Physics 161, spring 2015

Answer Key for HW # 6

The first two problems (PM 6.4 and 6.10) have answers in Chapter 12 of PM. The remaining four are also from Purcell and Morin.

3. PM 6.36.
4. PM 6.60
5. PM 6.63
6. PM 6.70

3. 6.36. **Field at different radii**

The radius is 2 cm, so 1/4 of the cross-sectional area, and hence current (so 2000 A), is enclosed within $r = 1$ cm. The current enclosed in both the $r = 2$ cm and $r = 3$ cm cases is 8000 A. So we have

$$\begin{aligned} B_1 &= \frac{\mu_0 I_1}{2\pi r_1} = \frac{(4\pi \cdot 10^{-7}\,\frac{\text{kg m}}{\text{C}^2})(2000\,\text{A})}{2\pi(0.01\,\text{m})} = 0.04\,\text{T}, \\ B_2 &= \frac{\mu_0 I_2}{2\pi r_2} = \frac{(4\pi \cdot 10^{-7}\,\frac{\text{kg m}}{\text{C}^2})(8000\,\text{A})}{2\pi(0.02\,\text{m})} = 0.08\,\text{T}, \\ B_3 &= \frac{\mu_0 I_3}{2\pi r_3} = \frac{(4\pi \cdot 10^{-7}\,\frac{\text{kg m}}{\text{C}^2})(8000\,\text{A})}{2\pi(0.03\,\text{m})} = 0.0533\,\text{T}. \end{aligned} \tag{435}$$

These fields are 400, 800, and 533 gauss, respectively.

4 6.60. **Zero field outside a solenoid**

Let the solenoid extend in the z direction. Consider one of the small patches. The current in this patch flows in some direction in the xy plane. That is, the $\boldsymbol{dl}$ vector in the Biot-Savart law lies in the xy plane. (We can slice the patch into many thin strips represented by $\boldsymbol{dl}$'s.) The $\boldsymbol{dl}$ vector gets crossed with the $\hat{\mathbf{r}}$ vector directed to the point P. Now, $\hat{\mathbf{r}}$ has a z component, but the $\hat{\mathbf{r}}$ vector associated with the corresponding patch defined by the thin cone on the other side of P has the opposite z component. These components therefore yield canceling contributions to the total magnetic field. So we need only worry about the component of $\hat{\mathbf{r}}$ that lies in the xy plane. Let's call this vector $\hat{\mathbf{r}}_{xy}$.

We need to compute the cross product of $\boldsymbol{dl}$ and $\hat{\mathbf{r}}_{xy}$, both of which lie in the xy plane. (The resulting cross product will therefore point in the z direction, so we have just proved that the B field from the solenoid must be longitudinal.) In general, $\boldsymbol{dl}$ has a component parallel to $\hat{\mathbf{r}}_{xy}$ and a component perpendicular to $\hat{\mathbf{r}}_{xy}$. The parallel component yields zero in the cross product $\boldsymbol{dl} \times \hat{\mathbf{r}}_{xy}$, so we need only worry about the component perpendicular to $\hat{\mathbf{r}}_{xy}$. In other words, if we project the area of the patch onto the (vertical) plane orthogonal to $\hat{\mathbf{r}}_{xy}$, then the cross product $\boldsymbol{dl} \times \hat{\mathbf{r}}_{xy}$ remains the same. We can do the same with the other patch in the same cone.

We therefore need to compare the Biot-Savart contributions from the two "projected" patches of area defined by a given cone. If the projected patches are distances r_1 and r_2 from the point P, then their areas are proportional to r_1^2 and r_2^2, because areas are proportional to length squared, and because the patches cut the line from P at the same angle (perpendicular, by construction).

Answer Key HW #6

Now, if we imagine a small rectangular patch (any patch can be built up from rectangles), the $I\,dl$ product in the Biot-Savart law is proportional to the area, because I is proportional to the height dh of the rectangle (since $I = \mathcal{J}\,dh$), and because $dh\,dl$ is the area of the rectangle. The numerators in the Biot-Savart law for the corresponding patches are therefore proportional to r_1^2 and r_2^2. These factors exactly cancel the r^2 in the denominator of the Biot-Savart law. So the magnitudes of the contributions from the two patches equal. And since the currents flow in opposite directions in the projected patches, the contributions therefore cancel. The entire solenoid can be considered to be built up from small patches subtended by cones, so the external field is zero, as desired.

If the solenoid isn't convex, then a given cone may define 4, 6, 8, etc., patches. But there will still be equal numbers of patches having currents in each direction (which can be traced to the fact that P has the property of being outside the solenoid), so the sum of the contributions will still be zero.

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6.63. Solenoids and superposition

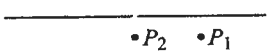

Figure 123

(a) Imagine adding a similar solenoid on the left, as shown in Fig. 123. This exactly doubles the field strength at P_2. But now the field strengths at P_2 and P_1 are approximately equal, because both points lie well inside a fairly long solenoid, the field at P_2 being slightly stronger. Therefore, the original field at P_2 must have been slightly more than half the field at P_1.

