Fluctuations in Inflation

Cosmological Perturbation Theory

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Abstract

Note on Units: Throughout this section, we work in reduced Planck units, where $M_P = 1$.

1 Introduction to Fluctuation in Cosmology

The introduction of cosmic inflation not only gives answers to problems such as flatness problem and horizon problem but also it can explain the origin of cosmological perturbation.

The cosmological perturbation can be classified in two categories the adiabatic perturbation and the entropy perturbation. The adiabatic perturbation refers to fluctuation along the same trajectory in phase space. These perturbations are described from the equation for energy density and pressure by the following equation:

$$\frac{\delta\rho}{\dot{\bar{\rho}}} = \frac{\delta p}{\dot{\bar{p}}} \tag{1}$$

where the perturbation is given from the scalar quantity x as $\delta x = \dot{\bar{x}}\delta t$.

The entropy perturbation perturb the solution of the background. The entropy perturbation is defined as

$$S = H\left(\frac{\delta p}{\dot{p}} - \frac{\delta\rho}{\rho}\right) \tag{2}$$

and the entropy perturbation between two quantities are given:

$$S_{xy} = H\left(\frac{\delta x}{\dot{x}} - \frac{\delta y}{\dot{y}}\right) \tag{3}$$

An other classification of cosmological perturbation can be give by fluctuation of the homogeneous FRW metric. Generally, the perturbed metric is given:

$$ds^2 = (1+2\Phi)dt^2 - 2a(t)B_{,i}dx^i dt - a^2[(1-2\Psi)\delta_{ij} + 2E_{ij}]dx^i dx^j \tag{4}$$

where Φ is the 3-scalar and it called lapse function, B_i is the 3-vector which is called shifted function, Ψ is the 3 scalar which corresponds to curvature perturbation, E_{ij} is the symmetric and traceless shear 3-tensor. The 10 degree of freedom, which is associated with the perturbed metric, can be decomposed as follows:

- 4 scalar components, Φ, B, Ψ and E which corresponds to 4 degree of freedom,
- 2 vector components B_i and E_i which corresponds to $2 \times (3-1) = 4$ degree of freedom,
- 1 tensor component E_{ij} which corresponds to $3 \times 2 - 1 - 3 = 2$ degree of freedom.

2 Fluctuation in Inflation

In this section we analyze the previously discussed decomposition of the perturbations of homogeneous metric into the categories: scalar, vector and tensor perturbations. For vector perturbations there is only a constraint equation which relates the gauge-invariant vector metric perturbation to the divergence-free velocity of the fluid, which vanishes in the presence of scalar fields. For this reason we discuss only scalar and tensor perturbations. In this section we work in mass Planck units.

2.1 Scalar Perturbations

2.1.1 Evaluating perturbations in multi field inflation

We consider the general form of action S for n scalar field φ_i, $(i = 1, n)$.

$$S = \int d^4x\sqrt{-|g|}\left[\frac{1}{2}G_{ij}\partial_\mu\varphi^i\partial^\mu\varphi^j - V\right] \tag{5}$$

where V is an arbitrary scalar potential and G_{ij} is the field metric. We consider the linear perturbation of a homogeneous unperturbed Universe. Specifically, we assume that the field ϕ_i have the following perturbations:

$$\varphi_i(t, \vec{x}) = \varphi_i(t) + \delta\varphi_i(t, \vec{x}) \tag{6}$$

and for the perturbed metric:

$$g_{\mu\nu}(t, \vec{x}) = g_{\mu\nu}(t) + h_{\mu\nu}(t, \vec{x}) \tag{7}$$

The unperturbed background equations of motion for n scalar fields are written by the following equation:

$$\ddot{\varphi}^i + \Gamma^i_{ij}\dot{\varphi}^i + 3H\dot{\varphi}^i + G^{ij}V_{,j} = 0 \tag{8}$$

which is equal to Eq. (**??**) in case of $n = 1$. For the Friedmann equation, defined in Eq. (**??**), we have:

$$H^2 = \frac{1}{3}\left[\frac{1}{2}G_{ij}\dot{\varphi}^i\dot{\varphi}^j + V\right] \tag{9}$$

and

$$\dot{H} = -\frac{1}{2}G_{ij}\dot{\varphi}^i\dot{\varphi}^j. \tag{10}$$

We consider the perturbed metric for scalar perturbation, which is given in Eq. (4) in the Newtonian gauge. In this gauge the scalar component B and E vanish. Therefore we have the following perturbed metric:

$$ds^2 = (1 + \Psi)dt^2 - a^2(1 - 2\Psi)dx^2 \tag{11}$$

where Φ and Ψ are the Bardeen potentials and we consider that $\Phi = \Psi$.

