Sampling of Gauss-Markov Processes over a Channel with Random Delay

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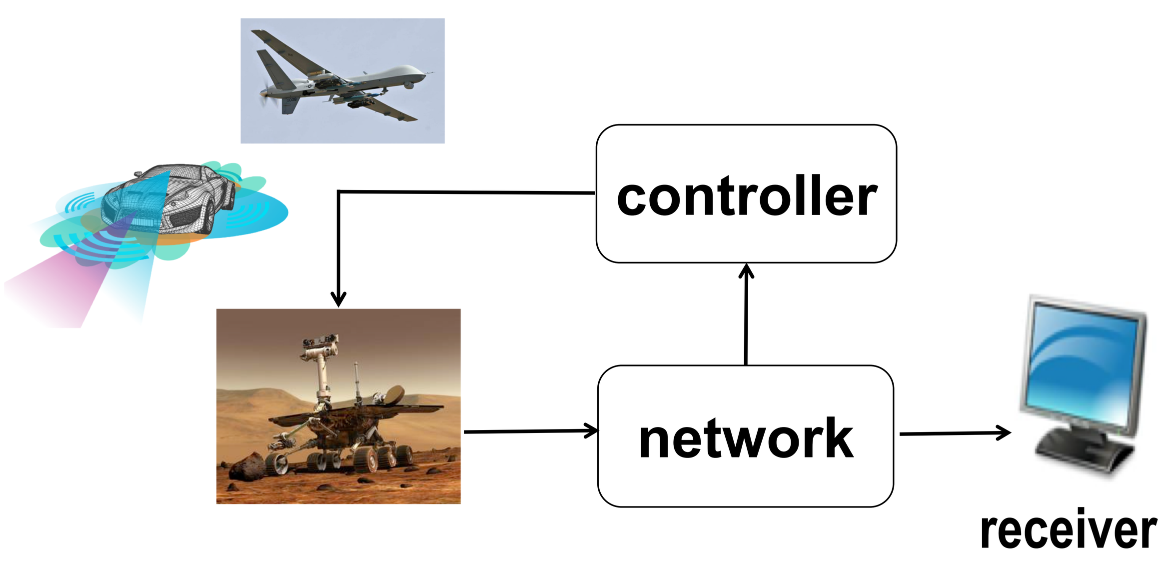
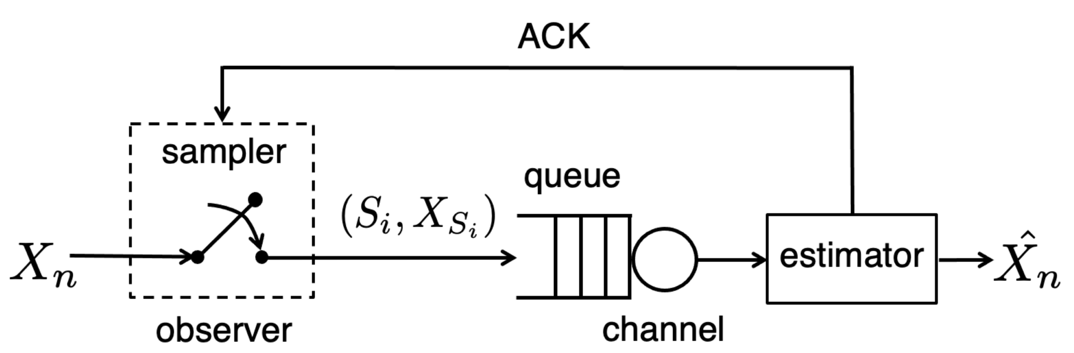
Fresh data in real-time applications



E.g., Sensor networks, IoT, CPS, robots

System Model



- Channel modeled as a **FIFO** queue with **discrete *i.i.d.*** service times Y_i
- Sample i is taken at time S_i and is delivered at time D_i
- MMSE Estimator: Use causally received samples estimate **real-time** value of X_n
- Gauss-Markov Signal Model:
  - Wiener process: Brownian motion
  - Ornstein-Uhlenbeck (OU) Process:
    $dX_n = \theta(\mu - X_n)dt + \sigma dW_n$
  - Stable ($\theta > 0$), Unstable ($\theta < 0$)

Signal-aware sampling

Gauss-Markov Signals:

- **Wiener Process/Brownian Motion:**
  $\theta = 0, \sigma = 1$
  e.g., physical processes, applied mathematics, economics, quantitative finance, evolutionary biology.
- **OU Process:**
  Stable → $\theta > 0$
  Unstable → $\theta < 0$
  e.g., interest rates, currency exchange rates, commodity prices, node-mobility in mobile ad-hoc networks, UAV systems, control systems.

