MSIAM M2 Thesis: Extreme quantile Bayesian inference

Tony Zheng

July 30, 2020

Problem

Given:

- Observations of some variable.
- Expert who has knowledge about this process.

Goal: How can we incorporate expert opinion to model quantiles of maxima?

Applications: Environmental variables such as rainfall, wind speed, river discharge etc.

Extreme value modelling 1

- Maxima are often modelled with the generalised extreme value (GEV) distribution with parameters $\mu, \sigma > 0, \xi \neq 0$.
- Its CDF is

$$F(x) = \exp\left(-\left\{1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right\}_+^{-\frac{1}{\xi}}\right),$$

where $\{x\}_+ = \max(x, 0)$.

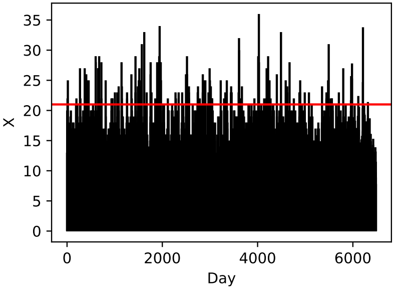
Extreme value modelling 2

- The GEV model can be generalised to a non-homogeneous Poisson point process with the same parameters.
- This model considers exceedances of a certain threshold as events occurring with intensity function

$$\lambda(x) = \frac{1}{\sigma}\left\{1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right\}_{+}^{-\frac{\xi+1}{\xi}} .$$

- Advantage: uses all observations above some threshold, not just block maxima.

Data



- Daily observations of average wind speed at Tours, France from 1981 to 2011.
- Chosen threshold in red.
- Goal: estimate quantiles (return levels) of the annual maxima.

Bayesian inference

- Bayes' theorem:

$$\pi(\mu, \sigma, \xi \mid \mathbf{x}) \propto \mathcal{L}(\theta \mid \mathbf{x})\pi(\mu, \sigma, \xi)\,,$$

  where $\mathcal{L}$ is the likelihood of the Poisson point process model and π is a prior on the parameters.
- Prior elicitation:

$$\text{Expert opinion} \longrightarrow \pi(\mu, \sigma, \xi)\,.$$

- Maximum entropy principle: given some constraints, choose a distribution with PDF f which maximises the Shannon entropy

$$\mathcal{E}(f) := -\int_0^{+\infty} f(x)\log(f(x))\,\mathrm{d}x\,.$$

Reparametrisation

- Let q_1, q_2, q_3 be the quantiles of annual maxima with probabilities $p_1 < p_2 < p_3$ respectively.
- There is an invertible transformation $(q_1, q_2, q_3) \to (\mu, \sigma, \xi)$.
- It is much easier for an expert to specify information about the quantiles than μ, σ, ξ.
- The problem becomes:

$$\text{Expert opinion} \longrightarrow \pi(q_1, q_2, q_3)\,.$$

Order constraint on quantiles

- The quantiles must satisfy the order constraint $q_1 < q_2 < q_3$.
- Solution 1: Positive priors on each quantile difference

$$\begin{aligned}\tilde{q}_1 &:= q_1\,,\\ \tilde{q}_2 &:= q_2 - q_1\,,\\ \tilde{q}_3 &:= q_3 - q_2\,,\end{aligned}$$

  with independent copula (Coles and Tawn 1996).
- Solution 2: Positive priors on each quantile, with some copula which satisfies order constraint. We choose the copula with maximum entropy, which has been derived in Butucea, Delmas, Dutfoy, and Fischer 2018.

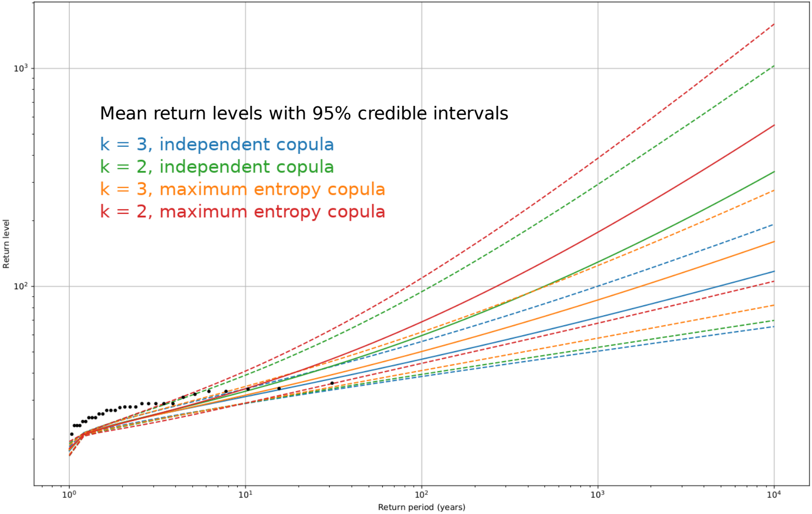
Expert specification

- In both solutions, we need to construct three positive distributions from the expert's opinion.
- We consider the case where the expert specifies the mean and variance of each distribution.
- In this case, the maximum entropy distribution is a truncated normal distribution.

Incomplete information

- So far, the expert has given information on $k = 3$ quantiles/quantile differences.
- We will also consider case $k = 2$, for which the expert specifies information on 2 quantiles/quantile differences.
- We use a reparametrisation $(q_1, q_2, \sigma) \to (\mu, \sigma, \xi)$, and place a uniform prior on $\log \sigma$.

Results



Mean return levels with 95% credible intervals
k = 3, independent copula
k = 2, independent copula
k = 3, maximum entropy copula
k = 2, maximum entropy copula
Return level
10^3
10^2
10^0
10^1
10^2
10^3
10^4
Return period (years)

Bibliography

[1] Cristina Butucea, Jean-François Delmas, Anne Dutfoy, and Richard Fischer. "Maximum entropy distribution of order statistics with given marginals". *Bernoulli* 24.1 (2018). DOI: 10.3150/16-BEJ868.

[2] Stuart G. Coles and Jonathan A. Tawn. "A Bayesian Analysis of Extreme Rainfall Data". *Journal of the Royal Statistical Society. Series C (Applied Statistics)* 45.4 (1996), pp. 463–478. DOI: 10.2307/2986068.