International Journal of Financial Engineering

Computing Option Prices Based on Heston Model to a Specified Tolerance*

Xiaoyang Zhao, Tianci Zhu and Fred J. Hickernell†
Full affiliations‡
,mailing addresses and telephone number

OTHER D. AUTHOR
Full affiliations
,mailing addresses and telephone number

Include a one-paragraph abstract of no more than 100 words. Do not include references, footnotes, or abbreviations in the abstract. Typeset the abstract in 8 pt Times Roman with baselineskip of 10 pt, making an indentation of $\frac{1}{4}$ inch on the left and right margins. Typeset similarly for keywords below.

Keywords: Enclose with each manuscript, on a separate page, from three to five keywords.

1. Introduction

We observed that in the financial market the volatility of asset prices may not be a constant. To have a more accurate pricing result, we need to implement an algorithm to simulate the volatility process. There are several well-known stochastic volatility models: the Hull-white model (1987), the Scott-Chesny model (1989), the Heston model (1993) and the SABR model (2002). We choose the Heston model as our approach since it is one of the most widely used stochastic volatility models. Upon solving the Heston model, the Quadratic Exponential (QE) scheme is used to simulate the volatility process and the Broadie-Kaya scheme is applied to the discretization of the asset price process.

However, the standard implementation gives inaccurate results when the volatility of the asset prices' volatility is set as zero. We identify this problem and fix it by a change of variables, which make it more accurate to calculate the Heston model. We compare the results with other simulation methods such as geometric Brownian motion and quasi-Monte Carlo, when setting the volatility of asset prices' volatility as zero, because in this case, the algorithm works like the one of asset price process with deterministic volatility. It shows that the new algorithm is accurate and fast. In addition, we implement the modified scheme in the Guaranteed Automatic Integration Library (GAIL) , which theoretically guarantees the result and stops algorithms automatically with user defined error tolerance .

The GAIL includes a suite of algorithms that applies the Monte Carlo methods for multidimensional integration and computation of means. The financial application module of GAIL is under construction, and our work is aimed to add algorithms of calculating stochastic volatility model in the asset path class.

*Typeset title in 10 pt Times Roman uppercase and boldface. Please write down in pencil a short title to be used as the running head.
†Typeset names in 8 pt Times Roman, uppercase and lightface. Use footnotes only to indicate if permanent and present addresses are different. Funding information should go in the Acknowledgement section.
‡Typeset affiliation and mailing addresses in 8pt Times italic.

The QE model was developed by [*Andersen*(2006)]. It is a market standard simulation method for the Heston model. Its attractiveness lies in its efficiency. It relies on simple probability density functions and needs a moderate amount of storage. Depending on the value of the volatility, the QE scheme approximates its distribution using either Gaussian or exponential distribution. After getting the value of the volatility for each time step, we discretize the asset price process. As we know, the computation of Broadie-Kaya algorithm is time consuming, but it is bias-free by construction. We discretize its scheme to get the simulation of the dynamic of asset price process. The Broadie-Kaya scheme does not satisfy an equivalent discrete-time martingale condition. The martingale property can be attained by adjusting a certain term in the scheme.

The setup of the paper is as follows: we first introduce Heston model, QE scheme and the brief idea behind GAIL. Then, we explain our new algorithm and the improved algorithm using variance reduction techniques. At last, we show its performance in comparison with various widespread used schemes and with different variance reduction applications

2. Options Modeled by the Heston Stochastic Volatility Model

2.1. *Heston Model*

Heston model is defined as

$$\mathrm{d}X = \mu X \,\mathrm{d}t + \sqrt{V} X \,\mathrm{d}W_1 \tag{2.1}$$

$$\mathrm{d}V = \kappa(\theta - V)\,\mathrm{d}t + \nu\sqrt{V}\,\mathrm{d}W_2 \tag{2.2}$$

The first equation simulates the asset price process and the second equation gives the evolution of the volatility process. X denotes the asset price. V is the stochastic volatility process. Two standard Brownian motions, $\mathrm{d}W_1$ and $\mathrm{d}W_2$, are set with a correlation constant ρ. μ, κ, θ, ν are constant parameters μ is the risk-free interest rate. κ is the speed of mean reversion. θ is the value of the long-term variance. ν is the volatility of volatility.

Applying the Ito's formula to (2.1), an equivalent form to simulate the process of asset price is shown in (2.3).

$$\mathrm{d}\ln(X) = (\mu - \frac{V}{2})\,\mathrm{d}t + \sqrt{V}\,\mathrm{d}W_1 \tag{2.3}$$

2.2. *Quadratic Exponential Scheme for Stochastic Volatility*

We applied the Quadratic Exponential Scheme illustrated in Andersen (2006) [?] to simulate the volatility process. Detailed steps are listed as follows:

(1) Given $\hat{V}(t)$, compute m and s^2 from following equations

$$m = \theta + (\hat{V}(t) - \theta)e^{-\kappa\Delta}$$

$$s^2 = \frac{\hat{V}(t)\nu^2 e^{-\kappa\Delta}}{\kappa}\left(1 - e^{-\kappa\Delta}\right) + \frac{\theta\nu^2}{2\kappa}\left(1 - e^{-\kappa\Delta}\right)^2$$

(2) Compute $\psi = s^2/m^2$

(3) Draw a uniform random number U_V

(4) If $\psi \le \psi_c$:

  (a) Compute a and b from following equations

$$b^2 = 2\psi^{-1} - 1 + \sqrt{2\psi^{-1}}\sqrt{2\psi^{-1} - 1} \ge 0, \qquad a = \frac{m}{1 + b^2}.$$

(b) Compute $Z_V = \Phi^{-1}(U_V)$
(c) Set $\hat{V}(t+\Delta) = a(b+Z_V)^2$

(5) Otherwise, if $\psi > \psi_c$

(a) Compute β and p according to equations

$$p = \frac{\psi - 1}{\psi + 1} \in [0, 1), \qquad \beta = \frac{1-p}{m} = \frac{2}{m(\psi+1)} > 0.$$

(b) set $\hat{V}(t+\Delta) = \Psi^{-1}(U_V; p, \beta)$

$$\Psi(x) = \Pr(\hat{V}(t+\Delta) \le x) = p + (1-p)(1-e^{-\beta x}),\ x \ge 0$$

$$\Psi^{-1}(u) = \Psi^{-1}(u; p, \beta) = \begin{cases} 0, & 0 \le u \le p, \\ \beta^{-1}\ln(\frac{1-p}{1-u}), & p < u \le 1 \end{cases}$$

2.3. *Broadie-Kaya Discretization Scheme for the Asset Prices*

We used the Broadie-Kaya scheme to simulate the asset price process. We give a brief derivation of this scheme here. Details are illustrated in Andersen (2006) [?].