Conclusions

1. Signal-aware sampling:

- Threshold on **instant** estimation error;
- Threshold = v(optimal objective value)
- Threshold function v(.) determined by signal model.

Is AoI a good freshness measure?

Age of Information (AoI), defined as

$$\Delta(t) = t - \max\{S_i : D_i \le t\}$$

Issue with AoI:

Some information varies **fast**

- E.g., car location, frequent update

Other information varies **slowly**

- E.g., temperature, infrequent update

Signal-aware sampling

Theorem:

The MSE-optimal sampling policy is

$$S_{i+1}(\beta) = \inf\Big\{n \ge D_i(\beta) : \underbrace{|X_n - \hat{X}_n|}_{\text{instantaneous estimation error}} \ge v(\beta)\Big\},$$

optimal β is determined by solving

$$\beta = \frac{\mathbb{E}\left[\int_{D_i(\beta)}^{D_{i+1}(\beta)} (X_n - \hat{X}_n)^2 dn\right]}{\mathbb{E}[D_{i+1}(\beta) - D_i(\beta)]} = \text{mse}_{\text{opt}}$$

β **= optimal objective value**

threshold function

$$v(\beta) = \begin{cases} \frac{\sigma}{\sqrt{\theta}} G^{-1}\left(\frac{\frac{\sigma^2}{2\theta} - \text{mse}_{Y_i}}{\frac{\sigma^2}{2\theta} - \beta}\right), & \text{if } \theta > 0, \\ \sqrt{3(\beta - \mathbb{E}[Y_i])}, & \text{if } \theta = 0, \\ \frac{\sigma}{\sqrt{-\theta}} K^{-1}\left(\frac{\frac{\sigma^2}{2\theta} - \text{mse}_{Y_i}}{\frac{\sigma^2}{2\theta} - \beta}\right), & \text{if } \theta < 0, \end{cases}$$

where
$$G(x) = \frac{e^{x^2}}{x}\int_0^x e^{-t^2} dt = \frac{e^{x^2}}{x}\frac{\sqrt{\pi}}{2}\text{erf}(x)$$

$$K(x) = \frac{e^{-x^2}}{x}\int_0^x e^{t^2} dt = \frac{e^{-x^2}}{x}\frac{\sqrt{\pi}}{2}\text{erfi}(x)$$

$$\text{mse}_{Y_i} = \begin{cases} \frac{\sigma^2}{2\theta}\mathbb{E}[1 - e^{-2\theta Y_i}], & \text{if } \theta \ne 0, \\ \sigma^2 \mathbb{E}[Y_i], & \text{if } \theta = 0, \end{cases}$$

$$\text{mse}_{\infty} = \frac{\sigma^2}{2\theta}$$

$$\text{mse}_{Y_i} \le \text{mse}_{\text{opt}} \le \beta \le \text{mse}_{\infty}$$

- Optimal threshold **changes** with **signal** structure.
- Optimal threshold is a **function** of the **optimal objective value** in every cases.

Conclusions

2. Signal-ignorant sampling:

- Threshold on **expected** estimation error;
- Threshold = optimal objective value

3. Optimal sampler follows a deterministic or randomized threshold

Sampling for minimizing est. error

$$\text{mse}_{\text{opt}} = \min_{\pi \in \Pi} \limsup_{T \to \infty} \frac{1}{T}\mathbb{E}\left[\int_0^T (X_n - \hat{X}_n)^2 dn\right]$$

- Sampling Policy: $\pi = (S_1, S_2, \ldots)$
- **Causal** policy space Π: Sampling time S_i decided by history of **signal** & **channel**

