International Journal of Financial Engineering

Computing Option Prices Based on Heston Model to a Specified Tolerance*

Xiaoyang Zhao,† Tianci Zhu‡and Fred J. Hickernell§¶
‖
10 W. 32rd Street, Chicago IL 60616
Department of Applied Mathematics
Illinois Institute of Technology

The Quadratic Exponential (QE) model is a market standard simulation method for the Heston stochastic volatility model. We identify certain numerical problems with the standard discretization and modify the original method to correct these problems. We implement our modified QE scheme for the Heston model in the Guaranteed Automatic Integration Library (GAIL)—a suite of algorithms that includes Monte Carlo and quasi-Monte Carlo methods for multidimensional integration and computation of means. GAIL computes answers to satisfy user-defined error tolerances. We also implement variance reduction techniques for our modified QE scheme in GAIL. The numerical results show that our modified scheme is fast and accurate, and satisfies the user-defined error tolerances.

Keywords: Enclose with each manuscript, on a separate page, from three to five keywords.

1. Introduction

In the financial market, observed volatility of the asset prices is not a constant. For more accurate option pricing, we need an algorithm that simulates the volatility process. There are several well-known stochastic volatility models: the Hull-white model [$HullandWhite(1987)$], the Scott-Chesny model [$ChesneyandScott(1989)$], the Heston model [$Heston(1993)$] and the SABR model [$Haganet\ al.(2002)Hagan, Kumar, Lesniewski, andWoodward$]. We focus on the Heston model since it is one of the most widely used stochastic volatility models. Specifically, the Quadratic Exponential (QE) scheme is used to simulate the volatility process and the Broadie-Kaya scheme gives the discretization of the asset price process.

However, the standard implementation gives inaccurate results when the volatility of the asset prices' variance is very small or zero. Thus inaccuracy is accentuated when the initial variance does not start from the value of the long-term variance. We identify this problem and fix it by finding a more accurate approximation of the time integral of the volatility. We also make a change of variables in the QE scheme to avoid round-off errors in the discretization scheme of the asset price process. We compare the results with different options at different strike prices. Our numerical experiments show that the new algorithm is accurate and fast.

In addition, we implement the modified scheme in the Guaranteed Automatic Integration Library (GAIL), which theoretically guarantees the result and stops algorithms automatically with user de-

*Typeset title in 10 pt Times Roman uppercase and boldface. Please write down in pencil a short title to be used as the running head.
†xzhao31@hawk.iit.edu
‡tianci.zhu26@gmail.com
§hickernell@iit.edu, 312.567.8983
¶Typeset names in 8 pt Times Roman, uppercase and lightface. Use footnotes only to indicate if permanent and present addresses are different. Funding information should go in the Acknowledgement section.
‖Typeset affiliation and mailing addresses in 8pt Times italic.

fined error tolerance. The GAIL includes a suite of algorithms that applies the Monte Carlo methods for multidimensional integration and computation of means. The financial application modules of GAIL are under construction, and our work is aimed to add efficient and accurate algorithms of calculating stochastic volatility model in the asset path class.

The QE model was developed by [*Andersen(2006)*]. It is a market standard simulation method for the Heston model. Its attractiveness lies in its efficiency. It relies on simple probability density functions and needs a moderate amount of storage. Depending on the value of the volatility of the variance, the QE scheme approximates its distribution using either Gaussian or exponential distribution. After obtaining the value of the volatility at each time step, the Broadie-Kaya scheme is used to simulate the dynamics of the asset price process. The Broadie-Kaya scheme does not satisfy an equivalent discrete-time martingale condition. The martingale property can be attained by adjusting the Broadie-Kaya scheme.

The setup of the paper is as follows: we first introduce Heston model, QE scheme and the Broadie-Kaya scheme. Then, we derive our modified QE algorithm and provide its justification. After that, we explain variance reduction techniques and present numerical tests. At last, we summarize our work and discuss possible future work.

2. Options Modeled by the Heston Stochastic Volatility Model

2.1. *The Heston model*

Steven L. Heston introduced a stochastic volatility model which describes the evolution of the volatility of an asset price process [*Heston(1993)*]. Compared to fixed volatility models, stochastic volatility models more accurately explain data from the real world. The Heston model assumes the following dynamics of stock price and volatility processes:

$$\mathrm{d}X_t = \mu X_t\,\mathrm{d}t + \sqrt{V_t}X_t\,\mathrm{d}W_t^1 \tag{2.1}$$

$$\mathrm{d}V_t = \kappa(\theta - V_t)\,\mathrm{d}t + \nu\sqrt{V_t}\,\mathrm{d}W_t^2 \tag{2.2}$$

The first equation describes the asset price process X_t and the second equation gives the evolution of the variance process V_t. Two standard Brownian motions, $\mathrm{d}W_1$ and $\mathrm{d}W_2$, are set with a correlation constant ρ. The drift, the mean reversion speed of the variance, the long-term variance, and the volatility of the variance are denoted by μ, κ, θ and ν respectively.

Applying the Ito's formula to Eq.(2.1), we have an equivalent form of asset price process:.

$$\mathrm{d}\ln(X) = (\mu - \frac{V}{2})\,\mathrm{d}t + \sqrt{V}\,\mathrm{d}W_1 \tag{2.3}$$

The asset price process X is governed by a geometric Brownian motion stochastic volatility.

2.2. *The quadratic exponential scheme for stochastic volatility*

We apply the Quadratic Exponential Scheme illustrated in Andersen (2006) [?] to simulate the volatility process. Let $\hat{V}$ denote the discrete-time approximation of V. Detailed steps are listed below.

(1) Given $\hat{V}_t$, compute m and s^2 from following equations

$$m = \theta + (\hat{V}_t - \theta)e^{-\kappa\Delta}$$

$$s^2 = \frac{\hat{V}_t\nu^2 e^{-\kappa\Delta}}{\kappa}\left(1 - e^{-\kappa\Delta}\right) + \frac{\theta\nu^2}{2\kappa}\left(1 - e^{-\kappa\Delta}\right)^2.$$

(2) Compute $\psi = s^2/m^2$.

