PHYS 161: Homework 5

Due on Wednesday February 25, 2015

Professor Landee 11:00am

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March 1, 2015

Problem 1

(6.10) *Rings with opposite currents*
Two parallel rings have the same axis and are separated by a small distance ϵ. They have the same radius a, and they carry the same current I but in opposite directions. Consider the magnetic field at points on the axis of the rings. The field is zero midway between the rings, because the contributions from the rings cancel. And the field is zero very far away. So it must reach a maximum value at some point in between. Find this point. Work in the approximation where $\epsilon \ll a$.
Solution
From the equation for field on axis

$$B_z = \frac{\mu_0 I b^2}{2(b^2+z^2)^{3/2}} \tag{6.53}$$

the field for this set up on the z axis would be

$$B_z = \frac{\mu_0 I a^2}{2}\left(\frac{1}{(a^2+(z-\epsilon/2)^2)^{3/2}} - \frac{1}{(a^2+(z+\epsilon/2)^2)^{3/2}}\right)$$

the first order Taylor expansion is

$$B_z \approx \frac{\mu_0 I a^2}{2}\frac{3z\epsilon}{(a^2+z^2)^{5/2}}$$

Now let constant $k = \frac{\mu_0 I a^2}{2}$,

$$\frac{B_z}{k} = \frac{3z\epsilon}{(a^2+z^2)^{5/2}}$$

So take derivative,

$$\frac{d}{dz}\frac{B_z}{k} = \frac{3\epsilon\left(a^2-4z^2\right)}{(a^2+z^2)^{7/2}}$$

and let it equals zero. It yields that $z = a/2$. Put this result into the equation, the maximum value is:

$$\frac{24\epsilon\mu_0 I}{25\sqrt{5}a^2}$$

Problem 2

(6.44) *Line integral along the axis*
Consider the magnetic field of a circular current ring, at points on the axis of the ring, given by Eq. (6.53). Calculate explicitly the line integral of the field along the axis from $-\infty$ to ∞, to check the general formula

$$\int \mathbf{B}\cdot d\mathbf{s} = \mu_0 I \tag{6.97}$$

Why may we ignore the "return" part of the path which would be necessary to complete a closed loop?
Solution
Let us mention the equation

$$B_z = \frac{\mu_0 I b^2}{2(b^2+z^2)^{3/2}} \tag{6.53}$$

So the integral

$$\int \mathbf{B}\cdot d\mathbf{s} = \int_{-\infty}^{\infty}\frac{\mu_0 I b^2}{2(b^2+z^2)^{3/2}}\,dz = \frac{\mu_0 I}{2}\int_{-\infty}^{\infty}\frac{b^2}{(b^2+z^2)^{3/2}}\,dz = \frac{\mu_0 I}{2}\left.\frac{z}{\sqrt{b^2+z^2}}\right|_{-\infty}^{\infty} = \mu_0 I$$

This integral goes from $-\infty$ to ∞, therefore the return part would be infinitesimally small, which could be ignored.

Problem 3

(6.54) *Force between a wire and a loop*
Figure 6.47 shows a horizontal infinite straight wire with current I_1 pointing into the page, passing a height z above a square horizontal loop with side length l and current I_2... (problem omitted)

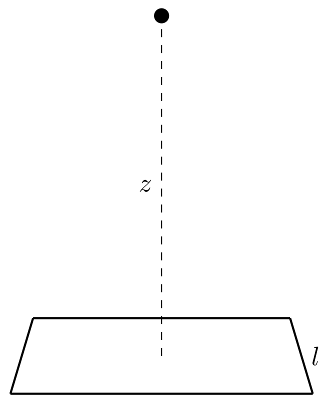
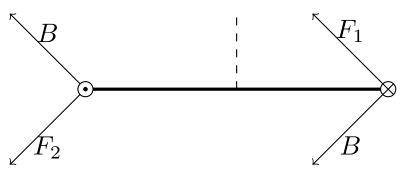


