Spin Model of Two Random Walkers in Complex Networks

Wenshuo Guo, Juntao Wang and Kwok Yip Szeto[1]

Abstract In this paper, we design a spin model to optimize the search in complex networks using two random walkers by reducing wasted time in revisiting nodes. A good measure of the global performance of the searching process is the Mean Cover Time (MCT), which is the time needed from the start of the searching process that all sites in the network are reached by at least one walker. We use a three-state spin model to minimize the MCT for two random walkers. In this model, each site in the complex network is described by a three-state spin, with unvisited sites having a spin state defined by white color, and the visited sites in spin states with non-white colors (red and blue in the case of two walkers). The visit of a site by a walker changes the state of the spin. We introduce a repulsive interaction between spin to model the interaction between walkers. Numerical results using Erdös–Rényi (ER) network and Watts–Strogatz (WS) network and three small-world like real network data sets show satisfactory results in reducing the MCT. For small-world network, both the artificial WS network and real-world network datasets show the existence of the critical repulsion strength which minimizes the MCT. We also provide a heuristic explanation for the presence of the critical repulsion that minimizes MCT for the WS network and its absence in the ER networks. Our model provides guidance to future research on multiple random walkers on complex networks, with potential application for efficient information spreading in social networks.

Keywords Random walk, Complex network, Spin model, Search

[1] W. Guo • J. Wang • K. Y. Szeto*
Department of Physics, the Hong Kong University of Science and Technology,
Kowloon, Hong Kong
*e-mail: phszeto@ust.hk

1 Introduction

Network science is an important multi-disciplinary research area due to its successful description of many complex systems, with applications in many real-world networks such as the human brain [16], the banking network ecosystem [12], the traffic network [18] and the Internet [11]. The interplay between dynamics and geometry is also a fascinating topic for physicists due to the availability of numerical simulation of dynamics taking place in evolving networks and the large body of works on the structural complexity of the networks [3, 5] such as the communities and hierarchical structures [1, 6, 14, 19].

Random walk has been an old research area with many applications in science and engineering, with introductory materials well documented in standard textbooks [13, 17]. Despite the long history of research in random walk, we still find many recent articles written on the subject, probably due to the increased importance of search in complex network. In the frontier research in search engine, random walk again finds itself a good paradigm due to its simplicity of implementation as well as its readily amenable for rigorous mathematical analysis. Examples include random walk on regular lattice [15], in bounded domains [7], in small-world networks or real-world scenarios [2, 21], walks with trappings [10], and the degree-biased walk or the popularity-biased walk [4, 24], etc. However, increasing the intelligence of the random walker inevitably raises cost due to increase in communication and information collection. Thus, a natural way to shorten the information delay and achieve faster information spreading for search is to increase the total amount of random walkers. Another issue of cost will be the waste of searching time due to the redundant effort in revisiting sites already visited by the walkers. One method of solution will be the employment of walkers practicing self-avoiding walks. However, walkers with ability to prevent visiting sites already visited has been treated in physics and it remains a difficult problem. Different characteristic times with different focuses to evaluate the performance of searching process have been defined and analyzed, including the mean return times, mean first passage times, and the mean cover time (MCT) that is our focused topic in this work [4, 8]. Recent work on multiple random walkers searching on networks includes T. Wang et al's study on a method of using harmonic law to predict search time of several independent random walkers [23], Ding and Szeto's work on optimization of the selection of the starting points of walkers [9].

In this paper, we address the employment of two random walkers for search, with a low requirement on the intelligence of the random walker and the global structural information of the network. Based on the idea of spin model, we designed a model to minimize the mean cover time (MCT) of the complex network of two random walkers, where each site in the network has a spin with different states. We assume our walker has some intelligences in gathering the local information on the spin-state of his nearest neighbors. The two walkers also interact with a repulsive force to control their probability of avoidance of sites already visited. We perform numerical results on artificial network datasets (Erdös–Rényi (ER) and Watts–Strogatz (WS) network) as well as some real network datasets. We find that we can reduce the MCT substantially with two intelligent walkers. For small-world like networks, we also find the existence of a critical repulsion parameter that can minimize the MCT. We expect the insights in this study can be useful to information spreading and advocating in real-world networks.