(3) Draw a uniform random number U_V.
(4) If $\psi \le 1.5$[1]:
  (a) Compute a and b from following equations

$$b^2 = 2\psi^{-1} - 1 + \sqrt{2\psi^{-1}}\sqrt{2\psi^{-1} - 1} \ge 0, \qquad a = \frac{m}{1+b^2}.$$

  (b) Compute $Z_V = \Phi^{-1}(U_V)$, where Phi is the standard normal distribution function.
  (c) Set $\hat{V}_{t+\Delta} = a(b + Z_V)^2$.
(5) Otherwise, if $\psi > 1.5$:
  (a) Compute β and p according to equations

$$p = \frac{\psi - 1}{\psi + 1} \in [0, 1), \qquad \beta = \frac{1-p}{m} = \frac{2}{m(\psi+1)} > 0.$$

  (b) Set $\hat{V}_{t+\Delta} = \Psi^{-1}(U_V; p, \beta)$, where Ψ is the exponential distribution with Dirac measure.

$$\Psi(x) = \Pr(\hat{V}(t+\Delta) \le x) = p + (1-p)(1 - e^{-\beta x}),\ x \ge 0$$

$$\Psi^{-1}(u) = \Psi^{-1}(u; p, \beta) = \begin{cases} 0, & 0 \le u \le p, \\ \beta^{-1} \ln(\frac{1-p}{1-u}), & p < u \le 1 \end{cases}$$

2.3. *The Broadie-Kaya discretization scheme for the asset price process*

We used the Broadie-Kaya discretization scheme to simulate the asset price process. To begin with, we give a brief derivation of such scheme. Details are presented in [*Andersen*(2006)].

First we integrate the SDE for V_t to have a bias-free scheme,

$$V_{t+\Delta} = V_t + \int_t^{t+\Delta} \kappa(\theta - V_u)\,\mathrm{d}u + \nu \int_t^{t+\Delta} \sqrt{V_u}\,\mathrm{d}W_u^2,$$

which can be written as

$$\int_t^{t+\Delta} \sqrt{V_u}\,\mathrm{d}W_u^2 = \nu^{-1}\left(V_{t+\Delta} - V_t - \int_t^{t+\Delta} \kappa(\theta - V_u)\,\mathrm{d}u\right). \tag{2.4}$$

Recall (2.3), by the Cholesky decomposition, we have

$$\mathrm{d}\ln X_t = (\mu - \frac{1}{2}V_t)\,\mathrm{d}t + \sqrt{V_t}\big(\rho\,\mathrm{d}W_t^2 + \sqrt{1-\rho^2}\,\mathrm{d}W_t\big),$$

where W is a Brownian motion independent of W^2. An integral form of the above equation is

$$\ln X_{t+\Delta} = \ln X_t + \mu\Delta - \frac{1}{2}\int_t^{t+\Delta} V_u\,\mathrm{d}u + \rho\int_t^{t+\Delta} \sqrt{V_u}\,\mathrm{d}W_u^2 + \sqrt{1-\rho^2}\int_t^{t+\Delta} \sqrt{V_u}\,\mathrm{d}W_u.$$

Substituting (2.4) into the above equation, we obtain

$$\begin{aligned} \ln X_{t+\Delta} = \ln X_t + \mu\Delta + \frac{\rho}{\nu}(Vt + \Delta - V_t - \kappa\theta\Delta) + \left(\frac{\kappa\rho}{\nu} - \frac{1}{2}\right)\int_t^{t+\Delta} V_u\,\mathrm{d}u \\ + \sqrt{1-\rho^2}\int_t^{t+\Delta} \sqrt{V_u}\,\mathrm{d}W_u. \end{aligned} \tag{2.5}$$

[1]The constant 1.5 is a typical value allowed in the QE scheme. The exact choice for this constant doesn't effect the quality of the overall simulation significantly [*Andersen*(2006)].

Next, we need to find appropriate approximations for the integrals in (2.5). Inspired by the Trapezoidal rule, we write

$$\int_t^{t+\Delta} V_u \,\mathrm{d}u \approx \Delta[\gamma V_t + (1-\gamma)V_{t+\Delta}] \tag{2.6}$$

where the parameter γ remains to be specified. In the Broadie-Kaya discretization scheme, the coefficient γ is usually taken to be $1/2$, but we propose an alternative in Section 3.1. The Itô integral $\int_t^{t+\Delta} \sqrt{V_u}\,\mathrm{d}W_u$ is Gaussian with mean zero and variance $\int_t^{t+\Delta} V_u \,\mathrm{d}u$. Hence, we have

$$\int_t^{t+\Delta} \sqrt{V_u}\,\mathrm{d}W_u = \sqrt{\int_t^{t+\Delta} V_u \,\mathrm{d}u}\cdot Z \approx \sqrt{\Delta[\gamma V_t + (1-\gamma)V_{t+\Delta}]}\cdot Z, \tag{2.7}$$

where Z is a standard Gaussian random variable which is independent of V. Approximation (2.7), combined with (2.5) and (2.6) and Section 2.2, provides us the following discretization scheme

$$\ln\hat{X}_{t+\Delta} = \ln\hat{X}_t + K_0 + K_1\hat{V}_t + K_2\hat{V}_{t+\Delta} + \sqrt{K_3\hat{V}_t + K_4\hat{V}_{t+\Delta}}\cdot Z \tag{2.8}$$

Parameters K_0, K_1, K_2, K_3 and K_4 are given by

$$K_0 = -\frac{\rho\kappa\theta}{\nu}\Delta, \qquad K_1 = \gamma\Delta\left(\frac{\kappa\rho}{\nu}-\frac{1}{2}\right)-\frac{\rho}{\nu}, \qquad K_2 = (1-\gamma)\Delta\left(\frac{\kappa\rho}{\nu}-\frac{1}{2}\right)+\frac{\rho}{\nu},$$
$$K_3 = \gamma\Delta(1-\rho^2), \qquad K_4 = (1-\gamma)\Delta(1-\rho^2).$$

Moreover, $\hat{X}$ denotes the discrete-time approximation of X. Given the simulation scheme for V, the discretization scheme for $\ln X$ can be generated in the following fashion:

(1) Given $\hat{V}_t$, generate $\hat{V}_{t+\Delta}$ using the QE scheme.
(2) Given $\ln\hat{X}_t$, $\hat{V}_t$ and the value for $\hat{V}_{t+\Delta}$ computed in Step 2, compute $\ln\hat{X}_{t+\Delta}$ from (2.8) using $Z = \Phi^{-1}(U)$, where U is independent of U_V used for $\hat{V}_{t+\Delta}$.

2.4. *Martingale correction*

In mathematical finance, the existence of a risk-neutral measure (or equivalent martingale measure) corresponds to an arbitrage-free market. Under the equivalent martingale measure, each price process equals to the expectation of its discounted price process. From the numerical point of view, the influence whether the asset price process X is a martingale or not is minor. Given its significant importance in mathematical finance, we present the martingale correction due to Andersen [*Andersen*(2006)] as follows:

$$\ln\hat{X}_{t+\Delta} = \ln\hat{X}_t + K_0^* + K_1\hat{V}_t + K_2\hat{V}_{t+\Delta} + \sqrt{K_3\hat{V}_t + K_4\hat{V}_{t+\Delta}}\cdot Z$$

where

$$K_0^* = \begin{cases} -\frac{Ab^2a}{1-2Aa} + \frac{1}{2}\ln(1-2Aa) - (K_1+\frac{1}{2}K_3)\hat{V}_t, & \psi \le \psi_c, \\ -\ln\left(\frac{\beta(1-p)}{\beta-A}\right) - (K_1+\frac{1}{2}K_3)\hat{V}_t, & \psi > \psi_c \end{cases}$$

and $A = K_2 + \frac{1}{2}K_4$.