First we integrate the SDE for $V(t)$ to have a bias-free scheme,

$$V(t+\Delta) = V(t) + \int_t^{t+\Delta} \kappa(\theta - V(u))\,\mathrm{d}u + \nu \int_t^{t+\Delta} \sqrt{V(u)}\,\mathrm{d}W_V(u)$$

and it can be written as

$$\int_t^{t+\Delta} \sqrt{V(u)}\,\mathrm{d}W_V(u) = \nu^{-1}\Big(V(t+\Delta) - V(t) - \int_t^{t+\Delta} \kappa(\theta - V(u))\,\mathrm{d}u\Big) \tag{2.4}$$

Recall (2.3), by Cholesky decomposition, we have

$$\mathrm{d}\ln X(t) = (\mu - \frac{1}{2}V(t))\,\mathrm{d}t + \sqrt{V(t)}\big(\rho\,\mathrm{d}W_V(t) + \sqrt{1-\rho^2}\,\mathrm{d}W(t)\big)$$

where W is a Brownian motion independent of W_V.

Now we integrate the above equation,

$$\begin{aligned}\ln X(t+\Delta) = &\ln X(t) + \mu\Delta - \frac{1}{2}\int_t^{t+\Delta} V(u)\,\mathrm{d}u + \rho\int_t^{t+\Delta}\sqrt{V(u)}\,\mathrm{d}W_V(u) \\ &+ \sqrt{1-\rho^2}\int_t^{t+\Delta}\sqrt{V(u)}\,\mathrm{d}W(u)\end{aligned}$$

Substituting Eq.(2.4) into it, we get

$$\begin{aligned}\ln X(t+\Delta) = &\ln X(t) + \mu\Delta + \frac{\rho}{\nu}(V(t+\Delta) - V(t) - \kappa\theta\Delta) + \Big(\frac{\kappa\rho}{\nu} - \frac{1}{2}\Big)\int_t^{t+\Delta} V(u)\,\mathrm{d}u \\ &+ \sqrt{1-\rho^2}\int_t^{t+\Delta}\sqrt{V(u)}\,\mathrm{d}W(u).\end{aligned} \tag{2.5}$$

We need to find appropriate approximation for the integrals in Eq.(2.5). For now, we simply write

$$\int_t^{t+\Delta} V(u)\,\mathrm{d}u \approx \Delta[\gamma_1 V(t) + \gamma_2 V(t+\Delta)] \tag{2.6}$$

Conditioning on $V(t)$ and $\int_t^{t+\Delta} V(u)\,\mathrm{d}u$, the Itô integral $\int_t^{t+\Delta} \sqrt{V(u)}\,\mathrm{d}W(u)$ is Gaussian with mean zero and variance $\int_t^{t+\Delta} V(u)\,\mathrm{d}u$. So, we write

$$\int_t^{t+\Delta} \sqrt{V(u)}\,\mathrm{d}W(u) \approx \Delta\sqrt{\gamma_1 V(t) + \gamma_2 V(t+\Delta)} \cdot Z$$

Therefore, we have the following discretiztion scheme

$$\ln\hat{X}(t+\Delta) = \ln\hat{X}(t) + K_0 + K_1\hat{V}(t) + K_2\hat{V}(t+\Delta) + \sqrt{K_3\hat{V}(t) + K_4\hat{V}(t+\Delta)} \cdot Z \tag{2.7}$$

where Z is a standard Gaussian random variable, independent of $\hat{V}$, and $K_0, \ldots, K_4$ are given by

$$K_0 = -\frac{\rho\kappa\theta}{\nu}\Delta, \qquad K_1 = \gamma_1\Delta\left(\frac{\kappa\rho}{\nu} - \frac{1}{2}\right) - \frac{\rho}{\nu}, \qquad K_2 = \gamma_2\Delta\left(\frac{\kappa\rho}{\nu} - \frac{1}{2}\right) + \frac{\rho}{\nu},$$

$$K_3 = \gamma_1\Delta(1-\rho^2), \qquad K_4 = \gamma_2\Delta(1-\rho^2).$$

where $\gamma_1 = \gamma_2 = \frac{1}{2}$ in Anderson (2006).

2.3.1. *Implementation steps*

*Maybe delete implementation steps here and add those after our modification

With given values of γ_1 and γ_2 and combined with the simulation scheme of V, the discretization scheme for $\ln X$ can be generated in the following fashion:

(1) Given $\hat{V}(t)$, generate $\hat{V}(t+\Delta)$ using QE schemes
(2) Draw a uniform random number U, independent of all random numbers used for $\hat{V}(t+\Delta)$
(3) Set $Z = \Phi^{-1}(U)$
(4) Given $\ln\hat{X}(t)$, $\hat{V}(t)$ and the value for $\hat{V}(t+\Delta)$ computed in Step 1, compute $\ln\hat{X}(t+\Delta)$ from Eq.(2.7)

Under the QE scheme, a martingale correction scheme is illustrated in Andersen (2006) as follows:

$$\ln\hat{X}(t+\Delta) = \ln\hat{X}(t) + K_0^* + K_1\hat{V}(t) + K_2\hat{V}(t+\Delta) + \sqrt{K_3\hat{V}(t) + K_4\hat{V}(t+\Delta)} \cdot Z$$

where

$$K_0^* = \begin{cases} -\frac{Ab^2a}{1-2Aa} + \frac{1}{2}\ln(1-2Aa) - (K_1 + \frac{1}{2}K_3)\hat{V}(t), & \psi \le \psi_c, \\ -\ln\left(\frac{\beta(1-p)}{\beta - A}\right) - (K_1 + \frac{1}{2}K_3)\hat{V}(t), & \psi > \psi_c \end{cases}$$

and $A = K_2 + \frac{1}{2}K_4$.

