HW #2 solutions

Thursday, November 5, 2020 4:44 PM

① Let the i^{th} clause be referred to as C_i

$\Delta = (X \vee Y)$, 1
$(X \vee Z)$, 2
$(\neg Y \vee W)$, 3
$(\neg Z \vee \neg W)$, 4
$(X \vee \neg Y)$, 5
$(\neg X \vee W)$, 6
$(\neg X \vee \neg W \vee V)$, 7
$(\neg X \vee \neg W \vee \neg V)$, 8
$(Z \vee \neg X)$, 9
$(\neg Z \vee W)$. 10

11. $\neg X \vee \neg W$ (7, 8)
12. $\neg X$ (6, 11)
13. X (1, 5)
14. $\perp$ (12, 13)

Hence $\Delta' = \{C_7, C_8, C_6, C_1, C_5\}$ is unsatisfiable.
Moreover, for each $c \in \Delta'$, $(\Delta' \setminus c) | \neg c$ is valid and hence $\Delta' \setminus c$ is satisfiable
Hence Δ' is min unsatisfiable core of Δ

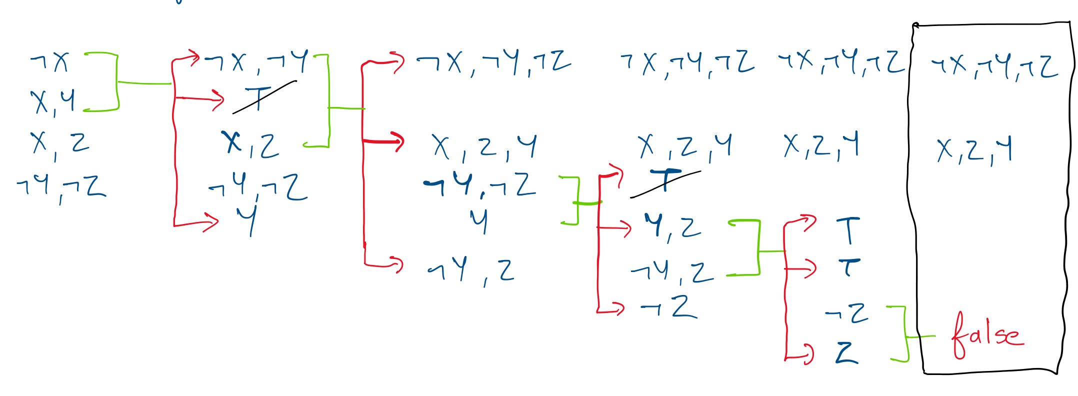
② (A)

$\Delta = (\neg X), (X \vee Y), (X \vee Z), (\neg Y \vee \neg Z)$.
1 2 3 4

X	Y	Z	C_1	C_2	C_3	C_4	cost
0	0	0	1	0	0	1	2
0	0	1	1	0	1	1	1
0	1	0	1	1	0	1	1
0	1	1	1	1	1	0	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	0	1	1	0	2

Optimal solution: Any of (rows 2–7)
Optimal cost = 1

(B) Using variable order X, Y, Z :-



Hence, optimal cost = 1, residual clauses : $\{X, Y, Z\}, \{\neg X, \neg Y, \neg Z\}$

