PHYS 161: Homework 7

Due on Wednesday March 18, 2015

Professor Landee 11:00am

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Problem 1

(5.2) *Maximum horizontal force*
A charge q_1 is at rest at the origin, and a charge q_2 moves with speed βc in the x direction, along the line $z = b$. For what angle θ shown in Fig. 5.27 will the horizontal component of the force on q_1 be maximum? What is θ in the $\beta \approx 1$ and $\beta \approx 0$ limits?
Solution
The radial field of the charge q_2 is

$$E = \frac{q_2}{4\pi\epsilon_0 r^2} \frac{1-\beta^2}{\left(1-\beta^2\sin^2\theta\right)^{3/2}}$$

where $r = b/\sin\theta$. So the x component on the force on q_1 is

$$F_x = \frac{q_1 q_2 \sin^2\theta}{4\pi\epsilon_0 b^2} \frac{1-\beta^2}{\left(1-\beta^2\sin^2\theta\right)^{3/2}} \cos\theta$$

so

$$F_x{}' = \frac{\left(\beta^2-1\right) q_1 q_2 \sin\theta \left(\left(\beta^2-3\right)\cos(2\theta) - \beta^2 - 1\right)}{\sqrt{2}\pi\beta^2\epsilon_0 \left(\beta^2\cos(2\theta) - \beta^2 + 2\right)^{5/2}}$$

Let $F_x{}' = 0$, the solution is $\theta = k\pi$, or $\theta = \pm\frac{1}{2}\cos^{-1}\left(\frac{\beta^2+1}{\beta^2-3}\right) + k\pi$, where $k \in \mathbb{Z}$. Identify that in the range of $(0, \pi/2)$ the root $\theta = \frac{1}{2}\cos^{-1}\left(\frac{\beta^2+1}{\beta^2-3}\right)$ gives the maximum.
When $\beta \approx 1$, $\theta = \pi/2 = 90^\circ$. When $\beta \approx 0$, $\theta = \frac{1}{2}\cos^{-1}(-1/3) \approx 54.7^\circ$.

Problem 2

(5.5) *E from a line of moving charges*
An essentially continuous stream of point charges moves with speed v along the x axis. The stream extends from $-\infty$ to $+\infty$. Let the charge density per unit length be λ, as measured in the lab frame. We know from using a cylindrical Gaussian surface that the electric field a distance r from the x axis is $E = \lambda/2\pi\epsilon_0 r$. Derive this result again by using Eq. (5.15) and integrating over all of the moving charges. You will want to use a computer or the integral table in Appendix K.
Solution
Consider a small section dq of the stream, making an angle $d\theta$ to the point P. The angle made between the perpendicular line from P to the stream, r, and the line between P and dq is θ.

$$dq = \lambda \frac{r}{\cos^2\theta}\, d\theta$$

The field is

$$dE = \frac{dq}{4\pi\epsilon_0 (r/\cos\theta)^2} \frac{1-\beta^2}{\left(1-\beta^2\cos^2\theta\right)^{3/2}}$$

and by symmetry of the setup, only y component of the field exists in the result. So

$$E = \int_{-\pi/2}^{\pi/2} \frac{\lambda \frac{r}{\cos^2\theta}\, d\theta}{4\pi\epsilon_0 (r/\cos\theta)^2} \frac{1-\beta^2}{\left(1-\beta^2\cos^2\theta\right)^{3/2}} \cos\theta$$

Using a computer algebraic system, the result of this integral is exactly

$$\frac{\lambda}{2\pi\epsilon_0 r}$$

Problem 3

(5.8) *Finding the magnetic field*
Consider the second scenario in the example at the end of Section 5.8. Show that the total force in frame F' equals the sum of the electric and magnetic forces, provided that there is a magnetic field pointing out of the page with magnitude $\gamma v E_2/c^2$.
Solution
In the example, total force is qE_2/γ, and the electric force is $\gamma q E_2$, repulsive. So the attractive magnetic force must be of the magnitude

$$\gamma q E_2 - \frac{qE_2}{\gamma} = \gamma q E_2 \left(1 - \frac{1}{\gamma^2}\right) = \gamma q E_2 \left(\frac{v^2}{c^2}\right) = qv\frac{\gamma v E_2}{c^2}$$

Because of the formula $F_B = qvB$, $B = \gamma v E_2/c^2$.

