# VV156 RC6
# Parametric Equations, Polar Coordinates and Series

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1. Calculus with Parametric Curves and Polar Coordinates
   - Calculus with Parametric Curves
   - Calculus with Polar Coordinates

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   - Infinite Sequences
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# Tangents

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if } \frac{dx}{dt} \neq 0$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

# Area and Arc Length

## Area

We know that the area under a curve $y = F(x)$ from $a$ to $b$ is $A = \int_a^b F(x)dx$, where $F(x) \geqslant 0$. If the curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

$$A = \int_a^b y dx = \int_\alpha^\beta g(t)f'(t)dt \quad \left[\text{or } \int_\beta^\alpha g(t)f'(t)dt\right]$$

# Area and Arc Length

## Arc Length

If a curve $C$ is described by the parametric equations $x = f(t)$, $y = g(t), \alpha \leqslant t \leqslant \beta$, where $f'$ and $g'$ are continuous on $[\alpha, \beta]$ and $C$ is traversed exactly once as $t$ increases from $\alpha$ to $\beta$, then the length of $C$ is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

# Area and Arc Length

## Surface Area

In the same way as for arc length, we can adapt to obtain a formula for surface area. If the curve given by the parametric equations $x = f(t), y = g(t), \alpha \leqslant t \leqslant \beta$, is rotated about the $x$-axis, where $f', g'$ are continuous and $g(t) \geqslant 0$, then the area of the resulting surface is given by

$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

# Ex 1

## Tangents

Find equations of the tangents to the curve $x = 3t^2 + 1$, $y = 2t^3 + 1$ that pass through the point $(4, 3)$

# Ex 1

## Solution

$x = 3t^2 + 1, y = 2t^3 + 1, \frac{dx}{dt} = 6t, \frac{dy}{dt} = 6t^2$, so $\frac{dy}{dx} = \frac{6t^2}{6t} = t$

So at the point corresponding to parameter value $t$, an equation of the tangent line is $y - \left(2t^3 + 1\right) = t\left[x - \left(3t^2 + 1\right)\right]$.

If this line is to pass through $(4, 3)$, we must have

$3 - \left(2t^3 + 1\right) = t\left[4 - \left(3t^2 + 1\right)\right] \Leftrightarrow 2t^3 - 2 = 3t^3 - 3t \Leftrightarrow$

$t^3 - 3t + 2 = 0 \Leftrightarrow (t-1)^2(t+2) = 0 \Leftrightarrow t = 1$ or $-2$.

Hence, the desired equations are $y - 3 = x - 4$, or $y = x - 1$, tangent to the curve at $(4, 3)$, and $y - (-15) = -2(x - 13)$, or $y = -2x + 11$, tangent to the curve at $(13, -15)$.

# Ex 2

## Area and Arc Length

1. Find the exact length of the curve

$$x = 1 + 3t^2, \quad y = 4 + 2t^3, \quad 0 \leqslant t \leqslant 1$$

2. Find the area enclosed by the $x$-axis and the curve

$$x = 1 + e^t, y = t - t^2$$

3. Find the exact area of the surface obtained by rotating the given curve about the $x$-axis.

$$x = 3t - t^3, \quad y = 3t^2, \quad 0 \leqslant t \leqslant 1$$

# Ex 2

## Solution

1.

$x = 1 + 3t^2, \quad y = 4 + 2t^3, \quad 0 \le t \le 1. dx/dt = 6t$ and

$dy/dt = 6t^2$, so $(dx/dt)^2 + (dy/dt)^2 = 36t^2 + 36t^4$

Thus, $L = \int_0^1 \sqrt{36t^2 + 36t^4}dt = \int_0^1 6t\sqrt{1+t^2}dt$

$= 6\int_1^2 \sqrt{u}\left(\frac{1}{2}du\right) \quad \left[u = 1 + t^2, du = 2tdt\right]$

$= 3\left[\frac{2}{3}u^{3/2}\right]_1^2 = 2\left(2^{3/2} - 1\right) = 2(2\sqrt{2} - 1)$

# Ex 2

## Solution

2. The curve $x = 1 + e^t, y = t - t^2 = t(1-t)$ intersects the $x$-axis when $y = 0$, that is, when $t = 0$ and $t = 1$. The corresponding values of $x$ are 2 and $1 + e$. The shaded area is given by

$$\begin{aligned}
\int_{x=2}^{x=1+e} (y_T - y_B)\, dx &= \int_{t=0}^{t=1} [y(t) - 0]x'(t)dt = \int_0^1 \left(t - t^2\right) e^t dt \\
&= \int_0^1 te^t dt - \int_0^1 t^2 e^t dt \\
&= \int_0^1 te^t dt - \left[t^2 e^t\right]_0^1 + 2\int_0^1 te^t dt \\
&= 3\int_0^1 te^t dt - (e - 0) = 3\left[(t-1)e^t\right]_0^1 - e \\
&= 3[0 - (-1)] - e = 3 - e
\end{aligned}$$

