A Whittle Index Policy for the Remote Estimation of Gauss-Markov Processes over Multiple Channels

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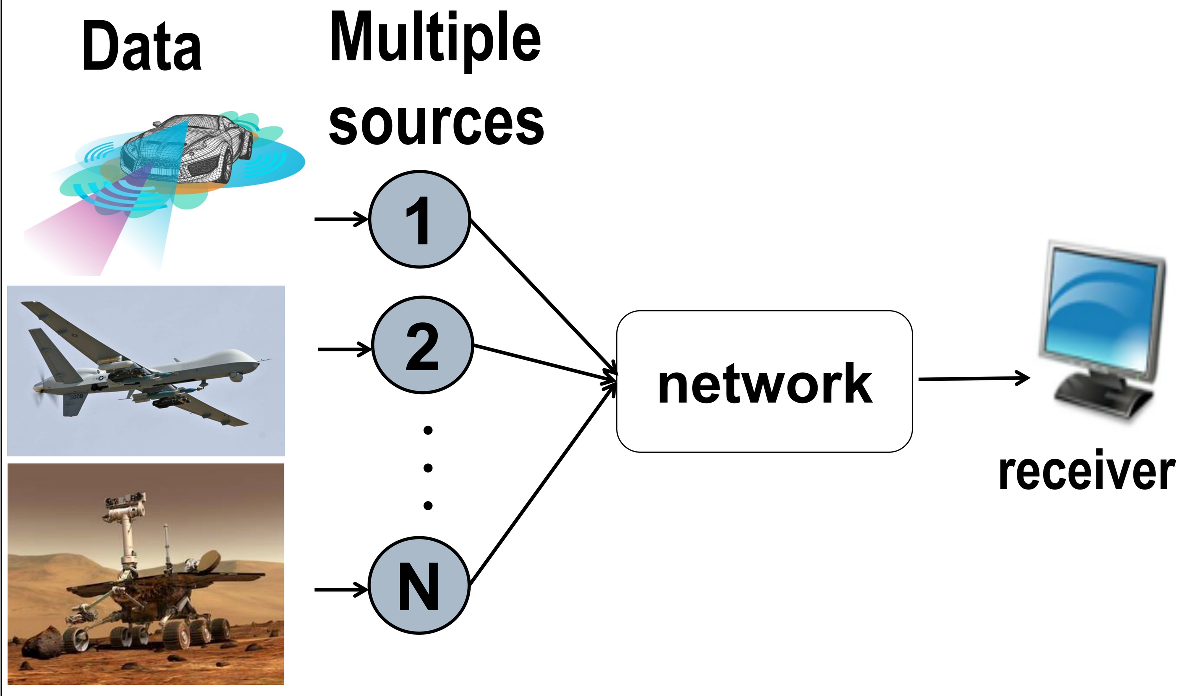
Fresh data in real-time applications



E.g., Sensor networks, IoT, CPS, robots

Is AoI a good freshness measure?

Age of Information (AoI), defined as

$$\Delta(t) = t - \max\{S_i : D_i \leq t\}$$

Issue with AoI:

