PHYS 161: Homework 3

Due on Wednesday February 4, 2015

Professor Landee 11:00am

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February 3, 2015

Problem 1

(II-20) In cylindrical coordinates... (problem omitted)

For the top and bottom surfaces,

$$\iint_{S_3} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{S_3} F_z\,dS \simeq F_z\left(r,\theta,z+\frac{\Delta z}{2}\right) r\Delta r\Delta\theta$$

$$\iint_{S_4} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = -\iint_{S_4} F_z\,dS \simeq -F_z\left(r,\theta,z-\frac{\Delta z}{2}\right) r\Delta r\Delta\theta$$

so

$$\frac{1}{\Delta V}\iint_{S_3+S_4} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS \simeq \frac{1}{\Delta z}\left[F_z\left(r,\theta,z+\frac{\Delta z}{2}\right) - F_z\left(r,\theta,z-\frac{\Delta z}{2}\right)\right]$$

So as $\Delta z \to 0$, this becomes $\frac{\partial F_z}{\partial z}$. For the side surfaces,

$$\iint_{S_5} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{S_5} F_\theta\,dS \simeq F_\theta\left(r,\theta+\frac{\Delta\theta}{2},z\right) \Delta r\Delta z$$

$$\iint_{S_6} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = -\iint_{S_6} F_\theta\,dS \simeq -F_\theta\left(r,\theta-\frac{\Delta\theta}{2},z\right) \Delta r\Delta z$$

so

$$\frac{1}{\Delta V}\iint_{S_5+S_6} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS \simeq \frac{1}{r\Delta\theta}\left[F_\theta\left(r,\theta+\frac{\Delta\theta}{2},z\right) - F_\theta\left(r,\theta-\frac{\Delta\theta}{2},z\right)\right]$$

So as $\Delta\theta \to 0$, this becomes $\frac{1}{r}\frac{\partial F_\theta}{\partial\theta}$.

Problem 2

(II-21) Repeat Problem II-20 to... (problem omitted)

In spherical coordinates, $\Delta V = r^2\sin\phi\Delta r\Delta\phi\Delta\theta$. For the surfaces in r direction,

$$\iint_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{S_1} F_r\,dS \simeq F_r\left(r+\frac{\Delta r}{2},\phi,\theta\right)\left(r+\frac{\Delta r}{2}\right)^2 \Delta r\Delta\theta\sin\phi$$

$$\iint_{S_2} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = -\iint_{S_2} F_r\,dS \simeq -F_r\left(r-\frac{\Delta r}{2},\phi,\theta\right)\left(r-\frac{\Delta r}{2}\right)^2 \Delta r\Delta\theta\sin\phi$$

so

$$\frac{1}{\Delta V}\iint_{S_1+S_2} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS \simeq \frac{1}{r^2\Delta r}\left[\left(r+\frac{\Delta r}{2}\right)^2 F_r\left(r+\frac{\Delta r}{2},\phi,\theta\right) - \left(r-\frac{\Delta r}{2}\right)^2 F_r\left(r-\frac{\Delta r}{2},\phi,\theta\right)\right]$$

So as $\Delta r \to 0$, this becomes $\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r)$.
For the surfaces in ϕ direction,

$$\iint_{S_3} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{S_3} F_\phi\,dS \simeq F_\phi\left(r,\phi+\frac{\Delta\phi}{2},\theta\right) r\Delta r\Delta\theta\sin\left(\phi+\frac{\Delta\phi}{2}\right)$$

$$\iint_{S_4} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = -\iint_{S_4} F_\phi\,dS \simeq -F_\phi\left(r,\phi-\frac{\Delta\phi}{2},\theta\right) r\Delta r\Delta\theta\sin\left(\phi+\frac{\Delta\phi}{2}\right)$$

Problem 2 continued on next page...