# Ex 2

## Solution

3.

$$x = 3t - t^3, y = 3t^2, 0 \le t \le 1.$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(3 - 3t^2\right)^2 + (6t)^2 = 9\left(1 + 2t^2 + t^4\right)$$

$$= \left[3\left(1 + t^2\right)\right]^2$$

$$S = \int_0^1 2\pi \cdot 3t^2 \cdot 3\left(1 + t^2\right) dt = 18\pi \int_0^1 \left(t^2 + t^4\right) dt$$

$$= 18\pi \left[\frac{1}{3}t^3 + \frac{1}{5}t^5\right]_0^1$$

$$= \frac{48}{5}\pi$$

# Tangents

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

# Area and Arc Length

$$r = f(\theta), a \leqslant \theta \leqslant b$$

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta$$

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

# Ex 3

## Area and Tangents in Polar Coordinates

1. Find the slope of the tangent line to the given polar curve at the point specified by the value of $\theta$.

$$r = 2 - \sin\theta, \quad \theta = \pi/3$$

2. Find the area of the region enclosed by one loop of the curve

$$r = 2\sin 5\theta$$

# Ex 3

## Solution

1.

$$x = r\cos\theta = (2-\sin\theta)\cos\theta, y = r\sin\theta = (2-\sin\theta)\sin\theta \Rightarrow$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(2-\sin\theta)\cos\theta + \sin\theta(-\cos\theta)}{(2-\sin\theta)(-\sin\theta) + \cos\theta(-\cos\theta)}$$

$$= \frac{2\cos\theta - 2\sin\theta\cos\theta}{-2\sin\theta + \sin^2\theta - \cos^2\theta} = \frac{2\cos\theta - \sin 2\theta}{-2\sin\theta - \cos 2\theta}$$

When $\theta = \frac{\pi}{3}, \frac{dy}{dx} = \frac{2(1/2) - (\sqrt{3}/2)}{-2(\sqrt{3}/2) - (-1/2)} = \frac{2-\sqrt{3}}{1-2\sqrt{3}}$

# Ex 3

## Solution

2.

$$r = 0 \Rightarrow 2\sin 5\theta = 0 \Rightarrow \sin 5\theta = 0 \Rightarrow 5\theta = \pi n \Rightarrow \theta = \frac{\pi}{5}n.$$

$$A = \int_0^{\pi/5} \frac{1}{2}(2\sin 5\theta)^2 d\theta = \frac{1}{2}\int_0^{\pi/5} 4\sin^2 5\theta d\theta$$

$$= 2\int_0^{\pi/5} \frac{1}{2}(1 - \cos 10\theta) d\theta = \left[\theta - \frac{1}{10}\sin 10\theta\right]_0^{\pi/5} = \frac{\pi}{5}$$

# Conic Section

Conic Section Will NOT be Covered in Final Exam

# Infinite Sequences

A sequence $\{a_n\}$ has the limit $L$ and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if we can make the terms $a_n$ as close to $L$ as we like by taking $n$ sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence converges (or is convergent). Otherwise, we say the sequence diverges (or is divergent).

# Theorem

1. If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when $n$ is an integer, then $\lim_{n\to\infty} a_n = L$
2. $\lim_{n\to\infty} a_n = \infty$ means that for every positive number $M$ there is an integer $N$ such that if $n > N$ then $a_n > M$
3. If $a_n \leqslant b_n \leqslant c_n$ for $n \geqslant n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$
4. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$ 5. Every bounded, monotonic sequence is convergent.

