Physics 161, spring 2015

Key to HW # 5

For problems related to the curl, Schey's chapter III between pages 75 and 104 is an excellent reference. P&M's treatment is in chapter 2, sections 2.14 and 2.17.

1. P&M 6.10. Rings with opposite currents (Taylor series practice)
   Solution found in PM chapter 12.
2. P&M. 6.44 Line integral along the axis.
3. P&M. 6.54 Force between a wire and a loop.
4. The square coil.
   Consider a square coil with the length of each side is a. A current I is passed through the ring. Calculate the magnetic field on the vertical axis that passes through the center of the square. Compare this field to that of a circular loop with the same current and a diameter a.

5. Schey III-8., **6.** Schey III-9, **7.** Schey III-15
8. Schey III-16, **9.** Schey III-19, **10.** Schey III-22

6.44 Line integral along the axis

The magnetic field on the axis is $B_z = \mu_0 I b^2/2(b^2+z^2)^{3/2}$, so the given line integral is (using the integral table in Appendix K)

$$\int_{-\infty}^{\infty} B_z\,dz = \frac{\mu_0 I b^2}{2}\int_{-\infty}^{\infty}\frac{dz}{(b^2+z^2)^{3/2}} = \frac{\mu_0 I b^2}{2}\,\frac{z}{b^2(b^2+z^2)^{1/2}}\bigg|_{-\infty}^{\infty} = \frac{\mu_0 I b^2}{2}\,\frac{2}{b^2} = \mu_0 I, \tag{449}$$

as desired. If you want, you can derive this integral with a trig substitution, $z = b\tan\theta$.

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To see why the integral along the axis should indeed be equal to $\mu_0 I$, consider the closed path shown in Fig. 110, which involves a semicircle touching the points $z = \pm r$. Assume that $r \gg b$. Along the z axis, B_z behaves like $1/z^3$ for $z \gg b$. And $|\mathbf{B}|$ also behaves like $1/r^3$ along the (large) semicircle. Accepting that this is true (see below), then since the length of the semicircle is proportional to r, the line integral along the semicircle is at least as small (in order of magnitude) as $r/r^3 = 1/r^2$, which goes to zero as $r \to \infty$. We can therefore ignore the return semicircular path. So the line integral along the whole loop (which encloses a current I) equals the line integral along the z axis, in the $r \to \infty$ limit.

Let's now argue why $|\mathbf{B}|$ behaves like $1/r^3$ for large r. Consider the point at the "side" of the semicircle in Fig. 110. In order of magnitude, the field at this point, due to the ring, is the same as the field due to a square with side b. But the field due to the square has contributions from two opposite sides (the sides perpendicular to the $\hat{\mathbf{r}}$ vector) that nearly cancel, because the current moves in opposite directions along these sides. The Biot-Savart law says that each side gives a contribution of order $1/r^2$. Taking the difference of these contributions is essentially the same as taking a derivative, and the derivative of $1/r^2$ is proportional to $1/r^3$, as desired. Additionally, the two sides parallel to the $\hat{\mathbf{r}}$ vector also happen to produce a contribution of order $1/r^3$; see Problem 6.14. At points in between the axis and the "side" point on the semicircle, there will be various angles that come into play. But these simply bring in factors of order 1 that morph the $1/z^3$ result on the axis to the $1/r^3$ result at the side point, so they don't change the general $1/r^3$ result.

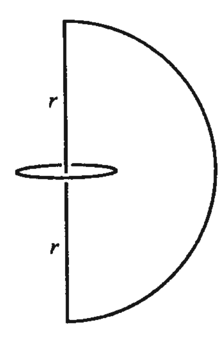


Figure 110

Key to HW #5

#3

6.54. Force between a wire and a loop

Consider a little segment in the right-hand side of the square. The current points into the page, and the magnetic field due to the infinite straight wire has magnitude $B_1 = \mu_0 I_1/2\pi R$ and points down to the left, as shown in Fig. 118. From the right-hand rule, the force $q\mathbf{v} \times \mathbf{B}$ on the charges in the current points up to the left, as shown (toward the infinite wire; parallel currents attract). In the left-hand side of the square, the current points out of the page, and the magnetic field due to the infinite wire points up to the left, as shown. The force $q\mathbf{v}\times\mathbf{B}$ on the charges in the current now points down to the left (away from the infinite wire; antiparallel currents repel). The vertical components of the preceding two forces cancel, but the leftward components add. So the net force is leftward, as desired. You can quickly show that the net force on each of the other two sides of the square is zero.

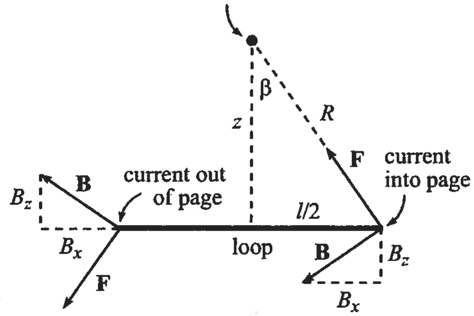


Figure 118

In short, it is the vertical component of **B** that matters, because this component changes sign from the right half to the left half of the square. And the direction of the square's current into and out of the page also changes sign. So these two negative signs cancel in $q\mathbf{v} \times \mathbf{B}$, yielding a net leftward force. In contrast, the horizontal component of **B** does *not* change sign, so the negation of the current causes a negation of the vertical force. The net vertical force is therefore zero.

Quantitatively, the general form of the force on a wire is $F = IB\ell$. The "B" we are concerned with here is the vertical component, which is $B \sin\beta$. The force comes from two sides, so the total horizontal force is

$$F = 2I_2(B_1 \sin\beta)\ell = 2I_2\left(\frac{\mu_0 I_1}{2\pi R}\cdot\frac{\ell/2}{R}\right)\ell = \frac{\mu_0 I_1 I_2 \ell^2}{2\pi R^2}, \tag{458}$$

where we have used $\sin\beta = (\ell/2)/R$.

