PHYS 161: Homework 10

Due on Wednesday April 22, 2015

Professor Landee 11:00am

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Problem 1

(10.2) *Force on a dielectric...* (problem omitted)
Solution

(a)

$$C = C_1 + C_2 = \frac{\epsilon_0 a(b + x\chi_e)}{s}$$

So stored energy

$$U = \frac{Q^2}{2C} = \frac{Q^2 s}{2\epsilon_0 a(b + x\chi_e)}$$

(b)

$$F = -\frac{dU}{dx} = \frac{Q^2 s\chi_e}{2\epsilon_0 a(b + x\chi_e)^2}$$

And U decreases as x increases, so the force pulls the dielectric in.

Problem 2

(10.8) *Force from an induced dipole...* (problem omitted)
Solution
For a charge q, its electric field will be $E = \frac{q}{4\pi\epsilon_0 r^2}$. The induced dipole moment of the atom will be $p = \alpha E$, which points from the ion to the atom. The field at the ion due to the atom will be $\frac{2p}{4\pi\epsilon_0 r^3}$. So force

$$F = \frac{2pq}{4\pi\epsilon_0 r^3} = \frac{2\alpha q^2}{(4\pi\epsilon_0)^2 r^5}$$

which would be attractive force. The potential

$$U = -\int_\infty^r F\, dr' = -\frac{\alpha q^2}{2(4\pi\epsilon_0)^2 r^4}$$

From this equation one could solve for r for the potential at 4×10^{-21} J, which would be 9.4×10^{-10} m.

Problem 3

(10.12) *Boundary conditions on D...* (problem omitted)
Solution
$D_\perp$ is continuous, because $\nabla \cdot \mathbf{D} = \rho_{\text{free}}$. No free charge so the divergence is zero. By divergence theorem, in flux would equal to out flux. So $D_{\perp,\text{in}}A = D_{\perp,\text{out}}A$, which means $D_\perp$ is continuous across the boundary. Because there is no discontinuity in $E_\parallel$, $D_\parallel$ would also be continuous. $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$, therefore discontinuity in $D_\parallel$ is the same as in $P_\parallel$. $\mathbf{P} = 0$ outside, so $\Delta D_\parallel = -D_{\perp,\text{in}}$.

Problem 4

(10.14) *Boundary conditions on E and B...* (problem omitted)
Solution
The equations without free charges or currents are:

$$\nabla \cdot \mathbf{D} = 0, \quad \nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t; \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \mu_0 \partial\mathbf{D}/\partial t;$$

Problem 4 continued on next page...

Our equations tell us that the net flux out of the volume is zero, so the perpendicular field on one side must equal the perpendicular field on the other. And for the parallel components, we can apply Stokes' theorem to the two "curl" equations, with the area chosen to be a thin rectangle, of vanishing area, spanning the surface. Our equations tell us that the line integral around the rectangle is zero, so the parallel field on one side must equal the parallel field on the other. So

$$D_{1,\perp} = D_{2,\perp}, \quad E_{1,\parallel} = E_{2,\parallel}; \quad B_{1,\perp} = B_{2,\perp}, \quad B_{1,\parallel} = B_{2,\parallel};$$

Since $\mathbf{D} = \epsilon\mathbf{E}$ for a linear dielectric, $\epsilon_1 E_{1,\perp} = \epsilon_2 E_{2,\perp}$, therefore $E_{2,\perp}$ is discontinuous. But other components are, including the entire $\mathbf{B}$ field.

Problem 5

(10.15) *Charge densities on a capacitor*... (problem omitted)
Solution
We know that because the plates are a whole, V must be the same for two regions. So $(C_1 + C_2)V = Q$, where $C_1 = \frac{\epsilon_0 a(b-x)}{s}$, and $C_2 = \frac{\kappa\epsilon_0 ax}{s}$. Therefore

$$E = \frac{V}{s} = \frac{Q}{\epsilon_0 a(b-x) + \kappa\epsilon_0 ax} = \frac{Q}{\epsilon_0 a(b + x\chi_e)}$$

So

$$\sigma_1 = \epsilon_0 E = \frac{Q}{a(b + x\chi_e)}$$

$$\sigma_2 = \kappa\epsilon_0 E = \frac{\kappa Q}{a(b + x\chi_e)}$$

which are indeed decreasing as x increases.

Problem 6

(10.17) *Maximum energy storage*... (problem omitted)
Solution
We know dielectric strength

$$E_{\max} = \frac{14\,\mathrm{kV}}{0.002\,54\,\mathrm{cm}} = 5.512 \times 10^8\,\mathrm{V/m}$$

So

$$U_{\max} = \int \frac{\epsilon {E_{\max}}^2}{2}\,dV = \frac{\epsilon {E_{\max}}^2 \rho}{2} = \frac{\kappa\epsilon_0 {E_{\max}}^2 \rho}{2} = 6.12 \times 10^9\,\mathrm{J/kg}$$