Some information varies **fast**

- **E.g.,** car location, frequent update

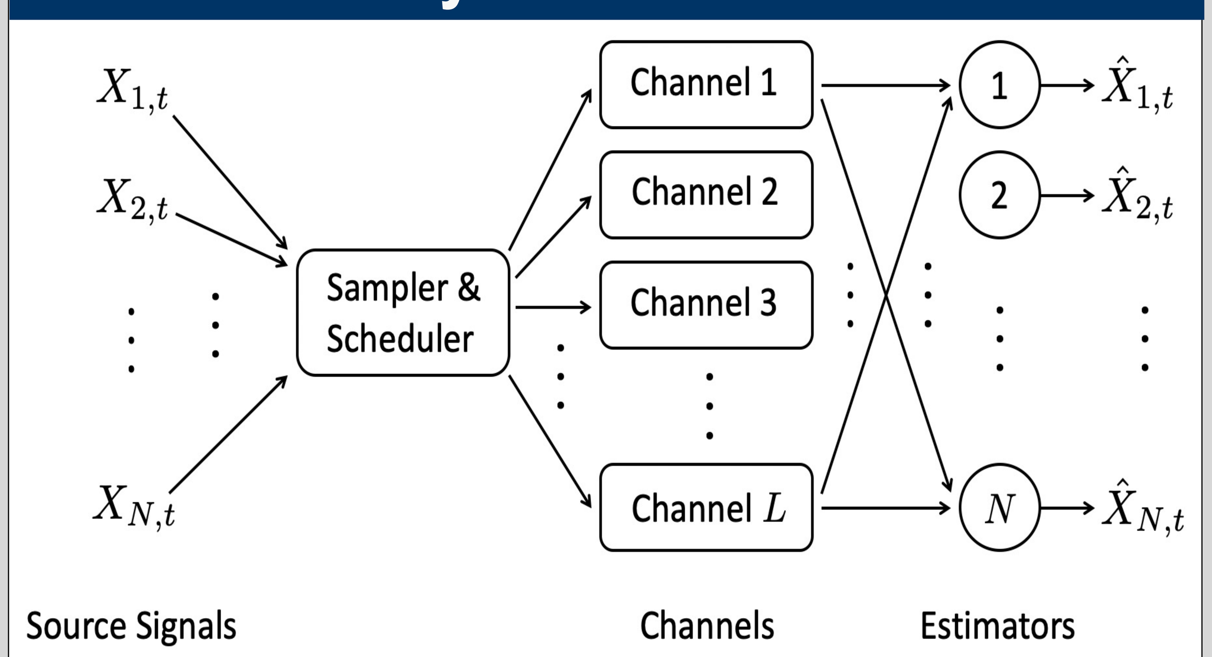
Other information varies **slowly**

- **E.g.,** temperature, infrequent update

Contributions

- For Remote Estimation of **multiple** and **multi-dimensional** Gauss-Markov processes with uncountable state space, we obtain
  - **indexability**
  - **analytical expression** of the Whittle index
- Unify **threshold-based** policy and **Whittle index** policy for single-source for signal-aware and signal-agnostic cases
- **Tool:** Free Boundary method for optimal stopping, Dynkin's formula

System Model



- N sources, L channels, $L < N$
- ***i.i.d.*** channel transmission times $Y_{n,i}$
- MMSE Estimator: Uses causally received samples estimate **real-time** value of $X_{n,t}$
- **Gauss-Markov** Signal Model:
  - **Wiener** process: Brownian motion
  - **Ornstein-Uhlenbeck (OU)** Process:
  $dX_{n,t} = \theta_n(\mu_n - X_{n,t})dt + \sigma_n dW_{n,t}$
  - Stable ($\theta_n > 0$), Unstable ($\theta_n < 0$)

Estimation Error Process

$$\varepsilon_n(t) = X_{n,t} - \hat{X}_{n,t} = \begin{cases} \frac{\sigma_n}{\sqrt{2\theta_n}} e^{-\theta_n(t-S_{n,i})} W_{n,e^{2\theta_n(t-S_{n,i})}-1}, & \text{if } \theta_n > 0, \\ \sigma_n (W_{n,t} - W_{S_{n,i}}), & \text{if } \theta_n = 0, \\ \frac{\sigma_n}{\sqrt{-2\theta_n}} e^{-\theta_n(t-S_{n,i})} W_{n,1-e^{2\theta_n(t-S_{n,i})}}, & \text{if } \theta_n < 0. \end{cases}$$