3. Overcoming Numerical Errors for Small Volatility of Variance

Before we introduce our modified QE scheme[2], we first show the problem of the original scheme with a plot. When ν is equal or close to zero, the discretization scheme introduced previously gives inaccurate answers, especially when the initial variance, V_0, does not equal to the long-term variance θ.

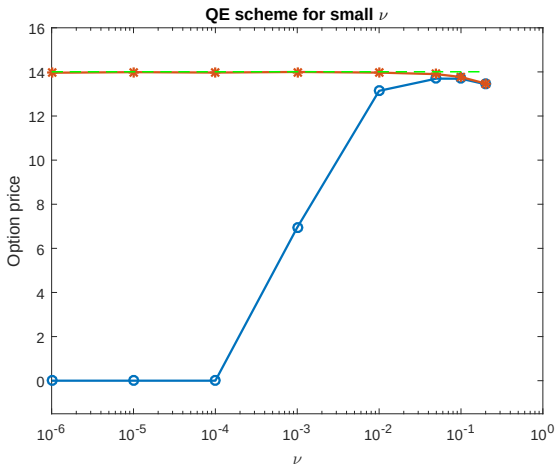

Fig. 1. European call option price calculate by the QE scheme and our modified QE scheme for an initial price of \$80, a strike price of \$100, zero interest rate, $T = 2$, $\kappa = 1$, $\rho = -0.3$, $\theta = 0.09$, $V_0 = 0.36$ and relative tolerance of 1%. Stars denote the prices calculated by our modified QE scheme when the relative tolerance of 1% is met. Circles denote the prices calculated by the original QE scheme. The dashed line denotes the option price calculated by the Black-Scholes formula.

We calculate the price of European call options with different values of ν. Neither the QE scheme nor the exact sampling works for small ν. When ν equals zero and the initial variance V_0 does not equal to the long-term variance θ, V_t is deterministic and changes over time. We set the variance in the Black-Scholes formula to be the same as the long-term variance θ and the correct option price should be close to this price when ν is small. However, the original QE scheme gives zero option price when ν equals zero and prices close to zero when ν is close to zero. It is clear from Figure 3.1 that the original QE scheme deviates from the prices calculated by the Black-Scholes model and our modified QE scheme when ν is smaller than approximately 0.05.

3.1. *Finding a γ that is accurate for the approximation of the time integral of V*

From (2.5) and (2.7), the Broadie-Kaya scheme in integral form is written as

$$\begin{aligned} \ln X_{t+\Delta} =& \ln X_t + \frac{\rho}{\nu}(V_{t+\Delta} - V_t - \kappa\theta\Delta) + \left(\frac{\kappa\rho}{\nu} - \frac{1}{2}\right)\int_t^{t+\Delta} V_u du \\ &+ \sqrt{(1-\rho^2)\int_t^{t+\Delta} V_u \, \mathrm{d}u} \cdot Z. \end{aligned} \tag{3.1}$$

We need to approximate the time-integral of V. Rather than simply setting $\gamma = 1/2$, we want to find the value of γ that makes (2.6) exact for $\nu = 0$ and show that the error for the approximation

[2]From this section, when we mention the QE scheme (modified QE scheme), we refer to the QE scheme (modified QE scheme) and the discretization scheme (modified discretization scheme) for X.

of $\int_t^{t+\Delta} V_u du$ is of order $o(\nu\Delta)$. The factor of ν in the error will then cancel the factor of $1/\nu$ in (3.1). So, the simulation will be accurate when ν is close to 0.

When $\nu = 0$, the stochastic partial differential equation of (2.2) becomes deterministic and has the solution

$$V_t = \theta + (V_0 - \theta)e^{-\kappa t}. \tag{3.2}$$

Then, we have

$$\begin{aligned} \int_t^{t+\Delta} V_u \, \mathrm{d}u &= \int_t^{t+\Delta} (V_0 - \theta)e^{-\kappa u} \, \mathrm{d}u + \theta\Delta \\ &= \frac{-(V_0 - \theta)e^{-\kappa(t+\Delta)} + (V_0 - \theta)e^{-\kappa t}}{\kappa} + \theta\Delta \end{aligned} \tag{3.3}$$

Setting $t = t + \Delta$ in (3.2) provides us an expression for θ as

$$\theta = \frac{V_t - e^{\kappa\Delta} V_{t+\Delta}}{1 - e^{\kappa\Delta}}$$

By substituting the above expression of θ into (3.3), we obtain

$$\begin{aligned} \int_t^{t+\Delta} V_u \, \mathrm{d}u &= \frac{V_t - V_{t+\Delta}}{\kappa} + \Delta \frac{V_t - e^{\kappa\Delta} V_{t+\Delta}}{1 - e^{\kappa\Delta}} \\ &= \Delta[\gamma V_t + (1 - \gamma)V_{t+\Delta}], \end{aligned}$$

where

$$\gamma = \frac{1 - e^{\kappa\Delta} + \kappa\Delta}{\kappa\Delta(1 - e^{\kappa\Delta})}, \quad 0 \le \gamma \le 1. \tag{3.4}$$

As $\kappa\Delta \to 0$, $\gamma \to \frac{1}{2}$. When $\kappa\Delta$ goes to zero, our scheme becomes the Trapezoidal rule.

We apply the value of γ in (3.4) for all values of ν to approximate the integral $\int_t^{t+\Delta} V_u \, \mathrm{d}u$ and

$$\int_t^{t+\Delta} V_u \, \mathrm{d}u = \Delta[\gamma V_t + (1 - \gamma)V_{t+\Delta}] + O(\nu\Delta^{1.5}).$$

We want to show the error of our approximation for $\int_t^{t+\Delta} V_u \, \mathrm{d}u$ is $o(\nu\Delta)$. First, recall the process

$$\mathrm{d}V_t = \kappa(\theta - V_t) \, \mathrm{d}t + \nu\sqrt{V_t} \, \mathrm{d}W_t, \quad V_0 \text{ given},$$

defined in (2.2). It is not easy to prove directly from process V. So, we define a new process U and let U_t satisfy a similar stochastic differential equation:

$$\mathrm{d}U_t = -\kappa U_t \, \mathrm{d}t + \nu\sqrt{V_t} \, \mathrm{d}W_t, \quad U_0 = 0. \tag{3.5}$$