Figure 1: Figure 6.47

Solution
From this front view



the net force $F_1 + F_2$ on the square is indeed to the left. Because the circular magnetic field due to I_1 is

$$B = \frac{\mu_0 I_1}{2\pi R}$$

the force F_1 on the right side is

$$F_1 = I_2 l B = \frac{\mu_0 I_1 I_2 l}{2\pi R}$$

with the angle θ to the x direction, where $\cos\theta = \frac{l/2}{R}$. So its x component is

$$F_{1x} = \frac{\mu_0 I_1 I_2 l}{2\pi R}\frac{l/2}{R} = \frac{\mu_0 I_1 I_2 l^2}{4\pi R^2}$$

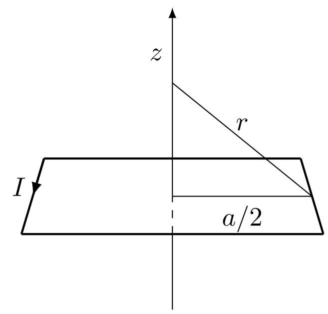
and F_{2x} could be shown as the same. Therefore the net force equals

$$F = F_{1x} + F_{2x} = \frac{\mu_0 I_1 I_2 l^2}{2\pi R^2}$$

Problem 4

The square coil.

Consider a square coil with the length of each side is a. A current I is passed through the ring. Calculate the magnetic field on the vertical axis that passes through the center of the square. Compare this field to that of a circular loop with the same current and a diameter a.
Solution



The Biot-Savart law says that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\mathbf{l}\times\mathbf{r}}{|\mathbf{r}|^3}$$

where $\mathbf{r}$ is the displacement vector. Now consider the right edge. $\mathbf{r} = (0-a/2)\hat{x} + (0-y)\hat{y} + (z-0)\hat{z}$. $d\mathbf{l} = dy\hat{y}$. Therefore $d\mathbf{l}\times\mathbf{r} = \left(z\hat{x} + \frac{a}{2}\hat{z}\right)dy$. So the integral

$$\mathbf{B}_1 = \frac{\mu_0 I}{4\pi}\left(z\hat{x} + \frac{a}{2}\hat{z}\right)\int_{-a/2}^{a/2}\frac{dy}{\left(\frac{a^2}{4} + y^2 + z^2\right)^{3/2}} = \frac{\mu_0 I}{4\pi}\left(z\hat{x} + \frac{a}{2}\hat{z}\right)\frac{4a}{\sqrt{\frac{a^2}{2} + z^2}\,(a^2 + 4z^2)}$$

By symmetry of the configuration, the segment on left would contribute the same $\hat{z}$ component but opposite $\hat{x}$ component. By the same reason, the two top and bottom segments will do the same. Therefore the total field would be

$$\mathbf{B} = 4\times\frac{\mu_0 I}{4\pi}\frac{a}{2}\hat{z}\frac{4a}{\sqrt{\frac{a^2}{2} + z^2}\,(a^2 + 4z^2)} = \frac{2a^2\mu_0 I}{\pi\sqrt{\frac{a^2}{2} + z^2}\,(a^2 + 4z^2)}\hat{z}$$

Comparing to the field on axis for a circular loop which is

$$\mathbf{B}' = \frac{a^2\mu_0 I}{\left(a^2 + z^2\right)^{3/2}}\hat{z}$$

Their ratio

$$\lim_{z\to\infty}\frac{\mathbf{B}}{\mathbf{B}'} = \lim_{z\to\infty}\frac{2\left(a^2 + z^2\right)^{3/2}}{\pi\sqrt{\frac{a^2}{2} + z^2}\,(a^2 + 4z^2)} = \frac{1}{2\pi}$$

Problem 5

(III-8) In the text (pages 82-86) we obtained... (problem omitted)
Solution
For the r coordinate, mark the side on the left going down as 1, and so on.

$$\oint_{C_1}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds \simeq -F_z\left(r, \theta - \frac{\Delta\theta}{2}, z\right)\Delta z$$

$$\oint_{C_3}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds \simeq F_z\left(r, \theta + \frac{\Delta\theta}{2}, z\right)\Delta z$$

