Physics 161, spring 2015

Answer Key for HW # 3

The first four problems are from Schey's chapter two. The following five are from Purcell and Morin (three from chapter two and two from chapter 3), while the last problem arose in class on Friday.

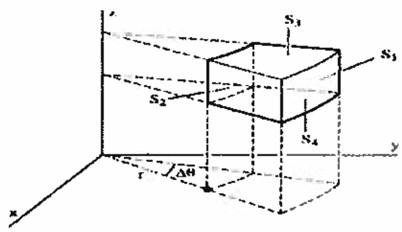
1. Schey II-20.

20. For the side S_1 we have

$$\iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS \cong F_\theta\left(r,\theta+\frac{\Delta\theta}{2},z\right)\Delta r\Delta z,$$

and for S_2,

$$\iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,dS \cong -F_\theta\left(r,\theta-\frac{\Delta\theta}{2},z\right)\Delta r\Delta z.$$



Thus

$$\frac{1}{\Delta V}\iint_{S_1+S_2} \mathbf{F}\cdot\mathbf{n}\,dS \cong \frac{1}{r\Delta\theta\Delta r\Delta z}\left[F_\theta\left(r,\theta+\frac{\Delta\theta}{2},z\right) - F_\theta\left(r,\theta-\frac{\Delta\theta}{2},z\right)\right]$$

$$= \frac{1}{r}\,\frac{F_\theta\left(r,\theta+\frac{\Delta\theta}{2},z\right) - F_\theta\left(r,\theta-\frac{\Delta\theta}{2},z\right)}{\Delta\theta} \rightarrow \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} \text{ as } \Delta\theta\rightarrow 0.$$

For S_3,

$$\iint_{S_3} \mathbf{F}\cdot\mathbf{n}\,dS \cong F_z\left(r,\theta,z+\frac{\Delta z}{2}\right)r\Delta\theta\Delta r,$$

and for S_4,

$$\iint_{S_4} \mathbf{F}\cdot\mathbf{n}\,dS \cong -F_z\left(r,\theta,z-\frac{\Delta z}{2}\right)r\Delta\theta\Delta r.$$

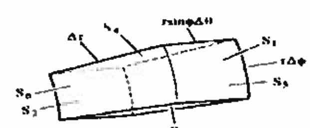
Hence

$$\frac{1}{\Delta V}\iint_{S_3+S_4} \mathbf{F}\cdot\mathbf{n}\,dS \cong \frac{F_z\left(r,\theta,z+\frac{\Delta z}{2}\right) - F_z\left(r,\theta,z-\frac{\Delta z}{2}\right)}{\Delta z} \rightarrow \frac{\partial F_z}{\partial z} \text{ as } \Delta z\rightarrow 0.$$

2. Schey II-21.

21. For sides S_1 and S_2 we have

$$\iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS \cong F_r\left(r+\frac{\Delta r}{2},\phi,\theta\right)r^2\sin\phi\Delta\phi\Delta\theta$$



and

$$\iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,dS \cong -F_r\left(r-\frac{\Delta r}{2},\phi,\theta\right)r^2\sin\phi\Delta\phi\Delta\theta.$$

Hence

$$\frac{1}{\Delta V}\iint_{S_1+S_2} \mathbf{F}\cdot\mathbf{n}\,dS \cong \frac{F_r\left(r+\frac{\Delta r}{2},\phi,\theta\right) - F_r\left(r-\frac{\Delta r}{2},\phi,\theta\right)}{\Delta r} \rightarrow \frac{\partial F}{\partial r} \text{ as } \Delta r\rightarrow 0.$$

For sides S_3 and S_4 we have

$$\iint_{S_3} \mathbf{F}\cdot\mathbf{n}\,dS \cong F_\phi\left(r,\phi+\frac{\Delta\phi}{2},\theta\right)r\sin\left(\phi+\frac{\Delta\phi}{2}\right)\Delta r\Delta\theta,$$

and

$$\iint_{S_4} \mathbf{F}\cdot\mathbf{n}\,dS \cong -F_\phi\left(r,\phi-\frac{\Delta\phi}{2},\theta\right)r\sin\left(\phi-\frac{\Delta\phi}{2}\right)\Delta r\Delta\theta.$$