Let $Y_t = V_t - U_t$, and note that Y_t satisfies a differential equation:

$$\mathrm{d}Y_t = \kappa(\theta - Y_t) \, \mathrm{d}t, \quad Y_0 = V_0.$$

Moreover, $Y_t = V_t$ when $\nu = 0$. So, we apply similar deduction to Y_t as in Section 3.1:

$$\int_t^{t+\Delta} Y_u \, \mathrm{d}u = \Delta[\gamma Y_t + (1 - \gamma)Y_{t+\Delta}],$$

where $\gamma = (1 - e^{\kappa\Delta} + \kappa\Delta)/\kappa\Delta(1 - e^{\kappa\Delta})$. Now we have the integral of V_t written in terms of V and U as follows:

$$\begin{aligned} \int_t^{t+\Delta} V_u \, \mathrm{d}u &= \int_t^{t+\Delta} (Y_u + U_u) \, \mathrm{d}u \\ &= \Delta[\gamma V_t + (1 - \gamma)V_{t+\Delta}] - \left\{ \Delta[\gamma U_t + (1 - \gamma)U_{t+\Delta}] - \int_t^{t+\Delta} U_u \, \mathrm{d}u \right\}. \end{aligned} \tag{3.6}$$

Our problem is to show that the error in approximating $\int_t^{t+\Delta} U_u \, \mathrm{d}u$ by $\Delta[\gamma_1 U_t + (1-\gamma_1)U_{t+\Delta}]$ is $o(\nu\Delta)$.

We can write the solution of U_t as

$$U_t = \nu e^{-\kappa t} \int_0^t e^{\kappa s} \sqrt{V_s} \, \mathrm{d}W_s.$$

Now we rewrite the integral of U_t by using integration by parts twice and (3.5):

$$\begin{aligned}
\int_t^{t+\Delta} U_u \, \mathrm{d}u &= U_u(u-t-\Delta\gamma)\big|_t^{t+\Delta} - \int_t^{t+\Delta} (u-t-\Delta\gamma) \, \mathrm{d}U_u \\
&= \Delta\big[\gamma U_t + (1-\gamma)U_{t+\Delta}\big] + \kappa U_{t+\Delta}\Delta^2\left(\frac{1}{2} - \gamma\right) \\
&\quad + \int_t^{t+\Delta} (u-t)\left[\frac{(u-t)}{2} - \Delta\gamma\right]\kappa^2 U_u \, \mathrm{d}u \\
&\quad - \nu \int_t^{t+\Delta} h(u)\sqrt{V_u} \, \mathrm{d}W_u, \qquad (3.7)
\end{aligned}$$

where $h(u) = \kappa(u-t)\big[(u-t)/2 - \Delta\gamma\big] + u - t - \Delta\gamma$. Since $U_t = O(\nu)$, it is easy to see that the second term in (3.7) is $O(\nu\Delta^2)$. By Hölder's inequality, the third term of (3.7) is

$$\begin{aligned}
&\int_t^{t+\Delta} \left(\frac{(u-t)^2}{2} - (u-t)\Delta\gamma\right)\kappa^2 U_u \, \mathrm{d}u \\
&\leq \Delta\gamma\kappa^2 \sqrt{\int_t^{t+\Delta} \left(\frac{(u-t)^2}{2} - (u-t)\Delta\gamma\right)^2 \mathrm{d}u \int_t^{t+\Delta} U_u^2 \, \mathrm{d}u} \\
&= O(\nu\Delta^3).
\end{aligned}$$

Now we want to show that

$$I = \nu \int_t^{t+\Delta} h(u)\sqrt{V_u} \, \mathrm{d}W_u = O(\nu\Delta^{1.5}).$$

Since V_t is a Cox-Ingersoll-Ross process, we know that its expectation given V_0 is $\mathbb{E}(V_t) = V_0 e^{-\kappa u} + \theta(1-e^{-\kappa u})$ [*Dufresne*(2001)], which is O(1). Also, since $\max_{t\leq u\leq t+\Delta} |h(u)| = O(\Delta)$,

$$\begin{aligned}
\mathbb{E}\left(\nu^2 \int_t^{t+\Delta} [h(u)]^2 V_u \, \mathrm{d}u\right) &= \nu^2 \int_t^{t+\Delta} [h(u)]^2 \mathbb{E}(V_u) \, \mathrm{d}u \\
&= \nu^2 \int_t^{t+\Delta} O(\Delta^2) \, \mathrm{d}u \\
&= O(\nu^2\Delta^3) \\
&< \infty,
\end{aligned}$$

we have $I \in \mathcal{L}^2$ and I is an Itô integral. By Theorem 4.7 of [*Klebaner*(2005)], Itô integral I is a continuous zero mean square integrable martingale. Thus, by the Itô isometry,

$$\begin{aligned}
\mathrm{Var}(I) &= \mathbb{E}(I^2) - [\mathbb{E}(I)]^2 \\
&= \mathbb{E}\left(\nu^2 \int_t^{t+\Delta} [h(u)]^2 V_u \, \mathrm{d}u\right) \\
&= O(\nu^2\Delta^3).
\end{aligned}$$

So, I is a normal distributed random variable with mean zero and variance of order $O(\nu^2\Delta^3)$. The Itô integral I is $O(\nu\Delta^{1.5})$. This implies that

$$\int_t^{t+\Delta} U_u \,\mathrm{d}u = \Delta\big(\gamma U_t + (1-\gamma)U_{t+\Delta}\big) + O(\nu\Delta^{1.5})$$

in distribution, and so by (3.6)

$$\int_t^{t+\Delta} V_u \,\mathrm{d}u = \Delta\big(\gamma V_t + (1-\gamma)V_{t+\Delta}\big) + O(\nu\Delta^{1.5}),$$

also in distribution.

3.2. *Change of variables to avoid overflow*

Another problem with the formula (3.1) is that ν appears in the denominator. This may lead to numerical error when ν is small. To avoid this error we make a change of variables of s^2, ψ, b^{-2}, a, b, K_0, K_1, K_2, K_3, and K_4. We first make a change of variables in the QE scheme

$$\begin{aligned}
\tilde{s}^2 = \nu^{-2}s &= \frac{\hat{V}_t e^{-\kappa\Delta}}{\kappa}\big(1-e^{-\kappa\Delta}\big) + \frac{\theta}{2\kappa}\big(1-e^{-\kappa\Delta}\big)^2,\\
\tilde{\psi} = \nu^{-2}\psi &= \frac{\frac{\hat{V}_t e^{-\kappa\Delta}}{\kappa}(1-e^{-\kappa\Delta}) + \frac{\theta}{2\kappa}(1-e^{-\kappa\Delta})^2}{\big(\theta + (\hat{V}_t-\theta)e^{-\kappa\Delta}\big)^2},\\
\tilde{b}^{-2} = \nu^{-2}b^{-2} &= \frac{\tilde{\psi}}{2\sqrt{1-\frac{\tilde{\psi}\nu^2}{2}}\left(1+\sqrt{1-\frac{\tilde{\psi}\nu^2}{2}}\right)},\\
\tilde{a} = \nu^{-2}a &= \frac{m\tilde{b}^{-2}}{1+\nu^2\tilde{b}^{-2}}.
\end{aligned}$$