And the area enclosed is $r\Delta\theta\Delta z$. So

$$\frac{1}{\Delta S}\int_{C_1+C_3}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds = \frac{1}{r\Delta\theta}\left[F_z\left(r,\theta+\frac{\Delta\theta}{2},z\right)-F_z\left(r,\theta-\frac{\Delta\theta}{2},z\right)\right]$$

Take the limit to $\Delta\theta\to 0$, this becomes

$$\frac{1}{r}\frac{\partial F_z}{\partial\theta}$$

Then

$$\oint_{C_2}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds \simeq F_\theta\left(r,\theta,z-\frac{\Delta z}{2}\right)r\Delta\theta$$

$$\oint_{C_4}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds \simeq -F_\theta\left(r,\theta,z+\frac{\Delta z}{2}\right)r\Delta\theta$$

so

$$\frac{1}{\Delta S}\int_{C_2+C_4}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds = -\frac{1}{\Delta z}\left[F_\theta\left(r,\theta,z+\frac{\Delta z}{2}\right)-F_\theta\left(r,\theta,z-\frac{\Delta z}{2}\right)\right]$$

Take the limit to $\Delta z\to 0$, this becomes

$$-\frac{\partial F_\theta}{\partial z}$$

Therefore we could conclude

$$(\nabla\times\mathbf{F})_r = \frac{1}{r}\frac{\partial F_z}{\partial\theta}-\frac{\partial F_\theta}{\partial z}$$

Similarly for the θ coordinate, mark the side closest to z axis going up as 1, and so on.

$$\oint_{C_1}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds \simeq F_z\left(r-\frac{\Delta r}{2},\theta,z\right)\Delta z$$

$$\oint_{C_3}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds \simeq -F_z\left(r+\frac{\Delta r}{2},\theta,z\right)\Delta z$$

And the area enclosed is $\Delta r\Delta z$. So

$$\frac{1}{\Delta S}\int_{C_1+C_3}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds = -\frac{1}{\Delta r}\left[F_z\left(r+\frac{\Delta r}{2},\theta,z\right)-F_z\left(r-\frac{\Delta r}{2},\theta,z\right)\right]$$

Take the limit to $\Delta r\to 0$, this becomes

$$-\frac{\partial F_z}{\partial r}$$

Then

$$\oint_{C_2}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds \simeq F_r\left(r,\theta,z+\frac{\Delta z}{2}\right)\Delta r$$

$$\oint_{C_4}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds \simeq -F_r\left(r,\theta,z-\frac{\Delta z}{2}\right)\Delta r$$

so

$$\frac{1}{\Delta S}\int_{C_2+C_4}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds = \frac{1}{\Delta z}\left[F_r\left(r,\theta,z+\frac{\Delta z}{2}\right)-F_r\left(r,\theta,z+\frac{\Delta z}{2}\right)\right]$$

Take the limit to $\Delta z\to 0$, this becomes

$$\frac{\partial F_r}{\partial z}$$

Therefore we could conclude

$$(\nabla\times\mathbf{F})_\theta = \frac{\partial F_r}{\partial z}-\frac{\partial F_z}{\partial r}$$

Problem 6

(III-9) Following the procedure... (problem omitted)
Solution
For the r coordinate,

$$\oint_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}\,ds \simeq F_\phi\left(r,\phi,\theta-\frac{\Delta\theta}{2}\right)r\Delta\phi$$

$$\oint_{C_3} \mathbf{F}\cdot\hat{\mathbf{t}}\,ds \simeq -F_\phi\left(r,\phi,\theta+\frac{\Delta\theta}{2}\right)r\Delta\phi$$

And the area enclosed is $r^2\Delta\theta\Delta\phi\sin\phi$. So

$$\frac{1}{\Delta S}\int_{C_1+C_3}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds = -\frac{1}{r\Delta\theta\sin\phi}\left[F_\phi\left(r,\phi,\theta+\frac{\Delta\theta}{2}\right)-F_\phi\left(r,\phi,\theta-\frac{\Delta\theta}{2}\right)\right]$$

