Linear Dispersive Waves in Optical Fibers

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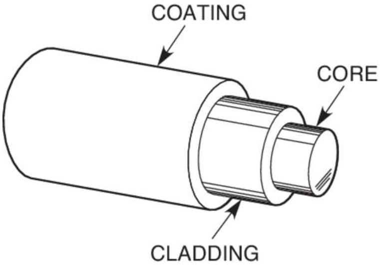
Physical Properties of an Optical Fiber

- Silica glass core surrounded by cladding.
- Glass naturally charge neutral at rest and assumed to remain neutrally charged as light passes through it.
- Core is isotropic, homogeneous and non-birefringent.
- Core-cladding boundary at radius a.



https://www.newport.com/t/fiber-optic-basics

- Refractive index of the core and cladding are n_1 and n_0, respectively.
- Cladding of arbitrary radius.

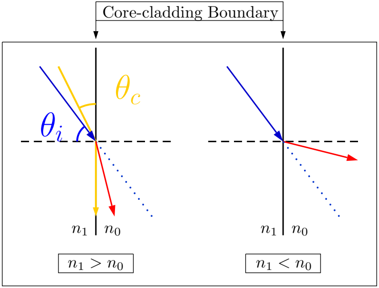
Boundary conditions necessary for light confinement in the core are required. We investigate the properties of light at the core-cladding boundary.

- Snell's Law is derived from Descarte's Law.
- **Descarte's Law:** Sufficiently small variations in the path of light implies light travels in a straight line.

- Reflection occurs only if

$$n_1 > n_0,$$

$$\frac{\pi}{2} - \theta_i > \theta_c = \sin^{-1}\left(\frac{n_0}{n_1}\right).$$

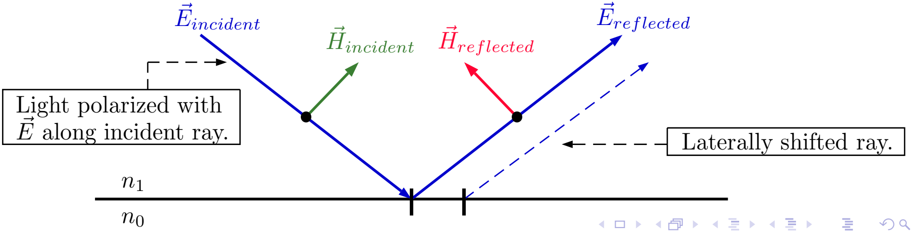


Properties of Light: Snell's Law and Goos-Hanchen Shift

Conclusion from Snell's law: Cylindrical symmetry and trigonometric identities give the maximum acceptance angle,

$$\theta_{max} \leq \sin^{-1} \sqrt{n_1^2 + n_0^2}.$$

- **Goos-Hanchen shift** is a lateral shift of reflected rays.
- Electromagnetic waves have complex phase.
- Reflected ray no longer has same complex phase.
- Phase difference implies a lateral shift occurs.



Electromagnetic Theory

- **Maxwell's equations:**

$$\vec{\nabla} \circ \vec{D} = \rho \qquad (1) \qquad\qquad \vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \qquad (3)$$

$$\nabla \circ \vec{B} = 0 \qquad (2) \qquad\qquad \vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad (4)$$

- Combine to obtain **Maxwell's wave equations**;

$$\nabla^2 \vec{E} = \mu_0 n^2(r) \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{and} \quad \nabla^2 \vec{H} = \mu_0 n^2(r) \varepsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2}.$$

- Solutions are monochromatic phasors w.r.t. time;

$$\vec{E}(\vec{r}, t) = \vec{E}(r, \theta, z) e^{i\omega t} \quad \text{and} \quad \vec{H}(\vec{r}, t) = \vec{H}(r, \theta, z) e^{i\omega t}. \qquad (5)$$

Hyperbolic PDEs: The Cauchy Problem

- **Cauchy Problem:** n^{th} order partial differential equation (PDE) with m independent variables and specified boundary conditions (BCs).
- Conditions specified on solution and derivatives of order $< (n-1)$.
- **Maxwell's wave equations:** 2^{nd} order linear PDEs; independent variables (r, θ, z, t).
- **Boundary conditions:**

$$E_T^{(core)} = E_T^{(cladding)} \quad \text{and} \quad H_T^{(core)} = H_T^{(cladding)},$$

$\forall T \in \{r, \theta, z, t\}$ satisfying $r = a$.

