PHYS 161: Homework 2

Due on Wednesday January 28, 2015

Professor Landee 11:00am

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Problem 1

(II-1) Find a unit vector $\hat{\mathbf{n}}$ normal to each of the following surfaces.

(a) $z = 2 - x - y$

(b) $z = (x^2 + y^2)^{1/2}$

(c) $z = (1 - x^2)^{1/2}$

(d) $z = x^2 + y^2$

(e) $z = (1 - x^2/a^2 - y^2/a^2)^{1/2}$

Using the principle ideal of (II-4),

(a) $f(x, y, z) = x + y + z - 2 = 0$. So $\mathbf{n} = \nabla f = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. $\hat{\mathbf{n}} = \begin{bmatrix} 1/3^{1/3} \\ 1/3^{1/3} \\ 1/3^{1/3} \end{bmatrix}$.

(b) For $z = (x^2 + y^2)^{1/2}$, $\mathbf{n} = \begin{bmatrix} \frac{\partial}{\partial x}(x^2 + y^2)^{1/2} \\ \frac{\partial}{\partial y}(x^2 + y^2)^{1/2} \\ -1 \end{bmatrix} = \begin{bmatrix} x(x^2 + y^2)^{-1/2} \\ y(x^2 + y^2)^{-1/2} \\ -1 \end{bmatrix}$. $\hat{\mathbf{n}} = \begin{bmatrix} x((x^2 + y^2)/2)^{-1/2} \\ y((x^2 + y^2)/2)^{-1/2} \\ -1/\sqrt{2} \end{bmatrix}$.

(c) $\hat{\mathbf{n}} = \dfrac{\hat{\mathbf{i}}\frac{\partial f}{\partial x} + \hat{\mathbf{j}}\frac{\partial f}{\partial y} - \hat{\mathbf{k}}}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}} = \dfrac{-x(1 - x^2)^{-1/2}\hat{\mathbf{i}} - \hat{\mathbf{k}}}{\sqrt{\frac{1}{1-x^2}}} = -x\hat{\mathbf{i}} - (1 - x^2)^{1/2}\hat{\mathbf{k}} = -x\hat{\mathbf{i}} - z\hat{\mathbf{k}}$

(d) $\hat{\mathbf{n}} = \dfrac{\hat{\mathbf{i}}\frac{\partial f}{\partial x} + \hat{\mathbf{j}}\frac{\partial f}{\partial y} - \hat{\mathbf{k}}}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}} = \dfrac{2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} - \hat{\mathbf{k}}}{\sqrt{1 + 4x^2 + 4y^2}} = \dfrac{2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} - \hat{\mathbf{k}}}{\sqrt{1 + 4z}}$

(e)

$$\begin{aligned}
\hat{\mathbf{n}} &= \frac{\hat{\mathbf{i}}\frac{\partial f}{\partial x} + \hat{\mathbf{j}}\frac{\partial f}{\partial y} - \hat{\mathbf{k}}}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}} \\
&= \frac{-x/a^2(1 - x^2/a^2 - y^2/a^2)^{-1/2}\hat{\mathbf{i}} - y/a^2(1 - x^2/a^2 - y^2/a^2)^{-1/2}\hat{\mathbf{j}} - \hat{\mathbf{k}}}{\sqrt{1 + (4x^2 + 4y^2)/(a^4(1 - x^2/a^2 - y^2/a^2))}} \\
&= \frac{-x\hat{\mathbf{i}} - y\hat{\mathbf{j}} - a^2 z\hat{\mathbf{k}}}{a\sqrt{a^2 + 4(x^2 + y^2)}}
\end{aligned}$$

Problem 2

(II-2)

(a) Show that the unit vector normal to the plane

$$ax + by + cz = d$$

is given by

$$\hat{\mathbf{n}} = \pm(\hat{\mathbf{i}}a + \hat{\mathbf{j}}b + \hat{\mathbf{k}}c)/(a^2 + b^2 + c^2)^{1/2}$$

Problem 2 continued on next page...

(b) Explain in geometric terms why this expression for $\hat{\mathbf{n}}$ is independent of the constant d.