As we see, new variables $\tilde{s}^2$ and $\tilde{\psi}$ are free of ν. Moreover, as $\nu \to 0$, $\tilde{a}$ and $\tilde{b}$ have finite limits. After making the change of variables, when ν goes to 0, our new variables have finite limits. Now we use new variables that we defined to calculate $\hat{X}$. Recall (2.8):

$$\ln\hat{X}_{t+\Delta} = \ln\hat{X}_t + K_0 + K_1\hat{V}_t + K_2\hat{V}_{t+\Delta} + \sqrt{K_3\hat{V}_t + K_4\hat{V}_{t+\Delta}}\cdot Z.$$

Observe the expression of parameters we discussed in Section **??**, the terms including ν^{-1} will cause overflow when ν is close or equal to zero. We want to define a new variable of stochastic volatility process V in a way that there is no ν^{-1} in our modified discretization scheme.

Define $\tilde{V}_t = \hat{V}_t - \theta$ and write $\hat{V}_{t+\Delta}$ in terms of new variables:

$$\hat{V}_{t+\Delta} = \frac{\theta + \tilde{V}_t e^{-\kappa\Delta}}{1+\nu^2\tilde{b}^{-2}}(1+\nu\tilde{b}^{-1}Z_V)^2.$$

Then, we have the expression of $\tilde{V}_{t+\Delta}$ as

$$\tilde{V}_{t+\Delta} = \frac{\theta\nu[2\tilde{b}^{-1}Z_V + \nu\tilde{b}^{-2}(Z_V^2-1)] + \tilde{V}_t e^{-\kappa\Delta}(1+\nu\tilde{b}^{-1}Z_V)^2}{1+\nu^2\tilde{b}^{-2}}.$$

Define $\mathring{V}_{t+\Delta} = (\tilde{V}_{t+\Delta} - \tilde{V}_t e^{-\kappa\Delta})/\nu$,

$$\mathring{V}_{t+\Delta} = (\theta + \tilde{V}_t e^{-\kappa\Delta})\left[\frac{2\tilde{b}^{-1}Z_V}{1+\nu^2\tilde{b}^{-2}} + \nu\frac{\tilde{b}^{-2}(Z_V^2-1)}{1+\nu^2\tilde{b}^{-2}}\right].$$

In order to calculate $\ln\hat{X}$, we note that

$$\gamma\tilde{V}_t + (1-\gamma)\tilde{V}_{t+\Delta} = \frac{1-e^{-\kappa\Delta}}{\kappa\Delta}\tilde{V}_t + (1-\gamma)\nu\mathring{V}_{t+\Delta}.$$

Substituting the above equation into the following equation:

$$K_0 + K_1\hat{V}_t + K_2\hat{V}_{t+\Delta} = -\theta\frac{\Delta}{2} - \frac{1-e^{-\kappa\Delta}}{2\kappa}\tilde{V}_t - \left[\frac{\nu(1-e^{\kappa\Delta}+\kappa\Delta e^{\kappa\Delta})}{2\kappa(1-e^{\kappa\Delta})} + \frac{\rho\kappa\Delta e^{\kappa\Delta}}{1-e^{\kappa\Delta}}\right]\mathring{V}_{t+\Delta}. \tag{3.8}$$

Moreover,

$$K_3\hat{V}_t + K_4\hat{V}_{t+\Delta} = \Delta(1-\rho^2)\left\{\theta + \left[\frac{1-e^{-\kappa\Delta}}{\kappa\Delta}\tilde{V}_t + (1-\gamma)\nu\mathring{V}_{t+\Delta}\right]\right\}. \tag{3.9}$$

Now we can rewrite the discretization scheme. For the discretization scheme of X when $\nu^2\tilde{\psi} \le \psi_c$:

$$\begin{aligned}\ln\hat{X}_{t+\Delta} =& \ln\hat{X}_t - \theta\frac{\Delta}{2} - \frac{1-e^{-\kappa\Delta}}{2\kappa}\tilde{V}_t - \left[\frac{\nu(1-e^{\kappa\Delta}+\kappa\Delta e^{\kappa\Delta})}{2\kappa(1-e^{\kappa\Delta})} + \frac{\rho\kappa\Delta e^{\kappa\Delta}}{1-e^{\kappa\Delta}}\right]\mathring{V}_{t+\Delta} \\ &+ \sqrt{\Delta(1-\rho^2)\left\{\theta + \left[\frac{1-e^{-\kappa\Delta}}{\kappa\Delta}\tilde{V}_t + (1-\gamma)\nu\mathring{V}_{t+\Delta}\right]\right\}}\cdot Z.\end{aligned} \tag{3.10}$$

Since the discretization scheme is inaccurate only when ν is small, we will only apply our modifications to $\nu^2\tilde{\psi} \le \psi_c$. For the $\nu^2\tilde{\psi} > \psi_c$ case, the scheme remains unchanged:

$$\ln\hat{X}_{t+\Delta} = \ln\hat{X}_t + K_0 + \frac{1}{\nu}K_1\hat{V}_t + \frac{1}{\nu}K_2\hat{V}_{t+\Delta} + \sqrt{K_3\hat{V}_t + K_4\hat{V}_{t+\Delta}}\cdot Z. \tag{3.11}$$

3.3. *The modified QE and Broadie-Kaya discretization scheme*

Now we can propose our modified scheme for the Heston model:

(1) Given $\hat{V}_t$, compute m and $\tilde{s}^2$ from the following equations

$$\begin{aligned} m &= \Theta + \tilde{V}_t e^{-\kappa\Delta}, \\ \tilde{s}^2 &= \frac{(\tilde{V}_t+\theta)e^{-\kappa\Delta}}{\kappa}\left(1-e^{-\kappa\Delta}\right) + \frac{\theta}{2\kappa}\left(1-e^{-\kappa\Delta}\right)^2. \end{aligned}$$

(2) Compute $\tilde{\psi} = \tilde{s}^2/m^2$.

(3) Generate two independent normal random variables Z_V and Z from GAIL.

(4) If $\nu^2\tilde{\psi} \le \psi_c$:

(a) Compute $\tilde{a}$ and $\tilde{b}$ from the following equations

$$\begin{aligned} \tilde{b}^{-2} &= \frac{\tilde{\psi}}{2\sqrt{1-\frac{1}{2}\tilde{\psi}\nu^2}\left(1+\sqrt{1-\frac{1}{2}\tilde{\psi}\nu^2}\right)}, \\ \tilde{a} &= \frac{m\tilde{b}^{-2}}{1+\nu^2\tilde{b}^{-2}}. \end{aligned}$$

(b) Set $\tilde{V}_{t+\Delta} = -\theta + \tilde{a}(\tilde{b}+\nu Z_V)^2$.