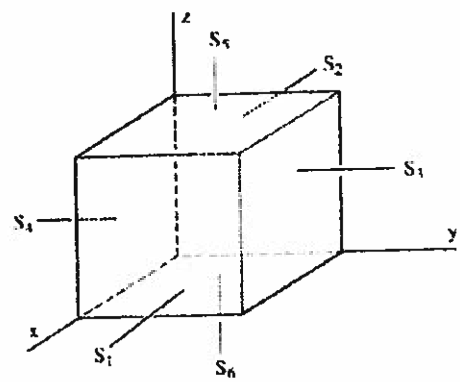
Thus

$$\frac{1}{\Delta V}\iint_{S_3+S_4} \mathbf{F}\cdot\mathbf{n}dS \cong \frac{F_\phi\left(r,\phi+\frac{\Delta\phi}{2},\theta\right)\sin\left(\phi+\frac{\Delta\phi}{2}\right) - F_\phi\left(r,\phi-\frac{\Delta\phi}{2},\theta\right)\sin\left(\phi-\frac{\Delta\phi}{2}\right)}{r\sin\theta\Delta\theta}$$

$$\to \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}(F_\phi \sin\phi) \quad \text{as } \Delta\phi\to 0.$$

Schey II-23.

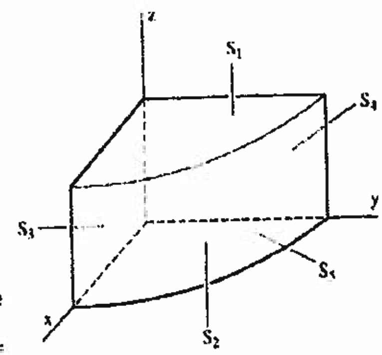
23. a. For faces S_1 and S_2 we have



$\iint_{S_1} \mathbf{F}\cdot\mathbf{n}dS = \iint_{S_1} xdydx = b\iint_{S_1} dydx =$ b^3, and $\iint_{S_2} \mathbf{F}\cdot\mathbf{n}dS = -\iint_{S_2} xdydx = 0$ because $x = 0$ on S_2. In exactly the same way S_3 and S_5 each yield b^3 and S_4 and S_6 both give 0. Hence $\iint_S \mathbf{F}\cdot\mathbf{n}dS$ $= 3b^3$. But

$$\nabla\cdot\mathbf{F} = \nabla\cdot(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z) = 3. \text{ Thus } \iiint_V \nabla\cdot\mathbf{F}\ dV = 3\iiint_V dV = 3b^3.$$

b. On S_1 we have



$\iint_{S_1} \mathbf{F}\cdot\mathbf{n}dS = \iint_{S_1} zdS = h\iint_{S_1} dS = \pi R^2h/4,$

because $z = h$ on S_1. On S_2, $\iint_{S_2} \mathbf{F}\cdot\mathbf{n}dS =$ $-\iint_{S_2} z\ dS = 0$, because $z = 0$ on S_2.

Next, $\iint_{S_3} \mathbf{F}\cdot\mathbf{n}dS = \iint_{S_4} \mathbf{F}\cdot\mathbf{n}dS = 0$ because $\hat{\mathbf{n}} = \pm\hat{\mathbf{e}}_\theta$ on S_3 and S_4 and $\mathbf{F}\cdot\hat{\mathbf{n}} = F_\theta =$ 0. Finally,

$\iint_{S_5} \mathbf{F}\cdot\mathbf{n}dS = \iint_{S_5} \mathbf{F}\cdot\hat{\mathbf{e}}_r\ dS = \iint_{S_5} r\ dS = R\iint_{S_5} dS = R\left[\frac{2\pi Rh}{4}\right] = \frac{\pi R^2h}{2}$. Adding the non-zero contributions from S_1 and S_5, we get $\iint_S \mathbf{F}\cdot\mathbf{n}dS = \frac{3\pi R^2h}{4}$.