③ $\Delta = (A \vee B \vee C), (\neg A \vee D \vee E)$

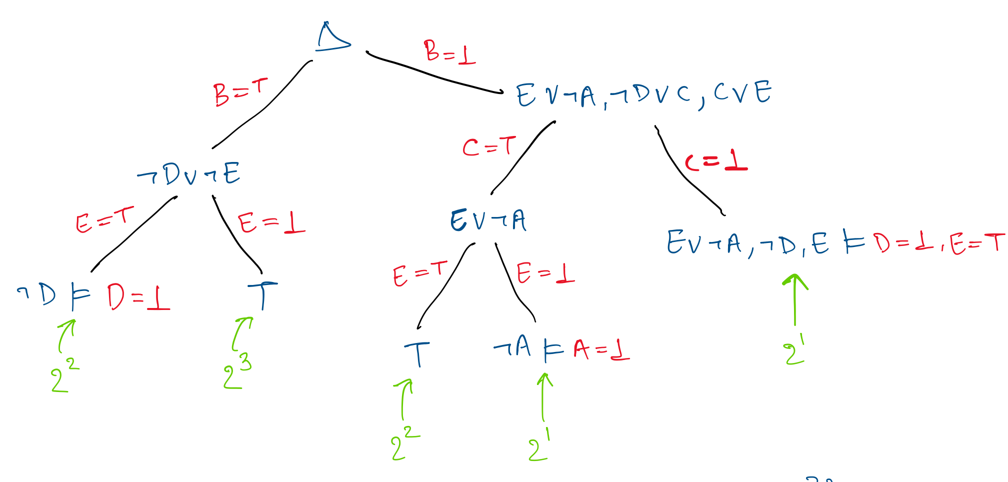
Resolving both clauses:
A, B, C, ¬D
A, B, C, D, ¬E
¬A, D, E, ¬B
¬A, D, E, B, ¬C
B, C, D, E

No resolution left wrt A. No resolution wrt B, C, D, E as well.

④ $\Delta = (\neg D \vee \neg E \vee B), (\neg B \vee E \vee \neg A), (\neg D \vee C \vee \neg B), (\neg B \vee C \vee E)$
1 2 3 4

Total number of assignments $= 2^5 = 32$

(A) EDPLL using order B, C, E, D, A :-



Satisfying assignments $= 4 + 8 + 4 + 2 + 2 = 20 > \frac{32}{2}$

Hence, MAJ-SAT $(\Delta) = 1$

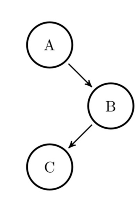
(B) $\Delta | \neg B, A, C = \neg D \vee \neg E$ has $3 > \frac{2^2}{2}$ satisfying assignments

Hence, E-MAJ-SAT $(\Delta, \{A, B, C\}, \{D, E\}) = 1$

(C) $\Delta | \neg B = \neg D \vee \neg E$ and MAJ-SAT $(\neg D \vee \neg E) = 1 \rightarrow 2^2$
$\Delta | B, C, \neg A = T$ and MAJ-SAT $(T) = 1 \rightarrow 1$
$\Delta | B, C, A = E$ and MAJ-SAT $(E) = 0 \rightarrow 1$
$\Delta | B, \neg C = \neg D \wedge E$ and MAJ-SAT $(\neg D \wedge E) = 0 \rightarrow 2^1$

So 5 out of 8 assignments give MAJ-SAT sub-functions.
Hence MAJ-MAJ-SAT $(\Delta, \{A, B, C\}, \{D, E\}) = 1$

⑤



A	B = T	B = F
F	$\theta_{B\mid\neg A}$	$\theta_{\neg B\mid\neg A}$
T	$\theta_{B\mid A}$	$\theta_{\neg B\mid A}$

B	C = T	C = F
F	$\theta_{C\mid\neg B}$	$\theta_{\neg C\mid\neg B}$
T	$\theta_{C\mid B}$	$\theta_{\neg C\mid B}$

A = T	A = F
θ_A	$\theta_{\neg A}$

$\Delta = P_A \Leftrightarrow A,\ P_{\neg A} \Leftrightarrow \neg A,\ P_{B|A} \Leftrightarrow A \wedge B,\ P_{\neg B|A} \Leftrightarrow A \wedge \neg B,$
$P_{B|\neg A} \Leftrightarrow \neg A \wedge B,\ P_{\neg B|\neg A} \Leftrightarrow \neg A \wedge \neg B,\ P_{C|B} \Leftrightarrow B \wedge C,\ P_{\neg C|B} \Leftrightarrow B \wedge \neg C,$
$P_{C|\neg B} \Leftrightarrow \neg B \wedge C,\ P_{\neg C|\neg B} \Leftrightarrow \neg B \wedge \neg C$