# Properties

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and $c$ is a constant, then

$$\lim_{n\to\infty}(a_n+b_n)=\lim_{n\to\infty}a_n+\lim_{n\to\infty}b_n$$

$$\lim_{n\to\infty}(a_n-b_n)=\lim_{n\to\infty}a_n-\lim_{n\to\infty}b_n$$

$$\lim_{n\to\infty}ca_n=c\lim_{n\to\infty}a_n \qquad \lim_{n\to\infty}c=c$$

$$\lim_{n\to\infty}(a_nb_n)=\lim_{n\to\infty}a_n\cdot\lim_{n\to\infty}b_n$$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\frac{\lim_{n\to\infty}a_n}{\lim_{n\to\infty}b_n} \quad \text{if } \lim_{n\to\infty}b_n\neq 0$$

$$\lim_{n\to\infty}a_n^p=\left[\lim_{n\to\infty}a_n\right]^p \text{ if } p>0 \text{ and } a_n>0$$

# Definition

Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let $s_n$ denote its $n$th partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called convergent and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s$$

The number $s$ is called the sum of the series. If the sequence $\{s_n\}$ is divergent, then the series is called divergent.

# Geometric Series

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad |r| < 1$$

If $|r| \geqslant 1$, the geometric series is divergent.

# Harmonic Series

The harmonic series is the divergent infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

# Grandi's Series

The divergent infinite series $1 - 1 + 1 - 1 + \cdots$, also written

$$\sum_{n=0}^{\infty} (-1)^n$$

One obvious method to attack the series

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots$$

is to treat it like a telescoping series and perform the subtractions in place:

$$(1 - 1) + (1 - 1) + (1 - 1) + \ldots = 0 + 0 + 0 + \ldots = 0$$

On the other hand, a similar bracketing procedure leads to the apparently contradictory result

$$1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \ldots = 1 + 0 + 0 + 0 + \ldots = 1$$

# Grandi's Series

Treating Grandi's series as a divergent geometric series and using the same algebraic methods that evaluate convergent geometric series to obtain a third value:

$$S = 1 - 1 + 1 - 1 + \dots, \text{ so}$$
$$1 - S = 1 - (1 - 1 + 1 - 1 + \dots) = 1 - 1 + 1 - 1 + \dots = S$$
$$1 - S = S$$
$$1 = 2S_1$$

resulting in $S = \frac{1}{2}$. In fact, both of these statements can be made precise and formally proven, but only using well-defined mathematical concepts that arose in the 19th century. After the late 17th-century introduction of calculus in Europe, but before the advent of modern rigor, the tension between these answers fueled what has been characterized as an "endless" and "violent" dispute between mathematicians.

# Fourier Series (Optional, but you will see it in VV256 & VE216

$$s_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left( a_n \cos\left(\frac{2\pi nx}{P}\right) + b_n \sin\left(\frac{2\pi nx}{P}\right) \right)$$

# Test for Divergence Theorem

1. If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
2. If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

# Properties

If $\Sigma a_n$ and $\Sigma b_n$ are convergent series, then so are the series $\Sigma c a_n$ (where $c$ is a constant), $\Sigma (a_n + b_n)$, and $\Sigma (a_n - b_n)$, and

(i)

$$\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$$

(ii)

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

(iii)

$$\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

# Ex 4

## Sequences and Series

Let $a_n = \frac{2n}{3n+1}$.

1. Determine whether $\{a_n\}$ is convergent.
2. Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.

# Ex 4

**Solution**

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{3n+1} = \frac{2}{3}$, so the sequence $\{a_n\}$ is convergent
2. Since $\lim_{n\to\infty} a_n = \frac{2}{3} \neq 0$, the series $\sum_{n=1}^{\infty} a_n$ is divergent by the Test for Divergence.

# Ex 5

## Convergence and Divergence

Find the values of $x$ for which the series converges. Find the sum of the series for those values of $x$

1.
$$\sum_{n=0}^{\infty}(-4)^n(x-5)^n$$

2.
$$\sum_{n=0}^{\infty}\frac{\sin^n x}{3^n}$$

# Ex 5

## Solution

1. $\sum_{n=0}^{\infty}(-4)^n(x-5)^n = \sum_{n=0}^{\infty}[-4(x-5)]^n$ is a geometric series with $r = -4(x-5)$, so the series converges $\Leftrightarrow$ $|r| < 1 \Leftrightarrow |-4(x-5)| < 1 \Leftrightarrow |x-5| < \frac{1}{4} \Leftrightarrow -\frac{1}{4} < x-5 < \frac{1}{4} \Leftrightarrow$ $\frac{19}{4} < x < \frac{21}{4}$. In that case, the sum of the series is $\frac{a}{1-r} = \frac{1}{1-[-4(x-5)]} = \frac{1}{4x-19}$.

