VV156 RC5
Problems in Mid2 and Parametric Equations

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1. Mid 2
2. Parametric Equations and Polar Coordinates
3. Q&A

Problems in Mid 2

Ex 2.1 ⋆ ⋆ ⋆ ⋆ ⋆ ⋆

$$\int \sqrt{\tan x}\,\mathrm{d}x$$

Problems in Mid 2

Ex 2.1 Solution

Let $u=\sqrt{\tan x}$, then $u^2=\tan x, x=\arctan\left(u^2\right)$, and $\mathrm{d}x=\frac{2u}{1+u^4}\,\mathrm{d}u$ Thus

$$\begin{aligned}
\int \sqrt{\tan x}\,\mathrm{d}x &= \int \frac{2u^2}{1+u^4}\,\mathrm{d}u \\
&= \frac{1}{\sqrt{2}}\int\left(\frac{u}{u^2-\sqrt{2}u+1}-\frac{u}{u^2+\sqrt{2}u+1}\right)\mathrm{d}u \\
&= \frac{1}{2\sqrt{2}}\int \frac{2u-\sqrt{2}}{u^2-\sqrt{2}u+1}+\frac{1}{2}\int\frac{\mathrm{d}u}{u^2-\sqrt{2}u+1} \\
&\quad -\frac{1}{2\sqrt{2}}\int\frac{2u+\sqrt{2}}{u^2+\sqrt{2}u+1}+\frac{1}{2}\int\frac{\mathrm{d}u}{u^2+\sqrt{2}u+1}
\end{aligned}$$

Problems in Mid 2

Ex 2.1 Solution

$$
\begin{aligned}
=&\frac{1}{2\sqrt{2}}\ln\left(u^2-\sqrt{2}u+1\right)-\frac{1}{2\sqrt{2}}\ln\left(u^2+\sqrt{2}u+1\right)\\
&+\frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}u+1\right)+\frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}u-1\right)+C\\
=&\frac{1}{2\sqrt{2}}\ln\left(\tan x-\sqrt{2\tan x}+1\right)-\frac{1}{2\sqrt{2}}\ln\left(\tan x+\sqrt{2\tan x}+1\right)\\
&+\frac{1}{\sqrt{2}}\arctan\left(\sqrt{2\tan x}+1\right)+\frac{1}{\sqrt{2}}\arctan\left(\sqrt{2\tan x}-1\right)+C
\end{aligned}
$$

(Note that the arguments of ln are already positive.)

Problems in Mid 2

Ex 2.2 ⋆⋆⋆

$$\int \frac{\mathrm{d}x}{(x^2+2x+2)^2}$$

Problems in Mid 2

Ex 2.2 Solution 1

Let $u = x + 1$, then

$$\int \frac{\mathrm{d}x}{\left(x^2+2x+2\right)^2} = \int \frac{\mathrm{d}x}{\left((x+1)^2+1\right)^2} = \int \frac{\mathrm{d}u}{\left(u^2+1\right)^2}$$

Now using integration by parts,

$$\begin{aligned}
\int \frac{\mathrm{d}u}{\left(u^2+1\right)^2} &= -\frac{1}{2}\int \frac{1}{u}\,\mathrm{d}\left(\frac{1}{1+u^2}\right) \\
&= -\frac{1}{u\left(1+u^2\right)} + \frac{1}{2}\int \left(\frac{1}{1+u^2} - \frac{1}{u^2}\right)\mathrm{d}u \\
&= -\frac{1}{u\left(1+u^2\right)} + \frac{1}{2}\arctan u + \frac{1}{2u} + C \\
&= \frac{1}{2}\arctan u + \frac{u}{2\left(u^2+1\right)} + C
\end{aligned}$$

Problems in Mid 2

Ex 2.2 Solution 1

Hence

$$\int \frac{\mathrm{d}x}{(x^2+2x+2)^2} = \frac{1}{2}\arctan(x+1) + \frac{1}{2}\frac{x+1}{(x+1)^2+1} + C$$