Take the limit to $\Delta\theta\to 0$, this becomes

$$-\frac{1}{r\sin\phi}\frac{\partial F_\phi}{\partial\theta}$$

Then

$$\oint_{C_2} \mathbf{F}\cdot\hat{\mathbf{t}}\,ds \simeq -F_\theta\left(r,\phi-\frac{\Delta\phi}{2},\theta\right)r\sin\left(\phi-\frac{\Delta\phi}{2}\right)\Delta\pi$$

$$\oint_{C_4} \mathbf{F}\cdot\hat{\mathbf{t}}\,ds \simeq F_\theta\left(r,\phi+\frac{\Delta\phi}{2},\theta\right)r\sin\left(\phi+\frac{\Delta\phi}{2}\right)\Delta\phi$$

so

$$\frac{1}{\Delta S}\int_{C_2+C_4}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds = \frac{1}{r\Delta\theta\sin\phi}\left[F_\theta\left(r,\phi+\frac{\Delta\phi}{2},\theta\right)\sin\left(\phi+\frac{\Delta\phi}{2}\right)-F_\theta\left(r,\phi-\frac{\Delta\phi}{2},\theta\right)\sin\left(\phi-\frac{\Delta\phi}{2}\right)\right]$$

Take the limit to $\Delta\theta\to 0$, this becomes

$$\frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi F_\theta)$$

Therefore we could conclude

$$(\nabla\times\mathbf{F})_r = \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi F_\theta) - \frac{1}{r\sin\phi}\frac{\partial F_\phi}{\partial\theta}$$

For the ϕ coordinate, enclosed area is $r\Delta r\Delta\theta\sin\phi$

$$\oint_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}\,ds \simeq -F_\theta\left(r+\frac{\Delta r}{2},\phi,\theta\right)\left(r+\frac{\Delta r}{2}\right)\Delta\theta\sin\phi$$

$$\oint_{C_2} \mathbf{F}\cdot\hat{\mathbf{t}}\,ds \simeq -F_r\left(r,\phi,\theta-\frac{\Delta\theta}{2}\right)\Delta r$$

$$\oint_{C_3} \mathbf{F}\cdot\hat{\mathbf{t}}\,ds \simeq F_\theta\left(r-\frac{\Delta r}{2},\phi,\theta\right)\left(r-\frac{\Delta r}{2}\right)\Delta\theta\sin\phi$$

$$\oint_{C_4} \mathbf{F}\cdot\hat{\mathbf{t}}\,ds \simeq F_r\left(r,\phi,\theta+\frac{\Delta\theta}{2}\right)\Delta r$$

so

$$\frac{1}{\Delta S}\int_{C_1+C_3}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds = -\frac{1}{r\Delta r}\left[F_\theta\left(r+\frac{\Delta r}{2},\phi,\theta\right)\left(r+\frac{\Delta r}{2}\right)-F_\theta\left(r-\frac{\Delta r}{2},\phi,\theta\right)\left(r-\frac{\Delta r}{2}\right)\right]$$

$$\frac{1}{\Delta S}\int_{C_2+C_4}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds = \frac{1}{r\Delta\theta\sin\phi}\left[F_r\left(r,\phi,\theta+\frac{\Delta\theta}{2}\right)-F_r\left(r,\phi,\theta-\frac{\Delta\theta}{2}\right)\right]$$

Take the limits, we could conclude

$$(\nabla \times \mathbf{F})_\phi = \frac{1}{r\sin\phi}\frac{\partial F_r}{\partial \theta} - \frac{1}{r}\frac{\partial}{\partial \theta}(rF_\theta)$$