The above reasoning shows where the two factors of R in the denominator come from. One comes from the distance to the wire, and the other comes from the fact that the **B** field becomes more horizontal (which means that the vertical component decreases) as R gets large.

We weren't concerned with torques in this exercise, but from looking at the vertical forces on the left and right sides (which come from the horizontal component of **B**), it is clear that there is a torque on the square. It will rotate counterclockwise when viewed from the side. This is consistent with conservation of angular momentum, because the straight wire will gain angular momentum (relative to, say, an origin chosen to be the center of the square) as it moves to the right. This angular momentum will have a clockwise sense, consistent with the fact that the total angular momentum of the system remains constant.

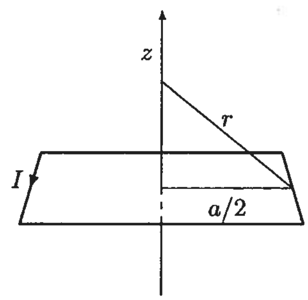
10)

Solution due to Colin Tan

#4

Consider a square coil with the length of each side is a. A current I is passed through the ring. Calculate the magnetic field on the vertical axis that passes through the center of the square. Compare this field to that of a circular loop with the same current and a diameter a.

Solution



The Biot-Savart law says that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l} \times \mathbf{r}}{|\mathbf{r}|^3}$$

where $\mathbf{r}$ is the displacement vector. Now consider the right edge. $\mathbf{r} = (0 - a/2)\hat{x} + (0 - y)\hat{y} + (z - 0)\hat{z}$. $d\mathbf{l} = dy\hat{y}$. Therefore $d\mathbf{l} \times \mathbf{r} = \left(z\hat{x} + \frac{a}{2}\hat{z}\right) dy$. So the integral

$$\mathbf{B}_1 = \frac{\mu_0 I}{4\pi}\left(z\hat{x} + \frac{a}{2}\hat{z}\right)\int_{-a/2}^{a/2} \frac{dy}{\left(\frac{a^2}{4} + y^2 + z^2\right)^{3/2}} = \frac{\mu_0 I}{4\pi}\left(z\hat{x} + \frac{a}{2}\hat{z}\right)\frac{4a}{\sqrt{\frac{a^2}{2} + z^2}\,(a^2 + 4z^2)}$$

By symmetry of the configuration, the segment on left would contribute the same $\hat{z}$ component but opposite $\hat{x}$ component. By the same reason, the two top and bottom segments will do the same. Therefore the total field would be

$$\mathbf{B} = 4 \times \frac{\mu_0 I}{4\pi}\frac{a}{2}\hat{z}\frac{4a}{\sqrt{\frac{a^2}{2} + z^2}\,(a^2 + 4z^2)} = \frac{2a^2\mu_0 I}{\pi\sqrt{\frac{a^2}{2} + z^2}\,(a^2 + 4z^2)}\hat{z}$$

Comparing to the field on axis for a circular loop which is

$$\mathbf{B}' = \frac{a^2\mu_0 I}{(a^2 + z^2)^{3/2}}\hat{z}$$

Their ratio

$$\lim_{z\to\infty}\frac{\mathbf{B}}{\mathbf{B}'} = \lim_{z\to\infty}\frac{2\left(a^2 + z^2\right)^{3/2}}{\pi\sqrt{\frac{a^2}{2} + z^2}\,(a^2 + 4z^2)} = \frac{1}{2\pi}$$

Problem 5

(III-8) In the text (pages 82-86) we obtained... (problem omitted)

Solution

For the r coordinate, mark the side on the left going down as 1, and so on.

$$\oint_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}\,ds \simeq -F_z\left(r, \theta - \frac{\Delta\theta}{2}, z\right)\Delta z$$

$$\oint_{C_3} \mathbf{F}\cdot\hat{\mathbf{t}}\,ds \simeq F_z\left(r, \theta + \frac{\Delta\theta}{2}, z\right)\Delta z$$

Problem 5 continued on next page...

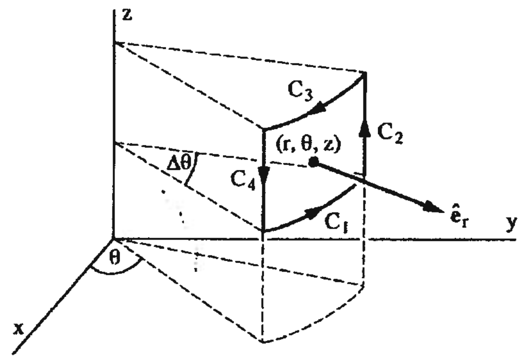
#5 8. To get the radial component of the curl we proceed as follows.

$$\int_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds \cong F_\theta\left(r,\theta,z - \frac{\Delta z}{2}\right)r\Delta\theta,$$

$$\int_{C_2} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds \cong F_z\left(r,\theta + \frac{\Delta\theta}{2},z\right)\Delta z,$$

$$\int_{C_3} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds \cong -F_\theta\left(r,\theta,z + \frac{\Delta z}{2}\right)r\Delta\theta,$$

$$\int_{C_4} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds \cong -F_z\left(r,\theta - \frac{\Delta\theta}{2},z\right)\Delta z.$$



Adding these four integrals and dividing by $\Delta A = r\Delta\theta\Delta z$ we get

$$\frac{1}{\Delta A}\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\, ds \cong \frac{F_z\left(r,\theta + \frac{\Delta\theta}{2},z\right) - F_z\left(r,\theta - \frac{\Delta\theta}{2},z\right)}{r\Delta\theta} - \frac{F_\theta\left(r,\theta,z + \frac{\Delta z}{2}\right) - F_\theta\left(r,\theta,z - \frac{\Delta z}{2}\right)}{\Delta z}$$

$$\to \frac{1}{r}\frac{\partial F_z}{\partial\theta} - \frac{\partial F_\theta}{\partial z} = (\nabla \times \mathbf{F})_r \quad \text{as } \Delta\theta \text{ and } \Delta z \to 0.$$

