Physics 160, HW #10

Chapter 10

Electric fields in matter

Solutions manual for *Electricity and Magnetism, 3rd edition*, E. Purcell, D. Morin.
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#5 10.15. **Densities on a capacitor**

The charge density σ_2 on the right part of each plate is κ times the charge density σ_1 on the left part. So

$$\sigma_1(b-x)a + \sigma_2 xa = Q \implies \sigma_1(b-x)a + (\kappa\sigma_1)xa = Q. \tag{639}$$

The two charge densities, σ_1 and $\sigma_2 = \kappa\sigma_1$, are therefore given by

$$\sigma_1 = \frac{Q/a}{b+(\kappa-1)x}, \qquad \sigma_2 = \frac{\kappa Q/a}{b+(\kappa-1)x}. \tag{640}$$

Since $\kappa > 1$, both of these densities decrease as x increases. It is possible for both densities to decrease while the total charge remains at the given value Q, because the charge in the right region increases (while the charge in the left region decreases), but it does so at a slower rate than the area increases; so the density decreases. We *would* have a paradox if the areas stayed the same.

An analogy: 10 people each have the same amount of money. 20 other people each have the same amount, but it is smaller than what the first 10 have. One of these 20 people takes some money from each of the first 10, and also from each of the other 19, so that she now has the same amount as the first 10. The total amount of money held by all 30 people is still the same, but the average amounts in the two groups (now with 11 and 19 people) have both decreased.

10.16. **Leyden jar**

Assume that the jar is cylindrical, with the height being twice the diameter d (the result will depend somewhat on the proportions assumed). Then the volume is $\pi(d/2)^2 \cdot (2d)$. Setting this equal to $10^{-3}\,\text{m}^3$ gives $d = 0.086\,\text{m}$. The area of the capacitor (assuming it has no top) is $A = \pi(d/2)^2 + \pi d(2d) = 9\pi d^2/4 = 0.052\,\text{m}^2$. So the capacitance is

$$C = \frac{\kappa\epsilon_0 A}{s} = \frac{(4)\left(8.85\cdot 10^{-12}\,\frac{\text{s}^2\,\text{C}^2}{\text{kg}\,\text{m}^3}\right)(0.052\,\text{m}^2)}{0.002\,\text{m}} = 9.2\cdot 10^{-10}\ \text{F}. \tag{641}$$

If we had chosen the height to instead be four times the diameter, then the capacitance would be about 20% larger. As long as the jar isn't too squat (in which case it would be better called a tray) or too tall (in which case it would be better called a tube), the dependence of the capacitance on the exact dimensions is fairly weak. (If the height is $h = nd$, then you can show that the capacitance is proportional to $(n+1/4)/n^{2/3}$.)

The capacitance of a sphere is $4\pi\epsilon_0 r$, so a sphere will have a capacitance of $9.2\cdot 10^{-10}$ F if $r = 8.3\,\mathrm{m}$. The diameter is then 16.6 m, or about 54 feet.

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10.17. Maximum energy storage

The maximum field is 14 kilovolts/mil, which in volts/meter equals

$$E_{\max} = \frac{1.4\cdot 10^4\,\mathrm{V}}{2.54\cdot 10^{-5}\,\mathrm{m}} = 5.5\cdot 10^8\ \mathrm{V/m}. \tag{642}$$

The capacitance of the Mylar-filled capacitor is $\kappa\epsilon_0 A/s$. The energy stored is still $C\phi^2/2$, so the maximum possible energy density is

$$\begin{aligned} \frac{\text{energy}}{\text{volume}} &= \frac{1}{2}C\phi^2\frac{1}{V} = \frac{1}{2}\frac{\kappa\epsilon_0 A}{s}(Es)^2\frac{1}{As} = \frac{1}{2}\kappa\epsilon_0 E^2 \\ &= \frac{1}{2}(3.25)\Big(8.85\cdot 10^{-12}\,\frac{\mathrm{s^2\,C^2}}{\mathrm{kg\,m^3}}\Big)(5.5\cdot 10^8\ \mathrm{V/m})^2 = 4.4\cdot 10^6\ \mathrm{J/m^3}. \end{aligned} \tag{643}$$

The maximum energy per kilogram of Mylar is therefore

$$\frac{4.4\cdot 10^6\,\mathrm{J/m^3}}{1400\,\mathrm{kg/m^3}} = 3100\ \mathrm{J/kg}. \tag{644}$$

To determine how high the capacitor could lift itself, let the entire mass of the capacitor be m. Then $3m/4$ of this is Mylar, so conservation of energy gives

$$E = mgh \implies (3100\,\mathrm{J/kg})(3m/4) = mgh \implies h = \frac{(3/4)((3100\,\mathrm{J/kg})}{9.8\,\mathrm{m/s^2}} = 240\ \mathrm{m}. \tag{645}$$

The D cell in Exercise 4.41 had an energy storage of $1.8\cdot 10^5$ J/kg, which is about 60 times as much as the Mylar capacitor. However, the capacitor can deliver all the stored energy in less than a microsecond!