Problems in Mid 2

Ex 2.2 Solution 2

Alternatively, we can use trigonometric substitution, let $u = \tan\theta$, then $\mathrm{d}u = \sec^2\theta \mathrm{d}\theta$, thus

$$\begin{aligned}\int \frac{\mathrm{d}u}{(u^2+1)^2} &= \int \frac{\sec^2\theta}{(\tan^2\theta+1)^2}\,\mathrm{d}\theta = \int \frac{1}{\sec^2\theta}\mathrm{d}\theta \\ &= \int \cos^2\theta \mathrm{d}\theta = \frac{1}{4}\sin 2\theta + \frac{1}{2}\theta + C \\ &= \frac{1}{2}\frac{\tan\theta}{1+\tan^2\theta} + \frac{1}{2}\theta + C \\ &= \frac{1}{2}\frac{u}{1+u^2} + \frac{1}{2}\arctan u + C\end{aligned}$$

The rest follows.

Problems in Mid 2

Ex 3 ⋆⋆⋆

Evaluate the following improper integral

$$\int_0^\infty \frac{1 - e^{-x^2}}{x^2}\,\mathrm{d}x$$

(You may use the fact that $\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$.)

Problems in Mid 2

Ex 3 Solution

Using integration by parts,

$$\begin{aligned}\int \frac{1-e^{-x^2}}{x^2}\,\mathrm{d}x &= \int \left(1-e^{-x^2}\right)\mathrm{d}\left(-\frac{1}{x}\right)\\ &= -\frac{1}{x}\left(1-e^{-x^2}\right) - \int \left(-\frac{1}{x}\right)\mathrm{d}\left(1-e^{-x^2}\right)\\ &= -\frac{1}{x}\left(1-e^{-x^2}\right) + \int \frac{1}{x}\cdot 2xe^{-x^2}\\ &= -\frac{1}{x}\left(1-e^{-x^2}\right) + 2\int e^{-x^2}\end{aligned}$$

hence

$$\int_0^\infty \frac{1-e^{-x^2}}{x^2}\,\mathrm{d}x = -\frac{1}{x}\left(1-e^{-x^2}\right)\Big|_0^\infty + 2\int_0^\infty e^{-x^2} = \sqrt{\pi}$$

Problems in Mid 2

Ex 4 ⋆⋆⋆⋆⋆

Evaluate the following limit (where $n \in \mathbb{N}$)

$$\lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{1/n}$$

Problems in Mid 2

Ex 4 Solution

Let L be the limit, then

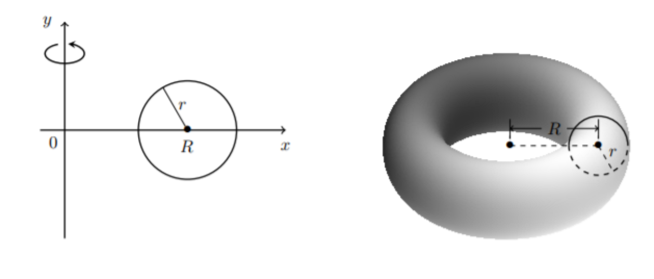
$$\begin{aligned}
\ln L &= \lim_{n\to\infty} \frac{1}{n} \ln\left(\frac{n!}{n^n}\right) = \lim_{n\to\infty} \frac{1}{n} \ln\left(\frac{1}{n}\cdot\frac{2}{n}\cdots\frac{n}{n}\right) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln\left(\frac{k}{n}\right) \\
&= \int_0^1 \ln(x)\mathrm{d}x = -\int_0^\infty e^{-y}\,\mathrm{d}y = -1
\end{aligned}$$

Therefore $L = e^{-1}$.

Problems in Mid 2

Ex 5 ⋆

Calculate the volume of a torus obtained by rotating the circle of radius r located at $(R, 0)$ about the y-axis. $(R > r > 0)$



Problems in Mid 2

Ex 5 Solution 1

Representing the circle as enclosed by the curve $x = R + \sqrt{r^2 - y^2} = f(y)$ and $x = R - \sqrt{r^2 - y^2} = g(y)$, the volume is given by

$$
\begin{aligned}
V &= \pi \int_{-r}^{r} \left\{ [f(y)]^2 - [g(y)]^2 \right\} \mathrm{d}y \\
&= \pi \int_{-r}^{r} [\left(R^2 + 2R\sqrt{r^2 - y^2} + r^2 - y^2\right) \\
&- \left(R^2 - 2R\sqrt{r^2 - y^2} + r^2 - y^2\right)]\mathrm{d}y \\
&= 4\pi R \int_{-r}^{r} \sqrt{r^2 - y^2}\, \mathrm{d}y = 4\pi R \cdot \frac{1}{2}\pi r^2 = 2\pi^2 r^2 R
\end{aligned}
$$

where the integral from the last step represents the area of a half circle.

