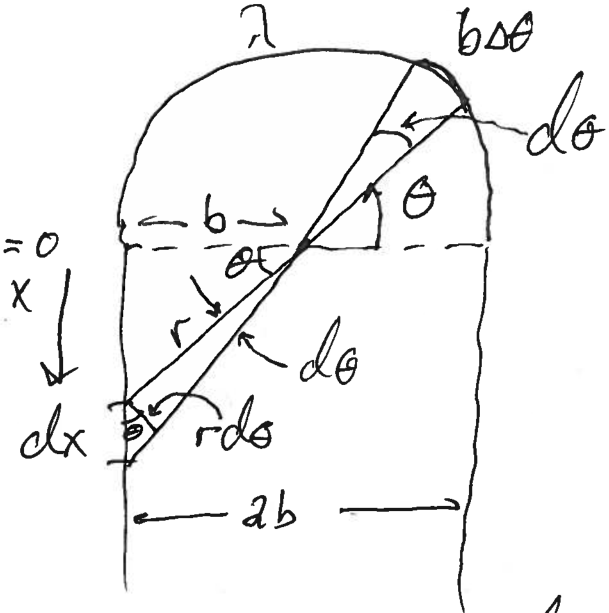
Physics 161, Spring 2015

HW #1. P&M Exercises 1.54
1.59
1.66
1.72
1.83



1.54 Semicircle and wires

a) Two charged wires capped by a semicircle of radius b. (Wires separated by distance 2b.) Find $\vec{E}$ at point C if there is a linear charge density λ along the wires.

$$\left| dE_{\substack{\text{circular}\\ \text{arc}}} \right| = \frac{1}{4\pi\varepsilon_0} \frac{(dq = \lambda b\, d\theta)}{b^2} = \frac{1}{4\pi\varepsilon_0} \frac{\lambda\, d\theta}{b}.$$

$$r = \frac{b}{\cos\theta}, \quad \cos\theta = \frac{r\,d\theta}{dx} \Rightarrow dx = \frac{r\,d\theta}{\cos\theta}.$$

$$\left| dE_{dx} \right| = \frac{1}{4\pi\varepsilon_0} \frac{(dq = \lambda\, r\,d\theta/\cos\theta)}{r^2 = \frac{b^2}{\cos^2\theta}} = \frac{1}{4\pi\varepsilon_0} \frac{\lambda\, d\theta}{b}.$$

The two contributions point in opposite directions and cancel exactly, for any value of θ. $\therefore$ $\underline{\vec{E}\,(\text{at } C) = 0}$

b) Consider the analogous 2D problem of a cylinder (diameter = 2b) and a hemispherical end cap of radius b, all with a surface charge density σ. Does the E-field a C still vanish? It appears so, at first, because we can extend the calculation in part a by rotating our increments of charge around the axis of symmetry, yielding a circular strip from the cylinder and a circular arc on top of the hemisphere

The increment of charge on the cylindrical strip equals:

$$dq = \sigma \, da_{strip} = \sigma\left(\frac{r\,d\theta}{\cos\theta}\right)(b\,d\phi) = \frac{b^2}{\cos^2\theta}\,d\theta\,d\phi = \frac{2\pi b^2}{\cos^2\theta}\,d\theta$$

$$dE_{strip} = \frac{1}{4\pi\varepsilon_0}\,\frac{dq}{\left(r^2 = \frac{b^2}{\cos^2\theta}\right)} = \frac{\sigma\, 2\pi \cancel{b^2}\,d\theta / \cancel{\cos^2\theta}}{\cancel{b^2}/\cancel{\cos^2\theta}} = \frac{\sigma\,d\theta}{2\varepsilon_0}$$

Increment of charge on circular arc $da = \sigma(b\,d\theta) \times \boxed{(b\cos\theta\,d\phi)}$

$$= 2\pi\sigma b^2 \cos\theta\,d\theta$$

$$dE_{arc} = \frac{1}{4\pi\varepsilon_0}\,\frac{dq}{b^2} = \frac{2\pi\sigma\,\cancel{b^2}\cos\theta\,d\theta\,d\phi}{4\pi\varepsilon_0 \times \cancel{b^2}} = \frac{\sigma\cos\theta\,d\theta}{2\varepsilon_0}$$

The field from the strip is always larger than the field from the arc at any angle, so the net electric field at C <u>points up</u>, towards the cap.

