PHYS 161: Homework 1

Due on Wednesday January 21, 2015

Professor Landee 11:00am

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Problem 1

(1.54) Semicircle and wires

(a) Two long, thin parallel rods, adistance $2b$ apart, are joined by a semicircular piece of radius b, as shown in Fig. 1.44. Charge of uniform linear density λ is deposited along the whole filament. Show that the field $\mathbf{E}$ of this charge distribution vanishes at the point C. Do this by comparing the contribution of the element at A to that of the element at B which is defined by the same values of θ and $d\theta$.

(b) Consider the analogous two-dimensional setup involving a cylinder and a hemispherical end cap, with uniform surface charge density σ. Using the result from part (a), do you think that the field at the analogous point C is directed upward, downward, or is zero? (No calculations needed!)

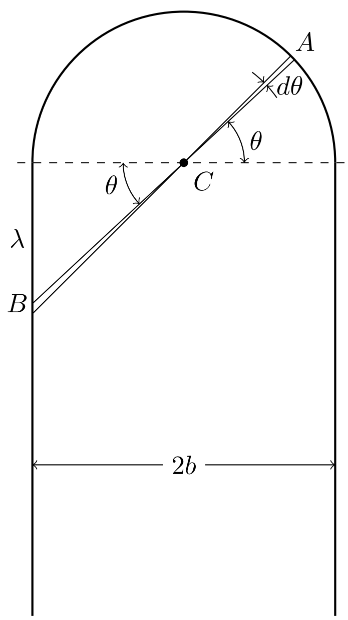


Figure 1.44

(a) For element A, $dl = b\,d\theta$, and $dQ = \lambda b\,d\theta$. $dE = \frac{dQ}{4\pi\epsilon_0 b^2} = \frac{\lambda\,d\theta}{4\pi\epsilon_0 b}$.

For element B, the radius is now $b/\cos\theta$. Because the length on rod has angle θ relative to the direction perpendicular to radius, $dl = b\,d\theta/\cos^2\theta$. so $dQ = \lambda b\,d\theta/\cos^2\theta$. $dE = \frac{dQ}{4\pi\epsilon_0 (b/\cos\theta)^2} = \frac{\lambda\,d\theta}{4\pi\epsilon_0 b}$.

Therefore, the contribution of the two elements A and B has the same magnitude and opposite direction. They cancel each other. Field $\mathbf{E}$ vanished at point C.

(b) Analogous to the one-dimensional setup, the field at point C should add up to be zero.

Problem 2

(1.59) Zero field in a cylindrical shell

Consider a distribution of charge in the form of a hollow circular cylinder, like a long charged pipe. In the spirit of Problem 1.17, show that the electric field inside the pipe is zero.

Consider two small circular patches on the cylindrical shell with radius R, defined by arcs projected from one point with the angle $d\theta$ onto the shell. One of them has the contribution to $\mathbf{E}$:

$$dQ = \pi \left(\frac{r\,d\theta}{2} \right)^2 \sigma$$

$$dE = \frac{dQ}{4\pi\epsilon_0 r^2} = \frac{(d\theta)^2\sigma}{16\epsilon_0}$$

The other one has the contribution:

$$dQ' = \pi \left(\frac{(R-r)\,d\theta}{2} \right)^2 \sigma$$

$$dE' = \frac{dQ'}{4\pi\epsilon_0 (R-r)^2} = \frac{(d\theta)^2\sigma}{16\epsilon_0}$$

They have the same magnitude, opposite direction, so they cancel each other. This explains why the electric field inside the pipe is zero.

Problem 3

(1.66) Force between two strips

(a) The two strips of charge shown in Fig. 1.47 have width b, infinite height, and negligible thickness (in the direction perpendicular to the page). Their charge densities per unit area are $\pm\sigma$. Find the magnitude of the electric field due to one of the strips, a distance x away from it (in the plane of the page).

(b) Show that the force (per unit height) between the two strips equals $\sigma^2 b(\ln 2)/\pi\epsilon_0$. Note that this result is finite, even though you will find that the field due to a strip diverges as you get close to it.