x	wt(x)	wt($\neg x$)
P_A	θ_A	1
$P_{B\mid A}$	$\theta_{B\mid A}$	1
$P_{B\mid\neg A}$	$\theta_{B\mid\neg A}$	1
$P_{C\mid B}$	$\theta_{C\mid B}$	1
$P_{C\mid\neg B}$	$\theta_{C\mid\neg B}$	1
A	1	1
B	1	1

x	wt(x)	wt($\neg x$)
$P_{\neg A}$	$\theta_{\neg A}$	1
$P_{\neg B\mid A}$	$\theta_{\neg B\mid A}$	1
$P_{\neg B\mid\neg A}$	$\theta_{\neg B\mid\neg A}$	1
$P_{\neg C\mid B}$	$\theta_{\neg C\mid B}$	1
$P_{\neg C\mid\neg B}$	$\theta_{\neg C\mid\neg B}$	1
C	1	1
	1	1

$$P(A=T, C=F) = \frac{WMC(\Delta \cup \{A, \neg C\})}{WMC(\Delta)}$$

Since $\theta_{\alpha|\beta} + \theta_{\neg\alpha|\beta} = 1$ in Bayesian nets, $WMC(\Delta) = 1$

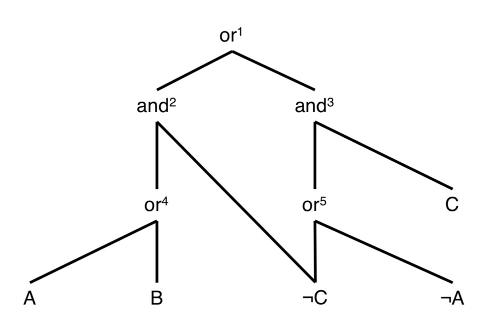
Hence, $P(A=T, C=F) = WMC(\underbrace{\Delta \cup \{A, \neg C, B\}}_{\Delta_1}) + WMC(\underbrace{\Delta \cup \{A, \neg C, \neg B\}}_{\Delta_2})$

Δ_1 has just 1 model where $A, \neg C, B, P_A, P_{B|A}, P_{\neg C|B}$ are true and hence having weight $\theta_A \theta_{B|A} \theta_{\neg C|B}$
Hence, $WMC(\Delta_1) = \theta_A \theta_{B|A} \theta_{\neg C|B}$

Similarly, $WMC(\Delta_2) = \theta_A \theta_{\neg B|A} \theta_{\neg C|\neg B}$

So, $WMC(\Delta) = \theta_A \theta_{B|A} \theta_{\neg C|B} + \theta_A \theta_{\neg B|A} \theta_{\neg C|\neg B}$
$= 0.6 \times (0.8 \times 0.75 + 0.2 \times 0.7) = 0.444$

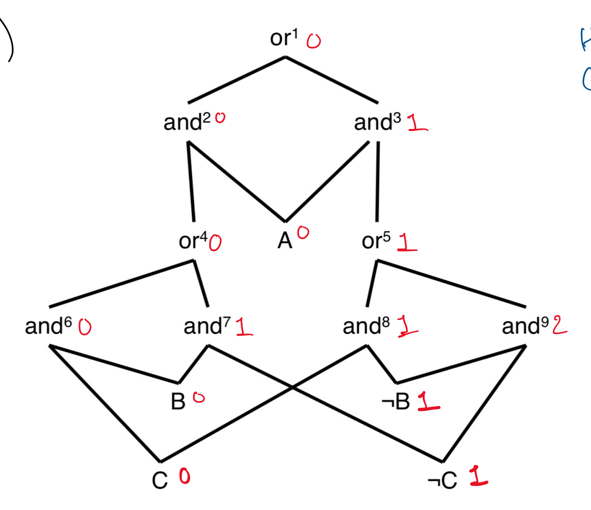
⑥



or^5 inputs have a common model $\{\neg A, \neg C, B\}$. or^4 inputs have $\{A, B, C\}$ in common.
So circuit is not deterministic.