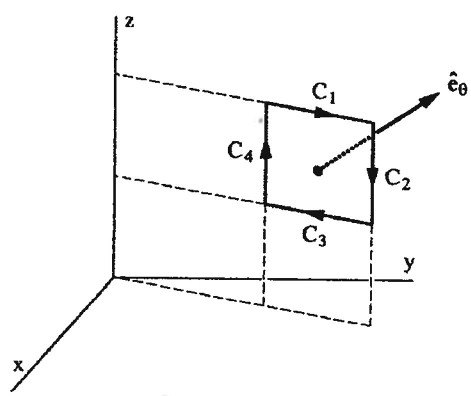
To get the θ component of the curl we proceed as follows.

$$\int_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds \cong F_r\left(r,\theta,z + \frac{\Delta z}{2}\right)\Delta r$$

$$\int_{C_2} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds \cong -F_z\left(r + \frac{\Delta r}{2},\theta,z\right)\Delta z$$

$$\int_{C_3} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds \cong -F_r\left(r,\theta,z - \frac{\Delta z}{2}\right)\Delta r$$

$$\int_{C_4} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds \cong F_z\left(r - \frac{\Delta r}{2},\theta,z\right)\Delta z$$



Adding these four integrals and dividing by $\Delta A = \Delta r\Delta z$ we get

$$\frac{1}{\Delta A}\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\, ds \cong \frac{F_r\left(r,\theta,z + \frac{\Delta z}{2}\right) - F_r\left(r,\theta,z - \frac{\Delta z}{2}\right)}{\Delta z} - \frac{F_z\left(r + \frac{\Delta r}{2},\theta,z\right) - F_z\left(r - \frac{\Delta r}{2},\theta,z\right)}{\Delta r}$$

$$\rightarrow \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} = (\nabla \times \mathbf{F})_\theta \quad \text{as } \Delta z \text{ and } \Delta r \rightarrow 0.$$

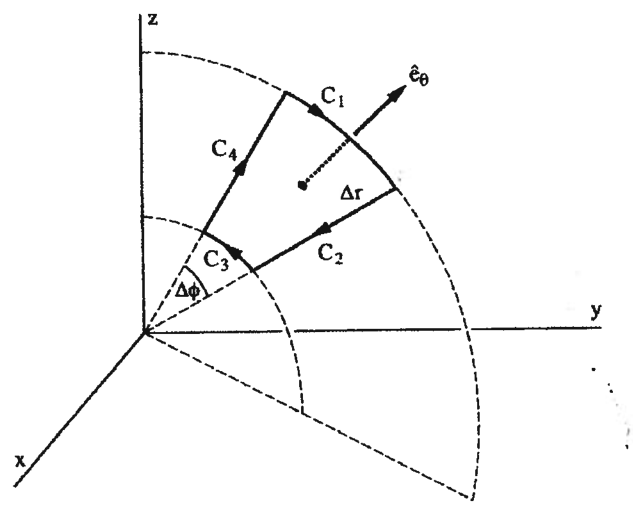
#6 9. To get the θ component of the curl we proceed as follows (see figure below):

$$\int_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = \int_{C_1} F_\phi ds \cong F_\phi\left(r + \frac{\Delta r}{2}, \phi, \theta\right)\left(r + \frac{\Delta r}{2}\right)\Delta\phi$$

$$\int_{C_2} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = -\int_{C_2} F_r\, ds \cong -F_r\left(r, \phi + \frac{\Delta\phi}{2}, \theta\right)\Delta r$$

$$\int_{C_3} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = -\int_{C_3} F_\phi\, ds \cong -\, F_\phi\left(r - \frac{\Delta r}{2}, \phi, \theta\right)\left(r - \frac{\Delta r}{2}\right)\Delta\phi$$

$$\int_{C_4} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = \int_{C_4} F_r\, ds \cong F_r\left(r, \phi - \frac{\Delta\phi}{2},\right)\Delta r$$



Adding the contributions from C_1 and C_3 and dividing by $\Delta A = r\Delta\theta\Delta r$, we get

$$\frac{1}{\Delta A}\int_{C_1+C_3} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds \cong \frac{F_\phi\left(r + \frac{\Delta r}{2}, \phi, \theta\right)\left(r + \frac{\Delta r}{2}\right) - F_\phi\left(r - \frac{\Delta r}{2}, \phi, \theta\right)\left(r - \frac{\Delta r}{2}\right)}{r\Delta r}$$

$$\rightarrow \frac{1}{r}\frac{\partial}{\partial r}(rF_\phi) \quad \text{as } \Delta r \rightarrow 0$$

Adding the contributions of C_2 and C_4 and dividing by ΔA we get

$$\frac{1}{\Delta A}\int_{C_2+C_4} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds \cong -\frac{F_r\left(r,\phi+\frac{\Delta\phi}{2},\theta\right) - F_r\left(r,\phi-\frac{\Delta\phi}{2},\theta\right)}{r\Delta\theta}$$

$$\to -\frac{1}{r}\frac{\partial F_r}{\partial\phi} \quad \text{as } \Delta r \to 0.$$

Combining these two results we get

$$(\nabla \times \mathbf{F})_\theta = \frac{1}{r}\frac{\partial}{\partial r}(rF_\phi) - \frac{1}{r}\frac{\partial F_r}{\partial \phi}$$