3. Overcoming Numerical Errors for Small Volatility of Volatility

3.1. *Modification of QE scheme without martingale correction*

When ν equal or close to zero, the QE scheme we introduced above can be fragile. Our numerical results show that the option price given by QE scheme deviate from that calculated by exact sampling[a] and our modified scheme even further when the initial volatility V_0 not equals to long-term variance θ.

[a]Introduced by Broadie, M. and Kaya, O. (2006) for the Heston stochastic volatility model.They applied the numerical inversion of a cumulative distribution using the characteristic function. The estimator of an asset price generate using the sample stock price and variance from the exact distribution is unbiased. The scheme is computationally expensive, so we used it as benchmarks for result comparison.

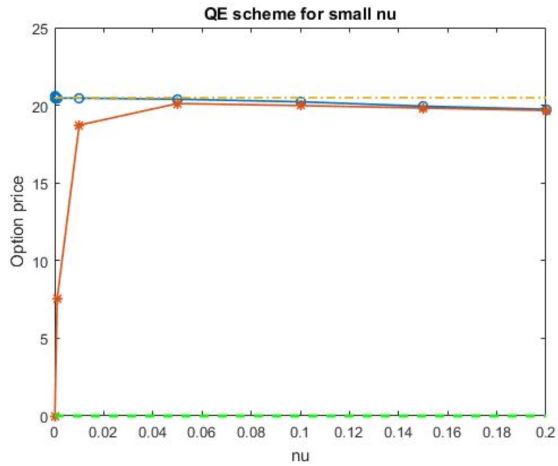
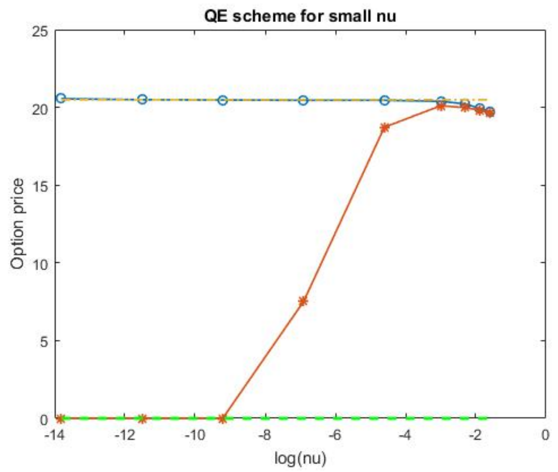


Fig. 1. European call option price calculated by QE scheme and our modified QE scheme. Initial price = 80, strike = 100, interest = 0, $\kappa = 1$, $\nu = 0.5$, $\rho = -0.3$, $\theta = 0.09$ and $V_0 = 0.36$. Small circles denote the price calculated by our modified QE scheme when the relative tolerance = 0.01 was met. The stars denote the price calculated by original QE scheme. The dot-dashed line denotes the option price calculated by Black-Scholes formula when $\nu = 0$. The dashed line at the bottom denotes the option price given by QE scheme when $\nu = 0$.



3.1.1. *Find more exact γ_1 and γ_2*

Recall Eq.(2.5), the Broadie-Kaya scheme in integral form is written as

$$\begin{aligned}\ln X(t+\Delta) =& \ln X(t) + \frac{\rho}{\nu}(V(t+\Delta) - V(t) - \kappa\theta\Delta) + \left(\frac{\kappa\rho}{\nu} - \frac{1}{2}\right)\int_t^{t+\Delta} V(u)du \\ &+ \sqrt{1-\rho^2}\int_t^{t+\Delta}\sqrt{V(u)}\,\mathrm{d}W(u).\end{aligned}$$

We need to handle the time-integral of V. Rather than simply setting $\gamma_1 = \gamma_2 = \frac{1}{2}$, we want to find the exact value of γ_1 and γ_2 that make the following equation holds.
When $\nu = 0$, the stochastic partial differential equation of V (Eq.(2.2)) has a solution in form

$$V(t) = \theta + (V(0) - \theta)e^{-\kappa t} \tag{3.1}$$

Denote $C = V(0) - \theta$,then $V(t) - \theta = Ce^{-\kappa t}$.
We have

$$\begin{aligned}\int_t^{t+\Delta} V(u)\,\mathrm{d}u &= \int_t^{t+\Delta} (V(u) - \theta)\,\mathrm{d}u + \theta\Delta \\ &= \int_t^{t+\Delta} Ce^{-\kappa u}\,\mathrm{d}u + \theta\Delta \\ &= \frac{-Ce^{-\kappa(t+\Delta)} + Ce^{-\kappa t}}{\kappa} + \theta\Delta\end{aligned} \tag{3.2}$$

