CS267A: Homework #6

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Problem 1

1. **Solution:** 1. $\Pr(\exists x.R(x) \wedge T(x))$
$= 1- \Pr[\forall x.\neg R(x) \vee \neg T(x)]$
$= 1 - \prod_i \Pr[\neg R(i) \vee \neg T(i)]$
$= 1 - \prod_i (1- \Pr[R(i) \wedge T(i)])$
$= 1 - \prod_i (1- \Pr[R(i)]\Pr[T(i)])$.

2. **Solution:** 2. $\Pr(\exists x.\exists y.S(x, y) \wedge R(x))$
$= 1-\Pr(\forall x.\forall y.\neg S(x, y) \vee \neg R(x))$
$= 1 - \prod_i \Pr[\forall y.\neg S(i, y) \vee \neg R(i)]$
$= 1 - \prod_i \Pr[\neg R(i) \vee \forall y.\neg S(i, y)]$
$= 1 - \prod_i (1-\Pr[R(i) \wedge \exists y.S(i, y)])$
$= 1 - \prod_i (1-\Pr[R(i)]\wedge\Pr[\exists y.S(i, y)])$
$= 1 - \prod_i \{1-\Pr[R(i)] \wedge (1-\Pr[\forall y.\neg S(i, y)])\}$
$= 1 - \prod_i \{1-\Pr[R(i)] \wedge (1 - \prod_j \Pr[\neg S(i, j)])\}$
$= 1 - \prod_i \{1-\Pr[R(i)] \wedge (1 - \prod_j (1-\Pr[S(i, j)]))\}$.

3. **Solution:** 3. It is not possible
$\Pr([\exists x.\exists y.R(x) \wedge S(x, y) \wedge T(y)] \vee [\exists x.U(x)])$
$= \Pr([\exists x.\exists y.R(x) \wedge S(x, y) \wedge T(y)])+\Pr([\exists x.U(x)])-\Pr([\exists x_1.\exists y.R(x_1) \wedge S(x_1, y) \wedge T(y)] \wedge [\exists x_2.U(x_2)])$.
The first clause can not be applied with lifted inference rule.

4.
Solution: 4. $\Pr(\exists x_1.\exists x_2.\exists y_1.\exists y_2.R(x_1) \wedge S(x_1,y_1) \wedge T(x_2) \wedge S(x_2,y_2))$
= $1-\Pr(\forall x_1.\forall x_2.\forall y_1.\forall y_2.\neg R(x_1) \vee \neg S(x_1,y_1) \vee \neg T(x_2) \vee \neg S(x_2,y_2))$
= $1-\Pr([\forall x_1.\forall y_1.\neg R(x_1) \vee \neg S(x_1,y_1)] \vee [\forall x_2.\forall y_2.\neg T(x_2) \vee \neg S(x_2,y_2)])$
= $1-\Pr([\forall x_1.\forall y_1.\neg R(x_1) \vee \neg S(x_1,y_1)] \vee [\forall x_1.\forall y_1.\neg T(x_1) \vee \neg S(x_1,y_1)])$
= $1-\prod_i \Pr(\forall y.\neg R(i) \vee \neg S(i,y) \vee \neg T(i))$
= $1-\prod_i \Pr(\neg R(i) \vee \neg T(i) \vee \forall y.\neg S(i,y))$
= $1-\prod_i [1-\Pr(R(i) \wedge T(i) \wedge \exists y.S(i,y))]$
= $1-\prod_i [1-\Pr(R(i))\times\Pr(T(i))\times\Pr(\exists y.S(i,y))]$
= $1-\prod_i [1-\Pr(R(i))\times\Pr(T(i))\times[1-\Pr(\forall y.\neg S(i,y))]]$
= $1-\prod_i [1-\Pr(R(i))\times\Pr(T(i))\times(1-\prod_j [1-\Pr(S(i,j))])]$