Problems in Mid 2

Ex 5 Solution 2

The width of the "washer"/annulus at height y is given by $2\sqrt{r^2 - y^2}$, hence its area is given by $\pi R \cdot 2\sqrt{r^2 - y^2}$, thus the volume of the torus is given by

$$V = \int_{-r}^{r} 2\pi R \cdot 2\sqrt{r^2 - y^2}\,\mathrm{d}y = 4\pi R \int_{-r}^{r} \sqrt{r^2 - y^2}\,\mathrm{d}y = 2\pi^2 r^2 R$$

Problems in Mid 2

Ex 5 Solution 3

The surface area of the cylindrical shell crossing x is given by $2\pi x \cdot 2\sqrt{r^2-(x-R)^2}$ thus the volume of the torus is given by

$$\begin{aligned} V &= \int_{R-r}^{R+r} 2\pi x \cdot 2\sqrt{r^2-(x-R)^2}\,\mathrm{d}x \\ &= \int_{-r}^{r} 4\pi(u+R)\sqrt{r^2-u^2}\,\mathrm{d}u \quad [\text{ Let } u = x-R] \\ &= 4\pi R\int_{-r}^{r}\sqrt{r^2-u^2}\,\mathrm{d}u + 4\pi\underbrace{\int_{-r}^{r} u\sqrt{r^2-u^2}\,\mathrm{d}u}_{=0} = 2\pi^2 r^2 R \end{aligned}$$

where the second integral vanishes since the integrand is odd.

Problems in Mid 2

Ex 5 Wrong Solution

Expand the torus as a cylinder and thus the volume of the torus is given by

$$V = 2\pi R \cdot \pi r^2 = 2\pi^2 r^2 R$$

Theorem of Pappus

Let $\mathscr{R}$ be a plane region that lies entirely on one side of a line l in the plane. If $\mathscr{R}$ is rotated about l, then the volume of the resulting solid is the product of the area A of $\mathscr{R}$ and the distance d traveled by the centroid of $\mathscr{R}$.

The circle has area $A = \pi r^2$. By the symmetry principle, its centroid is its center and so the distance traveled by the centroid during a rotation is $d = 2\pi R$. Therefore, by the Theorem of Pappus, the volume of the torus is

$$V = Ad = (2\pi R)\left(\pi r^2\right) = 2\pi^2 r^2 R$$

Problems in Mid 2

Ex 6 $\star\star\star$

Find the surface area of the solid obtained by rotating the function $y = \sin(2x), 0 \leq x \leq \frac{\pi}{8}$ about the x-axis.

Problems in Mid 2

Ex 6 Solution

The surface area is given by

$$\begin{aligned} S &= \int_0^{\pi/8} 2\pi y \sqrt{1 + (y')^2}\, \mathrm{d}x \\ &= \int_0^{\pi/8} 2\pi \sin(2x)\sqrt{1 + 4\cos^2(2x)}\mathrm{d}x \end{aligned}$$

Let $u = \cos(2x)$, then note that $u = 1$ if $x = 0$ and $u = 1/\sqrt{2}$ if $x = \pi/8$, thus

$$\begin{aligned} S &= -\pi \int_1^{1/\sqrt{2}} \sqrt{1 + 4u^2}\, \mathrm{d}u \\ &= \frac{\pi}{2} u\sqrt{1 + 4u^2} + \frac{\pi}{4} \ln\left(2u + \sqrt{1 + 4u^2}\right)\Big|_{1/\sqrt{2}}^{1} \\ &= \frac{\pi\sqrt{5}}{2} - \frac{\pi}{2}\sqrt{\frac{3}{2}} - \frac{1}{4}\pi\log\left(\sqrt{2} + \sqrt{3}\right) + \frac{1}{4}\pi\log\left(\sqrt{5} + 2\right) \end{aligned}$$

Problems in Mid 2

Ex 7 ⋆⋆⋆⋆⋆

Assume that everything is convergent in this problem, that is, usual computations are valid.

