CS264A: Automated Reasoning

Fall 2020
Homework1
Due Date: Wednesday, Oct 21

1. [8 pts] Show that the following sentences are consistent by identifying a world which satisfies each sentence:

   - $(\neg A \Rightarrow B) \wedge (A \Rightarrow \neg B)$.
   - $(A \wedge B) \Rightarrow (\neg A \vee \neg B)$.

   SOLUTION: For the first sentence, $A = false$, $B = true$ or $A = true$, $B = false$ satisfies it. For the second sentence, any world with $A = false$ or $B = false$ satisfies the sentence.

2. [8 pts] Show that the following sentences are valid by showing that each is true at every world:

   - $(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$.
   - $((A \vee B) \wedge (A \Rightarrow C)) \Rightarrow (B \vee C)$.

   SOLUTION: Use true table to illustrate both sentenses are true at every worlds.

3. [8 pts] Prove from the definitions of Boolean quantifiers $\exists$ and $\forall$ that (a) $\exists P(\Delta \vee \Gamma)$ is equivalent to $(\exists P\Delta) \vee (\exists P\Gamma)$, and (b) $\forall P(\Delta \wedge \Gamma)$ is equivalent to $(\forall P\Delta) \wedge (\forall P\Gamma)$. SOLUTION:

$$\begin{aligned}
\exists P(\Delta \vee \Gamma) &= (\Delta \vee \Gamma)|P \vee (\Delta \vee \Gamma)|\neg P \\
&= (\Delta|P \vee \Gamma|P) \vee (\Delta|\neg P \vee \Gamma|\neg P) \\
&= \Delta|P \vee \Gamma|P \vee \Delta|\neg P \vee \Gamma|\neg P \\
&= (\Delta|P \vee \Gamma|\neg P) \vee (\Gamma|P \vee \Delta|\neg P) \\
&= \exists P\Delta \vee \exists P\Gamma
\end{aligned}$$

$$\begin{aligned}
\forall P(\Delta \wedge \Gamma) &= (\Delta \wedge \Gamma)|P \wedge (\Delta \wedge \Gamma)|\neg P \\
&= (\Delta|P \wedge \Gamma|P) \wedge (\Delta|\neg P \wedge \Gamma|\neg P) \\
&= \Delta|P \wedge \Gamma|P \wedge \Delta|\neg P \wedge \Gamma|\neg P \\
&= (\Delta|P \wedge \Gamma|\neg P) \wedge (\Gamma|P \wedge \Delta|\neg P) \\
&= \forall P\Delta \wedge \forall P\Gamma
\end{aligned}$$

4. [8 pts] Convert the following knowledge base to clausal form:

$$\Delta = \quad A \Rightarrow B, \;\; \neg A \Rightarrow (\neg B \wedge C), \;\; (B \vee C) \Rightarrow D.$$

   SOLUTION:

$$A \Rightarrow B = \neg A \vee B$$

$$\begin{aligned} \neg A \Rightarrow (\neg B \wedge C) &= A \vee (\neg B \wedge C) \\ &= (A \vee \neg B) \wedge (A \vee C) \end{aligned}$$

$$\begin{aligned} (B \vee C) \Rightarrow D &= \neg(B \vee C) \vee D \\ &= (\neg B \wedge \neg C) \vee D \\ &= (\neg B \vee D) \wedge (\neg C \vee D) \end{aligned}$$

Therefore,

$$\Delta = (\neg A \vee B), (A \vee \neg B), (A \vee C), (\neg B \vee D), (\neg C \vee D).$$

5. [8 pts] Show that if we have a polynomial procedure for model counting, and another for clausal entailment on a knowledge base Γ, then we have a polynomial procedure for testing the equivalence between Γ and any CNF Δ.