(c) Compute $\mathring{V}_{t+\Delta} = (\theta + \tilde{V}_t e^{-\kappa\Delta})\left[\frac{2\tilde{b}^{-1}Z_V}{1+\nu^2\tilde{b}^{-2}} + \frac{\nu\tilde{b}^{-2}(Z_V^2-1)}{1+\nu^2\tilde{b}^{-2}}\right]$.

(d) Given $\tilde{V}_t$, $\ln \hat{X}_t$ and the value for $\mathring{V}_{t+\Delta}$, compute $\ln \hat{X}_{t+\Delta}$ from (3.10).

(5) Otherwise, if $\nu^2\tilde{\psi} > \psi_c$

(a) Compute β and p according to equations

$$p = \frac{\nu^2\tilde{\psi} - 1}{\nu^2\tilde{\psi} + 1} \in [0, 1),$$
$$\beta = \frac{1-p}{\tilde{m}} = \frac{2}{m(\nu^2\tilde{\psi} + 1)} > 0.$$

(b) Draw a uniform random number U_V.

(c) Set $\tilde{V}_{t+\Delta} = -\theta + \Psi^{-1}(U_V; p, \beta)$, where Ψ is the exponential distribution with Dirac measure.

(d) Given $\tilde{V}_t$, $\ln\hat{X}_t$ and the value for $\tilde{V}_{t+\Delta}$, compute $\ln\hat{X}_{t+\Delta}$ from (3.11).

3.4. *Canceling ν^{-1} out of the Broadie-Kaya discretization scheme with martingale correction*

The difference of the scheme without and with martingale correction lies in the K_0 term. Recall

$$K_0^* = -\frac{Ab^2a}{1-2Aa} + \frac{1}{2}\ln(1-2Aa) - (K_1 + \frac{1}{2}K_3)\hat{V}_t\,, \quad \nu^2\tilde{\psi} \le \psi_c,$$

where $A = K_2 + \frac{1}{2}K_4$.

In order to cancel out ν^{-1} in the scheme with martingale correction, we calculate terms $K_0^* + K_1\hat{V}_t + K_2\hat{V}_{t+\Delta}$ all together. First, we rewrite the first part of K_0^* in terms of new parameters defined in Section **??** and substitute $m = \theta + \tilde{V}_{t+\Delta} - \nu\mathring{V}_{t+\Delta}$ into it:

$$\begin{aligned}
-\frac{Ab^2a}{1-2Aa} &= -\frac{1}{\nu}\frac{A\tilde{a}\tilde{b}^2}{1-2\nu A\tilde{a}} \\
&= -\frac{1}{\nu}\frac{\rho\big((1-\gamma)\Delta\kappa+1\big)(\theta+\tilde{V}(t+\Delta))}{1+\nu^2\tilde{b}^{-2}-2\nu\tilde{b}^{-2}m\{\rho[(1-\gamma)\Delta\kappa+1]-\frac{1}{2}(1-\gamma)\Delta\nu\rho^2\}} \\
&\quad + \frac{\frac{1}{2}(1-\gamma)\Delta\rho^2(\theta+\tilde{V}(t+\Delta))}{1+\nu^2\tilde{b}^{-2}-2\nu\tilde{b}^{-2}m\{\rho[(1-\gamma)\Delta\kappa+1]-\frac{1}{2}(1-\gamma)\Delta\nu\rho^2\}} \\
&\quad + \frac{\mathring{V}(t+\Delta)\big[\rho\big((1-\gamma)\Delta\kappa+1\big)-\frac{1}{2}(1-\gamma)\Delta\nu\rho^2\big]}{1+\nu^2\tilde{b}^{-2}-2\nu\tilde{b}^{-2}m\{\rho[(1-\gamma)\Delta\kappa+1]-\frac{1}{2}(1-\gamma)\Delta\nu\rho^2\}}.
\end{aligned}$$

Then,

$$\begin{aligned}
&K_0^* + K_1\hat{V}_t + K_2\hat{V}_{t+\Delta} \\
=& -\frac{Ab^2a}{1-2Aa} + \frac{1}{2}\ln(1-2Aa) - (K_1+\frac{1}{2}K_3)\hat{V}_t + K_1\hat{V}_t + K_2\hat{V}_{t+\Delta} \\
=& \frac{\frac{1}{2}(1-\gamma)\Delta\rho^2(\theta+\tilde{V}_{t+\Delta})}{1+\nu^2\tilde{b}^{-2}-2\nu\tilde{b}^{-2}m\{\rho[(1-\gamma)\Delta\kappa+1]-\frac{1}{2}(1-\gamma)\Delta\nu\rho^2\}} \\
&+ \frac{\mathring{V}_{t+\Delta}\big[\rho\big((1-\gamma)\Delta\kappa+1\big)-\frac{1}{2}(1-\gamma)\Delta\nu\rho^2\big]}{1+\nu^2\tilde{b}^{-2}-2\nu\tilde{b}^{-2}m\{\rho[(1-\gamma)\Delta\kappa+1]-\frac{1}{2}(1-\gamma)\Delta\nu\rho^2\}} \\
&+ \frac{1}{2}\ln(1-2\nu A\tilde{a}) - \frac{1}{2}\gamma_1\Delta(1-\rho^2)(\theta+\tilde{V}_t) - \frac{1}{2}(1-\gamma)\Delta(\theta+\tilde{V}_{t+\Delta}) \\
&+ \rho\big((1-\gamma)\Delta\kappa+1\big)(\theta+\tilde{V}_{t+\Delta})\frac{\nu\tilde{b}^{-2}-2\tilde{b}^{-2}m\big[\rho\big((1-\gamma)\Delta\kappa+1\big)-\frac{1}{2}(1-\gamma)\Delta\nu\rho^2\big]}{1+\nu^2\tilde{b}^{-2}-2\nu\tilde{b}^{-2}m\{\rho[(1-\gamma)\Delta\kappa+1]-\frac{1}{2}(1-\gamma)\Delta\nu\rho^2\}}.
\end{aligned}$$

For $K_3\hat{V}_t + K_4\hat{V}_{t+\Delta}$, it is the same as the Broadie-Kaya discretization scheme without martingale correction.