In the following we choose to work in efold time, which is related to the cosmic time by the equation:

$$N(t) = \int H(t)dt \tag{12}$$

and the derivative of the field in respect to the number of efold is given as:

$$\frac{d\varphi}{dt} = H\frac{d\varphi}{dN} \tag{13}$$

The Eq. (8) in respect to the efold is given as follows:

$$\boxed{\varphi^{i\prime\prime} - \left(\frac{1}{2}G_{ij}\varphi^{i\prime}\varphi^{j\prime} - 3\right)\varphi^{k\prime} + \Gamma^i_{jk}\varphi^{j\prime}\varphi^{k\prime} = -G^{ij}\frac{V_{,j}}{V}\left(3 - \frac{1}{2}G_{ij}\varphi^{i\prime}\varphi^{j\prime}\right)} \tag{14}$$

where primes denote parameter in e-fold time. The Hubble parameter is given by the expression:

$$H^2 = \frac{V}{3 - \frac{1}{2}\sigma^2} \tag{15}$$

where the adiabatic field is introduced as follows:

$$\sigma = \sqrt{\frac{1}{2}G_{ij}\varphi^{i\prime}\varphi^{j\prime}} \tag{16}$$

which describes the evolution of all the fields along the classical trajectory. Applying the Eq. (6) in Eq. (14) we get for the perturbed field the following equation:

$$\boxed{\begin{aligned}\delta\varphi^{i\prime\prime} + \left(3 - \frac{1}{2}G_{kj}\varphi^{k\prime}\varphi^{j\prime}\right)\delta\varphi^{i\prime} + 2\Gamma^i_{jk}\varphi^{i\prime}\delta\varphi^{k\prime} + \left(\frac{k}{a^2H^2}\right)^2\delta\varphi^i + \\ + \left(\frac{(G^{ij}V_{,j})_{,k}}{H^2} + \Gamma^i_{jl,k}\varphi^{l\prime}\varphi^j\right)\delta\varphi^k = 4\varphi^{i\prime}\Psi' + 2G^{ik}V_{,k}\Psi\end{aligned}} \tag{17}$$

In order to find how the Bardeen potential Ψ varies, we consider the constraints from Einstein equations. Hence we can find the following equation:

$$\Psi'' + \left(7 - \frac{1}{2}G_{ij}\varphi^{i\prime}\varphi^{j\prime}\right)\Psi' = \left(\frac{k^2}{a^2H^2} - \frac{2V}{H^2}\right)\Psi - \frac{V_{,k}}{H^2}\delta\varphi^k. \tag{18}$$

With k we denote the comoving wavenumber and the Eqs. (17) and (18) is written in Fourier space.

The equations for the perturbation of the field can be written in respect to the gauge invariant Mukhanov-Sassaki variable Q. This quantity is defined as follows:

$$Q^i = \delta\varphi^i + \varphi^{i\prime}\Psi \tag{19}$$

In this new variable the Eqs. (17) and (18) take the form:

$$\begin{aligned}Q^{i\prime\prime} + \left(3 - \frac{1}{2}G_{kj}\varphi^{k\prime}\varphi^{j\prime}\right)Q^{i\prime} + 2\Gamma i_{jk}\varphi^{j\prime}Q^{k\prime} + \frac{k^2}{a^2H^2}Q^i + \left(\frac{(G^{ij}V_{,j})_{,k}}{H^2} + \Gamma^i_{jl,k}\varphi^{j\prime}\varphi^{l\prime}\right)Q^{k\prime} \\ + 3G_{kj}\varphi^{i\prime}\varphi^{j\prime}Q^k - \frac{1}{2}G_{jk}G_{lm}\varphi^{i\prime}\varphi^{j\prime}\varphi^{l\prime}\varphi^{m\prime}Q^k + G^{ij}G_{lk}\frac{\varphi^{l\prime}}{H^2}V_{,j}Q^k + \varphi^{i\prime}\frac{V_{,k}}{H^2}Q^k = 0\end{aligned} \tag{20}$$

and in case of a single scalar field we reduce the above equation in the Mukhanov-Sassaki equation:

$$u'' + \left(1 - \frac{1}{2}\varphi'\right)u' + u\left(\frac{k^2}{a^2H^2} - 2 + \frac{V_{,\varphi\varphi}}{H^2} + 3\varphi'^2 - \frac{1}{2}\varphi'^4 + 2\varphi'\frac{V_{,\varphi}}{H^2}\right) = 0 \tag{21}$$

which it reduces to the following form:

$$u'' + (1 - \epsilon_H)u' + \left[\frac{k^2}{\mathcal{H}^2} + (1 + \epsilon_H - \eta_H)(\eta_H - 2) - \frac{d(\epsilon_H - \eta_H)}{dN}\right]u_k = 0 \tag{22}$$

where $u = -aQ$.

In order to evaluate the fields perturbation (17) and (18) we assume that we are initially in Bunch-Davies vacuum. This vacuum is defined as follows: At sufficient early times all modes of interest were deep inside the horizon ($k \gg aH$). This means that in remote past all observable modes have time independent frequencies. Therefore, the Mukhanov-Sassaki in Minkowski space is:

$$\frac{d^2 u_k}{d\tau^2} + k^2 u_k = 0$$

where the solution is $u_k \propto e^{\pm ik\tau}$. In particular, the initial conditions are taken in subhorizon scales, where $k \gg aH$:

$$aQ \to \frac{e^{-ik\tau}}{\sqrt{2k}}. \tag{23}$$

The limit Eq. (23) comes from the Mukhanov-Sassaki equation (21), when $\tau \to \infty$ and defines the Bunch-Davies vacuum. Therefore, the initial conditions are given for the perturbed fields by the equations:

$$\begin{aligned} \delta\varphi_{ic} &= \frac{1}{\sqrt{2kG_{aa}(n_{ic})}} \\ \delta\varphi'_{ic} &= -\frac{1}{\sqrt{2kG_{aa}(n_{ic})}}\left(1 + i\frac{k}{a_{ic}H_{ic}}\right) \end{aligned} \tag{24}$$

and for the Bardeen potential for the expressions bellow:

$$\begin{aligned} \Psi_{ic} &= \frac{1}{2\left(\varepsilon_{H,ic} - \frac{k^2}{a_{ic}^2 H_{ic}^2}\right)}\left(\left(\frac{d\chi}{dN}\right)_{ic}\left(\frac{d\delta\chi}{dN}\right)_{ic} + \delta\chi_{ic}\left[3\left(\frac{d\chi}{dN}\right)_{ic} + \frac{1}{H_{ic}^2}\left(\frac{dV}{d\chi}\right)_{ic}\right]\right) \\ \left(\frac{d\Psi}{dN}\right)_{ic} &= \frac{1}{2}\left(\frac{d\chi}{dN}\right)_{ic}\delta\chi_{ic} - \Psi_{ic} \end{aligned} \tag{25}$$

where with the subscription ic, we denote the initial condition for each quantity.

2.1.2 Power Spectrum

The quantities which we calculate and compare with the observable constraints is the expectation values or in other words the power spectrum. In general, with the term of power spectrum we denote the two point correlation function in the quantum state, where the scalar fields is put by the cosmological evolution. For an arbitrary variable X in space, we define the power spectrum in Fourier space for the form:

$$\langle X(t_0,\ \mathbf{k})X^*(t_0,\ \mathbf{k}')\rangle = (2\pi)^3\delta(\ \mathbf{k}\ -\ \mathbf{k}')P_X(k) \tag{26}$$

where the Dirac delta function impose the independence of different mode.

The matter power spectrum can be evaluated by using the variance of fluctuation, which in homogeneous background is pictured by the comoving curvature perturbation R. So the primordial power spectrum is measured by:

$$\left\langle \hat{R}_{\mathbf{k}}\hat{R}_{\mathbf{k}'}\right\rangle = (2\pi)^3\delta^3(k + k')(2\pi^2/k^3)P_R(k) \tag{27}$$

where $\hat{R}$ is the 3 dimensional Fourier transform of $R(\mathbf{k})$. By solving the equations of perturbation, described in the previous section, we can calculate the dimensionless power spectrum of co-moving perturbation from:

$$P_R = \frac{k^3}{2\pi^2} |R_k|^2, \tag{28}$$

which reaches a constant value after Hubble crossing, $k \ll aH$. The co-moving wavenumber k is associated with the Mukhanov-Sassaki variable Q:

$$R_k = Q \frac{dN}{d\chi}. \tag{29}$$

The curvature perturbation ζ is releated to the curvature perturbation R by the expression:

$$-\zeta = R + \frac{2\rho}{9(\rho + p)} \left(\frac{k}{aH} \right) \Psi \tag{30}$$

and in large scale $R = -\zeta$.