(i)

$$\text{Show that } \int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right) \mathrm{d}x = \int_{-\infty}^{\infty} f(x) \mathrm{d}x$$

(ii)

$$\text{Evaluate the integral } \int_{-\infty}^{\infty} \exp\left(-x^2 - \frac{\alpha}{x^2}\right) \mathrm{d}x$$

Where $\alpha > 0$ is a constant.

(You may use the fact that $\int_{-\infty}^{\infty} e^{-x^2}\, \mathrm{d}x = \sqrt{\pi}$.)

Problems in Mid 2

Ex 7.1 Solution 1

Let $y = -1/x$, then

$$I := \int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right) \mathrm{d}x = \int_{-\infty}^{\infty} f\left(y - \frac{1}{y}\right) \frac{1}{y^2} \mathrm{d}y$$

hence

$$2I = \int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right) \left(1 + \frac{1}{x^2}\right) \mathrm{d}x$$

Now let $u = x - 1/x$, then

$$\begin{aligned} I &= \frac{1}{2} \int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right) \mathrm{d}\left(x - \frac{1}{x}\right) \\ &= \frac{1}{2} \left[\int_{-\infty}^{0} f(u)\mathrm{d}u + \int_{0}^{\infty} f(u)\mathrm{d}u\right] \\ &= \int_{-\infty}^{\infty} f(u)\mathrm{d}u \end{aligned}$$

Problems in Mid 2

Ex 7.1 Solution 2

Let $x = e^{-t}$, then

$$\begin{aligned}\int_0^{\infty} f\left(x - \frac{1}{x}\right) \mathrm{d}x &= \int_{-\infty}^{\infty} f\left(e^t - e^{-t}\right) \mathrm{d}\left(e^t\right) \\ &= \int_{-\infty}^{\infty} f\left(e^t - e^{-t}\right) e^t \, \mathrm{d}t\end{aligned}$$

Similarly, let $x = -e^{-t}$, then

$$\begin{aligned}\int_{-\infty}^{0} f\left(x - \frac{1}{x}\right) \mathrm{d}x &= \int_{-\infty}^{\infty} f\left(-e^t + e^{-t}\right) \mathrm{d}\left(-e^{-t}\right) \\ &= \int_{-\infty}^{\infty} f\left(-e^t + e^{-t}\right) e^{-t} \, \mathrm{d}t\end{aligned}$$

Problems in Mid 2

Ex 7.1 Solution 2

Now

$$\begin{aligned}
\int_{-\infty}^{\infty} f\left(x-\frac{1}{x}\right)\mathrm{d}x &= \int_{0}^{\infty} f\left(x-\frac{1}{x}\right)\mathrm{d}x + \int_{-\infty}^{0} f\left(x-\frac{1}{x}\right)\mathrm{d}x \\
&= \int_{-\infty}^{\infty} f\left(e^{t}-e^{-t}\right)\left(e^{t}+e^{-t}\right)\mathrm{d}t \\
&= \int_{-\infty}^{\infty} f\left(e^{t}-e^{-t}\right)\mathrm{d}\left(e^{t}-e^{-t}\right) \\
&= \int_{-\infty}^{\infty} f(x)\mathrm{d}x
\end{aligned}$$

Problems in Mid 2

Ex 7.2 Solution

First note that if we let $x = \alpha^{1/2} y$, then

$$\int_{-\infty}^{\infty} f\left(x - \frac{\alpha}{x}\right) \mathrm{d}x = \int_{-\infty}^{\infty} f\left(\alpha^{1/2} y - \frac{\alpha^{1/2}}{y}\right) \mathrm{d}\left(\alpha^{1/2} y\right)$$

$$= \int_{-\infty}^{\infty} f\left(\alpha^{1/2} y\right) \mathrm{d}\left(\alpha^{1/2} y\right) = \int_{-\infty}^{\infty} f(x) \mathrm{d}x$$

hence

$$\int_{-\infty}^{\infty} \exp\left(-x^2 - \frac{\alpha}{x^2}\right) \mathrm{d}x = \int_{-\infty}^{\infty} \exp\left\{-\left(x - \frac{\sqrt{\alpha}}{x}\right)^2 - 2\sqrt{\alpha}\right\} \mathrm{d}x$$

$$= \int_{-\infty}^{\infty} e^{-x^2 - 2\sqrt{\alpha}} \mathrm{d}x = e^{-2\sqrt{\alpha}} \sqrt{\pi}$$