10.18. Partially filled capacitors

The second capacitor in the figure consists of two capacitors in series; you can imagine the boundary between them to be two plates with charge Q and $-Q$ superposed. Both of these capacitors have plate separation $s/2$ and area A, so the capacitances are (with the two halves labeled by "v" for vacuum and "d" for dielectric) $C_\mathrm{v} = \epsilon_0 A/(s/2)$ and $C_\mathrm{d} = \kappa\epsilon_0 A/(s/2)$. Since $C_0 = \epsilon_0 A/s$, we have $C_\mathrm{v} = 2C_0$ and $C_\mathrm{d} = 2\kappa C_0$. Problem 3.18 gives the rule for adding capacitors in series, so the desired capacitance is (with "S" for series)

$$\frac{1}{C_\mathrm{S}} = \frac{1}{C_\mathrm{v}} + \frac{1}{C_\mathrm{d}} = \frac{1}{2C_0} + \frac{1}{2\kappa C_0} \implies C_\mathrm{S} = \frac{2\kappa}{\kappa+1}C_0. \tag{646}$$

The third capacitor in the figure consists of two capacitors in parallel. They both have plate separation s and area $A/2$, so the capacitances are $C_\mathrm{v} = \epsilon_0(A/2)/s$ and $C_\mathrm{d} = \kappa\epsilon_0(A/2)/s$. These can be written as $C_\mathrm{v} = C_0/2$ and $C_\mathrm{d} = \kappa C_0/2$. Problem 3.18 gives the rule for adding capacitors in parallel, so the desired capacitance is (with "P" for parallel)

$$C_\mathrm{P} = C_\mathrm{v} + C_\mathrm{d} = \frac{C_0}{2} + \frac{\kappa C_0}{2} = \frac{1+\kappa}{2}C_0. \tag{647}$$

This yields $r = 0.037$ m. The coordinates of the lower right point are then

$$(y, z) = (r\sin\theta, -r\cos\theta) = (0.030, -0.021)\,\text{m}. \tag{655}$$

The upper left point has the negative of these coordinates.

#7

10.24. Field lines

Let the dipole point in the z direction. If $r = r_0 \sin^2\theta$, then

$$\begin{aligned} x &= r\sin\theta = r_0\sin^3\theta, \\ z &= r\cos\theta = r_0\sin^2\theta\cos\theta. \end{aligned} \tag{656}$$

Therefore,

$$\begin{aligned} \frac{dx}{d\theta} &= 3r_0\sin^2\theta\cos\theta, \\ \frac{dz}{d\theta} &= r_0(2\sin\theta\cos^2\theta - \sin^3\theta) = r_0\sin\theta(2\cos^2\theta - \sin^2\theta) \\ &= r_0\sin\theta(3\cos^2\theta - 1). \end{aligned} \tag{657}$$

The ratio of these derivatives gives the slope of the tangent to the $r = r_0\sin^2\theta$ curve as

$$\frac{dz}{dx} = \frac{3\cos^2\theta - 1}{3\sin\theta\cos\theta}. \tag{658}$$

This is the same as the ratio E_z/E_x as given by Eq. (10.17). The tangent to the $r = r_0\sin^2\theta$ curve therefore points in the same direction as the **E** field, as we wanted to show.

Alternatively, we can work with polar coordinates, as we did in Section 2.7.2. With respect to the local $\hat{\mathbf{r}}$-$\hat{\boldsymbol{\theta}}$ basis, the slope of the field-line curve is

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{r_0\sin^2\theta}\cdot 2r_0\sin\theta\cos\theta = \frac{2\cos\theta}{\sin\theta}. \tag{659}$$

But from Eq. (10.18) this equals E_r/E_θ. So again, the tangent to the $r = r_0\sin^2\theta$ curve points in the same direction as the **E** field.

10.25. Average dipole field on a sphere

From Fig. 10.6, we quickly see that the average E_x value is zero, because for every field line pointing one way, there is another field line with the opposite E_x value. This holds for any vertical plane containing the z axis, so it holds for E_y too. Hence the averages of both E_x and E_y over the whole surface of a sphere are zero.