4. Determining the Number of Samples Required to Meet a Specified Error Tolerance

As we mentioned in the introduction, this modified algorithm is implemented in GAIL, which automatically determines the number of samples required to meet a specified error tolerance. For IID sampling, the proofs of the success of the two-stage algorithm are illustrated in the [*Hickernellet al.*(2013)*Hickernell, Jiang, Liu, andOwen*]. We give a brief introduction of the procedure of the two-stage algorithm here. The main assumption needed is that the kurtosis of the random sample Y does not exceed some user-specified bound. The assumption they use is an upper bound on the modified kurtosis of the random variable Y:

$$\tilde{\kappa} = \frac{\mathbb{E}[(Y-\mu)^4]}{\sigma^4} \leq \tilde{\kappa}_{\max}.$$

Under this assumption, the first stage is that generates a conservative upper bound on the variance and the second stage is that uses the variance bound generated in the first stage and a Berry-Esseen Theorem to determine the sample size n.

Let $Y_1, ..., Y_n$ be IID sampling, the goal is to obtain a confidence interval

$$\Pr[|\mu - \hat{\mu} \leq \epsilon] \geq 1 - \alpha,$$

where $\mu = \mathbb{E}(Y)$, $\hat{\mu} = \hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$, ϵ is the half-width of the confidence interval and α is the level of uncertainty. In the first stage, the user specifies four quantities: an initial sample size $n_\sigma \in \{2, 3, ...\}$ for variance estimation, a variance inflation factor $\mathfrak{C} \in (1, \infty)$, $\alpha \in (0, 1)$ and $\epsilon > 0$. The IID sampled Y_1,..., Y_{n_σ} are independent from the same distribution as Y. Then the conservative variance estimate $\hat{\sigma}^2 = \mathfrak{C}^2 s^2_{n_\sigma}$ is computed in terms of the sample variance $s^2_{n_\sigma}$, where $\mathfrak{C}^2 > 1$ is a "variance inflation factor" that will reduce the probability of underestimating $\sigma^2 = \mathrm{Var}(Y)$. The modified kurtosis of Y is assumed to be less than or equal to $\tilde{\kappa}_{\max}(\alpha, n_\sigma, \mathfrak{C})$. Now we can get the sample size n_μ for stage two by Chebychev's inequality and Berry-Esseen inequality with parameters $\epsilon, \hat{\sigma}, \tilde{\kappa}_{\max}$ and $\tilde{\alpha} = 1 - \sqrt{1-\alpha}$. After the sample size n_μ obtained, the second stage is to sample $Y_{n_\sigma+1}$, ..., $Y_{n_\sigma+n_\mu}$, which is independent of Y as well as Y_1,..., Y_{n_σ}. Hence, the algorithm returns the sample mean

$$\hat{\mu} = \frac{1}{n_\mu} \sum_{i=n_\sigma+1}^{n_\sigma+n_\mu} Y_i.$$

which has theoretical guarantee of satisfaction for the user-defined error tolerance.

In addition to the IID sampling, another algorithm is implemented in the GAIL to satisfy the error critetion for Sobol sequences. Details on this algorithm can be found in [?]. The main idea is to assume our integrands lie in a cone and bound $|\mu - \mu_n|$ in terms of function data, i.e., discrete Fourier coefficients. Then one can increase the sample size n until err_n is no greater than the absolute error tolerance.

5. Numerical Examples

We implement our modified QE scheme into GAIL and test our modified QE scheme by pricing European and American options. First we show test results for small ν compared with the original QE scheme and Broadie and Kaya's exact sampling. Then we compare the modified QE scheme

without martingale correction to the modified QE scheme with martingale correction. Our numerical results support our claim in Section 2.4 that the difference of option prices between the schemes with and without martingale correction is minor. However, the scheme with martingale correction is more computationally expensive. Then we introduce importance sampling to reduce variance and illustrate reduced runtimes and sample sizes in test results. Finally, we show how Sobol sampling with our modified scheme improves the computational efficiency in comparison to i.i.d. sampling.

5.1. *Tests for small ν*

We test our scheme by pricing an European call option. The parameters are setup as in Table 1. We set the relative error tolerance (relTol) as 0.5% in GAIL. GAIL will return prices that satisfy our error tolerance [*Hickernellet al.*(2012)*Hickernell, Jiang, Liu, andOwen*]. The parameter K denotes strike price. In Table 2, the exact price means the price calculated by exact sampling[3]. This scheme

T	Δ	S_0	K	V_0	θ	κ	ρ	r	relTol
5	0.2	100	90	0.04	0.09	1	-0.3	0	0.5%

is bias-free yet computationally expensive, so we use it as a benchmark. The exact sampling returns error messages for small values of ν, so we use "-" to indicate that an error occurred. The MATLAB codes for the exact sampling are provided in Kienitz [*KienitzandWetterau*(2012)]. The reason why the difference between the exact price and the QE price is larger than 1% is that the sampling size we used for computing the exact price is limited by the computers' memory. The result will be more accurate if we are able to increase its sample size. From the test result we can see that our modified

ν	0	1e-6	1e-5	1e-4	1e-3	1e-2	0.05	0.1	0.5
Exact price	-	-	-	-	-	-	-	-	27.782
QE	0.000	35.797e+22	14.193e+3	81.095	32.380	29.191	28.915	28.811	27.493
Modified QE	28.859	28.874	28.870	28.926	28.901	28.923	28.849	28.783	27.542
Diff	28.859	35.797e+22	14.164e+3	52.169	3.479	0.268	0.065	0.028	0.049

QE scheme works for all test values of ν. For given parameters, we consider $\nu = 0.01$ as the break point for the original QE scheme. When ν is smaller than 0.01, the relative difference between the original QE scheme and the modified QE scheme is larger than two times of the 0.5% relative error tolerance. When $\nu = 0.01$, the difference is barely within the twice of the relative error tolerance. The original QE scheme gives satisfactory results for ν larger than 0.01.

5.2. *Compare the modified QE scheme to the modified QE scheme with martingale correction*

Now we price European call options and compare test results of our modified schemes with and without martingale correction. The parameters are setup as in Table 3. We test for different strike prices and results shown in Table 4. For each case, we repeat calculation for five times and take the

[3]The exact sampling was introduced by M. Broadie and Ö. Kaya for the Heston stochastic volatility model. They applied the numerical inversion of a cumulative distribution using the characteristic function. The estimator of an asset price generate using the sample stock price and variance from the exact distribution is unbiased.

average of price, time and sample size. The scheme with martingale correction takes a third more runtime than that without martingale correction while the difference of average prices are within

T	Δ	S_0	V_0	θ	ν	κ	ρ	r	relTol
5	0.2	60	0.5	0.16	0.4	1	-0.3	0	1%

Option	Price (relTol=0.01)			Time (s)			Sample Size		
	K=20	K=60	K=100	K=20	K=60	K=100	K=20	K=60	K=100
Modified QE	42.782	23.505	14.208	8.33	17.70	29.42	1.08e+6	2.28e+6	3.75e+6
Modified QE_m	42.727	23.469	14.213	13.08	23.83	39.73	1.21e+6	2.22e+6	3.65e+6
Diff	0.055	0.036	0.005	-	-	-	-	-	-

0.2%. Because the martingale correction is more time consuming, we apply the modified QE scheme without martingale correction for the rest of numerical tests.