Next we have $\nabla\cdot\mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(r\cdot r) + \frac{1}{r}\frac{\partial}{\partial\theta}(0) + \frac{\partial}{\partial z}(z) = 3.$ Thus the volume integral is

$$\iiint_V \nabla\cdot\mathbf{F}\ dV = 3\iiint_V dV = 3\left(\frac{\pi r^2h}{4}\right) = \frac{3\pi R^2h}{4}.$$

Answer Key #3, p2

c. $\iint_S \mathbf{F}\cdot\mathbf{n}dS = \iint_S r^2\hat{\mathbf{e}}_r\cdot\hat{\mathbf{e}}_r dS = R^2\iint_S dS = 4\pi R^2$. But $\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\cdot r^2)$

$= 4r$. Therefore $\iiint_V \nabla\cdot\mathbf{F}\ dV = \iiint_V 4r dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} 4r\cdot r^2\sin\theta\ dr\ d\theta\ d\phi =$

$4\pi R^2$.

Problem 3: Schey II-24

24\. a. Using the divergence theorem, $\iint_S \mathbf{B}\cdot\hat{\mathbf{n}}\ dS = \iiint_V \nabla\cdot\mathbf{B}\ dV = 0$ because $\nabla\cdot\mathbf{B} = 0$.

b. From (a) $\iint_S \mathbf{B}\cdot\hat{\mathbf{n}}\ dS = \iint_{CS} \mathbf{B}\cdot\hat{\mathbf{n}}\ dS + \iint_{Base} \mathbf{B}\cdot\hat{\mathbf{n}}\ dS = 0$, where CS means the curved surface of the cone. It follows then that $\iint_{CS} \mathbf{B}\cdot\hat{\mathbf{n}}dS$

$= -\iint_{Base} \mathbf{B}\cdot\hat{\mathbf{n}}\ dS$. But on the base $\mathbf{B}\cdot\hat{\mathbf{n}} = -B$, because the normal points outward from the volume. Hence

$$\iint_{CS} \mathbf{B}\cdot\hat{\mathbf{n}}\ dS = B\iint_{Base} dS = \pi R^2 B.$$

Problem 5: PM 2.31 (next page)
Problem 6: PM 2.43

2.43. Potential from a rod

At point P_1 in Fig. 36 we have

$$\phi_1 = \frac{1}{4\pi\epsilon_0}\int_{-d}^{d}\frac{\lambda\, dz}{2d-z} = -\frac{\lambda}{4\pi\epsilon_0}\ln(2d-z)\Big|_{-d}^{d} = -\frac{\lambda}{4\pi\epsilon_0}\ln\frac{d}{3d} = \frac{\lambda}{4\pi\epsilon_0}\ln 3. \quad (124)$$

At point P_2 with a general x value, we have (using the integral table in Appendix K)

$$\phi_2 = \frac{1}{4\pi\epsilon_0}\int_{-d}^{d}\frac{\lambda\, dz}{\sqrt{x^2+z^2}} = \frac{\lambda}{4\pi\epsilon_0}\ln\left(\sqrt{x^2+z^2}+z\right)\Big|_{-d}^{d} = \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{\sqrt{x^2+d^2}+d}{\sqrt{x^2+d^2}-d}\right). \quad (125)$$

These two potentials are equal when

$$\frac{\sqrt{x^2+d^2}+d}{\sqrt{x^2+d^2}-d} = 3 \implies 4d = 2\sqrt{x^2+d^2} \implies x = \sqrt{3}\,d. \quad (126)$$

Problem 5: PM2.31

2.31. Finding the potential

The line integral along the first path is (we'll suppress the z component of the argument)

$$\begin{aligned}\int_{(0,0)}^{(x_1,y_1)} \mathbf{E}\cdot d\mathbf{s} &= \int_0^{x_1} E_x(x,0)\,dx + \int_0^{y_1} E_y(x_1,y)\,dy \\ &= 0 + \int_0^{y_1} (3x_1^2 - 3y^2)\,dy = 3x_1^2 y_1 - y_1^3. \end{aligned} \tag{99}$$

The line integral along the second path is

$$\begin{aligned}\int_{(0,0)}^{(x_1,y_1)} \mathbf{E}\cdot d\mathbf{s} &= \int_0^{y_1} E_y(0,y)\,dy + \int_0^{x_1} E_x(x,y_1)\,dx \\ &= \int_0^{y_1} (0 - 3y^2)\,dy + \int_0^{x_1} 6xy_1\,dx = -y_1^3 + 3x_1^2 y_1. \end{aligned} \tag{100}$$