It isn't obvious from Fig. 10.6 that the average E_z value is zero. We can't use a quick symmetry argument, so we need to actually integrate E_z over a spherical shell. The area element of a horizontal ring on the sphere is $da = (2\pi r\sin\theta)(r\,d\theta)$. Using the form of E_z given in Eq. (10.17), and ignoring the r's and all the other constant factors, we have

$$E_z^{\text{avg}} \propto \int_0^\pi (3\cos^2\theta - 1)\sin\theta\,d\theta = (-\cos^3\theta + \cos\theta)\Big|_0^\pi = 0, \tag{660}$$

as desired.

A mole of something with molecular weight w has a mass of w grams. Equivalently, since the proton mass is $1.67 \cdot 10^{-24}$ g, it takes $1/1.67 \cdot 10^{-24} = 6 \cdot 10^{23}$ protons to make a gram. This number is essentially Avogadro's number.

Water has a molecular weight of 18, so the number of water molecules in 1 gram is $(6 \cdot 10^{23}/\text{mole})/(18\,\text{g/mole}) = 3.33 \cdot 10^{22}\,\text{g}^{-1}$. The number of molecules per cm^3 is then obtained by multiplying by the mass density, $\rho = 1.00\,\text{g/cm}^3$ (which doesn't change the number in the case of water). Finally, to obtain the number of molecules per m^3, N, we must multiply by 10^6. The resulting N's for the three substances are shown in the table below. The dipole moments, p, from Fig. 10.14 are also listed. The calculated values of $C = (\kappa - 1)\epsilon_0 kT/Np^2$ are listed in the righthand column.

	κ	kT	N	p	C
H_2O	80	$4.0 \cdot 10^{-21}$ J	$3.3 \cdot 10^{28}\,\text{m}^{-3}$	$6.13 \cdot 10^{-30}$ C-m	2.3
NH_3	23	$3.3 \cdot 10^{-21}$ J	$2.9 \cdot 10^{28}\,\text{m}^{-3}$	$4.76 \cdot 10^{-30}$ C-m	1.0
CH_3OH	34	$4.0 \cdot 10^{-21}$ J	$2.5 \cdot 10^{28}\,\text{m}^{-3}$	$5.66 \cdot 10^{-30}$ C-m	1.5

10.36. Discontinuity in $E_\perp$

The internal field is $\mathbf{E} = -\mathbf{P}/3\epsilon_0$, so $E_\perp^{\text{in}} = P\cos\theta/3\epsilon_0$. (As a double check, the factor here is indeed $\cos\theta$ because the field at the north pole is $P/3\epsilon_0$, pointing downward.) This component points *inward* in the upper hemisphere (and outward in the lower hemisphere), because $\mathbf{E}$ points downward. The perpendicular external field is the radial field from a dipole, $E_\perp^{\text{out}} = E_r = p_0\cos\theta/2\pi\epsilon_0 r_0^3$. This component points *outward* in the upper hemisphere (and inward in the lower hemisphere). The effective dipole moment p_0 equals $(4\pi r_0^3/3)P$. Hence $E_\perp^{\text{out}} = 2P\cos\theta/3\epsilon_0$. Due to the different inward/outward directions of the vectors, the discontinuity in $E_\perp$ is $2P\cos\theta/3\epsilon_0 - (-P\cos\theta/3\epsilon_0) = P\cos\theta/\epsilon_0 = P_\perp/\epsilon_0$, as desired.

#8

10.37. E at the center of a polarized sphere

Consider a horizontal ring at an angle θ down from the top of the sphere, with angular span $d\theta$. The area of this ring is $2\pi(R\sin\theta)(R\,d\theta)$. Since the density is $\sigma = P\cos\theta$, the charge in the ring is $q = 2\pi PR^2\sin\theta\cos\theta\,d\theta$. A little bit of charge dq in a ring in the upper hemisphere creates a diagonally downward field of $dq/4\pi\epsilon_0 R^2$ at the center of the sphere. But by symmetry we are concerned only with the vertical component, which brings in a factor of $\cos\theta$. Integrating over all the dq's in a ring simply gives the total charge q in the ring. The net field from the ring therefore points downward with magnitude

$$\frac{2\pi PR^2\sin\theta\cos\theta\,d\theta}{4\pi\epsilon_0 R^2}\cos\theta = \frac{P\sin\theta\cos^2\theta\,d\theta}{2\epsilon_0}. \tag{679}$$

You can verify that this expression is valid for rings in the lower hemisphere too; all contributions to the field point downward. Integrating over θ from 0 to π gives a total magnitude of

$$E = \int_0^\pi \frac{P\sin\theta\cos^2\theta\,d\theta}{2\epsilon_0} = -\frac{P\cos^3\theta}{6\epsilon_0}\bigg|_0^\pi = \frac{P}{3\epsilon_0}, \tag{680}$$

as desired. The direction is downward.