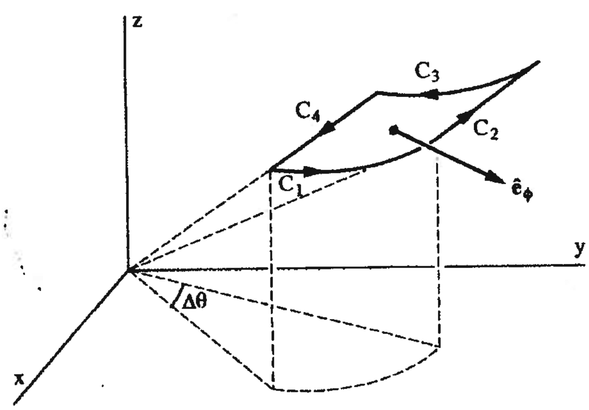
We next obtain $(\nabla \times \mathbf{F})_\phi$ (see figure below):

$$\int_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = \int_{C_1} F_\theta ds \cong \theta\left(r-\frac{\Delta r}{2},\phi,\theta\right)\left(r-\frac{\Delta r}{2}\right)\sin\phi\Delta\theta$$

$$\int_{C_2} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = \int_{C_2} F_r ds \cong F_r\left(r,\phi,\theta+\frac{\Delta\theta}{2}\right)\Delta r$$

$$\int_{C_3} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = -\int_{C_1} F_\theta ds \cong -F_\theta\left(r+\frac{\Delta r}{2},\phi,\theta\right)\left(r+\frac{\Delta r}{2}\right)\sin\phi\Delta\theta$$

$$\int_{C_4} \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = -\int_{C_2} F_r ds \cong -F_r\left(r,\phi,\theta-\frac{\Delta\theta}{2}\right)\Delta r$$



Adding the contributions from C_1 and C_3 and dividing by $\Delta A = r\sin\phi\Delta\theta\Delta r$, we get

$$\frac{1}{\Delta A}\int_{C_1+C_3} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds \cong -\frac{1}{r}\frac{F_\theta\left(r+\frac{\Delta r}{2},\phi,\theta\right)\left(r+\frac{\Delta r}{2}\right) - F_\theta\left(r-\frac{\Delta r}{2},\phi,\theta\right)\left(r-\frac{\Delta r}{2}\right)}{\Delta r}$$

$$\to -\frac{1}{r}\frac{\partial}{\partial r}(rF_\theta) \quad \text{as } \Delta r \to 0.$$

Adding the contributions from C_2 and C_4 and dividing by ΔA we get

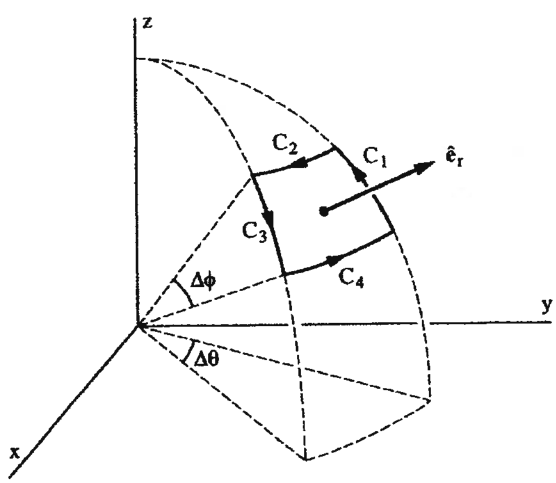
$$\frac{1}{\Delta A}\int_{C_2+C_4} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds \cong \frac{F_r\left(r,\phi,\theta+\frac{\Delta\theta}{2}\right) - F_r\left(r,\phi,\theta-\frac{\Delta\theta}{2}\right)}{r\sin\phi\Delta\theta}$$

$$\to \frac{1}{r\sin\phi}\frac{\partial F_r}{\partial\theta} \quad \text{as } \Delta\theta \to 0.$$

Combining these two results we find

$$(\nabla\times\mathbf{F})_\phi = \frac{1}{r\sin\phi}\frac{\partial F_r}{\partial\theta} - \frac{1}{r}\frac{\partial}{\partial r}(rF_\theta).$$

Finally we obtain $(\nabla\times\mathbf{F})_r$ (see figure below):



$$\int_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = -\int_{C_1} F_\phi ds \cong -F_\phi\left(r,\phi,\theta+\frac{\Delta\theta}{2}\right)r\Delta\phi$$

$$\int_{C_2} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = -\int_{C_2} F_\phi ds \cong -F_\theta\left(r,\phi-\frac{\Delta\phi}{2},\theta\right)r\sin\left(\phi-\frac{\Delta\phi}{2}\right)\Delta\phi$$

$$\int_{C_3} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = \int_{C_3} F_\phi ds \cong F_\phi\left(r,\phi,\theta-\frac{\Delta\theta}{2}\right)r\Delta\phi$$

$$\int_{C_4} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = \int_{C_4} F_\theta ds \cong F_\theta\left(r,\phi + \frac{\Delta\phi}{2},\theta\right) r\sin\left(\phi + \frac{\Delta\phi}{2}\right)\Delta\theta$$

Adding the integrals over C_1 and C_3 and dividing by $\Delta A = r^2\sin\phi\Delta\phi\Delta\theta$ we get

$$\frac{1}{\Delta A}\int_{C_1+C_3} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds \cong -\frac{F_\phi\left(r,\phi,\theta + \frac{\Delta\theta}{2}\right) - F_\phi\left(r,\phi,\theta - \frac{\Delta\theta}{2}\right)}{r\sin\phi\Delta\theta}$$

$$\rightarrow -\frac{1}{r\sin\phi}\frac{\partial F_\phi}{\partial\theta} \quad \text{as } \Delta\theta \rightarrow 0.$$

Adding the integrals over C_2 and C_4 and dividing by ΔA we get

$$\frac{1}{\Delta A}\int_{C_2+C_4} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds \cong$$

$$\frac{F_\theta\left(r,\phi + \frac{\Delta\phi}{2},\theta\right)\sin\left(\phi + \frac{\Delta\phi}{2}\right) - F_\theta\left(r,\phi - \frac{\Delta\phi}{2},\theta\right)\sin\left(\phi - \frac{\Delta\phi}{2}\right)}{r\mathrm{si}\phi\Delta\phi}$$

$$\rightarrow \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\ F_\theta) \quad \text{as } \Delta\phi \rightarrow 0.$$