2 Model

We formulate a model of search by many random walkers in a complex network with the objective to optimize the mean cover time (MCT). Specifically, we like to compare the performance of two "intelligent" random walkers performing random walk in a noisy environment by computing the mean cover time of the entire search area. Here the word "intelligent" refers to the walkers who can avoid sites already visited. We construct a spin model for computing the probability of avoiding sites so that in one limit, the walker is the simpleton who wanders around like a traditional random walker without memory, and in another limit he has a long memory so that he recognizes sites that have been visited and avoid them. Physically, we assume an intelligent walker has a charge, and has the ability to turn on a similar charge on the site that he has just visited. The repulsion between charges will deter the intelligent walker to go to those sites already visited. In our model, each node of the complex network is associated with a spin that has three possible states. The visit of each walker could lead to a physical change of the spin state of the node. For unvisited sites, we assume their spin states are white which can turn into either red or blue, depending on the color of the visitor. The visit of the red (blue) walker could change the visited site into red (blue) state. The mean cover time problem is to find the time steps that the two random walkers require, on the average, to make the spin states of each site in a complex network that are white to non-white, so that the entire network is covered. The optimization problem is to reduce the overlap among the paths of walkers. To avoid waste of resource in search, we introduce a model for interaction between spin states to describe the intelligence of the walkers.

The simplest model is to assume that the red walker performing simple random walk will be interacting only with the blue walker. In another word, the walker will not have any self-avoidance in the sense that a red (blue) walker will not avoid a node with spin state that is red (blue). However, if the red (blue) walker wants to go to a node with spin state that is blue (red), it will experience a repulsion. We assume that the probability that the red (blue) walker will go to the blue (red) node has a probability proportional to $0 \leq e^{-\beta} \leq 1$. Thus, for $\beta = \infty$, the red walker will not visit a blue neighboring node. On the other hand, if $\beta = 0$, then the red walker will treat the blue neighboring node as any other node, implying that the walker is not avoiding sites already visited and is a simple normal random walker. In general, a red (blue) walker with $0 < \beta < \infty$ has some intelligence in avoiding nodes already visited by blue (red) walker. The result is that the network will have community of red and blue nodes, and these communities will have boundaries which crossing by the walkers will cost them energy. We expect that this simple three-spin states model of two random walkers travelling on a complex network will be a simple model for resource allocation with multiple intelligent walkers searching for a general target in a complex network, with neither the prior knowledge of the target nor the network structures. The Boltzmann factor $e^{-\beta}$ refers to a corrected probability of crossing, with β being the inverse temperature that we use to model thermal noise. When $\beta = 0$, which is the simplest model of repulsion between different walkers, a red (blue) walker at a certain node *i* at time *t*, he can choose a site from the nearest neighbor sites of node *i* as the destination at time $(t+1)$ with equal probability, except those nodes of which current states are blue (red). If all of the neighboring nodes of the red node are blue, then the red (blue) walker will randomly choose one to go. On the other hand, when $\beta = \infty$, the red (blue) walker will not cross the boundary and will choose to go to another node to achieve his maximum.

Our model for the spin-spin repulsion reduces redundancy in the set of overlapped nodes in their search. In the three-spin state model, a red (blue) walker at node i with degree k_i will have the probability to go to a neighboring node in blue (red) state

$$p_i^c = \frac{e^{-\beta}}{k_i} \tag{1}$$

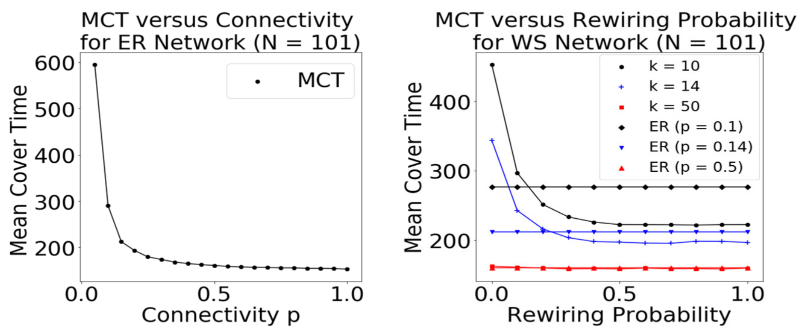
The subscript c refers to “crossing” from red (blue) to blue (red) node. On the other hand, the probability for the red (blue) walker to go to a neighbor node that is not in blue (red) state is:

$$p_i^r = \frac{1-n_i p_i^c}{k_i-n_i} = \frac{k_i-n_i e^{-\beta}}{k_i(k_i-n_i)} \tag{2}$$

Here n_i is the number of site i’s neighbor that are in blue (red) state. The subscript r refers to “repulsive walk” between red (blue) and blue (red) node. The maximum repulsion model is the special case with β goes to infinity.