For the θ coordinate, enclosed area is $r\Delta r\Delta\phi$

$$\oint_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}\,ds \simeq F_\phi\left(r+\frac{\Delta r}{2},\phi,\theta\right)\left(r+\frac{\Delta r}{2}\right)\Delta\phi$$

$$\oint_{C_2} \mathbf{F}\cdot\hat{\mathbf{t}}\,ds \simeq -F_r\left(r,\phi-\frac{\Delta\phi}{2},\theta\right)\Delta r$$

$$\oint_{C_3} \mathbf{F}\cdot\hat{\mathbf{t}}\,ds \simeq -F_\phi\left(r-\frac{\Delta r}{2},\phi,\theta\right)\left(r-\frac{\Delta r}{2}\right)\Delta\phi$$

$$\oint_{C_4} \mathbf{F}\cdot\hat{\mathbf{t}}\,ds \simeq F_r\left(r,\phi+\frac{\Delta\phi}{2},\theta\right)\Delta r$$

so

$$\frac{1}{\Delta S}\int_{C_1+C_3}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds = \frac{1}{r\Delta r}\left[F_\phi\left(r+\frac{\Delta r}{2},\phi,\theta\right)\left(r+\frac{\Delta r}{2}\right) - F_\phi\left(r-\frac{\Delta r}{2},\phi,\theta\right)\left(r-\frac{\Delta r}{2}\right)\right]$$

$$\frac{1}{\Delta S}\int_{C_2+C_4}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds = -\frac{1}{r\Delta\phi}\left[F_r\left(r,\phi+\frac{\Delta\phi}{2},\theta\right) - F_r\left(r,\phi-\frac{\Delta\phi}{2},\theta\right)\right]$$

Take the limits, we could conclude

$$(\nabla \times \mathbf{F})_\theta = \frac{1}{r}\frac{\partial}{\partial r}(rF_\phi) - \frac{1}{r}\frac{\partial F_r}{\partial \phi}$$

Problem 7

(III-15) Verify Stokes' theorem

$$\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\,ds = \iint_S \hat{\mathbf{n}}\cdot\nabla\times\mathbf{F}\,dS$$

in each of the following cases... (problem omitted)

Solution

(a)

$$\nabla\times\mathbf{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = 2z\hat{\mathbf{j}}$$

Therefore, at $y=0$

$$\iint_S \hat{\mathbf{n}}\cdot\nabla\times\mathbf{F}\,dS = \int_0^1\int_0^1 \hat{\mathbf{j}}\cdot 2z\hat{\mathbf{j}}\,dx\,dz = \int_0^1\int_0^1 2z\,dx\,dz = 1$$

And

$$\begin{aligned}\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\,ds &= \int_0^1 \mathbf{F}(x,y,z=1)\cdot\hat{\mathbf{i}}\,dx - \int_0^1 \mathbf{F}(x=1,y,z)\cdot\hat{\mathbf{k}}\,dz \\ &- \int_0^1 \mathbf{F}(x,y,z=0)\cdot\hat{\mathbf{i}}\,dx + \int_0^1 \mathbf{F}(x=0,y,z)\cdot\hat{\mathbf{k}}\,dz \\ &= 1-0-0+0 = 1\end{aligned}$$

As shown, Stokes' theorem holds in this case.

Problem 7 continued on next page...

(b)

$$\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\,ds = \int_{C_1}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds + \int_{C_2}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds + \int_{C_3}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds$$

Parametrize the paths.

For C_1, $r=1$, $\theta=\pi/2$, so $\hat{\mathbf{r}}(\phi)=(\cos\phi,\sin\phi,0)$, where $0\le\phi\le\pi/2$. And $\mathbf{f}(\phi)=(\sin\phi,0,\cos\phi)$. The integral becomes

$$\int_{C_1}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds = \int_0^{\pi/2}\mathbf{f}\cdot\hat{\mathbf{r}}'\,d\phi = \int_0^{\pi/2} -\sin^2\phi\,d\phi = -\frac{\pi}{4}$$

For C_2, $r=1$, $\phi=\pi/2$, so $\hat{\mathbf{r}}(\theta)=(0,\cos\theta,\sin\theta)$, where $0\le\theta\le\pi/2$. And $\mathbf{f}(\theta)=(\cos\theta,\sin\theta,0)$. The integral becomes

$$\int_{C_2}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds = \int_0^{\pi/2}\mathbf{f}\cdot\hat{\mathbf{r}}'\,d\theta = \int_0^{\pi/2} -\sin^2\theta\,d\theta = -\frac{\pi}{4}$$