so

$$\frac{1}{\Delta V}\iint_{S_3+S_4}\mathbf{F}\cdot\hat{\mathbf{n}}\,dS$$
$$\simeq\frac{1}{r\sin\phi\Delta\phi}\left[F_\phi\left(r,\phi+\frac{\Delta\phi}{2},\theta\right)r\Delta r\Delta\theta\sin\left(\phi+\frac{\Delta\phi}{2}\right)-F_\phi\left(r,\phi-\frac{\Delta\phi}{2},\theta\right)r\Delta r\Delta\theta\sin\left(\phi+\frac{\Delta\phi}{2}\right)\right]$$

So as $\Delta\phi\to 0$, this becomes $\frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi F_\phi)$.
For the surfaces in θ direction,

$$\iint_{S_5}\mathbf{F}\cdot\hat{\mathbf{n}}\,dS=\iint_{S_5}F_\theta\,dS\simeq F_\theta\left(r,\phi,\theta+\frac{\Delta\theta}{2}\right)r\Delta r\Delta\phi$$
$$\iint_{S_6}\mathbf{F}\cdot\hat{\mathbf{n}}\,dS=-\iint_{S_6}F_\theta\,dS\simeq -F_\theta\left(r,\phi,\theta-\frac{\Delta\theta}{2}\right)r\Delta r\Delta\phi$$

so

$$\frac{1}{\Delta V}\iint_{S_5+S_6}\mathbf{F}\cdot\hat{\mathbf{n}}\,dS\simeq\frac{1}{r\sin\phi\Delta\theta}\left[F_\theta\left(r,\phi,\theta+\frac{\Delta\theta}{2}\right)r\Delta r\Delta\phi-F_\theta\left(r,\phi,\theta-\frac{\Delta\theta}{2}\right)r\Delta r\Delta\phi\right]$$

So as $\Delta\theta\to 0$, this becomes $\frac{1}{r\sin\phi}\frac{\partial F_\theta}{\partial\theta}$.

Problem 3

(II-23) Verify the divergence theorem... (problem omitted)

(a) The surface integral could be devided into three parts, each at b of the three coordinates.

$$\iint_S\mathbf{F}\cdot\hat{\mathbf{n}}\,dS=\int_0^b\int_0^b b\,dy\,dz+\int_0^b\int_0^b b\,dx\,dz+\int_0^b\int_0^b b\,dx\,dy=3b^3$$
$$\iiint_V\nabla\cdot\mathbf{F}\,dV=3\iiint_V dV=3b^3$$

(b) The surface integral could be devided into two parts, the top of the cylinder, with unit normal vector $\hat{\mathbf{e}}_z$ and the outer curved surface, with unit normal vector $\hat{\mathbf{e}}_r$. The two surfaces on xz- and yz-plane is obviously zero.

$$\iint_S\mathbf{F}\cdot\hat{\mathbf{n}}\,dS=\iint_{S_1}h\,dS+\iint_{S_2}R\,dS=\frac{1}{4}\pi R^2h+\frac{1}{4}\times 2\pi R\times R=\frac{3\pi R^2h}{4}$$

For this function in cylindrical coordinates,

$$\text{div }\mathbf{F}=\frac{1}{r}\frac{\partial}{\partial r}(rF_r)+\frac{\partial F_z}{\partial z}=2+1=3$$
$$\iiint_V\nabla\cdot\mathbf{F}\,dV=3\iiint_V dV=\frac{3\pi R^2h}{4}$$

Problem 3 continued on next page. . .

(c)
$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iint_S R^2\, dS = 4\pi R^4$$

For this function in spherical coordinates,

$$\text{div } \mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) = 4r$$

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = 4\iiint_V r\, dV = 4\int_0^{2\pi}\int_0^{\pi}\int_0^{R} r \cdot r^2 \sin\theta\, dr\, d\theta\, d\phi = 4\pi R^4$$

Problem 4

(II-24) (a) One of Maxwell's equations... (problem omitted)