Hyperbolic PDEs: Method of Characteristics

Definition: Hyperbolic PDE

A PDE is hyperbolic for sets of points where the Cauchy problem has a unique solution in the neighbourhood of the points on any non-characteristic hyper-plane passing though them. Unique solutions are hyper-surfaces such that the hyper-planes have at least one less independent variable.

- Method of characteristics used to find unique solution along light rays.
- Known parametrization of solution used to determine unique solutions **or** hyperbolic regions for solutions are determined by unique parametrized hyper-planes satisfying solution uniquely.

Applying the Method of Characteristics: Example

- **1-dimensional wave equation:**

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0,$$

$$x \in [0, L] \subset \mathbb{R}$$

$$t \in [0, \infty) \subset \mathbb{R}$$

$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$
$$u(0,t) = 0$$
$$u(L,t) = 0$$

- Equivalent to 2^{nd} order PDE;

$$\mathcal{L}[u] = a(x,t)u_{xx} + b(x,t)u_{tt} + d(x,t)u_{xt} = 0,$$

with $a = 1,\ b = c^2$ and $d = 0$.

- Apply parametrization

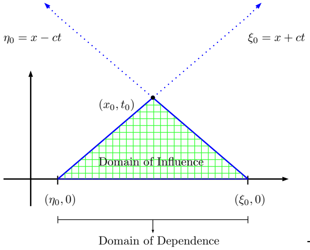
$$(x,t) \mapsto (\xi, \eta); \quad \xi(x,t) = x + ct, \quad \eta(x,t) = x - ct.$$

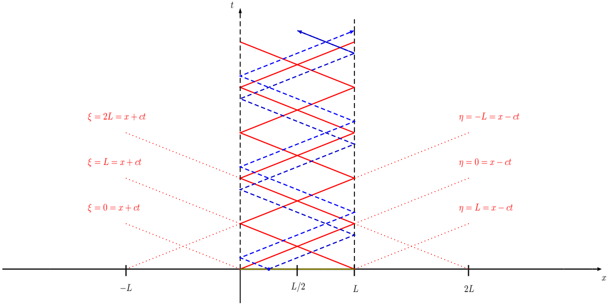
- Solutions necessarily hyperbolic since $b^2 - 4ac = c^4 > 0$.

Applying the Method of Characteristics: Example

- Solution obtain by inputting parametrization into PDE is called d'Alembert's solution;

$$u(x,t) = p(x+ct) - q(x-ct) = \frac{1}{2}\Big(f(\xi) + f(\eta)\Big) + \frac{1}{2c}\int_{\eta}^{\xi} g(s)ds.$$





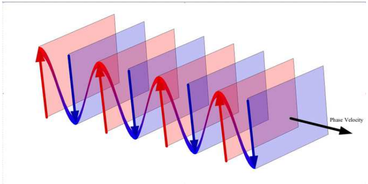
Dispersion: Definition

- Consider a homogeneous PDE with 4 independent variables (x_r, x_θ, x_z, t).
- **Dispersion** defined by solutions $u(\vec{x}, t) = Ae^{i\vec{\kappa}\circ\vec{x} - i\omega t}$, $\vec{\kappa}$, ω constants.
- Input into PDE to obtain dispersion relation $G(\omega, \kappa_r, \kappa_\theta, \kappa_z) = 0$.
- Solutions, ω, are called dispersion modes.
- Real solutions are $Re(u) = |A| \cos(\vec{\kappa} \circ \vec{x} - \omega t + ArgA)$.

Dispersion: Phase Velocity

- For each dispersion mode ω, consider phase surfaces $\vartheta = \vec{\kappa} \circ \vec{x} - \omega t$.
- **Phase velocity** is determined by the **average** number of wavelengths per period in the direction normal to phase surfaces with respect to space.
- For fixed phase surface $\vartheta =$ constant, this direction is $\vec{\kappa}$.
- Average wavelength is $\lambda = 2\pi / \ |\vec{\kappa}|$ and the average period is $2\pi/\omega$.
- Phase velocity is given by

$$c(\vec{\kappa}) = \frac{W(\vec{\kappa})}{|\vec{\kappa}|}\hat{\kappa} = \frac{\omega}{|\vec{\kappa}|}\hat{\kappa}.$$



https://courses.lumenlearning.com/boundless-physics/chapter/waves/

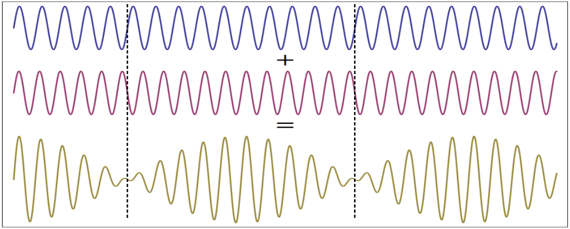
Dispersion: Group Velocity

- **Group velocity** is defined by considering **local** wavenumber and frequency given by

$$\kappa_i = \frac{\partial u}{\partial x_i} \quad \text{and} \quad \omega = -\frac{\partial u}{\partial t,}, \quad \text{respectively.}$$