Parametric Equations

Definition

Suppose that x and y are both given as functions of a third variable t (called a parameter) by the equations

$$x = f(t) \quad y = g(t)$$

Parametric Equations

$$x = \cos t \quad y = \sin t \quad 0 \leqslant t \leqslant 2\pi$$

If we plot points, it appears that the curve is a circle. We can confirm this impression by eliminating t. Observe that

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

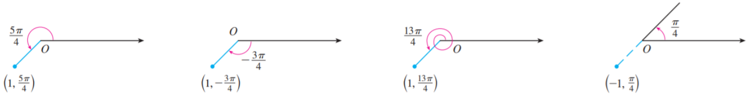
Thus the point (x, y) moves on the unit circle $x^2 + y^2 = 1$. Notice that in this example the parameter t can be interpreted as the angle (in radians). As t increases from 0 to 2π, the point $(x, y) = (\cos t, \sin t)$ moves once around the circle in the counterclockwise direction starting from the point $(1, 0)$

Polar Coordinates

We choose a point in the plane that is called the pole (or origin) and is labeled O. Then we draw a ray (half-line) starting at O called the polar axis. This axis is usually drawn horizontally to the right and corresponds to the positive x-axis in Cartesian coordinates.

If P is any other point in the plane, let r be the distance from O to P and let θ be the angle (usually measured in radians) between the polar axis and the line OP as in Figure 1 . Then the point P is represented by the ordered pair (r, θ) and r, θ are called polar coordinates of P. We use the convention that an angle is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction. If $P = O$, then $r = 0$ and we agree that $(0, \theta)$ represents the pole for any value of θ.

Polar Coordinates



In fact, since a complete counterclockwise rotation is given by an angle 2π, the point represented by polar coordinates (r, θ) is also represented by

$$(r, \theta + 2n\pi) \quad \text{and} \quad (-r, \theta + (2n+1)\pi)$$

Polar Coordinates and Cartesian coordinates

$$x = r\cos\theta \quad y = r\sin\theta$$
$$r^2 = x^2 + y^2 \quad \tan\theta = \frac{y}{x}$$

Cartesian coordinates, Cylindrical coordinates and Spherical coordinates (Optional, you will see them in VV255)

From cylindrical to Cartesian coordinates:

$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$z = z$$

The inverse relations (from Cartesian to cylindrical coordinates) are

$$r = \sqrt{x^2 + y^2}$$
$$\phi = \tan^{-1}\frac{y}{x}$$
$$z = z$$

Cartesian coordinates, Cylindrical coordinates and Spherical coordinates (Optional, you will see them in VV255)

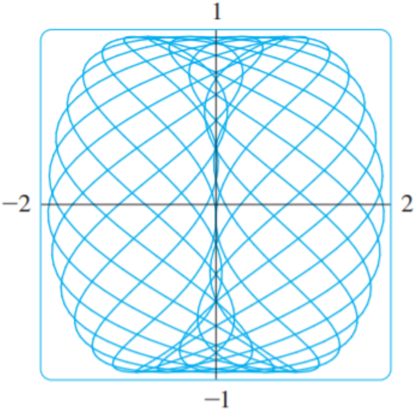
From spherical to Cartesian coordinates:

$$x = R\sin\theta\cos\phi$$
$$y = R\sin\theta\sin\phi$$
$$z = R\cos\theta$$

The inverse relations (from Cartesian to spherical coordinates) are

$$R = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}$$
$$\phi = \tan^{-1}\frac{y}{x}$$

