Physics 161, HW # 2

Answer Key

1. Schey II-1.

1. a. $\dfrac{\partial z}{\partial x} = -1,\ \dfrac{\partial z}{\partial y} = -1,\ \hat{\mathbf{n}} = (\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}$.

b. $\dfrac{\partial z}{\partial x} = \dfrac{x}{z},\ \dfrac{\partial z}{\partial y} = \dfrac{y}{z},\ \hat{\mathbf{n}} = \left(-\mathbf{i}\dfrac{x}{z}-\mathbf{j}\dfrac{y}{z}+\mathbf{k}\right)/\sqrt{(x/z)^2+(y/z)^2+1}$

$$= -\frac{\mathbf{i}x + \mathbf{j}y - \mathbf{k}z}{\sqrt{2}\,z}.$$

c. $\dfrac{\partial z}{\partial x} = -\dfrac{x}{z},\ \dfrac{\partial z}{\partial y} = 0,\ \hat{\mathbf{n}} = (\mathbf{i}x/z+\mathbf{k})/\sqrt{(x/z)^2+1} = (\mathbf{i}x+\mathbf{k}z)/\sqrt{x^2+z^2}$

$= \mathbf{i}x + \mathbf{k}z$ since $x^2 + z^2 = 1$.

d. $\dfrac{\partial z}{\partial x} = 2x,\ \dfrac{\partial z}{\partial y} = 2y,\ \hat{\mathbf{n}} = (-2\mathbf{i}x-2\mathbf{j}y+\mathbf{k})/\sqrt{1+4x^2+4y^2}$

$$= \frac{-2\mathbf{i}x - 2\mathbf{j}y + \mathbf{k}}{\sqrt{1 + 4z}}.$$

e. $\dfrac{\partial z}{\partial x} = -\dfrac{x/a^2}{z},\ \dfrac{\partial z}{\partial y} = -\dfrac{y/a^2}{z},\ \hat{\mathbf{n}} = \left(\mathbf{i}\dfrac{x}{a^2z}+\mathbf{j}\dfrac{y}{a^2z}+\mathbf{k}\right)/\sqrt{\dfrac{x^2}{a^4z^2}+\dfrac{y^2}{a^4z^2}+1}$

$$= \frac{\mathbf{i}x+\mathbf{j}y+\mathbf{k}a^2z}{a\sqrt{1-(a^2-1)z^2}}.$$

2. Schey II-2.

2. a. $z = (d-ax-by)/c,\ \dfrac{\partial z}{\partial x} = -a/c,\ \dfrac{\partial z}{\partial y} = -b/c,\ \hat{\mathbf{n}} = \dfrac{\mathbf{i}a/c+\mathbf{j}b/c+\mathbf{k}}{\sqrt{a^2/c^2+b^2/c^2+1}}$

$$= \frac{\mathbf{i}a+\mathbf{j}b+\mathbf{k}c}{\sqrt{a^2+b^2+c^2}}.$$

b. As d varies with a, b, and c fixed, a family of *parallel* planes is generated. Because they are parallel they all have the same normal.

3. Schey II-4.

4. a. $z = f(x,y) = 1 - x - y$ so $\partial f/\partial x = \partial f/\partial y = -1$. Thus

$$\iint_S G\,dS = \iint_R z\sqrt{3}\ dxdy = \sqrt{3}\iint_R (1 - x - y)\ dxdy$$

where R is the triangle in the xy-plane bounded by the coordinate axes and the line $x + y = 1$. Hence the integral is

$$\sqrt{3}\int_0^1\int_0^{1-x} dxdy - \sqrt{3}\int_0^1\int_0^{1-x} xdydx - \sqrt{3}\int_0^1\int_0^{1-x} ydydx =$$

$$\frac{\sqrt{3}}{2} - \sqrt{3}\int_0^1 x(1 - x)dx - \frac{\sqrt{3}}{2}\int_0^1 (1 - x)^2dx = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{6} - \frac{\sqrt{3}}{6} = \frac{\sqrt{3}}{6}.$$

b. $z = f(x,y) = x^2 + y^2$ so $\partial f/\partial x = 2x$ and $\partial f/\partial y = 2y$. Thus

$$\iint_S G dS = \iint_R \frac{1}{1 + 4(x^2 + y^2)} \sqrt{1 + 4x^2 + 4y^2}\, dxdy = \iint_R \frac{dxdy}{\sqrt{1 + 4(x^2 + y^2)}}$$

where R is the circle of radius 1 lying in the xy-plane with its center at the origin. Transforming to polar coordinates we get