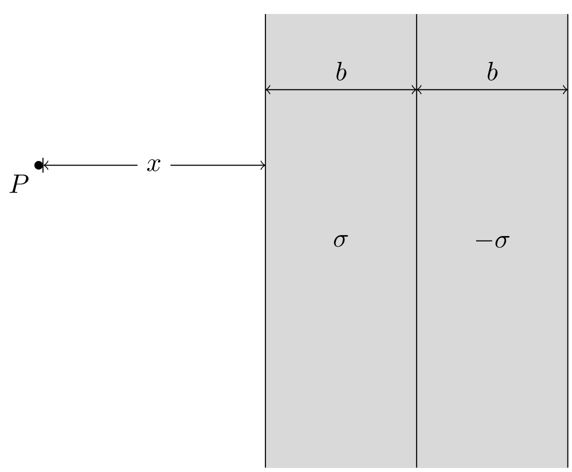


Figure 1.47

(a) Consider a point P on the left of the strips, distance x away from the left edge of the strips, and consider the left strip only. Each vertical line of charge with width dx contributes $E = \frac{\lambda}{2\pi\epsilon_0 x}$ to the electric field, where its linear charge density could be obtained as $\lambda = \sigma\, dx$. The total $\mathbf{E}$ is:

$$\int_{x+b}^{x} \frac{\sigma\, dx'}{2\pi\epsilon_0 x'} = \frac{\sigma}{2\pi\epsilon_0}\int_{x+b}^{x} \frac{dx'}{x} = \frac{\sigma}{2\pi\epsilon_0}\ln\frac{x}{x+b}$$

(b) A charge dQ experience a force $\frac{\sigma\, dQ}{2\pi\epsilon_0}\ln\frac{x}{x+b}$ at a distance x. So a horizontal strip of charge on the other strip with charge density $-\sigma$ experiences force of:

$$F = \int_0^b \frac{\sigma\, dQ}{2\pi\epsilon_0}\ln\frac{x}{x+b}dx = \frac{\sigma\, dQ}{2\pi\epsilon_0}b\ln 4 = \frac{b\sigma\, dQ}{\pi\epsilon_0}\ln 2$$

And dQ is simply the charge density σ, neglecting directions. So the force is $\frac{b\sigma^2}{\pi\epsilon_0}\ln 2$.

Problem 4

(1.72) A plane and a slab

An infinite plane has uniform surface charge density σ. Adjacent to it is an infinite parallel layer of charge of thickness d and uniform volume charge density ρ, as shown in Fig. 1.50. All charges are fixed. Find $\mathbf{E}$ everywhere.

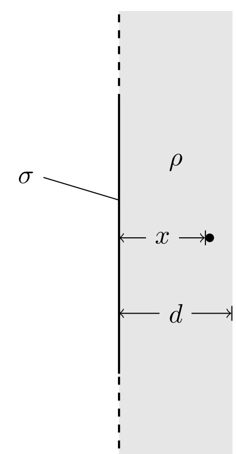


Figure 1.50

For the infinite plane, the $E = \frac{\sigma}{2\epsilon_0}$.
For the layer of charge, each layer with thickness dx has the charge density $\rho\, dx$, and has the contribution to $E = \frac{\rho\, dx}{2\epsilon_0}$.
For region to the left of the plane, the layer has

$$E = \int_0^d \frac{\rho\, dx}{2\epsilon_0} = \frac{\rho d}{2\epsilon_0}$$

So the $E = \frac{\sigma}{2\epsilon_0} + \frac{\rho d}{2\epsilon_0}$.
For the region to the right of the plane, positioned $x < d$ from the plane, the layer has