5.3. *Test options with importance sampling*

For out of the money options, we need much larger sample sizes to meet the user-defined error tolerance. In order to increase accuracy and reduce sample size, we apply importance sampling—a variance reduction technique.

As defined in (2.1), the asset price process X_t is driven by a Brownian motion. Or more precisely, our modified simulation scheme (3.10) and (3.11) is driven by a Gaussian random variable. So we can shift the mean of the Gaussian random variable to sample the asset price process more at where it gives positive payoffs [*Glasserman*(2003)]. We implement importance sampling in GAIL. Given parameters in Table 3, we set the value of mean shifts for tests of importance sampling as in Table 5. The numerical results in the following subsections will show the difference in prices, runtimes and sample sizes.

	European Call (EC)	European Put (EP)	American Put (AP)
IID Sampling	2	-0.8	-0.8
Sobol Sequences	1	-1	-0.6

5.3.1. *Simulations with IID sampling*

First we test our modified QE scheme with i.i.d. sampling. Since American call option is never optimal to exercise early, it is just like European call option. So, we omit the test for it. The test results for European call/put option and American put option are shown in Table 6. The ECQE (EPQE) means European call (put) option calculated by the original QE scheme. The EC/EP means European call/ put option. The AP is short for American put option. Since American call option is just like European call option, The IS is short for importance sampling.

After applying importance sampling to at the money and out of the money options, the average runtimes are reduced, especially for European call option. The average runtime is nearly a constant rather than significant correlated to strike prices. The average prices calculated with different

Option	Price (relTol=0.01)			Time (s)			Sample Size		
	K=20	K=60	K=100	K=20	K=60	K=100	K=20	K=60	K=100
ECQE[a]	42.728	23.464	14.218	3.19	3.19	3.22	1.00e+6	1.00e+6	1.00e+6
ECw/oIS	42.750	23.497	14.196	8.22	18.40	30.34	1.06e+6	2.35e+6	3.85e+6
ECw/IS	-	23.493	14.210	-	4.85	4.67	-	6.27e+5	6.04e+5
EPQE[a]	2.782	23.498	54.255	3.18	3.18	3.19	1.00e+6	1.00e+6	1.00e+6
EPw/oIS	2.772	23.460	54.141	9.40	3.07	1.78	1.21e+6	4.03e+5	2.41e+5
EPw/IS	2.773	23.493	-	5.14	2.21	-	6.52e+5	2.94e+5	-
APw/oIS	2.771	23.476	54.176	9.95	3.90	2.44	1.13e+6	3.82e+5	2.25e+5
APw/IS	2.775	23.460	-	6.78	3.86	-	7.07e+5	3.80e+5	-

[a] This is calculated by the original QE scheme without guaranteed error tolerance.

schemes are very close for the same option with the same strike price. Our modified QE scheme gives approximation within user-defined error tolerance with constant runtime and as fast as the original QE scheme.

5.3.2. *Simulations with Sobol sequences*

Sobol sequences is a commonly used low-discrepancy sequences and is one of the sampling methods implemented in GAIL. For Sobol sequences, the time step size relax to $\Delta = 0.5$. The average runtimes are about 1/100 of those for i.i.d. sampling for the same relative error tolerance 1%. So, in Table 7, we change the relative error tolerance to 0.1%. However, when we try various mean shifts for American put option, there is no obvious reduction in runtime. This is might because that the sample size of the Sobol sequences is a factor of 2. We don't have a good explanation for this now.

Option	Price (relTol=0.01)			Time (s)			Sample Size		
	K=20	K=60	K=100	K=20	K=60	K=100	K=20	K=60	K=100
ECw/oIS	42.798	23.502	14.230	0.40	0.87	2.30	1.31e+05	2.62e+05	6.29e+05
ECw/IS	-	23.501	14.230	-	0.20	0.40	-	6.55e+04	1.31e+05
EPw/oIS	2.771	23.473	54.187	0.39	0.06	0.04	1.31e+05	1.64e+04	9.83e+03
EPw/IS	2.777	-	-	0.20	-	-	6.55e+04	-	-
APw/oIS	2.779	23.444	54.306	1.01	0.31	0.17	1.31e+05	3.28e+04	1.64e+04
APw/IS	2.774	-	-	1.05	-	-	1.31e+05	-	-

We apply importance sampling to at the money and out of the money call option, and out of the money put option. The average runtimes for at the money European call options are reduced four times and for out of the money European call options are reduced five times. The average runtimes of out of the money European put option are reduced to a half of that without importance sampling. Though Sobol sequences is a very efficient way of sampling without importance sampling, the reduction in runtime will be more obvious when calculating a large number of options. Therefore, the test results show that our modified QE scheme gives fast and accurate approximation using Sobol sequences in GAIL, which also satisfies user-defined error tolerance.

6. Discussion

Appendix A. Appendices

Acknowledgments

References

Leif B. G. Andersen. Efficient simulation of the heston stochastic volatility model. 2006.

Marc Chesney and Louis Scott. Pricing european currency options: A comparison of the modified black-scholes model and a random variance model. *The Journal of Financial and Quantitative Analysis*, 24(3): 267–284, September 1989.

Daniel Dufresne. The integrated square-root process. 2001.

Paul Glasserman. *Monte Carlo Methods in Financial Engineering*, chapter 4, pages 267–268. Springer, 2003.

Patrick S. Hagan, Deep Kumar, Andrew S. Lesniewski, and Diana E. Woodward. Managing smile risk. *The Best of Wilmott 1: Incorporating the Quantitative Finance Review*, pages 249–296, 2002.

Steven L. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of financial studies*, 6(2):327–343, 1993.

Fred J. Hickernell, Lan Jiang, Yuewei Liu, and Art B. Owen. Guaranteed conservative fixed width confidence intervals via monte carlo sampling. In *Monte Carlo and Quasi Monte Carlo Methods*, pages 105–128, 2012.

Fred J. Hickernell, Lan Jiang, Yuewei Liu, and Art B. Owen. Guaranteed conservative fixed width confidence intervals via monte carlo sampling. *Monte Carlo and Quasi-Monte Carlo Methods 2012*, pages 105–128, 2013.

John Hull and Alan White. The pricing of options on assets with stochastic volatilities. *The journal of finance*, 42(2):281–300, June 1987.

Jörg Kienitz and Daniel Wetterau. *Financial Modelling Theory, Implementation and Practice (with Matlab source)*, chapter 7, page 296. Wiley, 2012.

Fima C Klebaner. *Introduction to Stochastic Calculus with Applications*. Imperial College Press, 2005.