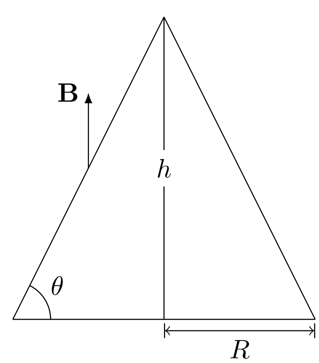
(a) Using the divergence theorem,

$$\iint_S \mathbf{B} \cdot \hat{\mathbf{n}}\, dS = \iiint_V \nabla \cdot \mathbf{B}\, dV = 0$$

(b) For the base surface,

$$\iint_{S_1} \mathbf{B} \cdot \hat{\mathbf{n}}\, dS = -\pi R^2 B$$

For the pointy surface, the surface area $A = \pi R\sqrt{R^2 + h^2}$. Look at this intersection



So $\cos\theta = \frac{R}{\sqrt{R^2+h^2}}$.

$$\iint_{S_2} \mathbf{B} \cdot \hat{\mathbf{n}}\, dS = B\cos\theta \cdot A = \pi R^2 B$$

The two integrals add up to be zero.

Problem 5

(2.31) Finding the potential... (problem omitted)

Through waypoint $(x_1, 0, 0)$

$$\int_{(0,0,0)}^{(x_1,0,0)} 6xy\,dx = 0$$

$$\int_{(x_1,0,0)}^{(x_1,y_1,0)} (3x^2 - 3y^2)\,dy = 3{x_1}^2 y_1 - {y_1}^3$$

Through waypoint $(0, y_1, 0)$

$$\int_{(0,0,0)}^{(0,y_1,0)} (3x^2 - 3y^2)\,dy = -{y_1}^3$$

$$\int_{(0,y_1,0)}^{(x_1,y_1,0)} 6xy\,dx = 3{x_1}^2 y_1$$

So the sums are the same! Now $\phi(x, y, z) = 3x^2 y - y^3$.

$$\nabla\phi(x, y, z) = (6xy, 3x^2 - 3y^2, 0)$$

Components are of the exact same form.

Problem 6

(2.43) Potential from a rod... (problem omitted)

For the point $(0, 0, 2d)$,

$$\phi(0, 0, 2d) = \int \frac{\rho(x', y', z')\,dx'\,dy'\,dz'}{4\pi\epsilon_0 r} = \int_{-d}^{d} \frac{\lambda\,dz}{4\pi\epsilon_0(2d - z)} = \frac{\lambda \ln 3}{4\pi\epsilon_0}$$

For the point $(x, 0, 0)$,

$$\phi(x, 0, 0) = \int_{-d}^{d} \frac{\lambda\,dz}{4\pi\epsilon_0\sqrt{x^2 + z^2}} = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{d + \sqrt{x^2 + d^2}}{x}$$

So

$$\ln 3 = 2 \ln \frac{d + \sqrt{x^2 + d^2}}{x} \Rightarrow x = \sqrt{3}d$$

Problem 7

(2.61) Dipole field on the axes... (problem omitted)

Problem 7 continued on next page. . .

On z axis,

$$E(r) = \left(\frac{1}{4\pi\epsilon_0}\frac{q}{(r-l/2)^2} - \frac{1}{4\pi\epsilon_0}\frac{q}{(r+l/2)^2}\right)\hat{\mathbf{z}} = \frac{q}{4\pi\epsilon_0 r^2}\left(\frac{1}{(1-l/2r)^2} - \frac{1}{(1+l/2r)^2}\right)\hat{\mathbf{z}}$$

Using the approximation $1/(1\pm\epsilon) \approx 1 \mp \epsilon$ this equals

$$\frac{q}{4\pi\epsilon_0 r^2}\left((1+l/2r)^2 - (1-l/2r)^2\right)\hat{\mathbf{z}} = \frac{ql}{2\pi\epsilon_0 r^3}\hat{\mathbf{z}}$$

Exactly the same with Equation 2.36 in the case that $\theta = 0$.
On x axis, $r = \sqrt{x^2+z^2}$ where $z = l/2$. $\sin\theta = \frac{z}{\sqrt{r^2+z^2}} = l/2r$. From q

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}(\cos\theta\hat{\mathbf{x}} - \sin\theta\hat{\mathbf{z}})$$

from $-q$

$$E = \frac{1}{4\pi\epsilon_0}\frac{-q}{r^2}(\cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{z}})$$

So add these two together,

$$E = -\frac{1}{2\pi\epsilon_0}\frac{q}{r^2}\sin\theta\hat{\mathbf{z}} = -\frac{ql}{4\pi\epsilon_0 r^3}\hat{\mathbf{z}}$$

Exactly the same with Equation 2.36 in the case that $\theta = \pi/2$.