- Solving for $\kappa_i(x_i, t)$ defines the group velocity component-wise as

$$C_i(\vec{\kappa}) = \frac{\partial W(\vec{\kappa}, \vec{x}, t)}{\partial \kappa_i}.$$



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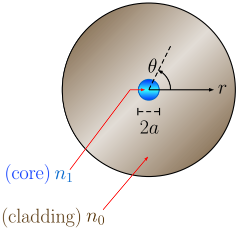
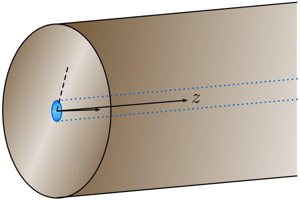
Deriving a Solution: Plane Wave Solution





$$\nabla^2 \vec{E} = \mu_0 n^2(r)\varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{H} = \mu_0 n^2(r)\varepsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2}$$

$$E_T^{(core)} = E_T^{(cladding)}$$

$$H_T^{(core)} = H_T^{(cladding)},$$

$$\forall T \in \{r, \theta, z, t\}, \quad r = a$$

Plane wave solution for dispersive light rays propagating in the z-direction:

$$\vec{E}(r,\theta,z,t) = \vec{E}(r,\theta)e^{i(\omega t - \beta z)} \quad \text{and} \quad \vec{H}(r,\theta,z,t) = \vec{H}(r,\theta)e^{i(\omega t - \beta z)}.$$

Deriving a Solution: Necessary Equations

- In vacuum, $c = \omega/k$ =**speed of light**, $k = \omega\sqrt{\mu_0\varepsilon_0}$ =**wavenumber**.
- Input $\vec{E}$ and $\vec{H}$ into Maxwell's wave equations to obtain:

$$\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \theta^2} + \left(k^2 n^2(r) - \beta^2\right) E_z = 0 \tag{6}$$

$$\frac{\partial^2 H_z}{\partial r^2} + \frac{1}{r}\frac{\partial H_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 H_z}{\partial \theta^2} + \left(k^2 n^2(r) - \beta^2\right) H_z = 0. \tag{7}$$

- Equations for $\vec{\nabla} \times \vec{E}$ and $\vec{\nabla} \times \vec{H}$ in cylindrical components used in Maxwell's equations (3) and (4), respectively, gives

$$\begin{aligned} \frac{1}{r}\frac{\partial E_z}{\partial \theta} + i\beta E_\theta &= -\mu_0 i\omega H_r \\ -i\beta E_r - \frac{\partial E_z}{\partial r} &= -\mu_0 i\omega H_\theta \\ \frac{1}{r}\left(\frac{\partial (rE_\theta)}{\partial r} - \frac{\partial E_r}{\partial \theta}\right) &= -\mu_0 i\omega H_z \end{aligned} \quad \text{and} \quad \begin{aligned} \frac{1}{r}\frac{\partial H_z}{\partial \theta} + i\beta H_\theta &= i\varepsilon_0\omega n^2 E_r \\ -i\beta H_r - \frac{\partial H_z}{\partial r} &= i\varepsilon_0\omega n^2 E_\theta \\ \frac{1}{r}\left(\frac{\partial (rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta}\right) &= i\varepsilon_0\omega n^2 E_z \end{aligned}$$

Deriving a Solution: Bessel's Equation

- Boundary value problems (6) and (7) are well-posed only if

$$E_z = R_E(r)\Theta_E(\theta), \quad \text{and} \quad H_z = R_H(r)\Theta_H(\theta).$$

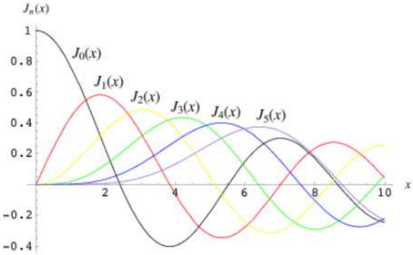
- W.l.o.g., R and Θ inputted into wave equation to obtain

