Fluctuations in Inflation

Cosmological Perturbation Theory

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Abstract

Note on Units: Throughout this section, we work in reduced Planck units, where $M_P = 1$.

1 Introduction

We analyze the decomposition of perturbations of the homogeneous metric into scalar, vector, and tensor categories.

Vector perturbations are governed by a constraint equation relating the gauge-invariant vector metric perturbation to the divergence-free velocity of the fluid. In the presence of scalar fields, this velocity vanishes, and vector modes decay rapidly. Therefore, we focus exclusively on **scalar** and **tensor** perturbations.

2 Scalar Perturbations

2.1 Multi-Field Inflation Action

We consider the general action S for n scalar fields φ_i (where $i = 1, \dots, n$):

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}G_{ij}\partial_\mu\varphi^i\partial^\mu\varphi^j - V(\varphi)\right] \tag{1}$$

Here, V is the scalar potential and G_{ij} is the field space metric. We analyze linear perturbations around a homogeneous unperturbed Universe. The field and metric are decomposed as:

$$\varphi_i(t,\vec{x}) = \varphi_i(t) + \delta\varphi_i(t,\vec{x}) \tag{2}$$

$$g_{\mu\nu}(t,\vec{x}) = \bar{g}_{\mu\nu}(t) + h_{\mu\nu}(t,\vec{x}) \tag{3}$$

2.2 Background Equations

The unperturbed background equations of motion for the n scalar fields are:

$$\ddot{\varphi}^i + \Gamma^i_{jk}\dot{\varphi}^j\dot{\varphi}^k + 3H\dot{\varphi}^i + G^{ij}V_{,j} = 0 \tag{4}$$

The Friedmann equations governing the expansion rate H are:

$$H^2 = \frac{1}{3}\left[\frac{1}{2}G_{ij}\dot{\varphi}^i\dot{\varphi}^j + V\right], \qquad \dot{H} = -\frac{1}{2}G_{ij}\dot{\varphi}^i\dot{\varphi}^j \tag{5}$$

2.3 The Newtonian Gauge

We adopt the **Newtonian gauge**, where the scalar metric perturbations B and E vanish. Assuming no anisotropic stress ($\Phi = \Psi$), the perturbed line element is:

$$ds^2 = -(1+2\Psi)dt^2 + a^2(1-2\Psi)\delta_{ij}dx^i dx^j \tag{6}$$

where Ψ represents the Bardeen potential.

2.3.1 Evolution in e-fold Time

It is often convenient to work with the number of e-folds, N, rather than cosmic time t, defined by $dN = Hdt$. The field equations transform as:

$$\frac{d^2\varphi^i}{dN^2} + (3-\epsilon)\frac{d\varphi^i}{dN} + \frac{1}{H^2}G^{ij}V_{,j} = 0 \tag{7}$$

where $\epsilon = -\dot{H}/H^2$.

2.4 Mukhanov-Sasaki Equation

To solve for the perturbations, we utilize the gauge-invariant **Mukhanov-Sasaki variable** Q^i, which combines field and metric perturbations:

$$Q^i = \delta\varphi^i + \frac{\dot{\varphi}^i}{H}\Psi \tag{8}$$

For a single field, the mode equation for $u_k = -aQ_k$ is:

$$u_k'' + \left(k^2 - \frac{z''}{z}\right)u_k = 0 \tag{9}$$

where primes denote derivatives with respect to conformal time τ, and $z = a\dot{\varphi}/H$.

Initial Conditions: Bunch-Davies Vacuum

We assume the fields originate in the **Bunch-Davies vacuum**. In the remote past, modes were deep inside the horizon ($k \gg aH$). The initial condition for the mode function u_k is:

$$u_k(\tau) \to \frac{e^{-ik\tau}}{\sqrt{2k}} \quad \text{as} \quad \tau \to -\infty \tag{10}$$

3 Power Spectrum

The primary observable is the power spectrum of the comoving curvature perturbation, $\mathcal{R}$. The dimensionless power spectrum $P_{\mathcal{R}}(k)$ is defined by:

$$\langle \mathcal{R}_{\mathbf{k}}\mathcal{R}^*_{\mathbf{k}'}\rangle = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')\frac{2\pi^2}{k^3}P_{\mathcal{R}}(k) \tag{11}$$

Analytically,

$$P_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2}|\mathcal{R}_k|^2 \tag{12}$$

where $\mathcal{R}_k$ is evaluated after the mode exits the horizon ($k \ll aH$). In the slow-roll approximation, this simplifies to:

$$P_{\mathcal{R}} \approx \left.\frac{H^2}{8\pi^2\epsilon}\right|_{k=aH} \tag{13}$$

4 Numerical Procedure for Scalar Power Spectrum

While slow-roll approximations are useful, multi-field models often require exact numerical integration. The procedure to derive the exact power spectrum for n scalar fields is as follows:

1. **Setup:** Define the number of scalar fields, n, and the field metric G_{ij}. If $n > 1$, diagonalize the field metric matrix if necessary.

2. **Initial Conditions (Background):** Set the initial field values. Calculate initial velocities assuming the attractor solution:
$$\dot{\varphi}^i\Big|_{ic} = -\frac{V_{,i}}{3H}\Big|_{ic} \tag{14}$$

3. **Background Evolution:** Solve the coupled background equations (Eq. 4) numerically until the end of inflation, defined by the condition $\epsilon_H = 1$.

4. **Mode Definition:** Select the comoving wavenumber k of interest. It is defined relative to the horizon scale:
$$k = C \cdot (aH)_{ic} \tag{15}$$
where we choose $C \gg 1$ to ensure the mode starts deep inside the horizon (sub-horizon).

5. **Perturbation Integration:** Solve the system of background equations and perturbation equations simultaneously.
   - Use the **Bunch-Davies** conditions (Eq. 24 in previous notes) to set initial values for $\delta\varphi$ and Ψ.
   - Integrate until the mode is well outside the horizon ($k \ll aH$) and the solution for the curvature perturbation freezes out.

6. **Evaluate Observables:** Calculate the curvature perturbation $\mathcal{R}$ and isocurvature modes $\mathcal{S}$ at the end of the integration. Compute the power spectrum using:
$$P_{\mathcal{R}} = \frac{k^3}{2\pi^2}|\mathcal{R}_k|^2 \tag{16}$$