Thus

$$(\nabla \times \mathbf{F})_r = \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\ F_\theta) - \frac{1}{r\sin\phi}\frac{\partial F_\phi}{\partial\theta}.$$

10\. a. $\mathbf{F} = -\mathbf{i}yz + \mathbf{j}xz$

$$= -(\hat{\mathbf{e}}_r\cos\theta - \hat{\mathbf{e}}_\theta\sin\theta)zr\sin\theta + (\hat{\mathbf{e}}_r\sin\theta + \hat{\mathbf{e}}_\theta\cos\theta)zr\cos\theta$$

$$= rz\hat{\mathbf{e}}_\theta$$

Thus

$$(\nabla \times \mathbf{F})_r = -\frac{\partial}{\partial z}(rz) = -r$$

$$(\nabla \times \mathbf{F})_\theta = 0$$

$$(\nabla \times \mathbf{F})_z = \frac{1}{r}\frac{\partial}{\partial r}(r\cdot rz) = 2z$$

and so

$$\nabla \times \mathbf{F} = -r\hat{\mathbf{e}}_r + 2z\hat{\mathbf{e}}_z = -r(\mathbf{i}\cos\theta + \mathbf{j}\sin\theta) + 2\mathbf{k}z$$

$$= -\mathbf{i}x - \mathbf{j}y + 2\mathbf{k}z.$$

13. The ones with zero curl, (d) and (h).

14. In Stokes' theorem put $\mathbf{F} = \mathbf{c}$ where $\mathbf{c}$ is an arbitrary constant vector. Then Stokes' theorem reads

$$\oint_C \mathbf{c}\cdot\hat{\mathbf{t}}\ ds = \iint_S \mathbf{n}\cdot\nabla \times \mathbf{c}\ dS = 0$$

because $\nabla \times \mathbf{c} = 0$. Hence $\mathbf{c}\cdot\oint_C \hat{\mathbf{t}}\ ds = 0$. But because $\mathbf{c}$ is an arbitrary vector this implies that $\oint_C \hat{\mathbf{t}}\ ds = 0$.

#7

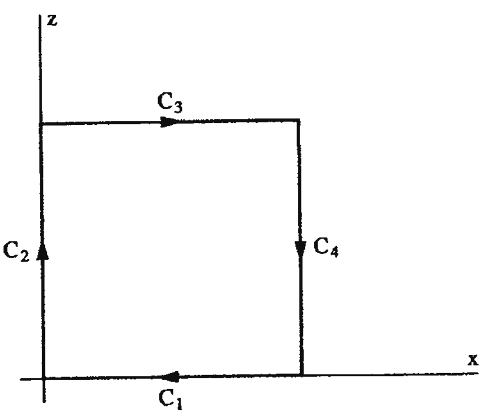
15. a. $\mathbf{F} = \mathbf{i}z^2 - \mathbf{j}y^2$. Thus $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = \oint_C z^2dx - y^2dy = \oint_C z^2dx$ because $y = 0$ on C. Now

$\int_{C_1} z^2dx = 0$, because $z = 0$ on C_1

$\int_{C_2} z^2dx = 0$, because $dx = 0$ on C_2

$\int_{C_3} z^2dx = \int_0^1 dx = 1$, because $z = 1$ on C_3

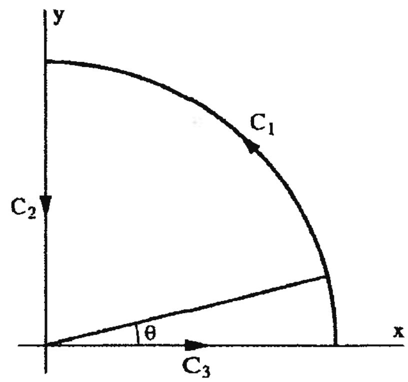
$\int_{C_4} z^2dx = 0$, because $dx = 0$ on C_4



$\therefore\ \oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\ ds = 1.$

It is easy to show that $\nabla \times \mathbf{F} = 2\mathbf{j}z$ which implies that $\hat{\mathbf{n}}\cdot\nabla \times \mathbf{F} = 0$ on all surfaces except S_5. But on S_5 $\hat{\mathbf{n}} = \mathbf{j}$ so $\hat{\mathbf{n}}\cdot\nabla \times \mathbf{F} = 2z$. Thus $\iint_{S_5} \hat{\mathbf{n}}\cdot\nabla \times \mathbf{F}\ dS = 2\int_0^1\int_0^1 z\ dxdz = 2\int_0^1 z\ dz = 1$, in agreement with the line integral given above.

b. $\mathbf{F} = \mathbf{i}y + \mathbf{j}z + \mathbf{k}x$. Hence $\int_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = \int_{C_1} y dx$ (because $z = 0$ on C_1). Letting $x = \cos\theta$, $y = \sin\theta$, this integral becomes $-\int_0^{\pi/2} \sin^2\theta d\theta = -\pi/4$. The integrals over C_2 and C_3 are treated in exactly the same way and both yield $-\pi/4$. Hence $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = -3\pi/4$.