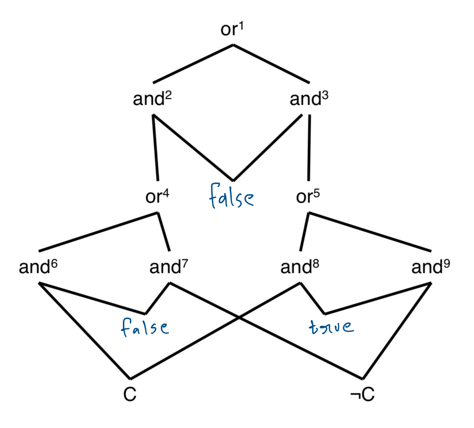
Both and^3 inputs depend on C.
So, circuit is not decomposable.

⑦ (A)



Hence, min cardinality = 0
Corresponding assignment :- A=T, B=T, C=T

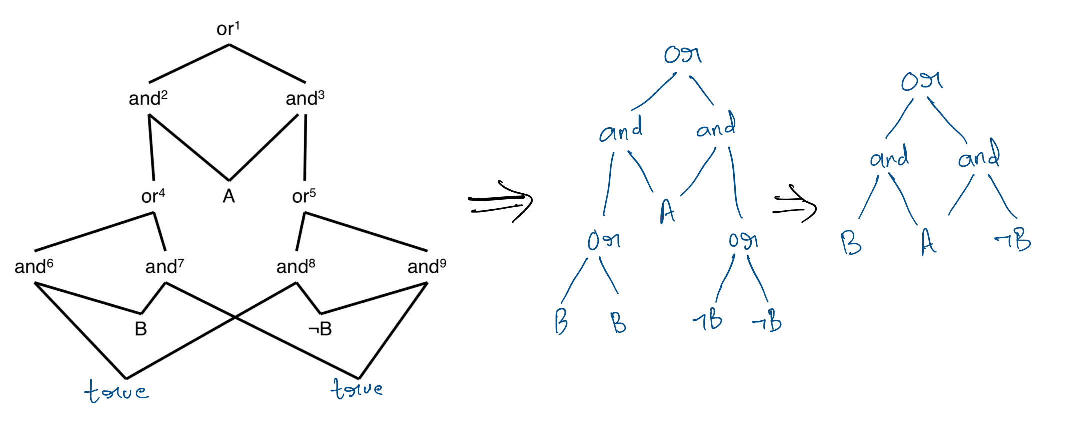
(B) $\Delta \models A \vee B$ iff $\Delta | \neg A, \neg B$ is inconsistent. Circuit for $\Delta | \neg A, \neg B$ can be obtained by replacing all A and B nodes with $\perp$



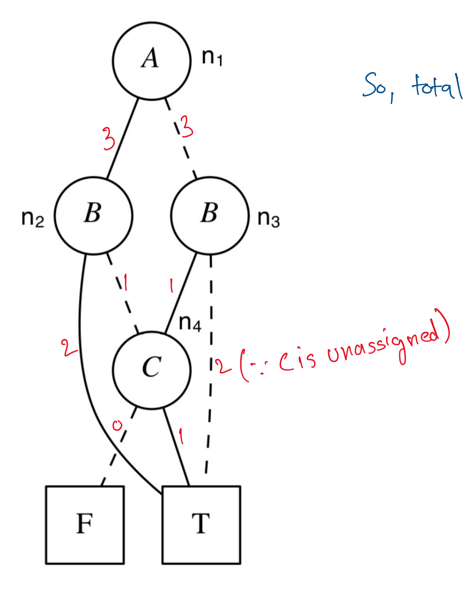
Since and^2 and and^3 have a false input, they are inconsistent.
Hence, or^1 is inconsistent.
So, $\Delta | \neg A, \neg B$ is inconsistent.

So, $\Delta \models A \vee B$

(C) Since all and gates are decomposible, the circuit is decomposable and we can push $\exists C$. all the way down to leaf nodes.