$$\int_0^{2\pi}\int_0^1 \frac{rdrd\theta}{\sqrt{1 + 4r^2}} = 2\pi\int_0^1 \frac{rdr}{\sqrt{1 + 4r^2}} = 2\pi\left(\frac{1}{4}\right)(1 + 4r^2)^{1/2}\Big|_0^1 = \frac{\pi}{2}\left(\sqrt{5} - 1\right).$$

c. $z = f(x,y) = (1 - x^2 - y^2)^{1/2}$ so $\partial f/\partial x = -x/z$ and $\partial f/\partial y = -y/z$. Hence

$$\iint_S G dS = \iint_R (1 - x^2 - y^2)^{3/2}\sqrt{1 + \frac{x^2 + y^2}{z^2}}\, dxdy =$$

$$\iint_R (1 - x^2 - y^2)^{3/2} \frac{1}{(1 - x^2 - y^2)^{1/2}}\, dxdy = \iint_R (1 - x^2 - y^2)\, dxdy,$$

where R is the circle of radius 1 lying in the xy-plane with its center at the origin. Transforming to polar coordinates we get

$$\int_0^{2\pi}\int_0^1 (1 - r^2) rdrd\theta = 2\pi\int_0^1 (1 - r^2)rdr = 2\pi\int_0^1 (r - r^3)\, dr = \frac{\pi}{2}.$$

4. Schey II-5.

5. a. $z = f(x,y) = 1 - x/2 - y/2$ so $\partial f/\partial x = \partial f/\partial y = -1/2$. Hence

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_R \left[-\left(-\frac{1}{2}\right)x - z\right] dxdy = \iint_R \left[\frac{x}{2} - \left(1 - \frac{x}{2} - \frac{y}{2}\right)\right] dxdy =$$

$$= \iint_R \left(x + \frac{y}{2} - 1\right) dxdy,$$

where R is the region in the xy-plane bounded by the coordinate axes and the line $x + y = 2$. Thus the integral is

$$\int_0^2\int_0^{2-x} xdydx + \frac{1}{2}\int_0^2\int_0^{2-x} ydydx - 2 = \int_0^2 x(2 - x)dx + \frac{1}{4}\int_0^2 (2 - x)^2\, dx - 2$$

$$= 4/3 + 2/3 - 2 = 0.$$

b. $z = f(x,y) = \sqrt{a^2 - x^2 - y^2}$ so $\partial f/\partial x = -x/z$ and $\partial f\partial y = -y/z$. Hence

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_R \left[-x\left(-\frac{x}{z}\right) - y\left(-\frac{y}{z}\right) + z\right] dxdy = \iint_R \frac{x^2 + y^2 + z^2}{z} dxdy =$$

$$a^2\iint_R \frac{dxdy}{\sqrt{a^2 - x^2 - y^2}},$$

where R is the circle of radius a lying in the xy-plane with its center at the origin. Transforming to polar coordinates we get

$$a^2\int_0^{2\pi}\int_0^a \frac{rdrd\theta}{\sqrt{a^2 - r^2}} = 2\pi a^2\int_0^a \frac{rdr}{\sqrt{a^2 - r^2}} = 2\pi a^2[-(a^2 - r^2)^{1/2}]\Big|_0^a = 2\pi a^3.$$

c. $z = f(x,y) = 1 - x^2 - y^2$ so $\partial f/\partial x = -2x$ and $\partial f/\partial y = -2y$. Thus

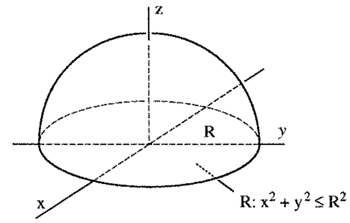
$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_R [-y(-2y) + 1]dxdy = \iint_R (1 + 2y^2)\, dxdy,$$

where R is the circle of radius 1 lying in the xy-plane with its center at the origin. Therefore the integral is

$$\iint_R dxdy + 2\iint_R y^2 dxdy = \pi + 2\int_0^{2\pi}\int_0^1 r^2\sin^2\theta\; rdrd\theta = \pi + 2\pi\int_0^1 r^3 dr$$