10.38. Uniform field via superposition

(a) Applying Gauss's law to a sphere of radius r inside the given sphere yields

$$4\pi r^2 E = \frac{q}{\epsilon_0} \implies 4\pi r^2 E = \frac{(4\pi r^3/3)\rho}{\epsilon_0} \implies E = \frac{\rho r}{3\epsilon_0}. \tag{681}$$

concerned, from the reasoning near the beginning of Section 10.9. From Eq. (10.18) the tangential field of a dipole is $p\sin\theta/4\pi\epsilon_0 a^3$, which gives $E_0\sin\theta$ here. But this is exactly what is needed to cancel the tangential component of the original uniform field E_0 (you can check that the direction is correct). The tangential component of the total external field is therefore zero, as desired.

The field strength inside the dielectric sphere, which from Eq. (10.53) is $3E_0/(2+\kappa)$, goes to zero as $\kappa \to \infty$. The sphere therefore becomes an equipotential, which is correct for a conducting sphere. A sketch of the field lines outside the sphere is shown in Fig. 160.

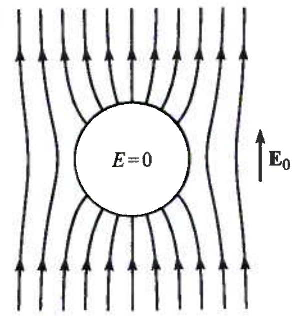

Figure 160

Using the above value of p, we see that the polarizability α, defined by $\mathbf{p} = \alpha\mathbf{E}_0$, of a perfectly conducting sphere equals $4\pi\epsilon_0 a^3$. The quantity that we normally work with, $\alpha/4\pi\epsilon_0$, therefore equals a^3 for our conducting sphere of radius a. Since $\alpha/4\pi\epsilon_0 = 0.66\cdot 10^{-30}\,\text{m}^3$ for hydrogen, a conducting sphere of equal polarizability has a radius of $(0.66\cdot 10^{-30}\,\text{m}^3)^{1/3} = 8.7\cdot 10^{-11}\,\text{m}$. This is very close to the Bohr radius, $5.3\cdot 10^{-11}\,\text{m}$.

10.40. **Continuity of D**

The **E** field due to the slab is the same as the **E** field due to two capacitor plates with surface charge densities $\pm P$. Both **E** and **P** are zero outside the slab, so the external **D** is likewise zero. Our task is therefore to show that **D** is zero inside the slab. And indeed, $\mathbf{E} = -\mathbf{P}/\epsilon_0$ (this is the field between two plates with densities $\pm P$), so $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon_0(-\mathbf{P}/\epsilon_0) + \mathbf{P} = 0$.

#9

10.41. **Discontinuity in $D_\parallel$**

Inside the sphere, we have $\mathbf{E} = -\mathbf{P}/3\epsilon_0$, so the displacement vector is $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon_0(-\mathbf{P}/3\epsilon_0) + \mathbf{P} = 2\mathbf{P}/3$. The tangential component of this is

$$D_\parallel^{\text{in}} = D_\theta^{\text{in}} = -\frac{2P\sin\theta}{3}. \tag{688}$$

The minus sign here comes from the fact that **P** points toward the north pole, whereas the positive $\hat{\theta}$ direction is defined to point away from the north pole.

Outside the sphere, **E** is the field due to a dipole with $\mathbf{p}_0 = (4\pi R^3/3)\mathbf{P}$. From Eq. (10.18) the tangential component of the dipole field is $E_\theta = p_0\sin\theta/4\pi\epsilon_0 R^3$. In terms of P this becomes $E_\theta = P\sin\theta/3\epsilon_0$. Since $\mathbf{P} = 0$ outside the sphere, the external **D** is obtained by simply multiplying the external **E** by ϵ_0. Therefore

$$D_\parallel^{\text{out}} = D_\theta^{\text{out}} = \frac{P\sin\theta}{3}. \tag{689}$$

Comparing this with the $D_\parallel^{\text{in}}$ in Eq. (688), we see that $D_\parallel$ has a discontinuity of $P\sin\theta$. $D_\parallel$ increases by $P\sin\theta$ in going from inside to outside. This makes sense, because we know that E_θ is continuous across the boundary, so the discontinuity in the tangential component of $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$ is simply the discontinuity in the tangential component of **P**.