A straightforward calculation gives $\nabla \times \mathbf{F} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$, while the equation of the surface is $z = \sqrt{1 - x^2 - y^2}$. Hence $\partial f/\partial x = -x/z$ and $\partial f/\partial y = -y/z$. We therefore have

$$\iint_S \hat{\mathbf{n}}\cdot\nabla \times \mathbf{F}\, dS = \iint_R \left(-\frac{x}{z} - \frac{y}{z} - 1\right) dxdy$$

where R is the quarter circle of radius 1 lying in the xy-plane and centered at the origin. The integral can be written

$$-\iint_R \frac{x}{\sqrt{1 - x^2 - y^2}} dxdy - \iint_R \frac{y}{\sqrt{1 - x^2 - y^2}} dxdy - \iint_R dxdy.$$

But

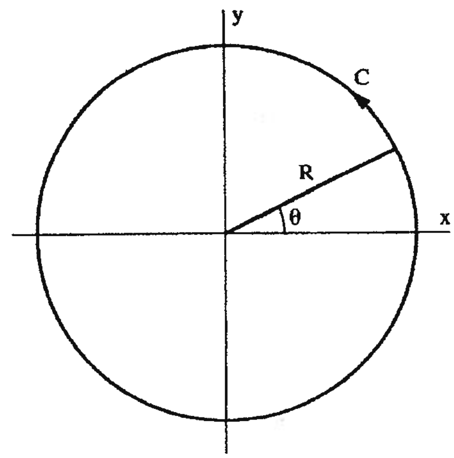
$$\iint_R \frac{x}{\sqrt{1 - x^2 - y^2}} dxdy = \int_0^{\pi/2}\int_0^1 \frac{r^2\cos\theta\ drd\theta}{\sqrt{1 - r^2}} = \int_0^1 \frac{r^2\ dr}{\sqrt{1 - r^2}} = \frac{\pi}{4}.$$

The second integral above can be treated in exactly the same way and also yields $\pi/4$. The third integral is just the area of the quarter-circle and thus also equals $\pi/4$. It follows then that

$$\iint_S \hat{\mathbf{n}}\cdot\nabla \times \mathbf{F}\, dS = -\frac{3\pi}{4},$$

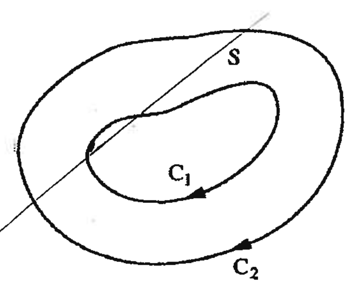
in agreement with the line integral calculated above.

c. $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = \oint_C y dx - x dy + z dz =$ $\oint_C y dx - x dy$, because $z = 0$ on C. But $\oint_C y dx = -R^2\int_0^{2\pi} \sin^2\theta d\theta = -\pi R^2$ where we have put $x = R\cos\theta$ and $y = R\sin\theta$, as shown in the figure. Using the same transformation we find $\oint_C x dy = -R^2\int_0^{2\pi} \cos^2\theta d\theta = -\pi R^2$. Adding these two



results we get $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = -2\pi R^2$. A straightforward calculation gives $\nabla \times \mathbf{F} = -2\hat{\mathbf{k}}$. On the curved surface of the cylinder $\hat{\mathbf{n}}\cdot\nabla \times \mathbf{F} = -2\hat{\mathbf{n}}\cdot\mathbf{k} = 0$. On the top of the cylinder $\hat{\mathbf{n}} = \mathbf{k}$ so $\hat{\mathbf{n}}\cdot\nabla \times \mathbf{F} = \mathbf{k}\cdot(-2\mathbf{k}) = -2$. Therefore we have $\iint_S \hat{\mathbf{n}}\cdot\nabla \times \mathbf{F}\, dS = -2\iint_S dS = -2(\pi R^2) = -2\pi R^2$, in agreement with the line integral calculated above.

#8 16. a. C_1 and C_2 together constitute a curve (in two parts) which encloses a surface S (see figure; note that the orientation of C_1 is the oppposite of that in the figure in the text). Applying Stokes'theorem, $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = \oint_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds + \oint_{C_2} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = \iint_S \mathbf{n}\cdot\nabla \times \mathbf{F}\, dS = 0$,

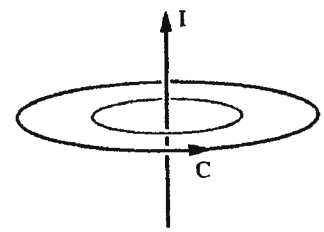


because $\nabla \times \mathbf{F} = 0$ on C and S. Thus $\oint_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = -\oint_{C_2} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds$. Changing the direction of C_1 to conform to the diagram in the text we get $\oint_{C_1} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = \oint_{C_2} \mathbf{F}\cdot\hat{\mathbf{t}}\, ds$.

#8 continued

b. Using cylindrical coordinates, $\nabla \times \left[\frac{\mu_0 I}{2\pi r}\hat{\mathbf{e}}_\theta\right] = \frac{\mu_0 I}{2\pi}\nabla \times \frac{\hat{\mathbf{e}}_\theta}{r} = \frac{\mu_0 I}{2\pi}\hat{\mathbf{e}}_z\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{1}{r}\right) = 0$. Note that this result does not hold at $r = 0$ where $\mathbf{B}$ is undefined.

c. $\oint_{\text{Circle}} \mathbf{B}\cdot\hat{\mathbf{t}}\, ds = \oint_{\text{Circle}} \frac{\mu_0 I}{2\pi r}\hat{\mathbf{e}}_\theta\cdot\hat{\mathbf{t}}\, ds$. But on the circle $\hat{\mathbf{t}} = \hat{\mathbf{e}}_\theta$ so $\oint_{\text{Circle}} \frac{\mu_0 I}{2\pi r}\hat{\mathbf{e}}_\theta\cdot\hat{\mathbf{t}}\, ds = \frac{\mu_0 I}{2\pi R}\oint_{\text{Circle}} ds = \frac{\mu_0 I}{2\pi R}(2\pi R) = \mu_0 I$. Now consider any closed curve C enclosing the line of current.



Construct a circular path lying entirely within C (see figure). Since $\nabla \times \mathbf{B} = 0$ $(r \neq 0)$ the result from (a) applies and gives $\oint_C \mathbf{B}\cdot\hat{\mathbf{t}}\, ds = \oint_{\text{Circle}} \mathbf{B}\cdot\hat{\mathbf{t}}\, ds = \mu_0 I$.