5. Schey II-6.

6. $m = \iint_S \sigma(x,y,z)\, dS = \frac{\sigma_0}{R^2}\iint_S (x^2 + y^2)\, dS$ where S is the surface $z = f(x,y) = \sqrt{R^2 - x^2 - y^2}$. Hence $\partial f/\partial x = -x/z$ and $\partial f/\partial y = -y/z$, so



$$m = \frac{\sigma_0}{R^2}\iint_R (x^2+ y^2)\sqrt{1 + (x/z)^2 + (y/z)^2}\, dxdy$$

where $\mathbf{R}$ is the disc $x^2 + y^2 \le R^2$. Thus,

$$m = \frac{\sigma_0}{R^2}\iint_R (x^2 + y^2)\sqrt{\frac{x^2 + y^2 + z^2}{z^2}}\, dxdy = \iint_R \frac{x^2 + y^2}{\sqrt{R^2 - x^2 - y^2}}\, dxdy$$

$$= \frac{\sigma_0}{R}\int_0^{2\pi}\int_0^{R} \frac{r^3\, drd\theta}{\sqrt{R^2 - r^2}} = \frac{2\pi\sigma_0}{R}\int_0^R \frac{r^3 dr}{\sqrt{R^2 - r^2}} = 2\pi\sigma_0 R^2\int_0^1 \frac{w^3 dw}{\sqrt{1 - w^2}}.$$

This integral can be done by elementary methods; its value is 2/3. Hence we have, finally, $m = 4\pi\sigma_0 R^2/3$.

6. Schey II-10.

10. a. On the face in the yz-plane, $\hat{\mathbf{n}} = \pm\mathbf{i}$, so $\mathbf{F}\cdot\hat{\mathbf{n}} = \pm x = 0$ (because $x = 0$ in the yz-plane). The other two faces can be handled in the same way. Hence $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\, dS = 0$.

b. On the circular top and bottom, $\hat{\mathbf{n}} = \pm\mathbf{k}$ and $\mathbf{F}\cdot\hat{\mathbf{n}} = 0$. On the curved surface $\hat{\mathbf{n}} = (\mathbf{i}x + \mathbf{j}y)/R$ and $x^2 + y^2 = R^2$. Hence

$$\mathbf{F}\cdot\hat{\mathbf{n}} = \frac{x^2 + y^2}{R}\ln(x^2 + y^2) = R\ln R^2 = 2R\ln R.$$

Thus $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\, dS = (2R\ln R)(2\pi Rh) = 4\pi R^2 h\ln R$.

c. On the spherical surface, $\hat{\mathbf{n}} = (\mathbf{i}x + \mathbf{j}y + \mathbf{k}z)/R$ and $x^2 + y^2 + z^2 = R^2$. Hence

$$\mathbf{F}\cdot\hat{\mathbf{n}} = \frac{x^2 + y^2 + z^2}{R}e^{-(x^2 + y^2 + z^2)} = \frac{R^2 e^{-R^2}}{R} = Re^{-R^2}.$$

Thus $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\, dS = Re^{-R^2}(4\pi R^2) = 4\pi R^3 e^{-R^2}$.

d. The only surfaces to contribute to the surface integral are the one at $x = 0$ and the one at $x = b$. At $x = 0$, $\hat{\mathbf{n}} = -\mathbf{i}$ and so $\mathbf{F}\cdot\hat{\mathbf{n}} = -E(x) = -E(0)$. At $x = b$, $\hat{\mathbf{n}} = \mathbf{i}$ and $\mathbf{F}\cdot\hat{\mathbf{n}} = E(x) = E(b)$. Thus

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\, dS = [E(b) - E(0)]b^2.$$

7. Schey II-14.

14. (a). $\frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial y}y^2 + \frac{\partial}{\partial z}z^2 = 2(x + y + z)$.

(b). $\frac{\partial}{\partial x}yz + \frac{\partial}{\partial y}xz + \frac{\partial}{\partial z}xy = 0$.

(c). $\frac{\partial}{\partial x}e^{-x} + \frac{\partial}{\partial y}e^{-y} + \frac{\partial}{\partial z}e^{-z} = -(e^{-x} + e^{-y} + e^{-z})$.

(d). $\frac{\partial}{\partial x}1 + \frac{\partial}{\partial y}(-3) + \frac{\partial}{\partial z}z^2 = 2z$.

(e). $\dfrac{\partial}{\partial x}\left[-\dfrac{xy}{x^2 + y^2}\right] + \dfrac{\partial}{\partial y}\left[\dfrac{xy}{x^2 + y^2}\right] = -\dfrac{y}{x^2 + y^2}.$

(f). $\dfrac{\partial}{\partial z}\sqrt{x^2 + y^2} = 0.$

(g). $\dfrac{\partial}{\partial x}x + \dfrac{\partial}{\partial y}y + \dfrac{\partial}{\partial z}z = 3.$

(h). $\dfrac{\partial}{\partial x}\left[-\dfrac{y}{\sqrt{x^2 + y^2}}\right] + \dfrac{\partial}{\partial y}\left[\dfrac{x}{\sqrt{x^2 + y^2}}\right] =$

$$xy(x^2 + y^2)^{-3/2} - xy(x^2 + y^2)^{-3/2} = 0.$$

8. Schey II-15.

15. (a). In the following we evaluate the function at the center of the relevant face of the cube.