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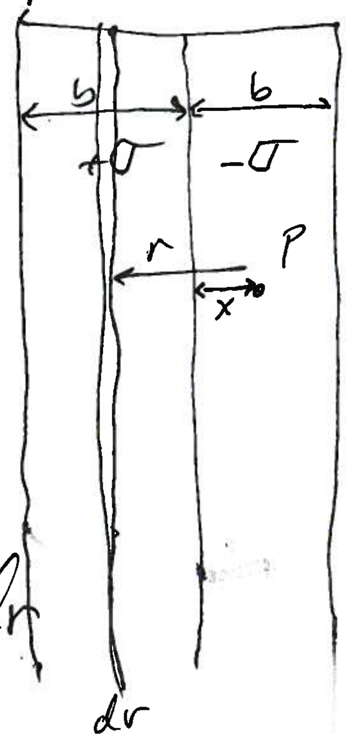
(2) 1.59 Zero field inside a cylindrical shell

See solution to Problem 1.17.

(3) 1.66 Force between charge strips.

The strips are infinitely long and have charge densities ~~+σ~~ and $-\sigma$.



a) Find the electric field due to one of the strips at a point P, x into the other strip.

Break the left strip into a collection of narrow ~~rectangles~~ wires, ~~with~~ width dr and linear charge density $\lambda = \sigma dr$

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The field of a charged wire = $E = \dfrac{\lambda}{2\pi\varepsilon_0 r}$ (eqn 1.38) and is $\perp$ wire.

The total E-field at P will be the sum (integral) of the fields of all the wires in the left strip.

$$E_{at\,P} = \int_{left\,strip} dE = \frac{\sigma}{2\pi\varepsilon_0}\int_x^{x+b} \frac{dr}{r} = \frac{\sigma}{2\pi\varepsilon_0} \ln\left(\frac{x+b}{x}\right).$$

b) Find the force ~~of~~ of attraction between the two strips.

Consider the right strip to be a collection of wires of width increment of dx and $\lambda = -\sigma dx$. The ~~net~~ force on this strip will be

$$dF = (E@x) \times dy$$

③ Continued $F = \frac{\sigma^2}{2\pi\varepsilon_0}\int\left[\ln\left(\frac{x+b}{x}\right)dx = \int(\ln(x+b) - \ln x)\,dx\right]$

$$\int \ln(x+b)\,dx = (x+b)\ln(x+b) - (x+b)$$

$$\therefore F = \frac{\sigma^2}{2\pi\varepsilon_0}\left\{\left[(x+b)\ln(x+b) - (x+b)\right]\Big|_0^b - \left[x\ln x - x\right]\Big|_0^b\right\}$$

$$= \frac{\sigma^2}{2\pi\varepsilon_0}\left[2b\ln(2b) - \cancel{2b} - \left[b\ln b - \cancel{b}\right] - \left\{\left(b\ln b - \cancel{b}\right) - 0\right\}\right]$$

$$= \frac{\sigma^2}{2\pi\varepsilon_0}\left[2b\ln 2b - 2b\ln b = 2b\left(\ln(2b) - \ln(b) = \ln(2)\right)\right]$$

$$= \frac{\sigma^2 \cancel{2} b\ln(2)}{\cancel{2}\pi\varepsilon_0} = \frac{\sigma^2 b\ln(2)}{\pi\varepsilon_0}.$$