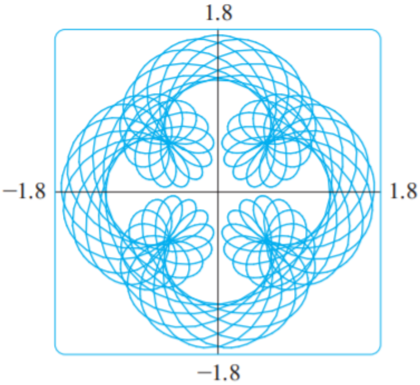
Some Curves



$$x = \sin t - \sin 2.3t$$

$$y = \cos t$$

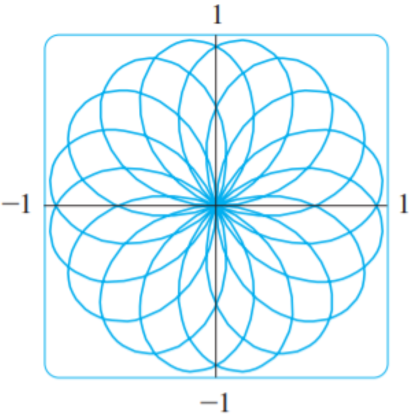
Some Curves



$$x = \sin t + \frac{1}{2}\sin 5t + \frac{1}{4}\cos 2.3t$$

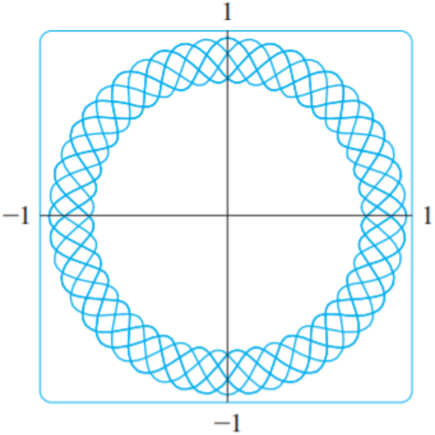
$$y = \cos t + \frac{1}{2}\cos 5t + \frac{1}{4}\sin 2.3t$$

Some Curves



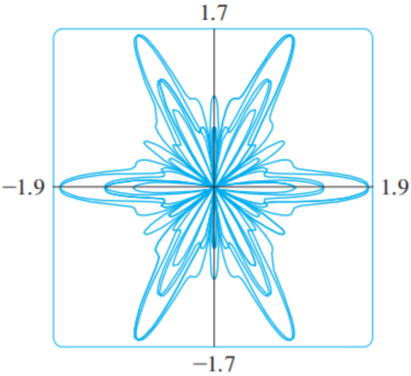
$$r = \sin(8\theta/5)$$

Some Curves



$$r = \sin^2(2.4\theta) + \cos^4(2.4\theta)$$

Some Curves



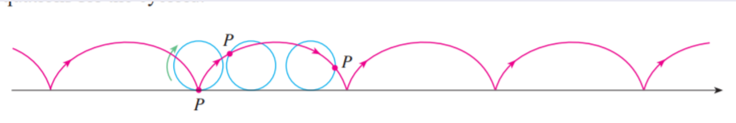
$$r = \sin^2(1.2\theta) + \cos^3(6\theta)$$

Cycloid （摆线）

Definition

The curve traced out by a point on the circumference of a circle as the circle rolls along a straight line is called a cycloid .
Therefore parametric equations of the cycloid are

$$x = r(\theta - \sin\theta) \quad y = r(1 - \cos\theta) \quad \theta \in \mathbb{R}$$

