PHYS 161: Homework 9

Due on Wednesday April 15, 2015

Professor Landee 11:00am

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Problem 1

(9.1) *The missing term*

Due to the contradiction between Eqs. (9.2) and (9.5), we know that there must be an extra term in the $\nabla \times \mathbf{B}$ relation, as we found in Eq. (9.10). Call this term $\mathbf{W}$. In the text, we used the Lorentz transformations to motivate a guess for $\mathbf{W}$. Find $\mathbf{W}$ here by taking the divergence of both sides of $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mathbf{W}$. Assume that the only facts you are allowed to work with are (1) $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$, (2) $\nabla \cdot \mathbf{B} = 0$, (3) $\nabla \cdot \mathbf{J} = -\partial\rho/\partial t$, and (4) $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ in the case of steady currents.

Solution

$$\begin{aligned}
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mathbf{W} \\
\nabla \cdot (\nabla \times \mathbf{B}) &= \nabla \cdot (\mu_0 \mathbf{J} + \mathbf{W}) \\
0 &= \mu_0 \nabla \cdot \mathbf{J} + \nabla \cdot \mathbf{W} \\
\nabla \cdot \mathbf{W} &= -\mu_0 \nabla \cdot \mathbf{J} \\
\nabla \cdot \mathbf{W} &= \mu_0 \frac{\partial \rho}{\partial t} \\
\nabla \cdot \mathbf{W} &= \mu_0 \epsilon_0 \frac{\partial}{\partial t} \nabla \cdot \mathbf{E} \\
\mathbf{W} &= \mu_0 \epsilon_0 \frac{\partial}{\partial t} \mathbf{E}
\end{aligned}$$

Problem 2

(9.3) *A charge and a half-infinite wire*

A half-infinite wire carries current I from negative infinity to the origin, where it builds up at a point charge with increasing q (so $dq/dt = I$). Consider the circle shown in Fig. 1, which has radius b and subtends an angle 2θ with respect to the charge. Calculate the integral $\int \mathbf{B} \cdot d\mathbf{s}$ around this circle. Do this in three ways.

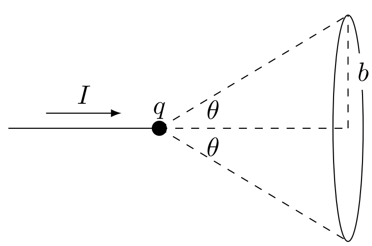

Figure 1

(a) Find the $\mathbf{B}$ field at a given point on the circle by using the Biot-Savart law to add up the contributions from the different parts of the wire.

(b) Use the integrated form of Maxwell's equation (that is, the generalized form of Ampère's law including the displacement current),

$$\int_C \mathbf{B} \cdot d\mathbf{s} = \mu_0 I + \mu_0 \epsilon_0 \int_S \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{a} \tag{9.59}$$

with S chosen to be a surface that is bounded by the circle and doesn't intersect the wire, but is otherwise arbitrary. (You can invoke the result from Problem 1.15.)

(c) Use the same strategy as in (b), but now let S intersect the wire.

Solution

(a) For a point P on the circle, let the angle between verticle direction and the line connecting P and points on the wire be ϕ. By BSL

$$B = \frac{\mu_0 I}{4\pi} \int_{\theta}^{\pi/2} \frac{\cos\phi}{r^2}\, dl = \frac{\mu_0 I}{4\pi b}(1-\cos\theta)$$

tangential to the circle. So

$$\int_C \mathbf{B}\cdot d\mathbf{s} = 2\pi b \frac{\mu_0 I}{4\pi b}(1-\cos\theta) = \frac{\mu_0 I}{2}(1-\cos\theta)$$

(b) $\mu_0 I = 0$ becuase the surface chosen does not intersect with the wire. From Problem 1.15, flux of field through the circle is

$$\Phi_E = \frac{q}{2\epsilon_0}(1-\cos\theta)$$

so

$$\mu_0\epsilon_0 \int_S \frac{\partial \mathbf{E}}{\partial t}\cdot d\mathbf{a} = \mu_0\epsilon_0 \frac{\partial}{\partial t}\int_S \mathbf{E}\cdot d\mathbf{a} = \mu_0\epsilon_0 \frac{\partial \Phi_E}{\partial t} = \frac{\mu_0 I}{2}(1-\cos\theta)$$