We want find a expression of θ written in terms of $V(t)$ and $V(t + \Delta)$.
Set time to be $t + \Delta$ in Eq.(3.1) and multiply $e^{\kappa\Delta}$ on both sides, we have

$$e^{\kappa\Delta}(V(t + \Delta) - \theta) = Ce^{-\kappa t}$$

Combine with $V(t) - \theta = Ce^{-\kappa t}$, we have the expression of θ as

$$\theta = \frac{V(t) - e^{\kappa\Delta}V(t + \Delta)}{1 - e^{\kappa\Delta}}$$

Substitute the expression of θ into equation (3.2), we get

$$\begin{aligned}\int_t^{t+\Delta} V(u)\mathrm{d}u &= \frac{V(t) - V(t+\Delta)}{\kappa} + \Delta\frac{V(t) - e^{\kappa\Delta}V(t + \Delta)}{1 - e^{\kappa\Delta}} \\ &= \frac{V(t)(1 - e^{\kappa\Delta} + \kappa\Delta) - V(t+\Delta)(1 - e^{\kappa\Delta} + \kappa\Delta e^{\kappa\Delta})}{\kappa(1 - e^{\kappa\Delta})} \\ &= \Delta[\gamma_1 V(t) + \gamma_2 V(t + \Delta)]\end{aligned}$$

where $\gamma_1 = \frac{1-e^{\kappa\Delta}+\kappa\Delta}{\kappa\Delta(1-e^{\kappa\Delta})}$, $\gamma_2 = -\frac{1-e^{\kappa\Delta}+\kappa\Delta e^{\kappa\Delta}}{\kappa\Delta(1-e^{\kappa\Delta})}$. We know $\gamma_1 \geq 0$ and $\gamma_2 \geq 0$.
As $\kappa\Delta \to 0$,

$$\begin{aligned}\gamma_1 &\to \frac{\frac{1}{2}(\kappa\Delta)^2}{\kappa\Delta} = \frac{1}{2} \\ \gamma_2 &\to \frac{-(1 + \kappa\Delta + \frac{1}{2}(\kappa\Delta)^2)(1 - \kappa\Delta) + 1}{\kappa\Delta(\kappa\Delta)} \to \frac{-(1 - (\kappa\Delta)^2) - \frac{1}{2}(\kappa\Delta)^2 + \frac{1}{2}(\kappa\Delta)^3 + 1}{(\kappa\Delta)^2} \to \frac{1}{2}\end{aligned}$$

When $\kappa\Delta$ goes to zero, our scheme just like the Euler scheme.

3.1.2. *Change of variables*

We want to do change of variable of $s^2, m, \psi, b^{-2}, a, b, K_0, K_1, K_2, K_3,$ and K_4 in a way that ν becomes multiplier rather than divisor.
For parameters in QE scheme, we do change of variables. As we see, new variables $\tilde{s}^2, \tilde{m}$ and $\tilde{\psi}$ are free of ν.

$$s^2 = \nu^2\tilde{s}^2 = \nu^2\left(\frac{\hat{V}(t)e^{-\kappa\Delta}}{\kappa}\left(1 - e^{-\kappa\Delta}\right) + \frac{\theta}{2\kappa}\left(1 - e^{-\kappa\Delta}\right)^2\right), \quad m = \tilde{m} = \theta + \hat{V}(t)e^{-\kappa\Delta},$$

$$\psi = \nu^2\tilde{\psi} = \nu^2\left(\frac{\frac{\hat{V}(t)e^{-\kappa\Delta}}{\kappa}(1 - e^{-\kappa\Delta}) + \frac{\theta}{2\kappa}(1 - e^{-\kappa\Delta})^2}{(\theta + (\hat{V}(t) - \theta)e^{-\kappa\Delta})^2}\right),$$

$$b^{-2} = \nu^2\tilde{b}^{-2} = \nu^2\frac{\tilde{\psi}}{2\sqrt{1 - \frac{\tilde{\psi}\nu^2}{2}}\left(1 + \sqrt{1 - \frac{\tilde{\psi}\nu^2}{2}}\right)}, \quad a = \nu^2\tilde{a} = \nu^2\frac{m\tilde{b}^{-2}}{1 + \nu^2\tilde{b}^{-2}}.$$

Similarly, for parameters in Broadie-Kaya discretization scheme, we have

$$K_0 = \frac{1}{\nu}\tilde{K}_0 = \frac{1}{\nu}(-\rho\kappa\theta\Delta), \quad K_1 = \frac{1}{\nu}\tilde{K}_1 = \frac{1}{\nu}\left[\rho[\kappa\gamma_1\Delta - 1] - \frac{\nu\gamma_1\Delta}{2}\right], \quad K_2 = \frac{1}{\nu}\tilde{K}_2 = \frac{1}{\nu}\left[\rho[\kappa\gamma_2\Delta - 1] - \frac{\nu\gamma_2\Delta}{2}\right],$$
$$K_3 = \tilde{K}_3 = \gamma_1\Delta(1-\rho^2), \quad K_4 = \tilde{K}_4 = \gamma_2\Delta(1-\rho^2).$$

After doing change of variables, when ν goes to 0, our new variables will be certain numbers rather than go to infinity.
Now we use new variables we defined to calculate $\hat{X}$. Recall Eq.(2.7):

$$\ln\hat{X}(t+\Delta) = \ln\hat{X}(t) + K_0 + K_1\hat{V}(t) + K_2\hat{V}(t+\Delta) + \sqrt{K_3\hat{V}(t) + K_4\hat{V}(t+\Delta)}\cdot Z$$

Observe the expression of parameters we discussed in section 2.3, the terms including $\frac{1}{\nu}$ will cause unstable of the scheme when ν is close or equal to zero. We want to define a new variable of stochastic volatility process V in a way that there is no $\frac{1}{\nu}$ in our modified discretization scheme.
Write $\hat{V}(t+\Delta)$ in terms of new variables that we defined before,

$$\hat{V}(t+\Delta) = \frac{\theta + \tilde{V}(t)e^{-\kappa\Delta}}{1+\nu^2\tilde{b}^{-2}}(1+\nu\tilde{b}^{-1}Z_V)^2$$

Define $\tilde{V}(t) = \hat{V}(t) - \theta$. Substitute the above equation into the following equation,

$$\tilde{V}(t+\Delta) = \frac{\theta\nu(2\tilde{b}^{-1}Z_V + \nu\tilde{b}^{-2}(Z_V^2-1)) + \tilde{V}e^{-\kappa\Delta}(1+\nu\tilde{b}^{-1}Z_V)^2}{1+\nu^2\tilde{b}^{-2}}$$

Define $\nu\mathring{V}(t+\Delta) = \tilde{V}(t+\Delta) - \tilde{V}(t)e^{-\kappa\Delta}$.

