Computational methods and applications (AMS 147)

Homework 2 - Due Friday, February 2

Please submit to CANVAS a .zip file that includes the following functions:

`chord_method.m`
`Newton_method.m`
`test_zero.m`

For the extra credit part, please scan your notes into a PDF file `proof.pdf` and attach it to your submission.

Exercise 1 Write two functions `chord_method.m` and `Newton_method.m` implementing, respectively, the chord and the Newton methods to find the zeros of nonlinear (scalar) equations. The functions should be of the following form

```
function [z0,iter,res,his] = chord_method(fun,a,b,tol,Nmax)
```

```
function [z0,iter,res,his] = Newton_method(fun,dfun,x0,tol,Nmax)
```

Inputs:
`fun`: function handle representing $f(x)$
`a`, `b`: interval $[a, b]$ in which we believe there is a zero
`tol`: maximum tolerance allowed for the increment $|x^{(k+1)} - x^{(k)}|$
`Nmax`: maximum number of iterations allowed
`dfun`: function handle representing $df(x)/dx$ (Newton method)
`x0`: initial guess for the zero (Newton method)

Outputs:
`z0`: approximation of the zero z_0
`iter`: number of iterations to obtain z_0
`res`: residual at z_0 (i.e., $|f(z_0)|$)
`his`: vector collecting all the elements of the sequence $\{x^{(k)}\}_{k=0,1,..}$ (convergence history)

Both functions should return the numerical approximation of the zero when the increment at iteration $k+1$ is such that $|x^{(k+1)} - x^{(k)}| <$ `tol` or when the iteration number reaches the maximum value `Nmax`.

Exercise 2 Use the functions of Exercise 1 to compute an approximation of the smallest zero of the fifth-order Chebyshev polynomial

$$f(x) = 16x^5 - 20x^3 + 5x, \qquad x \in [-1, 1]. \tag{1}$$

To this end, set `tol`$=10^{-15}$, `Nmax`$=20000$, `a`$=-0.99$, `b`$=-0.9$, `x0`$=-0.99$ and write a function `test_zero.m` that returns the aforementioned approximate zero by using the chord and the Newton methods. The function should be of the form

```
function [zc,zn,ec,en,x,f] = test_zero()
```

Outputs:
`zc`, `zn`: zero obtained with the chord method (`zc`) and the Newton method (`zn`).
`ec`, `en`: error vectors with components $|x^{(k)} - z_0|$ $(k = 0, 1, ...)$ generated by the cord method (`ec`) and the Newton method (`en`): $z_0 = \cos(9\pi/10)$ is the exact zero of (1) in the interval $[-1, -0.9]$, while $x^{(k)}$ is the sequence converging to z_0 generated by the chord or the Newton method.
`x`: row vector of 1000 evenly spaced nodes in $[-1, 1]$ including the endopoints.
`f`: row vector representing the function (1) evaluated at `x`.

The function `test_zero()` should also produce the following three figures

1. The graph of the function (1) in `figure(1)`.
2. The plots of the convergence histories, i.e., the errors $e_k = |x^{(k)} - z_0|$ versus k for the chord and the Newton methods. These two plots should be in the same `figure(2)`, and in a semi-log scale (use the command `semilogy`).
3. The plots of $e_{k+1} = |x^{(k+1)} - z_0|$ (y-axis) versus $e_k = |x^{(k)} - z_0|$ (x-axis) in a log-log scale (use the command `loglog`) for the chord and the Newton methods. These plots should be in the same `figure(3)`. Remember, for sufficiently large k, the slope of the curves in such log-log plots represents the convergence order of the sequences.

Extra Credit Let $f \in C^{(\infty)}([a, b])$ be a real valued function, $\alpha \in [a, b]$ such that $f(\alpha) = 0$ and $f'(\alpha) \neq 0$ (simple zero). By using the theory of fixed point iterations prove that the convergence order of the sequence

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})} - \frac{f\left(x^{(k)} - \dfrac{f(x^{(k)})}{f'(x^{(k)})}\right)}{f'(x^{(k)})} \tag{2}$$

is 3 in a suitable neighborhood of α.