

A Comparison Test for Improper Integrals

A Comparison Test for Improper Integrals

Sometimes it is impossible to find the exact value of an improper integral and yet it is important to know whether it is convergent or divergent.

In such cases the following theorem is useful. Although we state it for Type 1 integrals, a similar theorem is true for Type 2 integrals.

Comparison Theorem Suppose that f and g are continuous functions with $f(x) \geqslant g(x) \geqslant 0$ for $x \geqslant a$.

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

A Comparison Test for Improper Integrals

We omit the proof of the Comparison Theorem, but Figure 12 makes it seem plausible.

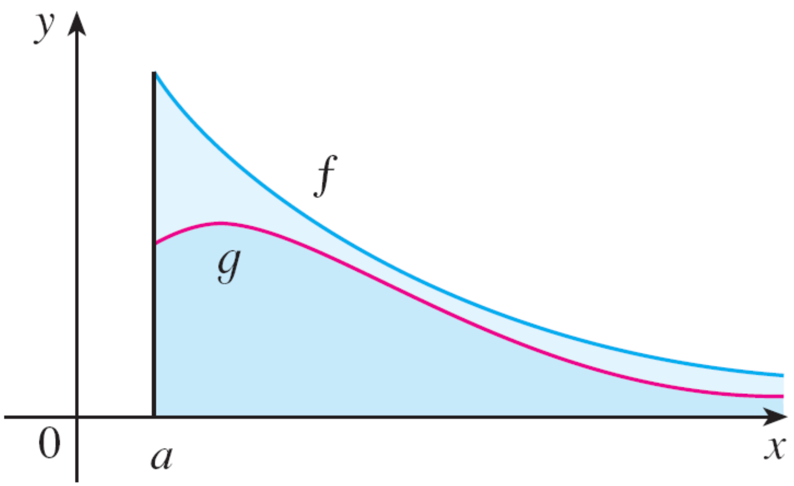


Figure 12

If the area under the top curve $y = f(x)$ is finite, then so is the area under the bottom curve $y = g(x)$.

A Comparison Test for Improper Integrals

And if the area under $y = g(x)$ is infinite, then so is the area under $y = f(x)$. [Note that the reverse is not necessarily true: If $\int_a^\infty g(x)\,dx$ is convergent, $\int_a^\infty f(x)\,dx$ may or may not be convergent, and if $\int_a^\infty f(x)\,dx$ is divergent, $\int_a^\infty g(x)\,dx$ may or may not be divergent.]

Example 9

Show that $\int_0^\infty e^{-x^2} dx$ is convergent.

Solution:

We can't evaluate the integral directly because the antiderivative of e^{-x^2} is not an elementary function.

We write

$$\int_0^\infty e^{-x^2} dx = \int_0^1 e^{-x^2} dx + \int_1^\infty e^{-x^2} dx$$

and observe that the first integral on the right-hand side is just an ordinary definite integral.

Example 9 – *Solution*

cont'd

In the second integral we use the fact that for $x \geq 1$ we have $x^2 \geq x$, so $-x^2 \leq -x$ and therefore $e^{-x^2} \leq e^{-x}$. (See Figure 13.)

The integral of e^{-x} is easy to evaluate:

$$\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_1^t e^{-x}\,dx$$

$$= \lim_{t\to\infty}(e^{-1} - e^{-t})$$

$$= e^{-1}$$

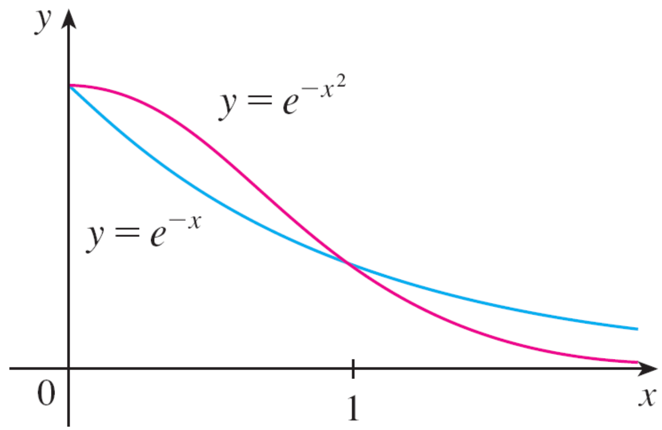


Figure 13

Example 9 – *Solution*

cont'd

Thus, taking $f(x) = e^{-x}$ and $g(x) = e^{-x^2}$ in the Comparison Theorem, we see that $\int_1^\infty e^{-x^2}\,dx$ is convergent.

It follows that $\int_0^\infty e^{-x^2}\,dx$ is convergent.