17. a. $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = \oint_C \frac{\hat{\mathbf{e}}_\theta\cdot\hat{\mathbf{t}}}{r}ds$. But $\hat{\mathbf{t}} = \hat{\mathbf{e}}_\theta$ and $r = R$ on C so $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = \frac{1}{R}(2\pi R) = 2\pi$. On the other hand, $\nabla \times \frac{\hat{\mathbf{e}}_\theta}{r} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{1}{r}\right) = 0$ so that $\iint_S \hat{\mathbf{n}}\cdot\nabla \times \frac{\hat{\mathbf{e}}_\theta}{r}\, dS = 0$. Thus Stokes' theorem fails. The reason is that Stokes' theorem requires $\mathbf{F}$ to be smooth on C and S, but $\mathbf{F} = \frac{\hat{\mathbf{e}}_\theta}{r}$ is not defined at $r = 0$.

b. $\mathbf{F}$ is smooth in D and Stokes' theorem holds. D is not simply connected.

18. Since $\mathcal{E} = -\frac{d\Phi}{dt}$ we have $\oint_C \mathbf{E}\cdot\hat{\mathbf{t}}\, ds = -\frac{d}{dt}\iint_S \mathbf{B}\cdot\hat{\mathbf{n}}\, dS = -\iint_S \hat{\mathbf{n}}\cdot\frac{\partial \mathbf{B}}{\partial t}\, dS$. Applying Stokes' theorem to the line integral, we find $\iint_S \hat{\mathbf{n}}\cdot\nabla \times \mathbf{E}\, dS = -\iint_S \hat{\mathbf{n}}\cdot\frac{\partial \mathbf{B}}{\partial t}\, dS$. Because this result holds for any capping surface S, this implies that $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$.

#19

19. A simple calculation shows that $\nabla \times \mathbf{F} = 0$ so that the line integral is independent of path. We may therefore replace the complicated path given in the statement of the problem by a simple one. We choose $x = w$, $y = w$, $z = w$ $(0 \le w \le 1)$. Then $\int_C \mathbf{F}\cdot\hat{\mathbf{t}}\, ds =$

$$\int_0^1 (e^{-w} - we^{-w})dw + \int_0^1 (e^{-w} - we^{-w})dw + \int_0^1 (e^{-w} - we^{-w})dw = 3\int_0^1 (e^{-w} - we^{-w})dw =$$

$3/e$.

20. Take the divergence of the fourth equation recalling that $\nabla\cdot\nabla \times \mathbf{F} = 0$. Then $\nabla\cdot\nabla \times \mathbf{B} = 0 = \varepsilon_0\mu_0\nabla\cdot\frac{\partial \mathbf{E}}{\partial t} + \mu_0\nabla\cdot\mathbf{J}$. But using the first equation we get $\nabla\cdot\frac{\partial \mathbf{E}}{\partial t} = \frac{\partial}{\partial t}(\nabla\cdot\mathbf{E}) = \frac{\partial}{\partial t}\left(\frac{\rho}{\varepsilon_0}\right)$. Hence $\varepsilon_0\mu_0\frac{\partial}{\partial t}\left(\frac{\rho}{\varepsilon_0}\right) + \nabla\cdot\mathbf{J} = 0$, or $\frac{\partial \rho}{\partial t} + \nabla\cdot\mathbf{J} = 0$. This equation asserts that electric charge is conserved.

21. From the second of Maxwell's equations we have $\mathbf{B}\cdot\nabla \times \mathbf{E} = -\mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t}$ $= -\frac{\partial}{\partial t}\left(\frac{B^2}{2}\right)$, and from the fourth, $\mathbf{E}\cdot\nabla \times \mathbf{B} = \varepsilon_0\mu_0\mathbf{E}\cdot\frac{\partial \mathbf{E}}{\partial t} + \mu_0\mathbf{J}\cdot\mathbf{E} =$ $\varepsilon_0\mu_0\frac{\partial}{\partial t}\left(\frac{E^2}{2}\right) + \mu_0\mathbf{J}\cdot\mathbf{E}$. Subtracting these two equations we get

$$\mathbf{E}\cdot(\nabla \times \mathbf{B}) - \mathbf{B}\cdot(\nabla \times \mathbf{E}) = \frac{\partial}{\partial t}\left(\frac{\varepsilon_0\mu_0 E^2 + B^2}{2}\right) + \mu_0\mathbf{J}\cdot\mathbf{E}$$

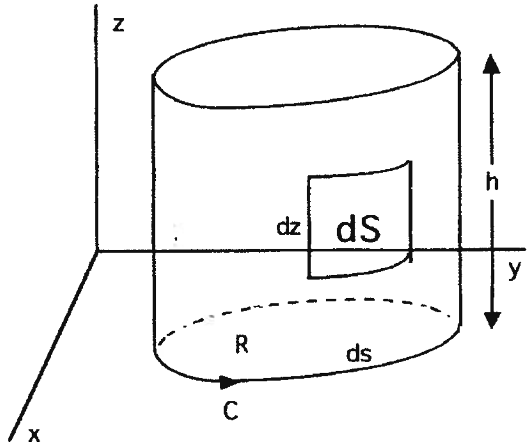
Using the fourth identity on the inside front cover of text we find that the left hand side of this last equation can be written $\nabla\cdot(\mathbf{B} \times \mathbf{E})$. Thus

$$\frac{\partial}{\partial t}\left(\frac{\varepsilon_0 E^2 + B^2/\mu_0}{2}\right) + \nabla\cdot\left(\frac{\mathbf{E} \times \mathbf{B}}{\mu_0}\right) = -\mathbf{J}\cdot\mathbf{E}.$$

