

# Fluctuations in Inflation

Cosmological Perturbation Theory

January 28, 2026

---

## Abstract

**Note on Units:** Throughout this section, we work in reduced Planck units, where  $M_P = 1$ .

## 1 Introduction

We analyze the decomposition of perturbations of the homogeneous metric into scalar, vector, and tensor categories.

**Vector perturbations** are governed by a constraint equation relating the gauge-invariant vector metric perturbation to the divergence-free velocity of the fluid. In the presence of scalar fields, this velocity vanishes, and vector modes decay rapidly. Therefore, we focus exclusively on **scalar** and **tensor** perturbations.

## 2 Scalar Perturbations

### 2.1 Multi-Field Inflation Action

We consider the general action  $S$  for  $n$  scalar fields  $\varphi_i$  (where  $i = 1, \dots, n$ ):

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} G_{ij} \partial_\mu \varphi^i \partial^\mu \varphi^j - V(\varphi) \right] \quad (1)$$

Here,  $V$  is the scalar potential and  $G_{ij}$  is the field space metric. We analyze linear perturbations around a homogeneous unperturbed Universe. The field and metric are decomposed as:

$$\varphi_i(t, \vec{x}) = \varphi_i(t) + \delta\varphi_i(t, \vec{x}) \quad (2)$$

$$g_{\mu\nu}(t, \vec{x}) = \bar{g}_{\mu\nu}(t) + h_{\mu\nu}(t, \vec{x}) \quad (3)$$

### 2.2 Background Equations

The unperturbed background equations of motion for the  $n$  scalar fields are:

$$\ddot{\varphi}^i + \Gamma_{jk}^i \dot{\varphi}^j \dot{\varphi}^k + 3H\dot{\varphi}^i + G^{ij}V_{,j} = 0 \quad (4)$$

The Friedmann equations governing the expansion rate  $H$  are:

$$H^2 = \frac{1}{3} \left[ \frac{1}{2} G_{ij} \dot{\varphi}^i \dot{\varphi}^j + V \right], \quad \dot{H} = -\frac{1}{2} G_{ij} \dot{\varphi}^i \dot{\varphi}^j \quad (5)$$

## 2.3 The Newtonian Gauge

We adopt the **Newtonian gauge**, where the scalar metric perturbations  $B$  and  $E$  vanish. Assuming no anisotropic stress ( $\Phi = \Psi$ ), the perturbed line element is:

$$ds^2 = -(1 + 2\Psi)dt^2 + a^2(1 - 2\Psi)\delta_{ij}dx^i dx^j \quad (6)$$

where  $\Psi$  represents the Bardeen potential.

### 2.3.1 Evolution in e-fold Time

It is often convenient to work with the number of e-folds,  $N$ , rather than cosmic time  $t$ , defined by  $dN = Hdt$ . The field equations transform as:

$$\frac{d^2\delta\varphi^i}{dN^2} + (3 - \epsilon)\frac{d\delta\varphi^i}{dN} + \frac{1}{H^2}G^{ij}V_{,j} = 0 \quad (7)$$

where  $\epsilon = -\dot{H}/H^2$ .

## 2.4 Mukhanov-Sasaki Equation

To solve for the perturbations, we utilize the gauge-invariant **Mukhanov-Sasaki variable**  $Q^i$ , which combines field and metric perturbations:

$$Q^i = \delta\varphi^i + \frac{\dot{\varphi}^i}{H}\Psi \quad (8)$$

For a single field, the mode equation for  $u_k = -aQ_k$  is:

$$u_k'' + \left(k^2 - \frac{z''}{z}\right)u_k = 0 \quad (9)$$

where primes denote derivatives with respect to conformal time  $\tau$ , and  $z = a\dot{\varphi}/H$ .

#### Initial Conditions: Bunch-Davies Vacuum

We assume the fields originate in the **Bunch-Davies vacuum**. In the remote past, modes were deep inside the horizon ( $k \gg aH$ ). The initial condition for the mode function  $u_k$  is:

$$u_k(\tau) \rightarrow \frac{e^{-ik\tau}}{\sqrt{2k}} \quad \text{as } \tau \rightarrow -\infty \quad (10)$$

## 3 Power Spectrum

The primary observable is the power spectrum of the comoving curvature perturbation,  $\mathcal{R}$ . The dimensionless power spectrum  $P_{\mathcal{R}}(k)$  is defined by:

$$\langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'}^* \rangle = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') \frac{2\pi^2}{k^3} P_{\mathcal{R}}(k) \quad (11)$$

Analytically,

$$P_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2} |\mathcal{R}_k|^2 \quad (12)$$

where  $\mathcal{R}_k$  is evaluated after the mode exits the horizon ( $k \ll aH$ ). In the slow-roll approximation, this simplifies to:

$$P_{\mathcal{R}} \approx \frac{H^2}{8\pi^2 \epsilon} \Big|_{k=aH} \quad (13)$$

## 4 Numerical Procedure for Scalar Power Spectrum

While slow-roll approximations are useful, multi-field models often require exact numerical integration. The procedure to derive the exact power spectrum for  $n$  scalar fields is as follows:

1. **Setup:** Define the number of scalar fields,  $n$ , and the field metric  $G_{ij}$ . If  $n > 1$ , diagonalize the field metric matrix if necessary.
2. **Initial Conditions (Background):** Set the initial field values. Calculate initial velocities assuming the attractor solution:

$$\dot{\varphi}^i \Big|_{ic} = -\frac{V_i}{3H} \Big|_{ic} \quad (14)$$

3. **Background Evolution:** Solve the coupled background equations (Eq. 4) numerically until the end of inflation, defined by the condition  $\epsilon_H = 1$ .
4. **Mode Definition:** Select the comoving wavenumber  $k$  of interest. It is defined relative to the horizon scale:

$$k = C \cdot (aH)_{ic} \quad (15)$$

where we choose  $C \gg 1$  to ensure the mode starts deep inside the horizon (sub-horizon).

5. **Perturbation Integration:** Solve the system of background equations and perturbation equations simultaneously.
  - Use the **Bunch-Davies** conditions (Eq. 24 in previous notes) to set initial values for  $\delta\varphi$  and  $\Psi$ .
  - Integrate until the mode is well outside the horizon ( $k \ll aH$ ) and the solution for the curvature perturbation freezes out.
6. **Evaluate Observables:** Calculate the curvature perturbation  $\mathcal{R}$  and isocurvature modes  $\mathcal{S}$  at the end of the integration. Compute the power spectrum using:

$$P_{\mathcal{R}} = \frac{k^3}{2\pi^2} |\mathcal{R}_k|^2 \quad (16)$$