2. $\sum_{n=0}^{\infty} \frac{\sin^n x}{3^n} = \sum_{n=0}^{\infty} \left(\frac{\sin x}{3}\right)^n$ is a geometric series with $r = \frac{\sin x}{3}$, so the series converges $\Leftrightarrow$ $|r| < 1 \Leftrightarrow \left|\frac{\sin x}{3}\right| < 1 \Leftrightarrow |\sin x| < 3$, which is true for all $x$. Thus, the sum of the series is $\frac{a}{1-r} = \frac{1}{1-(\sin x)/3} = \frac{3}{3-\sin x}$

# Several Methods

Divergence Test Theorem
Integral Test
Comparison Test
Absolute Convergence Test
Ratio Test
Root Test
Alternating Series Test

# Integral Test

Suppose $f$ is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent. In other words:

(i) If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
(ii) If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

## p-series

For what values of $p$ is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

SOLUTION: If $p < 0$, then $\lim_{n\to\infty} (1/n^p) = \infty$. If $p = 0$, then $\lim_{n\to\infty} (1/n^p) = 1$. In either case $\lim_{n\to\infty} (1/n^p) \neq 0$, so the given series diverges by the Test for Divergence (11.2.7) If $p > 0$, then the function $f(x) = 1/x^p$ is clearly continuous, positive, and decreasing on $[1, \infty)$. We found in Chapter 7 [see (7.8.2)] that

$$\int_1^{\infty} \frac{1}{x^p} dx \text{ converges if } p > 1 \text{ and diverges if } p \leqslant 1$$

The $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

# Remainder Estimate for the Integral Test

Suppose $f(k) = a_k$, where $f$ is a continuous, positive, decreasing function for $x \geqslant n$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)dx \leqslant R_n \leqslant \int_{n}^{\infty} f(x)dx$$

# Ex 6

## Determine whether the series is convergent or divergent

1.

$$\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$$

2.

$$\sum_{n=1}^{\infty} \frac{n}{n^4+1}$$

# Ex 6

## Solution

1. $f(x) = \frac{x^2}{x^3+1}$ is continuous and positive on $[2, \infty)$, and also decreasing since $f'(x) = \frac{x\left(2-x^3\right)}{\left(x^3+1\right)^2} < 0$ for $x \geq 2$ so we can use the Integral Test [note that $f$ is not decreasing on $[1, \infty)$].
$\int_2^\infty \frac{x^2}{x^3+1} dx = \lim_{t\to\infty} \left[\frac{1}{3} \ln\left(x^3+1\right)\right]_2^t =$
$\frac{1}{3} \lim_{t\to\infty} \left[\ln\left(t^3+1\right) - \ln 9\right] = \infty$, so the series $\sum_{n=2}^\infty \frac{n^2}{n^3+1}$ diverges, and so does the given series, $\sum_{n=1}^\infty \frac{n^2}{n^3+1}$

# Ex 6

## Solution

2. The function $f(x) = \frac{x}{x^4+1}$ is positive, continuous, and decreasing on $[1, \infty)$. [Note that $f'(x) = \frac{x^4+1-4x^4}{(x^4+1)^2} = \frac{1-3x^4}{(x^4+1)^2} < 0$ on $[1, \infty)$.] Thus, we can apply the Integral Test.

$\int_1^\infty \frac{x}{x^4+1} dx = \lim_{t\to\infty} \int_1^t \frac{\frac{1}{2}(2x)}{1+(x^2)^2} dx = \lim_{t\to\infty} \left[\frac{1}{2}\tan^{-1}\left(x^2\right)\right]_1^t = \frac{1}{2}\lim_{t\to\infty}\left[\tan^{-1}\left(t^2\right) - \tan^{-1} 1\right] = \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{8}$ so the series $\sum_{n=1}^\infty \frac{n}{n^4+1}$ converges.

# Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.
(i) If $\sum b_n$ is convergent and $a_n \leqslant b_n$ for all $n$, then $\sum a_n$ is also convergent.
(ii) If $\sum b_n$ is divergent and $a_n \geqslant b_n$ for all $n$, then $\sum a_n$ is also divergent.
In using the Comparison Test we must, of course, have some known series $\sum b_n$ for the purpose of comparison. Most of the time we use one of these series:
- A $p$-series $[\sum 1/n^p$ converges if $p > 1$ and diverges if $p \leqslant 1]$
- A geometric series $[\sum ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geqslant 1]$

# The Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where $c$ is a finite number and $c > 0$, then either both series converge or both diverge.