$$\begin{aligned}\mathring{V}(t+\Delta) &= \frac{1}{\nu}[\hat{V}(t+\Delta) - (\theta + \tilde{V}(t)e^{-\kappa\Delta})] \\ &= (\theta + \tilde{V}(t)e^{-\kappa\Delta})\left[\frac{2\tilde{b}^{-1}Z_V}{1+\nu^2\tilde{b}^{-2}} + \nu\frac{\tilde{b}^{-2}(Z_V^2-1)}{1+\nu^2\tilde{b}^{-2}}\right]\end{aligned}$$

In order to calculate $\ln\hat{X}$,we do the following calculation first.

$$\gamma_1\tilde{V}(t) + \gamma_2\tilde{V}(t+\Delta) = \frac{1-e^{-\kappa\Delta}}{\kappa\Delta}\tilde{V}(t) + \gamma_2\nu\mathring{V}(t+\Delta).$$

Substituting the above equation into the following equations, we get

$$\begin{aligned}&K_0 + K_1\hat{V}(t) + K_2\hat{V}(t+\Delta) \\ =&(K_0 + \theta K_1 + \theta K_2) + K_1\tilde{V}(t) + K_2\tilde{V}(t+\Delta) \\ =&-\theta\frac{\Delta}{2} - \frac{\Delta}{2}(\gamma_1\tilde{V}(t) + \gamma_2\tilde{V}(t+\Delta)) + \frac{\rho\kappa\Delta e^{\kappa\Delta}}{e^{\kappa\Delta}-1}\mathring{V}(t+\Delta) \\ =&-\theta\frac{\Delta}{2} - \frac{1-e^{-\kappa\Delta}}{2\kappa}\tilde{V}(t) - \left(\frac{\nu(1-e^{\kappa\Delta}+\kappa\Delta e^{\kappa\Delta})}{2\kappa(1-e^{\kappa\Delta})} + \frac{\rho\kappa\Delta e^{\kappa\Delta}}{1-e^{\kappa\Delta}}\right)\mathring{V}(t+\Delta)\end{aligned} \tag{3.3}$$

Moreover,

$$\begin{aligned}&K_3\hat{V}(t) + K_4\hat{V}(t+\Delta) \\ =&\Delta(1-\rho^2)[\theta + (\gamma_1\tilde{V}(t) + \gamma_2\tilde{V}(t+\Delta))] \\ =&\Delta(1-\rho^2)\left[\theta + \left(\frac{1-e^{-\kappa\Delta}}{\kappa\Delta}\tilde{V}(t) + \gamma_2\nu\mathring{V}(t+\Delta)\right)\right]\end{aligned} \tag{3.4}$$

Now we can rewrite the discretization scheme. For X when $\nu^2\tilde{\psi} \le \psi_C$:

$$\ln \hat{X}(t+\Delta) = \ln \hat{X}(t) - \theta\frac{\Delta}{2} - \frac{1-e^{-\kappa\Delta}}{2\kappa}\tilde{V}(t) - \left(\frac{\nu(1-e^{\kappa\Delta}+\kappa\Delta e^{\kappa\Delta})}{2\kappa(1-e^{\kappa\Delta})} + \frac{\rho\kappa\Delta e^{\kappa\Delta}}{1-e^{\kappa\Delta}}\right)\mathring{V}(t+\Delta)$$
$$+ \sqrt{\Delta(1-\rho^2)\left[\theta + \left(\frac{1-e^{-\kappa\Delta}}{\kappa\Delta}\tilde{V}(t) + \gamma_2\nu\mathring{V}(t+\Delta)\right)\right]} \cdot Z \tag{3.5}$$

For X when $\nu^2\tilde{\psi} > \psi_C$:

$$\ln\hat{X}(t+\Delta) = \ln\hat{X}(t) + \tilde{K}_0 + \frac{1}{\nu}\tilde{K}_1\hat{V}(t) + \frac{1}{\nu}\tilde{K}_2\hat{V}(t+\Delta) + \sqrt{\tilde{K}_3\hat{V}(t) + \tilde{K}_4\hat{V}(t+\Delta)} \cdot Z \tag{3.6}$$

Since the discretization scheme is unstable only when ν is very small, for the $\nu^2\tilde{\psi} > \psi_C$ case, the scheme remains unchanged. We just rewrite it in terms of new variables.

3.1.3. *The new algorithm for modified QE scheme and the discretization scheme of X*

Summary of QE algorithm and the discretization scheme of X:

(1) Given $\hat{V}(t)$, compute m and S^2 from following equations

$$\tilde{m} = \Theta + \tilde{V}(t)e^{-\kappa\Delta}$$
$$\tilde{s}^2 = \frac{(\tilde{V}(t)+\theta)e^{-\kappa\Delta}}{\kappa}\left(1-e^{-\kappa\Delta}\right) + \frac{\theta}{2\kappa}\left(1-e^{-\kappa\Delta}\right)^2$$

(2) Compute $\tilde{\psi} = \tilde{s}^2/\tilde{m}^2$

(3) Generate two Brownian motion random variables Z_V and Z from GAIL

(4) If $\psi \le \psi_c$:
   (a) Compute a and b from following equations

$$\tilde{b}^{-2} = \frac{\tilde{\psi}}{2\sqrt{1-\frac{\tilde{\psi}\nu^2}{2}}\left(1+\sqrt{1-\frac{\tilde{\psi}\nu^2}{2}}\right)}$$
$$\tilde{a} = \frac{\tilde{m}\tilde{b}^{-2}}{1+\nu^2\tilde{b}^{-2}}$$

   (b) Set $\tilde{V}(t+\Delta) = -\theta + \tilde{a}(\tilde{b}+\nu Z_V)^2$
   (c) Compute $\mathring{V}(t+\Delta) = (\theta + \tilde{V}(t)e^{-\kappa\Delta})\left[\frac{2\tilde{b}^{-1}Z_V}{1+\nu^2\tilde{b}^{-2}} + \frac{\nu\tilde{b}^{-2}(Z_V^2-1)}{1+\nu^2\tilde{b}^{-2}}\right]$
   (d) Given $\tilde{V}(t)$, generate $\tilde{V}(t+\Delta)$
   (e) Given $\ln\hat{X}(t)$, $\tilde{V}(t)$ and the value for $\mathring{V}(t+\Delta)$, compute $\ln\hat{X}(t+\Delta)$ from Eq.(3.5)