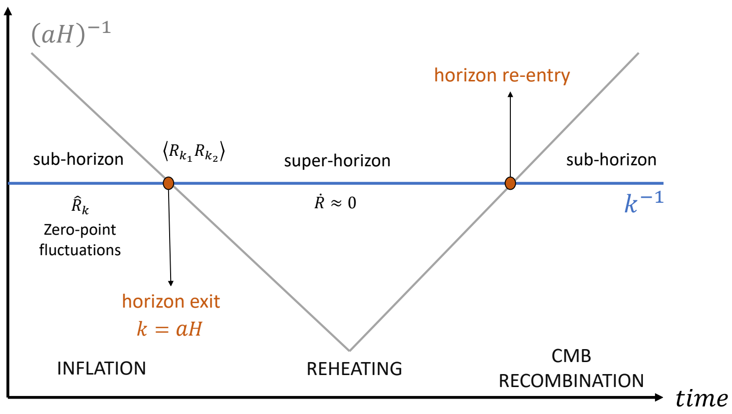

Figure 1: How the curvature perturbations evolve.

In Fig. 1 we show the evolution of the curvature perturbations during and after inflation. The comoving wavenumber exit the horizon at early times and re-enter at later times before CMB recombination. At the horizon crossing the comoving wavenumber take the value $k = aH$. The curvature perturbations does not evolve outside the horizon, $R \simeq 0$. In this figure we can also notice that the horizon $(aH)^{-1}$ shrinks during the inflation and increases after the reheating.

If the slow-roll approximation is valid, one can derive the scalar power spectrum without the evaluation of the fields perturbation. Specifically, in slow-roll approximation the power spectrum is given:

$$P_R = \frac{H^2}{4\pi^2} \left(\sum_i \varphi^{i\prime 2} \right). \tag{31}$$

Hence one need to solve only the background equations. However, the slow-roll approximation can be violated and hence the exact numerical procedure for the evaluation of power spectrum is necessary.

The source of perturbations comes from many models only for modes of curvature perturbation. However, in multi field inflation we need also to evaluate the isocurvature perturbation modes. The isocurvature power spectrum is defined by the isocurvature perturbation modes, or in other words the entropy modes, which is given as follows:

$$S = \frac{Q_\sigma}{\sigma'} \tag{32}$$

In case of single field inflation the entropy modes are suppressed. However in the study of multi-field inflation the contribution of entropy modes affect the adiabatic perturbation in super-horizon evolution.

3 Numerical Procedure For Scalar Power Spectrum

Although the slow-roll approximation provides us with easily evaluated result for the scalar power spectrum, it is imperative in such inflationary models the exact evaluation of fields perturbation. In this subsection we describe how one can derive numerically the exact power spectrum for n scalar fields with a general field metric. The algorithm is given as follows:

- We define the number of the scalar fields, n, and the field metric, $G_{i,j}$. We diagonize the matrix of the field metric in case of a general $n \times n$ matrix with $n > 1$.
- We define the intial condition for the fields. The intial velocities are evaluated assuming the attractor:

$$\varphi^{i\prime}\Big|_{ic} = -\frac{d \ln V(\varphi^i)}{d\varphi^i}\Big|_{ic}. \tag{33}$$

- We solve numerically the background equations for n fields from Eq. (14) until the end of inflation, $\varepsilon_H = 1$.
- We define the comoving wavenumber k by the expression:

$$k = C\mathcal{H} \tag{34}$$

  where $\mathcal{H} = aH$. We assume $C \gg 1$ in order to be initially in subhorizon mode.
- We solve the background equations from Eq. (14), the equations of the fields perturbations from Eq. (17) and the equation of Bardeen potential from Eq. (18) simultaneously for each k mode of interest. For the initial conditions of the Eqs. (17) and (18) we use the Eqs. (24) and (25) accordingly. The integration stops when the solution converges.
- We evaluate the quantities R, S and ζ for each k mode at the point when the integration stops. For the scalar power spectrum we can use the expression (28).