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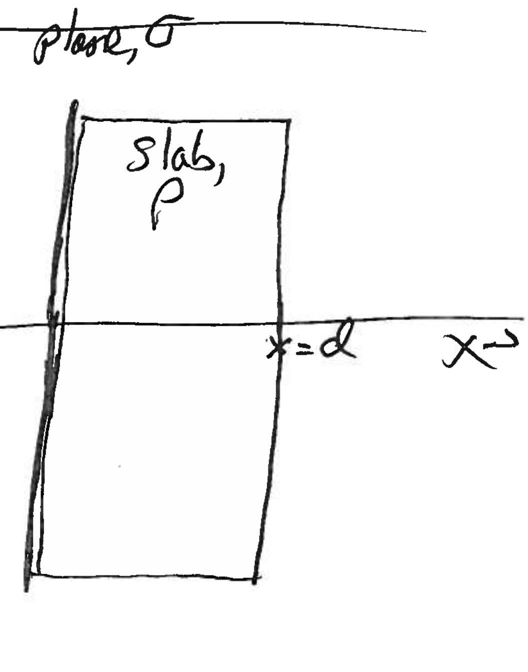
④ 1.72 a plane and a slab.

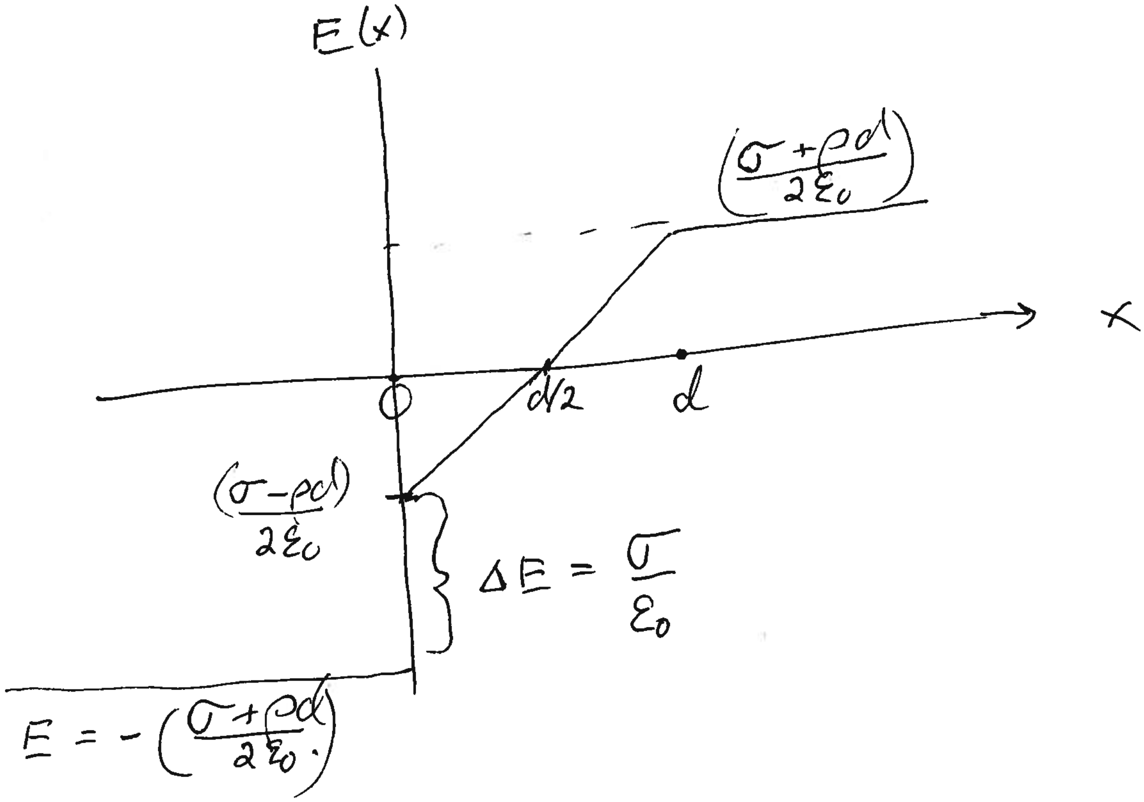
Find the electric field E_x arising from an yzplane at $x=0$ with surface charge density σ, and a slab of charge with thickness d and volume charge density ρ.