(5) Otherwise, if $\nu^2\tilde{\psi} > \psi_c$
   (a) Compute β and p according to equations

$$p = \frac{\nu^2\tilde{\psi}-1}{\nu^2\tilde{\psi}+1} \in [0,1)$$
$$\beta = \frac{1-p}{\tilde{m}} = \frac{2}{\tilde{m}(\nu^2\tilde{\psi}+1)} > 0$$

(b) Draw a uniform random number U_V
(c) Set $\tilde{V}(t+\Delta) = -\theta + \Psi^{-1}(U_V; p, \beta)$
(d) Given $\tilde{V}(t)$, generate $\tilde{V}(t+\Delta)$
(e) Given $\ln\hat{X}(t)$, $\tilde{V}(t)$ and the value for $\tilde{V}(t+\Delta)$, compute $\ln\hat{X}(t+\Delta)$ from Eq.(3.6)

3.2. *QE scheme with martingale correction*

3.2.1. *Cancel out $\frac{1}{\nu}$ in the expression of* $\ln X$

We calculate $K_0^* + K_1\hat{V}(t) + K_2\hat{V}(t+\Delta)$ first.
Recall $K_0^* = -\frac{Ab^2a}{1-2Aa} + \frac{1}{2}\ln(1-2Aa) - (K_1 + \frac{1}{2}K_3)\hat{V}(t)$ for case $\psi \le \psi_c$, where $A = K_2 + \frac{1}{2}K_4$. Rewrite the first part of K_0^* in terms of new parameters we defined and substituting $m = \theta + \tilde{V}(t+\Delta) - \nu\mathring{V}(t+\Delta)$ into it,

$$\begin{aligned}
-\frac{Ab^2a}{1-2Aa} &= -\frac{1}{\nu}\frac{\tilde{A}\tilde{a}\tilde{b}^2}{1-2\nu\tilde{A}\tilde{a}} \\
&= -\frac{1}{\nu}\frac{[\rho(\gamma_2\Delta\kappa+1) - \frac{1}{2}\gamma_2\Delta\nu\rho^2]m}{1+\nu^2\tilde{b}^{-2} - 2\nu\tilde{b}^{-2}m[\rho(\gamma_2\Delta\kappa+1) - \frac{1}{2}\gamma_2\Delta\nu\rho^2]} \\
&= -\frac{1}{\nu}\frac{\rho(\gamma_2\Delta\kappa+1)(\theta+\tilde{V}(t+\Delta))}{1+\nu^2\tilde{b}^{-2} - 2\nu\tilde{b}^{-2}m[\rho(\gamma_2\Delta\kappa+1) - \frac{1}{2}\gamma_2\Delta\nu\rho^2]} \\
&\quad + \frac{\frac{1}{2}\gamma_2\Delta\rho^2(\theta+\tilde{V}(t+\Delta))}{1+\nu^2\tilde{b}^{-2} - 2\nu\tilde{b}^{-2}m[\rho(\gamma_2\Delta\kappa+1) - \frac{1}{2}\gamma_2\Delta\nu\rho^2]} \\
&\quad + \frac{\mathring{V}(t+\Delta)[\rho(\gamma_2\Delta\kappa+1) - \frac{1}{2}\gamma_2\Delta\nu\rho^2]}{1+\nu^2\tilde{b}^{-2} - 2\nu\tilde{b}^{-2}m[\rho(\gamma_2\Delta\kappa+1) - \frac{1}{2}\gamma_2\Delta\nu\rho^2]}
\end{aligned}$$

Therefore,

$$\begin{aligned}
&K_0^* + K_1\hat{V}(t) + K_2\hat{V}(t+\Delta) \\
=& -\frac{Ab^2a}{1-2Aa} + \frac{1}{2}\ln(1-2Aa) - (K_1 + \frac{1}{2}K_3)\hat{V}(t) + K_1\hat{V}(t) + K_2\hat{V}(t+\Delta) \\
=& \frac{\frac{1}{2}\gamma_2\Delta\rho^2(\theta+\tilde{V}(t+\Delta))}{1+\nu^2\tilde{b}^{-2} - 2\nu\tilde{b}^{-2}m[\rho(\gamma_2\Delta\kappa+1) - \frac{1}{2}\gamma_2\Delta\nu\rho^2]} \\
&+ \frac{\mathring{V}(t+\Delta)[\rho(\gamma_2\Delta\kappa+1) - \frac{1}{2}\gamma_2\Delta\nu\rho^2]}{1+\nu^2\tilde{b}^{-2} - 2\nu\tilde{b}^{-2}m[\rho(\gamma_2\Delta\kappa+1) - \frac{1}{2}\gamma_2\Delta\nu\rho^2]} \\
&+ \frac{1}{2}\ln(1-2\nu\tilde{A}\tilde{a}) - \frac{1}{2}\gamma_1\Delta(1-\rho^2)(\theta+\tilde{V}(t)) - \frac{1}{2}\gamma_2\Delta(\theta+\tilde{V}(t+\Delta)) \\
&+ \rho(\gamma_2\Delta\kappa+1)(\theta+\tilde{V}(t+\Delta))\frac{\nu\tilde{b}^{-2} - 2\tilde{b}^{-2}m[\rho(\gamma_2\Delta\kappa+1) - \frac{1}{2}\gamma_2\Delta\nu\rho^2]}{1+\nu^2\tilde{b}^{-2} - 2\nu\tilde{b}^{-2}m[\rho(\gamma_2\Delta\kappa+1) - \frac{1}{2}\gamma_2\Delta\nu\rho^2]}
\end{aligned}$$

For $K_3\hat{V}(t) + K_4\hat{V}(t+\Delta)$, it is the same as QE scheme without martingale correction.