(a) As indicated in book, for $f(x,y,z) = ax + by + cz + d$, $\hat{\mathbf{n}} = \pm\frac{\hat{\mathbf{i}}\frac{\partial f}{\partial x} + \hat{\mathbf{j}}\frac{\partial f}{\partial y} + \hat{\mathbf{k}}\frac{\partial f}{\partial z}}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}} = \pm\frac{\hat{\mathbf{i}}a + \hat{\mathbf{j}}b + \hat{\mathbf{k}}c}{(a^2+b^2+c^2)^{1/2}}$.

(b) d in the equation could be imagined as shifting the surface up or down along z axis, which do not change the shape of the surface. Therefore the unit normal vector does not change at all.

Problem 3

(II-4) In each of the following use Equation II-12 to evaluate the surface integral $\iint_S G(x,y,z)\,dS$.

(a)

$$G(x,y,z) = z$$

where S is the portion of the plane $x + y + z = 1$ in the first octant.

(b)

$$G(x,y,z) = \frac{1}{1 + 4(x^2+y^2)}$$

where S is the portion of the paraboloid $z = x^2 + y^2$ between $z = 0$ and $z = 1$.

(c)

$$G(x,y,z) = (1 - x^2 - y^2)^{3/2}$$

where S is the hemisphere $z = (1 - x^2 - y^2)^{1/2}$.

$$\iint_S G(x,y,z)\,dS = \iint_R G[x,y,f(x,y)] \cdot \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\,dx\,dy \tag{II-12}$$

(a) So $z = f(x,y) = 1 - x - y$. This gives $\frac{\partial f}{\partial x} = -1$, and $\frac{\partial f}{\partial y} = -1$. So $\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} = \sqrt{3}$. The region R becomes the area in xy-plane in the first quardrant that $x + y < 1$, a triangle. So

$$\iint_S z\,dS = \sqrt{3}\iint_R (1-x-y)\,dx\,dy = \sqrt{3}\int_0^{1-x}\int_0^1 (1-x-y)\,dx\,dy = \frac{\sqrt{3}}{6}$$

(b) $z = f(x,y) = x^2 + y^2$. This gives $\frac{\partial f}{\partial x} = 2x$, and $\frac{\partial f}{\partial y} = 2y$. So $\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} = \sqrt{1 + 4(x^2+y^2)}$. The region R becomes the circle in xy-plane with $x^2 + y^2 = 1$.

$$\iint_S z\,dS = \iint_R \frac{1}{\sqrt{1+4(x^2+y^2)}}\,dx\,dy = \int_0^1\int_0^{2\pi} \frac{1}{\sqrt{1+4r^2}} r\,d\theta\,dr = \frac{(\sqrt{5}-1)\pi}{2}$$

(c) $\frac{\partial f}{\partial x} = -x(1-x^2-y^2)^{-1/2}$, and $\frac{\partial f}{\partial y} = -y(1-x^2-y^2)^{-1/2}$. $\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2} = \sqrt{1+\frac{x^2+y^2}{1-x^2-y^2}} = (1-x^2-y^2)^{-1/2}$. The region R is the circle in xy-plane with $x^2+y^2=1$.

$$\iint_S z\,dS = \iint_R (1-x^2-y^2)\,dx\,dy = \int_0^1 \int_0^{2\pi} (1-r^2)r\,d\theta\,dr = \frac{\pi}{2}$$

Problem 4

(II-5) In each of the following... (problem omitted)

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_R \left\{-F_x[x,y,f(x,y)]\frac{\partial f}{\partial x} - F_y[x,y,f(x,y)]\frac{\partial f}{\partial y} + F_z[x,y,f(x,y)]\right\}dx\,dy \qquad \text{(II-13)}$$

(a) $z=f(x,y)=1-x/2-y/2$. $\frac{\partial f}{\partial x}=-1/2$, $\frac{\partial f}{\partial y}=-1/2$. $F_x=x$, $F_y=0$, $F_z=-z$. R is the triangle region in the first quadrant of the xy-plane. So

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \int_0^{2-x}\int_0^2 (x+y/2-1)\,dx\,dy = 0$$

(b) $\frac{\partial f}{\partial x} = -x(a^2-x^2-y^2)^{-1/2}$, $\frac{\partial f}{\partial y} = -y(a^2-x^2-y^2)^{-1/2}$.