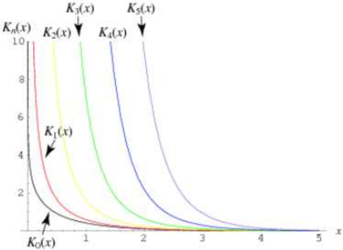
$$\frac{d^2\Theta}{d\theta^2} = -m^2\Theta \quad \text{and} \quad \frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\gamma^2 - \frac{m}{r^2}\right)R = 0.$$

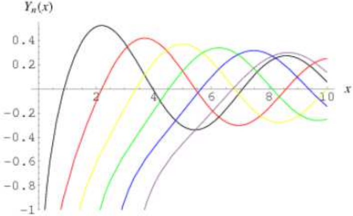
- ODE w.r.t. $R(r)$ is **Bessel's or modified Bessel's** equation of order m, such that $\gamma^2 = n^2k^2 - \beta^2$.
- $J_m(ur)$ and $K_m(wr)$ are the Bessel functions satisfying light confinement BCs whenever $r \to 0$ and $r \to \infty$, respectively.

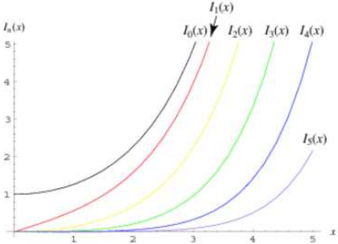
$$\begin{aligned} E_z &= AJ_m(ur)\cos(m\theta + \alpha), \\ H_z &= BJ_m(ur)\sin(m\theta + \alpha), \\ u^2 &= n_1^2k^2 - \beta^2. \end{aligned} \qquad \begin{aligned} E_z &= CK_m(wr)\cos(m\theta + \alpha), \\ H_z &= DK_m(wr)\sin(m\theta + \alpha), \\ w^2 &= \beta^2 - n_0^2k^2. \end{aligned}$$

Bessel's and Modified Bessel's Functions









http://mathworld.wolfram.com

Deriving a Solution: Bessel's Equation

Equations obtained from curls $\vec{\nabla} \times \vec{E}$ and $\vec{\nabla} \times \vec{H}$ are combined:

$$E_\theta^{(core)} = \frac{-i}{u^2}\left((Am\frac{\beta}{r}J_m(ur) + B\omega\mu_0\frac{\partial J_m(ur)}{\partial r}\right)\sin(m\theta+\alpha),$$

$$E_\theta^{(cladding)} = \frac{-i}{w^2}\left(Cm\beta\frac{\partial K_m(wr)}{\partial r} + D\frac{\omega\mu_0}{r}\frac{\partial K_m(wr)}{\partial \theta}\right)\sin(m\theta+\alpha),$$

$$H_\theta^{(core)} = \frac{-i}{u^2}\left(Am\frac{\beta}{r}J_m(wr) + B\omega\varepsilon_0 n_1^2\frac{\partial K_m(wr)}{\partial r}\right)\cos(m\theta+\alpha),$$

$$H_\theta^{(cladding)} = \frac{-i}{w^2}\left(Cm\beta J_m(wr) + D\frac{\omega\varepsilon_0 n_0^2}{r}\frac{\partial K_m(wr)}{\partial \theta}\right)\cos(m\theta+\alpha).$$

Applying BCs for Maxwell's wave equations obtains:

$$\begin{pmatrix} \frac{m\beta}{u^2a^2}J_m(ua) & \frac{k\omega\mu_0}{ua}J_m'(ua) & \frac{m\beta}{w^2a}K_m(wa) & -\frac{k\omega\mu_0}{w^2a}K_m'(wa) \\ \frac{kn_1^2}{ua}J_m'(ua) & \frac{\omega\mu_0 m}{u^2a^2}J_m(ua) & \frac{k^2n_0^2}{wa}K_m'(wa) & \frac{\omega\mu_0 m\beta}{w^2a^2}K_m(wa) \\ J_m(ua) & 0 & -K_m(wa) & 0 \\ 0 & J_m(ua) & 0 & K_m(wa) \end{pmatrix}\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Deriving a Solution: Characteristic Equation