3 Numerical Results: Strong Repulsion $\boldsymbol{\beta} = \infty$

We run simulations for the case of two walkers for the Erdös–Rényi (ER) and the Watts-Strogatz (WS)) network and analyze the networks with the same number of nodes and links. For the ER network, we vary the connectivity probability p and for the WS network, we vary the rewiring probability p_r. We first consider ER networks with 101 nodes with a given probability p. We average our simulation results over 100 trials on a given ER network, with 100 pairs of randomly chosen starting points for the pair of walkers, and then repeat that for 30 ER networks with the same connectivity p. We assume first the strong repulsion model, with $\beta = \infty$. We see from Fig.1(a) that MCT decreases with connectivity. This is reasonable since an ER network with higher connectivity has more edges, therefore the nodes are more accessible. The walkers have more choices to choose an unvisited site among all the neighbor sites when connectivity is high. We next consider WS network. For WS network with N nodes, the number of links is $L = Nk/2$ where k is the average degree. Its connectivity is the ratio $c = L/L_{max} = 2L/(N(N-1)) = k/(N-1)$.We consider three connected WS networks with 101 nodes and average degree k = 10/14/50 for numerical simulation, with corresponding connectivity 0.1/0.14/0.5 (Fig.1(b)). To see the impact of network structures on the dynamics of random walk process, the size and connectivity are fixed, and the rewiring probability of WS network is changed from 0 (perfect k-regular ring) to 1. For each value of the rewiring probability of WS network, we average the results over 100 trials on a given WS network, with 100 pairs of randomly chosen starting points for the pair of walkers, and then averaged over 30 WS networks with the same connectivity and rewiring probability.



(a) **(b)**

Fig. 1. Mean Cover Time (MCT) in the strong repulsion limit for networks with size N=101 **(a)** versus connectivity p for ER network **(b)** versus p_r for WS network at connectivity $c = 0.1/0.14/0.5$ and also for ER network at different connection probability p (0.1/0.14/0.5)

When connectivity is small (sparse network, e.g. p = 0.1/0.14 (or k=10/14) in Fig.1b.), we observe a clear difference between the MCT of WS and ER network. At small p_r, the MCT for WS network is above the corresponding ER network with the same connectivity. As p_r of WS network increases from 0 to 1, the MCT for WS network decreases and eventually smaller than the corresponding ER network with same connectivity. On the other hand, when the networks become dense, the difference between ER and WS becomes small. We expect this result since the structural complexity for dense WS network is similar to ER network at large rewiring probability [22].

4 Numerical Results: Tunable Repulsion

We perform simulations on ER and WS networks to analyze the impact of the repulsion strength between walkers on the cover time. We use ER networks with N=101 nodes, with different value of the probability p of connection and repulsion strength parameter β. For each pair of value (p, β), we perform simulation and the result is averaged over 30 different ER networks, with the results on each ER network averaged for 100 trials, each with different initial locations of the walkers. We observed in Fig.2(a) that the MCT decreases monotonically with the connectivity, which could be explained as an ER network with higher connectivity probability has more edges and the nodes are more accessible to the walkers. We also expect smaller overlapped paths between different walkers with finite β, leading to shorter cover time when we increase the repulsion parameter β between walkers. This theoretical expectation is verified numerically for ER networks.