Estimation error process $\varepsilon_n(t)$ is same as a time-shifted OU process $O_{n,t-S_{n,i}}$

$$\varepsilon_n(t) = O_{n,t-S_{n,i}}$$

$S_{n,i}$ → **Scheduling/Sampling time for *i*-th sample and source *n***

Problem Formulation

$$\inf_{\pi \in \Pi} \limsup_{T\to\infty} \sum_{n=1}^{N} w_n \mathbb{E}_\pi \left[\frac{1}{T}\int_0^T \varepsilon_n(t)dt\right],$$

$$\text{s.t. } \sum_{n=1}^{N} c_n(t) \leq L, c_n(t) \in \{0,1\}, t \in [0,\infty)$$

Continuous-time MDP

- Scheduling Policy: $\pi = (S_1, S_2, \ldots)$
- **Causal** policy space: Π
- weight for source n: w_n
- Channel occupancy: $c_n(t)$ at time t

Restless Multi-arm Bandit : For $\lambda \geq 0$

$$\inf_{\pi_n \in \Pi_n} \limsup_{T\to\infty} \mathbb{E}_{\pi_n}\left[\frac{1}{T}\int_0^T (w_n \varepsilon_n^2(t) + \lambda c_n(t))dt\right]$$

Signal-ignorant Case

If scheduling times are **independent** of signal

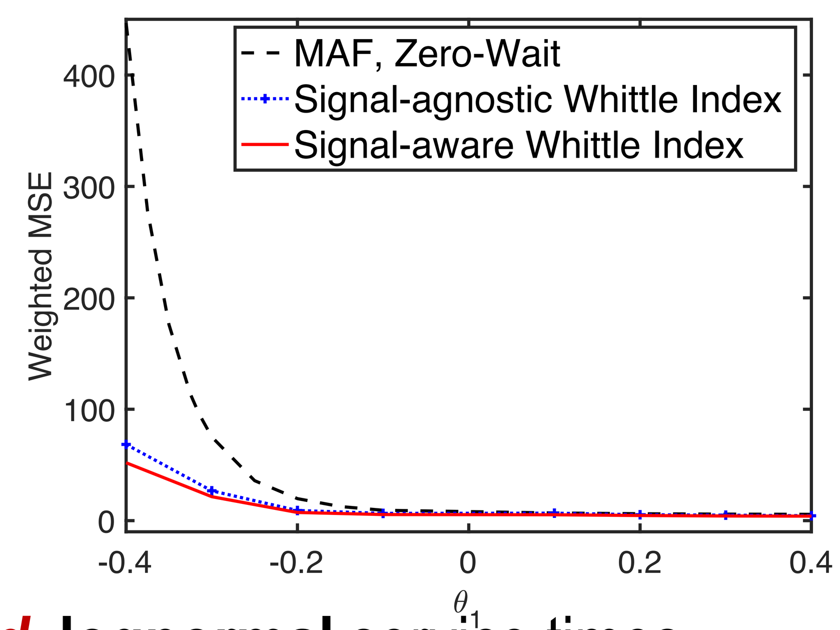
$$\mathbb{E}[\varepsilon_n^2(t)] = p_n(\Delta_n(t))$$

(MSE = Age penalty function)

Theorem: If Theorem 1 **holds**, and the scheduling times are **independent** of the signal, given $\Delta_n(t) = \delta$ the Whittle index is

$$W_n(\delta) = \frac{w_n}{\mathbb{E}[Y_{n,i}]}\left\{\mathbb{E}[D_{n,i+1}(\delta) - D_{n,i}(\delta)]\mathbb{E}[p_n(\delta + Y_{n,i+1})] - \mathbb{E}\left[\int_{D_{n,i}(\delta)}^{D_{n,i+1}(\delta)} p_n(s)ds\right]\right\}$$

Numerical Results



- ***i.i.d.*** **lognormal** service times
- Signal-aware policy performs **better** than other policies

Indexability and Whittle Index

Theorem 1: If the transmission times $Y_{n,i}$ are ***i.i.d.*** with $0 < \mathbb{E}[Y_{n,i}] < \infty$, then all arms are **indexable**