These two results are equal, as desired. The electric potential ϕ, if taken to be zero at $(0,0)$, is just the negative of our result, because we define ϕ by $\phi = -\int \mathbf{E}\cdot d\mathbf{s}$, or equivalently $\mathbf{E} = -\nabla\phi$. Hence $\phi(x,y) = y^3 - 3x^2y$. The negative gradient of this is

$$-\nabla\phi = -\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = (6xy,\, 3x^2 - 3y^2,\, 0), \tag{101}$$

which does indeed equal the given $\mathbf{E}$.

An alternative method of finding ϕ is to integrate the components of $\mathbf{E}$ to find the general form that ϕ must take. Since $-\partial\phi/\partial x$ equals $E_x = 6xy$, we see that $-\phi$ must take the form of $3x^2y + f(y,z)$, where $f(y,z)$ is an arbitrary function of y and z. Likewise, since $-\partial\phi/\partial y$ equals $E_y = 3x^2 - 3y^2$, we see that $-\phi$ must take the form of $3x^2y - y^3 + g(x,z)$. Finally, since $-\partial\phi/\partial z$ equals $E_z = 0$, we see that $-\phi$ must take the form of $0 + h(x,y)$, that is, ϕ is a function of only x and y. The only function consistent with all three of these forms is $-\phi = 3x^2y - y^3$ (plus a constant), in agreement with the above result.

Problem 7: PM2.61

2.61. Dipole field on the axes

With the charges q and $-q$ located at $z = \ell/2$ and $-\ell/2$, consider a distant point on the positive z axis with $z = r$. The charge q is slightly closer than the charge $-q$ is to this point, so the upward field due to the charge q is slightly stronger than the downward field due to the charge $-q$. The net field will therefore point upward, and it has magnitude (with $k \equiv 1/4\pi\epsilon_0$)

$$\begin{aligned} E = \frac{kq}{(r-\ell/2)^2} - \frac{kq}{(r+\ell/2)^2} &= \frac{kq}{r^2}\left(\frac{1}{(1-\ell/2r)^2} - \frac{1}{(1+\ell/2r)^2}\right) \\ &\approx \frac{kq}{r^2}\left(\frac{1}{1-\ell/r} - \frac{1}{1+\ell/r}\right), \end{aligned} \tag{175}$$

where we have dropped terms of order ℓ^2/r^2. Using $1/(1\pm\epsilon) \approx 1 \mp \epsilon$, we obtain

$$E \approx \frac{kq}{r^2}\left(\left(1+\frac{\ell}{r}\right) - \left(1-\frac{\ell}{r}\right)\right) = \frac{2kq\ell}{r^3}. \tag{176}$$

This field points in the positive $\hat{\mathbf{r}}$ direction, so it agrees with the result in Eq. (2.36),

$$\frac{kq\ell}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right), \tag{177}$$

when $\theta = 0$.

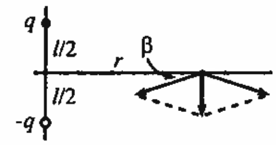


Figure 47

In the transverse direction, we have the situation shown in Fig. 47. The magnitudes of the two fields are equal. The horizontal components cancel, but the downward components add. The distances from the given point to the two charges are essentially equal to r, so the magnitudes of the fields are kq/r^2. The (negative) vertical components are obtained by multiplying by $\sin\beta$, which is approximately equal to $(\ell/2)/r$ in the small-angle approximation. The vertical field is therefore directed downward with magnitude

$$E \approx 2\left(\frac{kq}{r^2}\right)\frac{\ell/2}{r} = \frac{kq\ell}{r^3}. \tag{178}$$

This agrees with the result in Eq. (2.36) when $\theta = \pi/2$, because the $\hat{\boldsymbol{\theta}}$ vector points downward at the given point (in the direction of increasing θ, which is measured down from the vertical). This field is half as large as the field on the vertical axis, for a given value of r.