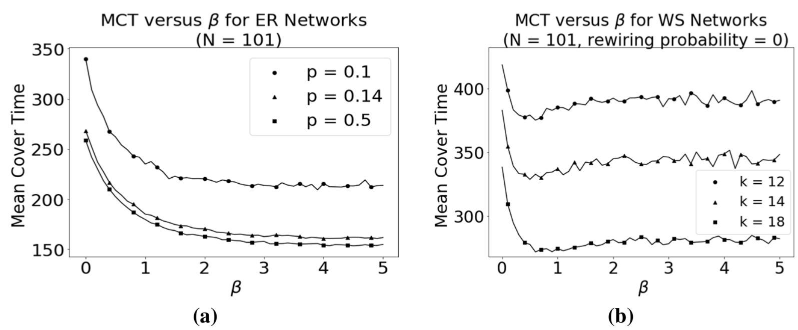


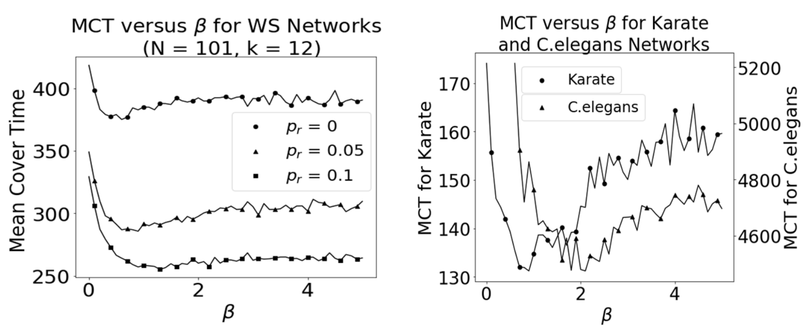
Fig. 2. Mean Cover Time (MCT) versus repulsion parameter β for **(a)** ER network (N = 101, p = 0.1/0.14/0.5) **(b)** WS network (N = 101, k = 12/14/18, rewiring probability = 0)

For WS network with N=101 nodes and different average degree k and rewiring probability p_r, we perform simulation and with result averaged over 30 different WS networks. For each network, we average the results for 100 trials each with different initial locations of the walkers. For perfect WS network (i.e. $p_r = 0$) at different k, we observe in Fig.2(b) the existence of a β_{min} at which the mean cover time is minimum. This value of β_{min} is increasing with the average degree k of WS network. The MCT also decreases with increasing repulsion strength, while β_{min} shifts to larger values as k increases. To understand this minimum in MCT as a function of β, we fix the average degree k of the WS network and change the rewiring probability p_r. From Fig.3(a), we observe β_{min} increases with p_r and the existence of β_{min} becomes less obvious for large p_r . We expect this as WS network with a larger rewiring probability is more like an ER network. In fact, for ER network there does not exist a β_{min} that minimizes the MCT.

Table. 1. Information of three real small-world like network datasets.

Network Datasets	N (# of Nodes)	K (Avg degree)	Clustering Coefficient
Karate	34	4.49	0.57
Football	115	10.66	0.40
C.elegans	297	14.46	0.29

To relate our results on WS networks to real networks, we select three real-world network datasets (Table.1): the karate social network (Zachary, 1977), the C. elegans neural network (Watts and Strogatz, 1998) and the American college football social network (Girvan and Newman, 2002) from Newman's network datasets (www.personal.umich.edu/~mejn/netdata/). These networks have been analyzed and classified to be small-world in the literature [20,22]. We perform random walk on these real networks and expect to see a finite $\beta_{\min}$ which implies that the repulsion could be tuned to minimize the MCT for real networks with small-world property (Fig.3(b)). This result is important for information spreading in application.



(a) **(b)**

Fig.3 Mean Cover Time versus repulsion parameter β **(a)** for WS network (N=101, k=12, rewiring probability=0/0.05/0.1) **(b)** for the Karate and C. elegans networks

We also generate artificial network data sets corresponding to the real world data set by tuning the rewiring probability of WS network with same size and connectivity, so that the cluster coefficient matches those of the real-world data set. Our simulation of repulsive walking process on these artificial WS networks confirms the existence of the minimum MCT at some critical repulsion strength β_{min} . We observe from Fig.4 that both the artificial WS network ($N = 115, k = 11, p_r = 0.18$) and the real data set follow a similar trend for MCT versus β.

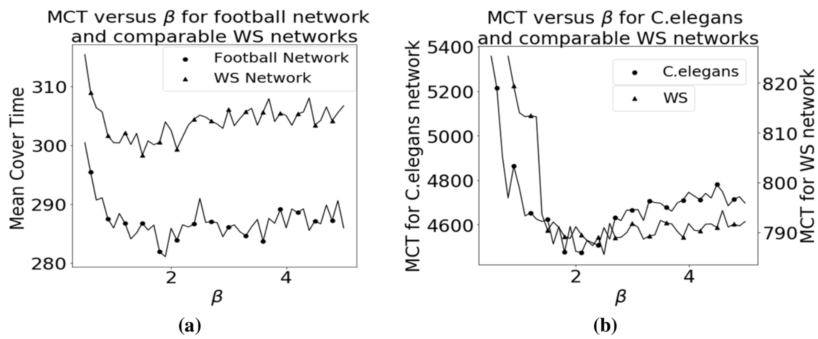


Fig.4 Comparison of the Mean Cover Time of artificial WS networks with approximately same size, connectivity and clustering coefficient as the real network versus repulsion parameter β **(a)** for the American football society social network **(b)** for the C.elegans neural network