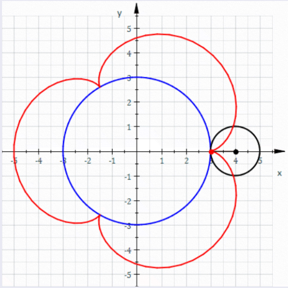


Epicycloid (Optional)（外摆线）

Definition

$$x(\theta) = (R + r)\cos\theta - r\cos\left(\frac{R+r}{r}\theta\right)$$

$$y(\theta) = (R + r)\sin\theta - r\sin\left(\frac{R+r}{r}\theta\right)$$

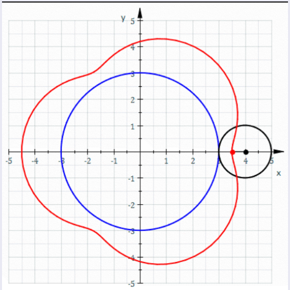


Epitrochoid (Optional) （外旋轮线）

Definition

$$x(\theta) = (R+r)\cos\theta - d\cos\left(\frac{R+r}{r}\theta\right)$$

$$y(\theta) = (R+r)\sin\theta - d\sin\left(\frac{R+r}{r}\theta\right)$$

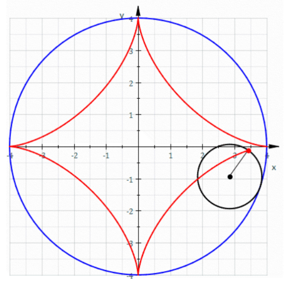


Hypocycloid (Optional)（内摆线）

Definition

$$x(\theta) = (R - r)\cos\theta - r\cos\left(\frac{R-r}{r}\theta\right)$$

$$y(\theta) = (R - r)\sin\theta - r\sin\left(\frac{R-r}{r}\theta\right)$$

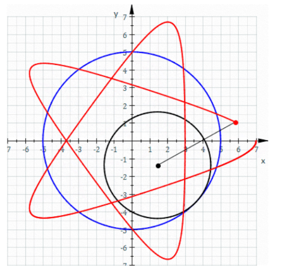


Hypotrochoid (Optional) （内旋轮线）

Definition

$$x(\theta) = (R - r)\cos\theta - d\cos\left(\frac{R - r}{r}\theta\right)$$

$$y(\theta) = (R - r)\sin\theta - d\sin\left(\frac{R - r}{r}\theta\right)$$



Cardioid（心脏线）

Definition

- parametric representation:

$$x(\varphi) = 2a(1 - \cos\varphi) \cdot \cos\varphi$$
$$y(\varphi) = 2a(1 - \cos\varphi) \cdot \sin\varphi, \quad 0 \leq \varphi < 2\pi$$

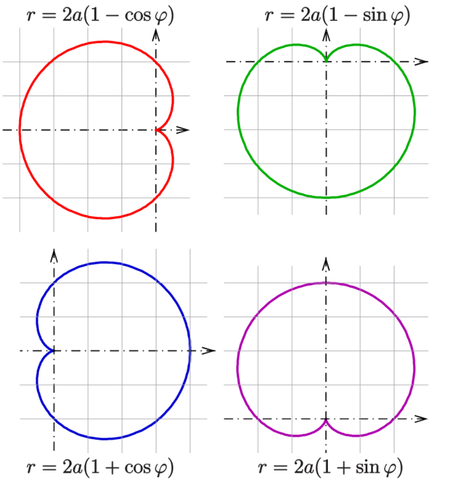
- polar coordinates:

$$r(\varphi) = 2a(1 - \cos\varphi)$$

Cardioid is special Cycloid and special Limaçon.

Cardioid（心脏线）

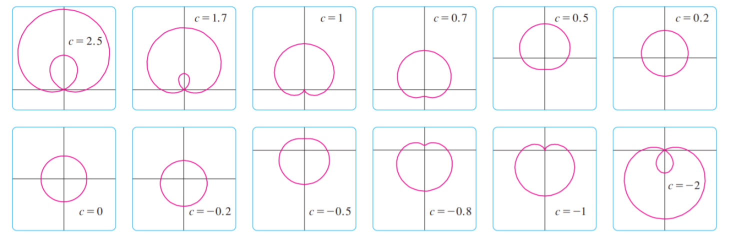
Cardioid



Limaçon （蚶线）

Definition

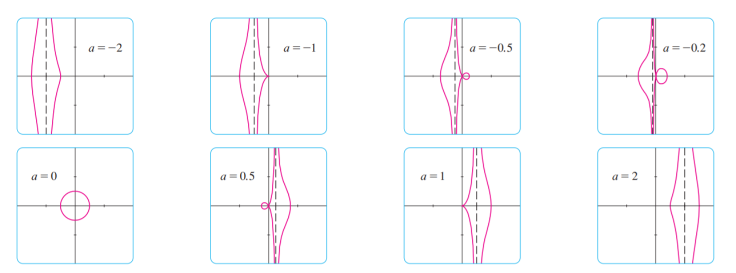
$$r = 1 + c\sin\theta$$



Conchoid（蚌线）

Definition

$$r = 1 + c\sec\theta$$



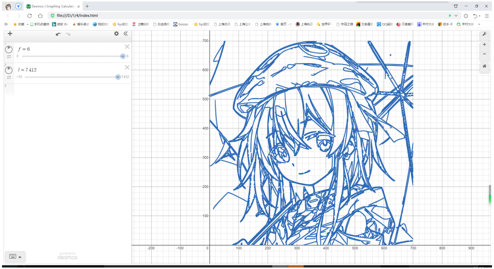
Bézier curves (Optional)