Connection between AoI and est. error

- If samples times are **independent** of signal
  $$\mathbb{E}\left[(X_t - \hat{X}_t)^2\right] = p(\Delta(t))$$
  (MSE = Age penalty function)
- However, if sampling times is determined based on **causal** knowledge of signal, MSE is **not** a function of age.
  - More knowledge, smaller MSE

Signal-ignorant Sampling

Theorem: (Age-Optimal Policy)

The MSE-optimal sampling policy is

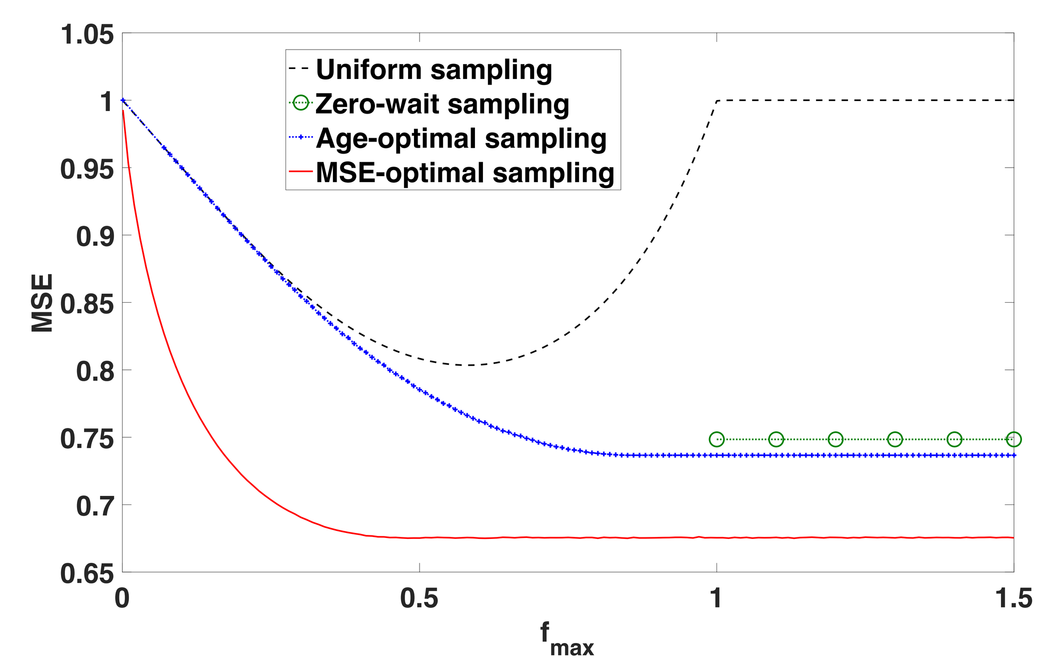
$$S_{i+1}(\beta) = \inf\{n \ge D_i(\beta) : \underbrace{\mathbb{E}[p(\Delta(n + Y_{i+1}))]}_{\text{expected age func.}} = \underbrace{\mathbb{E}[(X_{n+Y_{i+1}} - \hat{X}_{n+Y_{i+1}})^2]}_{\text{expected est. error}} \ge \beta\}$$

optimal β is determined by solving

$$\beta = \frac{\mathbb{E}\left[\int_{D_i(\beta)}^{D_{i+1}(\beta)} (X_t - \hat{X}_t)^2 dt\right]}{\mathbb{E}[D_{i+1}(\beta) - D_i(\beta)]} = \text{mse}_{\text{age-opt}}$$

β **= optimal objective value**

Numerical results



- MSE optimal policy performs better
- Simulation setup: *i.i.d.* exponential service times with $\mathbb{E}[Y_i] = 1$

References

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[2]. Tasmeen Zaman Ornee and Yin Sun, "Sampling for Remote Estimation through Queues: Age of Information and beyond", ***IEEE WiOpt***, 2019.

[3]. Tasmeen Zaman Ornee and Yin Sun, "Sampling and Remote Estimation for the Ornstein-Uhlenbeck Process through Queues: Age of Information and beyond", ***IEEE ToN***, 2021.

[4]. Tasmeen Zaman Ornee and Yin Sun, "Performance Bounds for Sampling and Remote Estimation of a Gauss-Markov Processes over a Noisy Channel with Random Delay", ***IEEE SPAWC***, 2021.