(c) Include $\mu_0 I$ now. Consider a cone-like shape that encloses the charge. The base would be the surface examined in part (b), and the side will be the one we are about to look at right now. ${\Phi_E}' = q/\epsilon_0 - \Phi_E$. So

$$\mu_0 I + \mu_0\epsilon_0 \int_S \frac{\partial \mathbf{E}}{\partial t}\cdot d\mathbf{a} = \mu_0 I + \mu_0\epsilon_0 \frac{\partial {\Phi_E}'}{\partial t} = \frac{\mu_0 I}{2}(1-\cos\theta)$$

Problem 3

(9.5) *Maxwell's equations for a moving charge*... (problem omitted)
Solution

$$E' = \frac{Q}{4\pi\epsilon_0 r'^2}\frac{1-\beta^2}{\left(1-\beta^2\sin^2\theta'\right)^{3/2}} \tag{5.13}$$

(a) So

$$\begin{aligned}\nabla\cdot\mathbf{B} &= \frac{1}{c^2}\nabla\cdot(\mathbf{v}\times\mathbf{E}) \\ &= \frac{1}{c^2}\left[\mathbf{E}\cdot(\nabla\times\mathbf{v}) - \mathbf{v}\cdot(\nabla\times\mathbf{E})\right]\end{aligned}$$

And $\nabla\times\mathbf{v} = 0$ because $\mathbf{v}$ is constant. $\nabla\times\mathbf{E}$ is perpendicular to $\mathbf{v}$, making their dot products zero as well. So $\nabla\cdot\mathbf{B} = 0$.

(b) From 5.13, with

$$D = (\gamma x)^2 + y^2 + z^2$$

the field becomes

$$\mathbf{E} = \frac{\gamma Q}{4\pi\epsilon_0 D^{3/2}}(x, y, z)$$

So

$$(\nabla\times\mathbf{E})_y = \frac{\gamma Q}{4\pi\epsilon_0}\left(\frac{-3xz}{D^{5/2}} + \frac{3\gamma^2 xz}{D^{5/2}}\right) = \frac{\gamma Q}{4\pi\epsilon_0}\frac{3\gamma^2 v^2 xz}{c^2 D^{5/2}}\hat{\mathbf{y}}$$

and the other components are zero.

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{1}{c^2}\frac{\partial}{\partial t}(\mathbf{v} \times \mathbf{E}) = \frac{1}{c^2}\mathbf{v} \times \frac{\partial \mathbf{E}}{\partial t}$$

Compute

$$\frac{\partial E_z}{\partial t} = \frac{\gamma Q}{4\pi\epsilon_0}\frac{3\gamma^2 xzv}{D^{5/2}}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{v}{c^2}\hat{\mathbf{x}} \times \frac{\gamma Q}{4\pi\epsilon_0}\frac{3\gamma^2 xzv}{D^{5/2}}\hat{\mathbf{z}} = -\nabla \times \mathbf{E}$$

Problem 4

(9.10) *Energy flow from a wire...* (problem omitted)
Solution
For the wire being considered as a tube, its magnetic field is

$$B = \frac{\mu_0 I}{2\pi b}$$

tangential to the wire. On the other hand for the sphere, its E field is radial

$$E = \frac{Q}{4\pi\epsilon_0 r^2}$$

So Poynting vector

$$S = \frac{EB}{\mu_0} = \frac{QI}{4\pi\epsilon_0 r^2 \times 2\pi b}$$

Integrate this over the surface of the tube, on which $da = 2\pi b\, dr$

$$\int S\, da = \int_R^\infty \frac{QI}{4\pi\epsilon_0 r^2 \times 2\pi b} 2\pi b\, dr = \frac{d}{dt}\frac{Q^2}{8\pi\epsilon_0 R}$$

To see energy stored in the electric field of the sphere, constant $\phi = Q/4\pi\epsilon_0 R$

$$U = \frac{1}{2}\int \rho\phi\, dv = \frac{1}{2}\phi\int \rho\, dv = \frac{1}{2}\frac{Q}{4\pi\epsilon_0 R}Q = \frac{Q^2}{8\pi\epsilon_0 R}$$

Therefore energy flow is equal to the energy stored.

Problem 5

(9.14) *Sphere with a hole...* (problem omitted)
Solution
Integral over the circumference C is

$$\int_C \mathbf{B} \cdot d\mathbf{s} = \mu_0 I$$

And integral over the surface S is

$$\int_S \left(\mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}\right) \cdot d\mathbf{a}$$

but the wire goes through the hole, so $\mathbf{J} = 0$. Therefore, becuase the charge $Q(t) = It$, and $E = \frac{Q}{4\pi\epsilon_0 r^2}$,

$$\mu_0\epsilon_0 \frac{\partial}{\partial t}\int_S \mathbf{E} \cdot d\mathbf{a} = \mu_0\epsilon_0 \frac{\partial}{\partial t}\left(\frac{It}{4\pi\epsilon_0 r^2} \cdot 4\pi r^2\right) = \mu_0 I$$

Therefore the integral form of Maxwell's equation is satisfied.