Problem 8: PM 3.75
Problem 9: PM 3.76

3.75. Average of six points

The Taylor expansions are

$$\begin{aligned}
\phi(x_0+\delta, y_0, z_0) &= \phi(x_0, y_0, z_0) + \delta\frac{\partial\phi}{\partial x} + \frac{\delta^2}{2!}\frac{\partial^2\phi}{\partial x^2} + \frac{\delta^3}{3!}\frac{\partial^3\phi}{\partial x^3} + \cdots,\\
\phi(x_0-\delta, y_0, z_0) &= \phi(x_0, y_0, z_0) - \delta\frac{\partial\phi}{\partial x} + \frac{\delta^2}{2!}\frac{\partial^2\phi}{\partial x^2} - \frac{\delta^3}{3!}\frac{\partial^3\phi}{\partial x^3} + \cdots,
\end{aligned} \tag{327}$$

and likewise for the $y_0 \pm \delta$ and $z_0 \pm \delta$ points. When we add up all six terms and divide by 6 to take the average, the terms with odd powers of δ cancel in pairs, and we are left with

$$\phi_{\text{avg}} = \frac{1}{6}\left[6\phi(x_0, y_0, z_0) + \delta^2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) + \mathcal{O}(\delta^4)\right]. \tag{328}$$

If $\nabla^2\phi = 0$, then the δ^2 term here is zero, so we have

$$\phi_{\text{avg}} = \phi(x_0, y_0, z_0) + \mathcal{O}(\delta^4), \tag{329}$$

which equals $\phi(x_0, y_0, z_0)$ through terms of order δ^3, as desired.

3.76. The relaxation method

After four iterations, the values of a through g, as they would appear in the array, are (at least for the order of my path running through the array):

29.2	61.9
26.2	56.5
18.4	37.5
9.2	

These values are close to the true values (which settle down after about 25 iterations), obtained from the computer program in Exercise 3.77:

29.2135	61.7978
25.8427	56.1798
17.9775	37.0787
8.98876	

Using either of the above sets of ϕ values, we see that the $\phi = 25$ and $\phi = 50$ equipotentials will look something like the ones shown in Fig. 84.

10. Friday's problem. We considered a potential $\varphi(x,y) = x^2 - y^2$. This scalar function satisfies LaPlace's equation so Theorem 2.1 in P&M applies. *For a scalar function that satisfies LaPlace's equation, the average value of the function on the surface of any sphere is equal to the value of the function at the center of the sphere.*

This is a 2D problem so a circle surrounding the point (x_0, y_0) serves as the sphere. Calculate the average value of ϕ on a circle of radius *r* centered at (x_0, y_0) by computing the value of the line integral $\oint \phi(x,y)dl$ around the circle and then dividing the value of the integral by the arc length $2\pi r$. (This integral is easier to do after converting to circular coordinates.)

Solution: consider a circle, centered at (x_0, y_0) with radius r. When the radius is inclined at the angle θ with respect to the horizontal axis, it is located a distance $x = r\cos\theta$ with respect to the center, or at location $x = x_0 + r\cos\theta$; likewise, the same position at the center has a v-value of $y = y_0 + r\sin\theta$. The function f(x, y) can then be written in cylindrical coordinates as the following:

$$f(x,y) = x^2 - y^2 = \left(x_0 + r\cos(\theta)\right)^2 - \left(y_0 + r\sin(\theta)\right)^2 =$$
$$\left(x_0^2 - y_0^2\right) + 2r\left(x_0\cos(\theta) - y_0\sin(\theta)\right) + r^2\left(\cos^2(\theta) - \sin^2(\theta)\right).$$

To compute the average of the function, we calculate the function at every place around a circle surrounding (x_0, y_0) and then do a line integral to add up the values, and finally divide the integral by the length of the circle, i.e. $2\pi r$.

There are three integrals to be done: the first integral is simple. $\oint\left(x_0^2 - y_0^2\right)r d\theta = 2\pi r\left(x_0^2 - y_0^2\right)$. The second integral contains the cosine and sine functions but integrating them over the range 0 to 2π leads to a value of zero. The third integral involves $\oint\cos^2(\theta)r d\theta = \oint\sin^2(\theta)r d\theta$, so taking the difference yields zero.