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_R \left(\frac{x^2+y^2}{(a^2-x^2-y^2)^{1/2}} + (a^2-x^2-y^2)^{1/2}\right)dx\,dy = \iint_R \frac{a^2}{(a^2-x^2-y^2)^{1/2}}\,dx\,dy$$

With a change in varibles,

$$\iint_R \frac{a^2}{(a^2-x^2-y^2)^{1/2}}\,dx\,dy = \int_0^a\int_0^{2\pi}\frac{a^2}{(a^2-r^2)^{1/2}}r\,d\theta\,dr = 2\pi a^3$$

(c) $\frac{\partial f}{\partial x}=-2x$, $\frac{\partial f}{\partial y}=-2y$. R again becomes the circle $x^2+y^2=1$ on the xy-plane. The integral then becomes

$$\iint_R (2y^2+1)\,dx\,dy = 4\int_0^{\sqrt{1-x^2}}\int_0^1 (2y^2+1)\,dx\,dy = \frac{3\pi}{2}$$

Problem 5

(II-6) The distribution of mass... (problem omitted)

Total mass of the shell is

$$\iint \sigma(x,y,z)\,ds = \iint \sigma(x,y)\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dx\,dy = \frac{1}{R}\iint\frac{\sigma_0(x^2+y^2)}{\sqrt{R^2-x^2-y^2}}$$

Problem 5 continued on next page. . .

A change of variables gives

$$\frac{\sigma_0}{R}\int_0^R\int_0^{2\pi}\frac{r^3}{\sqrt{R^2-r^2}}\,d\theta\,dr=\frac{4\pi R^2\sigma_0}{3}$$

Problem 6

(II-10) It sometimes happens that surface... (problem omitted)

(II-30)

(a) Take the S square in xy-plane as example. The integral becomes $\iint_S z\,dS$, which obviously equals 0. The same conclusion applies to the other two squares. So the integral is 0.

(b) The top and bottom contribute 0 to the integral. We only have to calculate the side surface. From

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS=\iiint_V \nabla\cdot\mathbf{F}\,dV \tag{II-30}$$

$\nabla\cdot\mathbf{F}=\ln(x^2+y^2)+\frac{2x^2}{x^2+y^2}+\ln(x^2+y^2)+\frac{2y^2}{x^2+y^2}=2\ln(x^2+y^2)+2$. So

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS=\iiint_V (2\ln(x^2+y^2)+2)\,dV$$

Change the variables into cylindrical coordinates.

$$\iiint_V (2\ln(x^2+y^2)+2)\,dV=\int_0^h\int_0^{2\pi}\int_0^R(2\ln(r^2)+2)r\,dr\,d\phi\,dz=4h\pi R^2\ln R$$

It is now obvious that the integral is actually $V\ln R^2$, where V is the volume of the cylinder and $R^2=x^2+y^2$.

(c) For the sphere with radius R, $\mathbf{F}=(\mathbf{i}x+\mathbf{j}y+\mathbf{k}z)e^{-R^2}$. $\nabla\cdot\mathbf{F}=3e^{-R^2}$. Therefore $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS=\iiint_V \nabla\cdot\mathbf{F}\,dV=e^{-R^2}\iiint_V\,dV=3e^{-R^2}\times\frac{4}{3}\pi R^3=4\pi R^3e^{-R^2}$.

(d) Only two surfaces parallel to the yz-plane has contribution to the integral. $\nabla\cdot\mathbf{F}=E'(x)$. Therefore $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS=\iiint_V \nabla\cdot\mathbf{F}\,dV=b^2\int_0^b E'(x)\,dx=b^2[E(b)-E(0)]$.

Problem 7

(II-14) Calculate the divergence... (problem omitted)

This is rather simple.

