Physics 161, spring 2015

Answer Key, HW # 4

Due Thursday 12 February by 4 pm.

I will be in my office on Tuesday afternoon.

There are only six problems in this set.

The solutions to problems 4 and 5 are found in Chapter 12 of PM.

1. P&M 2.34. Finding the extremum of a charge distribution
2. P&M 2.50. Dividing the charge.
3. P&M 2.58. Energy of a charged shell.
4. P&M 3.9. Grounding a shell
5. P&M 3.13. Image charge for a grounded spherical shell.
   (Remember the law of cosines.)
6. P&M 3.59. A coaxial capacitor.

1. P&M 2.34. Finding the extremum of a charge distribution

2.34. Extremum of ϕ

By symmetry, the **E** field at points on the y axis has no x or z component. And we know that E_y equals $-\partial\phi/\partial y$. So if ϕ has a local maximum or minimum at some point on the y axis, then $\partial\phi/\partial y$, and hence E_y, equals zero. The full vector **E** therefore also equals zero.

At the point $(0, y, 0)$ with $y > 1$, the E_y component equals (ignoring the factor of $1/4\pi\epsilon_0$, along with the units of the various quantities)

$$E_y = \frac{2}{y^2} - 2 \cdot \frac{1}{(y-1)^2 + 1^2} \cdot \frac{y-1}{\sqrt{(y-1)^2 + 1^2}}, \tag{107}$$

where the last factor gives the y component of the field from the two negative charges. Setting $E_y = 0$, moving one of the terms to the other side of the equation, and squaring, we find

$$y^4 = \frac{(y^2 - 2y + 2)^3}{(y-1)^2}. \tag{108}$$

Another way of obtaining this relation is to (as you can check) write down the potential (again ignoring the factor of $1/4\pi\epsilon_0$ and units),

$$\phi(0, y, 0) = \frac{2}{y} - 2 \cdot \frac{1}{\sqrt{(y-1)^2 + 1^2}}, \tag{109}$$

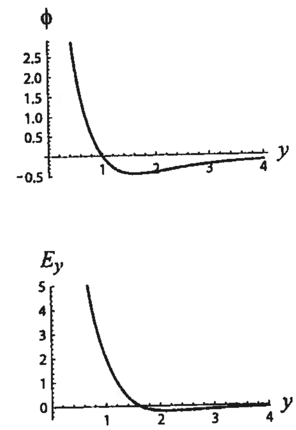

Figure 34

and then set $\partial\phi/\partial y = 0$. The result is Eq. (108), of course, because $E_y = -\partial\phi/\partial y$. We can solve Eq. (108) numerically; *Mathematica* gives the numerical result of $y = 1.621$. Plots of $\phi(y)$ and $E_y(y)$ (times $4\pi\epsilon_0$) for points on the y axis are shown in Fig. 34. For large y, we know that both ϕ and E_y must be negative, because for large y we have a charge 2 C at a distance y, and two -1 C charges at a distance essentially equal to $y - 1$. So the negative charges win out.

You can show that E_y reaches a maximum negative value at $y = 2.153$; you will again need to solve an equation numerically. The existence of such a point between $y = 1.621$ and $y = \infty$ follows from a continuity argument similar to the one involving ϕ: Since $E_y = 0$ at these two points, E_y must have a local maximum or minimum somewhere between.

Problem 2. P&M 250

2.50. Dividing the charge

The potential is constant over the surface of a given sphere, so we can pull the ϕ outside the integral in Eq. (2.32) and write the potential energy of a sphere as $U = (\phi/2)\int \rho\, dv = \phi q/2$. So if the spheres of radii R_1 and R_2 have charge q and $Q - q$, respectively, the sum of the two potential energies is

$$U = \frac{q}{4\pi\epsilon_0 R_1} \cdot \frac{q}{2} + \frac{Q-q}{4\pi\epsilon_0 R_2} \cdot \frac{Q-q}{2} = \frac{1}{4\pi\epsilon_0}\left(\frac{q^2}{2R_1} + \frac{(Q-q)^2}{2R_2}\right). \tag{145}$$

Minimizing this by setting the derivative with respect to q equal to zero yields

$$0 = \frac{dU}{dq} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{R_1} - \frac{(Q-q)}{R_2}\right). \tag{146}$$

Solving for q gives $q = QR_1/(R_1 + R_2)$. So there is charge $QR_1/(R_1 + R_2)$ on the first sphere and charge $QR_2/(R_1 + R_2)$ on the second sphere.

The two terms in Eq. (146) (without the minus sign in front of the second term) are simply the potentials of the two spheres. So the condition of minimum energy is equivalent to the condition of equal potentials. Note that the second derivative, $d^2U/d^2q = 1/R_1 + 1/R_2$, is positive, so the extremum is indeed a minimum of U, not a maximum. This is consistent with the special case where $R_1 = R_2$; equal division of the charge involves half as much total energy as piling all of Q on one sphere, from Eq. (145).

Problem 3. P&M 2.58

2.58. Energy of a shell

The relevant volume in the integral in Eq. (2.32) is all located right on the surface of the shell where the potential ϕ takes on the uniform value of $Q/4\pi\epsilon_0 R$. We can therefore take ϕ outside the integral, yielding

$$U = \frac{1}{2}\phi \int \rho\, dv = \frac{1}{2}\phi Q = \frac{1}{2}\left(\frac{Q}{4\pi\epsilon_0 R}\right) Q = \frac{Q^2}{8\pi\epsilon_0 R}. \tag{171}$$

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Problem 6. P&M 3.59

3.59. Coaxial capacitor

Neglecting end effects, we can assume that the charge $\pm Q$ is uniformly distributed along each cylinder. The field between the cylinders is that of a line charge with density $\lambda = Q/L$, so $E = \lambda/2\pi\epsilon_0 r = Q/2\pi\epsilon_0 Lr$. The magnitude of the potential difference between the cylinders is then

$$|\Delta\phi| = \int_b^a E\,dr = \int_b^a \frac{Q\,dr}{2\pi\epsilon_0 Lr} = \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{a}{b}\right). \qquad (265)$$

Since $C = Q/|\Delta\phi|$, the capacitance is given by $C = 2\pi\epsilon_0 L/\ln(a/b)$. If $a - b \ll b$, then we can use the Taylor series $\ln(1+\epsilon) \approx \epsilon$ to write

$$\ln\left(\frac{a}{b}\right) = \ln\left(1 + \frac{a-b}{b}\right) \approx \frac{a-b}{b}. \qquad (266)$$

So the capacitance becomes $C \approx 2\pi\epsilon_0 bL/(a-b)$. But $2\pi bL$ is the area A of the inner cylinder, and $a - b$ is the separation s between the cylinders. So the capacitance can be written as $C = \epsilon_0 A/s$, which agrees with the standard result for the parallel-plate capacitor.