Problem 4

(5.9) *"Twice" the velocity*
Suppose that the velocity of the test charge in Fig. 5.22 is chosen so that in its frame the electrons move backward with speed v_0.

(a) Show that the β associated with the test charge's velocity in the lab frame must be $\beta = 2\beta_0/\left(1+{\beta_0}^2\right)$.

(b) Using length contraction, find the net charge density in the test-charge frame, and check that it agrees with Eq. (5.24).

Solution

$$\lambda' = \gamma\beta\beta_0\lambda_0 \tag{5.24}$$

(a) Test charge seeing electrons moving backwards at v_0 is equivalent to electrons seeing test charge moving forward at v_0. $\beta_0 = v_0/c$. Velocity-addition formula states that

$$\beta = \frac{v}{c} = \frac{\frac{\beta_0 c + \beta_0 c}{1 + \frac{(\beta_0 c)^2}{c^2}}}{c} = \frac{2\beta_0}{1+{\beta_0}^2}$$

(b)

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1+{\beta_0}^2}{1-{\beta_0}^2}$$

The test-charge is contracted, so density becomes $\lambda_0\gamma$. Electrons move in opposite direction, but the speed is the same v_0. So density does not change. Total density then is

$$\lambda' = \lambda_0(\gamma - 1) = \lambda_0\frac{2{\beta_0}^2}{1-{\beta_0}^2} = \lambda_0\beta_0\frac{2\beta_0}{1+{\beta_0}^2}\frac{1+{\beta_0}^2}{1-{\beta_0}^2} = \gamma\beta\beta_0\lambda_0$$

Problem 5

(5.12) *Tilted sheet*
Redo the "Tilted sheet" example in Section 5.5 in terms of a general γ factor, to verify that Gauss's law holds for any choice of the relative speed of the two frames.
Solution

From Gauss's law $E = \frac{\sigma}{2\epsilon_0}$. For a general γ, we have

$$\sqrt{2}l\sigma = \sqrt{1+\left(\frac{1}{\gamma}\right)^2}\,l\sigma' \longrightarrow \sigma' = \frac{\sqrt{2}\gamma}{\sqrt{1+\gamma^2}}\sigma$$

$$E_{\parallel} = {E_{\parallel}}' = \frac{E}{\sqrt{2}}$$

$$\gamma E_{\perp} = {E_{\perp}}' = \frac{\gamma E}{\sqrt{2}}$$

So in the triangle

$$E' = \frac{E}{\sqrt{2}}\sqrt{1+\gamma^2}$$

$$\frac{{E_{\perp}}'}{{E_{\parallel}}'} = \gamma \longrightarrow \theta = \arctan\gamma$$

So

$$\begin{aligned}
{E_n}' &= E' \cos\left(2\theta - \frac{\pi}{2}\right) \\
&= \frac{E}{\sqrt{2}}\sqrt{1+\gamma^2}\cos\left(2\arctan\gamma - \frac{\pi}{2}\right) \\
&= \frac{E}{\sqrt{2}}\sqrt{1+\gamma^2}\sin\left(2\arctan\gamma\right) \\
&= \frac{E}{\sqrt{2}}\sqrt{1+\gamma^2}\frac{2\gamma}{1+\gamma^2} \\
&= \frac{\sqrt{2}\gamma}{\sqrt{1+\gamma^2}}E \\
&= \frac{\sqrt{2}\gamma}{\sqrt{1+\gamma^2}}\frac{\sigma}{2\epsilon_0} \\
&= \frac{\sigma'}{2\epsilon_0}
\end{aligned}$$

So Guass's law holds in frame F'.