5.
Solution: 5. It is not possible.
$\Pr((\exists x_1.\exists y_1.R(x_1) \wedge S(x_1,y_1)) \vee (\exists x_2.\exists y_2.T(y_2) \wedge S(x_2,y_2))$
= $\Pr((\exists x_1.R(x_1) \wedge \exists y_1.S(x_1,y_1)) \vee (\exists y_2.T(y_2) \wedge \exists x_2.S(x_2,y_2))$
= $\Pr((\exists x.R(x) \wedge \exists y.S(x,y)) \vee (\exists y.T(y) \wedge \exists x.S(x,y))$
= $1-\Pr((\forall x.\neg R(x) \vee \forall y.\neg S(x,y)) \wedge (\forall y.\neg T(y) \vee \forall x.\neg S(x,y))$
= $1-\Pr([(\forall x.\neg R(x) \vee \forall y.\neg S(x,y)) \wedge \forall y.\neg T(y)] \vee [(\forall x.\neg R(x) \vee \forall y.\neg S(x,y)) \wedge \forall x.\forall y.\neg S(x,y)])$
= $1-\Pr([\forall x.\forall y.\neg R(x) \wedge \neg T(y) \vee \neg S(x,y) \wedge \neg T(y)] \vee [\forall x.\forall y.\neg R(x) \wedge \neg S(x,y) \vee \neg S(x,y)])$
= $1-\Pr(\forall x.\forall y.[\neg R(x) \wedge \neg T(y) \vee \neg S(x,y) \wedge \neg T(y)] \vee [\forall x.\forall y.\neg S(x,y)])$
= $1-\Pr(\forall x.\forall y.\neg R(x) \wedge \neg T(y) \vee \neg S(x,y) \wedge \neg T(y) \vee \neg S(x,y))$
= $1-\Pr(\forall x.\forall y.\neg R(x) \wedge \neg T(y) \vee \neg S(x,y))$
= $1-\Pr(\forall x.\forall y.\neg R(x) \vee \neg S(x,y) \wedge \neg T(y) \vee \neg S(x,y))$
= $1-\Pr(\forall x.\forall y.\neg R(x) \vee \neg S(x,y))+\Pr(\forall x.\forall y.\neg T(y) \vee \neg S(x,y))-\Pr((\forall x.\forall y.\neg R(x) \vee \neg S(x,y)) \vee (\neg T(y) \vee \neg S(x,y)))$
= $1-\Pr(\forall x.\forall y.\neg R(x) \vee \neg S(x,y))+\Pr(\forall x.\forall y.\neg T(y) \vee \neg S(x,y))-\Pr((\forall x.\forall y.\neg R(x) \vee \neg T(y) \vee \neg S(x,y)))$
Lifted inference can not be applied on the last clause, and no possible cancellations.

6.
Solution: 6. It is not possible
$\Pr((\exists x_1.\exists y_1.R(x_1) \wedge S(x_1,y_1)) \vee (\exists x_2.\exists y_2.S(x_2,y_2) \wedge T(y_2)) \vee (\exists x_3.\exists y_3.R(x_3) \wedge T(y_3)))$
= $\Pr(R(x) \wedge S(x,y))+\Pr(S(x,y) \wedge T(y))+\Pr(R(x) \wedge T(y))-\Pr((R(x) \wedge S(x,y)) \wedge (S(x,y) \wedge T(y)))-\Pr((R(x) \wedge S(x,y)) \wedge (R(x) \wedge T(y)))-\Pr((S(x,y) \wedge T(y)) \wedge (R(x) \wedge T(y)))+\Pr((R(x) \wedge S(x,y)) \wedge (S(x,y) \wedge T(y)) \wedge (R(x) \wedge T(y)))$
= $\Pr(R(x) \wedge S(x,y))+\Pr(S(x,y) \wedge T(y))+\Pr(R(x) \wedge T(y))-\Pr(R(x) \wedge S(x,y) \wedge T(y))-\Pr(R(x) \wedge S(x,y) \wedge T(y))-\Pr(S(x,y) \wedge T(y) \wedge R(x))+\Pr(R(x) \wedge S(x,y) \wedge T(y))$
= $\Pr(R(x) \wedge S(x,y))+\Pr(S(x,y) \wedge T(y))+\Pr(R(x) \wedge T(y)) - 2\times\Pr(R(x) \wedge S(x,y) \wedge T(y))$
Lifted inference fails on the last term