We now provide a heuristic explanation for the existence of β_{min} for the WS network. When the repulsion is small, $\beta \to 0$, the paths of two walkers have a large overlap and will reduce the efficiency of search, leading to a large MCT. When repulsion is large, $\beta \to \infty$, the overlap of the paths of two walkers is reduced, but the self-overlap of each walker is increased. The walkers are trapped in clusters with nodes of the same color and the walker has to overcome the energy barrier to cross the red/blue boundary, and escape to search for the unvisited nodes. These two competing effects of trapping in cluster and minimizing path overlap of different walkers produces a finite repulsion at β_{min} where the MCT is minimized. While this explanation is plausible, we observe β_{min} only for the WS network, but not the ER network. We note that WS network in general has a higher clustering effect than the ER network, while at the same time have small characteristic path lengths, like random graphs [22]. Therefore, we expect the effect of self-overlap in WS network is larger than that in the ER network, as the walker is much easier to stay inside in a cluster and revisit nodes of the same color.

For ER network, the clustering effect for ER network is relatively small, so that the repulsion between walkers dominate over the self-overlap in cluster, resulting the monotonic decrease of MCT as the repulsion increases.

To confirm these heuristic explanations, we show the linear relationship between $\log(\beta_{min})$ and the rewiring probability p_r of WS network with given network size *N* and average degree *k* in Fig.5. The scattered data points are approximated by a straight line with confidence R^2=0.88. As the clustering coefficient $C(p_r)$ for a WS network decreases monotonically with increasing rewiring probability, $C(p_r) = C(p_r = 0) * (1 - p_r)^3$, we can deduce that β_{min} increases when the clustering coefficient decreases (or p_r increases) [22]. For small clustering coefficient (or large rewiring probability p_r), we have a WS network similar to the ER network and β_{min} effectively goes to infinity, implying a monotonic decrease of MCT for the ER network. This explains the presence of nonzero β_{min} in the WS networks, and its absence in the ER network.

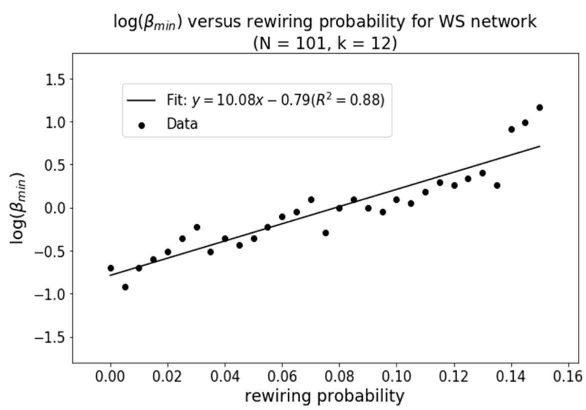


Fig.5 Linear relation of $\log(\beta_{min})$ versus p_r of WS network (N=101, *k*= 12)

5 Conclusion

We design a model to optimize the mean cover time for multiple random walkers in complex networks using only local information. We analyze the dynamics of two repulsive walkers using a spin model with three spin states defined on nodes on the complex network. The interaction between neighboring spins is introduced so that walkers of different spin state (color) will repel with a strength characterized by a Boltzmann factor $e^{-\beta}$. By tuning the repulsion strength parameter β in a model with two walkers in a network, we have a network with three colors (red, blue, and white for unvisited nodes), so that the mean cover time is the time that all the white color nodes become either red or blue. We monitor the effect of self-overlap in a cluster and the boundary effect between the red and the blue clusters. The tuning of the repulsion strength can reduce the wasted time in the redundant visits of nodes. In this work, we present both numerical experiments and theoretical reasoning for the presence of a critical repulsion strength that can minimize the mean cover time in WS network, and its absence in ER network. We argue that the existence of β_{min} in WS network is a result of the compromise between the reduction of wasted time in self-overlap walks and that of the mutual-overlap between different random walkers. Our numerical experiments on artificial ER network, WS networks and three small-world like real network data sets show satisfactory results in the reduction of the MCT using repulsive random walkers. This suggests plausible application in search in WS and ER networks. There are also potential applications to the industry, where efficient information spreading in social networks could lead to economic and social benefits. For future research, we can extend our model to M random walkers searching on complex networks using spin models with more spin states ($(M+1)$ colors with the unvisited sites colored white and the other visited sites in M different non-white colors). Our results show that the common intuition that the strongest repulsion among multiple walkers leads to the least overlapped paths and best search performance does not hold in all networks, especially for networks that are small-world like.