4. Determining the Number of Samples Required to Meet a Specified Error Tolerance

5. Numerical Examples

6. Discussion

Appendix A. Appendices

$$
\begin{aligned}
\mathrm{d}V_t &= \kappa(\theta - V_t)\mathrm{d}t + \nu\sqrt{V_t}\mathrm{d}W_t \\
\mathrm{d}U_t &= -\kappa U_t\mathrm{d}t + \nu\sqrt{V_t}\mathrm{d}W_t
\end{aligned}
\tag{A.1}
$$

For initial conditions, V_0 is given, $U_0 = 0$.

$$
\mathrm{d}(V_t - U_t) = \kappa(\theta - (V_t - U_t))\mathrm{d}t
$$

Denote $Y_t = V_t - U_t$, then we have the stochastic process Y_t has the same expression as V_t when $\nu = 0$. Also, the initial condition of Y_t is $Y_0 = V_0$. So, we apply similar deduction to Y_t as in Section 3.1.1.

$$
\begin{aligned}
\mathrm{d}Y_t &= \kappa(\theta - Y_t)\mathrm{d}t \\
\frac{\mathrm{d}Y_t}{\theta - Y_t} &= \kappa\mathrm{d}t \\
Y_t &= \theta + Ce^{-\kappa t}
\end{aligned}
$$

where $C = Y_0 - \theta$.

$$
\begin{aligned}
\int_t^{t+\Delta} Y_u\mathrm{d}u &= \theta\Delta + C\int_t^{t+\Delta} e^{-\kappa u}\mathrm{d}u \\
&= \theta\Delta + \frac{Ce^{-\kappa t} - Ce^{-\kappa(t+\Delta)}}{\kappa} \\
&= \theta\Delta + \frac{Y_t - Y_{t+\Delta}}{\kappa}
\end{aligned}
$$

From the result in Section 3.1.1, the θ has an expression as $\theta = \frac{Y(t) - e^{\kappa\Delta}Y(t+\Delta)}{1 - e^{\kappa\Delta}}$, then

$$
\int_t^{t+\Delta}(V_u - U_u)\mathrm{d}u = \int_t^{t+\Delta} Y_u\mathrm{d}u = \Delta(\gamma_1 Y_t + (1-\gamma_1)Y_{t+\Delta})
$$

where γ_1 and γ_2 are the same as in Section 3.1.1.
Now we have the integral of V_t written in terms of V and U as follows.

$$
\begin{aligned}
\int_t^{t+\Delta} V_u\mathrm{d}u &= \Delta(\gamma_1 Y_t + (1-\gamma_1)Y_{t+\Delta}) + \int_t^{t+\Delta} U_u\mathrm{d}u \\
&= \Delta(\gamma_1 V_t + (1-\gamma_1)V_{t+\Delta}) - \left(\Delta(\gamma_1 U_t + (1-\gamma_1)U_{t+\Delta}) + \int_t^{t+\Delta} U_u\mathrm{d}u\right)
\end{aligned}
$$

Our problem becomes to show that the approximation $\Delta(\gamma_1 U_t + (1-\gamma_1)U_{t+\Delta})$ of $\int_t^{t+\Delta} U_u\mathrm{d}u$ is of order $o(\nu\Delta)$.
First, we solve for U_t.

$$
\begin{aligned}
\mathrm{d}e^{\kappa t}U_t &= \kappa e^{\kappa t}U_t\mathrm{d}t + e^{\kappa t}\mathrm{d}U_t \\
&= \kappa e^{\kappa t}U_t\mathrm{d}t + e^{\kappa t}(-\kappa U_t\mathrm{d}t + \nu\sqrt{V_t}\mathrm{d}W_t) \\
&= \nu e^{\kappa t}\sqrt{V_t}\mathrm{d}W_t
\end{aligned}
$$

Integrate on both sides,

$$
U_t = \nu e^{-\kappa t}\int_0^t e^{\kappa s}\sqrt{V_s}\mathrm{d}W_s
$$

Now we start to working on the integral of U.

$$\begin{aligned}\int_t^{t+\Delta} U_u \mathrm{d}u &= \int_t^{t+\Delta} U_u \mathrm{d}(u-t-\Delta\gamma_1) \\ &= U_u(u-t-\Delta\gamma_1)|_t^{t+\Delta} - \int_t^{t+\Delta}(u-t-\Delta\gamma_1)\mathrm{d}U_u \\ &= U_u(u-t-\Delta\gamma_1)|_t^{t+\Delta} - \int_t^{t+\Delta}(u-t-\Delta\gamma_1)(-\kappa U_u\mathrm{d}u + \nu\sqrt{V_u}\mathrm{d}W_u) \\ &= \Delta(\gamma_1 U_t + (1-\gamma_1)U_{t+\Delta}) - \int_t^{t+\Delta}(u-t-\Delta\gamma_1)(-\kappa U_u)\mathrm{d}u \\ &- \int_t^{t+\Delta}(u-t-\Delta\gamma_1)\nu\sqrt{V_u}\mathrm{d}W_u \end{aligned} \tag{A.2}$$