For C_3, $r=1$, $\phi=0$, so $\hat{\mathbf{r}}(\theta)=(\cos\theta,0,\sin\theta)$, where $0\le\theta\le\pi/2$. And $\mathbf{f}(\theta)=(0,\sin\theta,\cos\theta)$. The integral becomes

$$\int_{C_3}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds = \int_0^{\pi/2}\mathbf{f}\cdot\hat{\mathbf{r}}'\,d\theta = \int_0^{\pi/2} -\cos^2\theta\,d\theta = -\frac{\pi}{4}$$

So adding these up,

$$\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\,ds = -\frac{3\pi}{4}$$

$\nabla\times\mathbf{F}=(-1,-1,-1)$. Change into spherical coordinates, $\nabla\times\mathbf{F}=\left(\sqrt{3},\pi-\tan^{-1}\left(\sqrt{2}\right),-\frac{3\pi}{4}\right)$. The normal vector $\hat{\mathbf{n}}=x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}$, therefore $\hat{\mathbf{n}}\cdot\nabla\times\mathbf{F}=-x-y-z$. So when $r=1$

$$\begin{aligned}\iint_S \hat{\mathbf{n}}\cdot\nabla\times\mathbf{F}\,dS &= \iint_S(-x-y-z)\,dS\\ &= -\int_0^{\pi/2}\int_0^{\pi/2}(x+y+z)r^2\sin\theta\,d\phi\,d\theta\\ &= -\int_0^{\pi/2}\int_0^{\pi/2}(\sin\phi\sin\theta+\cos\phi\sin\theta+\cos\theta)\sin\theta\,d\phi\,d\theta\\ &= -\frac{3\pi}{4}\end{aligned}$$

Stokes' theorem holds in this case.

(c) $\nabla\times\mathbf{F}=(0,0,-2)$, so in cylindrical coordinates

$$\begin{aligned}\iint_S \hat{\mathbf{n}}\cdot\nabla\times\mathbf{F}\,dS &= \iint_S \hat{\mathbf{n}}\cdot(0,0,-2)\,dS\\ &= \iint_{S_1}\hat{\rho}\cdot\nabla\times\mathbf{F}\,dS + \iint_{S_2}\hat{\mathbf{z}}\cdot\nabla\times\mathbf{F}\,dS\\ &= 0+\int_0^R\int_0^{2\pi} -2\rho\,d\phi\,d\rho\\ &= -2\pi R^2\end{aligned}$$

And then for the line integral, parametrize the path $\hat{\mathbf{r}}(\phi)=(R\cos\phi,R\sin\phi,0)$. And $\mathbf{f}=(R\sin\phi,-R\cos\phi,0)$. So

$$\int_C \mathbf{F}\cdot\hat{\mathbf{t}}\,ds = \int_0^{2\pi}\mathbf{f}\cdot\hat{\mathbf{r}}'\,d\phi = \int_0^{2\pi} -R^2\,d\phi = -2\pi R^2$$

Verified.

Problem 8

(III-16) (a) Consider a vector function with the property $\nabla \times \mathbf{F} = 0$... (problem omitted)
Solution

(a) Use Stokes' theorem. Because

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{t}}\, ds = \iint_S \hat{\mathbf{n}} \cdot \nabla \times \mathbf{F}\, dS = 0$$

on this dount shape, the upper and lower halves could be seen as having circulation of opposite directions, while C_1 or C_2 belongs to one of them.

$$\oint_{C_1} \mathbf{F} \cdot \hat{\mathbf{t}}\, ds - \oint_{C_2} \mathbf{F} \cdot \hat{\mathbf{t}}\, ds = 0$$

So

$$\oint_{C_1} \mathbf{F} \cdot \hat{\mathbf{t}}\, ds = \oint_{C_2} \mathbf{F} \cdot \hat{\mathbf{t}}\, ds$$

(b) When assuming $r > 0$, the curl

$$\nabla \times \mathbf{B} = \nabla \times \left(0, \frac{\mu_0 I}{2\pi r}, 0\right) = \left(0, 0, \frac{\mu_0 I}{2\pi r^2} - \frac{\mu_0 I}{2\pi r^2}\right) = (0, 0, 0)$$

When $r = 0$, the calculation becomes indeterminate.