# Ex 7

## Determine whether the series converges or diverges

1.
$$\sum_{n=1}^{\infty} \frac{1}{(n^2+2n+2)^2}$$

2.
$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

# Ex 7

## Solution

1. $\frac{1}{(n^2+2n+2)^2} < \frac{1}{(n^2)^2} = \frac{1}{n^4}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{1}{(n^2+2n+2)^2}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^4}$, which converges because it is a $p$-series with $p = 4 > 1$.
2. $\frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdots\cdots (n-1)n}{n \cdot n \cdot n \cdots\cdots n \cdot n} \leq \frac{1}{n} \cdot \frac{2}{n} \cdot 1 \cdot 1 \cdot \cdots \cdot 1$ for $n \geq 2$, so since $\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges [$p = 2 > 1$], $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges also by the Comparison Test.

# Absolute Convergence Test

**Definition**

A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.
A series $\sum a_n$ is called conditionally convergent if it is convergent but not absolutely convergent.

**Theorem**

If a series $\sum a_n$ is absolutely convergent, then it is convergent.

# Ratio Test

(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

# Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

# Ex 8

**Determine whether the series converges or diverges**

1.
$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

2.
$$\sum_{n=2}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$$

# Ex 8

## Solution

1.
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \lim_{n\to\infty}\frac{n^n}{(n+1)^n}$$
$$= \lim_{n\to\infty}\frac{1}{(1+1/n)^n} = \frac{1}{e} < 1, \text{ so the series } \sum_{n=1}^{\infty}\frac{n!}{n^n}$$
converges absolutely by the Ratio Test.

# Ex 8

## Solution

2.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left| \left( \frac{-2n}{n+1} \right)^{5n} \right|} = \lim_{n\to\infty} \frac{2^5 n^5}{(n+1)^5}$$

$$= 32 \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^5} = 32 \lim_{n\to\infty} \frac{1}{(1+1/n)^5}$$

$$= 32 > 1,$$

so the series $\sum_{n=2}^{\infty} \left( \frac{-2n}{n+1} \right)^{5n}$ diverges by the Root Test.

# Alternating Series Test

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \quad b_n > 0$$

satisfies

(i) $b_{n+1} \leqslant b_n \quad$ for all $n$

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

# Shanks Transformation（香克斯变换） (not in textbook, but will be covered in exam)

# Shanks Transformation

For each series $\sum_{n=0}^{\infty} a_n$, we can form the sequence of partial sums

$$A_n = \sum_{k=0}^{n} a_n$$

and

$$S_n = \frac{A_{n+1}A_{n-1} - A_n^2}{A_{n+1} + A_{n-1} - 2A_n}.$$

This new sequence, called the Shanks transformation of the series, will usually converge faster than the original series. It is denoted by $S(A_n)$, and works particular well on alternating series.

# Example

Iterated Shanks transformation for the series $\sum_{n=1}^{\infty}(.9)^n/n = \ln(10) \approx 2.302585093$. Shanks transformation improves convergence even though this is not an alternating series.

| $n$ | $A_n$ | $S(A_n)$ | $S^2(A_n)$ | $S^3(A_n)$ |
|---|---|---|---|---|
| 1 | 0.9 | | | |
| 2 | 1.305 | 1.9125 | | |
| 3 | 1.548 | 2.052692308 | 2.245159713 | |
| 4 | 1.712025 | 2.133803571 | 2.268413754 | 2.296053112 |
| 5 | 1.830123 | 2.184417 | 2.281042636 | 2.298856749 |
| 6 | 1.9186965 | 2.217632063 | 2.288432590 | 2.300349676 |
| 7 | 1.987024629 | 2.240240634 | 2.292993969 | 2.301192122 |
| 8 | 2.040833030 | 2.256066635 | 2.295924710 | |
| 9 | 2.083879751 | 2.267394719 | | |
| 10 | 2.118747595 | | | |

# Ex 9

## Shanks Transformation

First find $A_1$ through $A_7$ for the following sequences. Note that some sums begin at $m = 0$, causing $A_1$ to be the sum of two terms. Then apply the iterated Shanks transformation to find $S^2(A_n)$ for $n = 3$ to $n = 5$. How many digits of precision does $S^2(A_n)$ give in comparison to the given exact limit?

$$A_n = \sum_{m=1}^{n} \frac{(-1)^{m+1}}{m}, \quad \lim_{n\to\infty} A_n = \ln(2)$$

# Ex 9

## Solution

0.69327731, 0.69310576, 0.69316334; 3 places accuracy

1. Calculus with Parametric Curves and Polar Coordinates
   - Calculus with Parametric Curves
   - Calculus with Polar Coordinates

2. Infinite Sequences and Series
   - Infinite Sequences
   - Series
   - Divergence Test for Series

3. Q&A

# Q&A

Q&A