Problem 8

(3.75) Average of six points
Let $\phi(x,y,z)$ be any function that can be expanded in a power series around a point (x_0,y_0,z_0). Write a Taylor series expansion for the value of ϕ at each of the six points $(x_0+\delta,y_0,z_0)$, $(x_0-\delta,y_0,z_0)$, $(x_0,y_0+\delta,z_0)$, $(x_0,y_0-\delta,z_0)$, $(x_0,y_0,z_0+\delta)$, $(x_0,y_0,z_0-\delta)$, which symmetrically surround the point (x_0,y_0,z_0) at a distance δ. Show that, if ϕ satisfies Laplace's equation, the average of these six values is equal to (x_0,y_0,z_0) through terms of the third order in δ.

A Taylor expansion is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Laplace's equation is $\nabla^2\phi = 0$.
To take total derivative on $\phi(x,y,z)$ would complicate things. So for the points $(x_0+\delta,y_0,z_0)$ and $(x_0-\delta,y_0,z_0)$, the expansion could be taken on partial derivative respect to x.

$$\phi(x,y,z) = \phi(x_0,y_0,z_0) + \frac{\partial\phi(x_0,y_0,z_0)}{\partial x}(x-x_0) + \frac{1}{2}\frac{\partial^2\phi(x_0,y_0,z_0)}{\partial x^2}(x-x_0)^2 + \frac{1}{6}\frac{\partial^3\phi(x_0,y_0,z_0)}{\partial x^3}(x-x_0)^3 + \cdots$$

So

$$\frac{1}{2}(\phi(x_0+\delta,y_0,z_0) + \phi(x_0-\delta,y_0,z_0)) = \phi(x_0,y_0,z_0) + \frac{1}{2}\frac{\partial^2\phi(x_0,y_0,z_0)}{\partial x^2}\delta^2 + \frac{1}{24}\frac{\partial^4\phi(x_0,y_0,z_0)}{\partial x^4}\delta^4 + \cdots$$

And follow the exact same steps, the average on y and z directions could be derived. The average of all

Problem 8 continued on next page...

six points is then

$$\begin{aligned}
&\frac{1}{6}\left(\phi(x_0+\delta, y_0, z_0) + \phi(x_0-\delta, y_0, z_0) + \cdots + \phi(x_0, y_0, z_0-\delta)\right) \\
&= \phi(x_0, y_0, z_0) + \frac{1}{3}\left(\frac{1}{2}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right)\delta^2 + \frac{1}{24}\left(\frac{\partial^4\phi}{\partial x^4} + \frac{\partial^4\phi}{\partial y^4} + \frac{\partial^4\phi}{\partial z^4}\right)\delta^4 + \cdots\right) \\
&= \phi(x_0, y_0, z_0) + \frac{1}{3}\left(\frac{1}{2}\nabla^2\phi\delta^2 + \frac{1}{24}\nabla^4\phi\delta^4 + \cdots\right) = \phi(x_0, y_0, z_0)
\end{aligned}$$

Thus the average of these six values is equal to (x_0, y_0, z_0).

Problem 9

(3.76) The relaxation method... (problem omitted)

This problem is straightforward, once the idea is understood. The numbers are supposed to converge to a certain value that we want to know with each iteration of the specified operations. However the process of finding the average of sum of the neighbors for each point seems to be tedious. To reduce the labor, a Java program is written to do the calculation.