   SOLUTION: Notice that two propositional sentences Δ and Γ are equivalent iff $Mods(\Delta) = Mods(\Gamma)$. This test can be reduced to checking whether $Modes(\Gamma) \subseteq Mods(\Delta)$ and that $|Mods(\Gamma)| = |Mods(\Delta)|$.

6. [10 pts] Show using resolution that $D \vee E$ is entailed by the knowledge base:

$$\Delta = \ \neg A \Rightarrow B, \ \ A \Rightarrow \neg C, \ \ \neg D \Rightarrow \neg B \wedge \neg C, \ \ A \Rightarrow E.$$

   SOLUTION: We first convert each clasuse in Δ into clausal form:

$$\begin{aligned} \neg A \Rightarrow B &= A \vee B \\ A \Rightarrow \neg C &= \neg A \vee \neg C \\ \neg D \Rightarrow \neg B \wedge \neg C &= (D \vee \neg B) \wedge (D \vee \neg C) \\ A \Rightarrow E &= \neg A \vee E \end{aligned}$$

   Then we add $\neg D$ and $\neg E$ to Δ and show taht an empty clause can be derived.

1.	$\{A, B\}$	Δ
2.	$\{\neg A, \neg C\}$	Δ
3.	$\{\neg B, D\}$	Δ
4.	$\{\neg C, D\}$	Δ
5.	$\{\neg A, E\}$	Δ
6.	$\{\neg D\}$	Added
7.	$\{\neg E\}$	Added
8.	$\{\neg A\}$	5,7
9.	$\{B\}$	1,8
10.	$\{D\}$	3,9
11.	$\{\}$	6,10

Therefore, $D \vee E$ is implied by the knowledge base.

7. [12 pts] Show the termination tree for DPLL when run on the following KB, assuming that variables are tested according to the order A, B, C, D, E and *true* expanded before *false*:

$$\Delta = \begin{array}{l} 1.\ A \wedge D \Rightarrow E \\ 2.\ C \Rightarrow D \\ 3.\ D \Rightarrow \neg E \\ 4.\ B \wedge \neg C \Rightarrow D \end{array}$$

Note that DPLL does not use conflict–directed backtracking.

SOLUTION: See Figure 1. The solution found by DPLL is $A, \neg B, \neg C, \neg D$.

8. [12 pts] Show a trace of DPLL+ on the above KB, assuming that decisions are made according to the constraints given above. At each conflict, show the decision sequence, implication graph, conflict–drive clause, and its assertion level. Perform one trace of DPLL+which assuming that conflict–driven clauses are generated using the first UIP method of Section 3.6.2.

SOLUTION: Consider the following clausal form:

1.	$\{\neg A, \neg D, E\}$
2.	$\{\neg C, D\}$
3.	$\{\neg D, \neg E\}$
4.	$\{\neg B, C, D\}$

Using the first UIP method, the first contradiction is discovered at the decision sequence (A, B, C) leading to conflict-driven clause $\neg A \vee \neg D$ with assertion level 0; see Figure 2. Backtracking to this level and adding the learned clause leads to deriving $\neg D, \neg C, \neg B$ by unit resolution, which terminates the search.

9. [12 pts] Consider the following knowledge base.

$$\Delta = \quad A \Rightarrow B,\ \ \neg A \Rightarrow (\neg B \wedge C),\ \ (B \vee C) \Rightarrow D.$$

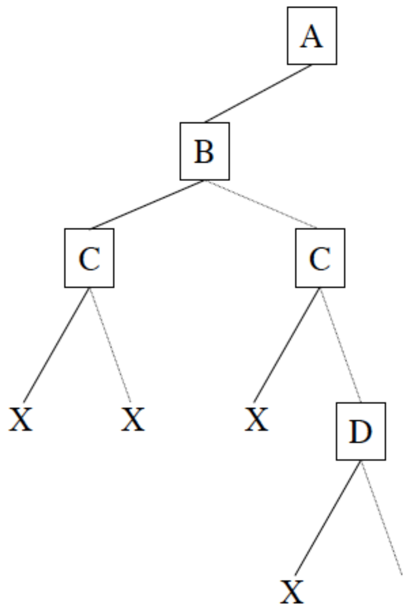

Figure 1: Termination tree. Solid edges correspond to true, while dotted edges correspond to false. Leaf nodes labelled with "X" represent failures.