Problem 6

(5.15) *Gauss's law for a moving charge*
Verify that Gauss's law holds for the electric field in Eq. (5.15). That is, verify that the flux of the field, through a sphere centered at the charge, is q/ϵ_0. Of course, we used this fact in deriving Eq. (5.15) in the first place, so we know that it must be true. But it can't hurt to double check. You'll want to use a computer or the integral table in Appendix K.
Solution

$$E' = \frac{Q}{4\pi\epsilon_0 {r'}^2}\frac{1-\beta^2}{\left(1-\beta^2\sin^2\theta'\right)^{3/2}} \tag{5.15}$$

So for charge q in $\hat{\mathbf{r}}$ direction

$$\begin{aligned}
\oint \mathbf{E}' \cdot d\mathbf{a}' &= \oint \frac{q}{4\pi\epsilon_0 r'^2} \frac{1-\beta^2}{\left(1-\beta^2\sin^2\theta'\right)^{3/2}} r'^2 \sin\theta' \, d\theta' \, d\phi' \\
&= \frac{q}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^{\pi} \frac{1-\beta^2}{\left(1-\beta^2\sin^2\theta'\right)^{3/2}} \sin\theta' \, d\theta' \, d\phi' \\
&= \frac{q}{2\epsilon_0} \int_0^{\pi} \frac{1-\beta^2}{\left(1-\beta^2\sin^2\theta'\right)^{3/2}} \sin\theta' \, d\theta' \\
&= \frac{q}{\epsilon_0}
\end{aligned}$$

Problem 7

(5.22) *Electron in an oscilloscope*

The deflection plates in a high-voltage cathode ray oscilloscope are two rectangular plates, 4 cm long and 1.5 cm wide, and spaced 0.8 cm apart. There is a difference in potential of 6000 V between the plates. An electron that has been accelerated through a potential difference of 250 kV enters this deflector from the left, moving parallel to the plates and halfway between them, initially. We want to find the position of the electron and its direction of motion when it leaves the deflecting field at the other end of the plates. We shall neglect the fringing field and assume the electric field between the plates is uniform right up to the end. The rest energy of the electron may be taken as 500 keV.

(a) First carry out the analysis in the lab frame by answering the following questions:

- What are the values of γ and β?
- What is p_x in units of mc?
- How long does the electron spend between the plates? (Neglect the change in horizontal velocity discussed in Exercise 5.25.)
- What is the transverse momentum component acquired, in units of mc?
- What is the transverse velocity at exit?
- What is the vertical position at exit?
- What is the direction of flight at exit?

(b) Now describe this whole process as it would appear in an inertial frame that moved with the electron at the moment it entered the deflecting region. What do the plates look like? What is the field between them? What happens to the electron in this coordinate system? Your main object in this exercise is to convince yourself that the two descriptions are completely consistent.

Solution

The electron is accelerated through 250 kV, so it gains $K = 250\,\text{keV}$. The total energy becomes $E = K + mc^2 = 750\,\text{kV} = 1.5mc^2$.

(a)
- So $\gamma = 1.5$. Then $\beta = \sqrt{1-(1/\gamma)^2} = 0.7454$.
- $p_x = \gamma\beta mc$. $p_x/mc = \gamma\beta = 1.1180$.
- $t = \frac{4\,\text{cm}}{\beta c} = 1.7901 \times 10^{-10}\,\text{s}$.
- $p_y = F_y t = eEt = eVt/s$. So $\frac{p_y}{mc} = \frac{eVt}{smc} = 0.0805$.

- $p_y = \gamma m v_y \longrightarrow v_y = 1.6089 \times 10^7\,\mathrm{m\,s^{-1}}$.
- $y = \frac{v_y t}{2} = 1.4400 \times 10^{-3}\,\mathrm{m}$.
- $\theta = \arctan \frac{p_y}{p_x} = 4.1183°$.

(b) In the frame where the electron is at rest, the planes are moving to $-x$ direction with $\beta = 0.7454$. The contracted length of the plates is $d' = d/\gamma = 2.6667\,\mathrm{cm}$. The electron takes $t' = d'/\beta$ to be accelerated in $E' = \gamma E$.

Therefore $p_y = eEt = eE'T'$, since y direction is not change in the transformation.

Problem 8

(5.26) *Charges in a wire*
In Fig. 5.22 the relative spacing of the black and gray dots was designed to be consistent with $\gamma = 1.2$ and $\beta_0 = 0.8$. Calculate $\beta_0{}'$. Find the value, as a fraction of λ_0, of the net charge density λ' in the test-charge frame.
Solution Because

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} = 0.552771$$

$$\beta_0{}' = \frac{\beta_0 - \beta}{1 - \beta\beta_0} = 0.443235$$

$$\lambda' = \gamma\beta\beta_0\lambda_0 = 0.53066\lambda_0$$