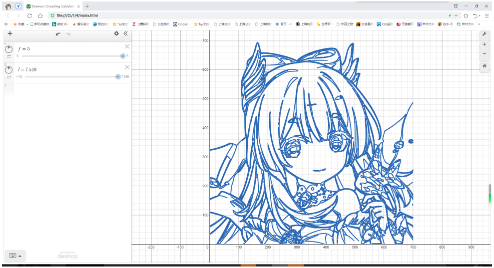
Definition

Bézier curves are used in computer-aided design and are named after the French mathematician Pierre Bézier (1910-1999), who worked in the automotive industry. A cubic Bézier curve is determined by four control points, $P_0\,(x_0, y_0)\,, P_1\,(x_1, y_1)\,, P_2\,(x_2, y_2)$, and $P_3\,(x_3, y_3)$, and is defined by the parametric equations

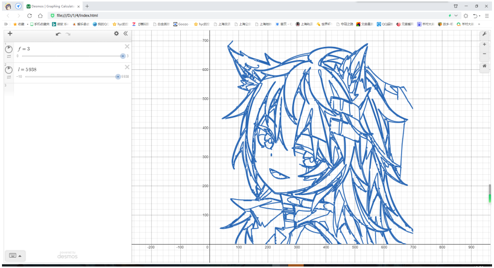
$$x = x_0(1-t)^3 + 3x_1 t(1-t)^2 + 3x_2 t^2(1-t) + x_3 t^3$$
$$y = y_0(1-t)^3 + 3y_1 t(1-t)^2 + 3y_2 t^2(1-t) + y_3 t^3$$

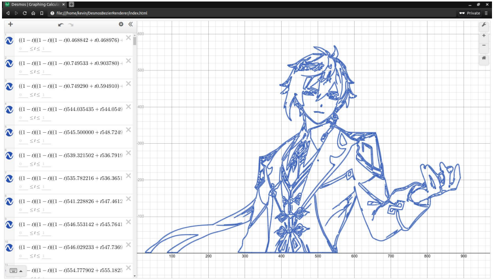
Application of Bézier curves in desmos



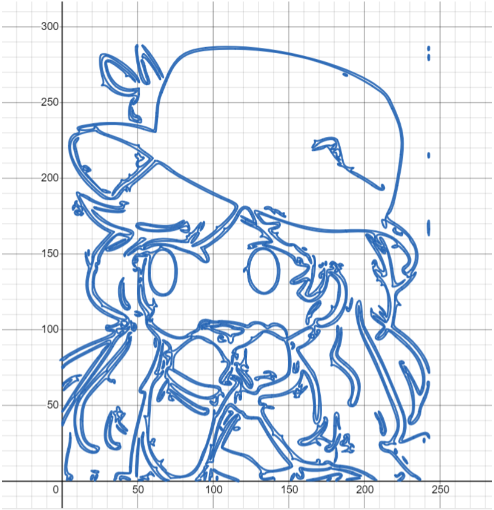


Application of Bézier curves in desmos

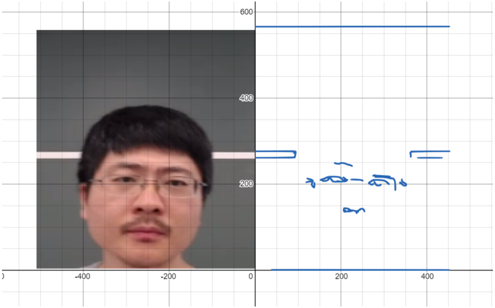




Application of Bézier curves in desmos



Application of Bézier curves in desmos



1. Mid 2
2. Parametric Equations and Polar Coordinates
3. Q&A

Q&A

Q&A