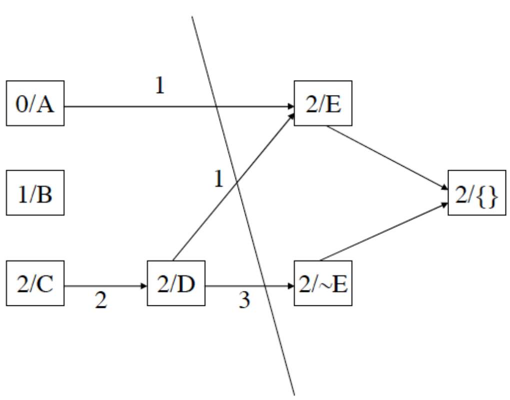

Figure 2: An implication graph and a cut defining a conflict-driven clause.

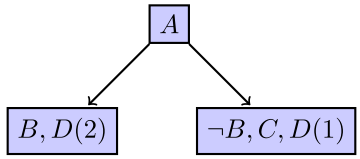

Figure 3: A termination tree for CDPLL.

Show how you can count the number of models of Δ using CDPLL and draw the termination tree. Assume that you are expanding variables according to the order A, B, C, D and always expand *true* before *false*.

SOLUTION: The Knowledge base is:

$$\Delta = (\neg A \vee B), (A \vee \neg B), (A \vee C), (\neg B \vee D), (\neg C \vee D).$$

Model count is 3; see Figure 3 for termination tree.

10. [14 pts] Consider the following knowledge base:

$$\Delta = P_1 \vee P_2 \vee P_3, \;\; P_1 \Rightarrow Q, \;\; P_2 \Rightarrow Q, \;\; P_3 \Rightarrow Q.$$

   a. Convert Δ into clausal form.
   b. Apply directed resolution to the clausal form using the order P_1, P_2, P_3, Q.
   c. Construct a decision tree for Δ and use it to count the number of models of Δ.

SOLUTION:

a. $\{P_1, P_2, P_3\}, \{\neg P_1, Q\}, \{\neg P_2, Q\}, \{\neg P_3, Q\}$.

b. We start with

$$\begin{array}{r|c|}
P_1 & \{P_1, P_2, P_3\}, \{\neg P_1, Q\} \\
P_2 & \{\neg P_2, Q\} \\
P_3 & \{\neg P_3, Q\} \\
Q & \\
\end{array}$$

The directed extension is:

$$\begin{array}{r|c|c}
P_1 & \{P_1, P_2, P_3\}, \{\neg P_1, Q\} & \\
P_2 & \{\neg P_2, Q\} & \{P_2, P_3, Q\} \\
P_3 & \{\neg P_3, Q\} & \{P_3, Q\} \\
Q & & \{Q\} \\
\end{array}$$

c. Model count is 7. See Figure 4.

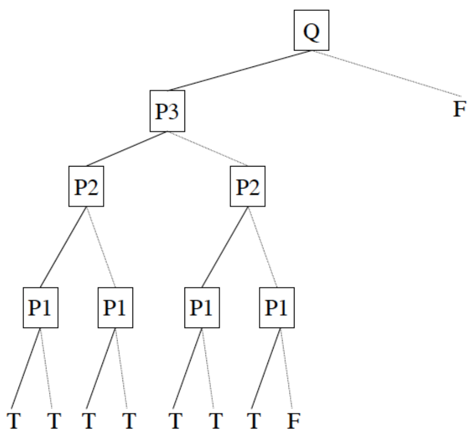

Figure 4: A decision tree generated from the extension of directed resolution.