The mass of the dielectric is 75% of the total mass. So

$$mgh = 0.75 U_{\max} m$$

and

$$h = \frac{3U_{\max}}{4g} = 4.68 \times 10^8\,\mathrm{m}$$

Problem 7

(10.24) *Field lines of a dipole*
A field line in the dipole field is described in polar coordinates by the very simple equation $r = r_0 \sin^2 \theta$, in which r_0 is the radius at which the field line passes through the equatorial plane of the dipole. Show that this is true by demonstrating that at any point on that curve the tangent has the same direction as the dipole field.
Solution
The equatorial plane is the plane bisecting the dipole. The equation $r = r_0 \sin^2 \theta \hat{r}$, which apperently should be pointing at the $\hat{r}$ direction, creates dumbell shapes around the dipole. So the tangent direction of the equation

$$\frac{\partial r}{\partial \theta} = r_0 \left(2 \sin \theta \cos \theta \hat{r} + \sin^2 \theta \frac{\partial \hat{r}}{\partial \theta} \right) = r_0 \sin \theta \left(2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right)$$

And the field far away from the dipole is

$$\mathbf{E} = \frac{P}{4\pi\epsilon_0 r^2} \left(2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right)$$

So they are pointing at the same direction.

Problem 8

(10.37) *E at the center of a polarized sphere*
If you don't trust the $\mathbf{E} = -\mathbf{P}/3\epsilon_0$ result we obtained in Section 10.9 for the field inside a uniformly polarized sphere, you will find it more believable if you check it in a special case. By direct integration of the contributions from the $\sigma = P \cos \theta$ surface charge density, show that the field at the center is directed downward (assuming $\mathbf{P}$ points upward) with magnitude $P/3\epsilon_0$.
Solution
Consider a ring on the spherical sheet. An element

$$\begin{aligned} dq &= \sigma \; da \\ &= P \cos \theta R \sin \theta R \; d\theta \; d\phi \\ &= PR^2 \sin \theta \cos \theta \; d\theta \; d\phi \\ &= 2\pi PR^2 \sin \theta \cos \theta \; d\theta \end{aligned}$$

So the vertical component of the field is

$$dE_z = \frac{1}{4\pi\epsilon_0 R^2} dq \cdot \cos \theta = \frac{P \sin \theta \cos^2 \theta \; d\theta}{2\epsilon_0}$$

The field is then

$$E_z = \int dE_z = \frac{P}{2\epsilon_0} \int_0^\pi \sin \theta \cos^2 \theta \; d\theta = \frac{P}{2\epsilon_0} \frac{2}{3} = \frac{P}{3\epsilon_0}$$

Problem 9

(10.41) *Discontinuity in $D_\parallel$*
Consider the polarized sphere from Section 10.9. Using the forms of the internal and external electric fields, find the discontinuity in $D_\parallel$ across the surface of the sphere, as a function of θ.

Problem 9 continued on next page. . .

Solution

$$\mathbf{D} = \epsilon\mathbf{E} + \mathbf{P}$$

For the inside $\mathbf{D} = \frac{2}{3}\mathbf{P}$, so the parallel component is

$$D_{\parallel,\text{in}} = \frac{2P\sin\theta}{3}\hat{\theta}$$

and the perpendicular component is

$$D_{\perp,\text{in}} = \frac{2P\cos\theta}{3}\hat{r}$$

The field outside is a dipole field

$$\mathbf{E}_{\text{out}} = \frac{P}{3\epsilon_0}\left(2\cos\theta\hat{r} + \sin\theta th\hat{e}ta\right)$$

and $\mathbf{D}_{\text{out}} = \epsilon_0\mathbf{E}_{\text{out}}$, so

$$D_{\parallel,\text{out}} = \frac{P\sin\theta}{3}\hat{\theta}$$

So the discontinuity is $\frac{P\sin\theta}{3}$.

Problem 10

(10.42) *Energy density in a dielectric*
By considering how the introduction of a dielectric changes the energy stored in a capacitor, show that the correct expression for the energy density in a dielectric must be $\epsilon E^2/2$. Then compare the energy stored in the electric field with that stored in the magnetic field in the wave studied in Section 10.15.

Solution

The energy stored in a capacitor is $\frac{1}{2}CV^2$, where in a parallel plate $C = \frac{\kappa\epsilon_0 A}{d}$, and $E = V/d$. So

$$U = \frac{1}{2}CV^2 = \frac{1}{2}\frac{\kappa\epsilon_0 A}{d}E^2d^2 = \frac{1}{2}\epsilon AE^2d$$

The energy density is U divided by volume Ad, which is exactly $\epsilon E^2/2$.
For the problem, the energy density of the electric field is

$$U_E = \frac{1}{2}\epsilon E^2$$

and of the magnetic field is

$$U_B = \frac{1}{2}\frac{B^2}{\mu_0}$$

The average value of the two fields are simply $\langle E\rangle = E_0/2$, and $\langle B\rangle = B_0/2$. Because in this field $E_0 = \pm\frac{B_0}{\sqrt{\mu_0\epsilon}}$,

$$U_E = \frac{1}{8}\epsilon\left(\frac{B_0}{\sqrt{\mu_0\epsilon}}\right)^2$$

$$U_B = \frac{1}{8}\frac{{B_0}^2}{\mu_0}$$

So the energy stored in each of the two fields turn out to be equal.