Problem 6

(9.15) *Field inside a discharging capacitor*… (problem omitted)
Solution
Imagine a circle C centered with the direction of the wire, with radius r, so that P lies on the circle. If the magnetic field B is constant,

$$\int_C \mathbf{B} \cdot d\mathbf{s} = 2\pi r B$$

On the other hand, no current passes through the circle, for which $\mathbf{J} = 0$. $\mathbf{E} = \frac{\sigma}{\epsilon_0} = \frac{Q}{\pi b^2 \epsilon_0}$. $a = 2\pi b^2$.

$$\int_S \left(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J} \right) \cdot d\mathbf{a} = \mu_0 \epsilon_0 \int_S \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{a} = \mu_0 \epsilon_0 \int_S \frac{\mathbf{I}}{\pi b^2 \epsilon_0} \cdot d\mathbf{a} = \frac{\mu_0 r^2 I}{b^2}$$

Therefore $2\pi r B = \frac{\mu_0 r^2 I}{b^2}$, or $B = \frac{\mu_0 I r}{2\pi b^2}$.

Problem 7

(9.20) *Kicked by a wave*… (problem omitted)
Solution

$$\mathbf{E} = \frac{E_0 \hat{\mathbf{y}}}{1 + \frac{(x+ct)^2}{l^2}}, \quad \mathbf{B} = \frac{-(E_0/c)\hat{\mathbf{z}}}{1 + \frac{(x+ct)^2}{l^2}} \tag{9.28}$$

The change in momentum, when x as well as B is negligible

$$\Delta \mathbf{p} = \int \mathbf{F} \, dt = e\hat{\mathbf{y}} \times 100 \, \mathrm{kV/m} \int_{-\infty}^{\infty} \frac{dt}{1 + \frac{(ct)^2}{l^2}} = \frac{e\pi l}{c} \hat{\mathbf{y}} \times 100 \, \mathrm{kV/m}$$

So velocity

$$v = \frac{p}{m} = \frac{e\pi l}{mc} \hat{\mathbf{y}} \times 100 \, \mathrm{kV/m}$$

So position

$$s = vt = \frac{e\pi l}{mc} \hat{\mathbf{y}} \times 1 \times 10^5 \, \mathrm{V/m} \times 1 \times 10^{-6} \, \mathrm{s} \approx 0.10 l \hat{\mathbf{y}}$$

Problem 8

(9.22) *Plane-wave pulse*… (problem omitted)
Solution

(a) Use Ampère's law

$$\oint \mathbf{B} \cdot d\mathbf{l} = \int_S \left(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J} \right) \cdot d\mathbf{a}$$

Only the section inside the slab contributes to the integral

$$\oint \mathbf{B} \cdot d\mathbf{l} = Bl$$

and

$$\int_S \left(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J} \right) \cdot d\mathbf{a} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \int_S \mathbf{E} \cdot d\mathbf{a} = \mu_0 \epsilon_0 \frac{\partial \Phi_E}{\partial t} = \mu_0 \epsilon_0 v E l$$

So $c^2 B = vE$.

Problem 8 continued on next page…

(b) Use Faraday's law

$$\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt}$$

This time let the loop be into the page, perpendicular to $\mathbf{B}$. Normal vector points upward.

$$-\frac{d\Phi_B}{dt} = Bvl$$

$$\oint \mathbf{E} \cdot d\mathbf{s} = El$$

So $E = Bv$

After all $v = c$.

