

Extreme quantile Bayesian inference

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**Given:**

- ▶ Observations of some variable.
- ▶ Expert who has knowledge about this process.

Goal: How can we incorporate expert opinion to model quantiles of maxima?

Applications: Environmental variables such as rainfall, wind speed, river discharge etc.



- ▶ Maxima are often modelled with the generalised extreme value (GEV) distribution with parameters $\mu, \sigma > 0, \xi \neq 0$.
- ▶ Its CDF is

$$F(x) = \exp \left(- \left\{ 1 + \xi \left(\frac{x - \mu}{\sigma} \right) \right\}_+^{-\frac{1}{\xi}} \right),$$

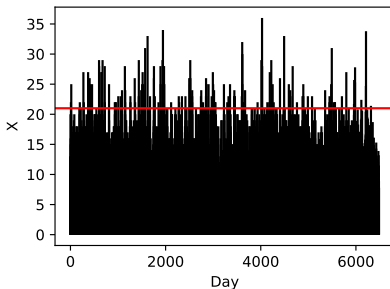
where $\{x\}_+ = \max(x, 0)$.



- ▶ The GEV model can be generalised to a non-homogeneous Poisson point process with the same parameters.
- ▶ This model considers exceedances of a certain threshold as events occurring with intensity function

$$\lambda(x) = \frac{1}{\sigma} \left\{ 1 + \xi \left(\frac{x - \mu}{\sigma} \right) \right\}_+^{-\frac{\xi+1}{\xi}}.$$

- ▶ Advantage: uses all observations above some threshold, not just block maxima.



- ▶ Daily observations of average wind speed at Tours, France from 1981 to 2011.
- ▶ Chosen threshold in red.
- ▶ Goal: estimate quantiles (return levels) of the annual maxima.



- Bayes' theorem:

$$\pi(\mu, \sigma, \xi \mid \mathbf{x}) \propto \mathcal{L}(\theta \mid \mathbf{x})\pi(\mu, \sigma, \xi),$$

where \mathcal{L} is the likelihood of the Poisson point process model and π is a prior on the parameters.

- Prior elicitation:

$$\text{Expert opinion} \longrightarrow \pi(\mu, \sigma, \xi).$$

- Maximum entropy principle: given some constraints, choose a distribution with PDF f which maximises the Shannon entropy

$$\mathcal{E}(f) := - \int_0^{+\infty} f(x) \log(f(x)) \, dx.$$



- ▶ Let q_1, q_2, q_3 be the quantiles of annual maxima with probabilities $p_1 < p_2 < p_3$ respectively.
- ▶ There is an invertible transformation $(q_1, q_2, q_3) \rightarrow (\mu, \sigma, \xi)$.
- ▶ It is much easier for an expert to specify information about the quantiles than μ, σ, ξ .
- ▶ The problem becomes:

Expert opinion $\longrightarrow \pi(q_1, q_2, q_3)$.



- ▶ The quantiles must satisfy the order constraint $q_1 < q_2 < q_3$.
- ▶ Solution 1: Positive priors on each quantile difference

$$\tilde{q}_1 := q_1 ,$$

$$\tilde{q}_2 := q_2 - q_1 ,$$

$$\tilde{q}_3 := q_3 - q_2 ,$$

with independent copula (Coles and Tawn 1996).

- ▶ Solution 2: Positive priors on each quantile, with some copula which satisfies order constraint. We choose the copula with maximum entropy, which has been derived in Butucea, Delmas, Dutfoy, and Fischer 2018.

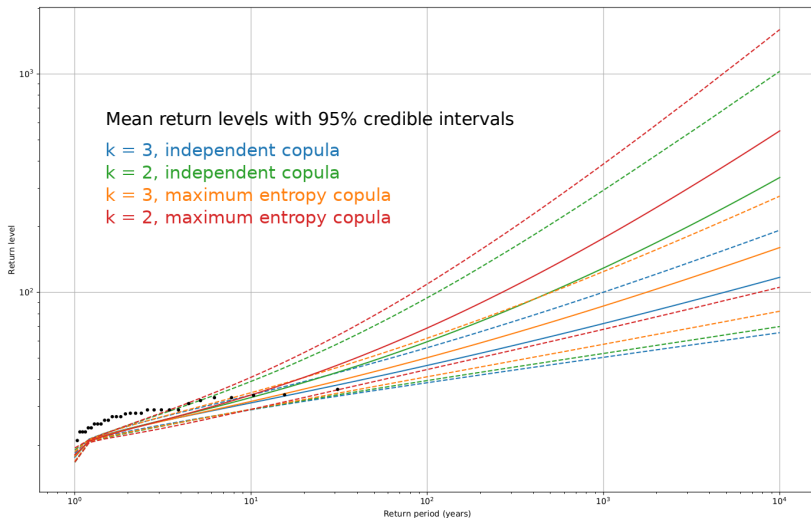


- ▶ In both solutions, we need to construct three positive distributions from the expert's opinion.
- ▶ We consider the case where the expert specifies the mean and variance of each distribution.
- ▶ In this case, the maximum entropy distribution is a truncated normal distribution.



- ▶ So far, the expert has given information on $k = 3$ quantiles/quantile differences.
- ▶ We will also consider case $k = 2$, for which the expert specifies information on 2 quantiles/quantile differences.
- ▶ We use a reparametrisation $(q_1, q_2, \sigma) \rightarrow (\mu, \sigma, \xi)$, and place a uniform prior on $\log \sigma$.

Results





- [1] Cristina Butucea, Jean-François Delmas, Anne Dutfoy, and Richard Fischer. “Maximum entropy distribution of order statistics with given marginals”. *Bernoulli* 24.1 (2018). DOI: 10.3150/16-BEJ868.
- [2] Stuart G. Coles and Jonathan A. Tawn. “A Bayesian Analysis of Extreme Rainfall Data”. *Journal of the Royal Statistical Society. Series C (Applied Statistics)* 45.4 (1996), pp. 463–478. DOI: 10.2307/2986068.