On S_1 $\mathbf{F}\cdot\hat{\mathbf{n}} = \mathbf{F}\cdot\mathbf{i} \cong (x_0 + s/2)^2$

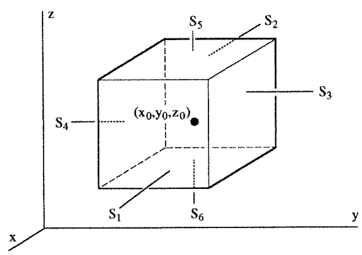
On S_2 $\mathbf{F}\cdot\hat{\mathbf{n}} = -\mathbf{F}\cdot\mathbf{i} \cong -(x_0 - s/2)^2$

On S_3 $\mathbf{F}\cdot\hat{\mathbf{n}} = \mathbf{F}\cdot\mathbf{j} \cong (y_0 + s/2)^2$

On S_4 $\mathbf{F}\cdot\hat{\mathbf{n}} = -\mathbf{F}\cdot\mathbf{j} \cong -(y_0 - s/2)^2$

On S_5 $\mathbf{F}\cdot\hat{\mathbf{n}} = \mathbf{F}\cdot\mathbf{k} \cong (z_0 + s/2)^2$

On S_6 $\mathbf{F}\cdot\hat{\mathbf{n}} = -\mathbf{F}\cdot\mathbf{k} \cong -(z_0 - s/2)^2$



Hence $\iint_{S_1}\mathbf{F}\cdot\hat{\mathbf{n}}\,dS + \iint_{S_2}\mathbf{F}\cdot\hat{\mathbf{n}}\,dS \cong s^2[(x_0 + s/2)^2 - (x_0 - s/2)^2] = 2x_0s^3,$ with analogous results for the othe two pairs of faces. Hence

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS \cong 2s^3(x_0 + y_0 + z_0).$$

(b). The volume of the cube is $V = s^3$ so

$$(1/V)\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \frac{2s^3(x_0 + y_0 + z_0)}{s^3} = 2(x_0 + y_0 + z_0).$$

By definition this is $\nabla\cdot\mathbf{F}$ at (x_0,y_0,z_0) and it agrees with Prob. II-14(a). [Note that there is no need to calculate the limit of this expression as $s \to 0$ since the result is independent of s.]

(c). For $\mathbf{F} = \mathbf{i}yz + \mathbf{j}xz + \mathbf{k}xy$ (evaluating $\mathbf{F}\cdot\hat{\mathbf{n}}$ at the center of the face),

On S_1 $\mathbf{F}\cdot\hat{\mathbf{n}} = yz$ so $\iint_{S_1}\mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{S_1} yz\,dS \cong y_0z_0s^2$

On S_2 $\mathbf{F}\cdot\hat{\mathbf{n}} = -yz$ so $\iint_{S_2}\mathbf{F}\cdot\hat{\mathbf{n}}\,dS = -\iint_{S_2} yz\,dS \cong -y_0z_0s^2.$

Note that these two results cancel. Calculations analogous to this one show that the other two pairs of faces also give cancelling results. Thus $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = 0$ and so $\nabla\cdot\mathbf{F} = 0$, which is the result obtained in Prob II-14(b).

For $\mathbf{F} = \mathbf{i}e^{-x} + \mathbf{j}e^{-y} + \mathbf{k}e^{-z}$ (evaluating $\mathbf{F}\cdot\hat{\mathbf{n}}$ at the center of the face), we find

$$\iint_{S_1}\mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{S_1} e^{-x}\,dS \cong e^{-(x_0 + s/2)}s^2$$

and

$$\iint_{S_2}\mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{S_2} e^{-x}\,dS \cong -e^{-(x_0 - s/2)}s^2.$$

Hence

$$\iint_{S_1 + S_2} e^{-x}\,dS \cong s^2[e^{-(x_0 + s/2)} - e^{-(x_0 - s/2)}]$$

Dividing this by the volume, s^3, gives

$$\frac{e^{-(x_0 + s/2)} - e^{-(x_0 - s/2)}}{s} \to e^{-x_0}$$

as $s \to 0$. The other two pairs of faces are treated in the same way and yield e^{-y_0} and e^{-z_0}. The sum of the three contributions is thus $e^{-x_0} + e^{-y_0} + e^{-z_0}$, which is the result of ProbII-14(c), evaluated at (x_0,y_0,z_0).