0.000	25.000	50.000	100.000
0.000	25.000	50.000	100.000
0.000	25.000	50.000	100.000
0.000	25.000	50.000	50.000
0.000	25.000	25.000	25.000
0.000	0.000	0.000	0.000

Table 1: Initial conditions

0.000	26.563	59.375	100.000
0.000	26.563	59.375	100.000
0.000	26.563	54.688	100.000
0.000	18.750	40.625	54.688
0.000	12.500	18.750	26.563
0.000	0.000	0.000	0.000

Table 3: Iteration 2

0.000	25.000	56.250	100.000
0.000	25.000	56.250	100.000
0.000	25.000	56.250	100.000
0.000	25.000	37.500	56.250
0.000	12.500	25.000	25.000
0.000	0.000	0.000	0.000

Table 2: Iteration 1

0.000	28.125	60.156	100.000
0.000	28.125	60.156	100.000
0.000	25.000	56.641	100.000
0.000	19.922	36.719	56.641
0.000	9.375	19.922	25.000
0.000	0.000	0.000	0.000

Table 4: Iteration 3

Problem 9 continued on next page...

0.000	28.320	61.230	100.000
0.000	28.320	61.230	100.000
0.000	26.172	55.469	100.000
0.000	17.773	38.281	55.469
0.000	9.961	17.773	26.172
0.000	0.000	0.000	0.000

Table 5: Iteration 4

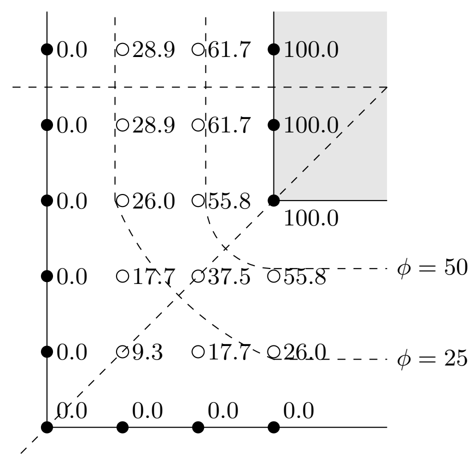
0.000	28.894	61.652	100.000
0.000	28.894	61.652	100.000
0.000	25.989	55.817	100.000
0.000	17.725	37.512	55.817
0.000	9.302	17.725	25.989
0.000	0.000	0.000	0.000

Table 7: Iteration 6

0.000	28.931	61.255	100.000
0.000	28.931	61.255	100.000
0.000	25.391	56.421	100.000
0.000	18.604	36.621	56.421
0.000	8.887	18.604	25.391
0.000	0.000	0.000	0.000

Table 6: Iteration 5

As shown above, after six iterations, the difference of each point between the current state and previous state is less than 1 unit. The sketch is shown below.



Problem 10

We considered a potential $\phi(x, y) = x^2 - y^2$. This scalar function satisfies LaPlace's equation so Theorem 2.1 in P&M applies. For a scalar function that satisfies LaPlace's equation, the average value of the function on the surface of any sphere is equal to the value of the function at the center of the sphere.

This is a 2D problem so a circle surrounding the point (x_0, y_0) serves as the sphere. Calculate the average value of on a circle of radius r centered at (x_0, y_0) by computing the value of the line integral $\oint \phi(x, y)\, dl$ around the circle and then dividing the value of the integral by the arc length $2r$. (This integral is easier to do after converting to circular coordinates.)

Theorem 2.1 If $\phi(x, y, z)$ satisfies Laplace's equation, then the average value of ϕ over the surface of any sphere (not necessarily a small sphere) is equal to the value of ϕ at the center of the sphere.
Convert the problem into polar coordinates. In other words, $x = r\cos\theta$, and $y = \sin\theta$. So

$$\oint \phi(x, y)\, dl = r^2 \oint (\cos^2\theta - \sin^2\theta)\, dl = r^2 \int_0^{2\pi} \cos 2\theta\, d\theta = 0$$

Which agrees with $\phi(0, 0) = 0$. The theorem is verified in this case.