(c) Let

$$a = \oint_C \mathbf{B} \cdot\, d\mathbf{l}$$

a small element of this is

$$da = \mathbf{B} \cdot\, d\mathbf{r} = B \cos\theta\, dr$$

When considering a cicular loop centered at z axis, $\theta = 0$,

$$B \cos\theta\, dr = B\, dr = \frac{\mu_0 I}{2\pi r}\, dr$$

So

$$\oint_C \mathbf{B} \cdot\, d\mathbf{l} = a = 2\pi r \frac{\mu_0 I}{2\pi r} = \mu_0 I$$

which is the Ampère's circuital law.

Problem 9

(III-19) Determine the value of the line integral $\int_C \mathbf{F} \cdot \hat{\mathbf{t}}, ds$ where

$$\mathbf{F} = (e^{-y} - ze^{-x})\hat{\mathbf{i}} + (e^{-z} - xe^{-y})\hat{\mathbf{j}} + (e^{-x} - ye^{-z})\hat{\mathbf{k}}$$

and C is the path... (problem omitted)
Solution
Firstly, calculated $\nabla \times \mathbf{F} = 0$. As shown in problem 8, for a field of zero curl, two closed curves enclosing one capping surface would have the same circulation. So the path C could be converted to $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, and $(1, 1, 1)$, and still gives the same result. Therefore at $y = z = 0$

$$\int_0^1 \left(e^{-y} - ze^{-x}\right)\, dx = 1$$

Problem 9 continued on next page...

At $x = 1$, $z = 0$

$$\int_0^1 \left(e^{-z} - xe^{-y}\right) \, dy = \frac{1}{e}$$

At $x = y = 1$,

$$\int_0^1 \left(e^{-x} - ye^{-z}\right) \, dz = -1 + \frac{2}{e}$$

So the line integral is the sum of these three results, $3/e$.

Problem 10

(III-22) (a) Apply the divergence theorem to the function

$$\mathbf{G}(x, y) = \mathbf{i}G_x(x, y) + \mathbf{j}G_y(x, y)$$

... (problem omitted)

Solution

(a) There is no z component, so

$$\nabla \cdot \mathbf{G} = \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y}$$

Therefore

$$\iiint_V \nabla \cdot \mathbf{G} \, dV = \iiint_V \left(\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y}\right) dV = h \iint_R \left(\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y}\right) dx \, dy$$

On the other hand, the top and bottom has no contribution to the surface integral. The surface integral does not change through the h, so

$$\begin{aligned}\iint_S \mathbf{G} \cdot \hat{\mathbf{n}} \, dS &= h \oint_C \mathbf{G} \cdot \hat{\mathbf{n}} \, dx \, dy \\ &= h \oint_C \left(\frac{\partial G}{\partial x} - \frac{\partial G}{\partial y}\right) dx \, dy \\ &= h \oint_C \left(\frac{G_x}{dx} - \frac{G_y}{dy}\right) dx \, dy \\ &= h \oint_C (G_x \, dy - G_y \, dx)\end{aligned}$$

So

$$\iint_R \left(\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y}\right) dx \, dy = \oint_C (G_x \, dy - G_y \, dx)$$

(b)

$$\nabla \times \mathbf{F} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$$

So

$$\iint_R \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_R \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dx \, dy$$

And

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (F_x \, dx + F_y \, dy)$$

So

$$\oint_C (F_x \, dx + F_y \, dy) = \iint_R \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dx \, dy$$

(c) With the result from the two problems above, just let $G_x = F_y$, and $G_y = -F_x$.