6 Acknowledgement

W. Guo and J. Wang acknowledge the support of the Hong Kong University of Science and Technology through the Undergraduate Research Opportunity Program (HKUST-UROP).

Reference:

[1] Ahn YY, Bagrow JP, Lehmann S (2009) Link communities reveal multiscale complexity in networks. arXiv preprint arXiv:09033178

[2] Almaas E, Kulkarni R, Stroud D (2003) Scaling properties of random walks on small-world networks. Physical Review E 68(5): 056,105

[3] Arenas A, Díaz-Guilera A, Kurths J, Moreno Y, Zhou C (2008) Synchronization in complex networks. Physics reports 469(3):93–153
[4] Bonaventura M, Nicosia V, Latora V (2014) Characteristic times of biased random walks on complex networks. Physical Review E 89(1): 012,803
[5] Castellano C, Fortunato S, Loreto V (2009) Statistical physics of social dynamics. Reviews of modern physics 81(2):591
[6] Clauset A, Moore C, Newman ME (2007) Structural inference of hierarchies in networks. In: Statistical network analysis: models, issues, and new directions, Springer, pp 1–13
[7] Condamin S, Bénichou O, Moreau M (2005) First-passage times for random walks in bounded domains. Physical review letters 95(26): 260,601
[8] Condamin S, Bénichou O, Moreau M (2007) Random walks and Brownian motion: A method of computation for first-passage times and related quantities in confined geometries. Physical Review E 75(2): 021,111
[9] Ding MC, Szeto KY (2017) Selection of random walkers that optimizes the global mean first-passage time for search in complex networks. Procedia Computer Science 108:2423–2427
[10] Evans JW, Nord R (1985) Random walks on finite lattices with multiple traps: Application to particle-cluster aggregation. Physical Review A 32(5):2926
[11] Fogel J, Nehmad E (2009) Internet social network communities: Risk taking, trust, and privacy concerns. Computers in human behavior 25(1):153–160
[12] Haldane AG, May RM (2011) Systemic risk in banking ecosystems. Nature 469(7330):351
[13] Hughes B (1995) Random Walks and Random Environments: Random walks. Vol. 1 in Oxford science publications, Clarendon Press.
[14] Leskovec J, Lang KJ, Dasgupta A, MahoneyMW (2009) Community structure in large networks: Natural cluster sizes and the absence of large well-defined clusters. Internet Mathematics 6(1):29–123
[15] Patel A, Rahaman MA (2010) Search on a hypercubic lattice using a quantum random walk. I. d > 2. Physical Review A 82(3): 032,330
[16] Power JD, Cohen AL, Nelson SM, Wig GS, Barnes KA, Church JA, Vogel AC, Laumann TO, Miezin FM, Schlaggar BL, et al (2011) Functional network organization of the human brain. Neuron 72(4):665–678
[17] Redner S (2001) A guide to first-passage processes. Cambridge University Press
[18] Rieser M, Nagel K (2008) Network breakdown at the edge of chaos in multiagent traffic simulations. The European Physics Journal B-Condensed Matter and Complex System 63(3):321–327
[19] Rosvall M, Bergstrom CT (2008) Maps of random walks on complex networks reveal community structure. Proceedings of the National Academy of Sciences 105(4):1118–1123
[20] Telesford QK, Joyce KE, Hayasaka S, Burdette JH, Laurienti PJ (2011) The ubiquity of small-world networks. Brain connectivity 1(5):367–375
[21] Volchenkov D, Blanchard P (2007) Random walks along the streets and canals in compact cities: Spectral analysis, dynamical modularity, information, and statistical mechanics. Physical Review E 75(2): 026104
[22] Watts D, Strogatz S (1998) Collective dynamics of small-world networks. Nature 393: 440–442. doi: 10.1038/30918
[23] Weng T, Zhang J, Small M, Hui P (2017) Multiple random walks on complex networks: A harmonic law predicts search time. Physical Review E 95(5): 052103
[24] Zhong M, Shen K (2006) Popularity-biased random walks for peer-to-peer search under the square-root principle. In: Proc. of the 5th International Workshop on Peer-to-Peer Systems,2006