- **Characteristic equations** give solutions along light rays; given by determinant

$$\begin{vmatrix} \frac{m\beta}{u^2a^2}J_m(ua) & \frac{k\omega\mu_0}{ua}J_m'(ua) & \frac{m\beta}{w^2a}K_m(wa) & -\frac{k\omega\mu_0}{w^2a}K_m'(wa) \\ \frac{kn_1^2}{ua}J_m'(ua) & \frac{\omega\mu_0 m}{u^2a^2}J_m(ua) & \frac{k^2n_0^2}{wa}K_m'(wa) & \frac{\omega\mu_0 m\beta}{w^2a^2}K_m(wa) \\ J_m(ua) & 0 & -K_m(wa) & 0 \\ 0 & J_m(ua) & 0 & K_m(wa) \end{vmatrix}.$$

- Families of solutions correspond to values of m w.r.t. **one** of the fields $\vec{E}$ or $\vec{H}$.
- Seek solution for modes satisfying $m = 0$, called the TE and TM modes.

Conclusion: TE and TM Modes

- When $m = 0$, the characteristic equation becomes

$$\left(\frac{J_0'(ua)}{uaJ_0(ua)} + \frac{K_0'(wa)}{waK_0(wa)}\right)\left(\frac{J_0'(ua)}{uaJ_0(ua)} + \frac{n_0^2}{n_1^2}\frac{K_0'(wa)}{waK_0(wa)}\right) = 0.$$

$$\text{with} \quad H_z = BJ_0(ur)\sin(\alpha), \qquad H_z = DK_0(wr)\sin(\alpha)$$

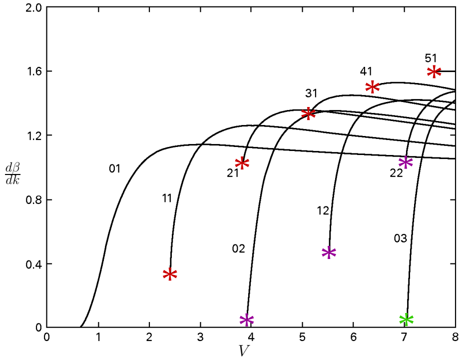
- **Normalized frequency** is

$$V = \sqrt{(wa)^2 + (ua)^2} = \left[a\sqrt{n_1^2 - n_0^2}\right]k = \left[a\sqrt{\mu_0\varepsilon_0(n_1^2 - n_0^2)}\right]\omega$$

- Since n_1 and n_0 are independent of frequency, the **normalized group velocity** is given by the product rule;

$$\frac{d\beta}{dk} = \frac{a\sqrt{n_1^2 - n_0^2}}{a\sqrt{n_1^2 - n_0^2}}\frac{d(\frac{V\beta}{k})}{dV} = V\frac{d\left(\frac{\beta}{k}\right)}{dV} + \frac{\beta}{k}.$$

Conclusion: TE and TM modes



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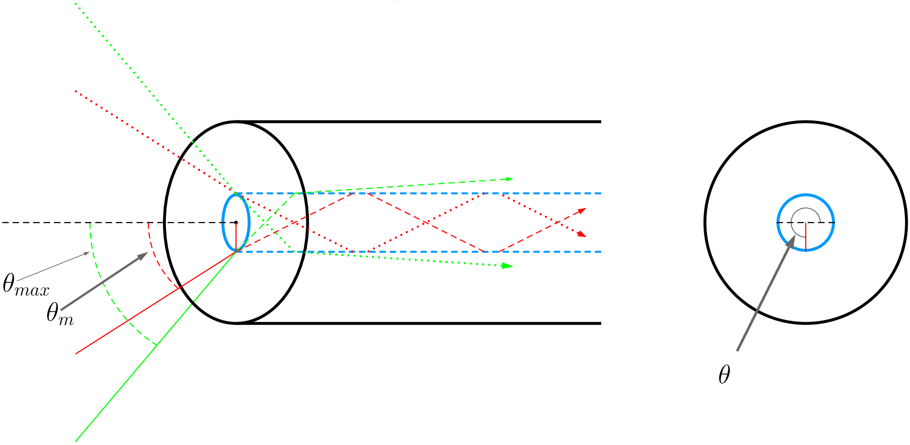
- If $V > 2.4$ then group velocity is non-zero for other modes with $m \neq 0$.
- For sufficiently small ω, only TE mode present.

- Non-zero group velocity implies the corresponding mode propagates with a particular group velocity determined by its normalized frequency.
- A dispersion mode with some value of ω in the solution propagates HE and EH modes with corresponding group velocities.

Conclusion: Hybrid Modes

Can interpret dispersive properties with group velocity behaviour of modes w.r.t. source frequency.



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