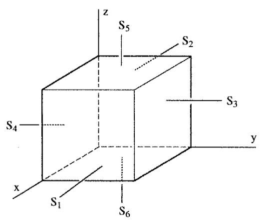
9. Schey II-16.

16. a. Let $f'(u) = df/du$. Then $\nabla\cdot\mathbf{F} = f'(x) + f'(y) + f'(-2z)(-2)$. With $(x,y,z) = (c,c,-c/2)$ we get $\nabla\cdot\mathbf{F} = f'(c) + f'(c) - 2f'(c) = 0$.

b. $\nabla\cdot\mathbf{G} = \dfrac{\partial}{\partial x}f(y,z) + \dfrac{\partial}{\partial y}g(x,z) + \dfrac{\partial}{\partial z}h(x,y) = 0.$

10. Schey II-23.

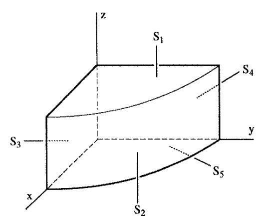
23. a. For faces S_1 and S_2 we have



$\iint_{S_1}\mathbf{F}\cdot\mathbf{n}dS = \iint_{S_1}xdydx = b\iint_{S_1}dydx = b^3$, and $\iint_{S_2}\mathbf{F}\cdot\mathbf{n}dS = -\iint_{S_2}xdydx = 0$ because $x = 0$ on S_2. In exactly the same way S_3 and S_5 each yield b^3 and S_4 and S_6 both give 0. Hence $\iint_S \mathbf{F}\cdot\mathbf{n}dS = 3b^3$. But

$\nabla\cdot\mathbf{F} = \nabla\cdot(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z) = 3$. Thus $\iiint_V \nabla\cdot\mathbf{F}\ dV = 3\iiint_V dV = 3b^3$.

b. On S_1 we have



$\iint_{S_1}\mathbf{F}\cdot\mathbf{n}dS = \iint_{S_1}zdS = h\iint_{S_1}dS = \pi R^2h/4$, because $z = h$ on S_1. On S_2, $\iint_{S_2}\mathbf{F}\cdot\mathbf{n}dS = -\iint_{S_2}z\ dS = 0$, because $z = 0$ on S_2. Next, $\iint_{S_3}\mathbf{F}\cdot\mathbf{n}dS = \iint_{S_4}\mathbf{F}\cdot\mathbf{n}dS = 0$ because $\hat{\mathbf{n}} = \pm\hat{\mathbf{e}}_\theta$ on S_3 and S_4 and $\mathbf{F}\cdot\hat{\mathbf{n}} = F_\theta = 0$. Finally,

$\iint_{S_5}\mathbf{F}\cdot\mathbf{n}dS = \iint_{S_5}\mathbf{F}\cdot\hat{\mathbf{e}}_r\ dS = \iint_{S_5}r\ dS = R\iint_{S_5}dS = R\left[\dfrac{2\pi Rh}{4}\right] = \dfrac{\pi R^2h}{2}$. Adding the non-zero contributions from S_1 and S_5, we get $\iint_S \mathbf{F}\cdot\mathbf{n}dS = \dfrac{3\pi R^2h}{4}$.

Next we have $\nabla\cdot\mathbf{F} = \dfrac{1}{r}\dfrac{\partial}{\partial r}(r\cdot r) + \dfrac{1}{r}\dfrac{\partial}{\partial\theta}(0) + \dfrac{\partial}{\partial z}(z) = 3$. Thus the volume integral is

$$\iiint_V \nabla\cdot\mathbf{F}\ dV = 3\iiint_V dV = 3\left(\frac{\pi r^2h}{4}\right) = \frac{3\pi R^2h}{4}.$$

c. $\iint_S \mathbf{F}\cdot\mathbf{n}dS = \iint_S r^2\hat{\mathbf{e}}_r\cdot\hat{\mathbf{e}}_r dS = R^2\iint_S dS = 4\pi R^2$. But $\nabla\cdot\mathbf{F} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2\cdot r^2) = 4r$. Therefore $\iiint_V \nabla\cdot\mathbf{F}\ dV = \iiint_V 4r dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} 4r\cdot r^2\sin\theta\ dr\ d\theta\ d\phi = 4\pi R^2$.