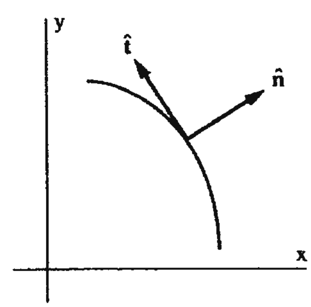
We interpret $\rho_E = \frac{\varepsilon_0 E^2 + B^2/\mu_0}{2}$ as the electromagnetic energy density, and $\mathbf{J}_E = \frac{\mathbf{E} \times \mathbf{B}}{\mu_0}$ as the electromagnetic energy current density. Thus our last equation reads $\frac{\partial \rho_E}{\partial t} + \nabla\cdot\mathbf{J}_E = -\mathbf{J}\cdot\mathbf{E}$. If the right-hand side of this equation were 0, the equation would assert that electromagnetic energy is conserved. However, the term $\mathbf{J}\cdot\mathbf{E}$ is the rate at which the electric field does work in moving electric charges. Thus the electromagnetic energy is not conserved: it decreases when the field does work on the charges, and it increases when the charges do work on the field.

#10

22. a. Let CS stand for curved surface and T&B for top and bottom. Then $\iint_S \mathbf{G}\cdot\hat{\mathbf{n}}\ dS = \iint_{CS} \mathbf{G}\cdot\hat{\mathbf{n}}\ dS + \iint_{T\&B} \mathbf{G}\cdot\hat{\mathbf{n}}\ dS$. However, the integral over the top and bottom is zero because $\hat{\mathbf{n}} = \pm\mathbf{k}$ and $\mathbf{G}\cdot\mathbf{k} = 0$. Now we note that on the curved surface $dS = ds dz$ (see figure). Hence



$\iint_{CS} \mathbf{G}\cdot\hat{\mathbf{n}}\ dS = \iint_{CS} \mathbf{G}\cdot\hat{\mathbf{n}}\ dsdz = h\oint_C \mathbf{G}\cdot\hat{\mathbf{n}}\ ds$, where h is the height of the cylinder. Now $\hat{\mathbf{n}}\cdot\hat{\mathbf{t}} = 0$ (see Figure b). Thus $n_x t_x + n_y t_y = 0$ so that $n_x = -n_y t_y/t_x$. Hence $\hat{\mathbf{n}} = (-n_y t_y/n_x)\mathbf{i} + n_y\mathbf{j} = (-n_y/t_x)(t_y\mathbf{i} - t_x\mathbf{j})$. To make this a unit vector we take $\hat{\mathbf{n}} = t_y\mathbf{i} - t_x\mathbf{j}$. Thus $\mathbf{G}\cdot\hat{\mathbf{n}} = G_x t_y - G_y t_x$, and so $\oint_C \mathbf{G}\cdot\hat{\mathbf{n}}ds = \oint_C (G_x t_y - G_y t_x)\ ds = \oint_C (G_x dy - G_y dx)$.



Thus we have

$$\iint_S \mathbf{G}\cdot\hat{\mathbf{n}}\ dS = h\oint_C (G_x dy - G_y dx) = \iiint_V \nabla\cdot\mathbf{G}\ dV = \iiint_V \left(\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y}\right)dxdydz$$

$$= h\iint_R \left(\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y}\right)dxdy.$$

$$\therefore \oint_C (G_x dy - G_y dx) = \iint_R \left(\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y}\right) dxdy.$$

b. $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = \oint_C F_x dx + F_y dy = \iint_S \hat{\mathbf{n}}\cdot\nabla \times \mathbf{F}\, dS$. But $\hat{\mathbf{n}} = \mathbf{k}$ so $\oint_C F_x dx + F_y dy = \iint_R \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dxdy.$

c. In the equation derived in (b) put $F_x = -G_y$ and $F_y = G_x$. Then we get $\oint_C G_x dy - G_y dx = \iint_R \left(\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y}\right) dxdy$, which is the equation derived in (a).

23. a. Using the result of Prob III22-(b), $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = \iint_S \hat{\mathbf{n}}\cdot\nabla \times \mathbf{F}\, dS$ $= \iint_R \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dxdy = A$ provided $\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 1.$

b. $\mathbf{F} = \mathbf{j}x$ or $\mathbf{F} = -\mathbf{i}y$.

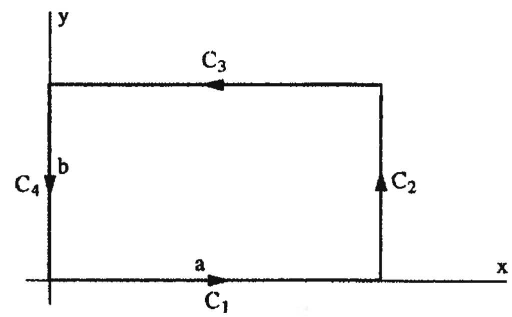
c. We use $\mathbf{F} = \mathbf{j}x$ so that $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\, ds = \oint_C x dy$.

i. $\int_{C_1} x dy = \int_{C_3} x dy = 0$ because $dy = 0$

$$\int_{C_2} x dy = a\int_0^b dy = ab$$

$$\int_{C4} x dy = 0 \text{ because } x = 0$$

$$\therefore A = ab$$



ii. $\int_{C_1} x dy = 0$ because $dy = 0$

$$\int_{C_2} x dy = h\int_0^b dy = hb$$

On C_3 $x/s = b/\sqrt{h^2 + b^2}$ so $x = bs/\sqrt{h^2 + b^2}$ and $y/s = h/\sqrt{h^2 + b^2}$ so $y = hs/\sqrt{h^2 + b^2}$. Thus $\int_{C_3} x dy = -\frac{hb}{b^2 + h^2}\int_0^{\sqrt{b^2 + h^2}} s ds = -\frac{hb}{2}$. Hence we have $A = hb - \frac{hb}{2} = \frac{hb}{2}$.