For the second term of Equation A.2,

$$\begin{aligned}\int_t^{t+\Delta}(u-t-\Delta\gamma_1)(-\kappa U_u)\mathrm{d}u &= \int_t^{t+\Delta}\frac{1}{2}(-\kappa U_u)\mathrm{d}(u-t-\Delta\gamma_1)^2 \\ &= -\int_t^{t+\Delta}\kappa U_u \mathrm{d}\Big(\frac{(u-t-\Delta\gamma_1)^2}{2} - \frac{\Delta^2\gamma_1^2}{2}\Big) \\ &= -\kappa U_u\Big(\frac{(u-t-\Delta\gamma_1)^2}{2} - \frac{\Delta^2\gamma_1^2}{2}\Big)\Big|_t^{t+\Delta} + \int_t^{t+\Delta}\Big(\frac{(u-t-\Delta\gamma_1)^2}{2} - \frac{\Delta^2\gamma_1^2}{2}\Big)\kappa \mathrm{d}U_u \\ &= \kappa U_{t+\Delta}\Big(\frac{(1-\gamma_1)^2-\gamma_1^2}{2}\Delta^2\Big) + \int_t^{t+\Delta}\Big(\frac{(u-t-\Delta\gamma_1)^2}{2} - \frac{\Delta^2\gamma_1^2}{2}\Big)\kappa \mathrm{d}U_u \\ &= -\kappa U_{t+\Delta}\Delta^2\left(\frac{1}{2}-\gamma_1\right) + \int_t^{t+\Delta}\left(\frac{(u-t)^2}{2} - (u-t)\Delta\gamma_1\right)\kappa\mathrm{d}U_u \\ &= O(\nu\Delta^2) + \int_t^{t+\Delta}\left(\frac{(u-t)^2}{2} - (u-t)\Delta\gamma_1\right)\kappa\mathrm{d}(-\kappa U_u\mathrm{d}u + \nu\sqrt{V_u}\mathrm{d}W_u) \\ &= O(\nu\Delta^2) - \int_t^{t+\Delta}\left(\frac{(u-t)^2}{2} - (u-t)\Delta\gamma_1\right)\kappa^2 U_u\mathrm{d}u \\ &+ \int_t^{t+\Delta}\left(\frac{(u-t)^2}{2} - (u-t)\Delta\gamma_1\right)\kappa\nu\sqrt{V_u}\mathrm{d}W_u \\ &= O(\nu\Delta^2)\end{aligned}$$

By Holder inequality

$$\begin{aligned}\int_t^{t+\Delta}(u-t)\Delta\gamma_1\kappa^2 U_u\mathrm{d}u &\le \Delta\gamma_1\kappa^2\sqrt{\int_t^{t+\Delta}(u-t)^2\mathrm{d}u\int_t^{t+\Delta}U_u^2\mathrm{d}u} \\ &= O(\nu\Delta^2)\end{aligned}$$

Likewise, integral $\int_t^{t+\Delta}\frac{(u-t)^2}{2}\kappa^2 U_u\mathrm{d}u$ is also of order $O(\nu\Delta^2)$. Now we want to show that for Ito integral $I = \int_t^{t+\Delta}\big(\frac{(u-t)^2}{2} - (u-t)\Delta\gamma_1\big)\kappa\nu\sqrt{V_u}\mathrm{d}W_u$. By Theorem 4.7 of [*Klebaner*(2005)], I is a continuous zero mean square integrable martingale. Since V_t is a Cox-Ingersoll-Ross process, we know its expectation given V_0 is $\mathbb{E}(V_t) = V_0 e^{-\kappa u} + \theta(1 - e^{-\kappa u})$.

$$\begin{aligned}
\mathbb{E}(I) &= 0 \\
\mathrm{Var}(I) &= \mathbb{E}(I^2) \\
\text{by Ito Isometry} &= \mathbb{E}\Big(\int_t^{t+\Delta} \big(\frac{(u-t)^2}{2} - \Delta\gamma_1(u-t)\big)^2 \kappa^2\nu^2 V_u \mathrm{d}u \Big) \\
&= \int_t^{t+\Delta} \big(\frac{(u-t)^2}{2} - \Delta\gamma_1(u-t)\big)^2 \kappa^2\nu^2 \mathbb{E}(V_u) \mathrm{d}u \\
&= \int_t^{t+\Delta} \big(\frac{(u-t)^2}{2} - \Delta\gamma_1(u-t)\big)^2 \kappa^2\nu^2 (V_0 e^{-\kappa u} + \theta(1 - e^{-\kappa u})) \mathrm{d}u \\
&= \kappa^2\nu^2 \Big[\int_t^{t+\Delta} \frac{1}{20}(V_0 e^{-\kappa u} + \theta(1 - e^{-\kappa u})) \mathrm{d}(u-t)^5 \\
&- \int_t^{t+\Delta} \frac{1}{4}\Delta\gamma_1(V_0 e^{-\kappa u} + \theta(1 - e^{-\kappa u})) \mathrm{d}(u-t)^4 \\
&+ \int_t^{t+\Delta} \frac{\Delta^2}{3}\gamma_1^2(V_0 e^{-\kappa u} + \theta(1 - e^{-\kappa u})) \mathrm{d}(u-t)^3 \Big] \\
&= \kappa^2\nu^2\Delta^5 \Big(\frac{1}{20}(V_0 e^{-\kappa(\xi^{\frac{1}{5}}+t)} + \theta(1 - e^{-\kappa(\xi^{\frac{1}{5}}+t)}) - \frac{1}{4}\gamma_1(V_0 e^{-\kappa(\xi^{\frac{1}{4}}+t)} + \theta(1 - e^{-\kappa(\xi^{\frac{1}{4}}+t)}) \\
&+ \frac{1}{3}\gamma_1^2(V_0 e^{-\kappa(\xi^{\frac{1}{3}}+t)} + \theta(1 - e^{-\kappa(\xi^{\frac{1}{3}}+t)})) \Big)
\end{aligned}$$

where $\xi \in [t, t+\Delta]$.
Therefore, I is a normal random variable with mean zero and variance of order $O(\nu^2\Delta^5)$. I is of order $O(\nu\Delta^{2.5})$.
Similarly, for the third term of Equation A.2, Ito integral $\int_t^{t+\Delta}(u - t - \Delta\gamma_1)\nu\sqrt{V_u}\mathrm{d}W_u$ is also a normal distributed random variable with mean zero and variance of order $O(\nu^2\Delta^3)$. So, the integral is of order $O(\nu\Delta^{1.5})$.
We proved the overall order of the approximation of $\int_t^{t+\Delta} V_u \mathrm{d}u$ is $O(\nu\Delta^{1.5})$.

Acknowledgments

References

Leif B. G. Andersen. Efficient simulation of the heston stochastic volatility model. 2006.
Fima C Klebaner. *Introduction to Stochastic Calculus with Applications.* Imperial College Press, 2005.