This cuts down the number of possible assignments of the complete twist considerably.

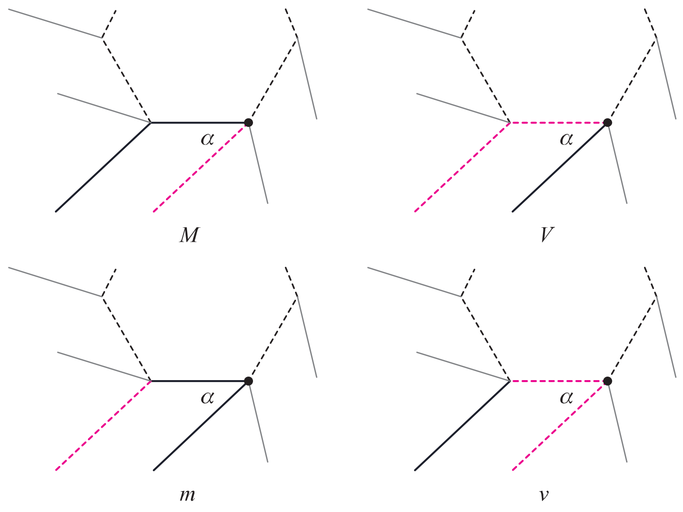
### ⋆ 3.5.3. Pleat Assignments

With each pleat we can associate three creases: the two creases of the pleat itself, and the crease from the central polygon that connects the two pleat creases. Taking into account the preceding result, there are four possible assignments of these three creases for each pleat. The polygon edge can be $M$ or $V$; the two creases of the pleat can be $MV$ or $VM$. Let us label a pleat primarily by the assignment of the polygon edge.

If the acute angle at the base of the pleat is anto, we will call the pleat an *anto pleat* and will label it with a capital $M$ or $V$. If the acute angle at the base of the pleat is iso, we will call it an *iso pleat* and will label it with lower case $m$ or $v$. We illustrate the four possibilities in Figure 3.33.

Now, every possible assignment of the creases around a *potentially* flat-foldable twist could be expressed as a string of symbols, one for each of the pleats of the twist, selected from the alphabet $\{M, V, m, v\}$. Each vertex of the central polygon is the junction of two pleats. Thus, we can examine the flat-foldability conditions at all possible vertices by considering all possible pairs of these four symbols and the implied crease assignment. There are $4 \times 4 = 16$ possibilities.

We can dispense with eight of them, however, quite readily. Consider the four combinations $\{Mm, Mv, mM, mV\}$, i.e., those




如果可折叠成平面则褶皱的峰谷性质相反，
所以只有四种可能

Figure 3.33. The four possible crease assignments for a single pleat. Top row: acute angle is anto. Bottom row: acute angle is iso.

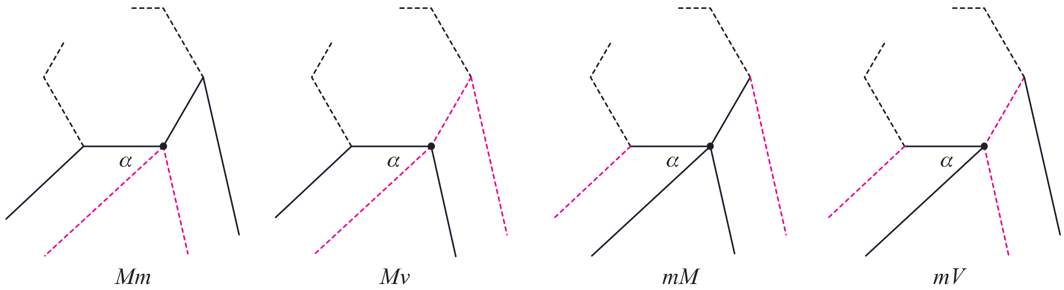



Figure 3.34.
Crease assignments for two consecutive pleats where one is anto and the other is iso.

combinations where an anto pleat follows an iso pleat or vice versa, as illustrated in Figure 3.34.