1.72. A plane and a slab.

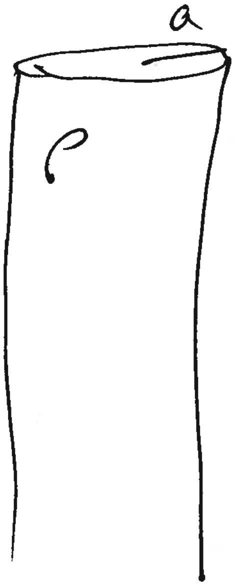
The plane (located at $x = 0$ and surface charge density σ) contributes a field $|E_{plane}| = \sigma/(2\varepsilon_0)$ on either side of the plane, resulting in a discontinuity of $\sigma/(\varepsilon_0)$ across the plane . The slab, thickness d and volume density ρ, has zero field in its center and a field that grows linearly from the center up to maximum field of $|E_{slab}| = \rho d/(2\varepsilon_0)$ on either side of the slab.

By superposition, the field to the left of the plane is $-(\rho+\sigma d)/(2\varepsilon_0)$, jumping to a value of $(\rho-\sigma d)/(2\varepsilon_0)$ at $x = 0 +\sigma$, then grows linearly to the final value of $(\rho+\sigma d)/(2\varepsilon_0)$ for $x \geq 0$.



1.83 Potential Energy of a cylinder.

Energy density $\left(\frac{J}{m^3}\right) = \frac{\varepsilon_0 E^2}{2}$.



Uniform cylinder, radius a, volume charge density ρ.

Inside the cylinder, the radial E-field grows linearly with distance : Gauss's Law $2\pi r E(r) = \rho \pi r^2 / \varepsilon_0$

$$\Rightarrow E_{in}(r) = \frac{\rho r}{2\varepsilon_0}, \quad dU = \frac{\varepsilon_0}{2}\left(\frac{\rho r}{2\varepsilon_0}\right)^2 dV$$

$$U_{inside} = \left(\frac{\varepsilon_0}{2}\right)\frac{\rho^2}{4\varepsilon_0^2}\left[\int_0^a r^2 \times 2\pi r\, dr = 2\pi \frac{a^4}{4}\right] = \frac{\pi \rho^2 a^4}{16\varepsilon_0}.$$

Defining $\boxed{\lambda = \rho(\pi a^2)}$, $U_{inside} = \frac{\lambda^2}{8\pi\varepsilon_0}$.

$$E_{outside} = \frac{Q_{inside}}{\varepsilon_0 \times (2\pi r)} = \frac{\rho a^2}{2\varepsilon_0 r}$$

$$dU = \frac{\varepsilon_0 E_{out}^2}{2} dvol \qquad U_{outside} = \frac{\varepsilon_0}{2}\left(\frac{\rho a^2}{2\varepsilon_0}\right)^2 \int_a^R \frac{2\pi r\, dr}{r^2} = \frac{2\pi\varepsilon_0}{2}\left(\frac{\rho a^2}{2\varepsilon_0}\right)^2 \times \ln(R/a).$$

Converting to λ format.

$$U_{outside} = \frac{\lambda^2}{\pi\varepsilon_0}\ln(R/a).$$

$$\text{Total energy}\Big|_0^R = \frac{\lambda^2}{16\pi\varepsilon_0} + \frac{\lambda^2}{4\pi\varepsilon_0}\ln(R/a)$$

$$= \frac{\lambda^2}{4\pi\varepsilon_0}\left[\frac{1}{4} + \ln\left(\frac{R}{a}\right)\right].$$