#10

10.42. **Energy density in a dielectric**

With a dielectric present, the capacitance of a parallel-plate capacitor is $C = \kappa\epsilon_0 A/s \equiv \epsilon A/s$. The energy stored is still $C\phi^2/2$, because it equals $Q\phi/2$ for all the same reasons as in the vacuum case (imagine a battery doing work in transferring charge from one plate to the other). So the energy density is

$$\frac{\text{energy}}{\text{volume}} = \frac{1}{2}C\phi^2\frac{1}{V} = \frac{1}{2}\frac{\epsilon A}{s}(Es)^2\frac{1}{As} = \frac{\epsilon E^2}{2}, \tag{690}$$

10 continued

as desired. Since $\epsilon \equiv \kappa\epsilon_0$, this energy density is κ times the $\epsilon_0 E^2/2$ energy density without the dielectric. Basically, since C is κ times larger, so is the energy, and hence the energy density. To see physically why the energy is larger, consider the case of induced dipole moments, discussed in Section 10.5. The stretched atoms and molecules are effectively little springs that are stretched, so they store potential energy. This makes the total energy larger than it would be for the same equivalent charge on/at the capacitor plates (free charge plus bound-charge layer).

In an electromagnetic wave in a dielectric, the energy density of the magnetic field is still $B^2/2\mu_0$. (It would be $B^2/2\mu$ in a magnetized material, but we're assuming that the material here is only electrically polarizable.) But from Eq. (10.83) the amplitudes of the E and B fields are related by $B_0 = \sqrt{\mu_0\epsilon}\,E_0$. So $B^2/2\mu_0 = \epsilon E^2/2$. The E and B energy densities are therefore equal, just as they are in vacuum.

10.43. **Reflected wave**

The incident, transmitted, and reflected waves are, respectively (using $\mathbf{E}\times\mathbf{B}\propto\mathbf{v}$ to find the direction of the $\mathbf{B}$'s),

$$\begin{aligned}
\mathbf{E}_\text{i} &= \hat{\mathbf{z}}E_\text{i}\sin(ky-\omega t), & \mathbf{B}_\text{i} &= \hat{\mathbf{x}}B_\text{i}\sin(ky-\omega t),\\
\mathbf{E}_\text{r} &= \hat{\mathbf{z}}E_\text{r}\sin(ky+\omega t), & \mathbf{B}_\text{r} &= -\hat{\mathbf{x}}B_\text{r}\sin(ky+\omega t),\\
\mathbf{E}_\text{t} &= \hat{\mathbf{z}}E_\text{t}\sin(k'y-\omega t), & \mathbf{B}_\text{t} &= \hat{\mathbf{x}}B_\text{t}\sin(k'y-\omega t).
\end{aligned} \tag{691}$$

The total wave in the empty space $y < 0$ is the sum of the incident and reflected waves.

Let's apply continuity of $\mathbf{E}$ and $\mathbf{B}$ at $y = 0$. After setting $y = 0$, we can cancel all the $\sin\omega t$ terms (which means that our results will hold for all t), but we must be careful about the extra minus sign in $\sin(-\omega t)$. We obtain

$$-E_\text{i} + E_\text{r} = -E_\text{t} \quad \text{and} \quad -B_\text{i} - B_\text{r} = -B_\text{t}. \tag{692}$$

We also have

$$E_\text{i} = cB_\text{i}, \qquad E_\text{r} = cB_\text{r}, \qquad E_\text{t} = (c/n)B_\text{t}. \tag{693}$$

Given the "i" quantities, we have four equations in four unknowns (the "r" and "t" quantities). Eliminating the B's quickly turns Eq. (692) into

$$E_\text{i} - E_\text{r} = E_\text{t} \quad \text{and} \quad E_\text{i} + E_\text{r} = nE_\text{t}. \tag{694}$$

Solving these yields

$$E_\text{r} = \frac{n-1}{n+1}E_\text{i} \quad \text{and} \quad E_\text{t} = \frac{2}{n+1}E_\text{i}. \tag{695}$$

So $E_\text{r}/E_\text{i} = (n-1)/(n+1)$, and $E_\text{t}/E_\text{i} = 2/(n+1)$.

The energy is proportional to the square of E, so the faction of the incident energy that is reflected, with $n = 1.6$, is

$$\frac{E_\text{r}^2}{E_\text{i}^2} = \left(\frac{n-1}{n+1}\right)^2 = 0.053. \tag{696}$$