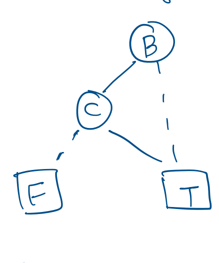
⑧ (A)



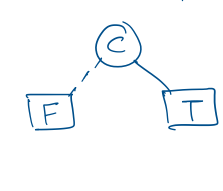
So, total number of satisfying assignments = 6

(B) Let Δ be the OBDD's NNF form. $\Delta \models A \vee \neg B$ iff $\Delta | \neg A, B$ is inconsistent

OBDD of $\Delta | \neg A$:



OBDD of $\Delta | \neg A, B$:



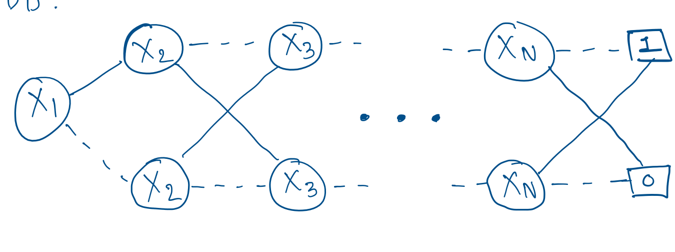
Since OBDD of $\Delta | \neg A, B$ is not $\boxed{F}$, $\Delta | \neg A, B$ is consistent
Hence $\Delta \not\models A \vee \neg B$

⑨ (A) Let the given function be $f_N(X_1, X_2, X_3, \dots, X_N)$
Let $m = \lfloor N/2 \rfloor$

Then, under an assignment $\alpha : \{X_1, X_2, \dots, X_m\} \rightarrow \{0, 1\}$,

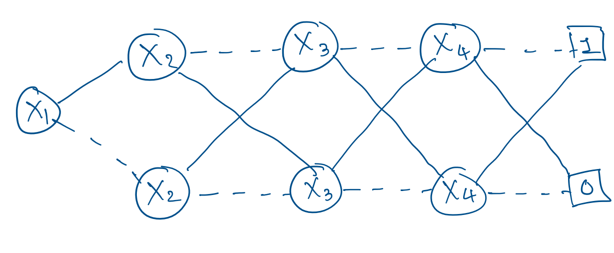
$$f_N | \alpha = \begin{cases} f_{N-m}(X_{m+1}, \dots, X_N) & \text{if } \sum_{i=1}^{m} \alpha(X_i) \text{ is even} \\ \neg f_{N-m}(X_{m+1}, \dots, X_N) & \text{if } \sum_{i=1}^{m} \alpha(X_i) \text{ is odd} \end{cases}$$

(B) ROBDD:



Hence, 2N+1 nodes

(C)



⑩ $\forall X (\varphi_1 \wedge \varphi_2) \equiv (\forall X \varphi_1) \wedge (\forall X \varphi_2)$ —— ①

Hence if we have circuits for $\forall X \varphi_1$ and $\forall X \varphi_2$, $\forall X (\varphi_1 \wedge \varphi_2)$ can easily be constructed.

$\forall X (\varphi_1 \vee \varphi_2) = (\varphi_1 \vee \varphi_2)|X \wedge (\varphi_1 \vee \varphi_2)|\neg X$
$= (\varphi_1|X \vee \varphi_2|X) \wedge (\varphi_1|\neg X \vee \varphi_2|\neg X)$

To apply distributivity, we need 1 formula in common.
So, we need either $\varphi_1|X \equiv \varphi_1|\neg X$ or $\varphi_2|X \equiv \varphi_2|\neg X$.
So, X can appear only in 1 branch of $\vee$. This can be ensured by requiring both the branches of $\wedge$ to depend on disjoint sets of variables

So, if this property holds for NNF $\varphi_1 \vee \varphi_2$,

$\forall X (\varphi_1 \vee \varphi_2) = (\forall X \varphi_1) \vee (\forall X \varphi_2)$ —— ②

Using ① and ②, we can propogate $\forall$ all the way down to literals. Base cases are as follow :-

$\forall X (Y) = Y$, $\forall X (\neg Y) = \neg Y$, $\forall X (X) = F$, $\forall X (\neg X) = F$

Hence, if the aforementioned property holds, $\forall X (\varphi)$ can be simply computed by replacing all X and $\neg X$ nodes in φ with F and simplifying, which can be done in linear time.