This "more than half" result is consistent with the extreme case where the solenoid is very short, basically just a ring. In this case the fields at P_2 and P_1 are essentially equal, both taking on the value of the field at the center of a ring, namely $\mu_0 I/2r$. So the field at P_2 is certainly more than half of the field at P_1.

A less elegant way of solving this exercise is to use Eq. (6.56). The field at the center is proportional to $2\cos\alpha_1$, and the field at the end is proportional to $\cos\alpha_2$ (plus $\cos 90°$, which is zero), where these angles are defined in Fig. 124. For small angles, both of these cosines are essentially equal to 1, hence the ratio of 1/2 in the fields. But $\cos\alpha_2 > \cos\alpha_1$, hence the "more than half."

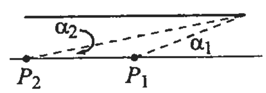

Figure 124

(b) Let's assume (in search of a contradiction) that there exists a field line that crosses the line GH with a vertical component, as shown in Fig. 125(a). Imagine flipping the solenoid upside down to obtain the situation in Fig. 125(b), and then reversing the direction of the current (so that it now has the same direction as in Fig. 125(a)) to obtain the situation in Fig. 125(c). Note that the field at the given point on the line GH has a downward component in both figures (a) and (c) (or upward in both, if we had initially drawn it upward).

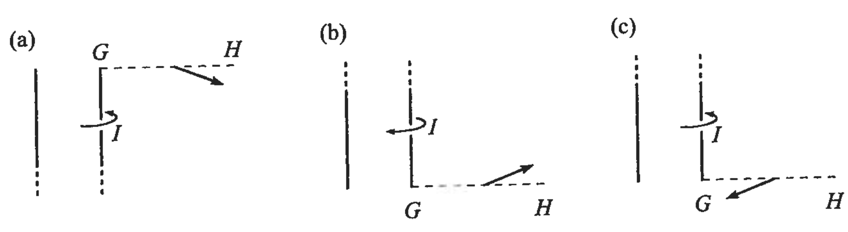

Figure 125

Now join the two semi-infinite solenoids in figures (a) and (c) end to end, thereby creating an infinite solenoid. By superposition, the fields simply add, so we end up with a downward component at the given point along GH. But this is a contradiction, because we know that the field of an infinite solenoid is zero outside the solenoid. We conclude that the field due to the semi-infinite solenoid

Answer Key, HW #6

6.63 conclusion

To find the magnitude of the field, we can use Ampere's law. By symmetry, the magnetic field must have the same magnitude B everywhere on a circle of radius r around the axis. The line integral of $\mathbf{B}$ around this circle equals μ_0 times the current enclosed. Since $\mathbf{B}$ points in the tangential direction, the line integral equals $2\pi r B$. If the circle doesn't lie inside the torus, the current enclosed is zero. This is true because either the disk defined by the circle doesn't intersect the torus, in which case the current enclosed is clearly zero; or the disk does intersect the torus, in which case a current of NI passes through the disk in one direction, but another NI also passes through in the other direction. Therefore, $\mathbf{B} = 0$ everywhere outside the torus.

On the other hand, if the circle lies inside the torus, the current enclosed is NI, because the disk defined by the circle intersects only the inner boundary of the torus. Therefore, $2\pi r B = \mu_0 NI \implies B = \mu_0 NI/2\pi r$ inside the torus. This expression for B holds for a torus of any (uniform) cross section. Note that B depends only on r, and not on the "height" inside the torus.

In the limit where $b - a \ll a$, the curvature of the torus is negligible, so we essentially have an infinite straight solenoid with rectangular cross section. The field should therefore equal $\mu_0 nI$ (see the solution to Problem 6.19), where n is the number of turns per unit length. And indeed, in the above result, $n = N/2\pi r$ is the number of turns per unit length (where r is essentially equal to both a and b), so we do obtain $B = \mu_0 nI$.

#6.

6.70. **Drifting motion**

If there is a frame in which the electric field is zero, then we know from Exercise 6.29 that the ion moves in a circle in that frame. Let F be the lab frame, and consider the frame F' that moves in the positive y direction with speed $v = (0.1\,\text{m})/(1\,\mu\text{s}) = 10^5\,\text{m/s} = c/3000$. Since F' moves with the average velocity of the ion, it is the only frame in which the ion could possibly be moving in a circle (because in any other frame the ion would drift away). F sees F' moving with velocity $v\hat{\mathbf{y}}$, so if we demand that the electric field be zero in F', then Eq. (6.76) tells us how $\mathbf{E}_\perp$ and $\mathbf{B}_\perp$ in the lab frame F must relate to each other:

$$\begin{aligned} \mathbf{E}'_\perp &= \gamma(\mathbf{E}_\perp + \mathbf{v} \times \mathbf{B}_\perp) \\ \implies 0 &= \gamma(\mathbf{E}_\perp + (v\hat{\mathbf{y}}) \times (0.6\,\text{T})\hat{\mathbf{z}}) \\ \implies \mathbf{E}_\perp &= -(10^5\,\text{m/s})(0.6\,\text{T})(\hat{\mathbf{y}} \times \hat{\mathbf{z}}) \end{aligned}$$

Note that $\mathbf{E}_\perp$ points in the (negative) x direction, whereas the drift of the motion is in the y direction. See Problem 6.26 for more details.