Problem 2

1.
Solution: The model count for this formula is 3.
Negate H_0 we have $\neg H_0 = \forall x.\forall y.\neg S(x) \vee \neg F(x,y) \vee \neg R(y)$
If $F(x,y) = 1$. Then, $\neg H_0 = \forall x.\forall y.\neg S(x) \vee \neg R(y)$.
The probability of $\neg S(x)$ and $\neg R(y)$ are 0.5, which means the probability of $S(x)$ and $R(y)$ are $1-0.5=0.5$.
Then, the model count is $K = (1-P(H_0))/p = (1-1/4)/1/4 = 3$.

Problem 2 continued on next page. . .

2. **Solution:** First take negation of H_0, then $\neg H_0 = \forall x.\forall y.\neg S(x) \vee \neg F(x,y) \vee \neg R(y)$ and Since it contains $\neg F(x,y)$ then we negate f, so we have:
$\neg f = (\neg x_1 \wedge \neg y_1) \vee (\neg x_2 \wedge \neg y_2) \vee (\neg x_3 \wedge \neg y_3) \vee (\neg x_1 \wedge \neg y_3)$.
$\neg F(x,y) = 0$ if $(x = \neg x_1, y = \neg y_1), (x = \neg x_2, y = \neg y_2), (x = \neg x_3, y = \neg y_3)$, or $(x = \neg x_1, y = \neg y_3)$
$\neg F(x,y) = 1$ otherwise.
Therefore, $\neg H_0 = \forall x.\forall y.\neg S(x) \vee \neg R(y)$ if and only if both variables occur in one of the four clauses.
There are $2^6 = 64$ scenarios since there are six vairables. $\neg x_1$ and $\neg y_3$ appears one more time than others which has 1.5 probabilities than others.
let s_i represent weight of $\neg x_i$ is from S(x) and r_i represents weight of $\neg y_i$ is from R(y)
So $P(H_0) = 1 - P(\neg H_0) = s_1 r_1 + s_2 r_2 + s_3 r_3 + s_1 r_3 = P(\neg f)$
Therefore $K = P(H_0)/p = (1 - P(\neg H_0))/p$ in this case p =1/64 and $s_1 = r_3 = \frac{3}{2}s_2 = \frac{3}{2}s_3 = \frac{3}{2}r_1 = \frac{3}{2}r_2$

Problem 3

1. **Solution:**

No. The sudoku puzzle can be solved by an SAT-Solver. So it means that the sudoku puzzle can be reduced to a SAT problem where SAT-solver is part of the solution. So sudoku puzzle is polynomial time reducible to SAT problem. It also means that SAT problem is at least as hard as sudoku puzzle. We have done the opposite which means sudoku is less hard or equal hard to NP-complete. We can not prove it is NP-hard.

Problem 4

1. **Solution:**

YES. In this case we want to reduce #PP3CNF counting problem to #PP2CNF. Since we have already know that #PP2CNF is #P-hard

First we create a PP2CNF whose variables V_1 are partitioned into 2 disjoint sets X and Y, i.e. $V_1 = X \cup Y, X \cap Y = \emptyset$. Let $Z = \emptyset$
Then for the previous PP3CNF $X \cap Y = Y \cap Z = X \cap Z = \emptyset$. When $Z = \emptyset$, $Y \cap Z = X \cap Z = \emptyset$ is satisfied for any X, Y. Furthermore, $X \cup Y \cup Z = X \cup Y = V_1$. Thus, this PP3CNF can be reduced to an equivalent PP2CNF which has $Z = \emptyset$. Since the #PP2CNF problem is #P-hard, it follows that the #PP3CNF is #P-hard as well.