$$E = \int_0^x \frac{\rho\, dx'}{2\epsilon_0} = \frac{\rho x}{2\epsilon_0}$$

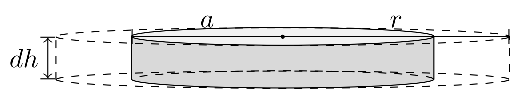
from its left and $\frac{\rho(d-x)}{2\epsilon_0}$ from its right. The total $E = -\frac{\sigma}{2\epsilon_0} - \frac{\rho x}{2\epsilon_0} + \frac{\rho(d-x)}{2\epsilon_0} = -\frac{\sigma}{2\epsilon_0} - \frac{\rho x}{\epsilon_0} + \frac{\rho d}{2\epsilon_0}$.
For the region to the right of the plane, positioned $x \geq d$ from the plane, similar to the region to left of the plane $E = -\frac{\sigma}{2\epsilon_0} - \frac{\rho d}{2\epsilon_0}$.
The direction of E would depend on the signs and magnitudes of σ and ρ, but the positive direction is set to be to the left.

Problem 5

(1.83) Potential energy of a cylinder
Problem 1.24 gives one way of calculating the energy per unit length stored in a solid cylinder with radius a and uniform volume charge density ρ. Calculate the energy here by using Eq. (1.53) to find the total energy per unit length stored in the electric field. Don't forget to include the field inside the cylinder.
You will find that the energy is infinite, so instead calculate the energy relative to the configuration where all the charge is initially distributed uniformly over a hollow cylinder with large radius R. (The field outside radius R is the same in both configurations, so it can be ignored when calculating the relative energy.) In terms of the total charge per unit length in the final cylinder, show that the energy per unit length can be written as $(\lambda^2/4\pi\epsilon_0)\,(1/4 + \ln(R/a))$.

$$U = \frac{\epsilon_0}{2}\int_{\substack{\text{entire}\\\text{field}}} E^2\, dv \tag{1.53}$$

The problem is equivalent to compressing charges from a disk of radius R to a smaller disk of radius a. Outside the cylinder, the field at radius r is $\frac{\lambda}{2\pi\epsilon_0 r}$. Consider a disk with height dh as shown below, whose $v = \pi r^2\, dh$, and thus $dv = 2\pi r\, dr\, dh$. It holds charge of $\rho\pi a^2\, dh$, which means the linear charge density $\lambda = \rho\pi a^2$, when it is considered from its outside.



The energy stored in the external field then is

$$U_{\text{ext}} = \frac{\epsilon_0}{2}\int_{r=a}^{r=R}\left(\frac{\rho\pi a^2}{2\pi\epsilon_0 r}\right)^2 2\pi r\, dr\, dh = \frac{\rho^2\pi a^4}{4\epsilon_0}\, dh \int_a^R \frac{dr}{r} = \frac{\rho^2\pi a^4}{4\epsilon_0}\ln\frac{R}{a}\, dh = \frac{\lambda^2}{4\pi\epsilon_0}\ln\frac{R}{a}\, dh$$

Inside the cylinder, also consider a disk with height dh and radius r, whose $v = \pi r^2\, dh$, and thus $dv = 2\pi r\, dr\, dh$. It holds charge of $\rho\pi r^2\, dh$, which means the linear charge density $\lambda = \rho\pi r^2$, when it is considered equivalently as a line charge.
The energy stored in the internal field is

$$U_{\text{int}} = \frac{\epsilon_0}{2}\int_{r=0}^{r=a}\left(\frac{\rho\pi r^2}{2\pi\epsilon_0 r}\right)^2 2\pi r\, dr\, dh = \frac{\rho^2\pi}{4\epsilon_0}\, dh\int_0^a r^3\, dr = \frac{\rho^2\pi a^4}{16\epsilon_0}\, dh = \frac{\lambda^2}{16\pi\epsilon_0}\, dh$$

Therefore it is shown that the total energy for a disk of thickness dh has the energy $\frac{\lambda^2}{4\pi\epsilon_0}\ln\frac{R}{a}\,dh + \frac{\lambda^2}{16\pi\epsilon_0}\,dh$. The energy per unit length is exactly

$$\frac{\lambda^2}{4\pi\epsilon_0}\left(\frac{1}{4}+\ln\frac{R}{a}\right)$$