(a) $2x + 2y + 2z$

(b) 0

(c) $-e^{-x} - e^{-y} - e^{-z}$

(d) $2z$

(e) $\frac{-y}{x^2+y^2}$

(f) 0

(g) 3

(h) 0

Problem 8

(II-15) (a) Calculate $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, dS$ for... (problem omitted)

(a)

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, dS &= \iiint_V \nabla \cdot \mathbf{F} \, dV \\
&= 2 \iiint_V (x + y + z) \, dV \\
&= 2 \int_{x_0 - s/2}^{x_0 + s/2} \int_{y_0 - s/2}^{y_0 + s/2} \int_{z_0 - s/2}^{z_0 + s/2} (x + y + z) \, dz \, dy \, dx \\
&= 2s^3(x_0 + y_0 + z_0)
\end{aligned}$$

(b) $2s^3(x_0 + y_0 + z_0)/s^3 = 2(x_0 + y_0 + z_0)$. It is the divergence evaluated at point (x_0, y_0, z_0).

(c) For II-14(b), div $\mathbf{F} = 0$, so $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = \iiint_V \nabla \cdot \mathbf{F} \, dV = 0$.

For II-14(c),

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, dS &= \iiint_V \nabla \cdot \mathbf{F} \, dV \\
&= \int_{x_0 - s/2}^{x_0 + s/2} \int_{y_0 - s/2}^{y_0 + s/2} \int_{z_0 - s/2}^{z_0 + s/2} (-e^{-x} - e^{-y} - e^{-z}) \, dz \, dy \, dx
\end{aligned}$$

Using CAS system, this result is $-2(e^{-x_0} + e^{-y_0} + e^{-z_0})s^2 \sinh(s/2)$. So this result devided by s^3 is $-2(e^{-x_0} + e^{-y_0} + e^{-z_0}) \sinh(s/2)/s$.

$$\lim_{s \to 0} \frac{-2(e^{-x_0} + e^{-y_0} + e^{-z_0}) \sinh(s/2)}{s} = 0$$

Problem 9

(II-16) (a) Calculate the divergence... (problem omitted)

Problem 9 continued on next page...

(a) In Cartesian coordinates, div $\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial f(x)}{\partial x} + \frac{\partial f(y)}{\partial y} + \frac{\partial f(-2z)}{\partial z} = f'(x) + f'(y) - 2f'(-2z)$. So at $(c, c, -c/2)$, div $\mathbf{F} = f'(c) + f'(c) - 2f'(c) = 0$.

(b) Obviously, each of the functions f, g, h does not have dependence on the corresponding coordinate. so div $\mathbf{G} = 0$.

Problem 10

(II-23) Verify the divergence theorem... (problem omitted)

(a) The surface integral could be devided into three parts, each at b of the three coordinates.

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \int_0^b \int_0^b b\, dy\, dz + \int_0^b \int_0^b b\, dx\, dz + \int_0^b \int_0^b b\, dx\, dy = 3b^3$$

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = 3 \iiint_V dV = 3b^3$$

(b) The surface integral could be devided into two parts, the top of the cylinder, with unit normal vector $\hat{\mathbf{e}}_z$ and the outer curved surface, with unit normal vector $\hat{\mathbf{e}}_r$. The two surfaces on xz- and yz-plane is obviously zero.

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iint_{S_1} h\, dS + \iint_{S_2} R\, dS = \frac{1}{4}\pi R^2 h + \frac{1}{4} \times 2\pi R \times R = \frac{3\pi R^2 h}{4}$$

For this function in cylindrical coordinates,

$$\text{div } \mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{\partial F_z}{\partial z} = 2 + 1 = 3$$

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = 3 \iiint_V dV = \frac{3\pi R^2 h}{4}$$

(c)

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iint_S R^2\, dS = 4\pi R^4$$

For this function in spherical coordinates,

$$\text{div } \mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) = 4r$$

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = 4 \iiint_V r\, dV = 4 \int_0^{2\pi} \int_0^{\pi} \int_0^R r \cdot r^2 \sin\theta\, dr\, d\theta\, d\phi = 4\pi R^4$$