Problem 9

(9.30) *Comparing the energy densities*
Consider the capacitor example in Section 9.6.2, but now let the current change in a way that makes the electric field inside the capacitor take the form of $E(t) = E_0 \cos \omega t$. The induced magnetic field is given in Eq. (9.46). Show that the energy density of the magnetic field is much smaller than the energy density of the electric field, provided that the time scale of ω (namely $2\pi/\omega$) is much longer than the time it takes light to travel across the diameter of the capacitor disks. (As in Problem 9.6, we are ignoring higher-order effects.)
Solution

$$B = \frac{\mu_0 \epsilon_0 r}{2} \frac{\partial E}{\partial t} = -\frac{\mu_0 \epsilon_0 r}{2} E_0 \sin \omega t$$

Energy density of the fields

$$U_E = \frac{\epsilon_0 E^2}{2} = \frac{\epsilon_0 {E_0}^2 \cos^2 \omega t}{2}$$

$$U_B = \frac{B^2}{2\mu_0} = \frac{{\mu_0}^2 {\epsilon_0}^2 r^2 {E_0}^2 \sin^2 \omega t}{4} \frac{1}{2\mu_0} = \frac{\mu_0 {\epsilon_0}^2 r^2 {E_0}^2 \sin^2 \omega t}{4}$$

So

$$U_B/U_E = \frac{\mu_0 \epsilon_0 r^2 \sin^2 \omega t}{2 \cos^2 \omega t} = \frac{\mu_0 \epsilon_0 r^2}{2} \tan^2 \omega t = \frac{r^2}{2c^2} \tan^2 \omega t$$

So the energy density of the magnetic field is much smaller than the energy density of the electric field.

Problem 10

(9.32) *A Lorentz invariant*
Starting from the field transformation given by Eq. (6.76), show that the scalar quantity $E^2 - c^2B^2$ is invariant under the transformation. In other words, show that ${E'}^2 - c^2{B'}^2 = E^2 - c^2B^2$. You can do this using only vector algebra, without writing out x, y, z components of anything. (The resolution into parallel and perpendicular vectors is convenient for this, since $\mathbf{E}_\perp \cdot \mathbf{E}_\parallel = 0$, $\mathbf{B}_\parallel \times \mathbf{E}_\parallel = 0$, etc.)
Solution
Use the notation

$$\mathbf{E} = \mathbf{E}_\parallel + \mathbf{E}_\perp, \quad \mathbf{E}' = \mathbf{E}'_\parallel + \mathbf{E}'_\perp$$

and the same applies to $\mathbf{B}$. Remeber that

$$\gamma^2 = \frac{1}{1 - v^2/c^2}$$

Problem 10 continued on next page...

and

$$(\mathbf{a} \times \mathbf{b})\cdot(\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

So:

$$\begin{aligned}
E'^2 - c^2 B'^2 &= {\mathbf{E}_\parallel}'^2 + {\mathbf{E}_\perp}'^2 - c^2 \left({\mathbf{B}_\parallel}'^2 + {\mathbf{B}_\perp}'^2\right) \\
&= {\mathbf{E}_\parallel}^2 + \gamma^2 \left(\mathbf{E}_\perp + \mathbf{v} \times \mathbf{B}_\perp\right)^2 - c^2 \left[{\mathbf{B}_\parallel}^2 + \gamma^2 \left(\mathbf{B}_\perp - \frac{\mathbf{v}}{c^2} \times \mathbf{E}_\perp\right)^2\right] \\
&= {\mathbf{E}_\parallel}^2 + \gamma^2 \left[{\mathbf{E}_\perp}^2 + 0 + (\mathbf{v} \times \mathbf{B}_\perp)^2\right] - c^2 \left\{{\mathbf{B}_\parallel}^2 + \gamma^2 \left[{\mathbf{B}_\perp}^2 + 0 + \left(\frac{\mathbf{v}}{c^2} \times \mathbf{E}_\perp\right)^2\right]\right\} \\
&= {\mathbf{E}_\parallel}^2 + \gamma^2 \left[{\mathbf{E}_\perp}^2 + \left(\mathbf{v}^2 {\mathbf{B}_\perp}^2 - 0\right)^2\right] - c^2 \left\{{\mathbf{B}_\parallel}^2 + \gamma^2 \left[{\mathbf{B}_\perp}^2 + \left(\frac{\mathbf{v}^2}{c^4} {\mathbf{E}_\perp}^2 - 0\right)^2\right]\right\} \\
&= {\mathbf{E}_\parallel}^2 + \left(\gamma^2 - \frac{\gamma^2 v^2}{c^2}\right) {\mathbf{E}_\perp}^2 - c^2 \left[{\mathbf{B}_\parallel}^2 + \left(\gamma^2 - \frac{\gamma^2 v^2}{c^2}\right) {\mathbf{B}_\perp}^2\right] \\
&= {\mathbf{E}_\parallel}^2 + {\mathbf{E}_\perp}^2 - c^2 \left({\mathbf{B}_\parallel}^2 + {\mathbf{B}_\perp}^2\right) \\
&= E^2 - c^2 B^2
\end{aligned}$$