In these assignments, there are either two mountain/two valley or four mountain/zero valley. These vertices therefore fail the Maekawa-Justin Condition and so cannot be flat-foldable for any possible assignment. The same is true if we invert all of the crease assignments, which takes care of the four heterogeneous possibilities $\{Vv, Vm, vV, vM\}$. Thus we have the following:

Theorem 20. *If a twist is flat-foldable, then all of its pleats are anto or all of them are iso.*

That leaves eight possible pairs: $\{MM, VV, MV, VM, mm, vv, vm, mv\}$. We now consider each of these individually. We note that since any crease pattern is flat-foldable if and only if the crease pattern obtained by reversing every crease is also flat-foldable, we only need to consider half of the combinations: $\{MM, MV, mm, mv\}$, since the same conditions will apply to their respective parity inversions.

## ⋆ 3.5.4. *mm*/*vv* Condition

Figure 3.35 shows an *mm* and an *mv* crease assignment for the polygon creases at a vertex of a regular polygonal twist.

We will assume going forward that the pleat angle $\alpha$ is non-obtuse (i.e., $\alpha \leq 90°$). Each vertex has two iso and two anto sectors; according to the BLBA Theorem, one of the two anto sectors must be the smallest angle at the vertex. For the *mm* vertex, the angle labeled $180° - \alpha$ can't be smallest (it's larger than $\alpha$); thus, the angle marked $360°/n$ must be as small or smaller than

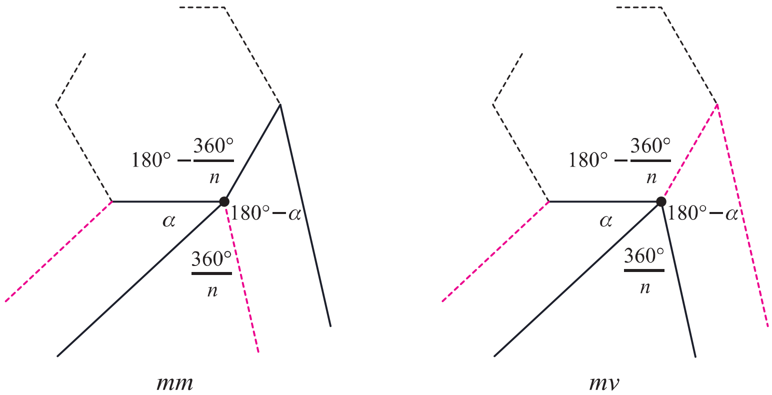



Figure 3.35.
Left: an *mm* vertex of a polygon twist.
Right: an *mv* vertex of a polygon twist.

any other angle at the vertex. Consequently, we have the following requirements:

$$\frac{360°}{n} \leq 180° - \frac{360°}{n} \quad \text{and} \quad \frac{360°}{n} \leq \alpha. \tag{3.2}$$

The first condition doesn't even depend on the twist angle, and so it sets an absolute condition on the number of sides, which is

$$n \geq 4. \tag{3.3}$$

The second condition sets a bound on the twist angle $\alpha$, namely that it is greater than $360°/n$. Since we have already made the assumption that $\alpha \leq 90°$, we can see that for there to be any possibility of an *mm* (or *vv*) vertex, the range of possible values of $\alpha$ is going to be rather narrow indeed.

## ⋆ 3.5.5. *mv*/*vm* Condition

Now considering the other half of Figure 3.35 for an *mv* or *vm* vertex. According to the BLBA Theorem, either $180°-\alpha$ or $180°-360°/n$ must be the smallest angle. As before, the assumption that $\alpha \leq 90°$ means that the former can't be the smallest, and so we must have

$$180° - \frac{360°}{n} \leq \frac{360°}{n} \quad \text{and} \quad 180° - \frac{360°}{n} \leq \alpha. \tag{3.4}$$

Once again, we have two inequalities. The first sets a limit on the absolute number of sides, which is

$$n \leq 4, \tag{3.5}$$

which is exactly the opposite result from the $mm/vv$ condition, and, in fact, is much more restrictive: there are only two possible values of $n$: 3 and 4.

So, consider the case of an $n = 3$ iso twist. Whether or not it contains any $mv$ or $vm$ vertices, because the number of sides is odd, there must be at least one $mm$ or $vv$ vertex. For that vertex, the $mm/vv$ condition applies, which requires that $n \geq 4$, which is a contradiction. Thus, there is no $n = 3$ regular twist composed of iso pleats.