Theorem 2: If (i) Theorem 1 **holds**, (ii) the **Whittle index** at state $\varepsilon_n(t) = \varepsilon$ is given by

- Case 1: If $\theta_n > 0$, then

$$W_n(\varepsilon) = \frac{w_n}{\mathbb{E}[Y_{n,i}]}\left\{\mathbb{E}[D_{n,i+1}(\varepsilon)] - \mathbb{E}[D_{n,i}(\varepsilon)]\frac{\sigma_n^2}{2\theta_n}\left(1 - \frac{\mathbb{E}[e^{-2\theta_n Y_{n,i}}]}{G(\frac{\sqrt{\theta_n}}{\sigma_n}|\varepsilon|)}\right) - \mathbb{E}\left[\int_{D_{n,i}(\varepsilon)}^{D_{n,i+1}(\varepsilon)} \varepsilon_n^2(s)ds\right]\right\},$$

- Case 2: If $\theta_n = 0$, then

$$W_n(\varepsilon) = \frac{w_n}{\mathbb{E}[Y_{n,i}]}\left\{\mathbb{E}[D_{n,i+1}(\varepsilon)] - \mathbb{E}[D_{n,i}(\varepsilon)]\left(\frac{\varepsilon^2}{3} + \sigma_n^2\mathbb{E}[Y_{n,i}]\right) - \mathbb{E}\left[\int_{D_{n,i}(\varepsilon)}^{D_{n,i+1}(\varepsilon)} \varepsilon_n^2(s)ds\right]\right\},$$

- Case 3: If $\theta_n < 0$, then

$$W_n(\varepsilon) = \frac{w_n}{\mathbb{E}[Y_{n,i}]}\left\{\mathbb{E}[D_{n,i+1}(\varepsilon)] - \mathbb{E}[D_{n,i}(\varepsilon)]\frac{\sigma_n^2}{2\theta_n}\left(1 - \frac{\mathbb{E}[e^{-2\theta_n Y_{n,i}}]}{K(\frac{\sqrt{-\theta_n}}{\sigma_n}|\varepsilon|)}\right) - \mathbb{E}\left[\int_{D_{n,i}(\varepsilon)}^{D_{n,i+1}(\varepsilon)} \varepsilon_n^2(s)ds\right]\right\},$$

where

$D_{n,i}$ → **delivery time of *i*-th sample**

$$G(x) = \frac{e^{x^2}}{x}\int_0^x e^{-t^2}dt = \frac{e^{x^2}}{x}\frac{\sqrt{\pi}}{2}\text{erf}(x),$$

$$K(x) = \frac{e^{-x^2}}{x}\int_0^x e^{t^2}dt = \frac{e^{-x^2}}{x}\frac{\sqrt{\pi}}{2}\text{erfi}(x),$$

$$\forall x \in [0,\infty)$$

- Whittle index **changes** with **signal** structure

Evaluation of the Expectations

In **Theorem 2**, it holds that

$$\mathbb{E}[D_{n,i+1}(\varepsilon) - D_{n,i}(\varepsilon)] = \mathbb{E}[R_{n,1}(\max\{|\varepsilon|, |O_{n,Y_{n,i}}|\})],$$

$$\mathbb{E}\left[\int_{D_{n,i}(\varepsilon)}^{D_{n,i+1}(\varepsilon)} \varepsilon_n^2(s)ds\right] = \mathbb{E}[R_{n,2}(\max\{|\varepsilon|, |O_{n,Y_{n,i}}|\} + O_{n,Y_{n,i+1}})] - \mathbb{E}[R_{n,2}(O_{n,Y_{n,i}})],$$

- Associated with $Y_{n,i}$ and $O_{n,Y_{n,i}}$
- Monte Carlo evaluation becomes **easier**
- **Tool**: Dynkin's formula
- $R_1(\cdot), R_2(\cdot)$ are generalized **hyper-geometric** functions