In conclusion, the average value of the function on a circle of radius r around point (x_0, y_0) is the value of the integral ($2\pi r\left(x_0^2 - y_0^2\right)$) divided by the length ($2\pi r$), leaving $x_0^2 - y_0^2$, the value at the center of the circle. QED. ■.

Bonus: Mathematic Solution.

3.77. **Relaxation method, numerical**

Here is a *Mathematica* program that gets the job done for an 18×18 array instead of the 9×9 array that appeared in Exercise 3.76:

```
ITS=150; (* number of iterations *)
n=6;     (* width of box in middle; 1/3 size of whole array *)
NN=3n+1; (* number of lattice points in each direction *)
T=Table[100, {i,1,NN},{j,1,NN}];  (* create matrix with all entries = 100 *)
(* create an initial array with equipotential squares varying linearly from 0
on the outer boundary to 100 on the inner square: *)
```

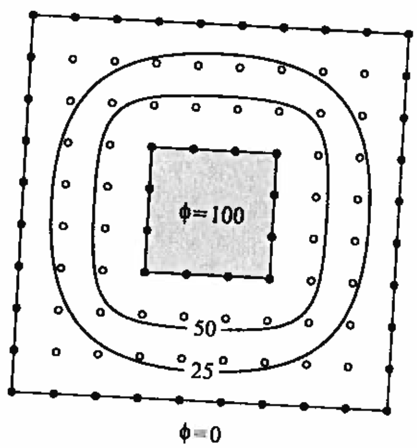


Figure 84

```
Do[Do[T[[i,j]]=(100./n)*(k-1), {i,k,NN-(k-1)},{j,k,NN-(k-1)}], {k,1,n+1}];
(* now do the iterative averaging: *)
Do[
 (* averaging for rows above middle box: *)
 Do[Do[T[[i,j]]=(T[[i,j-1]]+T[[i,j+1]]+T[[i-1,j]]+T[[i+1,j]])/4.,
       {j,2,NN-1}],{i,2,n}];
 (* averaging for rows left of middle box: *)
 Do[Do[T[[i,j]]=(T[[i,j-1]]+T[[i,j+1]]+T[[i-1,j]]+T[[i+1,j]])/4.,
       {j,2,n}],{i,n+1,2n+1}];
 (* averaging for rows right of middle box: *)
 Do[Do[T[[i,j]]=(T[[i,j-1]]+T[[i,j+1]]+T[[i-1,j]]+T[[i+1,j]])/4.,
       {j,2n+2,NN-1}],{i,n+1,2n+1}];
 (* averaging for rows below middle box: *)
 Do[Do[T[[i,j]]=(T[[i,j-1]]+T[[i,j+1]]+T[[i-1,j]]+T[[i+1,j]])/4.,
       {j,2,NN-1}],{i,2n+2,NN-1}],
{it,1,ITS}]     (* repeat averaging process ITS times *)
PaddedForm[MatrixForm[T], {6, 3}]     (* print matrix *)
```

The results for the "triangle" of entries analogous to those in Exercise 3.76 are:

14.357	29.179	44.917	61.945	80.423
14.124	**28.722**	44.271	**61.221**	79.872
13.416	27.313	42.224	58.796	77.846
12.227	**24.892**	38.515	**53.892**	72.717
10.601	21.512	33.052	45.541	59.129
8.666	**17.503**	26.641	**36.091**	
6.561	13.192	19.917		
4.386	**8.789**			
2.193				

The bold entries correspond to the seven entries in Exercise 3.76. The agreement is reasonably good. If the middle box is 48×48 instead of the above 6×6 or the 3×3 we had in Exercise 3.76, then, for example, the 36.091 entry becomes 35.2961. By looking at how this number changes with the size of the box, it appears to converge to approximately 35.2 in the continuum limit involving an infinite number of lattice points.

Interestingly, the computing time doesn't appear to be helped much by our choice of initial equipotentials that varied linearly from the outer boundary to the central square. If we had instead picked $\phi = 0$ on the outer boundary and $\phi = 100$ at every other point (*all* the other points, not just the ones in the middle square), then the computing time would be only slightly longer. The computing time increases by a factor of 16 for every doubling of the array's width, because there are 4 times as many points that each iteration needs to run through, and also it turns out that we need to do 4 times as many iterations to achieve a given accuracy.

Key #3, p8