For the remaining case, $n = 4$, we can have all four types of iso vertex, since $n = 4$ is permitted by both conditions. For all four types of vertex, the twist angle condition becomes

$$\alpha \geq 90°. \tag{3.6}$$

But then, we have already made the assumption that $\alpha \leq 90$. So the only possibility for $n = 4$ is a single twist angle: $\alpha = 90°$. At this twist angle, an $M$ pleat is equivalent to an $m$ pleat, since both angles at the base of the pleat are 90°. An $mvmv$ $\alpha = 90°$ twist is exactly the same as an $MVMV$ $\alpha = 90°$ twist, just with the base angles redefined. So anything that there is to say about an $n = 4$, $\alpha = 90°$ iso pleat assignment can be addressed by considering its anto equivalent.

So, the $mm/vv$ and $mv/vm$ conditions turn out to be quite restrictive. They stipulate the following:

- For $n = 3$, there are no iso pleat assignments.
- For $n = 4$, the only iso pleat assignments have a twist angle of 90°, which will have the same flat-foldability as their anto pleat assignment equivalents.
- For $n \geq 4$, the only possible iso pleat assignments are the fully cyclic ones, $m^n$ and $v^n$, with $\alpha \geq 360°/n$.

We now turn our attention to the all-anto pleat assignments.

## ⋆ 3.5.6. *MM/VV* Condition

Now, let us consider the case of an all-anto pleat assignment around a twist. Figure 3.36 shows an $MM$ and $MV$ crease assignment for the polygon creases at a vertex of a twist polygon. We continue with the assumption that $\alpha \leq 90°$.

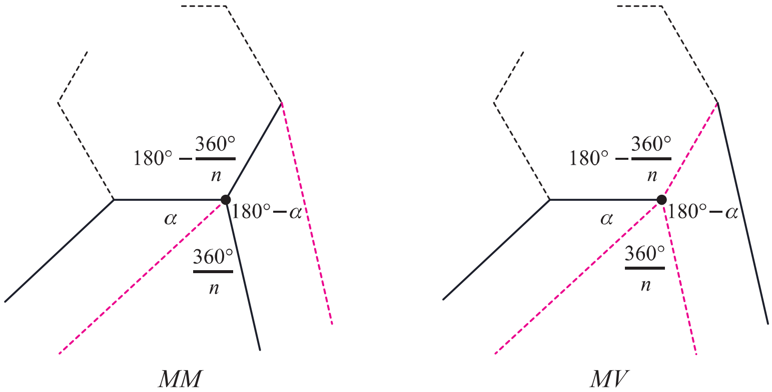



Figure 3.36.
Left: an $MM$ vertex of a polygon twist. Right: an $MV$ vertex of a polygon twist.

Let's look at the case of an $MM$ vertex. Now the two anto sector angles in the figure are the angles marked $\alpha$ and $360°/n$. So one of two things must be true: either (1) $\alpha$ is smaller than $360°/n$, or (2) $360°/n$ is smaller than $\alpha$. In this case, we don't know which angle is the smallest: it depends on the value of $n$. But we can consider both cases.

For case (1), if $\alpha$ is the smaller of the two anto angles, then it must be smaller than the two iso angles as well, and so,

$$\alpha \leq 180° - \frac{360°}{n}. \tag{3.7}$$

For case (2), $360°/n$ is the smaller of the two anto angles, so it must be smaller than the two iso angles as well, and so,

$$\frac{360°}{n} \leq 180° - \alpha, \tag{3.8}$$

which, when rearranged, gives exactly the same equation. Thus, no matter which of the two anto angles is smaller, for an $MM$ vertex, we must have that $\alpha \leq 180° - 360°/n$. And since the same analysis applies if we interchange $M$ and $V$, this is true for either $MM$ or $VV$ vertices.

## ⋆ 3.5.7. *MV*/*VM* Condition

Finally, let's look at the case of an $MV$ vertex. Again, the smallest angle in the figure must be one of the two anto angles, which now are $\alpha$ and the interior angle of the polygon, $180° - 360°/n$. (Of course, the way the figure is drawn, this interior angle doesn't look like the smallest angle, but for small $n$, the interior angles can be small—for $n = 3$, the interior angle is only 60°.)